Quantum Mechanics I

A Problem Text

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and

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Dedication

To all students of quantum mechanics past, present, and future, but especially to the first students who will learn quantum mechanics not as a mystery which cannot be understood, but as reality which must be experienced, explored, and harnessed.

Preface

The aim of this book is to enable students to solve problems in their first course in quantum mechanics. Student understanding of modern notation, appropriate application of the postulates of quantum mechanics, and appreciation of fundamental quantum mechanical concepts, all as a consequence of solving problems, are also goals. The emphasis on the student solving problems, with theory and rationale embedded in and around this focus on detailed solved problems, has led to this concept of a "problem text."

It is the outgrowth of a project started in 1997 to present material from the perspective of guided problem solving. It is largely the material presented by Larry Sorensen in a one-quarter, 400-level physics course as part of an evening Master of Science program at the University of Washington. The project originally intended only to typeset problems, posed both traditionally and then with guidance that Professor Sorensen dubbed his "garden path" version, and solutions. The stroll along the "garden path" was intended to make the homework sets accessible and an efficient use of the student's time. The population of his quantum mechanics courses was educationally diverse. Occasional doctoral candidates occupied the seats in his auditorium, as did occasional undergraduates. Most students were first time graduate students, many of whom had backgrounds in various fields of engineering. A majority of the clientele held professional positions. Some had limited preparation in physics and mathematics. Everyone appreciated Larry's orientation toward student learning, regardless of any person's starting point, and insistence on efficient use of the student's time and energy, particularly those with day jobs and families. The original concept seemed successful and was subsequently amplified and augmented.

Quantum Mechanics I may be used as a primary text, a supplementary text used with a standard textbook, or a self-study guide. Rudimentary knowledge of differential and integral calculus and differential equations is required. Comprehensive knowledge of these subjects is not required, rather, this volume may assist building competency within these areas. Prior knowledge of linear algebra is unnecessary, and again, use of this book will build competency within this area. Low-dimensional vectors and matrix operators are completely developed and extended to larger dimensions and functions. Indeed, the complete exposure to the appropriate arguments of linear algebra within the context of a first course in quantum mechanics may be responsible for its degree of student accessibility. Solutions immediately follow each prescript. Theory and rationale are addressed both in prescript and postscript narratives, and occasionally within the solutions themselves. The location of solutions immediately following each prescript, complete treatment of theory and rationale, unsolved exercises, and a higher degree of detail than is normally seen in the solutions distinguish Quantum Mechanics I from problem books intended solely as supplementary texts. Advantages of treating the theory within the focus of solved problems are repeated exposure to the notation and the mathematical techniques.

The student should attempt to solve each problem that offers challenge after reading the prescript. Solutions should be reviewed only after a problem has been completed, or in the event of an impasse. The prescripts provide varying amounts of information and explanation depending on the degree of complexity, the specific skills and techniques required, and the material previously encountered. Each prescript is intended to provide enough guidance to allow the student to obtain his/her own result. The solutions should be reviewed only after a result is obtained and for the purposes of verification, and learning new perspectives, skills, techniques, notation, and how the notation is used. Should the student's answer not be in agreement with that in the book, the conveniently located detailed solution presents the opportunity for immediate corrective learning. Of course, reading a portion of a solution to get "unstuck" may be an efficient use of the student's time and energy. Postscripts offer amplification, additional explanation, and alternate perspectives. All should be examined. Unsolved exercises generally parallel the solved problems and are intended as a second opportunity at a particular skill, technique, or concept; and are provided to be used in conjunction with the student's classwork.

Quantum Mechanics I deliberately emphasizes fundamental skills and techniques for a variety of reasons. Dirac notation, to cite one example among many possibilities, may be best learned in the context of something the student already knows. Fundamental and non-intuitive concepts seem best presented without the complexity that can serve to mask them. The educationally diverse clientele for whom this material was initially prepared often required significant review or were encountering it for the first time. The emphasis on fundamentals means that each student will not be challenged by every problem posed.

The postulates of quantum mechanics are beyond important. One view is that the entirety of quantum mechanics is the postulates. They are presented on the first page of chapter 1 to direct what is initially appropriately addressed. They are repeated on the first page of chapter 2 for emphasis and amplification. They are frequently explicitly revisited throughout the book. The postulates are realistically part of every problem. One mechanism for coping with the discomfiture of some of the counter-intuitive issues often met in the study of quantum mechanics is reconciliation with the postulates. If things are consistent with the postulates, then there is no actual issue. This is the mechanism I use to not be too bothered by quantum mechanics, and to support my standards of understanding and describing truth. I recommend it to you.

Dave DeBruyne

Prologue

The only way to learn physics is to do physics. However, almost all physics textbooks leave a huge gap between the level of the problems that they solve as examples, and the level of the problems that they assign to the students to do as homework and to thereby learn the physics. This book attempts to fill this gap for the first quantum mechanics course which our students find particularly difficult.

The level of our solved problems is the same as that of the solutions that we expect from our students. We try very hard not to leave out any of the "unnecessary" or "obvious" steps until the concepts are well rooted. Clearly no book can show you how to solve every problem. Our goal was to help you learn how to solve a representative subset of all beginning quantum mechanics problems.

We tried to select a minimum number of core concepts that all physicists would agree are essential for beginning quantum mechanics. We hope that you will find it possible to bridge the gap between these problems and the other problems that you want to solve. Most of our students have been able to make the necessary quantum leap.

There is a physics joke about the stages of learning quantum mechanics:

(1) You don't know what it means, you don't know how to calculate anything, and it doesn't bother you.
(2) You don't know what it means, you don't know how to calculate anything, and it bothers you.
(3) You don't know what it means, you know how to calculate things, and it bothers you.
(4) You don't know what it means, you know how to calculate things, and it doesn't bother you.

This book has been designed to help you learn to calculate. Our goal is to get you to stage (4).

We show you how to calculate by example: first we provide a set of paradigmatic problems and their complete solutions. By studying these detailed solutions, and by then using them to solve the additional practice problems we provide, our students have been able to master the fundamentals of quantum calculations. We consider these fundamentals to include the Dirac, Schrodinger, and Heisenberg formulations, which we treat with equal footing throughout the text.

Learning how to calculate is essential because the only language in which we can express, analyze, and discuss quantum mechanics is mathematics. As Willis Lamb put it in 1969:

> "I have taught graduate courses in quantum mechanics at Columbia, Stanford, Oxford, and Yale, and for almost all of them have dealt with measurement in the following manner. On beginning the lectures I told the students, 'You must first learn the rules of calculation in quantum mechanics, and then I will discuss the theory of measurement and discuss the meaning of the subject.' Almost invariably, the time allotted to the course ran out before I had to fulfill my promise."

As Weinberg put it

> "There is a good deal of confusion about this, because quantum mechanics can seem eerie if described in ordinary language."

According to Mermin, most physicists are at stage (3) or (4):

"...contemporary physicists come in two varieties. Type 1 physicists are bothered by EPR and Bell's Theorem. Type (2) (the majority) are not, but one has to distinguish two sub-varieties. Type 2a physicists explain why they are not bothered. Their explanations tend either to miss the point entirely (like Born's to Einstein) or to contain physical assertions that can be shown to be false. Type 2b are not bothered and refuse to explain why."

Of course the goal of physics is to reach stage (5): to know what it means to be able to calculate everything and not to be bothered by the way the universe works. Unfortunately, it has become traditional to teach quantum mechanics as a subject of great mystery that no one understands:

"If quantum mechanics hasn't profoundly shocked you, you cannot have understood it yet." (Bohr)

There is no reality in the absence of observation. (The Copenhagen Interpretation)

"Shut up and calculate." (Mermin's operational version of the Copenhagen Interpretation)

It seems crazy to us to continue to teach generation after generation of our students that they will not be able to understand quantum mechanics. All physicists eventually understand quantum mechanics to some extent. Of course, there are still open questions about the meaning of quantum mechanics; for example, Einstein's reservations about the meaning of quantum mechanics are legendary:

"I recall that during one walk Einstein suddenly stopped, turned to me and asked whether I really believed that the moon exists only when I look at it. The rest of this walk was devoted to a discussion of what a physicist should mean by the term *to exist*." (Pais)

"Quantum mechanics is very impressive. But an inner voice tells me that it is not yet the real thing. The theory yields a lot, but it hardly brings us any closer to the secret of the Old One. In any case I am convinced that He doesn't play dice." (Einstein)

"What nature demands from us is not a quantum theory or a wave theory; rather, nature demands from us a synthesis of these two views which thus far has exceeded the mental powers of physicists. I cannot seriously believe in the quantum theory because the theory is incompatible with the principle that physics is to represent reality in space and time, without spooky actions at a distance." (Einstein)

Even today, the "mysteries" of quantum mechanics continue to echo and to morph:

"No theory of reality compatible with quantum theory can require spatially separate events to be independent." (Bell)

"...experiments have now shown that what bothered Einstein is not a debatable point but the observed behavior of the real world." (Mermin)

"Anybody who's not bothered by Bell's theorem has to have rocks in his head."
(Wightman)

"Evidently, God not only plays dice but plays blind-folded, and, at times, throws them where you can't see them." (Hawking)

"And let no one use the Einstein-Podolsky-Rosen experiment to claim that information

> can be transmitted faster than light, or to postulate any 'quantum interconnectedness' between separate consciousnesses. Both are baseless. Both are mysticism. Both are moonshine." (Wheeler)

> "Niels Bohr brainwashed a whole generation of theorists into thinking that the job (interpreting quantum mechanics) was done 50 years ago." (Gell-Mann)

The formalism of quantum theory leads to results that agree with experiment with great accuracy and covers an extremely wide range of phenomena. As yet there are no experimental indications of any domain in which it might break down. Nevertheless, there still remain a number of basic questions concerning its fundamental significance which are obscure and confused.

> "Quantum mechanics, that mysterious, confusing discipline, which none of us really understands but which we know how to use." (Bohm and Hiley, quoting Gell-Mann)

> "Einstein said that if quantum mechanics is right, the world is crazy... Well, Einstein was right. The world is crazy." (Greenberger)

> "Most physicists are very naive; most still believe in real waves or real particles." (Zeilinger)

So, is there a problem, or isn't there? Note Feynman's changing perspective on this question:

> "...I think I can safely say that nobody understands quantum mechanics. So do not take the lecture too seriously, feeling that you have to understand in terms of some model what I am going to describe, but just relax and enjoy it. I am going to tell you what nature behaves like. If you will simply admit that she maybe does behave like this, you will find her a delightful, entrancing thing. Do not keep saying to yourself, if you can possibly avoid it 'But how can it be like that?' because you will get 'down the drain' into a blind alley from which nobody has yet escaped. Nobody knows how it can be like that." (1964)

> "We have always had a great deal of difficulty understanding the world view that quantum mechanics represents. At least I do, because I'm an old enough man that I haven't got to the point that this stuff is obvious to me. Okay, I still get nervous about it... You know how it always is, every new idea, it takes a generation or two until it is obvious that there's no real problem. I cannot define the real problem, therefore I suspect there's no real problem, but I'm not sure there's no real problem." (1982)

Quantum mechanics demands the most extraordinary change in our scientific world view of any physical theory. As a boy, Feynman studied Calculus Made Easy by S. P. Thompson that begins "What one fool can do, another can." He dedicated his book QED: The Strange Theory of Light and Matter to his readers with similar words: "What one fool can understand, another can." In a similar spirit, we hope that this book will help you start on your own journey towards understanding quantum mechanics.

As Feynman concludes his Lectures on Physics:

> "The quantum mechanics which was discovered in 1926 has had nearly 40 years of development, and rather suddenly it has begun to be exploited in many practical and real ways. I am sorry to say, gentlemen, that to participate in this adventure it is absolutely imperative that you learn quantum mechanics as soon as possible. I wanted most to give you some appreciation of the wonderful world and the physicist's way of

> looking at it, which, I believe, is a major part of the true culture of our times. Perhaps you will not only have some appreciation of this culture; it is even possible that you may want to join in the greatest adventure that the human mind has ever begun."

Perhaps it is only a matter of time:

> "A new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents eventually die, and a new generation grows up that is familiar with it." (Planck)

But surely, inevitably, the present drive for quantum computation and quantum cryptography will continue to force us towards a more realistic experimental practical understanding of quantum mechanics. You are the new generation–you are in the best of company–welcome to the inquiry!

Larry Sorensen

Table of Contents

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Chapter 1

Discrete Systems I The Postulates and Initial Essentials

One view is that the entirety of quantum mechanics is the postulates. The postulates expose symbology, terminology, and vocabulary for which familiarity is necessary. The discussion following the postulates identifies additional concepts, conditions, and implications. The essential goal of the first five chapters is an initial quantitative interpretation of the postulates.

1–1. What are the postulates of quantum mechanics?

The postulates direct what is initially appropriately addressed.

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1. The state of a system is represented by a vector $|\,\psi >$ in Hilbert space.

2. Every observable quantity is represented by a Hermitian operator.

3. A measurement of an observable quantity represented by the operator $\mathcal{A}$ can yield only the eigenvalues of $\mathcal{A}$.

4. If a system is in state $|\,\psi >$, a measurement of the observable quantity represented by the operator $\mathcal{A}$ that yields the eigenvalue α does so with the probability

$$P\,(\alpha) \quad \propto \quad \left|\,<\alpha\,|\,\psi>\,\right|^2,$$

where $|\,\alpha>$ is the eigenstate corresponding to the eigenvalue α.

5. A measurement of the observable quantity represented by the operator $\mathcal{A}$ with the result α changes the state of the system to the eigenstate $|\,\alpha>$.

6. The state vector obeys the time dependent Schrodinger equation

$$\mathcal{H}\,|\,\psi> \;=\; i\hbar\frac{d}{dt}\,|\,\psi>,$$

where $\mathcal{H}$ is the quantum mechanical Hamiltonian operator.

Though there is rationale to the order of presentation, there is no actual rank among the postulates so the numbering is largely cosmetic. The terms state vector, observable, eigenvalue, probability, eigenvector, and Schrodinger postulates are used for reference rather than a numbering system. Where do the postulates initially point?

The **state vector postulate:** The state vector encodes all information that can be known about the system. A state vector is also called a state function or wave function (or wavefunction), though those terms are generally reserved for continuous systems, and the term state vector also applies to continuous systems. Concepts such as eigenvalues, eigenvectors, probability, and expectation value are more straight forward in discrete systems, thus discrete systems are emphasized initially while building toward continuous systems. Also where practical, continuous systems are treated as generalizations of discrete systems to infinite dimensions. The symbol for the state vector, $|\,\psi>$, is written in Dirac notation. Familiarity with Dirac notation is a practical necessity for any modern investigation of quantum mechanical phenomena. This postulate indicates the necessity of understanding the concepts of a space in general and Hilbert space in particular. The spaces of interest to quantum mechanics are complex linear vector spaces. The field of complex numbers requires abstraction beyond classical mechanics and beyond the field of real numbers.

The **observables postulate:** An observable quantity is anything that can be measured. Position, momentum, energy, and angular momentum are the primary observable quantities addressed in this text. The outcome of a physical measurement is necessarily a real number. The objects of quantum mechanics possess intrinsically complex components and elements, thus, a challenge is presented in predicting outcomes with real number results. Operators may be expressed as matrix operators for discrete systems and differential operators for continuous systems, though an infinite by infinite matrix operator can contain the same information as a differential operator. Hermitian operators have four extraordinary and advantageous properties. It is beneficial to appreciate that (1) all Hermitian operators can be diagonalized, (2) the eigenvalues of Hermitian operators are real numbers, (3) the eigenvectors of all Hermitian operators are orthogonal, and (4) the eigenvectors of all Hermitian operators form a basis for the space in which they exist.

The **eigenvalue postulate:** An observable quantity is represented by an operator. Each operator has exactly one set of eigenvalues which are characteristics of the operator. Eigenvalues are comparable to normal modes in a mechanical system. For example, plucking an individual guitar string can initially set it vibrating unpredictably, however, it will quickly settle into a linear combination of the fundamental mode, the first overtone, the second overtone, and other overtones. Each possible mode of vibration is an eigenvalue of that guitar string. Obtaining eigenvalues is a dominant activity in quantum mechanics, because they are the only thing that **can** be measured. Eigenvalues of an observable quantity must be real. Imaginary or complex energy, for instance, are senseless concepts; a measured energy must be a real quantity. Eigenvalues, even for continuous systems, are discrete, limiting the possible values of any quantum mechanical measurement.

The **probability postulate:** The probability postulate features more Dirac notation and the interpretation of a state vector containing information about the probability of a measurement. A probability of zero means that an outcome cannot occur. A probability of one means an outcome is certain. Since only eigenvalues are possible results of a measurement, a calculation of the probabilities of all possibilities must sum to one. Notice that the probability is written as a proportionality. Normalization is the mathematical process of ensuring that the probabilities of all possibilities is one in order to obtain equalities instead of proportionalities. Notice also that the interpretation of state vector containing information about the probability of a measurement indicates a form of a product squared. The product is known as an inner product. The square of the norm is necessary because the objects of quantum mechanics possess intrinsically complex components and elements, and a probability is necessarily a real number. The probabilistic interpretation of quantum mechanics can be unsettling because it also limits what can be known and dictates that some things that can be known classically must remain unknown quantum mechanically.

The **eigenvector postulate:** An eigenstate is an eigenvector for a discrete system, and can be called an eigenvector even for a continuous system. The eigenvalues of an operator are properties of that operator. Each eigenvalue will have a corresponding eigenvector in a given basis. Should the basis change, the appearance of the operator changes, the eigenvectors change, but the eigenvalues of the operator remain the same. Ideally, each eigenvalue has a distinct eigenvector.

Some operators, however, have two or more eigenvalues that correspond to the same eigenvector in which case a complete set of commuting observables is required to remove or lift the degeneracy, which is to say uniquely identify the eigenstate. The eigenvector postulate embeds the mystery of the collapse of the wave function. Again, consider a guitar string. Hit an individual guitar string and it will settle into a combination of its normal modes quickly, but in a finite amount of time that could be measured. In non-relativistic quantum mechanics, the system settles into the state corresponding to the eigenvalue measured in the limit of zero time.

The **Schrodinger postulate:** Dirac notation is used to express the Schrodinger postulate in a differential equation. The Hamiltonian is the energy operator of classical mechanics, and the quantum mechanical Hamiltonian operator is the energy operator of quantum mechanics. It is surprising to some that the Schrodinger equation is a postulate. The Schrodinger equation is written in many forms and the form $\mathcal{H}\,|\,E> \;=\; E_n\,|\,E>$ is one of the most useful. Notice that the derivative is a time derivative. Thus, the Schrodinger equation governs the time evolution of a system. Notice also that time as a variable is limited to the Schrodinger postulate alone.

Postscript: The intent of introducing the postulates first is to motivate what follows in this chapter in particular, and in the rest of the text in general. The postulates seem to require knowledge of Dirac notation, a generalized concept of vectors, an appreciation of operators, Hermiticity, eigenvalues, eigenvectors, probability, and many other ingredients. These and other aspects of terminology, vocabulary, and mathematical technique are addressed in the problems that follow. Like problem 1–1, a few problems seek only to guide though most problems will have a prescript discussion with subsequent calculations intended to provide insight into the symbology and how it is used, and expose appropriate mathematical mechanics. Some problems have postscripts, like this one, intended to amplify, or further explain, and/or highlight particular portions of the problem.

Appendix A is the arithmetic of imaginary and complex numbers. Quantum mechanics is addressed in a complex linear vector space. The numbers, components, and elements of the scalars, vectors, and operators of quantum mechanics are intrinsically complex. Competency in quantum mechanical calculation requires mastery of complex arithmetic. Those desiring an introduction or some review should complete Appendix A before proceeding.

Linear independence is necessary to the general field of linear algebra to found the concept of a set of basis vectors. The observables postulate guarantees that anything that could be physically measured will be represented by a Hermitian operator. Hermitian operators have four advantageous properties. (1) All Hermitian operators can be diagonalized, (2) the eigenvalues of Hermitian operators are real numbers, (3) the eigenvectors of all Hermitian operators are orthogonal, and (4) the eigenvectors of all Hermitian operators form a basis for the space in which they exist. Property (4) states that the eigenvectors of a Hermitian operator necessarily form an appropriate basis, and are thus guaranteed to have the property of linear independence. Traditionally presented material on linear independence is included in appendix B for completeness.

Chapter 1 occasionally relies on results that are stated without proof, for instance, the four advantageous properties of Hermitian operators listed in the above paragraph. All claims will be supported as familiarity, technique, and background develop and evolve.

Again, one view is that the entirety of quantum mechanics is the postulates.

1–2. What is a ket?

A **ket** is a column vector written in **Dirac notation**. Dirac's ket is more general that the "magnitude and direction" concept of classical mechanics. A vector is simply a quantity requiring more than one number to describe. A force may be described as 10 Newtons at 30^o to the horizontal, which is two numbers. Another description of the same force is $5\sqrt{3}\,\widehat{x} + 5\,\widehat{y}$ Newtons, again two numbers are required to express a vector quantity in two dimensions. In three dimensions, three numbers are required. In four dimensions, four numbers are required. Four dimensions become difficult to imagine, though there are some routine uses for four dimensions. Five dimensions, however, requires five numbers, six dimensions requires six numbers and infinite dimensions requires infinite numbers. A vector is simply a quantity requiring more than one number to describe.

Dirac notation uses some symbol that uniquely identifies the ket, often the corresponding eigenvalue for an eigenvector, between the symbols "$|$" and "$>$", for instance, $|\,\psi>$, $|\,\alpha>$, $|\,1>$, $|\,\hbar>$, and $|\,i>$ are examples of how kets are written. Specific examples are seen below.

$$|\,b> \;\to\; \begin{pmatrix} 2i \\ 3-i \end{pmatrix}, \qquad |-1> \;\to\; \begin{pmatrix} -4 \\ i\cos(\phi) \\ 1+5i \end{pmatrix}, \qquad |\,\alpha> \;\to\; \begin{pmatrix} \sqrt{3}\,i \\ \sin(\theta) \\ e^{i\theta} \\ 2-5i \\ -8 \end{pmatrix}, \qquad |\,\lambda = 2> \;\to\; \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

Postscript: A ket is a column vector in Dirac notation. The symbol $|\,b>$ is a column vector in two dimensions, $|-1>$ is three dimensions, $|\,\alpha>$ is five dimensions, and $|\,\lambda = 2>$ indicates infinite dimensions. A vector is simply a quantity requiring more than one number to describe. The number of numbers required is not limited. The symbols between the "$|$" and "$>$" are not explained at this point, other than they must uniquely identify the ket from others in their system.

Notice that three of the four examples given have components with complex values. The last example is a unit vector. Whether of finite or infinite dimensions, unit vectors play a significant role in quantum mechanics.

There are an infinite number of ways to describe a force depending on the coordinate system chosen and the frame of reference. Similarly, there are an infinite number of ways to describe any vector. The symbol $|\,b>$ is a ket vector, but without reference to any coordinate system or frame of reference. The set of symbols $\begin{pmatrix} 2i \\ 3-i \end{pmatrix}$ is a **representation** of that ket vector for which a specific coordinate system and frame of reference are implied.

Notice that the symbol "$\to$" is used to specify the initial representation of $|\,b>$. Specifying a vector with a single arrow upon its initial representation denotes that $|\,b>$ is abstract and expressible in an infinite number of forms. The act of representing $|\,b>$ as a column vector with specific components fixes the coordinate system and frame of reference after which "$=$" symbols are appropriate for subsequent calculations. Many texts do not try to make this distinction, and use an equality symbol to denote their representations.

1–3. Form bras corresponding to the kets of problem 1–2.

A **bra** is a row vector that is the complex conjugate analogy of a ket. Make the column vector into a row vector and complex conjugate all of the components. The symbol identifying the ket is retained, but in a bra goes between a "$<$" and a "$|$."

$$<b\,| \;=\; \big(-2i\,,\;\; 3+i\big) \qquad <-1\,| \;=\; \big(-4,\;\; -i\cos(\phi)\,,\;\; 1-5i\big)$$

$$<\alpha\,| \;=\; \big(-\sqrt{3}\,i\,,\;\; \sin(\theta)\,,\;\; e^{-i\theta},\;\; 2+5i\,,\;\; -8\big) \qquad <\lambda=2\,| \;=\; \big(0,\;\; 0,\;\; 1,\;\; 0,\;\; 0,\cdots\big)$$

Postscript: There is no rank between bras an kets. Bras can be formed from kets, or kets can be formed from bras. A state vector, per the state vector postulate, in normally a ket. An eigenvector of an operator will be a ket in our development. Normally kets are encountered initially, then corresponding bras are formed as seen in this problem.

Terminology for making the column vector into a row vector and complex conjugating components is to form a **transpose conjugate** or **adjoint**. A bra is a transpose conjugate, or adjoint, of the corresponding ket, and a ket is a transpose conjugate, or adjoint, of the corresponding bra.

The origin of the terms "bra" and "ket" is Dirac's efficient manner of forming inner products, which look like $<\alpha\,|\,\beta>$, and which is called a bracket, or **braket**, or bra-ket. The separated parts, "bras" and "kets," are identifiable vector entities. Inner products will be addressed directly.

1–4. Add the following ket vectors where possible.

$$|\,\alpha> \;\to\; \begin{pmatrix} 2i \\ 3-4i \\ -6 \\ 2+7i \end{pmatrix} \qquad |\,5> \;\to\; \begin{pmatrix} 1 \\ 3-5i \\ 4 \\ -8i \end{pmatrix} \qquad |\,\hbar> \;\to\; \begin{pmatrix} 6+i \\ -6i \\ 5+4i \\ 3-2i \\ 3 \end{pmatrix} \qquad |\,e> \;\to\; \begin{pmatrix} 4-5i \\ -2+3i \\ 9-4i \\ 0 \\ -7i \end{pmatrix}$$

Kets and bras add just like vectors from introductory physics, simply add like component. Two kets must have the same number of components to be added, thus, $|\,\alpha>$ and $|\,5>$ can be added, $|\,\hbar>$ and $|\,e>$ can be added, but no other additions are possible among the four kets given.

$$|\,\alpha> \;+\; |\,5> \;=\; \begin{pmatrix} 2i \\ 3-4i \\ -6 \\ 2+7i \end{pmatrix} + \begin{pmatrix} 1 \\ 3-5i \\ 4 \\ -8i \end{pmatrix} = \begin{pmatrix} 2i+1 \\ 3-4i+3-5i \\ -6+4 \\ 2+7i-8i \end{pmatrix} = \begin{pmatrix} 1+2i \\ 6-9i \\ -2 \\ 2-i \end{pmatrix}$$

$$|\,\hbar> \;+\; |\,e> \;=\; \begin{pmatrix} 6+i \\ -6i \\ 5+4i \\ 3-2i \\ 3 \end{pmatrix} + \begin{pmatrix} 4-5i \\ -2+3i \\ 9-4i \\ 0 \\ -7i \end{pmatrix} = \begin{pmatrix} 6+i+4-5i \\ -6i-2+3i \\ 5+4i+9-4i \\ 3-2i+0 \\ 3-7i \end{pmatrix} = \begin{pmatrix} 10-4i \\ -2-3i \\ 14 \\ 3-2i \\ 3-7i \end{pmatrix}$$

Postscript: There is no significance to the symbols between the "$|$" and "$>$" at this point other than to uniquely identify the ket. The symbols between the "$|$" and "$>$" will often be eigenvalues.

1–5. Form bras from the kets given in problem 1–4 and add the resulting bras where possible.

Form row vectors from the column vectors given remembering to conjugate the components, then add like components. Again, two bras must have the same number of components to be added, thus, $<\alpha\,|$ and $<5\,|$ can be added, $<\hbar\,|$ and $<e\,|$ can be added, but no other additions are possible among the four bras formed.

$$
\begin{aligned}
<\alpha\,| \;+\; <5\,| \;&=\; \big(-2i\,,\;\; 3+4i\,,\;\; -6\,,\;\; 2-7i\big) \;+\; \big(1\,,\;\; 3+5i\,,\;\; 4\,,\;\; 8i\big) \\
&=\; \big(-2i+1\,,\;\; 3+4i+3+5i\,,\;\; -6+4\,,\;\; 2-7i+8i\big) \\
&=\; \big(1-2i\,,\;\; 6+9i\,,\;\; -2\,,\;\; 2+i\big)
\end{aligned}
$$

$$
\begin{aligned}
<\hbar\,| \;+\; <e\,| \;&=\; \big(6-i\,,\;\; 6i\,,\;\; 5-4i\,,\;\; 3+2i\,,\;\; 3\big) \;+\; \big(4+5i\,,\;\; -2-3i\,,\;\; 9+4i\,,\;\; 0\,,\;\; 7i\big) \\
&=\; \big(6-i+4+5i\,,\;\; 6i-2-3i\,,\;\; 5-4i+9+4i\,,\;\; 3+2i+0\,,\;\; 3+7i\big) \\
&=\; \big(10+4i\,,\;\; -2+3i\,,\;\; 14\,,\;\; 3+2i\,,\;\; 3+7i\big)
\end{aligned}
$$

Postscript: Notice that $<\alpha\,| + <5\,|$ is the bra (transpose conjugate or adjoint) corresponding to the ket $|\,\alpha> + |\,5>$, and that $<\hbar\,| + <e\,|$ and $|\,\hbar> + |\,e>$ also correspond.

1–6. Use the vectors of problems 1–4 and 1–5 to do these subtractions where possible.

(a) $<\alpha\,| \;-\; <5\,|$

(b) $|\,\hbar> \;-\; |\,e>$

(c) $|\,e> \;-\; |\,\hbar>$

(d) $<5\,| \;-\; <e\,|$

(e) $|\,e> \;-\; |\,\alpha>$

(f) $|\,\hbar> \;-\; <e\,|$

Again, like introductory physics, subtract like components. The vectors must have the same number of components to be added or subtracted. Additionally, they must be of the same type; bras cannot be added/subtracted to/from kets, and kets cannot be added/subtracted to/from bras. Only the subtractions in parts (a), (b), and (c) are possible.

$$
\begin{aligned}
\text{(a)}\qquad <\alpha\,| \;-\; <5\,| \;&=\; \big(-2i\,,\;\; 3+4i\,,\;\; -6\,,\;\; 2-7i\big) \;-\; \big(1\,,\;\; 3+5i\,,\;\; 4\,,\;\; 8i\big) \\
&=\; \big(-2i-1\,,\;\; 3+4i-3-5i\,,\;\; -6-4\,,\;\; 2-7i-8i\big) \\
&=\; \big(-1-2i\,,\;\; -i\,,\;\; -10\,,\;\; 2-15i\big)
\end{aligned}
$$

$$
\text{(b)}\qquad |\,\hbar> \;-\; |\,e> \;=\; \begin{pmatrix} 6+i \\ -6i \\ 5+4i \\ 3-2i \\ 3 \end{pmatrix} \;-\; \begin{pmatrix} 4-5i \\ -2+3i \\ 9-4i \\ 0 \\ -7i \end{pmatrix} \;=\; \begin{pmatrix} 6+i-4+5i \\ -6i+2-3i \\ 5+4i-9+4i \\ 3-2i-0 \\ 3+7i \end{pmatrix} \;=\; \begin{pmatrix} 2+6i \\ 2-9i \\ -4+8i \\ 3-2i \\ 3+7i \end{pmatrix}
$$

$$
\begin{aligned}
\text{(c)}\quad <e\,| \;-\; <\hbar\,| \;&=\; \big(4+5i\,,\;\; -2-3i\,,\;\; 9+4i\,,\;\; 0\,,\;\; 7i\big) \;-\; \big(6-i\,,\;\; 6i\,,\;\; 5-4i\,,\;\; 3+2i\,,\;\; 3\big) \\
&=\; \big(4+5i-6+i\,,\;\; -2-3i-6i\,,\;\; 9+4i-5+4i\,,\;\; 0-3-2i\,,\;\; 7i-3\big) \\
&=\; \big(-2+6i\,,\;\; -2-9i\,,\;\; 4+8i\,,\;\; -3-2i\,,\;\; -3+7i\big)
\end{aligned}
$$

(d) Similar types, but dissimilar number of components so not possible.

(e) Similar types, but dissimilar number of components so not possible.

(f) Similar number of components, but dissimilar types so not possible.

Postscript: A more general manner of stating the subtractions in parts (d), (e), and (f) are not possible is that the two objects are not in the same space.

1–7. (a) Multiply $|\,\chi\!> \rightarrow \begin{pmatrix} 2-i \\ 3i \end{pmatrix}, \quad |\,W\!> \rightarrow \begin{pmatrix} 4+3i \\ -5 \\ 6i \end{pmatrix}$ and corresponding bras by $\delta = 3-2i$.

(b) Find $<\!\chi\,\delta\,|$ and $<\!W\,\delta\,|$.

Scalar multiplication means to multiply each component by the scalar. The designation "scalar" is well conceived because scalar multiplication "scales" the vector, for instance

$$2\,|\,\chi\!> \;=\; 2\begin{pmatrix} 2-i \\ 3i \end{pmatrix} \;=\; \begin{pmatrix} 2(2-i) \\ 2(3i) \end{pmatrix} \;=\; \begin{pmatrix} 4-2i \\ 6i \end{pmatrix}$$

thus, the vector is scaled up by a factor of 2. The "scaling" is more difficult to see for complex scalars, though the process is the same.

$$\delta\,|\,\chi\!> \;=\; (3-2i)\begin{pmatrix} 2-i \\ 3i \end{pmatrix} \;=\; \begin{pmatrix} (3-2i)(2-i) \\ (3-2i)(3i) \end{pmatrix} \;=\; \begin{pmatrix} 6-3i-4i-2 \\ 9i+6 \end{pmatrix} \;=\; \begin{pmatrix} 4-7i \\ 6+9i \end{pmatrix}$$

$$\begin{aligned} <\!W\,|\,\delta \;&=\; \big(4-3i\,,\ -5\,,\ \ -6i\big)(3-2i) \;=\; \big((4-3i)(3-2i)\,,\ -5(3-2i)\,,\ \ (-6i)(3-2i)\big) \\ &=\; \big(12-8i-9i-6\,,\ -15+10i\,,\ -18i-12\big) \;=\; \big(6-17i\,,\ -15+10i\,,\ \ -12-18i\big). \end{aligned}$$

The meaning of $<\!W\,\delta\,|$ is different than $<\!W\,|\,\delta$. Given a scalar like δ, if it is outside of the bra, it means multiply by the given δ. However, if it is inside of the bra, $<\!W\,\delta\,|$ means $<\!W\,|\,\delta^*$, or

$$\begin{aligned} <\!W\,\delta\,| \;&=\; \big(4-3i\,,\ -5\,,\ \ -6i\big)(3+2i) \;=\; \big((4-3i)(3+2i)\,,\ -5(3+2i)\,,\ \ (-6i)(3+2i)\big) \\ &=\; \big(12+8i-9i+6\,,\ -15-10i\,,\ -18i+12\big) \;=\; \big(18-i\,,\ -15-10i\,,\ \ 12-18i\big). \end{aligned}$$

Just as the components of a ket are conjugated to form the corresponding bra, a scalar between the "$<$" and the "$|$" in a bra is implied to be conjugated.

$$\begin{aligned} <\!\chi\,|\,\delta \;&=\; \big(2+i\,,\ -3i\big)(3-2i) \;=\; \big((2+i)(3-2i)\,,\ \ (-3i)(3-2i)\big) \\ &=\; \big(6-4i+3i+2\,,\ -9i-6\big) \;=\; \big(8-i\,,\ -6-9i\big). \end{aligned}$$

$$\begin{aligned} <\!\chi\,\delta\,| \;&=\; \big(2+i\,,\ -3i\big)(3+2i) \;=\; \big((2+i)(3+2i)\,,\ \ (-3i)(3+2i)\big) \\ &=\; \big(6+4i+3i-2\,,\ -9i+6\big) \;=\; \big(4+7i\,,\ 6-9i\big). \end{aligned}$$

Notice that the components of $<\chi\,\delta\,|$ are complex conjugates to those of $\delta\,|\,\chi> \;=\; |\,\delta\,\chi>$, but $<\chi\,|\,\delta$ is unrelated to $\delta\,|\,\chi>$.

$$\delta\,|\,W> \;=\; (3-2i)\begin{pmatrix}4+3i\\-5\\6i\end{pmatrix} = \begin{pmatrix}(3-2i)(4+3i)\\(3-2i)(-5)\\(3-2i)6i\end{pmatrix} = \begin{pmatrix}12+9i-8i+6\\-15+10i\\18i+12\end{pmatrix} = \begin{pmatrix}18+i\\-15+10i\\12+18i\end{pmatrix}.$$

Notice that this ket has components that are complex conjugates of the bra $<W\,\delta\,|$, but is unrelated to $<W\,|\,\delta$.

Postscript: $<W\,\delta\,|$ means $<W\,|\,\delta^*$, though $|\,\delta\,W>$ and $\delta\,|\,W>$ have the same meaning.

1–8. What is a space, a subspace, a dual space, and what is the Hilbert space?

A **space** is a collection of all possible objects having the same number of components over a known field. The field is complex numbers in quantum mechanics. The number of components determine the dimension of the space. The ket vectors $|\,\alpha>$ and $|\,5>$ each have four components, those components are complex numbers, so they belong to the space $\mathbb{C}^4$. The vectors $|\,\hbar>$ and $|\,e>$ each have five complex components so belong to $\mathbb{C}^5$. The symbology

$$|\,\alpha>,\ |\,5> \;\in\; \mathbb{C}^4 \qquad \text{and} \qquad |\,\hbar>,\ |\,e> \;\in\; \mathbb{C}^5$$

denotes these facts, where the symbol "$\in$" is read "is an element of." Quantum mechanics is most generally done in $\mathbb{C}^\infty$ which is to indicate infinite dimensions of complex components.

Introductory physics is often addressed in $\mathbb{R}^3$, meaning three dimensions over the field of real numbers. Adding $2\widehat{x}+5\widehat{y}$ to $4\widehat{x}+3\widehat{y}+7\widehat{z}$ to obtain $6\widehat{x}+8\widehat{y}+7\widehat{z}$ is not mixing $\mathbb{R}^2$ and $\mathbb{R}^3$, rather this addition is done in $\mathbb{R}^3$ where the z component of the first vector is zero.

Real numbers are a subset of complex numbers. Every real number can be written $a \;=\; a+0i$. Introductory physics could be done in the larger space $\mathbb{C}^3$ by designating an imaginary component of zero for each real number, which is an unnecessary complication, and thus, introductory physics is generally addressed in the subspace of $\mathbb{C}^3$ denoted $\mathbb{R}^3$. A **subspace** is a subset of a larger space. Much of the remainder of chapter 1 is done in $\mathbb{C}^2$ or $\mathbb{C}^3$ which are subspaces of $\mathbb{C}^\infty$.

The bra vectors $<\alpha\,|$ and $<5\,|$ also belong to $\mathbb{C}^4$, but it is a different $\mathbb{C}^4$ than for $|\,\alpha>$ and $|\,5>$. The bra vectors $<\alpha\,|$ and $<5\,|$ belong to the $\mathbb{C}^4$ that is the dual space of the $\mathbb{C}^4$ containing the vectors $|\,\alpha>$ and $|\,5>$. A **dual space** is a transpose conjugate space in quantum mechanics. The bras $<\hbar\,|$ and $<e\,|$ are in the dual space of $|\,\hbar>$ and $|\,e>$. Similarly, the kets $|\,\hbar>$ and $|\,e>$ are in the dual space of the $\mathbb{C}^5$ containing $<\hbar\,|$ and $<e\,|$.

The **Hilbert space** is that space comprised by the sum of all $\mathbb{C}^n$, for all n. The Hilbert space contains all objects in $\mathbb{C}^1$, $\mathbb{C}^2$, $\mathbb{C}^3$,..., to $\mathbb{C}^\infty$ including all dual spaces.

Postscript: The Hilbert space contains not only all possible vectors, kets and bras, but also all possible operators, all possible functions, all possible spaces, or said more generally, all possibilities. A mathematician will address a Hilbert space as defined by a specific set of properties. A physicist includes all possibilities into one and speaks of the Hilbert space.

1–9. Find the norms of $|\beta> \rightarrow \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $|\chi> \rightarrow \begin{pmatrix} 2-i \\ 3i \end{pmatrix}$, $|W> \rightarrow \begin{pmatrix} 4+3i \\ -5 \\ 1-6i \end{pmatrix}$, and $<\hbar|$.

There are many concepts in $\mathbb{R}^3$ which simply do not translate completely to a space of any dimension over a complex field, and/or any dimension larger than 3. In fact, that is one reason to write, for example, $|v> = \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix}$ instead of $\vec{v} = 4\,\widehat{x} + 3\,\widehat{y} + 7\,\widehat{z}$. The difference in notation in $\mathbb{R}^3$ is cosmetic. Should the space be $\mathbb{C}^3$, $|W> = \begin{pmatrix} 4+3i \\ -5 \\ 1-6i \end{pmatrix}$ is clear, but $\vec{W} = (4+3i)\,\widehat{x} - 5\,\widehat{y} + (1-6i)\,\widehat{z}$ may not be so clear. For instance, what is the length of $\vec{W}$?

The length of the real valued $\vec{v} = 4\,\widehat{x} + 3\,\widehat{y} + 7\,\widehat{z}$ is the square root of the dot product,

$$|\vec{v}| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{4\times 4 \ + \ 3\times 3 \ + \ 7\times 7} = \sqrt{16+9+49} = \sqrt{74}$$

in whatever units are given. How can a “length” be obtained when the vector has complex components or is in four or more dimensions; does the term “length” even have meaning?

The concept of a length generalizes to the concept of a **norm** for a vector with complex components and/or more than three components. A norm is the generalization of the square root of a dot product, for instance,

$$|\vec{\beta}| = \sqrt{\vec{\beta}\cdot\vec{\beta}} = \sqrt{1\times 1 \ + \ 2\times 2 \ + \ 3\times 3 \ + \ 4\times 4} = \sqrt{1+4+9+16} = \sqrt{30}\,,$$

so the norm of $|\beta>$ is $\sqrt{30}$, denoted $\big|\,|\beta>\big| = \sqrt{30}$.

It is helpful to see that $\vec{\beta}$ and $|\beta>$ are just two different ways to express the same thing. Forming the bra of $|\beta>$, we write

$$<\beta|\beta> = \begin{pmatrix} 1, & 2, & 3, & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

and imagine rotating the column vector counter clockwise over the row vector and multiplying the corresponding components, and adding the products to obtain 30. Then the norm of $|\beta>$ is the square root, or $\big|\,|\beta>\big| = \sqrt{30}$. This is often done in two steps, a typical calculation is

$$\big|\,|\beta>\big|^2 = <\beta|\beta> = \begin{pmatrix} 1, & 2, & 3, & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = 1+4+9+16 = 30 \quad \Rightarrow \quad \big|\,|\beta>\big| = \sqrt{30}\,.$$

Remember that the product of any complex number and its complex conjugate is the sum of the square of the real part and the square of the imaginary part,

$$\big(a+bi\big)\big(a-bi\big) = a^2 - abi + abi + b^2 = a^2 + b^2\,. \text{ Thus}$$

$$\big|\,|\,\chi\!>\big|^2 \;=\; <\!\chi\,|\,\chi\!> \;=\; \big(2+i\,,\;\; -3i\big)\begin{pmatrix} 2-i \\ 3i \end{pmatrix} \;=\; 4+1+9 \;=\; 14 \qquad \Rightarrow \qquad \big|\,|\,\chi\!>\big| \;=\; \sqrt{14}$$

The products $<\beta\,|\,\beta>$ and $<\chi\,|\,\chi>$ are known as **inner products**. These are also known as **brakets** when using Dirac notation, but inner product is the more general term. An inner product is a generalization of a dot product to more than three dimensions and/or to a field allowing complex components. A norm is a generalization of length to more than three dimensions and/or to a field allowing complex components.

Notice that the probability postulate contains the inner product $<\alpha\,|\,\psi>$. Calculating inner products should become routine. Notice also that $<\alpha\,|\,\psi>$ is between two " | " symbols with a power of 2 in the probability postulate, meaning a norm squared.

Probability is a real number between 0 and 1. A complex number times its conjugate is a real number. Thus any component of a bra multiplied by its corresponding ket is a real number, and the sum of real numbers is a real number. A vector times its transpose conjugate is a real number.

Probabilities must be real numbers. The outcome of a measurement of an observable quantity such as position, momentum, energy, or angular momentum also must be a real number. The inner product, a vector times its transpose conjugate, illustrates one manner in which a real number is extracted from objects with complex components.

$$<\!W\,|\,W\!> \;=\; \big(4-3i\,,\;\; -5\,,\;\; 1+6i\big)\begin{pmatrix} 4+3i \\ -5 \\ 1-6i \end{pmatrix} \;=\; 16+9+25+1+36 \;=\; 87 \quad \Rightarrow \quad \big|\,|\,W\!>\big| \;=\; \sqrt{87}$$

$$\begin{aligned} <\!\hbar\,|\,\hbar\!> \;&=\; \big(6-i\,,\;\; 6i\,,\;\; 5-4i\,,\;\; 3+2i\,,\;\; 3\big)\begin{pmatrix} 6+i \\ -6i \\ 5+4i \\ 3-2i \\ 3 \end{pmatrix} \\ &=\; 36+1+36+25+16+9+4+9 \;=\; 136 \qquad \Rightarrow \qquad \big|\,|\,\hbar\!>\big| \;=\; \sqrt{136} \end{aligned}$$

1–10. Normalize $|\,\beta> \;\rightarrow\; \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $|\,\chi> \;\rightarrow\; \begin{pmatrix} 2-i \\ 3i \end{pmatrix}$, $|\,W> \;\rightarrow\; \begin{pmatrix} 4+3i \\ -5 \\ 1-6i \end{pmatrix}$, and $<\hbar\,|$.

Normalization is the process of adjusting the "length," or length analog, of a vector to 1.

The state vector contains all the information that can be known about a system. The direction, or direction analog, of the state vector is what contains this information. Any vector of any length (length analog) in the same direction (direction analog) contains the same information. The probability postulate uses a proportionality, $P\,(\alpha) \;\propto\; \big|\,<\alpha\,|\,\psi>\big|^2$, which may be written as an equality with a proportionality constant. The probability of certainty must be 1, a condition which may be addressed

$$P\,(\psi) \;=\; 1 \qquad \Rightarrow \qquad K\Big(\big|\,<\psi\,|\,\psi>\big|^2\Big) \;=\; 1 \qquad \Rightarrow \qquad K \;=\; \frac{1}{\big|\,<\psi\,|\,\psi>\big|^2}$$

thus the proportionality constant is the inverse of the norm squared. The usual procedure is to symmetrize by placing a factor of $\sqrt{K}$ with both the bra and the ket, for example

$$| \beta > \; = \; \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \qquad \Rightarrow \qquad < \beta | \; = \; \begin{pmatrix} 1, & 2, & 3, & 4 \end{pmatrix} \frac{1}{\sqrt{30}}$$

are the normalized vectors $| \beta >$ and $< \beta |$. Notice that

$$< \beta | \beta > \; = \; \begin{pmatrix} 1, & 2, & 3, & 4 \end{pmatrix} \frac{1}{\sqrt{30}} \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \; = \; \frac{1}{30} \big(1 + 4 + 9 + 16 \big) \; = \; \frac{30}{30} \; = \; 1 \, .$$

The normalization condition is that $< \psi | \psi > \; = \; 1$, and simply means that the probability of certainty is 1. Having found the norms of the last three vectors in the previous problem, the conclusion to this problem is simply writing the answers.

$$| \chi > \; = \; \frac{1}{\sqrt{14}} \begin{pmatrix} 2 - i \\ 3i \end{pmatrix} , \qquad | W > \; = \; \frac{1}{\sqrt{87}} \begin{pmatrix} 4 + 3i \\ -5 \\ 1 - 6i \end{pmatrix} ,$$

$$< \hbar | \; = \; \begin{pmatrix} 6 - i, & 6i, & 5 - 4i, & 3 + 2i, & 3 \end{pmatrix} \frac{1}{\sqrt{136}}$$

Postscript: Notice that the ket/bra notation does not change when the vector is normalized because, for instance, $| \chi > \; = \begin{pmatrix} 2 - i \\ 3i \end{pmatrix}$ and $| \chi > \; = \; \frac{1}{\sqrt{14}} \begin{pmatrix} 2 - i \\ 3i \end{pmatrix}$ are the same vector quantum mechanically. Only the direction (direction analog) contains information. The length (length analog), if not 1 initially, must be adjusted to be consistent with the probability postulate.

The factor $\sqrt{K} \; = \; \dfrac{1}{< \psi | \psi >}$ is called a **normalization constant**. It is conventionally placed to the left of a ket and to the right of a bra.

The probability postulate is occasionally written as an equality,

$$P \, (\alpha) \; = \; \frac{\big| < \alpha | \psi > \big|^2}{< \alpha | \alpha > < \psi | \psi >} ,$$

though it is more generally stated as a proportionality. Division of each vector $| i >$ by $< i | i >$ once, whether in the normalization process or the probability calculation, is necessary to obtain consistent probabilities.

It is not mandatory to normalize a state vector, however, doing so as a first step has the advantages of clarity, organizational consistency, and more accessible probability calculations. We will generally normalize a state vector as the first step in all calculations.

The time evolution of a state vector is governed solely by the Schrodinger postulate. Given a normalized state vector, that is a length analog or norm of 1, the only possible evolution over time is analogous to changing direction. The Schrodinger equation changes only the direction (direction analog) of the state vector.

1–11. Given $|\gamma> \;\rightarrow\; \begin{pmatrix} 3+2i \\ 4 \\ -i \end{pmatrix}$ and $|\omega> \;\rightarrow\; \begin{pmatrix} -5 \\ i \\ -4-3i \end{pmatrix}$,

(a) normalize each vector,

(b) obtain the inner products $<\gamma|\omega>$ and $<\omega|\gamma>$ using your normalized vectors,

(c) and compare $<\gamma|\omega>$ with $<\omega|\gamma>$.

Part (a) is a straightforward application of the normalization process. Of course, $<\gamma|\gamma> \;=\; 1$ and $<\omega|\omega> \;=\; 1$ following normalization. Part (c) asks does $<\gamma|\omega> \;=\; <\omega|\gamma>$?

(a) $$<\gamma|\gamma> \;=\; \big(3-2i\,,\;\; 4\,,\;\; i\big)\, N^* N \begin{pmatrix} 3+2i \\ 4 \\ -i \end{pmatrix} \;=\; \big|N\big|^2\big(9+4+16+1\big) \;=\; \big|N\big|^2\,30 \;=\; 1$$

$$\Rightarrow \qquad N = \frac{1}{\sqrt{30}} \qquad \Rightarrow \qquad |\gamma> \;=\; \frac{1}{\sqrt{30}}\begin{pmatrix} 3+2i \\ 4 \\ -i \end{pmatrix}$$ where N is a normalization constant.

$$<\omega|\omega> \;=\; \big(-5\,,\;\; -i\,,\;\; -4+3i\big)\, N^* N \begin{pmatrix} -5 \\ i \\ -4-3i \end{pmatrix} \;=\; \big|N\big|^2\big(25+1+16+9\big)$$

$$= \big|N\big|^2\,51 \;=\; 1 \qquad \Rightarrow \qquad N \;=\; \frac{1}{\sqrt{51}} \qquad \Rightarrow \qquad |\omega> \;=\; \frac{1}{\sqrt{51}}\begin{pmatrix} -5 \\ i \\ -4-3i \end{pmatrix}$$

(b) $$\begin{aligned} <\gamma|\omega> \;&=\; \big(3-2i\,,\;\; 4\,,\;\; i\big)\,\frac{1}{\sqrt{30}}\,\frac{1}{\sqrt{51}}\begin{pmatrix} -5 \\ i \\ -4-3i \end{pmatrix} \\ &= \frac{1}{\sqrt{5\cdot 3\cdot 2\cdot 17\cdot 3}}\Big((3-2i)(-5)+4(i)+i(-4-3i)\Big) \\ &= \frac{1}{3\sqrt{5\cdot 2\cdot 17}}\Big(-15+10i+4i-4i+3\Big) \;=\; \frac{1}{3\sqrt{170}}\big(-12+10i\big) \\ <\omega|\gamma> \;&=\; \big(-5\,,\;\; -i\,,\;\; -4+3i\big)\,\frac{1}{\sqrt{51}}\,\frac{1}{\sqrt{30}}\begin{pmatrix} 3+2i \\ 4 \\ -i \end{pmatrix} \\ &= \frac{1}{3\sqrt{170}}\Big((-5)(3+2i)-i(4)+(-4+3i)(-i)\Big) \\ &= \frac{1}{3\sqrt{170}}\big(-15-10i-4i+4i+3\big) \;=\; \frac{1}{3\sqrt{170}}\big(-12-10i\big) \end{aligned}$$

(c) $<\gamma|\omega> \neq <\omega|\gamma>$ but $<\gamma|\omega> = <\omega|\gamma>^*$; these are complex conjugates.

Postscript: Notice that **an inner product is a scalar**. A scalar is a tensor of zero rank, but a simpler explanation is that a scalar is an individual number that scales. Normalization constants are good examples. Given $|\chi> = \begin{pmatrix} 2-i \\ 3i \end{pmatrix}$, the normalized $|\chi> = \frac{1}{\sqrt{14}}\begin{pmatrix} 2-i \\ 3i \end{pmatrix}$ is scaled down by $\frac{1}{\sqrt{14}}$. Each component is $\frac{1}{\sqrt{14}}$ smaller than an original component, or each component is scaled to $\frac{1}{\sqrt{14}}$ of its original magnitude.

Recognizing that an inner product is a scalar also means that all properties of scalars, whether real or complex, apply. For instance, scalars commute so inner products commute, scalars can be removed from integrals, so inner products can be removed from integrals, etc.

In general, $<\gamma|\omega> = <\omega|\gamma>^*$, where the superscript " $*$ " indicates a complex conjugate. A proof in three dimensions is left as an exercise.

1–12. For $\beta = 2+3i$, $\mathcal{H} \to \begin{pmatrix} 2 & 3i & -4i \\ -3i & 0 & 5 \\ 4i & 5 & 6 \end{pmatrix}$, and $\mathcal{K} \to \begin{pmatrix} 2i & 5 & 1+i \\ 0 & 3i & -8 \\ 1 & -8 & -5i \end{pmatrix}$,

find (a) $\beta\mathcal{H}$, (b) $\beta\mathcal{K}$, (c) $\mathcal{H}+\mathcal{K}$, and (d) $\mathcal{H}-\mathcal{K}$.

This problem introduces matrix operators, demonstrates scalar multiplication of operators, and matrix operator addition and subtraction. The concepts conveyed in the subspaces of $\mathbb{C}^2$ and $\mathbb{C}^3$ can be extended to larger spaces including $\mathbb{C}^\infty$. Many topics, for instance orbital angular momentum and spin, can be addressed in $\mathbb{C}^2$ and $\mathbb{C}^3$.

A **matrix operator** is an array, known simply as a matrix in the language of linear algebra, but known as an operator in quantum mechanics. An operator may be a matrix operator or a differential operator. The postulates are most readily explained using matrix operators in $\mathbb{C}^2$ and $\mathbb{C}^3$ and then extended to $\mathbb{C}^\infty$, where all differential operators exist. Notice that four of the six postulates address operators. Operators represent physically measurable quantities, for instance energy, position, momentum, or angular momentum. All of the information of the system is contained in the state vector $|\psi>$. All of the information about the measurable quantity is contained in the operator representing that measurable quantity.

Scalar multiplication of operators is comparable to scalar multiplication of vectors, multiply each of the elements by the scalar, for instance

$$2\mathcal{H} = 2\begin{pmatrix} 2 & 3i & -4i \\ -3i & 0 & 5 \\ 4i & 5 & 6 \end{pmatrix} = \begin{pmatrix} 2\cdot 2 & 2\cdot 3i & 2(-4i) \\ 2(-3i) & 2\cdot 0 & 2\cdot 5 \\ 2\cdot 4i & 2\cdot 5 & 2\cdot 6 \end{pmatrix} = \begin{pmatrix} 4 & 6i & -8i \\ -6i & 0 & 10 \\ 8i & 10 & 12 \end{pmatrix}$$

thus, each of the elements is scaled larger by a factor of 2. Scaling by a factor of $\beta = 2+3i$ is more difficult to imagine, though the process is the same.

Addition and subtraction is accomplished by adding/subtracting like elements, for instance

$$\begin{pmatrix} 3 & 4+3i \\ 4-3i & 5 \end{pmatrix} + \begin{pmatrix} 2i & -7 \\ -7 & -2i \end{pmatrix} = \begin{pmatrix} 3+2i & 4+3i-7 \\ 4-3i-7 & 5-2i \end{pmatrix} = \begin{pmatrix} 3+2i & -3+3i \\ -3-3i & 5-2i \end{pmatrix},$$

$$\begin{pmatrix} 3 & 4+3i \\ 4-3i & 5 \end{pmatrix} - \begin{pmatrix} 2i & -7 \\ -7 & -2i \end{pmatrix} = \begin{pmatrix} 3-2i & 4+3i+7 \\ 4-3i+7 & 5+2i \end{pmatrix} = \begin{pmatrix} 3-2i & 11+3i \\ 11-3i & 5+2i \end{pmatrix}.$$

(a) $$(2+3i)\begin{pmatrix} 2 & 3i & -4i \\ -3i & 0 & 5 \\ 4i & 5 & 6 \end{pmatrix} = \begin{pmatrix} (2+3i)2 & (2+3i)3i & (2+3i)(-4i) \\ (2+3i)(-3i) & (2+3i)0 & (2+3i)5 \\ (2+3i)4i & (2+3i)5 & (2+3i)6 \end{pmatrix}$$

$$= \begin{pmatrix} 4+6i & -9+6i & 12-8i \\ 9-6i & 0 & 10+15i \\ -12+8i & 10+15i & 12+18i \end{pmatrix}$$

(b) $$(2+3i)\begin{pmatrix} 2i & 5 & 1+i \\ 0 & 3i & -8 \\ 1 & -8 & -5i \end{pmatrix} = \begin{pmatrix} (2+3i)2i & (2+3i)5 & (2+3i)(1+i) \\ (2+3i)0 & (2+3i)3i & (2+3i)(-8) \\ (2+3i)1 & (2+3i)(-8) & (2+3i)(-5i) \end{pmatrix}$$

$$= \begin{pmatrix} -6+4i & 10+15i & 2+2i+3i-3 \\ 0 & -9+6i & -16-24i \\ 2+3i & -16-24i & 15-10i \end{pmatrix} = \begin{pmatrix} -6+4i & 10+15i & -1+5i \\ 0 & -9+6i & -16-24i \\ 2+3i & -16-24i & 15-10i \end{pmatrix}$$

(c) $$\begin{pmatrix} 2 & 3i & -4i \\ -3i & 0 & 5 \\ 4i & 5 & 6 \end{pmatrix} + \begin{pmatrix} 2i & 5 & 1+i \\ 0 & 3i & -8 \\ 1 & -8 & -5i \end{pmatrix} = \begin{pmatrix} 2+2i & 3i+5 & -4i+1+i \\ -3i+0 & 0+3i & 5-8 \\ 4i+1 & 5-8 & 6-5i \end{pmatrix}$$

$$= \begin{pmatrix} 2+2i & 5+3i & 1-3i \\ -3i & 3i & -3 \\ 1+4i & -3 & 6-5i \end{pmatrix}$$

(d) $$\begin{pmatrix} 2 & 3i & -4i \\ -3i & 0 & 5 \\ 4i & 5 & 6 \end{pmatrix} - \begin{pmatrix} 2i & 5 & 1+i \\ 0 & 3i & -8 \\ 1 & -8 & -5i \end{pmatrix} = \begin{pmatrix} 2-2i & 3i-5 & -4i-1-i \\ -3i-0 & 0-3i & 5+8 \\ 4i-1 & 5+8 & 6+5i \end{pmatrix}$$

$$= \begin{pmatrix} 2-2i & -5+3i & -1-5i \\ -3i & -3i & 13 \\ -1+4i & 13 & 6+5i \end{pmatrix}$$

Postscript: Linear algebra addresses matrices of unequal dimensions, for instance, 2 X 3 or 5 X 4. Square matrix operators, for instance, 2 X 2, 3 X 3, 4 X 4,... dominate quantum mechanics. Hermitian operators in matrix form are necessarily square.

Operators must be of the same dimension to be added/subtracted, for instance a 2 X 2 matrix cannot be added/subtracted to/from a 3 X 3 matrix. Said another way, operators must be of the same space/subspace to be added/subtracted.

1–13. For $|v> \;\rightarrow\; \begin{pmatrix} 4 \\ 3i \\ 1-5i \end{pmatrix}, \quad \mathcal{H} \;\rightarrow\; \begin{pmatrix} 2 & 3i & -4i \\ -3i & 0 & 5 \\ 4i & 5 & 6 \end{pmatrix}, \quad$ and $\quad \mathcal{K} \;\rightarrow\; \begin{pmatrix} 2i & 5 & 1+i \\ 0 & 3i & -8 \\ 1 & -8 & -5i \end{pmatrix},$

find each operator/vector or vector/operator product possible.

First, notice the product $\mathcal{H}\,|\,\psi>$ in the Schrodinger postulate. This is an operator/vector product.

Operator/vector products or vector/operator products are possible only when the two objects are of the same space/subspace. A two dimensional vector cannot be multiplied by a three dimensional operator, nor can a two dimensional operator be multiplied by a three dimensional vector. A vector and an operator must have the same number of dimensions to be multiplied in any sense.

Operator/vector or vector/operator products are extensions of an inner product. A 2 X 2 example of an operator/vector product is

$$\begin{pmatrix} 3 & 4i \\ -4i & 5 \end{pmatrix} \begin{pmatrix} 6i \\ 2 \end{pmatrix}.$$

Again, imagine rotating the column vector counter clockwise over the matrix operator and forming an inner product with each row. That would look like

$$\begin{pmatrix} 3 & 4i \\ -4i & 5 \end{pmatrix} \begin{pmatrix} 6i \\ 2 \end{pmatrix} \;=\; \begin{pmatrix} 3\cdot 6i + 4i\cdot 2 \\ -4i\cdot 6i + 5\cdot 2 \end{pmatrix} \;=\; \begin{pmatrix} 18i + 8i \\ 24 + 10 \end{pmatrix} \;=\; \begin{pmatrix} 26i \\ 34 \end{pmatrix}.$$

An inner product is a scalar, but there is one scalar for each row, thus, an operator/vector product is a vector. Should this represent a Hamiltonian acting on a state vector per the product $\mathcal{H}\,|\,\psi>$ in the Schrodinger postulate, a changed state vector is the result.

The expressions

$$\begin{pmatrix} 6i \\ 2 \end{pmatrix} \begin{pmatrix} 3 & 4i \\ -4i & 5 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 3 & 4i \\ -4i & 5 \end{pmatrix} \big(-6i\,,\;\; 2 \big)$$

do not make sense under an extension of an inner product, however,

$$\big(-6i\,,\;\; 2 \big) \begin{pmatrix} 3 & 4i \\ -4i & 5 \end{pmatrix}$$

does make sense within the inner product analogy, thus

$$\big(-6i\,,\;\; 2 \big) \begin{pmatrix} 3 & 4i \\ -4i & 5 \end{pmatrix} \;=\; \big(-18i - 8i\,,\;\; 24 + 10 \big) \;=\; \big(-26i\,,\;\; 34 \big)$$

is an example of a vector/operator product.

There are four possible products between the vector and the two operators given. They are $\mathcal{H}\,|\,v>,\;\; \mathcal{K}\,|\,v>,\;\; <v\,|\,\mathcal{H},$ and $<v\,|\,\mathcal{K}.$

$$\mathcal{H}\,|\,v\!> \;=\; \begin{pmatrix} 2 & 3i & -4i \\ -3i & 0 & 5 \\ 4i & 5 & 6 \end{pmatrix} \begin{pmatrix} 4 \\ 3i \\ 1-5i \end{pmatrix} \;=\; \begin{pmatrix} 2(4) & + & 3i(3i) & + & -4i(1-5i) \\ -3i(4) & + & 0(3i) & + & 5(1-5i) \\ 4i(4) & + & 5(3i) & + & 6(1-5i) \end{pmatrix}$$

$$= \begin{pmatrix} 8-9-4i-20 \\ -12i+0+5-25i \\ 16i+15i+6-30i \end{pmatrix} = \begin{pmatrix} -21-4i \\ 5-37i \\ 6+i \end{pmatrix}$$

$$\mathcal{K}\,|\,v\!> \;=\; \begin{pmatrix} 2i & 5 & 1+i \\ 0 & 3i & -8 \\ 1 & -8 & -5i \end{pmatrix} \begin{pmatrix} 4 \\ 3i \\ 1-5i \end{pmatrix} \;=\; \begin{pmatrix} 2i(4) & + & 5(3i) & + & (1+i)(1-5i) \\ 0(4) & + & 3i(3i) & + & (-8)(1-5i) \\ 1(4) & + & -8(3i) & + & -5i(1-5i) \end{pmatrix}$$

$$= \begin{pmatrix} 8i+15i+1-5i+i+5 \\ 0-9-8+40i \\ 4-24i-5i-25 \end{pmatrix} = \begin{pmatrix} 6+19i \\ -17+40i \\ -21-29i \end{pmatrix}$$

$$<\!v\,|\,\mathcal{H} \;=\; \begin{pmatrix} 4\,, & -3i\,, & 1+5i \end{pmatrix} \begin{pmatrix} 2 & 3i & -4i \\ -3i & 0 & 5 \\ 4i & 5 & 6 \end{pmatrix}$$

$$= \Big(4(2) \;+\; -3i(-3i) \;+\; (1+5i)(4i)\,,$$
$$4(3i) \;+\; -3i(0) \;+\; (1+5i)(5)\,,$$
$$4(-4i) \;+\; -3i(5) \;+\; (1+5i)(6)\Big)$$

$$= \Big(8-9+4i-20\,, \quad 12i+0+5+25i\,, \quad -16i-15i+6+30i\Big)$$

$$= \Big(-21+4i\,, \quad 5+37i\,, \quad 6-i\Big)$$

$$<\!v\,|\,\mathcal{K} \;=\; \begin{pmatrix} 4\,, & -3i\,, & 1+5i \end{pmatrix} \begin{pmatrix} 2i & 5 & 1+i \\ 0 & 3i & -8 \\ 1 & -8 & -5i \end{pmatrix}$$

$$= \Big(4(2i) \;+\; -3i(0) \;+\; (1+5i)(1)\,,$$
$$4(5) \;+\; -3i(3i) \;+\; (1+5i)(-8)\,,$$
$$4(1+i) \;+\; -3i(-8) \;+\; (1+5i)(-5i)\Big)$$

$$= \Big(8i-0+1+5i\,, \quad 20+9-8-40i\,, \quad 4+4i+24i-5i+25\Big)$$

$$= \Big(1+13i\,, \quad 21-40i\,, \quad 29+23i\Big)$$

Postscript: Notice that $\mathcal{H}\,|\,v\!>$ and $<\!v\,|\,\mathcal{H}$ are corresponding ket and bra, but that $\mathcal{K}\,|\,v\!>$ and $<\!v\,|\,\mathcal{K}$ appear unrelated.

The terms operator/vector and vector/operator product are rarely encountered. Conventional terminology is that an operator operates to the right on a ket and to the left on a bra.

1–14. Find all possible operator/operator products for $\mathcal{C} \rightarrow \begin{pmatrix} 3 & 4i \\ -4i & 3 \end{pmatrix}$ and $\mathcal{D} \rightarrow \begin{pmatrix} 4 & -5i \\ 5i & 4 \end{pmatrix}$, and for the $\mathcal{H}$ and $\mathcal{K}$ given in the previous problem, ignoring powers.

Operators must have the same dimensions to be multiplied. $\mathcal{C}$ and $\mathcal{D}$ can be multiplied, $\mathcal{H}$ and $\mathcal{K}$ can be multiplied, but the 2 X 2 matrices cannot be multiplied the 3 X 3 matrices or vice versa.

Matrix multiplication, or an operator/operator product, is our final generalization of an inner product. A 2 X 2 example of an operator/operator product is

$$\mathcal{C}\,\mathcal{D} = \begin{pmatrix} 3 & 4i \\ -4i & 3 \end{pmatrix} \begin{pmatrix} 4 & -5i \\ 5i & 4 \end{pmatrix}.$$

First, imagine each column of the right operator as an individual ket. Then imagine rotating each individual column vector counter clockwise over the first operator and forming an inner product with each row for all columns. That is

$$\mathcal{C}\,\mathcal{D} = \begin{pmatrix} 3 & 4i \\ -4i & 3 \end{pmatrix} \begin{pmatrix} 4 & -5i \\ 5i & 4 \end{pmatrix} = \begin{pmatrix} 3(4) \;+\; 4i(5i) & 3(-5i) \;+\; 4i(4) \\ -4i(4) \;+\; 3(5i) & -4i(-5i) \;+\; 3(4) \end{pmatrix}$$

$$= \begin{pmatrix} 12-20 & -15i+16i \\ -16i+15i & -20+12 \end{pmatrix} = \begin{pmatrix} -8 & i \\ -i & -8 \end{pmatrix}$$

which is the operator/operator product. An inner product is a scalar, an operator/vector product is a vector, and an operator/operator product is an operator.

There are two operator/operator products possible for any two operators in the same space. The order in which the two operators are multiplied matters. In the other order, the product is

$$\mathcal{D}\,\mathcal{C} = \begin{pmatrix} 4 & -5i \\ 5i & 4 \end{pmatrix} \begin{pmatrix} 3 & 4i \\ -4i & 3 \end{pmatrix} = \begin{pmatrix} 4(3) \;+\; -5i(-4i) & 4(4i) \;+\; -5i(3) \\ 5i(3) \;+\; 4(-4i) & 5i(4i) \;+\; 4(3) \end{pmatrix}$$

$$= \begin{pmatrix} 12-20 & 16i-15i \\ 15i-16i & -20+12 \end{pmatrix} = \begin{pmatrix} -8 & i \\ -i & -8 \end{pmatrix}.$$

This operator/operator product is the same in both orders in this case, which is unusual.

$$\mathcal{H}\,\mathcal{K} = \begin{pmatrix} 2 & 3i & -4i \\ -3i & 0 & 5 \\ 4i & 5 & 6 \end{pmatrix} \begin{pmatrix} 2i & 5 & 1+i \\ 0 & 3i & -8 \\ 1 & -8 & -5i \end{pmatrix}$$

$$= \begin{pmatrix} 2(2i)+3i(0)-4i(1) & 2(5)+3i(3i)-4i(-8) & 2(1+i)+3i(-8)-4i(-5i) \\ -3i(2i)+0(0)+5(1) & -3i(5)+0(3i)+5(-8) & -3i(1+i)+0(-8)+5(-5i) \\ 4i(2i)+5(0)+6(1) & 4i(5)+5(3i)+6(-8) & 4i(1+i)+5(-8)+6(-5i) \end{pmatrix}$$

$$= \begin{pmatrix} 4i+0-4i & 10-9+32i & 2+2i-24i-20 \\ 6+0+5 & -15i+0-40 & -3i+3+0-25i \\ -8+0+6 & 20i+15i-48 & 4i-4-40-30i \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1+32i & -18-22i \\ 11 & -40-15i & 3-28i \\ -2 & -48+35i & -44-26i \end{pmatrix}$$

$$
\mathcal{K}\mathcal{H} = \begin{pmatrix} 2i & 5 & 1+i \\ 0 & 3i & -8 \\ 1 & -8 & -5i \end{pmatrix} \begin{pmatrix} 2 & 3i & -4i \\ -3i & 0 & 5 \\ 4i & 5 & 6 \end{pmatrix}
$$
$$
= \begin{pmatrix} 2i(2)+5(-3i)+(1+i)(4i) & 2i(3i)+5(0)+(1+i)(5) & 2i(-4i)+5(5)+(1+i)(6) \\ 0(2)+3i(-3i)-8(4i) & 0(3i)+3i(0)-8(5) & 0(-4i)+3i(5)-8(6) \\ 1(2)-8(-3i)-5i(4i) & 1(3i)-8(0)-5i(5) & 1(-4i)-8(5)-5i(6) \end{pmatrix}
$$
$$
= \begin{pmatrix} 4i-15i+4i-4 & -6+0+5+5i & 8+25+6+6i \\ 0+9-32i & 0+0-40 & 0+15i-48 \\ 2+24i+20 & 3i-0-25i & -4i-40-30i \end{pmatrix}
$$
$$
= \begin{pmatrix} -4-7i & -1+5i & 39+6i \\ 9-32i & -40 & -48+15i \\ 22+24i & -22i & -40-34i \end{pmatrix}
$$

Notice that none of the elements of the two product matrices are the same, and thus $\mathcal{H}\mathcal{K} \neq \mathcal{K}\mathcal{H}$.

Postscript: If the product of two objects is the same regardless of the order of the multiplication, the two objects are said to **commute**. All scalars, including complex scalars, commute with all other scalars. Problem 1–11 demonstrates that vectors do not commute with other vectors in general. Though $\mathcal{C}\mathcal{D} = \mathcal{D}\mathcal{C}$, the fact that $\mathcal{H}\mathcal{K} \neq \mathcal{K}\mathcal{H}$, demonstrates that operators do not commute with other operators in general.

1–15. Show by explicit multiplication in $\mathbb{C}^3$ that an identity operator multiplying

(a) a ket,

(b) a bra,

(c) a matrix operator from the left, and

(d) a matrix operator from the right results in the original ket, bra, or matrix operator.

The **identity operator** is the analogy of "1" in the real number system. One times a number is the original number. This problem introduces the identity operator but also intends to persuade that any legitimate multiplication by the identity operator results in the original object.

$$
\mathcal{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ in } \mathbb{C}^2 \qquad \text{and} \qquad \mathcal{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ in } \mathbb{C}^3.
$$

An identity operator is a matrix with 1's on the principal diagonal and zeros elsewhere. Assume

$$
|\,v> \;\rightarrow \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad \text{and} \qquad \mathcal{A} \rightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.
$$

Of course $\mathcal{I}\,|\,v> \;=\; |\,v>$, $<v\,|\,\mathcal{I} = \;<v\,|$, $\mathcal{I}\mathcal{A} = \mathcal{A}$, and $\mathcal{A}\mathcal{I} = \mathcal{A}$.

(a) $\mathcal{I}\,|\,v\!> \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \;=\; \begin{pmatrix} b_1+0+0 \\ 0+b_2+0 \\ 0+0+b_3 \end{pmatrix} \;=\; \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \;=\; |\,v\!>\,.$

(b) $<\!v\,|\,\mathcal{I} \;=\; \big(b_1,\, b_2,\, b_3\big) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;=\; \big(b_1{+}0{+}0,\; 0{+}b_2{+}0,\; 0{+}0{+}b_3\big) \;=\; \big(b_1,\, b_2,\, b_3\big) \;=\; <\!v\,|\,.$

(c) $\mathcal{I}\mathcal{A} \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$= \begin{pmatrix} a_{11}+0+0 & a_{12}+0+0 & a_{13}+0+0 \\ 0+a_{21}+0 & 0+a_{22}+0 & 0+a_{23}+0 \\ 0+0+a_{31} & 0+0+a_{32} & 0+0+a_{33} \end{pmatrix} \;=\; \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \;=\; \mathcal{A}\,.$$

(d) $\mathcal{A}\mathcal{I} \;=\; \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} a_{11}+0+0 & 0+a_{12}+0 & 0+0+a_{13} \\ a_{21}+0+0 & 0+a_{22}+0 & 0+0+a_{23} \\ a_{31}+0+0 & 0+a_{32}+0 & 0+0+a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \mathcal{A}\,.$$

Postscript: A scalar times an operator, from either side, is another operator. The meaning of

$$\alpha\mathcal{I} \quad \text{is} \quad \alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;=\; \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} \quad \text{in} \quad \mathbb{C}^3\,,$$

which is an operator with the scalar on the principal diagonal and zeros elsewhere. Operating on a vector or operator with this yields the same result as multiplication of the vector or operator by the scalar. The scalar times the identity operator is what is intended when an author sets a scalar equal to an operator. For instance,

$$\mathcal{A}\mathcal{B} \;=\; \alpha \quad \text{means} \quad \mathcal{A}\mathcal{B} \;=\; \alpha\mathcal{I}\,.$$

The concept of a diagonal operator is also introduced. The identity operator is diagonal. A **diagonal operator** is an operator with any non-zero elements on the principal diagonal and all off-diagonal elements of zero, like $\begin{pmatrix} -5 & 0 \\ 0 & 3 \end{pmatrix}$ in $\mathbb{C}^2$, and $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ in $\mathbb{C}^3$.

1–16. Given $\mathcal{L}^2 = \mathcal{L}_x^2 + \mathcal{L}_y^2 + \mathcal{L}_z^2$ and

$$\mathcal{L}_x \to \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar\,, \qquad \mathcal{L}_y \to \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\hbar\,, \qquad \mathcal{L}_z \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\hbar\,,$$

show that $\mathcal{L}^2 = 2\hbar^2\mathcal{I}$.

Powers are the operator multiplied by itself, for instance $\mathcal{H}^2 = \mathcal{H}\mathcal{H}$ or $\mathcal{C}^3 = \mathcal{C}\mathcal{C}\mathcal{C}$, or

$$\mathcal{L}_x^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar = \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1+1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

where elements that are all 1's or 0's make the matrix arithmetic particularly straightforward. The given matrices, $\mathcal{L}_x$, $\mathcal{L}_y$, and $\mathcal{L}_z$ are the component orbital angular momentum operators. They offer a convenient platform to consolidate operator arithmetic having realism, introduce operator powers, and employ an identity operator $\mathcal{I}$.

$$\mathcal{L}^2 = \mathcal{L}_x^2 + \mathcal{L}_y^2 + \mathcal{L}_z^2$$

$$\to \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar\,\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\hbar\,\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\hbar$$

$$+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\hbar$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1+1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\hbar^2 + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1+1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\hbar^2 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\hbar^2$$

$$= \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}\hbar^2 + \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix}\hbar^2 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\hbar^2$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\hbar^2 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\hbar^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\hbar^2 = 2\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2\hbar^2\mathcal{I}\,.$$

1–17. Use $\mathcal{C}$, $\mathcal{D}$, $\mathcal{H}$, and $\mathcal{K}$ from problems 1–13 and 1–14 to find $\left[\mathcal{C},\mathcal{D}\right]$ and $\left[\mathcal{H},\mathcal{K}\right]$.

The objects $\left[\mathcal{C},\mathcal{D}\right]$ and $\left[\mathcal{H},\mathcal{K}\right]$ are called commutators. In general, a **commutator** is defined

$$\left[\mathcal{A},\mathcal{B}\right] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$$

thus, if $\left[\mathcal{A},\mathcal{B}\right] = 0$, the operators $\mathcal{A}$ and $\mathcal{B}$ commute. The required product operators were calculated in problem 1–14.

$$\left[\mathcal{C},\mathcal{D}\right] = \mathcal{C}\mathcal{D} - \mathcal{D}\mathcal{C} = \begin{pmatrix} -8 & i \\ -i & -8 \end{pmatrix} - \begin{pmatrix} -8 & i \\ -i & -8 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \quad \text{in } \mathbb{C}^2 .$$

$$\left[\mathcal{H},\mathcal{K}\right] = \mathcal{H}\mathcal{K} - \mathcal{K}\mathcal{H} = \begin{pmatrix} 0 & 1+32i & -18-22i \\ 11 & -40-15i & 3-28i \\ -2 & -48+35i & -44-26i \end{pmatrix} - \begin{pmatrix} -4-7i & -1+5i & 39+6i \\ 9-32i & -40 & -48+15i \\ 22+24i & -22i & -40-34i \end{pmatrix}$$

$$= \begin{pmatrix} 4+7i & 2+27i & -57-28i \\ 2+32i & -15i & 51-43i \\ -24-24i & -48+57i & -4+8i \end{pmatrix} .$$

Postscript: The meaning of zero in the equation $\left[\mathcal{A},\mathcal{B}\right] = 0$ is a zero matrix, a matrix with all elements 0, such as $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ in $\mathbb{C}^2$. One purpose of the commutator is to show that two operators commute. The calculation for $\left[\mathcal{H},\mathcal{K}\right]$ is unlikely because it can be seen by inspection that the product matrices do not commute.

A commutator is an operator. The product of two operators is an operator, and the difference of two operators is an operator. It is important to understand that a commutator is an operator.

The operators for the observable quantities position and momentum are denoted $\mathcal{X}$ and $\mathcal{P}$ respectively. A significant statement of the relation between these two observable quantities is

$$\left[\mathcal{X},\mathcal{P}\right] = i\hbar$$

where $i\hbar$ means $i\hbar\mathcal{I}$ for the appropriate space. This is known as the **fundamental canonical commutator** when expressed for position and momentum. It is known as a **canonical commutator** when expressed for other observable quantities. The Heisenberg uncertainty relation applies to all observable quantities having a canonical commutation relation.

The identity operator $\mathcal{I}$ commutes with all other operators in the same space. Thus, a scalar multiple of an identity operator, like $\mathcal{L}^2$, commutes with all other operators in the same space.

1–18. Which of the following operators are Hermitian?

$$\mathcal{C} \rightarrow \begin{pmatrix} 3 & 4i \\ -4i & 3 \end{pmatrix}, \quad \mathcal{D} \rightarrow \begin{pmatrix} 4 & -5i \\ 5i & 4 \end{pmatrix}, \quad \mathcal{H} \rightarrow \begin{pmatrix} 2 & 3i & -4i \\ -3i & 0 & 5 \\ 4i & 5 & 6 \end{pmatrix}, \quad \mathcal{K} \rightarrow \begin{pmatrix} 2i & 5 & 1+i \\ 0 & 3i & -8 \\ 1 & -8 & -5i \end{pmatrix}.$$

Every observable quantity is represented by a **Hermitian operator**. The property of Hermiticity is defined $\mathcal{A} = \mathcal{A}^\dagger$ where the superscript dagger indicates an **adjoint** which means a transpose conjugate. Transpose means make all the rows columns, or make all the columns rows. Then complex conjugate each element to obtain the adjoint operator. For instance

$$\mathcal{C} \rightarrow \begin{pmatrix} 3 & 4i \\ -4i & 3 \end{pmatrix} \quad \Rightarrow \quad \mathcal{C}^T = \begin{pmatrix} 3 & -4i \\ 4i & 3 \end{pmatrix} \quad \Rightarrow \quad \mathcal{C}^{T*} = \mathcal{C}^\dagger = \begin{pmatrix} 3 & 4i \\ -4i & 3 \end{pmatrix} = \mathcal{C}$$

and since $\mathcal{C}^\dagger = \mathcal{C} \iff \mathcal{C} = \mathcal{C}^\dagger$, $\mathcal{C}$ is Hermitian. Another example is

$$\mathcal{Q} \rightarrow \begin{pmatrix} 1 & 2+i \\ 2-i & 3i \end{pmatrix} \quad \Rightarrow \quad \mathcal{Q}^T \rightarrow \begin{pmatrix} 1 & 2-i \\ 2+i & 3i \end{pmatrix} \quad \Rightarrow \quad \mathcal{Q}^{T*} \rightarrow \begin{pmatrix} 1 & 2+i \\ 2-i & -3i \end{pmatrix} \neq \mathcal{Q}$$

because though the other three elements are identical, the element in the second row and second column is not the same, therefore $\mathcal{Q}$ is not Hermitian.

$$\mathcal{D} \rightarrow \begin{pmatrix} 4 & -5i \\ 5i & 4 \end{pmatrix} \quad \Rightarrow \quad \mathcal{D}^T = \begin{pmatrix} 4 & 5i \\ -5i & 4 \end{pmatrix} \quad \Rightarrow \quad \mathcal{D}^{T*} = \mathcal{D}^\dagger = \begin{pmatrix} 4 & -5i \\ 5i & 4 \end{pmatrix} = \mathcal{D}$$

so $\mathcal{D}$ is Hermitian.

$$\mathcal{H} \rightarrow \begin{pmatrix} 2 & 3i & -4i \\ -3i & 0 & 5 \\ 4i & 5 & 6 \end{pmatrix} \Rightarrow \mathcal{H}^T = \begin{pmatrix} 2 & -3i & 4i \\ 3i & 0 & 5 \\ -4i & 5 & 6 \end{pmatrix} \Rightarrow \mathcal{H}^{T*} = \mathcal{H}^\dagger = \begin{pmatrix} 2 & 3i & -4i \\ -3i & 0 & 5 \\ 4i & 5 & 6 \end{pmatrix} = \mathcal{H}$$

thus $\mathcal{H}$ is Hermitian.

$$\mathcal{K} \rightarrow \begin{pmatrix} 2i & 5 & 1+i \\ 0 & 3i & -8 \\ 1 & -8 & -5i \end{pmatrix} \Rightarrow \mathcal{K}^T = \begin{pmatrix} 2i & 0 & 1 \\ 5 & 3i & -8 \\ 1+i & -8 & -5i \end{pmatrix} \Rightarrow \mathcal{K}^{T*} = \begin{pmatrix} -2i & 0 & 1 \\ 5 & -3i & -8 \\ 1-i & -8 & 5i \end{pmatrix} \neq \mathcal{K}$$

because seven of the nine elements differ, so $\mathcal{K}$ is not Hermitian.

Postscript: Observable quantities must be real numbers. The methods of calculating results that are real numbers from a space with intrinsically complex numbers, components, and elements are built into the postulates. A measurement of an observable quantity represented by an operator must be one of the eigenvalues of that operator. One of the advantageous properties of Hermitian operators is that their eigenvalues are real numbers!

A number times its complex conjugate is a real number. An inner product of a bra times its corresponding ket is a real number. Forming a bra from its corresponding ket is a multicomponent generalization of conjugating a complex number, and is in fact, transpose conjugation. Calling a bra the adjoint of its corresponding ket is entirely correct. The process of transpose conjugating an operator, forming an adjoint operator, is simply a further generalization.

Examine the main diagonal of all five examples. The process of transposing elements, making all rows columns or all columns rows, does not affect elements on the main diagonal. An operator with an imaginary or complex number on the main diagonal cannot be Hermitian because that element will remain in the same position during the process of transposition and then be changed by the process of conjugation. Hermitian operators must have real numbers on the principal diagonal.

Some of the most favorable Hermitian operators are diagonal operators, like $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ or $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, with real numbers on the main diagonal and zeros elsewhere.

Remember that real numbers are a subset of complex numbers that do not have an imaginary part, or have an imaginary part that is zero. One way to demonstrate that a number is real is to show that it is the same as its complex conjugate. That $\alpha = \alpha^*$ means α is a real number.

Hermitian operators are also known as **self adjoint** operators.

1–19. Calculate the determinants of

$$\mathcal{Q} \rightarrow \begin{pmatrix} 1 & 2+i \\ 2-i & 3i \end{pmatrix}, \qquad \mathcal{C} \rightarrow \begin{pmatrix} 3 & 4i \\ -4i & 3 \end{pmatrix}, \qquad \text{and} \qquad \mathcal{H} \rightarrow \begin{pmatrix} 2 & 3i & -4i \\ -3i & 0 & 5 \\ 4i & 5 & 6 \end{pmatrix}.$$

A measurement can only yield an eigenvalue and the measurement of any eigenvalue collapses the system to the eigenvector corresponding to that eigenvalue. Evaluating determinants is a skill that is central to the manual method of solving the eigenvalue/eigenvector problem.

An integral with limits is a scalar, (though it may take considerable effort to evaluate). Similarly, a **determinant** is a scalar. $\det \mathcal{A}$ is a scalar associated with a square matrix operator.

A determinant is a function of a square matrix that is the alternating sum of the products of the elements of any row or column and the respective cofactors.

Symbolically, $\mathcal{A} \rightarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \Rightarrow \quad \det \mathcal{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12},$ and

$$\mathcal{A} \rightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \Rightarrow \quad \det \mathcal{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}\big(a_{22}a_{33} - a_{32}a_{23}\big) - a_{21}\big(a_{12}a_{33} - a_{32}a_{13}\big) + a_{31}\big(a_{12}a_{23} - a_{22}a_{13}\big)$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} + a_{21}a_{32}a_{13} - a_{21}a_{12}a_{33} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13}.$$

This can be extended to arbitrary or infinite dimension. For evaluation of determinants beyond dimension 3, refer to Arfken[1], Boas[2], or a favorite linear algebra text.

$$\det \mathcal{Q} = \begin{vmatrix} 1 & 2+i \\ 2-i & 3i \end{vmatrix} = (1)(3i) - (2-i)(2+i) = 3i - 4 - 1 = -5 + 3i\,.$$

$$\det \mathcal{C} = \begin{vmatrix} 3 & 4i \\ -4i & 3 \end{vmatrix} = (3)(3) - (-4i)(4i) = 9 - 16 = -7\,.$$

$$\det \mathcal{H} = \begin{vmatrix} 2 & 3i & -4i \\ -3i & 0 & 5 \\ 4i & 5 & 6 \end{vmatrix}$$

$$= (2)(0)(6) - (2)(5)(5) + (-3i)(5)(-4i) - (-3i)(3i)(6) + (4i)(3i)(5) - (4i)(0)(-4i)$$

$$= 0 - 50 - 60 - 54 - 60 - 0 = -224\,.$$

Postscript: Notice that the determinant of the non-Hermitian operator is a complex number, but that the determinants of the two Hermitian operators are real numbers. The determinant of a Hermitian operator must be a real number.

If $\det \mathcal{A} = 0$, $\mathcal{A}$ is **singular**. The inverse of a singular operator does not exist.

1–20. Given the matrix operator $\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$,

(a) show that $|\,2> = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 2,

(b) and that $|\,3> = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector with eigenvalue 3.

(c) Interpret these facts geometrically.

This problem is a numerical introduction to the eigenvalue/eigenvector equation. Both eigenvalues and eigenvectors are central to interpreting the postulates.

The product of an operator and a vector is another vector, $\mathcal{A}\,|\,v> = |\,w>$. For some matrices and all Hermitian operators, there are products of the operator and special vectors such that the new vector is a product of a scalar and the original vector, *i.e.*,

$$\mathcal{A}\,|\,v_i> = \alpha_i\,|\,v_i> \,.$$

[1] Arfken *Mathematical Methods for Physicists,* Academic Press, 1970, chap 4.
[2] Boas *Mathematical Methods in the Physical Sciences,* John Wiley & Sons, 1983, pp. 87–94.

This **eigenvalue/eigenvector equation** actually describes a family of equations, thus subscripts are sometimes used, though $\mathcal{A}\,|\,v> \;=\; \alpha\,|\,v>$ has the same meaning as the above equation. There are as many matching scalars and vectors as the dimension of the space for operators having an eigenvalue/eigenvector relation. The scalars are known as **eigenvalues** and the vectors are known as **eigenvectors**. It is popular to place the eigenvalue between the $|$ and $>$ that indicate the ket to identify the corresponding eigenvector. Thus, demonstrate that

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \;=\; 2\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

for part (a), for instance. This is simply a numerical statement of the eigenvalue/eigenvector equation. Consider "length" and "direction" of the vectors for part (c).

(a) $\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \;=\; \begin{pmatrix} 1+1 \\ -2+4 \end{pmatrix} \;=\; \begin{pmatrix} 2 \\ 2 \end{pmatrix} \;=\; 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so for the operator $\mathcal{A} \;=\; \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$,

$$2 \text{ is an eigenvalue, and } \;|\,2> \;=\; \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is the corresponding eigenvector.}$$

(b) $\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \;=\; \begin{pmatrix} 1+2 \\ -2+8 \end{pmatrix} \;=\; \begin{pmatrix} 3 \\ 6 \end{pmatrix} \;=\; 3\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, so for the operator $\mathcal{A} \;=\; \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$,

$$3 \text{ is an eigenvalue, and } \;|\,3> \;=\; \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ is the corresponding eigenvector.}$$

In a 2 dimensional space, we expect exactly two eigenvalues and two eigenvectors.

(c) In part (a), the effect of the operator on its eigenvector is a vector twice as long and in exactly the same direction as the original vector. In part (b), the effect of the operator on its eigenvector is a vector three times as long as the original vector and in exactly the same direction. An operator acting on an eigenvector results in changing the "length" of the vector without rotating it. The length of the eigenvector is changed by a factor equal to its eigenvalue.

Postscript: The eigenvalue/eigenvector equation is some form of $\mathcal{A}\,|\,v> \;=\; \alpha\,|\,v>$. An operator acting on an eigenvector returns the same vector scaled.

The eigenvectors in this problem were deliberately not normalized to allow integer arithmetic. $|\,2> \;=\; \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $|\,3> \;=\; \dfrac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ when normalized.

Physicists use various methods to denote eigenvectors other than by placing the eigenvalue between the $|$ and $>$ that indicate the eigenvector or **eigenket**. For instance, $|\,v_1>$ or $|\,1>$ might identify the first eigenvector and $|\,v_2>$ or $|\,2>$ might denote the second eigenvector. Indices known as quantum numbers are often placed between the $|$ and $>$ that indicate the eigenket.

Again, the eigenvalue/eigenvector equation always denotes a family of equations. Subscripts are not always used. $\mathcal{A}\,|\,v> \;=\; \alpha\,|\,v>$ has the same meaning as $\mathcal{A}\,|\,v_i> \;=\; \alpha_i\,|\,v_i>$. There are as many eigenvalues and eigenvectors as the dimension of the operator under consideration.

The prefix "eigen" on the words "value" and "vector" is from German and translates roughly to the word "characteristic" in English. Thus, an eigenvalue is a characteristic value of the operator, and an eigenvector is a characteristic vector of the operator.

1–21. Find the eigenvalues and normalized eigenvectors of

$$\mathcal{F} \rightarrow \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}, \qquad \mathcal{C} \rightarrow \begin{pmatrix} 3 & 4i \\ -4i & 3 \end{pmatrix}, \qquad \text{and} \qquad \mathcal{H} \rightarrow \begin{pmatrix} 1 & 0 & -2i \\ 0 & 1 & 0 \\ 2i & 0 & 4 \end{pmatrix}.$$

Here is a roadmap to solve an eigenvalue/eigenvector problem manually.

(1) Set $\det\left(\mathcal{A} - \alpha\mathcal{I}\right) = 0$. This is known as the **characteristic equation**.

(2) Solve the characteristic equation. The solutions are the eigenvalues.

(3) Use the eigenvalue/eigenvector equation to solve for the eigenvectors.

(4) Normalize the eigenvectors.

There are some conventions associated with the eigenvalue/eigenvector problem. We will work from the eigenvalue of least magnitude to the eigenvalue of greatest magnitude. If the simultaneous equations that result from the eigenvalue/eigenvector equation are indeterminate, choose values that allow integer arithmetic, usually meaning choose the first non-zero element of the eigenket to be positive and real, most often 1. This is a convention that is popular but not universal[3]. The operator $\mathcal{H}$ in this problem can be addressed with integers if the last non-zero element of the eigenvector is chosen to be 1.

$$\det\left(\mathcal{F} - \alpha\mathcal{I}\right) = \det\left[\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} - \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}\right] = \det\begin{pmatrix} 1-\alpha & 1 \\ -2 & 4-\alpha \end{pmatrix} = (1-\alpha)(4-\alpha) - (1)(-2)$$

$= 4 - 5\alpha + \alpha^2 + 2 = \alpha^2 - 5\alpha + 6$. The characteristic equation is $\alpha^2 - 5\alpha + 6 = 0$

$\Rightarrow \quad (\alpha - 2)(\alpha - 3) = 0 \quad \Rightarrow \quad \alpha = 2,\ \alpha = 3,$ are the eigenvalues. $\mathcal{A}\,|\,v> \; = \alpha\,|\,v>$ for

$$\alpha = 2 \quad \text{is} \quad \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2\begin{pmatrix} a \\ b \end{pmatrix} \quad \Rightarrow \quad \begin{array}{rcrcl} a & + & b & = & 2a \\ -2a & + & 4b & = & 2b \end{array} \quad \Rightarrow \quad \begin{array}{rcl} b & = & a \\ -2a & = & -2b \end{array}$$

so let $a = 1 \quad \Rightarrow \quad b = 1 \quad \Rightarrow \quad |\,2> \; = N\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and normalizing

$$(1,\ 1)N^* N\begin{pmatrix} 1 \\ 1 \end{pmatrix} = |\,N\,|^2(1+1) = |\,N\,|^2(2) = 1 \quad \Rightarrow \quad N = \frac{1}{\sqrt{2}}$$

$\Rightarrow \quad |\,2> \; = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the normalized eigenvector. $\mathcal{A}\,|\,v> \; = \alpha\,|\,v>$ for $\alpha = 3$ is

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 3\begin{pmatrix} a \\ b \end{pmatrix} \quad \Rightarrow \quad \begin{array}{rcrcl} a & + & b & = & 3a \\ -2a & + & 4b & = & 3b \end{array} \quad \Rightarrow \quad \begin{array}{rcl} b & = & 2a \\ -2a & = & -b \end{array}$$

[3] Shankar, *Principles of Quantum Mechanics* (Plenum Press, New York, 1994), 2nd ed., p. 34.

then $\quad a \;=\; 1 \quad\Rightarrow\quad b \;=\; 2 \quad\Rightarrow\quad |\,3\!> \;=\; N\begin{pmatrix}1\\2\end{pmatrix},$ and normalizing

$$(1\,,\;\; 2)N^*N\begin{pmatrix}1\\2\end{pmatrix} \;=\; |\,N\,|^2(1+4) \;=\; |\,N\,|^2(5) \;=\; 1 \quad\Rightarrow\quad N \;=\; \frac{1}{\sqrt{5}}$$

$$\Rightarrow\quad |\,3\!> \;=\; \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix} \quad \text{is the normalized eigenvector.}$$

$$\det\left(\mathcal{C}-\alpha\mathcal{I}\right) = \det\left[\begin{pmatrix}3 & 4i\\-4i & 3\end{pmatrix} - \begin{pmatrix}\alpha & 0\\0 & \alpha\end{pmatrix}\right] = \det\begin{pmatrix}3-\alpha & 4i\\-4i & 3-\alpha\end{pmatrix} = (3-\alpha)(3-\alpha)-(4i)(-4i)$$

$= \; 9-6\alpha+\alpha^2-16 \;=\; \alpha^2-6\alpha-7\,.$ The characteristic equation is $\alpha^2-6\alpha-7 \;=\; 0$

$\Rightarrow\quad (\alpha+1)(\alpha-7)=0 \quad\Rightarrow\quad \alpha=-1\,,\ \alpha=7\,,$ are the eigenvalues. $\mathcal{C}\,|\,v\!> \;=\; \alpha\,|\,v\!>$ for $\alpha=-1\,,$

$$\begin{pmatrix}3 & 4i\\-4i & 3\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} \;=\; -1\begin{pmatrix}a\\b\end{pmatrix} \quad\Rightarrow\quad \begin{array}{rcrcr}3a & + & 4ib & = & -a\\-4ia & + & 3b & = & -b\end{array} \quad\Rightarrow\quad \begin{array}{rcr}ib & = & -a\\-ia & = & -b\end{array}$$

thus $\quad a \;=\; 1 \quad\Rightarrow\quad b \;=\; i \quad\Rightarrow\quad |\!-1\!> \;=\; N\begin{pmatrix}1\\i\end{pmatrix},$ and normalizing

$$(1\,,\;\; -i)N^*N\begin{pmatrix}1\\i\end{pmatrix} \;=\; |\,N\,|^2(1+1) \;=\; |\,N\,|^2(2) \;=\; 1 \quad\Rightarrow\quad N \;=\; \frac{1}{\sqrt{2}}$$

$\Rightarrow\quad |\!-1\!> \;=\; \dfrac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$ is the normalized eigenvector. $\mathcal{C}\,|\,v\!> \;=\; \alpha\,|\,v\!>$ for $\alpha \;=\; 7$ is

$$\begin{pmatrix}3 & 4i\\-4i & 3\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} \;=\; 7\begin{pmatrix}a\\b\end{pmatrix} \quad\Rightarrow\quad \begin{array}{rcrcr}3a & + & 4ib & = & 7a\\-4ia & + & 3b & = & 7b\end{array} \quad\Rightarrow\quad \begin{array}{rcr}ib & = & a\\-ia & = & b\end{array}$$

so $\quad a \;=\; 1 \quad\Rightarrow\quad b \;=\; -i \quad\Rightarrow\quad |\,7\!> \;=\; N\begin{pmatrix}1\\-i\end{pmatrix},$ and normalizing

$$(1\,,\;\; i)N^*N\begin{pmatrix}1\\-i\end{pmatrix} \;=\; |\,N\,|^2(1+1) \;=\; |\,N\,|^2(2) \;=\; 1 \quad\Rightarrow\quad N \;=\; \frac{1}{\sqrt{2}}$$

$\Rightarrow\quad |\,7\!> \;=\; \dfrac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$ is the second normalized eigenvector. Then for $\mathcal{H}\,,$

$$\det\begin{pmatrix}1-\alpha & 0 & -2i\\0 & 1-\alpha & 0\\2i & 0 & 4-\alpha\end{pmatrix} \;=\; (1-\alpha)^2(4-\alpha)-(2i)(1-\alpha)(-2i) \;=\; (1-2\alpha+\alpha^2)(4-\alpha)-4(1-\alpha)$$

$= \; 4-9\alpha+6\alpha^2-\alpha^3-4+4\alpha = -\alpha^3+6\alpha^2-5\alpha \quad\Rightarrow\quad \alpha^3-6\alpha^2+5\alpha \;=\; 0$ is the characteristic equation and can be factored $\alpha(\alpha-1)(\alpha-5) \;=\; 0 \quad\Rightarrow\quad \alpha \;=\; 0,\ 1,$ and 5 are the eigenvalues.

$$\mathcal{H}\,|\,v\!> = 0\,|\,v\!> \;\Rightarrow\; \begin{pmatrix}1 & 0 & -2i\\0 & 1 & 0\\2i & 0 & 4\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = 0\begin{pmatrix}a\\b\\c\end{pmatrix} \;\Rightarrow\; \begin{array}{rcrcr}a & + & -2ic & = & 0\\ & & b & = & 0\\2ia & + & 4c & = & 0\end{array} \;\Rightarrow\; \begin{array}{c}a = 2ic\\b = 0\\ia = -2c\end{array}$$

so $b=0$ and pick $c=1$ so that integers may be used $\quad\Rightarrow\quad a=2i \quad\Rightarrow\quad |\,0\!> \,=\, N\begin{pmatrix}2i\\0\\1\end{pmatrix}$.

$$\big(-2i\,,\,0\,,\,1\big)N^*N\begin{pmatrix}2i\\0\\1\end{pmatrix} = |\,N\,|^2(4{+}0{+}1) = |\,N\,|^2(5)=1 \;\Rightarrow\; N=\frac{1}{\sqrt{5}} \;\Rightarrow\; |\,0\!>=\frac{1}{\sqrt{5}}\begin{pmatrix}2i\\0\\1\end{pmatrix}.$$

$$\mathcal{H}\,|\,v\!> \,=\, 1\,|\,v\!> \,\Rightarrow \begin{pmatrix}1&0&-2i\\0&1&0\\2i&0&4\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = 1\begin{pmatrix}a\\b\\c\end{pmatrix} \;\Rightarrow\; \begin{array}{ccccc} a & + & -2ic & = & a\\ & b & & = & b\\ 2ia & + & 4c & = & c\end{array} \;\Rightarrow\; \begin{array}{c} c=0\\ b=b\\ 2ia=-3c\end{array}$$

so $c \,=\, 0 \quad\Rightarrow\quad a \,=\, 0$ and let $b \,=\, 1 \quad\Rightarrow\quad |\,1\!> \,=\, N\begin{pmatrix}0\\1\\0\end{pmatrix}$.

$$\big(0\,,\;1\,,\;0\big)N^*N\begin{pmatrix}0\\1\\0\end{pmatrix} \,=\, |\,N\,|^2(0{+}1{+}0) \,=\, |\,N\,|^2(1) \,=\, 1 \quad\Rightarrow\quad N=1 \quad\Rightarrow\quad |\,1\!> \,=\, \begin{pmatrix}0\\1\\0\end{pmatrix}.$$

$$\mathcal{H}\,|\,v\!> \,=\, 5\,|\,v\!> \,\Rightarrow \begin{pmatrix}1&0&-2i\\0&1&0\\2i&0&4\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = 5\begin{pmatrix}a\\b\\c\end{pmatrix} \;\Rightarrow\; \begin{array}{ccccc} a & + & -2ic & = & 5a\\ & b & & = & 5b\\ 2ia & + & 4c & = & 5c\end{array} \;\Rightarrow\; \begin{array}{c} -ic=2a\\ b=0\\ 2ia=c\end{array}$$

so $b \,=\, 0$ then let $a \,=\, 1 \quad\Rightarrow\quad c \,=\, 2i \quad\Rightarrow\quad |\,5\!> \,=\, N\begin{pmatrix}1\\0\\2i\end{pmatrix}$.

$$\big(1\,,\,0\,,\,-2i\big)N^*N\begin{pmatrix}1\\0\\2i\end{pmatrix} = |\,N\,|^2(1{+}0{+}4) = |\,N\,|^2(5)=1 \;\Rightarrow\; N=\frac{1}{\sqrt{5}} \;\Rightarrow\; |\,5\!>=\frac{1}{\sqrt{5}}\begin{pmatrix}1\\0\\2i\end{pmatrix}.$$

Postscript: The characteristic equation arises from the eigenvalue/eigenvector equation,

$$\mathcal{A}\,|\,a\!> \,=\, \alpha\,|\,a\!> \quad\Rightarrow\quad \mathcal{A}\,|\,a\!> - \,\alpha\,|\,a\!> \,=\, 0 \quad\Rightarrow\quad \big(\mathcal{A}-\alpha\mathcal{I}\big)\,|\,a\!> \,=\, 0$$

and $|\,a\!> \,\neq\, 0$ for a real system, so the only non-trivial solution is when $\det\big(\,\mathcal{A}-\alpha\mathcal{I}\,\big) \,=\, 0$.

The primary difficulty using this "manual" method is that the characteristic equation can be difficult to solve. The matrix operators in this text either represent physical systems or are designed to communicate concepts for which convenient eigenvalues are generally designed into the problems and exercises presented. Numerous computer applications are available to find eigenvalues and eigenvectors. A computer application may be a practical necessity to obtain eigenvalues and eigenvectors for an arbitrary matrix operator.

The method of obtaining eigenvalues and eigenvectors known as diagonalization will be developed using the manual method introduced in this problem. The method of diagonalization extends to $\mathbb{C}^\infty$ in which the manual method does not apply. The intent is to first expose diagonalization in $\mathbb{C}^2$ and $\mathbb{C}^3$. The genuine power of the method of obtaining eigenvalues and eigenvectors using diagonalization is its existence.

1–22. What are the eigenvalues and eigenvectors for $\mathcal{L}_y \rightarrow \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$?

Obtaining eigenvalues and eigenvectors in a realistic application such as a component orbital angular momentum operator (less scalar factors) seems appropriate before addressing diagonalization.

$$\det\left(\mathcal{L}_y - \alpha\mathcal{I}\right) = \det\begin{pmatrix} -\alpha & -i & 0 \\ i & -\alpha & -i \\ 0 & i & -\alpha \end{pmatrix} = (-\alpha)^3 - (-\alpha)(i)(-i) - (-i)(i)(-\alpha) = -\alpha^3 + \alpha + \alpha = 0$$

$$\Rightarrow \quad \alpha^3 - 2\alpha = 0 \quad \Rightarrow \quad \alpha(\alpha^2 - 2) = \alpha(\alpha - \sqrt{2})(\alpha + \sqrt{2}) = 0 \quad \Rightarrow \quad \alpha = -\sqrt{2},\ 0,\ \sqrt{2}$$

are the eigenvalues. Starting with the smallest eigenvalue to obtain eigenvectors,

$$\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \left(-\sqrt{2}\right)\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} & -bi & & = & -\sqrt{2}\,a \\ ai & - & ci & = & -\sqrt{2}\,b \\ & bi & & = & -\sqrt{2}\,c \end{matrix}$$

Adding the top and bottom equations, $-\sqrt{2}\,a - \sqrt{2}\,c = 0 \quad \Rightarrow \quad a = -c$. Using this in the middle equation, we get $ai + ai = -\sqrt{2}\,b \quad \Rightarrow \quad -\dfrac{2ai}{\sqrt{2}} = b \quad \Rightarrow \quad b = -\sqrt{2}\,ai$. If we choose $a = 1$, then $b = -\sqrt{2}\,i$, and $c = -1$, so the eigenket is

$$|-\sqrt{2}> \; = A\begin{pmatrix} 1 \\ -\sqrt{2}i \\ -1 \end{pmatrix} \quad \Rightarrow \quad <-\sqrt{2}\,|-\sqrt{2}> \; = \left(1,\ \sqrt{2}\,i,\ -1\right) A^* A \begin{pmatrix} 1 \\ -\sqrt{2}\,i \\ -1 \end{pmatrix}$$

$$= |\,A\,|^2\left(1+2+1\right) \quad \Rightarrow \quad 4\,|\,A\,|^2 = 1 \quad \Rightarrow \quad |-\sqrt{2}> \; = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2}\,i \\ -1 \end{pmatrix} \quad \text{for the eigenvalue } -\sqrt{2}\,.$$

To find the eigenvector corresponding to the next smallest eigenvalue 0,

$$\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} & -bi & & = & 0 \\ ai & - & ci & = & 0 \\ & bi & & = & 0 \end{matrix}$$

The top and bottom equations say $b = 0$, and the middle equation says $a = c$, so by convention, $a = 1 \quad \Rightarrow \quad c = 1$, and the eigenket is

$$|\,0> \; = A\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad <0\,|\,0> \; = \left(1,\ 0,\ 1\right) A^* A \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = |\,A\,|^2\left(1+0+1\right) = 2\,|\,A\,|^2 = 1$$

$$\Rightarrow \quad |\,0> \; = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{for the eigenvalue } 0\,.$$

For the eigenvector corresponding to the largest eigenvalue,

$$\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \sqrt{2} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} & -bi & & = & \sqrt{2}\,a \\ ai & - & ci & = & \sqrt{2}\,b \\ & bi & & = & \sqrt{2}\,c \end{matrix}$$

Adding the top and bottom equations, we get $\sqrt{2}\,a + \sqrt{2}\,c = 0 \quad \Rightarrow \quad a = -c$. Using this in the middle equation, we get

$$ai + ai = \sqrt{2}\,b \quad \Rightarrow \quad \frac{2ai}{\sqrt{2}} = b \quad \Rightarrow \quad b = \sqrt{2}\,ai\,.$$

If we choose $a = 1$, then $b = \sqrt{2}\,i$, and $c = -1$, the eigenket is

$$|\,\sqrt{2}> \;=\; A \begin{pmatrix} 1 \\ \sqrt{2}\,i \\ -1 \end{pmatrix} \quad \Rightarrow \quad <\sqrt{2}\,|\,\sqrt{2}> \;=\; \big(\,1,\ -\sqrt{2}\,i,\ -1\big)\, A^* A \begin{pmatrix} 1 \\ \sqrt{2}\,i \\ -1 \end{pmatrix}$$

$$= \;|\,A\,|^2\big(1+2+1\big) \quad \Rightarrow \quad 4\,|\,A\,|^2 = 1 \quad \Rightarrow \quad |\,\sqrt{2}> \;=\; \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2}i \\ -1 \end{pmatrix} \text{ for the eigenvalue } \sqrt{2}\,.$$

1–23. Find the normal modes of oscillation for a frictionless system of two identical masses connected to each other and outside walls by ideal springs with the same spring constant.

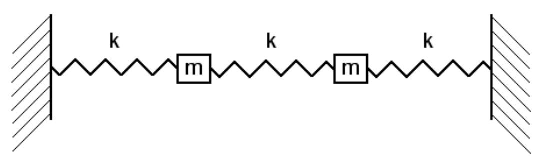

Figure 1–1. Two identical masses, three identical springs.

A brief excursion to classical mechanics is appropriate to explain the general meaning of an eigenvalue, and also to generally expose how the eigenvalue/eigenvector problem arises naturally from physical systems, including quantum mechanical systems.

Normal modes of oscillation are also known as natural frequencies, characteristic modes, and characteristic frequencies. Imagine displacing one or both masses. The masses may oscillate unpredictably at first, but given some time for the energy to be distributed throughout the system, this system will oscillate in one of two modes at one of two frequencies. These are the normal modes of oscillation, or just normal modes. Calculating eigenvalues in a quantum mechanical system is comparable to finding the normal modes in a classical mechanical system.

The ideal springs provide restoring forces $F = -kx$. Let the displacement of the mass on the left be x_1 and the mass on the right be x_2. The restoring force on the masses are

$$F_1 = -kx_1 + k\,(x_2 - x_1) \quad \Rightarrow \quad ma_1 = -2kx_1 + kx_2 \quad \Rightarrow \quad ma_1 + 2kx_1 - kx_2 = 0$$

$$F_2 = -kx_2 - k\,(x_2 - x_1) \quad \Rightarrow \quad ma_2 = kx_1 - 2kx_2 \quad \Rightarrow \quad ma_2 - kx_1 + 2kx_2 = 0$$

where the two equations on the right can be expressed as the matrix equation

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

which when multiplied by $\begin{pmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{m} \end{pmatrix}$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

and since $a_i = -\omega^2 x_i$, this is

$$\begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad \Rightarrow \quad \begin{pmatrix} \frac{2k}{m} - \omega^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} - \omega^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

which will have a non-trivial solution when the determinant of the matrix is zero, so

$$\det \begin{pmatrix} \frac{2k}{m} - \omega^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} - \omega^2 \end{pmatrix} = 0 \quad \Rightarrow \quad \left(\frac{2k}{m} - \omega^2 \right)^2 - \frac{k^2}{m^2} = 0$$

$$\Rightarrow \quad \frac{4k^2}{m^2} - \frac{4k}{m}\omega^2 + \omega^4 - \frac{k^2}{m^2} = 0 \quad \Rightarrow \quad \omega^4 - \frac{4k}{m}\omega^2 + \frac{3k^2}{m^2} = 0$$

$$\left(\omega^2 - \frac{3k}{m} \right) \left(\omega^2 - \frac{k}{m} \right) = 0 \quad \Rightarrow \quad \omega = \sqrt{\frac{3k}{m}}, \quad \omega = \sqrt{\frac{k}{m}}$$

are the normal modes of oscillation.

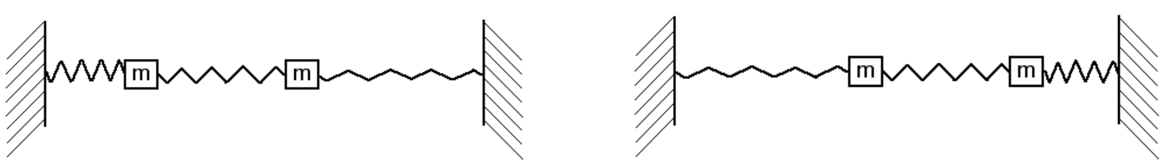

Figure 1–2. Normal mode for $\omega = \sqrt{\dfrac{k}{m}}$.

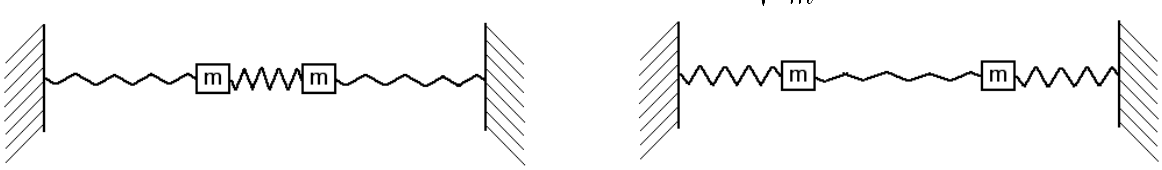

Figure 1–3. Normal mode for $\omega = \sqrt{\dfrac{3k}{m}}$.

Postscript: Notice that $\det\left(\mathcal{A} - \omega\mathcal{I}\right) = 0$ simply emerges. Its solutions are called normal modes, characteristics of the system, that are readily interpreted as eigenvalues. An analogy is that a quantum mechanical system exists only in a linear combination of states comparable to normal modes/eigenvalues, thus, only one of the "normal modes/eigenvalues" can be measured.

This system with only two normal modes lends itself directly to exposing how an eigenvalue/eigenvector problem naturally describes a physical system. The problem is more complicated for an individual guitar string. Plucking a guitar string can initially set it vibrating unpredictably, however, it will quickly settle into a linear combination of the fundamental mode, the first overtone, the second overtone, and other overtones. Each possible mode of vibration is a normal mode/eigenvalue of that guitar string. There are an infinite number of modes of vibration for an "ideal" guitar string. The eigenvalues of this two mass system exist in $\mathbb{R}^2$. The eigenvalues of an ideal guitar string exist in $\mathbb{R}^\infty$. The eigenvalues of a quantum mechanical system also exist in $\mathbb{R}^\infty$, though a subspace may be used for some systems. The components and elements of the vectors and operators from which the quantum mechanical eigenvalues are extracted exist in $\mathbb{C}^\infty$, though the eigenvalues exist in $\mathbb{R}^\infty$. The normal modes of a guitar string are likely a better overall analogy to the eigenvalues of a realistic quantum mechanical system than the two mass/three spring system because of the infinity of eigenvalues intrinsic.

Calculating eigenvalues in a quantum mechanical system is comparable to finding the normal modes in a classical mechanical system.

1–24. What are the eigenvalues and eigenvectors for $\mathcal{D} \rightarrow \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$?

The given $\mathcal{D}$ is a **diagonal operator**. This means that all non-zero elements are on the main diagonal, and all elements not on the main diagonal are zero. Notice that the non-zero elements are arbitrary. This problem provides a convenient and useful result.

$\det \begin{pmatrix} \alpha-\lambda & 0 & 0 \\ 0 & \beta-\lambda & 0 \\ 0 & 0 & \gamma-\lambda \end{pmatrix} = (\alpha-\lambda)(\beta-\lambda)(\gamma-\lambda)$ where the only non-zero contribution

to the determinant is from the main diagonal, so the characteristic equation can be written

$$(\alpha-\lambda)(\beta-\lambda)(\gamma-\lambda) = 0 \qquad \Rightarrow \qquad \lambda = \alpha, \;\; \lambda = \beta, \;\; \text{and} \;\; \lambda = \gamma .$$

This is a significant finding. The eigenvalues in a diagonal matrix are the elements on the main diagonal. The eigenvalues in a diagonal matrix are found by inspection. Eigenvectors are found

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad \Rightarrow \qquad \begin{aligned} \alpha a &= \alpha a \\ \beta b &= \alpha b \\ \gamma c &= \alpha c \end{aligned}$$

where the top equation says $a = a$, so let $a = 1$. The second and third equations say that $b = c = 0$ in general. The middle equation can be written $(\beta-\alpha)\,b = 0$, and in general $\beta \neq \alpha$, thus $b = 0$. The same argument applies to the bottom equation which can be written $(\gamma-\alpha)\,c = 0$, and in general $\gamma \neq \alpha$, thus $c = 0$ and the eigenvector is $|\,\alpha> = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, which is already normalized. For the eigenvalue β,

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \beta \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad \Rightarrow \qquad \begin{aligned} \alpha a &= \beta a \\ \beta b &= \beta b \\ \gamma c &= \beta c \end{aligned}$$