Mahmoud Filali (Ed.)

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Banach Algebras and Applications

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Dedication

In the last two years the Banach Algebra community has sadly lost two distinguished members, Philip Curtis and Eberhard Kaniuth. Both, Philip and Eberhard had a very strong and positive impact on the mathematics community. Phil Curtis together with Bill Bade organized the first conference in this series in 1974 at UCLA. Eberhard Kaniuth has been one of the main pillars in Banach algebra theory and abstract harmonic analysis, and had a deep and wide impact on the work taking place in these two fields. These proceedings are dedicated to their memories.

Preface

The 23rd conference on Banach Algebras and Applications was held in Oulu from July 3rd to July 11th, 2017. It is the most important scientific event in the Banach algebras community. The first conference in this sequence took place at the University of California, Los Angeles (UCLA) in 1974. This was followed by Leeds 1976, Long Beach 1981, Copenhagen 1985, Berkeley 1986, Leeds 1987, Berkeley 1988, Canberra 1989, Berkeley 1990, Cambridge 1991, Winnipeg 1993, Newcastle 1995, Blaubeuren 1997, Pomona 1999, Odense 2001, Edmonton 2003, Bordeaux 2005, Quebec 2007, Poznan 2009, Waterloo 2011, Gothenburg 2013, Fields Institute (Toronto) 2015.

With more than 70 participants from America, Asia, Europe and Africa, it was probably one of the biggest and most diverse events in Mathematics in Northern Finland. Banach algebras is a multilayered area in mathematics and has many ramifications. The diversity of the schools taking part in the conference made the event very successful and exciting. The talks reflected recent achievements in many areas contained in Banach algebra theory such as Banach Algebras over Groups, Abstract Harmonic Analysis, Group Actions, Amenability, Topological Homology, Semigroup Compactifications, Arens Irregularity, C*-Algebras and Dynamical Systems, Operator Theory, Operator Spaces, and more.

In fact the last decade has seen so much progress in many branches of Banach Algebra theory, and so much of these new achievements has been presented at the conference. We would like to thank the participants who came all the way to Oulu to attend the conference and share all the exciting mathematics they have been working on in recent years. The present volume contains sixteen refereed articles based on the high level expository talks presented at the conference. So, our thanks are due to the authors, there would be no proceedings without their contributions. We are also very thankful for the efforts and time spent by the referees checking thoroughly the papers, and for detailed reports and corrections sent to the authors. We believe the proceedings will ultimately serve as a platform for researchers in Banach algebras theory and related areas.

Partial support was made available to us by The Mathematics Foundation (The Finnish Academy of Science and Letters). This has helped to support a number of colleagues who did not have any financial support from their home institute or country. We are gratefully indebted to the Foundation and particularly to Olli Martio for his quick and positive answer. Kiitos Olli!

BusinessOulu with Riina Aikio and Helena Pikkarainen has played an important role for the success of the event and the comfort of our guests during the conference. These two wonderful ladies were behind the wonderful social program and

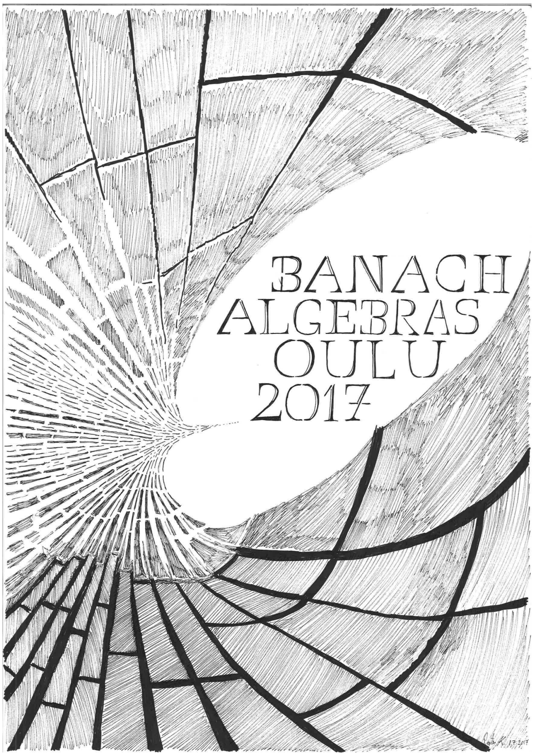

Fig. 1: Created by Peetu Karttunen

many other practical matters. Their help and enthusiasm with the conference are gratefully acknowledged.

The painstaking task of putting all the tex files sent by the authors into the form required by the journal was noticeably lifted off my shoulders by Nadja Schedensack, Project Editor of de Gruyter journal, and by our department technician Pekka Kangas. To both I wish to express my sincere gratitude for their continuous help and patience.

Finally thanks to my family and friends who had been helping me right from the beginning long before the conference too place. Without their encouragements and their help with the practical matters, this conference would not have taken place.

Mahmoud Filali
Oulu, September 30, 2019

Contents

Ali Baklouti and Mahmoud Filali

Beurling's Theorem on locally compact abelian groups

Abstract: We prove the analogue to Beurling's theorem for any locally compact abelian group. This generalizes an earlier work by the first author and Kaniuth on Hardy's uncertainty principle (cf. [1]).

Keywords: Beurling's theorem, locally compact abelian group.

Classification: Primary 22A05; Secondary 54E15 54H11.

1 Introduction

An attractive theorem of Beurling on Fourier transform pairs says that if $f \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |f(x)||\widehat{f}(y)|\exp(|xy|)dxdy < \infty, \tag{1}$$

then $f = 0$.

In other words, the trivial function is the only function in $L^1(\mathbb{R})$ for which $f\widehat{f}$ is integrable on $\mathbb{R}^2$ with respect to the measure $\exp(|xy|)dxdy$. Here, $\widehat{f}$ is the Fourier transform of f. The theorem is stated without a proof on page 372 of Beurling's collected works [2]. Based on the notes Hörmander took when Beurling explained the result to him in the mid-sixties, Hörmander reproduced the proof and published it in [5]. This was followed by a long list of papers extending Beurling's theorem to various groups. This list is too long to cite all the papers, but the reader may consult [3], where Beurling's Theorem was generalized to $\mathbb{R}^n$, and which is needed for our proofs.

In this note, we give an analogue to Beurling's assumption (1) and prove the analogue to Beurling's theorem for any locally compact abelian group G.

The second author wishes to express his sincere gratitude to *Sfax University* for the partial financial support.

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2 The theorem

Note first that Beurling's theorem does not hold if G is discrete. For instance, take the function $f = \delta_e$ and let Ψ be any function on $G \times \widehat{G}$ such that $\Psi(e, .)$ is measurable and bounded on $\widehat{G}$. Then $\widehat{f}$ is the constant function 1 on $\widehat{G}$ and

$$\sum_{x \in G} \int_{\widehat{G}} f(x)\widehat{f}(y)\Psi(x, y)dxdy = \int_{\widehat{G}} \Psi(e, y)dy < \infty.$$

So we are actually concerned with non-discrete groups. We denote the scalar product of x and y in $\mathbb{R}^n$ simply by xy. We use the structure theorem and write our group as $G = \mathbb{R}^n \times H$, where $n \geq 0$ and H is a locally compact abelian group which contains an open compact subgroup K, see for example [4, Theorems 24.29, 24.30]. Write u in G as $u = (x, s)$ with $x \in \mathbb{R}^n$ and $s \in H$. Write χ in $\widehat{G}$ as $\chi = (y, \xi)$ with $y \in \mathbb{R}^n$ and $\xi \in \widehat{H}$.

and let $\varphi : G \to \mathbb{R}^n$ and $\psi : \widehat{G} \to \mathbb{R}^n$ be given, respectively, by

$$\varphi(u) = \begin{cases} x, & \text{if} \quad n > 0 \\ 0, & \text{if} \quad n = 0 \end{cases} \qquad \text{and} \qquad \psi(\chi) = \begin{cases} y, & \text{if} \quad n > 0 \\ 0, & \text{if} \quad n = 0. \end{cases}$$

For the analogue to Beurling's assumption (1), we set

$$B(f) := \int_G \int_{\widehat{G}} |f(u)||\widehat{f}(\chi)| \exp(|\varphi(u)\varphi(\chi)|)dud\chi. \tag{2}$$

Then, the analogue to Beurling's theorem may be stated as follows:

Theorem 2.1. *Let G be a locally compact abelian group with dual group $\widehat{G}$ and let $f \in L^1(G)$. Then the implication $B(f) < \infty \Longrightarrow f = 0$ holds if and only if the connected component of the identity in G is not compact.*

Proof. Suppose that the component of the identity e in G is not compact, i.e., $n > 0$, and write as above our group as $G = \mathbb{R}^n \times H$.

Note first that the assumption $B(f) < \infty$ means that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_H \int_{\widehat{H}} |f(x, s)||\widehat{f}(y, \xi)| \exp(|xy|)dxdydsd\xi < \infty,$$

and so

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_H |f(x, s)||\widehat{f}(y, \xi)| \exp(|xy|)dxdyds < \infty \tag{3}$$

almost everywhere on $\widehat{H}$. Consider now for each $x \in \mathbb{R}^n$, the function

$$F(x, \xi) = \int_H f(x, s)\xi(s)ds,$$

i.e., $F(x, .)$ is the Fourier transform of the function f_x defined on H by $f_x(s) = f(x, s)$, and note that

$$|F(x, \xi)| \leq \int_H |f(x, s)\xi(s)|ds = \int_H |f(x, s)|ds. \tag{4}$$

Then for each $\xi \in \widehat{H}$, the function $F_\xi : \mathbb{R}^n \to \mathbb{R}$, given by $F_\xi(x) = F(x, \xi)$, is in $L^1(\mathbb{R}^n)$ since

$$\int_{\mathbb{R}^n} |F_\xi(x)|dx \leq \int_{\mathbb{R}^n} \int_H |f(x, s)\xi(s)|dsdx = \int_G |f(u)|du,$$

and its Fourier transform is given, for every $y \in \mathbb{R}^n$, by

$$\begin{aligned}\widehat{F_\xi}(y) = \int_{\mathbb{R}^n} F_\xi(x)\exp(ixy)dx &= \int_{\mathbb{R}^n} \left(\int_H f(x, s)\xi(s)ds \right) \exp(ixy)dx \\ &= \int_{\mathbb{R}^n} \int_H f(x, s)\xi(s)\exp(ixy)dxds = \widehat{f}(y, \xi).\end{aligned} \tag{5}$$

The observations (3), (4) and (5) lead to

$$\begin{aligned}&\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F_\xi(x)||\widehat{F_\xi}(y)|\exp(|xy|)dxdy \\ &\qquad \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_H |f(x, s)|\widehat{f}(y, \xi)\exp(|xy|)dxdyds < \infty\end{aligned}$$

almost everywhere on $\widehat{H}$.

Accordingly, we may apply Beurling's Theorem when $n = 1$ or [3] when $n > 1$ to deduce, that for almost every $\xi \in \widehat{H}$, $F_\xi(x) = \widehat{f}_x(\xi) = 0$ almost everywhere on $\mathbb{R}^n$. Therefore, $f_x = 0$ for almost every x in $\mathbb{R}^n$. This implies that $f(x, s) = 0$ almost everywhere on $G = \mathbb{R}^n \times H$, as required.

Conversely, suppose that the component of e in G is compact. Then, $G = \mathbb{Z}^m \times M$, where M contains an open compact subgroup K. Note that here $\varphi(u) = \psi(\chi) = 0$ for every $u \in G$ and $\chi \in \widehat{G}$ since $n = 0$.

We follow the notation in [4, Sections 23-24]. Let $A(\widehat{G}, K)$ be the annihilator of K in $\widehat{G}$, that is,

$$A(\widehat{G}, K) = \{\chi \in \widehat{G} : \chi_{|K} = 1\}.$$

By [4, Remarks 23.24 or 23.29], $A(\widehat{G},K)$ is a compact subgroup of $\widehat{G}$ since K is open. (The converse is also true as noted in 23.29).

Let the Haar measure on G be normalized on K, and let f be the function with values 1 on K and 0 elsewhere. Then f is a non-trivial function in $L^1(G)$. Moreover, for every $\chi \in \widehat{G}$,

$$\widehat{f}(\chi) = \int_G f(u)\chi(u)du = \int_K \chi(u)du = \begin{cases} 0, & \text{if } \chi_{|K} \neq 1 \\ 1 & \text{otherwise.} \end{cases}$$

Accordingly,

$$\begin{aligned} \int_G \int_{\widehat{G}} |f(u)||\widehat{f}(\chi)|\exp(|\varphi(u)\psi(\chi)|)dud\chi &= \int_K \int_{\widehat{G}} \widehat{f}(\chi)dud\chi = \left(\int_K du\right)\left(\int_{\widehat{G}} \widehat{f}(\chi)d\chi\right) \\ &= \left(\int_K du\right)\left(\int_{A(\widehat{G},K)} d\chi\right) < \infty \end{aligned}$$

since $A(\widehat{G},K)$ is compact. This completes the proof. □

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Tomasz Ciaś

Fréchet algebras with a dominating Hilbert algebra norm

Abstract: Let $\mathcal{L}^*(s)$ be the maximal $\mathcal{O}^*$-algebra of unbounded operators on ℓ_2 whose domain is the space s of rapidly decreasing sequences. This is a noncommutative topological algebra with involution which can be identified, for instance, with the algebra $\mathcal{L}(s) \cap \mathcal{L}(s')$ or the algebra of multipliers for the algebra $\mathcal{L}(s', s)$ of smooth compact operators. We give a simple characterization of unital commutative Fréchet *-subalgebras of $\mathcal{L}^*(s)$ isomorphic as Fréchet spaces to nuclear power series spaces $\Lambda_\infty(\alpha)$ of infinite type. It appears that many natural Fréchet *-algebras are closed *-subalgebras of $\mathcal{L}^*(s)$, for example, the algebras $C^\infty(M)$ of smooth functions on smooth compact manifolds and the algebra $\mathcal{S}(\mathbb{R}^n)$ of smooth rapidly decreasing functions on $\mathbb{R}^n$.

Keywords: Representations of commutative Fréchet algebras with involution, topological algebras of unbounded operators, nuclear Fréchet algebras of smooth functions, dominating norm, Hilbert algebra.

Classification: Primary: 46J25. Secondary: 46A11, 46A63, 46E25, 46K15, 47L60.

1 Introduction

Let s be the Fréchet space of rapidly decreasing complex sequences and let

$$\mathcal{L}^*(s) := \{x\colon s \to s : x \text{ is linear}, s \subset \mathcal{D}(x^*) \text{ and } x^*(s) \subset s\},$$

where $\mathcal{D}(x^*)$ is the domain of the adjoint of an unbounded operator x on ℓ_2. The class $\mathcal{L}^*(s)$ is known as the maximal $\mathcal{O}^*$-algebra with domain s and it can be seen as the largest *-algebra of unbounded operators on ℓ_2 with domain s – for details see the book of Schmüdgen [18, Section I.2.1]. The *-algebra $\mathcal{L}^*(s)$ can be topologised in several natural ways, as is shown in [18, Sections I.3.3 and I.3.5]. Here

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the space $\mathcal{L}^*(s)$ is considered with – the best from the functional analysis point of view – locally convex topology τ^* (for definition see Preliminaries and also Proposition 2.6). Indeed, standard tools of functional analysis, such as closed graph theorem, open mapping theorem or uniform boundedness principle, can be applied to $(\mathcal{L}^*(s), \tau^*)$ (see [8, Th. 4.5]). Furthermore, $\mathcal{L}^*(s)$ is a topological *-algebra – i.e. multiplication is separately continuous and involution is continous – but it is neither locally m-convex nor a $\mathcal{Q}$-algebra. The algebra $\mathcal{L}^*(s)$ is isomorphic as a topological *-algebra, for example, to the algebra $\mathcal{L}(s) \cap \mathcal{L}(s')$, the algebra of multipliers for the algebra $\mathcal{L}(s', s)$ of smooth compact operators and also to the matrix algebra

$$\Lambda(\mathscr{A}) := \left\{ x = (x_{ij}) \in \mathbb{C}^{\mathbb{N}^2} : \forall N \in \mathbb{N}\, \exists n \in \mathbb{N} \quad \sum_{i,j \in \mathbb{N}^2} |x_{ij}| \max\left\{ \frac{i^N}{j^n}, \frac{j^N}{i^n} \right\} < \infty \right\};$$

for details and more information about topological and algebraic properties of $\mathcal{L}^*(s)$ we refer the reader to [8].

The space s carries all the information about nuclear Fréchet (even locally convex) spaces. Indeed, by the Kōmura-Kōmura theorem, a Fréchet space is nuclear if and only if it is isomorphic to some closed subspace of $s^{\mathbb{N}}$ (see [14, Cor. 29.9]). What about closed subspaces of the space s itself? In [21] Vogt proved that a nuclear Fréchet space is isomorphic to a closed subspace of s if and only if it has the so-called property (DN). Moreover, quotients of s were characterised by Vogt and Wagner in [22] via the so-called property (Ω). Consequently, we have the following characterization: a nuclear Fréchet space is isomorphic to a complemented subspace of s if and only if it has the properties (DN) and (Ω). It is also well-known that a Fréchet space with (DN), (Ω) and a Schauder basis is isomorphic to a power series space $\Lambda_\infty(\alpha)$ of infinite type. However, it is still an open problem – a particular case of the famous Mityagin-Pełczyński problem – whether there is a complemented subspace of s without a basis.

In this paper, we are mainly interested in unital Fréchet algebras with involution which are isomorphic as Fréchet spaces to nuclear power series spaces of infinite type. We show that a large class of them – those algebras E which admit a dominating Hilbert norm $||\cdot|| = \sqrt{(\cdot,\cdot)}$ such that

$$(xy, z) = (y, x^* z) \tag{1}$$

for all $x, y, z \in E$ – can be embedded into $\mathcal{L}^*(s)$ as closed, even complemented, *-subalgebras (see Theorem 3.5 and Remark 3.17). In the commutative case we even have the following characterization: a unital commutative Fréchet *-algebra isomorphic as a Fréchet space to a nuclear power series space $\Lambda_\infty(\alpha)$ of infinite type is isomorphic as a Fréchet *-algebra to a closed *-subalgebra of $\mathcal{L}^*(s)$ if and only if it admits a dominating Hilbert norm satisfying condition (1) (see again Theorem 3.5). In Theorem 3.6 we also characterize commutative Fréchet unital *-subalgebras of $\mathcal{L}^*(s)$ consisting of bounded operators on ℓ_2 and isomorphic as Fréchet space to nuclear spaces $\Lambda_\infty(\alpha)$. It is worth noting that condition (1) appears in the definiton of Hilbert algebras playing an important role in the theory of von Neumann algebras (see [9, A.54]).

The above-mentioned results may be seen as a step towards an analogue – in the context of nuclear power series spaces of infinite type – of the celebrated commutative Gelfand-Naimark theorem. In the separable case it states that there is one to one correspondence (given by isometric *-isomorphisms) between Banach algebras $C(K)$ of continuous functions on compact Hausdorff metrizable spaces K and closed unital commutative *-subalgebras of the C^*-algebra $\mathcal{B}(\ell_2)$ of bounded operators on ℓ_2.

Our results are applicable. In the last section we give concrete examples of Fréchet *-algebras which can be represented in $\mathcal{L}^*(s)$ in the way described above. Among them there are: the algebras $C^\infty(M)$ of smooth functions on smooth compact manifolds, the algebras $\mathcal{E}(K)$ with Schauder basis of smooth Whitney jets on compact sets K with the extension property, the algebra $\mathcal{S}(\mathbb{R}^n)$ of smooth rapidly decreasing functions on $\mathbb{R}^n$, nuclear power series algebras $\Lambda_\infty(\alpha)$ of infinite type and the noncommutative algebra $\mathcal{L}(s', s)$ of compact smooth operators. We also provide one counterexample. We show that the unital commutative Fréchet *-algebra $A^\infty(\mathbb{D})$ of holomorphic functions on the open unit disc $\mathbb{D}$ with smooth boundary values is not isomorphic to any closed *-subalgebra of $\mathcal{L}^*(s)$.

2 Preliminaries

The canonical ℓ_2 norm and the corresponding scalar product will be denoted by $||\cdot||_{\ell_2}$ and $\langle\cdot,\cdot\rangle$, respectively.

For locally convex spaces E and F, we denote by $\mathcal{L}(E, F)$ the space of all continuous linear operators from E to F and we set $\mathcal{L}(E) := \mathcal{L}(E, E)$. These spaces will be considered with the topology $\tau_{\mathcal{L}(E,F)}$ of uniform convergence on bounded sets.

By a *topological *-algebra E* we mean a topological vector space endowed with at least separately continuous multiplication and continuous involution which make E a *-algebra. A *Fréchet *-algebra* is a topological *-algebra whose underlying topological vector space is a Fréchet space (i.e. metrizable complete locally convex space). We do not require a Fréchet *-algebra to be locally m-convex.

Let $\alpha = (\alpha_j)_{j\in\mathbb{N}}$ be a monotonically increasing sequence in $(0,\infty)$ such that $\lim_{j\to\infty}\alpha_j = \infty$. Then

$$\Lambda_\infty(\alpha) := \{(\xi_j)_{j\in\mathbb{N}} \subset \mathbb{C}^{\mathbb{N}} : |\xi|^2_{\alpha,q} := \sum_{j=1}^{\infty} |\xi_j|^2 e^{2q\alpha_j} < \infty \quad \text{for all } q \in \mathbb{N}_0\}$$

equipped with the norms $|\cdot|_{\alpha,q}$, $q \in \mathbb{N}_0$, is a Fréchet space and it is called a *power series space of infinite type*. It appears that the space $\Lambda_\infty(\alpha)$ is nuclear if and only if $\sup_{j\in\mathbb{N}} \frac{\log j}{\alpha_j} < \infty$ (see e.g. [14, Prop. 29.6]). In particular, for the sequence $\alpha_j := \log j$, $j \in \mathbb{N}$, we obtain the space s of *rapidly decreasing sequences*, i.e.

$$s := \{(\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |\xi|^2_q := \sum_{j\in\mathbb{N}} |\xi_j|^2 j^{2q} < \infty \quad \text{for all } q \in \mathbb{N}_0\}. \tag{2}$$

By s_n we denote the Hilbert space corresponding to the norm $|\cdot|_n$.

The strong dual of s – i.e. the space of all continuous linear functionals on s with the topology of uniform convergence on bounded subsets of s (see e.g. [14, Def. on p. 267]) – is isomorphic to the space

$$s' := \{(\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |\xi|^2_{-q} := \sum_{j\in\mathbb{N}} |\xi_j|^2 j^{-2q} < \infty \quad \text{for some } q \in \mathbb{N}_0\} \tag{3}$$

of *slowly increasing sequences* equipped with the inductive limit topology for the sequence $(s_{-n})_{n\in\mathbb{N}_0}$ of Hilbert spaces corresponding to the norms $|\cdot|_{-n}$. In other words, the locally convex topology on s' is given by the family $\{|\cdot|'_B\}_{B\in\mathcal{B}}$ of seminorms , $|\xi|'_B := \sup_{\eta\in B} |\langle\eta,\xi\rangle|$, where $\mathcal{B}$ denotes the class of all bounded subsets of s and, recall, $\langle\cdot,\cdot\rangle$ is the canonical scalar product on ℓ_2.

Definition 2.1. A Fréchet space E with a fundamental system $(\|\cdot\|_q)_{q\in\mathbb{N}_0})$ of seminorms

(1) has the *property* (DN) (cf. [14, Def. on p. 359]) if there is a continuous norm $\|\cdot\|$ on E – called a *dominating norm* – such that for all $q \in \mathbb{N}_0$ there is $r \in \mathbb{N}_0$ and $C > 0$ such that
$$\|x\|^2_q \leq C\|x\|\,\|x\|_r$$
for all $x \in E$;

(2) has the *property* (Ω) (cf. [14, Def. on p. 367]) if for all $p \in \mathbb{N}_0$ there is $q \in \mathbb{N}_0$ such that for all $r \in \mathbb{N}$ there are $\theta \in (0,1)$ and $C > 0$ with
$$\|y\|^*_q \leq C\|y\|^{*\,1-\theta}_p \|y\|^{*\,\theta}_r$$
for all $y \in E'$, where E' is the topological dual of E and
$$\|y\|^*_p := \sup\{|y(x)| : \|x\|_p \leq 1\}.$$

The properties (DN) and (Ω) are linear-topological invariants which play a key role in a structure theory of nuclear Fréchet spaces. The following Theorem is due to Vogt and Wagner.

Theorem 2.2. ([14, Ch. 31] and [21, 22]) *A Fréchet space is isomorphic to:*
(i) *a closed subspace of s if and only if it is nuclear and has the property* (DN)*;*
(ii) *a quotient of s if and only if it is nuclear and has the property* (Ω)*;*
(iii) *a complemented subspace of s if and only if it is nuclear and has the properties* (DN) *and* (Ω).

We also cite another result of Vogt which will be crucial for our futher considerations.

Theorem 2.3. *[23, Cor. 7.7] Let E be a Fréchet space isomorphic to a power series space $\Lambda_\infty(\alpha)$ of infinite type. Then for every dominating Hilbert norm $||\cdot||$ on E there is an isomorphism $u\colon E \to \Lambda_\infty(\alpha)$ such that $||u\xi||_{\ell_2} = ||\xi||$ for all $\xi \in E$.*

Let E be a Fréchet space with a continuous Hilbert norm $||\cdot||$. Let H be the completion of E in the norm $||\cdot||$ and let $(\cdot,\cdot)$ be the corresponding scalar product. Then we define

$$\mathcal{L}^*(E, ||\cdot||) := \{x\colon E \to E : x \text{ is linear}, E \subset \mathcal{D}(x^*) \text{ and } x^*(E) \subset E\},$$

where

$$\mathcal{D}(x^*) := \{\eta \in H : \ \exists \zeta \in H\, \forall \xi \in E \quad (x\xi, \eta) = (\xi, \zeta)\}$$

and $x^*\eta := \zeta$ for $\eta \in \mathcal{D}(x^*)$. In the case when E is a closed subspace of s or $E = \Lambda_\infty(\alpha)$ we write $\mathcal{L}^*(E)$ instead of $\mathcal{L}^*(E, ||\cdot||_{\ell_2})$. Since E is a dense linear subspace of H, each $x \in \mathcal{L}^*(E, ||\cdot||)$ can be considered as a dense unbounded operator in H with domain $\mathcal{D}(x) = E$, and thus it has the adjoint $x^*\colon \mathcal{D}(x^*) \to H$. By definition, the operator $x^*{}_{|E}$, for simplicity denoted again by x^*, is in $\mathcal{L}^*(E, ||\cdot||)$, as well. Moreover, by definition,

$$\mathcal{D}(xy) := \{\xi \in \mathcal{D}(y) : \ y\xi \in \mathcal{D}(x)\} = E$$

for all $x, y \in \mathcal{L}^*(E, ||\cdot||)$. This shows that $\mathcal{L}^*(E, ||\cdot||)$ is a *-algebra. In fact, the class $\mathcal{L}^*(E, ||\cdot||)$ can be seen as the largest *-algebra of unbounded operators on H with domain E and it is known as the *maximal $\mathcal{O}^*$-algebra* with domain E (see [18, 2.1] for details).

In the theory of maximal $\mathcal{O}^*$-algebras – and, more generally, of algebras of unbounded operators in Hilbert spaces – one consider the so-called *graph topology* ([18, Def. 2.1.1]). With E and $||\cdot||$ as above, the graph topology of $\mathcal{L}^*(E, ||\cdot||)$ on E is, by definition, given by the system of seminorms $(||\cdot||_a)_{a\in\mathcal{L}^*(E,||\cdot||)}$, $||\xi||_a := ||a\xi||$ for $\xi \in E$.

The following easy observation is kind of folklor – for completness we present here the proof.

Proposition 2.4. *Let E be a Fréchet space with a continuous Hilbert norm $||\cdot||$. Then the graph topology of $\mathcal{L}^*(E, ||\cdot||)$ on E is weaker than the Fréchet space toplogy.*

Proof. Let $(\cdot,\cdot)$ denote the scalar product corresponding to the Hilbert norm $||\cdot||$ and let H be the completion of E in the norm $||\cdot||$. We shall show that each $a \in \mathcal{L}^*(E, ||\cdot||)$ is a continuous map from the Fréchet space E to the Hilbert space H. Let $(\xi_j)_{j\in\mathbb{N}} \subset E$ be a sequence converging in the Fréchet space topology to 0 and assume that $a\xi_j$ converges in the norm $||\cdot||$ to some $\eta \in H$. We have, for all $\zeta \in E$,

$$\lim_{j\to\infty}(a\xi_j, \zeta) = (\eta, \zeta)$$

and, on the other hand,

$$\lim_{j\to\infty}(a\xi_j, \zeta) = \lim_{j\to\infty}(\xi_j, a^*\zeta) = 0.$$

Hence, $(\eta, \zeta) = 0$ for all $\zeta \in E$, and thus $\eta = 0$. Consequently, by the closed graph theorem for Fréchet spaces (cf. [14, Th. 24.31]), the map $a\colon E \to H$ is continuous, which is the desired conclusion. □

Sometimes the initial Fréchet space topology and the graph topology coincide.

Proposition 2.5. *Let E be a Fréchet space isomorphic to a power series space $\Lambda_\infty(\alpha)$ of infinite type and let $||\cdot||$ be a dominating Hilbert norm on E. Then the graph topology of $\mathcal{L}^*(E, ||\cdot||)$ on E coincides with the Fréchet space topology.*

Proof. Let $(\cdot,\cdot)$ denote the scalar product corresponding to the Hilbert norm $||\cdot||$. By [23, Cor. 7.7], there is an isomorphism $u\colon E \to \Lambda_\infty(\alpha)$ such that $||u\xi||_{\ell_2} = ||\xi||$ for all $\xi \in E$. Let $||\xi||_n := |u\xi|_{\alpha,n}$ for $\xi \in E$ and $n \in \mathbb{N}$. Then $(||\cdot||_n)_{n\in\mathbb{N}}$ is a fundamental sequence of dominating Hilbert norms on E. For $n \in \mathbb{N}$, we define the diagonal map $d_n\colon \Lambda_\infty(\alpha) \to \Lambda_\infty(\alpha)$, $d_n\xi := (e^{n\alpha_j}\xi_j)_{j\in\mathbb{N}}$. Clearly, each d_n is an automorphism of the Fréchet space $\Lambda_\infty(\alpha)$ and $||d_n\xi||_{\ell_2} = |\xi|_{\alpha,n}$ for all $\xi \in \Lambda_\infty(\alpha)$. Now, for $n \in \mathbb{N}$, let $a_n\colon E \to E$, $a_n := u^{-1}d_nu$. We have

$$(a_n\xi, \zeta) = (u^{-1}d_nu\xi, \zeta) = \langle d_nu\xi, u\zeta\rangle = \langle u\xi, d_nu\zeta\rangle = (\xi, u^{-1}d_nu\zeta) = (\xi, a_n\zeta)$$

for all $\xi, \zeta \in E$, whence $a_n \in \mathcal{L}^*(E, ||\cdot||)$. Consequently, since $||\xi||_n = ||a_n\xi||$ for all $\xi \in E$, i.e. $||\cdot||_n = ||\cdot||_{a_n}$, the graph topology of $\mathcal{L}^*(E, ||\cdot||)$ on E is finer than the Fréchet space toppology and thus, in view of Proposition 2.4, these topologies are equal. □

There are plenty natural topologies on the space $\mathcal{L}^*(E, ||\cdot||)$ (see [18, Sect. 3.3, 3.5]). Here we are interested in the locally convex topology τ^* on $\mathcal{L}^*(E, ||\cdot||)$ given by the seminorms

$$p^{a,B}(x) := \max\left\{\sup_{\xi\in B}\|ax\xi\|,\ \sup_{\xi\in B}\|ax^*\xi\|\right\},$$

where a and B run over $\mathcal{L}^*(E,\|\cdot\|)$ and the class of all bounded subsets of E equipped with the graph topology of $\mathcal{L}^*(E,\|\cdot\|)$, respectively (see [18, pp. 81–82]). It is well-known that $\mathcal{L}^*(E,\|\cdot\|)$ endowed with the topology τ^* is a topological *-algebra (cf. [18, Prop. 3.3.15 (i)]). If we, moreover, assume that E is isomorphic to a power series space of infinite type and $\|\cdot\|$ is a dominating Hilbert norm on E, then, by Proposition 2.5 and [18, Prop. 3.3.15 (iv)], $(\mathcal{L}^*(E,\|\cdot\|),\tau^*)$ is complete. The following characterization of the topology τ^* is a direct consequence of Proposition 2.5.

Proposition 2.6. *Let E be a Fréchet space isomorphic to a power series space of infinite type and let $\|\cdot\|$ be a dominating Hilbert norm on E. Let $(\|\cdot\|_n)_{n\in\mathbb{N}}$ be a fundamental sequence of norms on E. Then the topology τ^* on $\mathcal{L}^*(E,\|\cdot\|)$ is given by the seminorms $(p_{n,B})_{n\in\mathbb{N},B\in\mathcal{B}_E}$,*

$$p_{n,B}(x) := \max\{\sup_{\xi\in B}\|x\xi\|_n, \sup_{\xi\in B}\|x^*\xi\|_n\}, \tag{4}$$

where $\mathcal{B}_E$ denote the class of all bounded subsets of E.

3 Fréchet subalgebras of $\mathcal{L}^*(s)$

In this section we give abstract descriptions of two large classes of complemented commutative Fréchet *-subalgebras of $\mathcal{L}^*(s)$ (Theorems 3.5 and 3.6). Moreover, we provide a criterion for the existence of a "nice" embedding in $\mathcal{L}^*(s)$ of not necessarily commutative Fréchet *-algebras (see Remark 3.17).

Let us first recall the notion of Hilbert algebras.

Definition 3.1. (cf. [9, A.54]) A *Hilbert algebra* is a *-algebra E endowed with a Hilbert norm $\|\cdot\| := \sqrt{(\cdot,\cdot)}$ such that:

(α) $(xy,z) = (y,x^*z)$ for all $x,y,z\in E$;
(β) for all $x\in E$ there is $C>0$ such that $\|xy\|\le C\|y\|$ for all $y\in E$, i.e. the left multiplication maps $m_x\colon (E,\|\cdot\|)\to(E,\|\cdot\|)$, $m_x(y) := xy$, are bounded;
(γ) $(y^*,x^*) = (x,y)$ for all $x,y\in E$;
(δ) the linear span of the set $E^2 := \{ab :\ a,b\in E\}$ is dense in E.

Each norm $\|\cdot\|$ satisfying conditions (α)–(δ) is called a *Hilbert algebra norm*.

Remark 3.2. If E is unital, then condition (δ) in the above definition is trivially satisfied. If, moreover, E is commutative, then (α) implies (γ). Hence, every Hilbert norm on a unital commutative *-algebra satisfying condition (α) and (β) is already a Hilbert algebra norm.

Definition 3.3. A Fréchet *-algebra is called a DN-*algebra* if it admits a Hilbert dominating norm satysfying condition (α) in Definition 3.1. A DN-algebra is called a βDN-*algebra* if the corresponding Hilbert dominating norm satisfies conditions (α) and (β) simultaneously.

Remark 3.4. In [13, Def. 1.5] M. Măntoiu and R. Purice defined a Fréchet-Hilbert algebra as a Fréchet *-algebra admitting a continous Hilbert algebra norm (more precisely, in their definition the corresponding Hilbert algebra scalar product is predetermined). Hence, in view of Remark 3.2, every unital commutative βDN-algebra is a Fréchet-Hilbert algebra.

Our main results read as follows.

Theorem 3.5. *Let E be a unital commutative Fréchet *-algebra isomorphic as a Fréchet space to a nuclear power series space of infinite type. Then the following statements are equivalent.*

(i) *E is isomorphic as a Fréchet *-algebra to a complemented *-subalgebra of $\mathcal{L}^*(s)$.*
(ii) *E is isomorphic as a Fréchet *-algebra to a closed *-subalgebra of $\mathcal{L}^*(s)$.*
(iii) *E is a DN-algebra.*

Theorem 3.6. *Let E be a unital commutative Fréchet *-algebra isomorphic as a Fréchet space to a nuclear power series space of infinite type. Then the following statements are equivalent.*

(i) *E is isomorphic as a Fréchet *-algebra to a complemented *-subalgebra F of $\mathcal{L}^*(s)$ such that $F \subset \mathcal{L}(\ell_2)$.*
(ii) *E is isomorphic as a Fréchet *-algebra to a closed *-subalgebra F of $\mathcal{L}^*(s)$ such that $F \subset \mathcal{L}(\ell_2)$.*
(iii) *E is a βDN-algebra.*

We divide the proof into a sequence of lemmas. As a by-product, we obtain also three results which are interesting enough to be stated as “corollaries".

For every $N, n \in \mathbb{N}_0$ we define the space

$$\mathcal{L}(s_n, s_N) \cap \mathcal{L}(s_{-N}, s_{-n}) := \{x \in \mathcal{L}(s_n, s_N) : \exists \widetilde{x} \in \mathcal{L}(s_{-N}, s_{-n}) \quad \widetilde{x}_{|s_n} = x\}$$

with the norm

$$r_{N,n}(x) := \max\left\{ \sup_{|\xi|_n \le 1} |x\xi|_N, \sup_{|\xi|_{-N} \le 1} |\widetilde{x}\xi|_{-n} \right\}.$$

Formally, the space $\mathcal{L}(s_n, s_N) \cap \mathcal{L}(s_{-N}, s_{-n})$ is the projective limit of the Banach spaces $\mathcal{L}(s_n, s_N)$ and $\mathcal{L}(s_{-N}, s_{-n})$ with their standard norms, and thus it is a

Banach space itself. Since

$$\sup_{|\xi|_n\le 1}|x^*\xi|_N=\sup_{|\xi|_n\le 1}\sup_{|\eta|_{-N}\le 1}|\langle x^*\xi,\eta\rangle|=\sup_{|\xi|_n\le 1}\sup_{|\eta|_{-N}\le 1}|\langle \xi,\widetilde{x}\eta\rangle|=\sup_{|\eta|_{-N}\le 1}|\widetilde{x}\eta|_{-n},$$

we have

$$r_{N,n}(x)=\max\left\{\sup_{|\xi|_n\le 1}|x\xi|_N,\ \sup_{|\xi|_n\le 1}|x^*\xi|_N\right\},$$

where $x^*\in\mathcal{L}(s_n,s_N)$ is the hilbertian adjoint of the operator $\widetilde{x}$. Moreover, $\mathcal{L}^*(s)=\mathcal{L}(s)\cap\mathcal{L}(s')$ (see, e.g., [8, Prop. 3.7]), hence

$$\mathcal{L}^*(s)=\{x\colon s\to s:\ x \text{ linear and } \forall N\in\mathbb{N}_0\ \exists n\in\mathbb{N}_0\quad r_{N,n}(x)<\infty\}$$

as sets. Therefore, we can endow $\mathcal{L}^*(s)$ with the topology of the PLB-space (a countable projective limit of a countable inductive limit of Banach spaces)

$$\operatorname{proj}_{N\in\mathbb{N}_0}\operatorname{ind}_{n\in\mathbb{N}_0}\mathcal{L}(s_n,s_N)\cap\mathcal{L}(s_{-N},s_{-n}).$$

It appears that the topology τ^* and the PLB-topology on $\mathcal{L}^*(s)$ coincide.

Lemma 3.7. *We have*

$$\mathcal{L}^*(s)=\operatorname{proj}_{N\in\mathbb{N}_0}\operatorname{ind}_{n\in\mathbb{N}_0}\mathcal{L}(s_n,s_N)\cap\mathcal{L}(s_{-N},s_{-n})$$

as topological vector spaces.

Proof. By [8, Cor. 4.2], $\mathcal{L}^*(s)$ is ultrabornological and

$$\operatorname{proj}_{N\in\mathbb{N}_0}\operatorname{ind}_{n\in\mathbb{N}_0}\mathcal{L}(s_n,s_N)\cap\mathcal{L}(s_{-N},s_{-n})$$

is webbed as a PLB-space. Hence, by the open mapping theorem (see e.g. [14, Th. 24.30]), it is enough to show that the identity map

$$\iota\colon\ \operatorname{proj}_{N\in\mathbb{N}_0}\operatorname{ind}_{n\in\mathbb{N}_0}\mathcal{L}(s_n,s_N)\cap\mathcal{L}(s_{-N},s_{-n})\to\mathcal{L}^*(s)$$

is continuous. Let $N\in\mathbb{N}_0$ and let B be a bounded subset of s. For every $m\in\mathbb{N}_0$ choose a constant $\lambda_m>0$ such that $B\subset\{\xi\in s\colon |\xi|_m\le\lambda_m\}$. Then

$$\begin{aligned}p_{N,B}(x)=\max\left\{\sup_{\xi\in B}|x\xi|_N,\sup_{\xi\in B}|x^*\xi|_N\right\}&\le\lambda_m\max\left\{\sup_{|\xi|_m\le 1}|x\xi|_N,\ \sup_{|\xi|_m\le 1}|x^*\xi|_N\right\}\\&=\lambda_m r_{N,m}(x)\end{aligned}$$

for every $m\in\mathbb{N}_0$ and $x\in\mathcal{L}(s_m,s_N)\cap\mathcal{L}(s_{-N},s_{-m})$, and thus ι is continuous. □

Lemma 3.8. *For every Fréchet subspace F of $\mathcal{L}^*(s)$ there is $m\in\mathbb{N}_0$ such that $F\subset\mathcal{L}(s_m,\ell_2)\cap\mathcal{L}(\ell_2,s_{-m})$ and, moreover, for each such m,*

$$r_m\colon F\to[0,\infty),\quad r_m(x):=\max\{\sup_{|\xi|_m\le 1}\|x\xi\|_{\ell_2},\ \sup_{|\xi|_m\le 1}\|x^*\xi\|_{\ell_2}\},$$

is a dominating norm on F.

Proof. By the very definition of projective topology, the canonical embedding

$$\mathrm{proj}_{N\in\mathbb{N}_0}\,\mathrm{ind}_{n\in\mathbb{N}_0}\,\mathcal{L}(s_n,s_N)\cap\mathcal{L}(s_{-N},s_{-n})\hookrightarrow\mathrm{ind}_{n\in\mathbb{N}_0}\,\mathcal{L}(s_n,\ell_2)\cap\mathcal{L}(\ell_2,s_{-n})$$

is continuous and thus the identity map $\kappa\colon F\hookrightarrow\mathrm{ind}_{n\in\mathbb{N}_0}\,\mathcal{L}(s_n,\ell_2)\cap\mathcal{L}(\ell_2,s_{-n})$ is continuous, as well. Hence, by Grothendieck's factorization theorem [14, Th. 24.33], there is $m\in\mathbb{N}$ such that $\kappa(F)\subset\mathcal{L}(s_m,\ell_2)\cap\mathcal{L}(\ell_2,s_{-m})$. Since we can identify in a obvious way F with $\kappa(F)$, we get the first part of the thesis.

Now, fix an arbitrary $m\in\mathbb{N}_0$ such that $F\subset\mathcal{L}(s_m,\ell_2)\cap\mathcal{L}(\ell_2,s_{-m})$. Then r_m is a continuous seminorm on F. Since F is a Fréchet space, there is a sequence $(B_N)_{N\in\mathbb{N}}$, $B_N\subset B_{N+1}$, of bounded subsets of s such that $(p_N)_{N\in\mathbb{N}}$,

$$p_N(x):=\max\left\{\sup_{\xi\in B_N}|x\xi|_N,\ \sup_{\xi\in B_N}|x^*\xi|_N\right\}$$

for $x\in F$, is a fundamental sequence of seminorms on F. Moreover, for every $N\in\mathbb{N}$ there is $\lambda_N>0$ such that $B_N\subset\{\xi\in s:\ |\xi|_m\le\lambda_N\}$. Hence, for $x\in F$ and $N\in\mathbb{N}$, we obtain

$$\begin{aligned}
p_N^2(x)&=\max\left\{\sup_{\xi\in B_N}|x\xi|_N^2,\ \sup_{\xi\in B_N}|x^*\xi|_N^2\right\}\\
&\le\max\left\{\sup_{\xi\in B_N}(\|x\xi\|_{\ell_2}|x\xi|_{2N}),\ \sup_{\xi\in B_N}(\|x^*\xi\|_{\ell_2}|x^*\xi|_{2N})\right\}\\
&\le\max\left\{\sup_{\xi\in B_N}\|x\xi\|_{\ell_2}\cdot\sup_{\xi\in B_N}|x\xi|_{2N},\ \sup_{\xi\in B_N}\|x^*\xi\|_{\ell_2}\cdot\sup_{\xi\in B_N}|x^*\xi|_{2N}\right\}\\
&\le\lambda_N\max\left\{\sup_{|\xi|_m\le1}\|x\xi\|_{\ell_2}\cdot\sup_{\xi\in B_{2N}}|x\xi|_{2N},\ \sup_{|\xi|_m\le1}\|x^*\xi\|_{\ell_2}\cdot\sup_{\xi\in B_{2N}}|x^*\xi|_{2N}\right\},
\end{aligned}$$

where the first inequality follows from the Cauchy-Schwartz inequality. Finally, since

$$\max\{ab,cd\}\le\max\{a,c\}\cdot\max\{b,d\}$$

for all $a,b,c,d\ge0$, we obtain

$$\begin{aligned}
p_N^2(x)&\le\lambda_N\max\left\{\sup_{|\xi|_m\le1}\|x\xi\|_{\ell_2},\ \sup_{|\xi|_m\le1}\|x^*\xi\|_{\ell_2}\right\}\cdot\max\left\{\sup_{\xi\in B_{2N}}|x\xi|_{2N},\ \sup_{\xi\in B_{2N}}|x^*\xi|_{2N}\right\}\\
&=\lambda_N r_m(x)p_{2N}(x)
\end{aligned}$$

for all $x\in F$, and thus r_m is a dominating norm on F. □

Corollary 3.9. (i) *Every Fréchet subspace of $\mathcal{L}^*(s)$ is isomorphic to a closed subspace of s.*

(ii) *Every Fréchet quotient of $\mathcal{L}^*(s)$ is isomorphic to a quotient of s.*

(iii) *Every complemented Fréchet subspace of $\mathcal{L}^*(s)$ is isomorphic to a complemented subspace of s.*

Proof. First note that every closed subspace and quotient of $\mathcal{L}^*(s)$ is nuclear because $\mathcal{L}^*(s)$ is nuclear itself (see [8, Prop. 3.8 & Cor. 4.2]).

(i) This follows immediately from Lemma 3.8 and [14, Prop. 31.5].

(ii) Let E be a Fréchet quotient of $\mathcal{L}^*(s)$. It follows from [8, Prop. 4.7] and [2, Cor. 1.2(a) and (c)] that E, being isomorphic to a quotient of $\mathcal{L}^*(s)$, has the property (Ω). Therefore, by [14, Prop. 31.6], E is isomorphic to a quotient of s.

(iii) This is a direct consequence of the previous items and [14, Prop. 31.7]. □

Let e_j denote the j-th unit vector in $\mathbb{C}^{\mathbb{N}}$. If F is a Fréchet subspace of $\mathcal{L}^*(s)$ then, by Lemma 3.8, there is $m \in \mathbb{N}_0$ such that $F \subset \mathcal{L}(s_m, \ell_2) \cap \mathcal{L}(\ell_2, s_{-m})$ and r_m is a continuous (dominating) norm on F. Since, for all $x \in F$, we have

$$\begin{aligned}[x]_m := \Big(\sum_{j=1}^{\infty} \|xe_j\|_{\ell_2}^2 j^{-2m-2}\Big)^{1/2} &\leq \Big(\sum_{j=1}^{\infty} j^{-2}\Big)^{1/2} \cdot \sup_{|\xi|_m \leq 1} \|x\xi\|_{\ell_2} \\ &\leq \frac{\pi}{\sqrt{6}} \max\Big\{\sup_{|\xi|_m \leq 1} \|x\xi\|_{\ell_2}, \sup_{|\xi|_m \leq 1} \|x^*\xi\|_{\ell_2}\Big\} = \frac{\pi}{\sqrt{6}} r_m(x),\end{aligned}$$

the scalar product

$$[\cdot,\cdot]_m \colon F \times F \to \mathbb{C}, \quad [x,y]_m := \sum_{j=1}^{\infty} \langle xe_j, ye_j\rangle j^{-2m-2}, \tag{5}$$

is well-defined and $[\,\cdot\,]_m = \sqrt{[\cdot,\cdot]_m}$ is a continuous Hilbert norm on F.

Lemma 3.10. *Let F be a commutative Fréchet *-subalgebra of $\mathcal{L}^*(s)$ and let $m \in \mathbb{N}_0$ be such that $F \subset \mathcal{L}(s_m, \ell_2) \cap \mathcal{L}(\ell_2, s_{-m})$. Then the norm $[\,\cdot\,]_m$ defined by* (5) *is a Hilbert dominating norm on F satisfying condition* (α).

Proof. Since F is commutative, we have

$$\|x\xi\|_{\ell_2} = \|x^*\xi\|_{\ell_2}$$

for all $x \in E$ and all $\xi \in s$. Hence,

$$\begin{aligned} r_{m+2}(x) = \max\Big\{\sup_{|\xi|_{m+2} \leq 1} \|x\xi\|_{\ell_2}, \sup_{|\xi|_{m+2} \leq 1} \|x^*\xi\|_{\ell_2}\Big\} &= \sup_{|\xi|_{m+2} \leq 1} \|x\xi\|_{\ell_2} \\ = \sup_{|\xi|_{m+2} \leq 1} \Big\|\sum_{j=1}^{\infty} \xi_j j^{m+2} \cdot x(e_j j^{-m-2})\Big\|_{\ell_2} &\leq \sup_{|\xi|_{m+2} \leq 1} \sum_{j=1}^{\infty} |\xi_j| j^{m+2} \cdot \|xe_j\|_{\ell_2} \cdot j^{-m-2} \\ \leq \sum_{j=1}^{\infty} \|xe_j\|_{\ell_2} \cdot j^{-m-2} \leq \Big(\sum_{j=1}^{\infty} j^{-2}\Big)^{1/2} \cdot \Big(\sum_{j=1}^{\infty} \|xe_j\|_{\ell_2}^2 j^{-2m-2}\Big)^{1/2} &= \frac{\pi}{\sqrt{6}} [x]_m. \end{aligned}$$

Therefore,

$$[\,\cdot\,]_m \geq \frac{\sqrt{6}}{\pi} r_{m+2},$$

and, by Lemma 3.8, $[\cdot]_m$ is a dominating norm on F. Moreover, we have

$$[xy, z]_m = \sum_{j=1}^{\infty} \langle xye_j, ze_j\rangle j^{-2m-2} = \sum_{j=1}^{\infty} \langle ye_j, x^*ze_j\rangle j^{-2m-2} = [y, x^*z]_m,$$

which completes the proof. □

Definition 3.11. A closed subspace E of the space s is called *orthogonally complemented* in s if there is a continuous projection π in s onto E admitting the extension to the orthogonal projection in ℓ_2. Then we call π an *orthogonal projection* in s onto E.

Lemma 3.12. *Let E be a Fréchet space isomorphic to a nuclear power series space of infinite type and let $||\cdot||$ be a dominating Hilbert norm on E. Then there is an orthogonally complemented subspace G of s and an isomorphism $w\colon E \to G$ of Fréchet spaces such that $||w\xi||_{\ell_2} = ||\xi||$ for all $\xi \in E$.*

Proof. Since E is isomorphic to a nuclear power series space of infinite type, by [14, Lemma 29.2(3) & Lemma 29.11(3)], E has the properties (DN) and (Ω). Hence, by [14, Prop. 31.7], E is isomorphic to a complemented subspace of s. This means that there is a complemented subspace F of s with a continuous projection $\pi\colon s \to F$ and a Fréchet space isomorphism $\psi\colon E \to F$. Hence, $||\cdot||_\psi\colon F \to [0,\infty)$ defined by $||\xi||_\psi := ||\psi^{-1}\xi||$ is a dominating Hilbert norm on F. Since, $||\cdot||_{\ell_2}$ is also a dominating Hilbert norm on F, by [23, Cor. 7.7], there is an automorphism u of F such that $||u\xi||_{\ell_2} = ||\xi||_\psi$ for all $\xi \in F$. Moreover, by [23, Th. 7.2], there is an automorphism v of s such that $\rho := v\pi v^{-1}$ is the orthogonal projection in s onto $G := v(F)$ and a simple analysis of the proof of [23, Th. 7.2] shows that $||v\xi||_{\ell_2} = ||\xi||_{\ell_2}$ for all $\xi \in F$. Therefore, the operator $w := vu\psi$ has the desired properties. □

Lemma 3.13. *Let E be a Fréchet space isomorphic to a nuclear power series space of infinite type and let $||\cdot||$ be a dominating Hilbert norm on E. Let H denote the completion of E in the norm $||\cdot||$. Then there is a map $\varphi \in \mathcal{L}(H, \ell_2)$ and an orthogonally complemented subspace G of s such that*

(i) $\varphi(E) = G$;

(ii) $\varphi^*(s) = E$;

(iii) $||\varphi\xi||_{\ell_2} = ||\xi||$ *for all* $\xi \in E$;

(iv) $\varphi\phi^*$ *is the orthogonal projection in* ℓ_2 *with* $\varphi\phi^*(s) = G$.

Moreover, the map

$$\varphi\colon \mathcal{L}^*(E, ||\cdot||) \to \mathcal{L}^*(s), \quad x \mapsto \varphi x\varphi^*,$$

is a continuous injective **-algebra homomorphism with* $\operatorname{im}\varphi = \mathcal{L}^*(G)$ *and the map*

$$P\colon \mathcal{L}^*(s) \to \mathcal{L}^*(G), \quad x \mapsto \varphi\phi^*x\varphi\phi^*,$$

is a continuous projection onto $\mathcal{L}^*(G)$.

Proof. By Lemma 3.12, there is an orthogonally complemented subspace G of s and an isomorphism $w\colon E \to G$ of Fréchet spaces such that $\|w\xi\|_{\ell_2} = \|\xi\|$ for all $\xi \in E$. Let $\rho\colon s \to s$ be the orthogonal projection onto G. The operators w, ρ and the identity map $\iota\colon G \hookrightarrow s$ can be extended to the continuous linear operators between Hilbert spaces (for simplicity denoted by the same symbols): $w\colon H \to \overline{G}$, $\rho\colon \ell_2 \to \ell_2$ and $\iota\colon \overline{G} \hookrightarrow \ell_2$, where $\overline{G}$ is the closure of G in ℓ_2. Therefore, the Hermitian adjoints w^* and ι^* of the operators w and ι are well-defined. We have thus the following commutative diagram of continuous linear maps between Fréchet and Hilbert spaces

$$\begin{tikzcd} E \arrow[r, "w"] \arrow[d, hook] & G \arrow[r, hook, "\iota"] \arrow[d, hook] & s \arrow[d, hook] \\ H \arrow[r, "w"] & \overline{G} \arrow[r, hook, "\iota"] & \ell_2 \end{tikzcd}$$

and the diagram with the corresponding adjoint operators

$$\begin{tikzcd} \ell_2 \arrow[r, "\iota^*"] \arrow[dr, "\rho"'] & \overline{G} \arrow[r, "w^*"] \arrow[d, hook, "\iota"] & H \\ & \ell_2. & \end{tikzcd}$$

It follows easily that $\iota^*\colon \ell_2 \to \overline{G}$ is the orthogonal projection onto $\overline{G}$, whence $\iota^*(s) = \rho(s) = G$. Moreover, $w^*(G) = E$. Indeed, if $(\cdot,\cdot)$ denotes the scalar product on E corresponding to the Hilbert norm $\|\cdot\|$, then

$$(w^*w\xi, \eta) = \langle w\xi, w\eta\rangle = (\xi, \eta)$$

for all $\xi, \eta \in E$. Hence, E being dense in H, $w^*w = \mathrm{id}_H$, and so $w^*(G) = E$. Consequently, we have the following commutative diagram

$$\begin{tikzcd} s \arrow[r, "\iota^*"] \arrow[dr, "\rho"'] & G \arrow[r, "w^*"] \arrow[d, hook, "\iota"] & E \\ & s. & \end{tikzcd}$$

It is easy to check that $\varphi := \iota w$ satisfies conditions (i)–(iii) and a simple computation shows that $\varphi\phi^*$ is a self-adjoint projection (and thus orthogonal) on ℓ_2 with $\varphi\phi^*(s) = G$. In consequence, $\varphi\colon \mathcal{L}^*(E, \|\cdot\|) \to \mathcal{L}^*(s), x \mapsto \varphi x \varphi^*$, is an injective *-homomorphism with $\operatorname{im}\varphi = \mathcal{L}^*(G)$ and, moreover,

$$P\colon \mathcal{L}^*(s) \to \mathcal{L}^*(G), x \mapsto \varphi\phi^* x \varphi\phi^*$$