Richard J. Nowakowski, Bruce M. Landman, Florian Luca, Melvyn B. Nathanson,
Jaroslav Nešetřil, and Aaron Robertson (Eds.)
Combinatorial Game Theory

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Combinatorial Game Theory

A Special Collection in Honor of Elwyn Berlekamp, John H. Conway and Richard K. Guy

Edited by
Richard J. Nowakowski, Bruce M. Landman, Florian Luca, Melvyn B. Nathanson, Jaroslav Nešetřil, and Aaron Robertson

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Preface

What is $1 + 1 + 1$?
John H. Conway, 1973

Individually, each of Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy have received much, rightly deserved, praise. Each made lasting contributions to many areas of mathematics. This volume is dedicated to their work in combinatorial game theory. It is due to their efforts that combinatorial game theory exists as a subject.

Brief History of how *Winning Ways* came to be

Bouton first analyzed NIM [67], little realizing how central NIM was to be. In the next two decades, other researchers contributed the analysis of a few other, specific games. The chess champion Emanuel Lasker came close to a complete theory of impartial games. It was in the 1930s that Grundy [68] and Sprague [72] gave a complete analysis, now known as the Sprague–Grundy theory. Despite being an elegant theory and easy to apply, the subject languished because there was no clear direction in which to develop the theory. In the late 1940s, Richard K. Guy rediscovered the theory and defined the *octal* games. In 1956, Guy and C. A. B. Smith published *The G-values of various games* [42]. This gave the world an infinite number of impartial games and led to many interesting, easy to state, and yet still unsolved conjectures.

The analysis of partizan games looked out of reach. The Fields' medalist John Milnor [70] in 1953 published *Sums of positional games*. This only covered games in which players gained when they played and was not easy to apply. In 1960, John Conway met Michael Guy, Richard's son. Through this friendship, John met Richard and asked about partizan games. This turned out to be a recurring theme in their work in the next two decades. Also in 1960, Elwyn Berlekamp got roped into playing 3×3 DOTS-&-BOXES game against a computer. He lost, but knowing about the Sprague–Grundy theory, he analyzed the game. (Recently, Elwyn claimed that he had never lost a game since.) Elwyn met Richard at the 1967 Chapel Hill conference and suggested that they write a book. Richard agreed, got John and Elwyn together in 1969, and work began. The analysis of each nonimpartial game was well thought out but ad hoc. John, with his training in set theory, started to see a structure emerging when games were decomposed into components. He gave the names of 1 and 1/2 to two abstract games and was delighted (giggled like a baby was the phrase he used) when he discovered that, as games, $1/2 + 1/2 = 1$. He wrote *On Numbers and Games* [28] in a week. This caused some friction among the three, but, eventually, work restarted on *Winning Ways* [3, 4].

R. Austin, S. Devitt, D. Duffus, and myself, as graduate students at Calgary, scoured the early page-proofs. We suggested numerous jokes and puns. Fortunately, the authors rejected all of them.

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One other person deserves to be mentioned, Louise Guy, Richard's wife. A gracious lady made every visitor to their house feel welcome. Some people have asked why the combinatorial game players, Left and Right, are female and male, respectively. The original reasons have been forgotten, but after *Winning Ways* appeared, it became a mark of respect to remember them as **L**ouise and **R**ichard.

Why Elwyn, John, and Richard are important

Many books are written, enjoy a little success, and then are forgotten by all but a few. *On Numbers and Games* but especially *Winning Ways* [3, 4] are still popular today. This popularity is due to the personalities and their approach to mathematics. All were great ambassadors for mathematics, writing explanatory articles and giving many public lectures. More than that, they understood that mathematics needs a human touch. These days, it is easy to get a computer to play a game well, but how do you get a person to play well? This was one of their aims. *Winning Ways* is 800+ pages of puns, humor, easy-to-remember sayings, and verses. These provide great and memorable insights into the games and their structures, and the book is still a rich source of material for researchers. Mathscinet reports that *Winning Ways* is cited by over 300 articles, Google Scholar reports over 3000 citations. Yet, any reader will be hard pressed to find a single mathematical proof in the book. Elwyn, John, and Richard wrote it to entertain, draw in a reader, and give them an intuitive feeling for the games.

After the publication of *Winning Ways*, even though all were well known for their research outside of combinatorial game theory, they remained active in the subject. Each was interested in many parts of the subject, but, very loosely, their main interests were:

- Elwyn Berlekamp considered the problem of how to define and quantify the notion of the "urgency" of a move. He made great strides with his concept of an enriched environment [11, 24, 25]. He was also fascinated by GO [7, 8, 9, 10, 12, 11] and DOTS-&-BOXES [13, 18, 23].
- John Conway remained interested in pushing the theory of surreal numbers, particularly infinite games [30, 37, 41], games from groups and codes [32, 39], and misère games [35].
- Richard K. Guy retained an interest in subtraction and octal games, writing a book for inquisitive youngsters [52]. He continued to present the theory as it was [54, 57, 58, 59, 61] and also summarized the important problems [56, 60, 62, 64].

Standing on their shoulders

Most of the papers in this volume can be traced directly back to *Winning Ways* and *On Numbers and Games*, or to the continuing interests of the three. Several though, illustrate how far the subject has developed. A general approach of impartial misère games was only started by Plambeck [71]. A. Siegel (a student of Berlekamp), a major figure developing this theory, pushes this further in Chapter 20. The theory of partizan misère games was only started in 2007 [69]. Whilst playing in the context of all misère games, Chapter 10 analyzes a specific game. Chapter 16 contains important results for analyzing misère dead-ending games. In *Winning Ways*, DOTS-&-BOXES and TOP-ENTAILS do not fit into the theory, each in a separate way. They are only partially analyzed and that via ad hoc methods. Chapter 17 finds a normal play extension that covers both types of games. (The authors think this would have intrigued them but are not sure if they would have fully approved.)

Chapters 1, 5–9, 12, 15, 18, and 19 either directly extend the theory or consider a related game to ones given in *Winning Ways*. As is evidenced by Richard K. Guy's early contributions, it is also important to have new sources of games. These are presented in Chapters 2, 3, 11, 13, and 14.

Serendipity gave Chapter 4. This paper is the foundation of Chapters 1 and 5. It gives a simple, effect-for-humans, test for when games are numbers. The authors are sure that Elwyn, John, and Richard would have started it with a rhyming couplet that everyone would then remember.

Elwyn, John, and Richard gave freely of their time. Many people will remember the coffee-time and evenings at the MSRI and BIRS Workshops. Each would be at a large table fully occupied by anyone who wished to be there, discussing and sometimes solving problems. Students were especially welcome. All combinatorial games workshops now follow this inclusive model. A large number of papers originate at these workshops, have several coauthors, and include students. They shared their time outside of conferences and workshops. Many students will remember those offhand moments, with one or more of them, that often stretched to hours. I was a second-year undergraduate student when on meeting John, he immediately asked me what was $1 + 1 + 1$? Even after I answered "3", he still took the time to explain the intricacies of 3-player games. (The question is still unanswered.)

Their wit, wisdom, and willingness to play provided people with pleasure. They will be sorely missed, but their legacy lives on.

Richard J. Nowakowski

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Contents

Anthony Bonato, Melissa A. Huggan, and Richard J. Nowakowski

The game of FLIPPING COINS

Abstract: We consider FLIPPING COINS, a partizan version of the impartial game TURNING TURTLES, played on lines of coins. We show that the values of this game are numbers, and these are found by first applying a reduction, then decomposing the position into an iterated ordinal sum. This is unusual since moves in the middle of the line do not eliminate the rest of the line. Moreover, if G is decomposed into lines H and K, then $G = (H : K^R)$. This is in contrast to HACKENBUSH STRINGS, where $G = (H : K)$.

1 Introduction

In *Winning Ways*, Volume 3 [3], Berlekamp, Conway, and Guy introduced TURNING TURTLES and considered many variants. Each game involves a finite row of turtles, either on feet or backs, and a move is to turn one turtle over onto its back, with the option of flipping a number of other turtles, to the left, each to the opposite of its current state (feet or back). The number depends on the rules of the specific game. The authors moved to playing with coins as playing with turtles is cruel.

These games can be solved using the Sprague–Grundy theory for impartial games [2], but the structure and strategies of some variants are interesting. The strategy for MOEBIUS (flip up to five coins) played with 18 coins, involves *Möbius transformations*; for MOGUL (flip up to seven coins) on 24 coins, it involves the *miracle octad generator* developed by R. Curtis in his work on the Mathieu group M_{24} and the Leech lattice [6, 7]; TERNUPS [3] (flip three equally spaced coins) requires ternary expansions; and TURNING CORNERS [3], a two-dimensional version where the corners of a rectangle are flipped, needs nim-multiplication.

We consider a simple partizan version of TURNING TURTLES, also played with coins. We give a complete solution and show that it involves ordinal sums. This is somewhat surprising since moves in the middle of the line do not eliminate moves at the end. Compare this with HACKENBUSH STRINGS [2] and DOMINO SHAVE [5].

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We will denote heads by 0 and tails by 1. Our partizan version will be played with a line of coins, represented by a 0–1 sequence $d_1d_2\dots d_n$, where $d_i \in \{0,1\}$. With this position, we associate the binary number $\sum_{i=1}^{n} d_i 2^{i-1}$. Left moves by choosing some pair of coins d_i, d_j, $i < j$, where $d_i = d_j = 1$, and flips them over so that both coins are 0s. Right also chooses a pair d_k, d_ℓ, $k < \ell$, with $d_k = 0$ and $d_\ell = 1$, and flips them over. If j is the greatest index such that $d_j = 1$, then d_k, $k > j$, will be deleted. For example,

$$1011 = \{0001, 001, 1 \mid 1101, 111\}.$$

The game eventually ends since the associated binary number decreases with every move. We call this game FLIPPING COINS.

Another way to model FLIPPING COINS is to consider tokens on a strip of locations. Left can remove a pair of tokens, and Right is able to move a token to an open space to its left. We use the coin flipping model for this game to be consistent with the literature.

The game is biased to Left. If there are a nonzero even number of 1s in a position, then Left always has a move; that is, she will win. Left also wins any nontrivial position starting with 1. However, there are positions that Right wins. The two-part method to find the outcomes and values of the remaining positions can be applied to all positions. First, apply a modification to the position (unless it is all 1s), which reduces the number of consecutive 1s to at most three. After this reduction, build an iterated ordinal sum, by successively deleting everything after the third last 1, this deleted position determines the value of the next term in the ordinal sum. As a consequence, the original position is a Right win if the position remaining at the end is of the form $0\dots01$, and the value is given by the ordinal sum.

The necessary background for numbers is in Section 2. Section 3 contains results about outcomes and also includes our main results. First, we show that the values are numbers in Theorem 3.2. Next, an algorithm to find the value of a position is presented, and Theorem 3.3 states that the value given by the algorithm is correct.

The actual analysis is in Section 4. It starts by identifying the best moves for both players in Theorem 4.2. This leads directly to the core result Lemma 4.5, which shows that the value of a position is an ordinal sum. The ordinal sum decomposition of G is found as follows. Let G^L be the position after the Left move that removes the rightmost 1s. Let H be the string $G \setminus G^L$; that is, the substring eliminated by Left's move. Let H^R be the result of Right's best move in H. Now we have that $G = G^L : H^R$. In contrast, the ordinal sums for HACKENBUSH STRINGS and DOMINO SHAVE [5] involve the value of H not H^R.

The proof of Theorem 3.3 is given in Section 4.1. The final section includes a brief discussion of open problems.

Finally, we pose a question for the reader, which we answer at the end of Section 4.1: Who wins 0101011111 + 1101100111 + 0110110110111 and how?

2 Numbers

All the values in this paper are numbers, and this section contains all the necessary background to make the paper self-contained. For further details, consult [1, 8]. Positions are written in terms of their options; that is, $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$.

Definition 2.1 ([1, 2, 8]). Let G be a number whose options are numbers, and let G^L, G^R be the Left and Right options of the canonical form of G.
1. If there is an integer k, $G^L < k < G^R$, or if either G^L or G^R does not exist, then G is the integer, say n, closest to zero that satisfies $G^L < n < G^R$.
2. If both G^L and G^R exist and the previous case does not apply, then $G = \frac{p}{2^q}$, where q is the least positive integer such that there is an odd integer p satisfying $G^L < \frac{p}{2^q} < G^R$.

The properties of numbers required for this paper are contained in the next three theorems.

Theorem 2.2 ([1, 2, 8]). *Let G be a number whose options are numbers, and let G^L and G^R be the Left and Right options of the canonical form of G. If G' and G'' are any Left and Right options, respectively, then*

$$G' \leqslant G^L < G < G^R \leqslant G''.$$

Theorem 2.2 shows that if we know that the string of inequalities holds, then we need to only consider the unique best move for both players in a number.

We include the following examples to further illustrate these ideas:

(a) $0 = \{\,\mid\,\} = \{-9 \mid\,\} = \{-\frac{1}{2} \mid \frac{7}{4}\}$;

(b) $-2 = \{\,\mid -1\} = \{-\frac{5}{2} \mid -\frac{31}{16}\}$;

(c) $1 = \{0 \mid\,\} = \{0 \mid 100\}$;

(d) $\frac{1}{2} = \{0 \mid 1\} = \{\frac{3}{8} \mid \frac{17}{32}\}$.

For games G and H, to show that $G \geqslant H$, we need to show that $G - H \geqslant 0$, meaning that we need to show that $G - H$ is a Left win moving second. For more information, see Sections 5.1, 5.8, and 6.3 of [1].

Let G and H be games. The *ordinal sum* of G, the *base*, and H, the *exponent*, is

$$G : H = \{G^{\mathcal{L}}, G : H^{\mathcal{L}} \mid G^{\mathcal{R}}, G : H^{\mathcal{R}}\}.$$

Intuitively, playing in G eliminates H, but playing in H does not affect G. For ease of reading, if an ordinal sum is a term in an expression, then we enclose it in brackets.

Note that $x : 0 = x = 0 : x$ since neither player has a move in 0. We demonstrate how to calculate the values of other positions with the following examples:

(a) $1 : 1 = \{1 \mid\,\} = 2$;

(b) $1 : -1 = \{0 \mid 1\} = \frac{1}{2}$;
(c) $1 : \frac{1}{2} = \{0, (1:0) \mid (1:1)\} = \{0, 1 \mid \{1 \mid\}\} = \{1 \mid 2\} = \frac{3}{2}$;
(d) $\frac{1}{2} : 1 = \{0, (\frac{1}{2} : 0) \mid 1\} = \{0, \frac{1}{2} \mid 1\} = \{\frac{1}{2} \mid 1\} = \frac{3}{4}$;
(e) $(1 : -1) : \frac{1}{2} = (\frac{1}{2} : \frac{1}{2}) = \{0, (\frac{1}{2} : 0) \mid 1, (\frac{1}{2} : 1)\} = \{0, \frac{1}{2} \mid 1, \frac{3}{4}\} = \{\frac{1}{2} \mid \frac{3}{4}\} = \frac{5}{8}$.

Note that in all cases, when base and exponent are numbers, the players prefer to play in the exponent. In the remainder of this paper, all the exponents will be positive.

One of the most important results about ordinal sums was first reported in *Winning Ways*.

Theorem 2.3 (Colon Principle [2]). *If $K \geqslant K'$, then $G : K \geqslant G : K'$.*

The Colon Principle helps prove inequalities that will be useful in this paper.

Theorem 2.4. *Let G and H be numbers all of whose options are also numbers, and let $H \geqslant 0$.*
1. *If $H = 0$, then $G : H = G$. If $H > 0$, then $(G : H) > G$.*
2. $G^L < (G : H^L) < (G : H) < (G : H^R) < G^R$.

Proof. For item (1), the result follows immediately by Theorem 2.3.

For item (2), if $H \geqslant 0$ and all the options of G and H are numbers, then $G^L < G = (G : 0) \leqslant (G : H^L) < (G : H) < (G : H^R)$. The second, third, and fourth inequalities hold since H is a number and thus $0 \leqslant H^L < H < H^R$ and by applying the Colon Principle. To complete the proof, we need to show that $(G : H^R) < G^R$. To do so, we check that $G^R - (G : H^R) > 0$; in words, we check that Left can always win. Left moving first can move in the second summand to $G^R - G^R = 0$ and win. Right moving first has several options:
1. Moving to $G^R - G^L > 0$, since G and its options are numbers. Hence Left wins.
2. Moving to $G^R - (G : H^{RL}) > 0$ by induction.
3. Moving to $G^{RR} - G : H^R$, but Left can respond to $G^{RR} - G^R > 0$ since G and its options are numbers.

In all cases, Left wins moving second. The result follows. □

To prove that all the positions are numbers, we use results from [4]. A set of positions from a ruleset is called a *hereditarily closed set of positions of a ruleset* if it is closed under taking options. This game satisfies ruleset properties introduced in [4]. In particular, the properties are called the *F1 property* and the *F2 property*, which both highlight the notion of *First-move-disadvantage* in numbers and are defined formally as follows.

Definition 2.5 ([4]). Let S be a hereditarily closed ruleset. Given a position $G \in S$, the pair $(G^L, G^R) \in G^{\mathcal{L}} \times G^{\mathcal{R}}$ satisfies the *F1 property* if there is $G^{RL} \in G^{R\mathcal{L}}$ such that $G^{RL} \geqslant G^L$ or there is $G^{LR} \in G^{L\mathcal{R}}$ such that $G^{LR} \leqslant G^R$.

Definition 2.6 ([4]). Let S be a hereditarily closed ruleset. Given a position $G \in S$, the pair $(G^L, G^R) \in G^{\mathcal{L}} \times G^{\mathcal{R}}$ satisfies the *F2 property* if there are $G^{LR} \in G^{L\mathcal{R}}$ and $G^{RL} \in G^{R\mathcal{L}}$ such that $G^{RL} \geqslant G^{LR}$.

As proven in [4], if given any position $G \in S$, all pairs $(G^L, G^R) \in G^{\mathcal{L}} \times G^{\mathcal{R}}$ satisfy one of these properties, then the values of all positions are numbers. Furthermore, satisfying the F2 property implies satisfying the F1 property, and it was shown that all positions $G \in S$ are numbers if and only if for any $G \in S$, all pairs $(G^L, G^R) \in G^{\mathcal{L}} \times G^{\mathcal{R}}$ satisfy the F1 property. Combining these results gives the following theorem.

Theorem 2.7 ([4]). *Let S be a hereditarily closed ruleset. All positions $G \in S$ are numbers if and only if for any position $G \in S$, all pairs $(G^L, G^R) \in G^{\mathcal{L}} \times G^{\mathcal{R}}$ satisfy either the F1 or the F2 property.*

3 Main results

Before considering the values and associated strategies, we consider the outcomes, that is, we partially answer the question “Who wins the game?” The full answer requires an analogous analysis to finding the values.

Theorem 3.1. *Let $G = d_1 d_2 \ldots d_n$. If $d_1 d_2 \ldots d_n$ contains an even number of* 1s*, or if $d_1 = 1$ and there are least two* 1s*, then Left wins G.*

Proof. A Right move does not decrease the number of 1s in the position. Thus, if in G, Left has a move, then she still has a move after any Right move in G. Consequently, regardless of d_1, if there are an even number of 1s in G, then it will be Left who reduces the game to all 0s. Similarly, if $d_1 = 1$ and there are an odd number of 1s, then Left will eventually reduce G to a position with a single 1, that is, to $d_1 = 1$ and $d_i = 0$ for $i > 1$. In this case, Right has no move and loses. □

The remaining case, $d_1 = 0$ and an odd number of 1s, is more involved. The analysis of this case is the subject of the remainder of the paper. We first prove the following:

Theorem 3.2. *All* FLIPPING COINS *positions are numbers.*

Proof. Let G be a FLIPPING COINS position. If only one player has a move, then the game is an integer. Otherwise, let L be the Left move to change (d_i, d_j) from $(1,1)$ to $(0,0)$. Let R be the Right move to change (d_k, d_ℓ) from $(0,1)$ to $(1,0)$. No other digits are changed. If all four indices are distinct, then both L and R can be played in either order. In this case, $G^{LR} = G^{RL}$. Thus the F2 property holds. If there are only three distinct indices, then two of the bits are ones. If Left moves first, then $d_i = d_j = d_k = 0$. If Right moves first, then there are still two ones remaining after his move. After Left moves, we have $d_i = d_j = d_k = 0$, and hence $G^L = G^{RL}$. The F1 property holds.

There are no more cases since there must be at least three distinct indices. Since every position satisfies either the F1 or the F2 property, by Theorem 2.7 it follows that every position is a number. □

Given a position G, the following algorithm returns a value.

Algorithm Let G be a FLIPPING COINS position. Let $G_0 = G$.

---

1. Set $i = 0$.
2. Reductions: Let α and β be binary strings, and either can be empty.
   (a) If $G_0 = \alpha 01^{3+j}\beta$, $j \geqslant 1$, then set $G_0 = \alpha 101^j\beta$.
   (b) If $G_0 = \alpha 01^3\beta$ with β containing an even number of 1s, then set $G_0 = \alpha 10\beta$.
   (c) Repeat until neither case applies; then go to Step 3.
3. If G_i is 0^r1, $r \geqslant 0$, or $1^a0^{p_i}10^{q_i}1$, $a \geqslant 0$, and $p_i + q_i \geqslant 0$, then go to Step 5. Otherwise, $G_i = \alpha 01^a0^{p_i}10^{q_i}1$, $p_i + q_i \geqslant 1$, $a > 0$, and some α. Set
$$Q_i = 0^{p_i}10^{q_i}1,$$
$$G_{i+1} = \alpha 01^a.$$
   Go to Step 4.
4. Set $i = i + 1$. Go to Step 3.
5. If $G_i = 0^r1$, then set $v_i = -r$. If $G_i = 1^a0^{p_i}10^{q_i}1$, then set $v_i = \lfloor \frac{a}{2} \rfloor + \frac{1}{2^{2p_i+q_i}}$. Go to Step 6.
6. For j from $i - 1$ down to 0, set $v_j = v_{j+1} : \frac{1}{2^{2p_j+q_j-1}}$.
7. Return the number v_0.

The algorithm implicitly returns two different results:
1. For Step 3, the substrings $Q_0, Q_1, \ldots, Q_{i-1}, G_i$ partition the reduced version of G;
2. The value v_0.

First, we illustrate the algorithm with the following example. Consider the position $G = 10011110110110111011110011$. We highlight at each step which reduction is being applied to the underlined digits; 2(a) is denoted by †, whereas 2(b) is denoted by ‡. The algorithm gives that

$$\begin{aligned}10011110110110111011110011 &= 10011110110110111\underline{0111}10011(\dagger)\\ &= 1001111011011\underline{0111}1010011(\dagger)\\ &= 1001111011\underline{0111}01010011(\ddagger)\\ &= 1001111\underline{0111}001010011(\ddagger)\\ &= 10\underline{0111}1110001010011(\dagger)\\ &= 1010110001010011.\end{aligned}$$

Step 3 partitions the last expression into 101(011)(000101)(0011) so that the ordinal sum is given by

$$\begin{aligned} v_0 &= \left(\left(\frac{1}{2} : \frac{1}{2}\right) : \frac{1}{64}\right) : \frac{1}{8} \\ &= \frac{10257}{16348}. \end{aligned}$$

Now let $H = 01001110110111011101$. The reductions give that

$$\begin{aligned} 01001110110111011101 &= 0100111011\underline{0111}011101 \\ &= 01001110\underline{111}0011101 \\ &= 010\underline{0111}100011101 \\ &= 01010100011101. \end{aligned}$$

The last expression partitions into 01(0101)(00011)(101) so that

$$\begin{aligned} v_0 &= \left(\left(-1 : \frac{1}{4}\right) : \frac{1}{32}\right) : 1 \\ &= -\frac{893}{1024}. \end{aligned}$$

The next theorem is the main result of the paper.

Theorem 3.3 (Value theorem). *Let G be a* FLIPPING COINS *position. If v_0 is the value obtained by the algorithm applied to G, then $G = v_0$.*

In the next section, we derive several results that will be used to prove Theorem 3.3. The proof of Theorem 3.3 will appear in Section 4.1.

4 Best moves and reductions

The proofs in this section use induction on the options. An alternate but equivalent approach is to regard the techniques as induction on the associated binary number of the positions. The proofs require detailed examination of the positions, and we will use notation suitable to the case being considered. Often, a typical position will be written as a combination of generic strings and the substring under consideration. For example, 111011000110101 might be parsed as (11101)(100011)(0101) and written as $\alpha100011\beta$ or more compactly as $\alpha10^31^2\beta$.

We require several results before being able to prove Theorem 3.3. We begin by proving a simplifying reduction, followed by the best moves for each player, and then the remaining reductions used in the algorithm.

As an immediate consequence of Theorems 3.2 and 2.2, we have the following:

Corollary 4.1. *Let α, β, and γ be arbitrary binary strings. We then have that $\alpha 1\beta 0\gamma > \alpha 0\beta 1\gamma$. Moreover, for an integer $r \geqslant 0$, we have that $\beta 10^r 1 > \beta$.*

Proof. Recall that by Theorem 3.2 all FLIPPING COINS positions are numbers. Thus Theorem 2.2 applies.

A Right option of $\alpha 0\beta 1\gamma$ is $\alpha 1\beta 0\gamma$, and so we have that $\alpha 1\beta 0\gamma > \alpha 0\beta 1\gamma$. Similarly, a Left option of $\beta 10^r 1$ is β, and so we have that $\beta 10^r 1 > \beta$. □

Next, we prove the best moves for each player. Right wants to play the zero furthest to the right and the 1 adjacent to it. Left wants to play the two ones furthest to the right.

Theorem 4.2. *Let G be a FLIPPING COINS position, where in G, r and n, $r \neq n$, are the greatest indices such that $d_r = d_n = 1$. Let s be the greatest index such that $d_s = 0$. Left's best move is to play (d_r, d_n), and Right's best move is to play (d_s, d_{s+1}).*

Proof. We prove this theorem by induction on the options. Note that we use the equivalent binary representation of the game position. If there are three or fewer bits, then, by exhaustive analysis, the theorem is true.

Let G be $d_1 d_2 \dots d_n$. We begin by proving Left's best moves. Let r and n be the two largest indices, where $d_r = d_n = 1$, and thus $d_k = 0$ for $r < k < n$. Let i and j, $i < j$, be two indices with $d_i = d_j = 1$. We use the notation $G(d_i, d_j, d_r, d_n)$ to highlight the salient bits. The claimed best Left move is from $G(1,1,1,1)$ to $G(1,1,0,0)$. This must be compared to any other Left move, represented by moving from $G(1,1,1,1)$ to $G(0,0,1,1)$. That is, we need to show that $G(1,1,0,0) - G(0,0,1,1) \geqslant 0$.

For the moves to be different, at least three of i, j, r, n are distinct. We first assume that the four indices are distinct. In this case, we have that $i < j < r < n$. By applying Corollary 4.1 twice we have that

$$G(1,1,0,0) > G(1,0,0,1) > G(0,0,1,1).$$

We may assume then, without loss of generality, that $j = r$ or $j = n$. If $j = n$, then $i < r$, since there are two distinct moves. Now consider $G(d_i, d_r, d_n) = G(1,1,1)$. By Corollary 4.1 we have that if $j = r$, then $G(1,0,0) > G(0,0,1)$, and if $j = n$, then $G(1,0,0) > G(0,1,0)$.

We now prove Right's best move. There are more cases to consider. Let s be the largest index such that $d_s = 0$ and therefore $d_{s+1} = 1$. Let i, j, $i < j$ be indices with $d_i = 0$ and $d_j = 1$. The claimed best move is d_s, d_{s+1}, and this must be compared to the arbitrary Right move d_i, d_j. For the moves to be different, there must be at least three distinct indices.

The original position is either

$$G(d_i, d_j, d_s, d_{s+1}) = G(0,1,0,1), \quad i < s,$$

or

$$G(d_s, d_{s+1}, d_j) = G(0, 1, 1), \quad i = s, j > s + 1.$$

We need to show either $D = G(1, 0, 0, 1) - G(0, 1, 1, 0) \geqslant 0$ or $D = G(1, 1, 0) - G(1, 0, 1) \geqslant 0$, respectively. Suppose Right plays in the first summand of D. Note that, by induction, the best moves of Left and Right are known.

1. First, suppose $j < s$. By induction Right's best move in the first summand of D is to $D' = G(1, 0, 1, 0) - G(0, 1, 1, 0)$. Since $i < j$, it follows that $G(1, 0, 1, 0)$ is a Right option of $G(0, 1, 1, 0)$, and thus D' is positive by Corollary 4.1.
2. If $j = s + 1$, then there are only three distinct indices. The original game is $G(d_i, d_s, d_{s+1}) = G(0, 0, 1)$ and $D = G(1, 0, 0) - G(0, 1, 0)$. Since $G(1, 0, 0)$ is a Right option of $G(0, 1, 0)$, it follows that D is positive by Corollary 4.1.
3. Suppose $j > s + 1$.
   If $i < s$, then the original game is of the form

   $$G = \alpha d_i \beta d_s d_{s+1} 1^a d_j 1^b = \alpha 0 \beta 0 1 1^a 1 1^b, \quad a \geqslant 0, b \geqslant 0,$$

   and

   $$D = \alpha 1 \beta 0 1 1^a 0 1^b - \alpha 0 \beta 1 0 1^a 1 1^b.$$

   Two applications of Corollary 4.1 (applied to the highlighted terms) give

   $$\alpha \underline{1} \beta \underline{0} 1 1^a 0 1^b \geqslant \alpha 0 \beta 1 \underline{1} 1^a \underline{0} 1^b \geqslant \alpha 0 \beta 1 0 1^a 1 1^b.$$

   If $i = s$, then

   $$G = \alpha d_s d_{s+1} 1^a d_j 1^b = \alpha 0 1 1^a 1 1^b, \quad a \geqslant 0, b \geqslant 0,$$

   and

   $$D = \alpha 1 1 1^a 0 1^b - \alpha 1 0 1^a 1 1^b.$$

   One application of Corollary 4.1 (relevant terms again highlighted) gives

   $$\alpha 1 \underline{1} 1^a \underline{0} 1^b \geqslant \alpha 1 0 1^a 1 1^b.$$

   Thus $D \geqslant 0$.

Next, we consider Right moving in the second summand of $D = G(1, 0, 0, 1) - G(0, 1, 1, 0)$. Note that by the choices of the subscripts, $d_\ell = 1$ if $n \geqslant \ell \geqslant s + 1$.

1. If $n > s + 2$, then Right's best move in the second summand is to change d_{n-1}, d_n from $(1,1)$ to $(0,0)$. Left copies this move in the first summand, and the resulting difference game is nonnegative by induction.
2. Suppose $n = s + 2$.
   i. If $j < s + 1$, then $G(d_i, d_j, d_s, d_{s+1}, d_{s+2}) = G(0,1,0,1,1)$ and $D = G(1,0,0,1,1) - G(0,1,1,0,1)$. Right's best move is to $G(1,0,0,1,1) - G(0,1,0,0,0)$. Left moves to $G(1,0,0,0,0) - G(0,1,0,0,0)$. This is positive by Corollary 4.1, and Left wins. For the next two subcases, exactly two 1s will occupy two of the four indexed positions. Since Right is moving in the second summand, he is changing two 1s to two 0s. Thus Left's best response for each case is to move in the first summand, bringing the game to $G(0,0,0,0) - G(0,0,0,0) = 0$, and she wins. For these cases, we only list the original position. The strategy for both cases is as just described.
   ii. If $j = s + 1$, then $G(d_i, d_s, d_{s+1}, d_{s+2}) = G(0,0,1,1)$ and $D = G(1,0,0,1) - G(0,1,0,1)$.
   iii. If $j = s + 2$, then $G(d_i, d_s, d_{s+1}, d_{s+2}) = G(0,0,1,1)$ and $D = G(1,0,1,0) - G(0,1,0,1)$.
3. Now suppose $n = s + 1$.
   i. If $j < s + 1$, then let $\ell < s + 1$ be the largest index such that $d_\ell = 1$.
   If $j < \ell$, then we have $G(d_i, d_j, d_\ell, d_s, d_{s+1}) = G(0,1,1,0,1)$ and $D = G(1,0,1,0,1) - G(0,1,1,1,0)$. Right's best move is to $G(1,0,1,0,1) - G(0,1,0,0,0)$. Left moves to $G(1,0,0,0,0) - G(0,1,0,0,0)$, which is positive since $G(1,0,0,0,0)$ is a Right option of $G(0,1,0,0,0)$.
   If $j = \ell$, then $G(d_i, d_j, d_s, d_{s+1}) = G(0,1,0,1)$ and $D = G(1,0,0,1) - G(0,1,1,0)$. Right's best move is to $G(1,0,0,1) - G(0,0,0,0)$. Left moves to $G(0,0,0,0) - G(0,0,0,0) = 0$, and Left wins.
   ii. If $j = s + 1$, then $G(d_i, d_s, d_{s+1}) = G(0,0,1)$ and $D = G(1,0,0) - G(0,1,0)$. This is positive by Corollary 4.1.

In all cases, Left wins D moving second, proving the result. □

Suppose in a position that the bits of the best Right move are different from those of the best Left move. The next lemma essentially says that the positions before and after one move by each player are equal. It is phrased in a way that is useful for reducing the length of the position. Recall that a nontrivial position looks like $G = \alpha 01^a 0^p 10^q 1\beta$, where a, p, and q are nonnegative integers, and α and β are arbitrary binary strings. For the algorithm, it suffices to prove the result for β being empty. However, it is useful, certainly for a human, to reduce the length of the position as much as possible.

Lemma 4.3. *Let α be an arbitrary binary string. If $a \geqslant 0$, then we have that $\alpha 01111^a = \alpha 101^a$.*

Proof. Let $H = \alpha01111^a - \alpha101^a$. We need to show that $H = 0$. To simplify the proof, in some cases the second player will play suboptimal moves. We have several cases to consider.

1. If $a \geqslant 2$, then playing the same move in the other summand is a good response. After two such moves, we have either

$$\alpha01111^{a-2} - \alpha101^{a-2} = 0 \quad \text{by induction}$$

   or

$$\alpha10111^a - \alpha1101^{a-1} = \alpha1101^{a-1} - \alpha1101^{a-1} = 0 \quad \text{by induction.}$$

2. If $a = 1$, then $H = \alpha01111 - \alpha101$. The cases are:
   i. Left plays in the first summand to $\alpha011 - \alpha101$; then Right moves to $\alpha101 - \alpha101 = 0$.
   ii. Right plays in the second summand to $\alpha01111 - \alpha$; then Left moves to $\alpha011 - \alpha$. Since $(\alpha011)^L = \alpha$, we have $\alpha011 > \alpha$.
   iii. Right plays in the first summand to $\alpha10111 - \alpha101$; then Left responds to $\alpha101 - \alpha101 = 0$.
   iv. Left plays in the second summand to $\alpha01111 - \alpha11$; then Right moves to $\alpha10111 - \alpha11 = \alpha11 - \alpha11 = 0$ by induction.
3. If $a = 0$, then $H = \alpha0111 - \alpha1$. There are several cases to consider.
   i. If Left or Right plays in the first summand, then the response is in the first summand giving $\alpha1 - \alpha1 = 0$.
   ii. If Left plays in the second summand, then since there is a Left move, we have $\alpha = \beta01^b$, $b \geqslant 0$. If $b > 0$, then we have that $\beta01^b0111 - \beta01^b1$, and Left moves to $\beta01^b01^3 - \beta101^b$. Here Right responds to $\beta101^{b-1}01^3 - \beta101^b$, which by induction is equal to $\beta101^{b-1}1 - \beta101^b = 0$. If $b = 0$, then we have that $\beta01^b0111 - \beta01^b1 = \beta00111 - \beta01$, and we want to show that Right can win moving second. Left plays to $\beta00111 - \beta10$, and Right can respond to $\beta01110 - \beta1$, which, by induction, is equal to $\beta1 - \beta1 = 0$.
   iii. Right plays in the second summand. Then for a Right move to exist, $\alpha = \beta10^a$, $a \geqslant 0$. Thus $H = \beta10^a0111 - \beta10^a1$, and Right moves to $\beta10^a0111 - \beta$. Left responds by moving to $\beta00^a011 - \beta$. We then have that $(\beta00^a011)^L = \beta$, and thus $\beta00^a011 > \beta$. Hence we find that $\beta00^a011 - \beta > 0$.

In all cases the second player wins H thereby proving the result. □

There are reductions that can be applied to the middle of the position, but extra conditions are needed.

Lemma 4.4. *Let α and β be arbitrary binary strings where either* (a) *β starts with a* 1*, or* (b) *β starts with* 0 *and has an even number of* 1s*. We then have that*

$$\alpha 0111\beta = \alpha 10\beta.$$

Proof. Let $H = \alpha 0111\beta - \alpha 10\beta$. We need to show that $H = 0$. We have several cases to consider.

1. If β is empty or $\beta = 1^a$, then $H = 0$ by Lemma 4.3. Therefore we may assume that β has at least one 1 and one 0.
2. If $\beta = 1\gamma 1$ (β must end in a 1), then in both summands the best moves are pairs of bits in β and $-\beta$. If each player copies the opponent's move in the other summand, then this leads to

   $$\alpha 0111\beta - \alpha 10\beta \to \alpha 0111\beta' - \alpha 10\beta',$$

   and the latter expression is equal to 0 by induction.
3. If $\beta \neq 1\gamma 1$, then $\beta = 0\gamma 1$, and $\gamma 1$ has at least two 1s. The best moves are in β and $-\beta$ and are the best responses to each other. We then derive that

   $$\alpha 0111\beta - \alpha 10\beta \to \alpha 0111\beta' - \alpha 10\beta' = 0 \quad \text{by induction.}$$

In all cases, $H = 0$, and this concludes the proof. □

In Lemma 4.4 the conditions are necessary. An example is

$$3/8 = 011101 \neq 1001 = 1/4.$$

Here β starts with a 0 and has an odd number of 1s.

These reduction lemmas are important in evaluating a position. The reduced positions will end in 011 or 01. By considering the exact end of the string, specifically, if there are at least two 0s (in one special case, three 0s), then we can find an ordinal sum decomposition. The decomposition is determined by where the third rightmost 1 is situated.

The next result is the start of the ordinal sum decomposition of a position. The exponent is the value of the Right option of the substring being removed.

Lemma 4.5. *Let α be an arbitrary binary string. If $a \geqslant 1$ and p and q are nonnegative integers such that $p + q \geqslant 1$, then*

$$\alpha 01^a 0^p 10^q 1 = \alpha 01^a : \frac{1}{2^{2p+q-1}}.$$

Proof. We prove that

$$\alpha 01^a 0^p 10^q 1 - \left(\alpha 01^a : \frac{1}{2^{2p+q-1}}\right) = 0.$$

Note that in Theorem 2.4 we have that playing in the base of $\alpha 01^a : \frac{1}{2^{2p+q-1}}$ is worse than playing in the exponent. We have two cases to consider.

1. Left plays first in the first summand, and Right responds in the second summand, or Right plays first in the second summand, and Left responds in the first summand. In either case, Right has a move in the exponent (moves to 0) since $2p+q-1 \geqslant 0$. In either order the final position is given by

$$\alpha 01^a - (\alpha 01^a : 0) = \alpha 01^a - \alpha 01^a = 0.$$

2. Right plays first in the first summand, and Left responds in the second summand, or Left plays first in the second summand, and Right responds in the first summand. In either case, we consider

$$\alpha 01^a 0^p 10^q 1 - \left(\alpha 01^a : \frac{1}{2^{2p+q-1}}\right).$$

   We have two subcases.

   i. Assume that $2p+q-1 \neq 0$. After the two moves, we have the position

   $$\alpha 01^a 0^r 10^s 1 - \left(\alpha 01^a : \frac{1}{2^{2p+q-2}}\right),$$

   where $2r+s = 2p+q-1$. By induction we have that

   $$\begin{aligned} \alpha 01^a 0^r 10^s 1 &= \alpha 01^a : \frac{1}{2^{2r+s-1}} \\ &= \alpha 01^a : \frac{1}{2^{2p+q-2}}. \end{aligned}$$

   Thus $\alpha 01^a 0^r 10^s 1 - (\alpha 01^a : \frac{1}{2^{2p+q-2}}) = 0$.

   ii. Assume that $2p+q-1 = 0$, that is, $q = 1$ and $p = 0$. The original position is

   $$\alpha 01^a 101 - (\alpha 01^a : 1).$$

   After the two moves, we have the position $\alpha 01^a 11 - \alpha 101^{a-1}$ (note that Left has no move in the exponent). By Lemma 4.3, $\alpha 01^a 11 = \alpha 101^{a-1}$. Hence we have that $\alpha 01^a 11 - \alpha 101^{a-1} = 0$, and the result follows. □

The values of the positions not covered by Lemma 4.5 are given next.

Lemma 4.6. *Let a, p, and q be nonnegative integers. We then have that*

$$0^p 1 = -p, \quad \text{and} \quad 1^a 0^p 10^q 1 = \left\lfloor \frac{a}{2} \right\rfloor + \frac{1}{2^{2p+q}}.$$

Proof. Let $G = 0^p1$. Left has no moves, and Right has p. Note that in 1^a, Left has $\lfloor \frac{a}{2} \rfloor$ moves, and Right has none.

Now let $G = 1^a0^p10^q1$. We proceed by induction on $p + q$. In all cases, Left's move is to 1^a, that is, to $\lfloor \frac{a}{2} \rfloor$. If $p = 0$ and $q = 0$, then $G = 1^a11$, which has the value $\lfloor \frac{a}{2} \rfloor + \frac{1}{2^0} = \lfloor \frac{a}{2} \rfloor + 1$. Assume that $p + q = k$, $k > 0$. If $q > 0$, then $G = \{\lfloor \frac{a}{2} \rfloor \mid 1^a0^p10^{q-1}1\}$. By induction we have that

$$G = \left\{ \left\lfloor \frac{a}{2} \right\rfloor \,\middle|\, \left\lfloor \frac{a}{2} \right\rfloor + \frac{1}{2^{2p+q-1}} \right\} = \left\lfloor \frac{a}{2} \right\rfloor + \frac{1}{2^{2p+q}}.$$

If $q = 0$, then $G = \{\lfloor \frac{a}{2} \rfloor \mid 1^a0^{p-1}10^11\}$. By induction we have that

$$G = \left\{ \left\lfloor \frac{a}{2} \right\rfloor \,\middle|\, \left\lfloor \frac{a}{2} \right\rfloor + \frac{1}{2^{2(p-1)+1}} \right\} = \left\lfloor \frac{a}{2} \right\rfloor + \frac{1}{2^{2p}},$$

and the result follows. □

4.1 Proof of the value theorem

We now have all tools to prove Theorem 3.3.

Proof of Theorem 3.3. Let G be a FLIPPING COINS position. Step 2 reduces the binary string. The reductions in Step 2(a) are those of Lemma 4.3 and Lemma 4.4(a). The reductions in Step 2(b) are those of Lemma 4.4(b). In all cases, these lemmas show that each new reduced position is equal to G.

In Step 3, we claim $G_i \neq \beta 1^3$ for any β. This is true for $i = 0$ by Lemma 4.3. If $i > 0$, then at each iteration of Step 3, the last two 1s are removed from G_{i-1}. Now the original reduced position would be $G_0 = \beta 1^3 \gamma$, where γ has an even number of 1s. Lemma 4.4(b) would apply eliminating the three consecutive 1s. Now either G_i is one of 0^r1, $r \geqslant 0$, or $1^a0^{p_i}10^{q_i}1$, $a \geqslant 0$, $p_i + q_i \geqslant 0$, or $G_i = \alpha 01^a0^{p_i}10^{q_i}1$, $p_i + q_i \geqslant 1$, $a > 0$. In the latter case the index is incremented, and the algorithm goes back to Step 3.

Step 5 applies when Step 3 no longer applies, i. e., G_i is one of 0^r1, $r \geqslant 0$, or $1^a0^{p_i}10^{q_i}1$, $a \geqslant 0$, $p_i + q_i \geqslant 0$. Now v_i is the value of G_i as given in Lemma 4.6.

Lemma 4.5 shows that for each $j < i$, $G_j = G_{j+1} : \frac{1}{2^{2p_j+q_j}}$, the evaluation in Step 6. Thus the value of G is v_0, and the theorem follows. □

The question "Who wins $0101011111 + 1101100111 + 0110110110111$ and how?" from Section 1 can now be answered.

First, we have that

$$\begin{aligned} 0101011111 = 01011011 &= \left(01011 : \frac{1}{2}\right) = \left(\left(01 : \frac{1}{2}\right) : \frac{1}{2}\right) \\ &= \left(\left(-1 : \frac{1}{2}\right) : \frac{1}{2}\right) = -\frac{11}{16}, \end{aligned}$$

$$1101100111 = 1101101 = (1101 : 1) = \left(\frac{1}{2} : 1\right) = \frac{3}{4},$$
$$0110110110111 = 0110110111 = 0110111 = 0111 = 0.$$

Thus we have that

$$0101011111 + 1101100111 + 0110110110111 = -\frac{11}{16} + \frac{3}{4} + 0 = \frac{1}{16}.$$

Left's only winning move is to

$$01010111 + 1101100111 + 0110110110111 = -\frac{3}{4} + \frac{3}{4} + 0 = 0.$$

Her best move in the second summand gives a sum of $-\frac{11}{16} + \frac{5}{8} + 0 = -\frac{1}{16}$, and in the third, it gives $-\frac{11}{16} + \frac{3}{4} - \frac{1}{8} = -\frac{1}{16}$. Left loses both times.

5 Future directions

Natural variants of FLIPPING COINS involve increasing the number of coins that can be flipped from two to three or more. A brief computer search suggests that the only version where the values are numbers is the game in which Left flips a subsequence of all 1s and Right flips a subsequence of 0s ended by a 1. We conjecture that a similar ordinal sum structure will arise in these variants. Other variants have values that include *switches, tinies, minies,* and other three-stop games. However, some variants, when the reduced canonical values are considered, only seem to consist of numbers and switches. A more thorough investigation should shed light on their structures.

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The game of BLOCKING PEBBLES

Abstract: Graph pebbling is a well-studied single-player game on graphs. We introduce the game of BLOCKING PEBBLES, which adapts Graph Pebbling into a two-player strategy game to examine it within the context of combinatorial game theory. Positions with game values matching all integers, all nimbers, and many infinitesimals and switches are found. This game joins the ranks of other combinatorial games on graphs, games with discovered moves, and partisan games with impartial movement options. The computational complexity of the general case is shown to be PSPACE-hard.

1 Introduction

Graph pebbling is an area of current interest in graph theory. In an undirected graph G, a root vertex r is designated. Heaps of pebbles are placed on the vertices of G, with a legal move consisting of choosing a vertex v with at least two pebbles, removing two pebbles, and placing a single pebble on a neighbor of v. The goal is to *pebble* or place a pebble on the vertex r. The *pebbling number* of G, denoted $\pi(G)$, is the fewest number of pebbles necessary so that any initial distribution of $\pi(G)$ pebbles among the vertices of G, and any vertex of G chosen as the root has a sequence of moves resulting in the root being pebbled.

Introduced by Chung [5] in 1989, a number of results on pebbling of different families of graphs have been found. Of note are pebbling numbers of paths, cycles [13], and continuing work on a conjecture of Graham's on the Cartesian products of graphs [5]. Time complexity is also known, both for determination of $\pi(G)$ and for the minimum number of moves in a successful pebbling solution, for general graphs. See [9] for a survey of results in graph pebbling.

The results and language here are in reference to combinatorial game theory (CGT). The *nim sum*, also called the *digital sum*, of nonnegative integers is the result of

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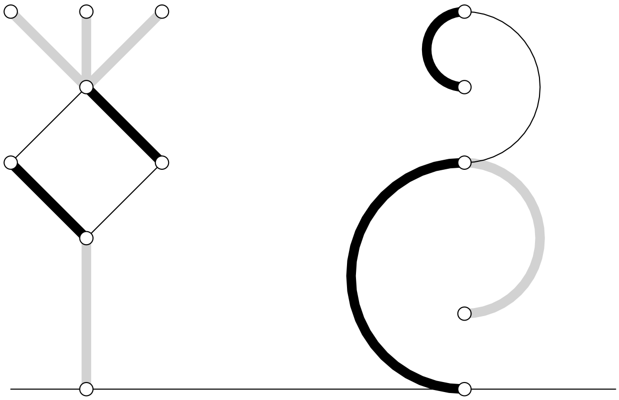

Figure 2.1: A BRG-HACKENBUSH position with blue, red, and green represented by thin, thick black, and grey lines, respectively.

their sum in binary without carry. This is denoted $x_1 \oplus x_2$ if there are only two numbers, and in the case of more, we use the notation $\sum \oplus x_i$. For more notation and background on the computation of CGT game values, we refer the reader to [3, 1].

In Section 2, we introduce a two-player combinatorial ruleset based on graph pebbling, with subsequent sections addressing results on both impartial and partisan positions. This game involves strategic play that results in blocking the moves of one's opponent. AMAZONS is another well-known game, which also involves a notion of blocking. However, in AMAZONS the blocking is always permanent (burnt square) or temporary (queen occupies square). Due to the standard pebbling toll in BLOCKING PEBBLES, each pebble only has mobility for a finite time.

There are several pebbling games that appear in the literature [10, 14, 11]. The one which is most similar to the game introduced here was originated by Lagarias and Saks in 1989 to solve a problem of Erdős. These games do not include the nontoll moves across an edge in the "wrong direction." This type of move is unique to BLOCKING PEBBLES (as far as we are aware). There are also other pebbling games older than that introduced by Lagarias and Saks [10]. These games bear no resemblance to BLOCKING PEBBLES and are used to study graph algorithm complexity.

2 Ruleset and play

A game of BLOCKING PEBBLES consists of a directed acyclic graph G and a 3-tuple (b, r, g) at each vertex of G, representing the numbers of blue, red, and green pebbles. Left may move blue and green pebbles, whereas Right may move red and green. This follows one convention of BRG-HACKENBUSH (see Fig. 2.1) wherein players may remove an edge of their own color or the neutral color green. In BRG-HACKENBUSH all dyadic

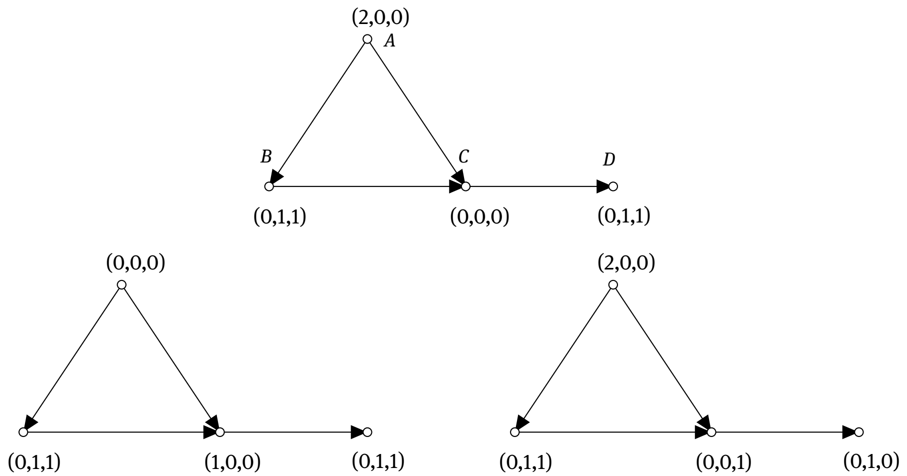

Figure 2.2: A position in BLOCKING PEBBLES and two of Left's options.

rationals and nimbers are achievable game values. In addition, when allowing for infinite positions, all real numbers and ordinals are achievable values, but switches are not. By contrast, in BLOCKING PEBBLES players may move any number of pebbles at a single vertex within certain constraints on the graph and pebble distribution. In this way, BLOCKING PEBBLES is similar to GRAPH NIM [8, 4].

Ruleset 1. Given a tuple of the form (b, r, g) at each vertex of a directed acyclic graph G, Left can make one of the following two moves from the vertex v.

1. Move a positive number of blue and/or green pebbles from v to an in-neighbor of v.
2. Remove two blue and/or green pebbles from v and place one on an out-neighbor of v and discard the other.

No blue pebbles can be moved to a vertex with a nonzero number of red pebbles. Right has the obvious symmetric moves.

Play proceeds following the normal play convention, where the last player to make a legal move wins.

Note that if Left removes one blue and one green pebble from v, then she may add the green to v's out-neighbor. However, it is always preferable to instead add the blue as this results in a position with more blue pebbles and increases the number of vertices blocked by Left.

As an example, consider the position in Figure 2.2. At the top is a position in BLOCKING PEBBLES. Note that Left cannot move any blue pebbles from vertex A to B since B already contains a red pebble. However, Left can move a single blue pebble from A to C at a cost of one blue pebble. She can also move the one green pebble from D to C.

An interesting property of this ruleset is the existence of *discovered moves*, similar to discovered attacks in chess. A player may be unable to move at one point in the game, but after their opponent moves, the game is once again playable by the first player. As an example, consider a simple out-star with two red pebbles on the source and a single blue pebble on a sink node. Left has no moves, but once Right moves, Left can move their pebble to the source. The presence of discovered moves precludes this being a strong placement game. For more on these types of games, see [7] and [12].

3 Blue-red-green blocking pebbles

In this section, we will address some families of game values that are achievable in BLOCKING PEBBLES. We will only address finite graphs, and hence we will not encounter nondyadic rationals. This is similar to BRG-HACKENBUSH, described in Section 1. Due to the complexity of analysis, we will also restrict our graphs to orientations of stars, paths, and small graphs.

We begin with a simple result.

Theorem 3.1. *For every $k \in \mathbb{Z}$, there is a position in* BLOCKING PEBBLES *with value k.*

Proof. Let G be a single arc directed from u to v. If $k > 0$, then place $2k$ blue pebbles and a single red pebble on u, and no pebbles on v. Switch red and blue pebbles if $k < 0$. This allows for k-many moves for Left by moving blue pebbles from u to v, but the presence of a red pebble on u prevents moving any blue pebbles in the reverse direction. Zero is trivially achieved by a graph with no pebbles or by any number of other pebble distributions. □

Regarding infinitesimals, $\downarrow$ is realized by an out-star with two leaves; that is, a vertex u with out-neighbors v_1, v_2. Vertex v_1 has a blue pebble, and v_2 has one red and one green pebble. Left can move the blue or green pebble to u, which is simple to identify as $*$. Right, however, can move the green to the source vertex u resulting in $*$, the red to u resulting in zero, or both red and green pebbles to u, which is also a zero position. The initial position is $\{* \mid 0, *\} = \downarrow$.

Due to the blocking rule, BLOCKING PEBBLES is relatively unique among partisan combinatorial games. In BRG-HACKENBUSH the presence of a move for one player does not inhibit moves for the other. In CLOBBER, another two-player partisan combinatorial game (see [2]), the presence of a red piece actually encourages movement for Left, and vice versa. This is a property common to all dicot games. However, in BLOCKING PEBBLES a single well-placed blue pebble, for example, can cut off many of Right's moves. The only other well-known ruleset with this property appears to be AMAZONS, which does not allow for discovered moves. It is natural, then, that many positions result in game values that are switches.

Parts (1) and (4) of the next result show that every integer switch is achievable with a specified pebbling configuration on the out-star $K_{1,2}$.

In the following lemma, we use the following notation for a bLue/Red pebbling configuration of the out-star $K_{1,2}$: $[(a,b),[c,d],[e,f]]$ is the configuration with a blue pebbles and b red pebbles on the central vertex, c blue pebbles and d red pebbles on one of the pendant vertices, and e blue pebbles and f red pebbles on the other pendant vertex.

Lemma 3.2. *The following results pertain to a given* BLOCKING PEBBLES *configuration on the out-star* $K_{1,2}$.

1. *For* $c \geq 1$, *the position* $[(a,b),[0,c],[0,0]]$ *has value* $-\lfloor \frac{b}{2} \rfloor$ *if* $a = 1$ *and value* $\{\lfloor \frac{a}{2} \rfloor - 1 \mid \lfloor \frac{a-b}{2} \rfloor + 1\}$ *if* $a \geq 2$,
2. *for* $a,b,c,d \geq 1$, *the position* $[(a,b),[c,0],[0,d]]$ *has value* $\lfloor \frac{a-b}{2} \rfloor$,
3. *for* $a,b,c,d,e \geq 1$, *the position* $[(a,b),[c,0],[d,e]]$ *has value* $\lfloor \frac{a-1}{2} \rfloor$,
4. *for* $a,b,c,d \geq 1$, *the position* $[(0,0),[a,b],[c,d]]$ *has value* $\{a+c-1 \mid -(b+d-1)\}$,
5. *for* $a,b,c \geq 1$, *the position* $[(0,0),[a,b],[0,c]]$ *has value* $\{a-1 \mid -(3(b+c)-5)\}$,
6. *for* $a,b \geq 1$, *the position* $[(0,0),[a,b],[0,0]]$ *has value* $\{3a-5 \mid -(3b-5)\}$,
7. *for* $b \geq 1$, $[(1,0),[0,b],[0,0]]$ *and* $[(2,0),[0,b],[0,0]]$ *are both zero positions.*

Proof. For `Case (1)`, the position $[(1,1),[c,0],[0,0]]$ is the zero position. It is also readily checked that the position $[(1,2),[c,0],[0,0]]$ has value 0.

If $b > 2$, then Left has no move from $[(1,b),[c,0],[0,0]]$. From $[(1,b),[c,0],[0,0]]$ Right may move to the position $[(1,b-2),[c,0],[0,1]]$, which has value $\lfloor \frac{-b+1}{2} \rfloor = -\lfloor \frac{b}{2} \rfloor + 1$ by induction. Hence $[(1,b),[c,0],[0,0]]$ has value $-\lfloor \frac{b}{2} \rfloor$ as required.

If $a \geq 2$, then Left's best move from $[(a,b),[c,0],[0,0]]$ is to $[(a-2,b),[c,0],[1,0]]$, which has value $\lfloor \frac{a-2}{2} \rfloor$ (Right has no move from this position, and Left has $\lfloor \frac{a-2}{2} \rfloor$ moves). Right's only move is to $[(a,b-2),[c,0],[0,1]]$, which has value $\lfloor \frac{a-b+2}{2} \rfloor$, also by induction. Hence $[(a,b),[c,0],[0,0]]$ has value $\{\lfloor \frac{a}{2} \rfloor - 1 \| \lfloor \frac{a-b}{2} \rfloor + 1\}$ when $a \geq 2$.

For `Case (2)`, it is clear that the position $[(1,1),[c,0],[0,d]]$ is a zero position. If $a \geq 2$, then from $[(a,1),[c,0],[0,d]]$ Left has a move to $[(a-2,1),[c+1,0],[0,d]]$, and Right has no move. Thus $[(a,1),[c,0],[0,d]]$ has value $\lfloor \frac{a-1}{2} \rfloor$ by induction.

A similar argument establishes the claim that $[(1,b),[c,0],[0,d]]$ has value $\lfloor \frac{1-b}{2} \rfloor$.

Now if $a,b \geq 2$, then from $[(a,b),[c,0],[0,d]]$ Left has the move to $[(a-2,b),[c+1,0],[0,d]]$, and Right has the move to $[(a,b-2),[c,0],[0,d+1]]$. By induction we see that $[(a,b),[c,0],[0,d]]$ has value

$$\left\{\left\lfloor \frac{a-2-b}{2} \right\rfloor \,\middle|\, \left\lfloor \frac{a-b+2}{2} \right\rfloor\right\} = \left\{\left\lfloor \frac{a-b}{2} \right\rfloor - 1 \,\middle|\, \left\lfloor \frac{a-b}{2} \right\rfloor + 1\right\} = \left\lfloor \frac{a-b}{2} \right\rfloor.$$

For `Case (3)`, note that if $a = 1$, then there are no moves for either player; the formula given correctly yields the game value 0. If $a = 2$, then from the position $[(2,b),[c,0],[d,e]]$ Left has the move to $[(0,b),[c+1,0],[d,e]]$. From here Left has no

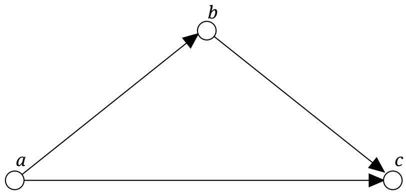

Figure 2.3: A transitive triple graph.

move, and Right has e moves. Thus the position $[(0,b),[c+1,0],[d,e]]$ has value $-e$. Hence $[(2,b),[c,0],[d,e]]$ has value 0, as required.

If $a>2$, then from $[(a,b),[c,0],[d,e]]$ Left can move to $[(a-2,b),[c+1,0],[d,e]]$, which has value $\lfloor\frac{a-3}{2}\rfloor = \lfloor\frac{a-1}{2}\rfloor - 1$ by induction. Right has no moves from $[(a,b),[c,0],[d,e]]$. Hence $[(a,b),[c,0],[d,e]]$ has value

$$\left\{\left\lfloor\frac{a-1}{2}\right\rfloor - 1 \,\middle|\, \right\} = \left\lfloor\frac{a-1}{2}\right\rfloor,$$

as desired.

For `Case (4)`, note that Left's only move from $[(0,0),[a,b],[c,d]]$ is to $[(1,0),[a-1,b],[c,d]]$. This last position has value $a+c-1$ by induction. Similarly, Right's only move from $[(0,0),[a,b],[c,d]]$ is to $[(0,1),[a,b-1],[c,d]]$. This position has value $-(b+d-1)$ by induction. It now follows that $[(0,0),[a,b],[c,d]]$ has value

$$\{a+c-1 \mid -(b+d-1)\}.$$

`Cases (5)` and `(6)` follow from the previous result, and `Case (7)` is trivial. □

We now consider a transitive 3-cycle graph (Fig. 2.3) with vertices a, b, c and arcs ab, ac, and bc. The pebbling configurations considered below are written so that the first array entry corresponds to the source vertex a, the second corresponds to b, and the third to the sink vertex c.

An interesting result concerning the transitive 3-cycle crops up from the somewhat unnatural starting position where 1 blue pebble and k red pebbles occupy the same starting vertex. Specifically, we prove the following:

Theorem 3.3. *For $k>1$, the pebbling configuration on the transitive 3-cycle given by $[[0,0],[0,0],[1,k]]$ has game value $(3-3k)+_{k-4}$.*

We see the game tree of the base case in Figure 2.4.

To prove this result, we consider several positions, which arise as subpositions of the above pebbling configuration.

Lemma 3.4. *Consider the following pebbling configurations of the transitive 3-cycle T. Then the position*

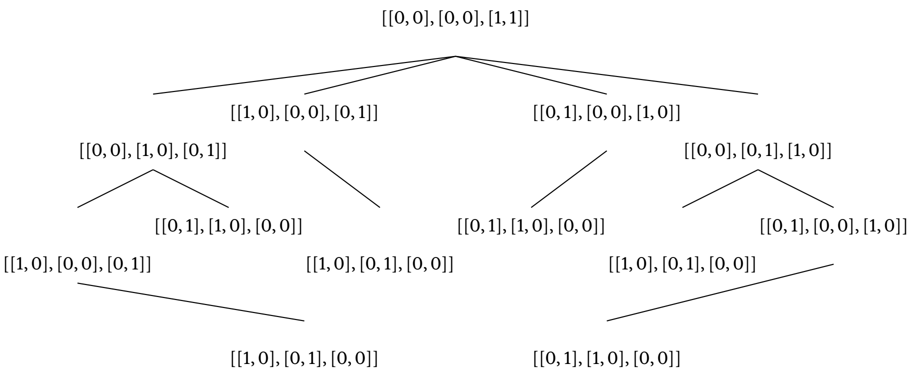

Figure 2.4: Game tree of the position $[[0,0],[0,0],[1,1]]$ on the transitive 3-cycle.

1. $[[1,0],[0,j],[0,k]]$ *has value* $-3k-2j+2$ *if at least one of* j *or* k *is* ≥ 1*;*
2. $[[0,j],[1,0],[0,k]]$ *has value* $-3k-2j+3$ *if* $j,k \geq 1$*;*
3. $[[0,0],[1,0],[0,k]]$ *has value* $-3k+3$ *if* $k \geq 2$ *and value* $-\frac{1}{2}$ *if* $k=1$*;*
4. $[[0,j],[1,0],[0,0]]$ *has value* $-2j+3$ *if* $j \geq 2$ *and value* 0 *if* $j=1$*;*
5. $[[0,j],[0,k],[1,0]]$ *has value* $-3k-2j+4$ *if* $j,k \geq 1$*;*
6. $[[0,j],[0,0],[1,0]]$ *has value* $-2j+4$ *if* $j \geq 2$ *and value* 1 *if* $j=1$*;*
7. $[[0,0],[0,k],[1,0]]$ *has value* $\{-2k+2 \mid -3k+5\}$ *if* $k \geq 2$ *and value* $\frac{1}{2}$ *if* $k=1$*;*
8. $[[0,0],[0,j],[1,k]]$ *has value* $\{-3k-2j+2 \mid -4k-3j+5\}$ *if* $j \geq 2$ *and* $k \geq 1$ *and value* $\{-3k \mid -4k+3\}$ *if* $j=1$ *and* $k \geq 1$*;*
9. $[[0,j],[0,0],[1,k]]$ *has value* $\{-3k-2j+3 \mid -4k-2j+5\}$ *if* $j,k \geq 1$*; and*
10. $[[0,\ell],[0,j],[1,k]]$ *has value* $-4k-3j-2\ell+4$ *if* $j,k,\ell \geq 1$.

Proof. All claims will be proven simultaneously using induction (on the height of the game tree). Base cases are easily checked and left to the interested reader.

`Case (1)`: From $[[1,0],[0,j],[0,k]]$ Left has no move; Right's best move is to $[[1,0],[0,j+1],[0,k-1]]$. By induction this position has value $-3k-2j+3$ by (1). Hence $[[1,0],[0,j],[0,k]]$ has value

$$\{\mid -3k-2j+3\} = -3k-2j+2,$$

as desired.

`Case (2)`: Left again has no move from the starting position. Right's best move is to $[[0,j+1],[1,0],[0,k-1]]$. If $k=1$, then this position has value $-2j+1$ by (4); if $k \geq 2$, then this position has value $-3k-2j+4$ by (2). In either case, $[[0,j],[1,0],[0,k]]$ has value $-3k-2j+3$.

`Case (3)`: First, suppose that $k \geq 2$. Left can move to $[[1,0],[0,0],[0,k]]$ from $[[0,0],[1,0],[0,k]]$. From (1), this position has value $-3k+2$. Right's best move is to $[[0,1],[1,0],[0,k-1]]$ with value $-3k+4$. Hence $[[0,0],[1,0],[0,k]]$ has value

$$\{-3k+2 \mid -3k+4\} = -3k+3.$$

If $k = 1$, then Left's only move becomes $[[1,0],[0,0],[0,1]]$, which has value -1 by (1), and Right's only move is to $[[0,1],[1,0],[0,0]]$ with value 0 by (4). Thus $[[0,0],[1,0],[0,1]]$ has value $-\frac{1}{2}$.

Case (4): For $j \geq 3$, Left has no move from $[[0,j],[1,0],[0,0]]$, and Right can move to $[[0,j-2],[1,0],[0,1]]$ with value $-2j+4$, by (2), giving $[[0,j],[1,0],[0,0]]$ the game value of $-2j+3$.

If $j = 2$, then Right's move is to $[[0,0],[1,0],[0,1]]$ with value $-\frac{1}{2}$. Thus $[[0,2],[1,0],[0,0]]$ has a value of $-1(= -2 \cdot 2 + 3)$.

Finally, if $j = 1$, then neither Left nor Right has a move from $[[0,1],[1,0],[0,0]]$, and so its value is 0.

Case (5): Let $k \geq 2$. Left has no move, and Right can move to $[[0,j+1],[0,k-1],[1,0]]$ (Right's best move). By induction this position has value $-3k-2j+5$ giving $[[0,j],[0,k],[1,0]]$ the value $-3k-2j+4$.

If $k = 1$, then, again, Left has no move. However, Right can move to $[[0,j+1],[0,0],[1,0]]$. By (6) this position has value $-2j+2$, thus giving $[[0,j],[0,1],[1,0]]$ the value $-2j+1$.

Case (6): First, we consider the case $j > 2$. Left's only move is to $[[0,j],[1,0],[0,0]]$. By (4) this position has value $-2j+3$. Right's only move is to $[[0,j-2],[0,1],[1,0]]$. By (5) this position has value $-2j+5$. Hence, if $j > 2$, then $[[0,j],[0,0],[1,0]]$ has value $-2j+4$.

If $j = 2$, $[[0,2],[1,0],[0,0]]$ has value -1 by (4). Right's move to $[[0,0],[0,1],[1,0]]$ has value $\frac{1}{2}$ by (7). Thus $[[0,2],[0,0],[1,0]]$ has value 0.

Finally, if $j = 1$, then Left's move $[[0,1],[1,0],[0,0]]$ has value 0, and Right has no moves. Thus $[[0,1],[0,0],[1,0]]$ has value 1.

Case (7): If $k \geq 2$, Left's move to $[[1,0],[0,k],[0,0]]$ has value $-2k+2$ by (1). In this case, Right's move to $[[0,1],[0,k-1],[1,0]]$ has value $-3k+5$ by (5). Therefore $[[0,0],[0,k],[1,0]]$ has value

$$\{-2k+2 \mid -3k+5\}.$$

If $k = 1$, then $[[1,0],[0,1],[0,0]]$ has value 0, and $[[0,1],[0,0],[1,0]]$ has value 1. Hence $[[0,0],[0,1],[1,0]]$ has value $\frac{1}{2}$.

Case (8): First suppose $j \geq 2$ and $k \geq 2$. Then Left's move to $[[1,0],[0,j],[0,k]]$ has value $-3k-2j+2$ by (1). Right has two sensible moves: one to $[[0,1],[0,j],[1,k-1]]$ and one to $[[0,1],[0,j-1],[1,k]]$. The former has value $-4k-3j+6$ by (10), and the latter has value $-4k-3j+5$, also by (10). Thus $[[0,0],[0,j],[1,k]]$ has value

$$\{-3k-2j+2 \mid -4k-3j+5\}.$$

Next, we look at the case where $j \geq 2$ and $k = 1$. Left's move to $[[1,0],[0,j],[0,1]]$ has value $-2j-1$ by (1). Right's move to $[[0,1],[0,j],[1,0]]$ has value $-3j+2$ by (5). Right's move to $[[0,1],[0,j-1],[1,1]]$ has value $-3j+1$ by (10). Hence $[[0,0],[0,j],[1,1]]$ has value

$$\{-2j-1 \mid -3j+1\}.$$

We now consider the case $j = 1$ and $k \geq 2$. Left's only move is to $[[1,0],[0,1],[0,k]]$. This position has value $-3k$ by (1). Right's move to $[[0,1],[0,0],[1,k]]$ has value $\{-3k+1 \mid -4k+3\}$ by (9), and his move to $[[0,1],[0,1],[1,k-1]]$ has value $-4k+3$ by (10). Therefore the position $[[0,0],[0,1],[1,k]]$ has value

$$\big\{-3k \mid \{-3k+1 \mid -4k+3\}, -4k+3\big\}.$$

It can be shown that the option $\{-3k+1 \mid -4k+3\}$ is reversible. Hence the canonical form of the position $[[0,0],[0,1],[1,k]]$ has value

$$\{-3k \mid -4k+3\}.$$

Finally, we consider the case $j = 1$ and $k = 1$. Left's move from $[[0,0],[0,1],[1,1]]$ to $[[1,0],[0,1],[0,1]]$ has value -3 by (2). Right's move to $[[0,1],[0,0],[1,1]]$ has value $\{-2 \mid -1\} = -\frac{3}{2}$. The move to $[[0,1],[0,1],[1,0]]$ has value -1 by (5). Thus the position $[[0,0],[0,1],[1,1]]$ has value $\{-3 \mid -\frac{3}{2}\} = -2$.

`Case (9)`: First, suppose that $k \geq 2$. Then Left's move from $[[0,j],[0,0],[1,k]]$ to $[[0,j],[1,0],[0,k]]$ has value $-3k-2j+3$ by (2). Right's best move is to $[[0,j],[0,1],[1,k-1]]$ with value $-4k-2j+5$, thus giving the position $[[0,j],[0,0],[1,k]]$ the game value of

$$\{-3k-2j+3 \mid -4k-2j+5\}.$$

If $k = 1$, then Left's only move has value $-2j$, again by (2). Right's move to $[[0,j],[0,1],[1,0]]$ has value $-2j+1$ by (5). Hence $[[0,j],[0,0],[1,1]]$ has value

$$\{-2j \mid -2j+1\} = (-4j+1)/2.$$

`Case (10)`: Let $j = 1$ and $k \geq 2$. Left has no move from this starting position, and Right has three sensible moves. Right can move to $[[0,\ell+1],[0,0],[1,k]]$ with value $\{-3k-2\ell+1 \mid -4k-2\ell+3\}$ by (9), or to $[[0,\ell+1],[0,1],[1,k-1]]$ with value $-4k-2\ell+3$

by (10), or to $[[0,\ell],[0,2],[1,k]]$ with value $-4k-2\ell+2$ by (10). The last move is optimal for Right, and hence the position $[[0,\ell],[0,1],[1,k]]$ has game value $-4k-2\ell+1$, as required.

If $j = 1$ and $k = 1$, then Left has no move from $[[0,\ell],[0,1],[1,1]]$, and Right has again three sensible moves. Right's move to $[[0,\ell+1],[0,0],[1,1]]$ has value $(-4\ell-3)/2$ by (9), Right's move to $[[0,\ell],[0,2],[1,0]]$ has value $-2\ell-2$ by (5), and Right's move to $[[0,\ell+1],[0,1],[1,0]]$ has value $-2\ell-1$ by (5). Therefore the position $[[0,\ell],[0,1],[1,1]]$ has value $-2\ell-3$.

If $j \geq 2$ and $k = 1$, then Left has no move from $[[0,\ell],[0,j],[1,1]]$, and Right has three moves, each not costing a pebble to make: Right can move to $[[0,\ell+1],[0,j],[1,0]]$ with value $-3j-2\ell+2$ by (5), Right can move to $[[0,\ell],[0,j+1],[1,0]]$ with value $-3j-2\ell+1$ by (5), and Right can move to $[[0,\ell+1],[0,j-1],[1,0]]$ with value $-3j-2\ell+1$ by (10). Hence the position $[[0,\ell],[0,j],[1,1]]$ has value $-3j-2\ell$.

Finally, if $j \geq 2$ and $k \geq 2$, then, as in every other subcase, Left has no move. Right has his usual three moves: Right can move to $[[0,\ell+1],[0,j],[1,k-1]]$ with value $-4k-3j-2\ell+6$ by (10), to $[[0,\ell],[0,j+1],[1,k-1]]$ with value $-4k-3j-2\ell+5$, or to $[[0,\ell+1],[0,j-1],[1,k]]$ with value $-4k-3j-2\ell+5$. Thus $[[0,\ell],[0,j],[1,k]]$ has game value $-4k-3j-2\ell+4$. □

With Lemma 3.4 in hand, we can now prove Theorem 3.3.

Proof. Left has two moves from the starting position $[[0,0],[0,0],[1,k]]$: Left can move to $[[1,0],[0,0],[0,k]]$ with value $-3k+2$ by Lemma 3.4(1) or to $[[0,0],[1,0],[0,k]]$ with value $-3k+3$ by 3.4(3). The latter move is clearly the optimal move for her.

There are two types of moves that Right can make: Right can move to $[[0,\ell],[0,0],[1,k-\ell]]$, where $1 \leq \ell \leq k$, or to $[[0,0],[0,j],[1,k-j]]$, where $1 \leq j \leq k$.

First suppose that Right moves to $[[0,\ell],[0,0],[1,k-\ell]]$, where $1 \leq \ell < k$. This position has value

$$\{-3k+\ell+3 \mid -4k+2\ell+5\} = (-3k+\ell+3) + \{0 \mid -k+\ell+2\}$$

by Lemma 3.4(9).

Next, suppose that Right moves to $[[0,k],[0,0],[1,0]]$. This position has value

$$-2k+4$$

by Lemma 3.4(6).

We now consider the other type of move for Right. Suppose that Right moves to $[[0,0],[0,j],[1,k-j]]$, where $1 < j < k$. This position has value

$$\{-3k+j+2 \mid -4k+j+5\} = (-3k+j+2) + \{0 \mid -k+3\}$$

by Lemma 3.4(8).

Next, suppose that Right moves to $[[0,0],[0,1],[1,k-1]]$. This position has value

$$\{-3k+3 \mid -4k+7\} = (-3k+3) + \{0 \mid -k+4\}$$

by Lemma 3.4(8).

Finally, suppose that Right moves to $[[0,0],[0,k],[1,0]]$. This position has value

$$\{-2k+2 \mid -3k+5\} = (-2k+2) + \{0 \mid -k+3\}$$

by Lemma 3.4(7).

We will now show that the move to $[[0,0],[0,1],[1,k-1]]$ is Right's optimal move. First note that since $\{0 \mid -k+4\} \leq 1$, it follows that

$$(-3k+3) + \{0 \mid -k+4\} \leq -3k+4 < -2k+4.$$

Next, observe that if $k = 2$, then

$$(-3 \cdot 2 + 3) + \{0 \mid -2+4\} = -2 < -\frac{3}{2} = (-2 \cdot 2 + 2) + \{0 \mid -2+3\}.$$

If $k \geq 3$, then $-k+2 < \{0 \mid -k+3\}$, and so it follows that

$$-3k+4 = (-2k+2) + (-k+2) < (-2k+2) + \{0 \mid -k+3\}.$$

Hence

$$(-3k+3) + \{0 \mid -k+4\} < (-2k+2) + \{0 \mid -k+3\}$$

for $k \geq 2$.

To show that

$$(-3k+3) + \{0 \mid -k+4\} < (-3k+\ell+3) + \{0 \mid -k+\ell+2\}, \quad \ell \geq 1,$$

it suffices to show that $\{\ell \mid -k+2\ell+2\} + \{k-4 \mid 0\} > 0$. To this end, note that Left's move to $\ell + \{k-4 \mid 0\}$ is a winning first move. Right's move to $-k+2\ell+2+\{k-4 \mid 0\}$ leads to $(-k+2\ell+2)+(k-4) = 2\ell-2 \geq 0$ after Left's response. Right's move to $\{\ell \mid -k+2\ell+2\}+0$ is not better, leading to $\ell + 0 = \ell \geq 1$.

Our last task is showing that

$$(-3k+3) + \{0 \mid -k+4\} < (-3k+j+2) + \{0 \mid -k+3\} \quad \text{for } j > 1.$$

This can be established by showing that $\{j-1 \mid -k+j+2\} + \{k-4 \mid 0\} > 0$. The proof of this fact is virtually identical to that of the similar statement in the preceding paragraph, and so it will be omitted.

It now follows that the value of the position $[[0,0],[0,0],[1,k]]$ is

$$\{-3k+3 \mid -3k+3+\{0 \mid -k+4\}\} = (-3k+3)+\{0 \mid \{0 \mid -(k-4)\}\} = (-3k+3)+_{k-4}.$$

□

In the table below, we present, without proof, other interesting game values achievable as bLue/Red Blocking Pebbles positions.

Underlying Digraph	Pebbling Configuration	Game Value
Transitive 3-Cycle	$[[1,0],[2,4],[0,0]]$	$1/4$
Transitive 3-Cycle	$[[3,1],[0,0],[0,1]]$	$1/2$
Transitive 3-Cycle	$[[2,3],[0,0],[1,0]]$	$3/4$
Transitive 3-Cycle	$[[4,4],[0,0],[0,0]]$	$\pm 1/2$
Transitive 3-Cycle	$[[3,5],[0,0],[1,0]]$	$\uparrow *$
Transitive 3-Cycle	$[[3,5],[2,0],[0,0]]$	$\uparrow^{[2]} *$
Directed P_3	$[[0,0],[2,2],[0,0]]$	$*2$

We end this section with a short discussion of the differences between BLOCKING PEBBLES and BRG-HACKENBUSH.

As noted above, the blocking mechanic of BLOCKING PEBBLES results in a preponderance of switches, whereas BRG-HACKENBUSH has no such positions. Also, while we would be surprised to find a dyadic that is not the game value for some BLOCKING PEBBLES position, we have found many dyadic rationals difficult to construct, even with the use of computational methods. BRG-HACKENBUSH positions, on the other hand, are easily constructed that have rational noninteger game values.

3.1 Green-only games

The game of BLOCKING PEBBLES restricted to green pebbles is an impartial game, with positions admitting only nimbers as game values. The interested reader will seek out [3, 1] for more on Sprague–Grundy theory and nimbers. Whereas there is no use for players to employ a blocking strategy, the game remains mathematically interesting for its connections to its roots in graph pebbling.

First, we consider in-stars and out-stars, with green pebble distributions denoted by $\rangle g_0, g_1, \dots, g_n\langle$ and $\langle g_0, g_1, \dots, g_n\rangle$, respectively. In each case, $g_i \geq 0$, and g_0 is the number of pebbles on the center vertex.

Theorem 3.5. *The value of an in-star with distribution $\rangle g_0, g_1, \dots, g_n\langle$ is $*g_0$.*

Proof. We will demonstrate this using induction on g_0. First, note that if $g_0 = 0$, then any move of a green pebble to the center from a leaf, resulting in the loss of a pebble, can be countered by returning it to the same leaf. Next, we note that any move from $\rangle g_0, g_1, \ldots, g_n\langle$ results in a change to g_0 and that there is a move from this position that results in any number of pebbles on the center node strictly less than g_0. Hence the in-star is equivalent to a NIM heap of size g_0. □

The *nim dimension* of a ruleset is the greatest integer k where a position in the ruleset has value $*2^{k-1}$ but no position has value $*2^k$. A ruleset in which the nim dimension is unbounded is said to have infinite nim dimension, as Santos and Silva [6] showed is true for KONANE. Theorem 3.5 implies that GREEN BLOCKING PEBBLES also has infinite nim dimension, whereas the nim dimension of BLUE-RED BLOCKING PEBBLES is still unknown.

The fact revealed in Theorem 3.5 that an in-star is equivalent to a single NIM heap can be generalized to multiple heaps with an out-star.

Theorem 3.6. *The value of an out-star with distribution* $\langle g_0, g_1, \ldots, g_n\rangle$ *is* $*(\sum_{i=1}^{n} \oplus g_i)$, *that is, the nim sum of all heaps.*

Proof. We note that this game is analogous to NIM, except that instead of removing pebbles from a heap, they are moved to the center at no cost. The player with the advantage simply plays the winning NIM strategy. Any move of a pebble from the center vertex to a leaf can immediately be reversed at a net cost of one pebble from the center. Thus the number of pebbles at the center does not contribute to the game value, which equals the nim sum of the leaf heaps. □

On a path, we get a similar result.

Theorem 3.7. *If* $(g_1, \ldots, g_n)$ *is a distribution of green pebbles along a path directed left to right, then the game value is* $*(\sum \oplus g_{2k})$.

Proof. An empty path is trivial, so let us assume that the claim is false and consider the set C of all counterexamples with the fewest total number of pebbles. From C let $(g_{0_1}, g_{0_2}, \ldots, g_{0_n})$ be the last when ordered lexicographically. Any move from this position either decreases the total number of pebbles or increases its lexicographic position. Therefore all options of $(g_{0_1}, g_{0_2}, \ldots, g_{0_n})$ are outside C, and hence the claim holds for them. Since each has a digital sum of even terms that differs from $(\sum_\oplus g_{2k})$, and all smaller sums are realized through NIM moves on the even heaps, we see that $(g_{0_1}, g_{0_2}, \ldots, g_{0_n})$ also satisfies the claim. Therefore, C is empty, and the claim is true. □

Note that in Theorems 3.5, 3.6, and 3.7 the strategy is equivalent to NIM. In fact, in these particular cases, BLOCKING PEBBLES is very similar to the game of POKER NIM, wherein players make NIM moves but retain any removed pebbles, and may add them to a heap instead of removing. Although POKER NIM is loopy and BLOCKING PEBBLES is not, both games played optimally have the same strategy and the same reciprocal moves for non-NIM moves.