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Maxwell's Equations

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Maxwell's Equations

Analysis and Numerics

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Preface

This volume contains 12 chapters that provide some recent developments in the analysis and numerics of Maxwell's equations. The contributions result from Workshop 1 on "Analysis and Numerics of Acoustic and Electromagnetic Problems" held at the Radon Institute for Computational and Applied Mathematics (RICAM) in Linz, Austria, October 17–22, 2016. This workshop was the first workshop within the Special Semester on "Computational Methods in Science and Engineering," which took place in Linz, October 10–December 16, 2016; see also the website:

https://www.ricam.oeaw.ac.at/specsem/specsem2016/

Maxwell's equations of electro-dynamics are of huge importance in mathematical physics, engineering, and especially in mathematics, leading since their discovery to interesting mathematical problems and even to new fields of mathematical research, particularly in the analysis and numerics of partial differential equations and applied functional analysis. The impact to science in general has been formulated by the famous physicist, Richard Feynman:

> *From a long view of the history of mankind – seen from, say, ten thousand years from now – there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics. The American Civil War will pale into provincial insignificance in comparison with this important scientific event of the same decade.*

The deep understanding of Maxwell's equations and the possibility of their numerical solution in complex geometries and different settings have led to very efficient and robust simulation methods in Computational Electromagnetics. Moreover, efficient simulation methods pave the way for optimizing electromagnetic devices and processes. Digital communication and e-mobility are two fields where simulation and optimization techniques that are based on Maxwell's equations play a deciding role.

More than 70 scientists from 14 countries participated in the workshop; see Figure 1. The workshop brought together different communities, namely people working in analysis of Maxwell's equations with those working in numerical analysis of Maxwell's equations and computational electromagnetics and acoustics. This collection of selected contributions contains original papers that are arranged in an alphabetical order. We are now going to give short description of these contributions.

In Chapter 1, Alonso Rodríguez, Bertolazzi, and Valli proposed and analyzed two variational saddle-point formulations of the curl-div system. Moreover, suitable Hilbert spaces and curl-free and divergence-free finite elements are employed. Finally, numerical tests illustrate the performance of the proposed approximation methods.

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Figure 1: Participants of the first workshop of the special semester 2016 at RICAM.

In Chapter 2, Bauer gives an asymptotic expansion of time dependent Maxwell's equations in terms of iterated div-curl systems in case that charge velocities are small in comparison with the speed of light.

In Chapter 3, Bauer, Pauly, and Schomburg prove that the space of differential forms with weak exterior- and co-derivative is compactly embedded into the space of square integrable forms. Mixed boundary conditions and weak Lipschitz domains are considered. Furthermore, canonical applications such as Maxwell estimates, Helmholtz decompositions, and static solution theories are shown.

In Chapter 4, Bonnet-Ben Dhia, Fliss, and Tjandrawidjaja considered the 2D Helmholtz equation with a complex wavenumber in the exterior of a convex polygonal obstacle with a Robin-type boundary condition using the principle of the half-space matching method. It is proved that this system is of Fredholm type and the theoretical results are supported by numerical experiments.

In Chapter 5, Cogar, Colton, and Monk present an approach to the problem of the possible non-uniqueness of solutions to inverse electromagnetic scattering problems in anisotropic media through the use of appropriate "target signatures," i. e., eigenvalues associated with the direct scattering problem that are accessible to measurement from a knowledge of the scattering data. In this contribution, three different sets of eigenvalues are utilized as target signatures.

In Chapter 6, Costabel and Dauge investigate Maxwell eigenmodes in three-dimensional bounded electromagnetic cavities that have the form of a product of lower dimensional domains in some system of coordinates such as Cartesian, cylindrical, and spherical variables. As application of their general formulas, explicit eigenpairs in a cuboid, in a circular cylinder, and in a cylinder with a coaxial circular hole are found.

In Chapter 7, Hiptmair and Pechstein show stable discrete regular decompositions for Nédélec's tetrahedral edge element spaces of any polynomial degree on a bounded Lipschitz domain. Such decompositions have turned out to be crucial in the numerical analysis of "edge" finite element methods for variational problems in computational electromagnetics. Key tools for these constructions are continuous regular decompositions, boundary-aware local co-chain projections, projection-based interpolations, and quasi-interpolations with low regularity requirements.

In Chapter 8, Kress presents a survey on uniqueness, that is, identifiability and on reconstruction issues for inverse obstacle scattering for time-harmonic acoustic and electromagnetic waves. New integral equation formulations for transmission eigenvalues that play an important role through their connections with the linear sampling method and the factorization method for inverse scattering problems for penetrable objects are given as well.

In Chapter 9, Nicaise and Tomezyk suggest a variational formulation of the time-harmonic Maxwell equation with impedance boundary conditions in polyhedral domains, and show existence and uniqueness of weak solutions by a compact perturbation argument. Corner and edge singularities are investigated and a wavenumber explicit error analysis is performed.

In Chapter 10, Osterbrink and Pauly investigate time-harmonic electro-magnetic scattering or radiation problems governed by Maxwell's equations in an exterior weak Lipschitz domain with mixed boundary conditions. A solution theory in terms of a Fredholm-type alternative using the framework of polynomially weighted Sobolev spaces, Eidus' principle of limiting absorption, and local compact embeddings is presented.

In Chapter 11, Picard considers a coupled system of Maxwell's equations and the equations of elasticity, where the coupling occurs not via material properties but through an interaction on an interface separating the two regimes. Evolutionary well-posedness in the sense of Hadamard well-posedness supplemented by causal dependence is shown for a natural choice of generalized interface conditions. The results are obtained in a Hilbert space setting (Picard's approach) incurring no regularity constraints on the boundary and the interface of the underlying regions.

In Chapter 12, Waurick addresses the continuous dependence of solutions to certain equations on the coefficients. Three examples are discussed: A homogenization problem for a Kelvin–Voigt model for elasticity, the discussion of continuous dependence of the coefficients for acoustic waves with impedance-type boundary conditions, and a singular perturbation problem for a mixed-type equation. By means of counterexamples optimality of these results are obtained.

The careful reviewing process was only possible with the help of the anonymous referees who did an invaluable work that helped the authors to improve their contributions. Furthermore, we would like to thank the administrative and technical staff of RICAM for their support during the special semester. Last but not least, we express our thanks to Apostolos Damialis and Nadja Schedensack from the Walter de Gruyter GmbH, Berlin/Boston, for continuing support and patience while preparing this volume.

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Contents

Ana Alonso Rodríguez, Enrico Bertolazzi, and Alberto Valli

1 The curl–div system: theory and finite element approximation

Abstract: We first propose and analyze two variational formulations of the curl–div system that rewrite it as a saddle-point problem. Existence and uniqueness results are then an easy consequence of this approach. Second, introducing suitable constrained Hilbert spaces, we devise other variational formulations that turn out to be useful for numerical approximation. Curl-free and divergence-free finite elements are employed for discretizing the problem, and the corresponding finite element solutions are shown to converge to the exact solution. Several numerical tests are also included, illustrating the performance of the proposed approximation methods.

Keywords: Curl–div system, well-posedness, finite element approximation

MSC 2010: 65N30, 35J56, 35Q35, 35Q60

1 Introduction

The curl–div system often appears in electromagnetism (electrostatics, magnetostatics) and in fluid dynamics (rotational incompressible flows, velocity–vorticity formulations). Let $\Omega \subset \mathbb{R}^3$ be a bounded domain (i. e., a bounded, open and connected set): depending on the boundary condition, in its most basic form it reads

$$\begin{cases} \operatorname{curl} \mathbf{u} = \mathbf{J} & \text{in } \Omega \\ \operatorname{div} \mathbf{u} = f & \text{in } \Omega \\ \mathbf{u} \times \mathbf{n} = \mathbf{a} & \text{on } \partial\Omega , \end{cases} \tag{1.1}$$

or

$$\begin{cases} \operatorname{curl} \mathbf{u} = \mathbf{J} & \text{in } \Omega \\ \operatorname{div} \mathbf{u} = f & \text{in } \Omega \\ \mathbf{u} \cdot \mathbf{n} = b & \text{on } \partial\Omega , \end{cases} \tag{1.2}$$

with in addition some topological conditions assuring uniqueness.

The aim of this paper is two-fold: first, at the theoretical level, we present a couple of saddle-point variational formulations of the curl–div system and show that they are

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well-posed; second, focusing on discretization, we devise other non-standard variational formulations of this problem which lead to simple and efficient finite element schemes for its numerical approximation.

Concerning the second issue, the main novelty resides in the functional framework we adopt: we look for the solution in the spaces of curl-free or divergence-free vector fields. For the sake of implementation, we also describe in detail how to construct a simple finite element basis for these vector spaces; convergence of the finite element approximations is then shown easily. A key point of our approach is a suitable tree–cotree decomposition of the graph given by the nodes and the edges of the mesh.

The paper is organized as follows. In Section 2, after having recalled some classical results, by means of a saddle-point approach we show that the curl–div system has a unique solution, for both types of boundary condition. Sections 3 and 4 are devoted to devising two other new variational formulations, that will be used for numerical approximation, and to prove that they are well-posed. In Section 5, we give an overview of some previous results related to the discretization of the curl–div system. In Sections 6 and 7, the finite element numerical approximation of the curl–div system based on the new variational formulations is described and analyzed. In the last section, we finally present several numerical results that illustrate the performance of the proposed approximation methods.

2 Theoretical results

Let us start with some notation. Let Ω be a bounded domain of $\mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$ and let $(\partial\Omega)_0, \dots, (\partial\Omega)_p$ be the connected components of $\partial\Omega$, $(\partial\Omega)_0$ being the external one. From the topological point of view, p is the rank of the second homology group of $\overline{\Omega}$, namely, the second Betti number $\beta_2(\Omega)$. The unit outward normal vector on $\partial\Omega$ is indicated by $\mathbf{n}$.

The space of infinitely differentiable functions with compact support in Ω is denoted by $C_0^\infty(\Omega)$. The classical Sobolev spaces are denoted by $H^s(\Omega)$ or $H^s(\partial\Omega)$, for $s \in \mathbb{R}$; for $s = 0$, we write $H^0(\Omega) = L^2(\Omega)$. The space of (essentially) bounded and measurable functions defined in Ω is denoted by $L^\infty(\Omega)$. Moreover, we define

$$H(\mathrm{curl};\Omega) = \{\mathbf{v} \in (L^2(\Omega))^3 \mid \mathrm{curl}\,\mathbf{v} \in (L^2(\Omega))^3\}\,,$$
$$H(\mathrm{curl}^0;\Omega) = \{\mathbf{v} \in (L^2(\Omega))^3 \mid \mathrm{curl}\,\mathbf{v} = \mathbf{0} \text{ in } \Omega\}\,,$$
$$H(\mathrm{div};\Omega) = \{\boldsymbol{\xi} \in (L^2(\Omega))^3 \mid \mathrm{div}\,\boldsymbol{\xi} \in L^2(\Omega)\}\,,$$
$$H(\mathrm{div}^0;\Omega) = \{\boldsymbol{\xi} \in (L^2(\Omega))^3 \mid \mathrm{div}\,\boldsymbol{\xi} = 0 \text{ in } \Omega\}\,.$$

The space of traces on $\partial\Omega$ of functions ϕ belonging to $H^1(\Omega)$ is the space $H^{1/2}(\partial\Omega)$ (whose dual space is the space $H^{-1/2}(\partial\Omega)$); the space of normal traces $\boldsymbol{\xi} \cdot \mathbf{n}$ on $\partial\Omega$ of vector fields $\boldsymbol{\xi}$ belonging to $H(\mathrm{div};\Omega)$ is $H^{-1/2}(\partial\Omega)$; the space of tangential traces $\mathbf{v}\times\mathbf{n}$ on $\partial\Omega$ of vector fields $\mathbf{v}$ belonging to $H(\mathrm{curl};\Omega)$ is denoted by $H^{-1/2}(\mathrm{div}_\tau;\partial\Omega)$ (for the interested reader, an intrinsic characterization of this space can be found in Buffa and Ciarlet [24, 25]; see also Alonso Rodríguez and Valli [8, Section A1]).

In the following, we also need to consider a set closed curves in $\overline{\Omega}$, denoted by $\{\sigma_n\}_{n=1}^{g}$, that are representatives of a basis of the first homology group (whose rank is therefore equal to g, the first Betti number $\beta_1(\Omega)$): in other words, this set is a maximal set of non-bounding closed curves in $\overline{\Omega}$. Let us recall that an explicit and efficient construction of the closed curves $\{\sigma_n\}_{n=1}^{g}$ is given by Hiptmair and Ostrowski [39]. For a more detailed presentation of the homological concepts that are useful in this context, see, e. g., Bossavit [20, Chap. 5], Hiptmair [37, Section 2 and Section 3], Gross and Kotiuga [35, Chapter 1 and Chapter 3]; see also Benedetti et al. [13], Alonso Rodríguez et al. [4].

2.1 The curl–div system with assigned tangential component on the boundary

Let $\boldsymbol{\eta}$ be a symmetric matrix, uniformly positive definite in Ω, with entries belonging to $L^\infty(\Omega)$. Given $\mathbf{J} \in (L^2(\Omega))^3$, $f \in L^2(\Omega)$, $\mathbf{a} \in H^{-1/2}(\mathrm{div}_\tau;\partial\Omega)$, $\boldsymbol{\alpha} \in \mathbb{R}^p$, we look for $\mathbf{u} \in (L^2(\Omega))^3$ such that

$$\begin{cases} \mathrm{curl}(\boldsymbol{\eta}\mathbf{u}) = \mathbf{J} & \text{in } \Omega \\ \mathrm{div}\,\mathbf{u} = f & \text{in } \Omega \\ (\boldsymbol{\eta}\mathbf{u}) \times \mathbf{n} = \mathbf{a} & \text{on } \partial\Omega \\ \displaystyle\int_{(\partial\Omega)_r} \mathbf{u}\cdot\mathbf{n} = \alpha_r & \text{for each } r = 1,\dots,p\,. \end{cases} \tag{2.1}$$

The data must satisfy the necessary conditions $\mathrm{div}\,\mathbf{J} = 0$ in Ω, $\int_\Omega \mathbf{J}\cdot\boldsymbol{\rho} + \int_{\partial\Omega} \mathbf{a}\cdot\boldsymbol{\rho} = 0$ for each $\boldsymbol{\rho} \in \mathcal{H}(m)$, where $\mathcal{H}(m)$ is the space of Neumann harmonic fields, namely,

$$\mathcal{H}(m) = \{\boldsymbol{\rho} \in (L^2(\Omega))^3 \mid \mathrm{curl}\,\boldsymbol{\rho} = \mathbf{0} \text{ in } \Omega, \mathrm{div}\,\boldsymbol{\rho} = 0 \text{ in } \Omega, \boldsymbol{\rho}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}\,, \tag{2.2}$$

whose dimension is known to be equal to g, the rank of the first homology group of $\overline{\Omega}$, and finally $\mathbf{J}\cdot\mathbf{n} = \mathrm{div}_\tau\,\mathbf{a}$ on $\partial\Omega$ (for a summary of the properties of the spaces of harmonic fields and for a definition of the tangential divergence operator div_τ; see, e. g., Alonso Rodríguez and Valli [8, Section A1 and Section A4]).

By means of a variational approach Saranen [59, 60] has shown that this problem has a unique solution (see also the results proved in Alonso Rodríguez and Valli [8, Section A3], and the more abstract approach by Picard [52, 53]). Let us briefly summarize the principal points of this procedure. The method is based on the Helmholtz decomposition, namely, a splitting of the solution in three terms, orthogonal with respect to the scalar product $\int_\Omega \boldsymbol{\eta}^{-1}\mathbf{v}\cdot\mathbf{w}$, that reads

$$\boldsymbol{\eta}\mathbf{u} = \boldsymbol{\eta}\,\mathrm{curl}\,\mathbf{q} + \mathrm{grad}\,\chi + \boldsymbol{\eta}\mathbf{h}\,.$$

Here, the vector field $\mathbf{q}$ satisfies $\mathrm{curl}(\boldsymbol{\eta}\,\mathrm{curl}\,\mathbf{q}) = \mathbf{J}$ in Ω and $(\boldsymbol{\eta}\,\mathrm{curl}\,\mathbf{q})\times\mathbf{n} = \mathbf{a}$ on $\partial\Omega$; χ is the solution to $\mathrm{div}(\boldsymbol{\eta}^{-1}\,\mathrm{grad}\,\chi) = f$ in Ω and $\chi = 0$ on $\partial\Omega$; $\mathbf{h}$ is a generalized Dirichlet harmonic field, namely, it is an element of the finite dimensional vector space

$$\begin{aligned}\mathcal{H}_\eta(e) = \{\boldsymbol{\pi}\in (L^2(\Omega))^3 \mid \mathrm{curl}(\boldsymbol{\eta}\boldsymbol{\pi}) = \mathbf{0} \text{ in } \Omega, \mathrm{div}\,\boldsymbol{\pi} = 0 \text{ in } \Omega,\\ (\boldsymbol{\eta}\boldsymbol{\pi})\times\mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}\,,\end{aligned} \tag{2.3}$$

whose dimension is known to be equal to p (precisely, $\mathbf{h}$ is the unique element of $\mathcal{H}_\eta(e)$ satisfying $\int_{(\partial\Omega)_r}\mathbf{h}\cdot\mathbf{n} = \alpha_r - \int_{(\partial\Omega)_r}\boldsymbol{\eta}^{-1}\,\mathrm{grad}\,\chi\cdot\mathbf{n}$ for each $r = 1,\dots,p$).

Since a solution $\mathbf{q}$ to $\mathrm{curl}(\boldsymbol{\eta}\,\mathrm{curl}\,\mathbf{q}) = \mathbf{J}$ in Ω and $(\boldsymbol{\eta}\,\mathrm{curl}\,\mathbf{q})\times\mathbf{n} = \mathbf{a}$ on $\partial\Omega$ is not unique ($\mathbf{q}+\mathrm{grad}\,\phi$ is still a solution), other equations have to be added. Typically, one imposes the gauge conditions $\mathrm{div}\,\mathbf{q} = 0$ in Ω, $\mathbf{q}\cdot\mathbf{n} = 0$ on $\partial\Omega$ and $\mathbf{q}\perp\mathcal{H}(m)$.

The approach we have just described has thus led to two variational problems: a standard Dirichlet boundary value problem for χ, and a constrained problem for $\mathbf{q}$ (the determination of the harmonic field $\mathbf{h}$ also needs some additional work, but it is an easy finite dimensional problem).

Numerical approaches for approximating these two problems are easily devised. In fact, the first one is a standard elliptic problem. Numerical approximation can be performed by scalar nodal elements in $H^1(\Omega)$, looking for the unknown χ and then computing its gradient, or by means of a mixed method in $H(\mathrm{div};\Omega)\times L^2(\Omega)$, in which $\mathrm{grad}\,\chi \in H(\mathrm{div};\Omega)$ is directly computed as an auxiliary unknown.

Concerning the problem related to the vector field $\mathbf{q}$, a first choice is to work in $H(\mathrm{curl};\Omega)\cap H(\mathrm{div};\Omega)$, hence with globally-continuous nodal finite elements for each component of $\mathbf{q}$; the drawback is that, in the presence of re-entrant corners, the solution is singular (it does not belong to $(H^1(\Omega))^3$) and $(H^1(\Omega))^3$ is a closed subspace of $H(\mathrm{curl};\Omega)\cap H(\mathrm{div};\Omega)$, hence in this case a finite element scheme cannot be convergent (see, e. g., Costabel et al. [29]).

An alternative method is to formulate the problem as a saddle-point problem for the vector field $\mathbf{q}$ in $H(\mathrm{curl};\Omega)$, in which the divergence constraint is imposed in a week sense, introducing a scalar Lagrange multiplier; in this way the number of degrees of freedom is rather high, as, besides an edge approximation of the vector field $\mathbf{q}$, one has also to consider a nodal approximation of the scalar Lagrange multiplier. The resulting algebraic problem is associated to an indefinite matrix; however, for its resolution efficient regularization techniques are known (see Hiptmair [37, Section 6.1]).

A way for avoiding the introduction of a Lagrange multiplier is to solve the equation $\mathrm{curl}(\boldsymbol{\eta}\,\mathrm{curl}\,\mathbf{q}) = \mathbf{J}$ in Ω by using edge elements without any gauge. Though the matrix to deal with is singular, the conjugate gradient method is known to be a viable tool for solving the associated algebraic problem (see the theoretical result by Kaasschieter [40]; see also Bossavit [20, Section 6.2], Bíró [16]); however, the computation of the right-hand side should be done with particular care (see Fujiwara et al. [33], Bíró et al. [17], Ren [56]), and, for problems with a large number of unknowns, it is not easy to devise an efficient preconditioner.

Summing up, the most classical variational formulations of the curl–div system are not completely satisfactory when numerical approximation has to be performed. We will present in Section 3 a new variational formulation of problem (2.1) that looks much more suitable for finite element discretization.

However, before coming to this point, we want to put the problem on a solid foundation, providing in this and in the following section a proof of the well-posedness of the curl–div system. Instead of reporting the classical result obtained by Saranen [59, 60], we propose a saddle-point formulation that to our knowledge has not been considered yet. With this approach, one does not introduce the potentials $\mathbf{q}$ and χ, keeps the original unknown $\mathbf{u}$ and imposes the curl constraint by means of a Lagrange multiplier: it could be seen as a least-squares formulation with a constraint on the curl of $\mathbf{u}$, or similarly, a Lagrangian method for a constrained optimization problem.

Let us derive step by step the variational problem we are interested in. Taking the gradient of the second equation in (2.1) we obtain $\operatorname{grad}\operatorname{div}\mathbf{u} = \operatorname{grad} f$. Multiplying for a test vector field $\boldsymbol{\xi}$, integrating in Ω and integrating by parts we obtain

$$-\int_\Omega (\operatorname{div}\mathbf{u} - f)\operatorname{div}\boldsymbol{\xi} + \int_{\partial\Omega} (\operatorname{div}\mathbf{u} - f)\boldsymbol{\xi}\cdot\mathbf{n} = 0\,.$$

The integral on the boundary will be omitted in the variational formulation, in order to impose in a suitable weak sense the condition $\operatorname{div}\mathbf{u} - f = 0$ on $\partial\Omega$.

Multiplying the first equation in (2.1) by $\mathbf{v}$, integrating in Ω and integrating by parts we find

$$\int_\Omega \mathbf{J}\cdot\mathbf{v} = \int_\Omega \operatorname{curl}(\boldsymbol{\eta}\mathbf{u})\cdot\mathbf{v} = \int_\Omega \boldsymbol{\eta}\mathbf{u}\cdot\operatorname{curl}\mathbf{v} + \int_{\partial\Omega} \mathbf{n}\times\boldsymbol{\eta}\mathbf{u}\cdot\mathbf{v}\,,$$

hence

$$\int_\Omega \boldsymbol{\eta}\mathbf{u}\cdot\operatorname{curl}\mathbf{v} = \int_\Omega \mathbf{J}\cdot\mathbf{v} + \int_{\partial\Omega} \mathbf{a}\cdot\mathbf{v}\,.$$

Then, introducing a Lagrange multiplier $\mathbf{p}$, we are led to consider the problem

$$\begin{aligned}
&\int_\Omega \operatorname{div}\mathbf{u}\operatorname{div}\boldsymbol{\xi} + \int_\Omega \boldsymbol{\eta}\boldsymbol{\xi}\cdot\operatorname{curl}\mathbf{p} = \int_\Omega f\operatorname{div}\boldsymbol{\xi}\\
&\int_\Omega \boldsymbol{\eta}\mathbf{u}\cdot\operatorname{curl}\mathbf{v} = \int_\Omega \mathbf{J}\cdot\mathbf{v} + \int_{\partial\Omega} \mathbf{a}\cdot\mathbf{v}\,.
\end{aligned}$$

Now the natural question is: which are the variational spaces for $\mathbf{u}$, $\boldsymbol{\xi}$, $\mathbf{p}$ and $\mathbf{v}$? Define the Hilbert spaces

$$\begin{aligned} \mathcal{W} &= \left\{ \boldsymbol{\xi} \in H(\mathrm{div};\Omega) \mid \int_{(\partial\Omega)_r} \boldsymbol{\xi}\cdot\mathbf{n} = 0 \text{ for each } r = 1,\dots,p \right\} \\ \mathcal{Q} &= \left\{ \mathbf{v} \in H(\mathbf{curl};\Omega) \mid \int_{\Omega} \mathbf{v}\cdot\mathbf{w} = 0 \text{ for each } \mathbf{w} \in H(\mathbf{curl}^0;\Omega) \right\}. \end{aligned} \tag{2.4}$$

We choose $\mathbf{u}, \boldsymbol{\xi} \in \mathcal{W}$ and $\mathbf{p}, \mathbf{v} \in \mathcal{Q}$. It is worth noting that the space $H(\mathbf{curl}^0;\Omega)$ can be described as

$$H(\mathbf{curl}^0;\Omega) = \mathbf{grad}\, H^1(\Omega) \overset{\perp}{\oplus} \mathcal{H}(m) \tag{2.5}$$

(see, e. g., Alonso Rodríguez and Valli [8, Section A3]). Therefore, by integration by parts, an element $\mathbf{v} \in \mathcal{Q}$ can be characterized as an element in $H(\mathbf{curl};\Omega)$ such that $\mathrm{div}\,\mathbf{v} = 0$ in Ω, $\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\Omega$ and $\mathbf{v} \perp \mathcal{H}(m)$.

Summing up, our variational problem is

$$\begin{aligned} &\text{find } \mathbf{u} \in \mathcal{W}, \mathbf{p} \in \mathcal{Q} : \\ &\quad \int_\Omega \mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\boldsymbol{\xi} + \int_\Omega \boldsymbol{\eta}\boldsymbol{\xi}\cdot\mathbf{curl}\,\mathbf{p} = \int_\Omega f\,\mathrm{div}\,\boldsymbol{\xi} \\ &\quad \int_\Omega \boldsymbol{\eta}\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v} = \int_\Omega \mathbf{J}\cdot\mathbf{v} + \int_{\partial\Omega} \mathbf{a}\cdot\mathbf{v} \\ &\text{for each } \boldsymbol{\xi} \in \mathcal{W}, \mathbf{v} \in \mathcal{Q}. \end{aligned} \tag{2.6}$$

Before analyzing this problem, we need an additional tool. It is known that it is possible to select a basis $\{\boldsymbol{\pi}_s^\eta\}_{s=1}^p$ of the space of harmonic fields $\mathcal{H}_\eta(e)$ defined in (2.3) with the properties

$$\int_{(\partial\Omega)_r} \boldsymbol{\pi}_s^\eta \cdot \mathbf{n} = \delta_{rs}$$

(see, e. g., Alonso Rodríguez and Valli [8, Section A4]; for $\boldsymbol{\eta} = \mathrm{Id}$ we simply write $\boldsymbol{\pi}_s$). Then, if $\mathbf{u}$ is a solution to problem (2.1) with $\alpha_r = 0$, $r = 1,\dots,p$, we check easily that $\mathbf{u} + \sum_{r=1}^p \alpha_r \boldsymbol{\pi}_r^\eta$ is a solution to problem (2.1) with given α_r.

This also says that a solution $\mathbf{u}$ of problem (2.1), if it exists, is unique. In fact, taking vanishing data, it follows from the first three equations that $\mathbf{u} \in \mathcal{H}_\eta(e)$, and consequently it can be written as $\mathbf{u} = \sum_{s=1}^p u_s \boldsymbol{\pi}_s^\eta$. Then, for each $r = 1,\dots,p$,

$$0 = \int_{(\partial\Omega)_r} \mathbf{u}\cdot\mathbf{n} = \sum_{s=1}^p u_s \int_{(\partial\Omega)_r} \boldsymbol{\pi}_s^\eta \cdot \mathbf{n} = u_r\,,$$

and in conclusion $\mathbf{u} = \mathbf{0}$.

Theorem 1. *If* $(\mathbf{u}, \mathbf{p})$ *is a solution to problem* (2.6) *then* $\mathbf{p} = \mathbf{0}$ *and* $\mathbf{u}$ *is a solution to problem* (2.1) *for* $\alpha_r = 0$, $r = 1, \ldots, p$.

Proof. By the Stokes theorem for closed surfaces, we know that $\operatorname{curl} \mathbf{v} \in \mathcal{W}$ for each $\mathbf{v} \in \mathcal{Q}$. Therefore, taking $\boldsymbol{\xi} = \operatorname{curl} \mathbf{p}$ in the first equation we find

$$\int_\Omega \boldsymbol{\eta} \operatorname{curl} \mathbf{p} \cdot \operatorname{curl} \mathbf{p} = 0\,,$$

hence $\operatorname{curl} \mathbf{p} = \mathbf{0}$; since the elements in $\mathcal{Q}$ are orthogonal to $H(\operatorname{curl}^0; \Omega)$ (with respect to the $L^2(\Omega)$-scalar product), it follows $\mathbf{p} = \mathbf{0}$.

Choosing $\boldsymbol{\xi} \in (C_0^\infty(\Omega))^3$ we find that $\operatorname{grad}(\operatorname{div} \mathbf{u} - f) = 0$ in Ω in the distributional sense, hence $(\operatorname{div} \mathbf{u} - f)$ is constant in Ω. Take $\widehat{\boldsymbol{\xi}} \in H(\operatorname{div}; \Omega)$ and define $\widehat{\xi}_r = \int_{(\partial\Omega)_r} \widehat{\boldsymbol{\xi}} \cdot \mathbf{n}$. Then $\boldsymbol{\xi} = \widehat{\boldsymbol{\xi}} - \sum_{r=1}^p \widehat{\xi}_r \boldsymbol{\pi}_r^\eta$ belongs to $\mathcal{W}$ and satisfies $\operatorname{div} \boldsymbol{\xi} = \operatorname{div} \widehat{\boldsymbol{\xi}}$. Hence the first equation in problem (2.6) is satisfied for each $\widehat{\boldsymbol{\xi}} \in H(\operatorname{div}; \Omega)$, and by integration by parts we find $\operatorname{div} \mathbf{u} - f = 0$ on $\partial\Omega$, hence $\operatorname{div} \mathbf{u} = f$ in Ω.

Let us prove that the second equation is indeed satisfied for each $\widehat{\mathbf{v}} \in H(\operatorname{curl}; \Omega)$. Let $P\widehat{\mathbf{v}}$ be the $L^2(\Omega)$-orthogonal projection of $\widehat{\mathbf{v}}$ on $H(\operatorname{curl}^0; \Omega)$. Then $P\widehat{\mathbf{v}} = \operatorname{grad} \widehat{\omega} + \widehat{\boldsymbol{\rho}}$, with $\widehat{\omega} \in H^1(\Omega)$ and $\widehat{\boldsymbol{\rho}} \in \mathcal{H}(m)$, $\mathbf{v} = (\widehat{\mathbf{v}} - P\widehat{\mathbf{v}}) \in \mathcal{Q}$, and $\operatorname{curl} \mathbf{v} = \operatorname{curl} \widehat{\mathbf{v}}$. Moreover,

$$\int_\Omega \mathbf{J} \cdot \mathbf{v} + \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{v} = \int_\Omega \mathbf{J} \cdot \widehat{\mathbf{v}} + \int_{\partial\Omega} \mathbf{a} \cdot \widehat{\mathbf{v}} - \int_\Omega \mathbf{J} \cdot P\widehat{\mathbf{v}} - \int_{\partial\Omega} \mathbf{a} \cdot P\widehat{\mathbf{v}}$$

and, by integrating by parts in Ω and on $\partial\Omega$,

$$\begin{aligned} \int_\Omega \mathbf{J} \cdot P\widehat{\mathbf{v}} + \int_{\partial\Omega} \mathbf{a} \cdot P\widehat{\mathbf{v}} &= \int_\Omega \mathbf{J} \cdot (\operatorname{grad} \widehat{\omega} + \widehat{\boldsymbol{\rho}}) + \int_{\partial\Omega} \mathbf{a} \cdot (\operatorname{grad} \widehat{\omega} + \widehat{\boldsymbol{\rho}}) \\ &= -\int_\Omega \operatorname{div} \mathbf{J} \widehat{\omega} + \int_{\partial\Omega} \mathbf{J} \cdot \mathbf{n} \widehat{\omega} + \int_\Omega \mathbf{J} \cdot \widehat{\boldsymbol{\rho}} - \int_{\partial\Omega} \operatorname{div}_\tau \mathbf{a} \widehat{\omega} + \int_{\partial\Omega} \mathbf{a} \cdot \widehat{\boldsymbol{\rho}} = 0\,, \end{aligned}$$

having used the compatibility conditions on the data $\mathbf{J}$ and $\mathbf{a}$.

Hence the second equation is satisfied for each $\widehat{\mathbf{v}} \in H(\operatorname{curl}; \Omega)$, and taking $\widehat{\mathbf{v}} \in (C_0^\infty(\Omega))^3$ it follows $\operatorname{curl}(\boldsymbol{\eta}\mathbf{u}) = \mathbf{J}$ in Ω in the distributional sense. Repeating the same procedure for $\widehat{\mathbf{v}} \in H(\operatorname{curl}; \Omega)$, integration by parts gives $\boldsymbol{\eta}\mathbf{u} \times \mathbf{n} = \mathbf{a}$ on $\partial\Omega$. □

The existence of a solution to problem (2.1) is therefore reduced to the proof of the existence of a solution to problem (2.6). This is a consequence of well-known results for saddle-point problems (see, e. g., Boffi et al. [19, Section 4.2]). In fact, the following two propositions permit us to apply the general well-posedness theory.

Proposition 1. *The bilinear form* $a(\boldsymbol{\psi}, \boldsymbol{\xi}) = \int_\Omega \operatorname{div} \boldsymbol{\psi} \operatorname{div} \boldsymbol{\xi}$ *is coercive in the space* $\mathcal{B}_0 \times \mathcal{B}_0$, *where*

$$\mathcal{B}_0 = \left\{ \boldsymbol{\xi} \in \mathcal{W} \mid \int_\Omega \boldsymbol{\eta}\boldsymbol{\xi} \cdot \operatorname{curl} \mathbf{v} = 0 \text{ for all } \mathbf{v} \in \mathcal{Q} \right\}.$$

Proof. Indeed, we have already seen that, if $\boldsymbol{\xi} \in \mathcal{B}_0$, then it follows that $\int_\Omega \boldsymbol{\eta}\boldsymbol{\xi} \cdot \operatorname{curl} \mathbf{v} = 0$ for all $\mathbf{v} \in H(\operatorname{curl};\Omega)$. Therefore, by integration by parts we deduce at once that $\operatorname{curl}(\boldsymbol{\eta}\boldsymbol{\xi}) = \mathbf{0}$ in Ω and $\boldsymbol{\eta}\boldsymbol{\xi} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$. Coercivity follows from the Friedrichs inequality: there exists a constant $C > 0$ such that for any vector field $\boldsymbol{\xi}$ belonging to $H(\operatorname{div};\Omega)$, with $\operatorname{curl}(\boldsymbol{\eta}\boldsymbol{\xi}) \in (L^2(\Omega))^3$, $\boldsymbol{\eta}\boldsymbol{\xi} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ and satisfying $\int_{(\partial\Omega)_r} \boldsymbol{\xi} \cdot \mathbf{n} = 0$ for each $r = 1, \dots, p$, it holds

$$\|\boldsymbol{\xi}\|_{L^2(\Omega)} \le C\big(\|\operatorname{curl}(\boldsymbol{\eta}\boldsymbol{\xi})\|_{L^2(\Omega)} + \|\operatorname{div} \boldsymbol{\xi}\|_{L^2(\Omega)}\big) .$$

This result can be shown by adapting in a straightforward way the proof presented, e. g., in Fernandes and Gilardi [32], using the fact that the space

$$\{\boldsymbol{\xi} \in H(\operatorname{div};\Omega) \mid \operatorname{curl}(\boldsymbol{\eta}\boldsymbol{\xi}) \in (L^2(\Omega))^3, \boldsymbol{\eta}\boldsymbol{\xi} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}$$

is compactly imbedded in $(L^2(\Omega))^3$ (see, e. g., Weber [64], Picard [54]). □

Proposition 2. *The bilinear form* $b(\boldsymbol{\xi}, \mathbf{v}) = \int_\Omega \boldsymbol{\eta}\boldsymbol{\xi} \cdot \operatorname{curl} \mathbf{v}$ *satisfies an inf–sup condition, namely, there exists* $\beta > 0$ *such that for each* $\mathbf{v} \in \mathcal{Q}$ *there exists* $\boldsymbol{\xi} \in \mathcal{W}$, $\boldsymbol{\xi} \neq \mathbf{0}$, *satisfying*

$$\int_\Omega \boldsymbol{\eta}\boldsymbol{\xi} \cdot \operatorname{curl} \mathbf{v} \ge \beta \|\boldsymbol{\xi}\|_{\mathcal{W}} \|\mathbf{v}\|_{\mathcal{Q}} .$$

Proof. If $\operatorname{curl} \mathbf{v} = \mathbf{0}$ in Ω, nothing has to be proved. Then suppose that $\operatorname{curl} \mathbf{v} \neq \mathbf{0}$. We have already seen that $\operatorname{curl} \mathbf{v} \in \mathcal{W}$ for each $\mathbf{v} \in \mathcal{Q}$, and that any vector field $\mathbf{v} \in \mathcal{Q}$ satisfies $\operatorname{div} \mathbf{v} = 0$ in Ω, $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega$ and $\mathbf{v} \perp \mathcal{H}(m)$. The thesis follows by choosing $\boldsymbol{\xi} = \operatorname{curl} \mathbf{v}$, as $\operatorname{div} \boldsymbol{\xi} = 0$ in Ω and the Friedrichs inequality

$$\|\mathbf{v}\|_{L^2(\Omega)} \le C \|\operatorname{curl} \mathbf{v}\|_{L^2(\Omega)}$$

is valid for $\mathbf{v} \in H(\operatorname{curl};\Omega) \cap H(\operatorname{div}^0;\Omega)$ satisfying $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega$ and $\mathbf{v} \perp \mathcal{H}(m)$ (see, e. g., Girault and Raviart [34, Section 3.5] if $\mathcal{H}(m) = \emptyset$, or Fernandes and Gilardi [32] if $\mathcal{H}(m) \neq \emptyset$). □

In conclusion, by means of these two propositions we have proved that the saddle-point problem (2.6) has a unique solution, and thus the same is true for problem (2.1).

2.2 The curl–div system with assigned normal component on the boundary

Let $\boldsymbol{\mu}$ be a symmetric matrix, uniformly positive definite in Ω, with entries belonging to $L^\infty(\Omega)$. Given $\mathbf{J} \in (L^2(\Omega))^3$, $f \in L^2(\Omega)$, $b \in H^{-1/2}(\partial\Omega)$, $\boldsymbol{\beta} \in \mathbb{R}^g$, we look for $\mathbf{u} \in (L^2(\Omega))^3$

such that

$$\begin{cases} \operatorname{curl}\mathbf{u} = \mathbf{J} & \text{in } \Omega \\ \operatorname{div}(\boldsymbol{\mu}\mathbf{u}) = f & \text{in } \Omega \\ \boldsymbol{\mu}\mathbf{u}\cdot\mathbf{n} = b & \text{on } \partial\Omega \\ \oint\limits_{\sigma_n} \mathbf{u}\cdot d\mathbf{s} = \beta_n & \text{for each } n = 1,\dots,g\,, \end{cases} \tag{2.7}$$

where the data satisfy the necessary conditions $\operatorname{div}\mathbf{J} = 0$ in Ω, $\int_\Omega f = \int_{\partial\Omega} b$; moreover, since we need to give a meaning to the line integral of $\mathbf{u}$ on σ_n, we follow the arguments in Alonso Rodríguez et al. [7, Section 2] and we also assume that $\mathbf{J}\cdot\mathbf{n} = 0$ on $\partial\Omega$ (which is more restrictive than the necessary condition $\int_{(\partial\Omega)_r} \mathbf{J}\cdot\mathbf{n} = 0$ for each $r = 1,\dots,p$).

The variational approach proposed by Saranen [59, 60] shows that this problem has a unique solution (see also Alonso Rodríguez and Valli [8, Section A3], and the results obtained by Picard [52, 53]). Again, the method is based on a orthogonal decomposition result, through which the solution is split as

$$\mathbf{u} = \boldsymbol{\mu}^{-1}\operatorname{curl}\mathbf{q} + \operatorname{grad}\chi + \mathbf{h}\,,$$

where the vector field $\mathbf{q}$ is a solution to $\operatorname{curl}(\boldsymbol{\mu}^{-1}\operatorname{curl}\mathbf{q}) = \mathbf{J}$ in Ω and $\mathbf{q}\times\mathbf{n} = \mathbf{0}$ on $\partial\Omega$; χ is the solution to $\operatorname{div}(\boldsymbol{\mu}\operatorname{grad}\chi) = f$ in Ω and $\boldsymbol{\mu}\operatorname{grad}\chi\cdot\mathbf{n} = b$ on $\partial\Omega$; $\mathbf{h}$ is a generalized Neumann harmonic field, namely, it is an element of the finite dimensional vector space

$$\begin{aligned} \mathcal{H}_\mu(m) = \{\boldsymbol{\rho} \in (L^2(\Omega))^3 \mid \operatorname{curl}\boldsymbol{\rho} = \mathbf{0} \text{ in } \Omega, \operatorname{div}(\boldsymbol{\mu}\boldsymbol{\rho}) = 0 \text{ in } \Omega, \\ \boldsymbol{\mu}\boldsymbol{\rho}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}\,, \end{aligned} \tag{2.8}$$

whose dimension is known to be equal to g (precisely, $\mathbf{h}$ is the unique element of $\mathcal{H}_\mu(m)$ satisfying $\oint_{\sigma_n} \mathbf{h}\cdot d\mathbf{s} = \beta_n - \oint_{\sigma_n} \boldsymbol{\mu}^{-1}\operatorname{curl}\mathbf{q}\cdot d\mathbf{s}$ for each $n = 1,\dots,g$).

Since a solution $\mathbf{q}$ to $\operatorname{curl}(\boldsymbol{\eta}\operatorname{curl}\mathbf{q}) = \mathbf{J}$ in Ω and $\mathbf{q}\times\mathbf{n} = \mathbf{0}$ on $\partial\Omega$ is not unique ($\mathbf{q} + \operatorname{grad}\phi$, with $\phi = 0$ on $\partial\Omega$, is still a solution), other equations have to be added. The standard gauge conditions are $\operatorname{div}\mathbf{q} = 0$ in Ω and $\mathbf{q}\perp\mathcal{H}(e)$, where $\mathcal{H}(e)$ is the space of Dirichlet harmonic vector fields, namely,

$$\begin{aligned} \mathcal{H}(e) = \{\boldsymbol{\pi} \in (L^2(\Omega))^3 \mid \operatorname{curl}\boldsymbol{\pi} = \mathbf{0} \text{ in } \Omega, \operatorname{div}\boldsymbol{\pi} = 0 \text{ in } \Omega, \\ \boldsymbol{\pi}\times\mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}\,. \end{aligned} \tag{2.9}$$

We do not specify the details of the proof of the existence of a solution $\mathbf{q}$ because here, as in the previous case, we base the theoretical analysis of the curl–div system (2.7) on a saddle-point variational formulation, quite close to that proposed by Kikuchi [42] (the limitations in that paper are that the domain has a simple topological shape, the boundary conditions are homogeneous and the coefficient $\boldsymbol{\mu}$ is a constant scalar parameter). With this approach, the introduction of the potentials $\mathbf{q}$ and χ is not needed, the original unknown $\mathbf{u}$ is kept and the equation related to the divergence is imposed by a means of Lagrange multiplier; more precisely, what we propose looks like a least-squares formulation with a constraint on the divergence of $\mathbf{u}$. Let us also point out that another variational formulation, more suitable for numerical approximation, will be introduced in Section 4.

We proceed as follows. Taking the curl of the first equation in (2.7) we obtain $\operatorname{curl}\operatorname{curl}\mathbf{u} = \operatorname{curl}\mathbf{J}$. Multiplying for a test vector field $\mathbf{v}$, integrating in Ω and integrating by parts we obtain

$$\int_\Omega (\operatorname{curl}\mathbf{u} - \mathbf{J}) \cdot \operatorname{curl}\mathbf{v} + \int_{\partial\Omega} \mathbf{n} \times (\operatorname{curl}\mathbf{u} - \mathbf{J}) \cdot \mathbf{v} = 0\,.$$

The integral on the boundary will be omitted in the variational formulation, in order to impose in a suitable weak sense the condition $\mathbf{n} \times (\operatorname{curl}\mathbf{u} - \mathbf{J}) = \mathbf{0}$ on $\partial\Omega$.

Multiplying the second equation in (2.7) by φ, integrating in Ω and integrating by parts we find

$$\int_\Omega f\varphi = \int_\Omega \operatorname{div}(\boldsymbol{\mu}\mathbf{u})\varphi = -\int_\Omega \boldsymbol{\mu}\mathbf{u} \cdot \operatorname{grad}\varphi + \int_{\partial\Omega} \boldsymbol{\mu}\mathbf{u} \cdot \mathbf{n}\varphi\,,$$

hence

$$\int_\Omega \boldsymbol{\mu}\mathbf{u} \cdot \operatorname{grad}\varphi = -\int_\Omega f\varphi + \int_{\partial\Omega} \boldsymbol{\mu}\mathbf{u} \cdot \mathbf{n}\varphi\,.$$

Then, introducing a Lagrange multiplier λ, we are led to consider the problem

$$\begin{aligned} &\int_\Omega \operatorname{curl}\mathbf{u} \cdot \operatorname{curl}\mathbf{v} + \int_\Omega \boldsymbol{\mu}\mathbf{v} \cdot \operatorname{grad}\lambda = \int_\Omega \mathbf{J} \cdot \operatorname{curl}\mathbf{v} \\ &\int_\Omega \boldsymbol{\mu}\mathbf{u} \cdot \operatorname{grad}\varphi = -\int_\Omega f\varphi + \int_{\partial\Omega} b\varphi\,. \end{aligned}$$

The variational spaces are

$$\begin{aligned} \mathcal{V} = \Big\{ &\mathbf{v} \in H(\operatorname{curl};\Omega) \mid \operatorname{curl}\mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega, \\ &\qquad \oint_{\sigma_n} \mathbf{v} \cdot d\mathbf{s} = 0 \text{ for each } n = 1,\dots,g \Big\} \\ \mathcal{R} = \Big\{ &\varphi \in H^1(\Omega) \mid \int_\Omega \varphi = 0 \Big\}\,, \end{aligned} \tag{2.10}$$

and the variational problem is

$$\begin{aligned}
&\text{find } \mathbf{u} \in \mathcal{V}, \lambda \in \mathcal{R} : \\
&\qquad \int_\Omega \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{v} + \int_\Omega \boldsymbol{\mu} \mathbf{v} \cdot \operatorname{grad} \lambda = \int_\Omega \mathbf{J} \cdot \operatorname{curl} \mathbf{v} \\
&\qquad \int_\Omega \boldsymbol{\mu} \mathbf{u} \cdot \operatorname{grad} \varphi = -\int_\Omega f\varphi + \int_{\partial\Omega} b\varphi \\
&\text{for each } \mathbf{v} \in \mathcal{V}, \varphi \in \mathcal{R} .
\end{aligned} \tag{2.11}$$

Let us select a basis $\{\boldsymbol{\rho}_m^\mu\}_{m=1}^g$ of the space of harmonic fields $\mathcal{H}_\mu(m)$ defined in (2.8) with the properties

$$\oint_{\sigma_n} \boldsymbol{\rho}_m^\mu \cdot d\mathbf{s} = \delta_{nm}$$

(see, e. g., Alonso Rodríguez and Valli [8, Section A4]; for $\boldsymbol{\mu} = \mathrm{Id}$ we simply write $\boldsymbol{\rho}_m$). Then, if $\mathbf{u}$ is a solution to problem (2.7) with $\beta_n = 0$, $n = 1, \ldots, g$, the vector field $\mathbf{u} + \sum_{n=1}^g \beta_n \boldsymbol{\rho}_n^\mu$ is a solution to problem (2.7) with assigned β_n.

A consequence of this remark is that a solution $\mathbf{u}$ of problem (2.7), if it exists, is unique. Taking in fact vanishing data, it follows from the first three equations that $\mathbf{u} \in \mathcal{H}_\mu(m)$, and thus it can be written as $\mathbf{u} = \sum_{n=1}^g u_n \boldsymbol{\rho}_n^\mu$. Then, for each $n = 1, \ldots, g$,

$$0 = \oint_{\sigma_n} \mathbf{u} \cdot d\mathbf{s} = \sum_{m=1}^g u_m \oint_{\sigma_n} \boldsymbol{\rho}_m^\mu \cdot d\mathbf{s} = u_n ,$$

and in conclusion $\mathbf{u} = \mathbf{0}$.

Theorem 2. *If* $(\mathbf{u}, \lambda)$ *is a solution to problem* (2.11)*, then* $\lambda = 0$ *and* $\mathbf{u}$ *is a solution to problem* (2.7) *for* $\beta_n = 0$, $n = 1, \ldots, g$.

Proof. For $\varphi \in \mathcal{R}$ it holds $\oint_{\sigma_n} \operatorname{grad} \varphi \cdot d\mathbf{s} = 0$ for each $n = 1, \ldots, g$, hence $\operatorname{grad} \varphi \in \mathcal{V}$ for each $\varphi \in \mathcal{R}$. Therefore, taking $\mathbf{v} = \operatorname{grad} \lambda$ in the first equation we find

$$\int_\Omega \boldsymbol{\mu} \operatorname{grad} \lambda \cdot \operatorname{grad} \lambda = 0 ,$$

hence $\operatorname{grad} \lambda = \mathbf{0}$ and $\lambda = \text{const}$ in Ω; since the elements in $\mathcal{R}$ have zero mean, it follows $\lambda = 0$.

Choosing $\mathbf{v} \in (C_0^\infty(\Omega))^3$ we find that $\operatorname{curl}(\operatorname{curl} \mathbf{u} - \mathbf{J}) = \mathbf{0}$ in Ω in the distributional sense. Moreover, integrating by parts we also find

$$\int_{\partial\Omega} (\operatorname{curl} \mathbf{u} - \mathbf{J}) \cdot \mathbf{n} \times \mathbf{v} = 0$$

for each $\mathbf{v} \in \mathcal{V}$. Since $(\operatorname{curl} \mathbf{u} - \mathbf{J})$ is curl-free, from (2.5) we know that it can be written as

$$\operatorname{curl} \mathbf{u} - \mathbf{J} = \operatorname{grad} \chi + \sum_{n=1}^g \zeta_n \boldsymbol{\rho}_n$$

for $\chi \in H^1(\Omega)$ and $\zeta_n \in \mathbb{R}$. Thus we have

$$0 = \int_{\partial\Omega} (\operatorname{curl} \mathbf{u} - \mathbf{J}) \cdot \mathbf{n} \times \mathbf{v} = \int_{\partial\Omega} \operatorname{grad} \chi \cdot \mathbf{n} \times \mathbf{v} + \sum_{n=1}^{g} \zeta_n \int_{\partial\Omega} \boldsymbol{\rho}_n \cdot \mathbf{n} \times \mathbf{v} .$$

In addition, we recall from Buffa [23], Hiptmair et al. [38] that the tangential trace of $\mathbf{v} \in \mathcal{V}$ can be written on $\partial\Omega$ as

$$\mathbf{n} \times \mathbf{v} = \mathbf{n} \times \operatorname{grad} \vartheta + \sum_{m=1}^{g} \eta_m \mathbf{n} \times \boldsymbol{\rho}'_m ,$$

where $\vartheta \in H^1(\Omega)$, $\eta_m \in \mathbb{R}$ and the vector fields $\boldsymbol{\rho}'_m$ satisfy the relations

$$\int_{\partial\Omega} \boldsymbol{\rho}_n \cdot \mathbf{n} \times \boldsymbol{\rho}'_m = \delta_{nm}$$

(see Hiptmair et al. [38], Alonso Rodríguez et al. [7, Lemmas 4 and 5]). By integration by parts on $\partial\Omega$, we find

$$\int_{\partial\Omega} \operatorname{grad} \chi \cdot \mathbf{n} \times \mathbf{v} = - \int_{\partial\Omega} \chi \operatorname{div}_\tau (\mathbf{n} \times \mathbf{v}) = 0 ,$$

as $\operatorname{div}_\tau(\mathbf{n} \times \mathbf{v}) = -\operatorname{curl} \mathbf{v} \cdot \mathbf{n}$ on $\partial\Omega$; similarly, $\int_{\partial\Omega} \boldsymbol{\rho}_n \cdot \mathbf{n} \times \operatorname{grad} \vartheta = 0$ for each $n = 1, \dots, g$. In conclusion, we have obtained

$$0 = \int_{\partial\Omega} (\operatorname{curl} \mathbf{u} - \mathbf{J}) \cdot \mathbf{n} \times \mathbf{v} = \sum_{n,m=1}^{g} \zeta_n \eta_m \int_{\partial\Omega} \boldsymbol{\rho}_n \cdot \mathbf{n} \times \boldsymbol{\rho}'_m = \sum_{n=1}^{g} \zeta_n \eta_n .$$

Since η_n are arbitrary, it follows $\zeta_n = 0$ for each $n = 1, \dots, g$, and consequently $\operatorname{curl} \mathbf{u} - \mathbf{J} = \operatorname{grad} \chi$ in Ω. On the other hand, from the assumptions on the data, $\operatorname{div}(\operatorname{curl} \mathbf{u} - \mathbf{J}) = 0$ in Ω and $(\operatorname{curl} \mathbf{u} - \mathbf{J}) \cdot \mathbf{n} = 0$ on $\partial\Omega$, hence $\operatorname{grad} \chi = \mathbf{0}$ in Ω.

Let us prove now that the second equation is indeed satisfied for each $\widehat{\varphi} \in H^1(\Omega)$. Let $\widehat{\varphi}_\Omega = \frac{1}{\operatorname{meas} \Omega} \int_\Omega \widehat{\varphi}$. Then $\varphi = (\widehat{\varphi} - \widehat{\varphi}_\Omega) \in \mathcal{R}$ and $\operatorname{grad} \widehat{\varphi} = \operatorname{grad} \varphi$. Moreover,

$$\begin{aligned} - \int_\Omega f \widehat{\varphi} + \int_{\partial\Omega} b \widehat{\varphi} &= - \int_\Omega f \varphi + \int_{\partial\Omega} b \varphi - \widehat{\varphi}_\Omega \Big(- \int_\Omega f + \int_{\partial\Omega} b \Big) \\ &= - \int_\Omega f \varphi + \int_{\partial\Omega} b \varphi , \end{aligned}$$

having used the compatibility conditions on the data f and b.

Hence the second equation is satisfied for each $\widehat{\varphi} \in H^1(\Omega)$, and taking $\widehat{\varphi} \in C_0^\infty(\Omega)$ it follows $\operatorname{div}(\boldsymbol{\mu} \mathbf{u}) = f$ in Ω in the distributional sense. Repeating the same procedure for $\widehat{\varphi} \in H^1(\Omega)$, integration by parts gives $\boldsymbol{\mu} \mathbf{u} \cdot \mathbf{n} = b$ on $\partial\Omega$. $\square$

As in the previous section, the existence of a solution to problem (2.7) is therefore reduced to the proof of the existence of a solution to a variational saddle-point problem, in this case problem (2.11). Applying the general theory reported, e. g., in Boffi et al. [19, Section 4.2], we prove that problem (2.11) has a unique solution. In fact, the following results hold true.

Proposition 3. *The bilinear form* $a(\mathbf{w},\mathbf{v}) = \int_\Omega \operatorname{curl}\mathbf{w}\cdot\operatorname{curl}\mathbf{v}$ *is coercive in the space* $\mathcal{D}_0\times\mathcal{D}_0$, *where*

$$\mathcal{D}_0 = \left\{\mathbf{v}\in\mathcal{V} \mid \int_\Omega \boldsymbol{\mu}\mathbf{v}\cdot\operatorname{grad}\varphi = 0 \textit{ for all } \varphi\in\mathcal{R}\right\}.$$

Proof. Indeed, we already know that, if $\mathbf{v}\in\mathcal{D}_0$, then it holds $\int_\Omega \boldsymbol{\mu}\mathbf{v}\cdot\operatorname{grad}\varphi = 0$ for all $\varphi\in H^1(\Omega)$. Therefore, by integration by parts we deduce at once that $\operatorname{div}(\boldsymbol{\mu}\mathbf{v}) = 0$ in Ω and $\boldsymbol{\mu}\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\Omega$. Coercivity follows from the Friedrichs inequality

$$\|\mathbf{v}\|_{L^2(\Omega)} \le C\big(\|\operatorname{curl}\mathbf{v}\|_{L^2(\Omega)} + \|\operatorname{div}(\boldsymbol{\mu}\mathbf{v})\|_{L^2(\Omega)}\big).$$

This inequality is valid for a vector field $\mathbf{v}$ belonging to $H(\operatorname{curl};\Omega)$, with $\operatorname{div}(\boldsymbol{\mu}\mathbf{v})\in L^2(\Omega)$, $\boldsymbol{\mu}\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\Omega$, and satisfying $\operatorname{curl}\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\Omega$ and $\oint_{\sigma_n}\mathbf{v}\cdot d\mathbf{s} = 0$ for each $n = 1,\dots,g$. This result can be shown by adapting in a straightforward way the proof presented, e. g., in Fernandes and Gilardi [32] (see also Alonso Rodríguez et al. [7, Lemma 9]), using the fact that the space

$$\{\mathbf{v}\in H(\operatorname{curl};\Omega) \mid \operatorname{div}(\boldsymbol{\mu}\mathbf{v})\in (L^2(\Omega))^3, \boldsymbol{\mu}\mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}$$

is compactly imbedded in $(L^2(\Omega))^3$ (see, e. g., Weber [64], Picard [54]). □

Proposition 4. *The bilinear form* $b(\mathbf{v},\varphi) = \int_\Omega \boldsymbol{\mu}\mathbf{v}\cdot\operatorname{grad}\varphi$ *satisfies an inf–sup condition, namely, there exists* $\beta > 0$ *such that for each* $\varphi\in\mathcal{R}$ *there exists* $\mathbf{v}\in\mathcal{V}$, $\mathbf{v}\neq\mathbf{0}$, *satisfying*

$$\int_\Omega \boldsymbol{\mu}\mathbf{v}\cdot\operatorname{grad}\varphi \ge \beta\|\mathbf{v}\|_{\mathcal{V}}\|\varphi\|_{\mathcal{R}}.$$

Proof. We can suppose $\operatorname{grad}\varphi\neq\mathbf{0}$. The thesis follows by choosing $\mathbf{v} = \operatorname{grad}\varphi$, as $\operatorname{curl}\mathbf{v} = \mathbf{0}$ in Ω and the Poincaré inequality

$$\|\varphi\|_{L^2(\Omega)} \le C\|\operatorname{grad}\varphi\|_{L^2(\Omega)}$$

is valid for $\varphi\in\mathcal{R}$ (see, e. g., Dautray and Lions [31, p. 127]). □

In conclusion, we have proved that the saddle-point problem (2.11) has a unique solution, and thus the same is true for problem (2.7).

Remark 1. The same existence result can be proved for the problem

$$\begin{cases} \operatorname{curl}\mathbf{u} = \mathbf{J} & \text{in } \Omega \\ \operatorname{div}(\boldsymbol{\mu}\mathbf{u}) = f & \text{in } \Omega \\ \boldsymbol{\mu}\mathbf{u}\cdot\mathbf{n} = b & \text{on } \partial\Omega \\ \int_\Omega \boldsymbol{\mu}\mathbf{u}\cdot\boldsymbol{\rho}_n^\mu = \beta_n & \text{for each } n = 1,\dots,g\,, \end{cases} \tag{2.12}$$

where the field $\mathbf{J}$ is only required to satisfy the necessary compatibility conditions $\operatorname{div}\mathbf{J} = 0$ in Ω and $\int_{(\partial\Omega)_r} \mathbf{J}\cdot\mathbf{n} = 0$ for each $r = 1,\dots,p$ (namely, the more restrictive assumption $\mathbf{J}\cdot\mathbf{n} = 0$ on $\partial\Omega$ has been dropped).

In the variational formulation, one has only to replace the space $\mathcal{V}$ by

$$\mathcal{V}_\sharp = \left\{\mathbf{v}\in H(\operatorname{curl};\Omega) \mid \int_\Omega \boldsymbol{\mu}\mathbf{v}\cdot\boldsymbol{\rho}_n^\mu = 0 \text{ for all } n = 1,\dots,g\right\},$$

keeping the other space $\mathcal{R}$ (that still satisfies $\operatorname{grad}\mathcal{R}\subset\mathcal{V}_\sharp$).

The proofs can be easily adapted: the only point that deserves some explanation is that now the variational solution $\mathbf{u}$ is shown to satisfy $\operatorname{curl}(\operatorname{curl}\mathbf{u}-\mathbf{J}) = \mathbf{0}$ in Ω, and moreover, $(\operatorname{curl}\mathbf{u}-\mathbf{J})\times\mathbf{n} = \mathbf{0}$ on $\partial\Omega$. This latter result follows from the fact that the first variational equation is indeed satisfied for all $\widehat{\mathbf{v}}\in H(\operatorname{curl};\Omega)$, and not only for $\mathbf{v}\in\mathcal{V}_\sharp$. In fact, let $P_\mu\widehat{\mathbf{v}}$ be the orthogonal projection of $\widehat{\mathbf{v}}$ on $\mathcal{H}_\mu(m)$ with respect to the scalar product $\int_\Omega \boldsymbol{\mu}\mathbf{v}\cdot\mathbf{w}$. Then $\mathbf{v} = (\widehat{\mathbf{v}}-P_\mu\widehat{\mathbf{v}})\in\mathcal{V}_\sharp$ and $\operatorname{curl}\mathbf{v} = \operatorname{curl}\widehat{\mathbf{v}}$, as the elements in $\mathcal{H}_\mu(m)$ are curl-free.

Thus $(\operatorname{curl}\mathbf{u}-\mathbf{J})\in\mathcal{H}(e)$, and the conditions $\int_{(\partial\Omega)_r}(\operatorname{curl}\mathbf{u}-\mathbf{J})\cdot\mathbf{n} = 0$ for each $r = 1,\dots,p$ permit to conclude that $\operatorname{curl}\mathbf{u}-\mathbf{J} = \operatorname{curl}\boldsymbol{\Phi}$ in Ω (see, e. g., Cantarella et al. [26]). Therefore,

$$\begin{aligned}\int_\Omega(\operatorname{curl}\mathbf{u}-\mathbf{J})\cdot(\operatorname{curl}\mathbf{u}-\mathbf{J}) &= \int_\Omega(\operatorname{curl}\mathbf{u}-\mathbf{J})\cdot\operatorname{curl}\boldsymbol{\Phi}\\ &= \int_\Omega \operatorname{curl}(\operatorname{curl}\mathbf{u}-\mathbf{J})\cdot\boldsymbol{\Phi} + \int_{\partial\Omega}(\operatorname{curl}\mathbf{u}-\mathbf{J})\cdot\mathbf{n}\times\boldsymbol{\Phi} = 0\,,\end{aligned}$$

namely, $\operatorname{curl}\mathbf{u} = \mathbf{J}$ in Ω.

3 A new variational formulation for problem (2.1)

The discussion at the beginning of Section 2.1 should have explained why our aim here is to find a different variational formulation for problem (2.1), a formulation that turns out to be more suitable for numerical approximation.

In our procedure, the first step is to find a vector field $\mathbf{u}^\star \in (L^2(\Omega))^3$ satisfying

$$\begin{cases} \operatorname{div} \mathbf{u}^\star = f & \text{in } \Omega \\ \displaystyle\int_{(\partial\Omega)_r} \mathbf{u}^\star \cdot \mathbf{n} = \alpha_r & \text{for each } r = 1, \dots, p\,. \end{cases} \tag{3.1}$$

Such a vector field does exist: for instance, one can think to take $\mathbf{J} = \mathbf{0}$ and $\mathbf{a} = \mathbf{0}$ in (2.1), or any choice of $\mathbf{J}$ and $\mathbf{a}$ satisfying the compatibility conditions (indeed, we will not assume in the sequel that $\operatorname{curl}(\boldsymbol{\eta}\mathbf{u}^\star) = \mathbf{0}$ or $(\boldsymbol{\eta}\mathbf{u}^\star) \times \mathbf{n} = \mathbf{0}$).

The vector field $\mathbf{W} = \mathbf{u} - \mathbf{u}^\star$ satisfies

$$\begin{cases} \operatorname{curl}(\boldsymbol{\eta}\mathbf{W}) = \mathbf{J} - \operatorname{curl}(\boldsymbol{\eta}\mathbf{u}^\star) & \text{in } \Omega \\ \operatorname{div} \mathbf{W} = 0 & \text{in } \Omega \\ (\boldsymbol{\eta}\mathbf{W}) \times \mathbf{n} = \mathbf{a} - (\boldsymbol{\eta}\mathbf{u}^\star) \times \mathbf{n} & \text{on } \partial\Omega \\ \displaystyle\int_{(\partial\Omega)_r} \mathbf{W} \cdot \mathbf{n} = 0 & \text{for each } r = 1, \dots, p\,, \end{cases} \tag{3.2}$$

and the second step of the procedure is finding a simple variational formulation of this problem.

Multiplying the first equation by a test function $\mathbf{v} \in H(\operatorname{curl}; \Omega)$, integrating in Ω and integrating by parts, we find:

$$\begin{aligned} \int_\Omega \mathbf{J} \cdot \mathbf{v} &= \int_\Omega \operatorname{curl}[\boldsymbol{\eta}(\mathbf{W} + \mathbf{u}^\star)] \cdot \mathbf{v} \\ &= \int_\Omega \boldsymbol{\eta}(\mathbf{W} + \mathbf{u}^\star) \cdot \operatorname{curl} \mathbf{v} - \int_{\partial\Omega} [\boldsymbol{\eta}(\mathbf{W} + \mathbf{u}^\star) \times \mathbf{n}] \cdot \mathbf{v} \\ &= \int_\Omega \boldsymbol{\eta}\mathbf{W} \cdot \operatorname{curl} \mathbf{v} + \int_\Omega \boldsymbol{\eta}\mathbf{u}^\star \cdot \operatorname{curl} \mathbf{v} - \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{v}\,. \end{aligned}$$

Let us introduce the space

$$\begin{aligned} \mathcal{W}_0 = \Big\{ \boldsymbol{\xi} \in H(\operatorname{div}; \Omega) \mid \operatorname{div} \boldsymbol{\xi} = 0 \text{ in } \Omega, \\ \int_{(\partial\Omega)_r} \boldsymbol{\xi} \cdot \mathbf{n} = 0 \text{ for each } r = 1, \dots, p \Big\}\,. \end{aligned} \tag{3.3}$$

Note that this space can be written as $\mathcal{W}_0 = \operatorname{curl}[H(\operatorname{curl}; \Omega)]$: in fact, the inclusion $\operatorname{curl}[H(\operatorname{curl}; \Omega)] \subset \mathcal{W}_0$ is obvious, while the inclusion $\mathcal{W}_0 \subset \operatorname{curl}[H(\operatorname{curl}; \Omega)]$ is a classical result concerning vector potentials (see, e. g., Cantarella et al. [26]). The vector field $\mathbf{W}$ is thus a solution to

$$\begin{aligned} \mathbf{W} \in \mathcal{W}_0 \ : \ \int_\Omega \boldsymbol{\eta}\mathbf{W} \cdot \operatorname{curl} \mathbf{v} = \int_\Omega \mathbf{J} \cdot \mathbf{v} - \int_\Omega \boldsymbol{\eta}\mathbf{u}^\star \cdot \operatorname{curl} \mathbf{v} \\ + \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{v} \quad \forall\, \mathbf{v} \in H(\operatorname{curl}; \Omega)\,. \end{aligned} \tag{3.4}$$

More precisely, $\mathbf{W}$ is the *unique* solution of that problem: in fact, assuming $\mathbf{J} = \mathbf{u}^* = \mathbf{a} = \mathbf{0}$, and taking $\mathbf{v}$ such that $\operatorname{curl} \mathbf{v} = \mathbf{W}$, it follows at this point $\int_\Omega \boldsymbol{\eta} \mathbf{W} \cdot \mathbf{W} = 0$, hence $\mathbf{W} = \mathbf{0}$.

Let us remark at once that, due to the identity $\mathcal{W}_0 = \operatorname{curl}[H(\operatorname{curl}; \Omega)]$, an edge finite element scheme related to this variational formulation leads to a well-structured stiffness matrix: the one of the curl curl operator (for a suitable set of the basis functions, see (6.9) and Proposition 6).

Remark 2. Let us consider the electrostatic problem in a domain with simple topological shape, namely, problem (2.1) with $\mathbf{J} = \mathbf{0}$ in Ω, $\mathbf{a} = \mathbf{0}$ on $\partial\Omega$, and $p = 0$. We have already seen in Section 2.1 that $\boldsymbol{\eta}\mathbf{u} = \operatorname{grad} \chi$ in Ω, where the potential χ satisfies $\operatorname{div}(\boldsymbol{\eta}^{-1} \operatorname{grad} \chi) = f$ in Ω and $\chi = 0$ on $\partial\Omega$. In this situation, the simplest way for determining the approximate solution is clearly to solve this Dirichlet boundary value problem by using nodal finite elements.

4 A new variational formulation for problem (2.7)

The variational formulation of the curl–div system with assigned normal component on the boundary that we present here is similar to the one we have proposed in Alonso Rodríguez et al. [4] for the problem of magnetostatics. However, we think it can be interesting for its particular simplicity, as here we will formulate the problem in the space $\mathcal{V}_0 = \operatorname{grad}[H^1(\Omega)]$, while in [4] it was set in the space $H(\operatorname{curl}^0; \Omega)$, which in the general topological case is more complicated to discretize.

Also in this case, we need a preliminary step: to find a vector field $\mathbf{u}^* \in (L^2(\Omega))^3$ satisfying

$$\begin{cases} \operatorname{curl} \mathbf{u}^* = \mathbf{J} & \text{in } \Omega \\ \displaystyle\oint_{\sigma_n} \mathbf{u}^* \cdot d\mathbf{s} = \beta_n & \text{for each } n = 1, \dots, g\,. \end{cases} \tag{4.1}$$

This vector field does exist: for instance, one can choose $f = 0$ and $b = 0$ in (2.7), or any choice of f and b satisfying the compatibility condition (indeed, we do not need to assume in the sequel that $\operatorname{div}(\boldsymbol{\mu}\mathbf{u}^*) = 0$ or $(\boldsymbol{\mu}\mathbf{u}^*) \cdot \mathbf{n} = 0$).

The vector field $\mathbf{V} = \mathbf{u} - \mathbf{u}^*$ satisfies

$$\begin{cases} \operatorname{curl} \mathbf{V} = \mathbf{0} & \text{in } \Omega \\ \operatorname{div}(\boldsymbol{\mu}\mathbf{V}) = f - \operatorname{div}(\boldsymbol{\mu}\mathbf{u}^*) & \text{in } \Omega \\ (\boldsymbol{\mu}\mathbf{V}) \cdot \mathbf{n} = b - (\boldsymbol{\mu}\mathbf{u}^*) \cdot \mathbf{n} & \text{on } \partial\Omega \\ \displaystyle\oint_{\sigma_n} \mathbf{V} \cdot d\mathbf{s} = 0 & \text{for each } n = 1, \dots, g\,, \end{cases} \tag{4.2}$$

and now we only have to find a variational formulation of this problem.

Multiplying the second equation by a test function $\phi \in H^1(\Omega)$, integrating in Ω and integrating by parts we find:

$$\begin{aligned}\int_\Omega f\phi &= \int_\Omega \operatorname{div}[\boldsymbol{\mu}(\mathbf{V}+\mathbf{u}^*)]\phi \\ &= -\int_\Omega \boldsymbol{\mu}(\mathbf{V}+\mathbf{u}^*)\cdot \operatorname{grad}\phi + \int_{\partial\Omega} [\boldsymbol{\mu}(\mathbf{V}+\mathbf{u}^*)\cdot \mathbf{n}]\phi \\ &= -\int_\Omega \boldsymbol{\mu}\mathbf{V}\cdot \operatorname{grad}\phi - \int_\Omega \boldsymbol{\mu}\mathbf{u}^*\cdot \operatorname{grad}\phi + \int_{\partial\Omega} b\phi \,.\end{aligned}$$

Let us introduce the space

$$\begin{aligned}\mathcal{V}_0 = \Big\{\mathbf{v} \in H(\mathrm{curl};\Omega) \mid \operatorname{curl}\mathbf{v} = \mathbf{0} \text{ in } \Omega, \\ \oint_{\sigma_n} \mathbf{v}\cdot d\mathbf{s} = 0 \text{ for each } n = 1,\dots,g\Big\}\,.\end{aligned} \tag{4.3}$$

Note that this space can be written as $\mathcal{V}_0 = \operatorname{grad}[H^1(\Omega)]$: in fact, the inclusion $\operatorname{grad}[H^1(\Omega)] \subset \mathcal{V}_0$ is obvious, while the inclusion $\mathcal{V}_0 \subset \operatorname{grad}[H^1(\Omega)]$ is a classical result concerning scalar potentials (see, e. g., Cantarella et al. [26]). The vector field $\mathbf{V}$ is thus a solution to

$$\begin{aligned}\mathbf{V} \in \mathcal{V}_0 \ : \ \int_\Omega \boldsymbol{\mu}\mathbf{V}\cdot \operatorname{grad}\phi = -\int_\Omega f\phi - \int_\Omega \boldsymbol{\mu}\mathbf{u}^*\cdot \operatorname{grad}\phi \\ + \int_{\partial\Omega} b\phi \quad \forall\, \phi \in H^1(\Omega)\,.\end{aligned} \tag{4.4}$$

It is easy to see that $\mathbf{V}$ is indeed the *unique* solution of that problem: in fact, assuming $f = b = 0$, $\mathbf{u}^* = \mathbf{0}$, and taking ϕ such that $\operatorname{grad}\phi = \mathbf{V}$, it follows at once $\int_\Omega \boldsymbol{\mu}\mathbf{V}\cdot\mathbf{V} = 0$, hence $\mathbf{V} = \mathbf{0}$.

Also in this case we remark that, due to the identity $\mathcal{V}_0 = \operatorname{grad}[H^1(\Omega)]$, a nodal finite element scheme related to this variational formulation leads to a very simple and nice stiffness matrix: the one of the Laplace operator $-\Delta$ (for all the basis functions except one, see (7.7)).

5 Finite element approximation: generalities

Without pretending to be exhaustive, in this section we give a general overview of the methods that have been proposed for the finite element numerical approximation of the curl–div problem (mainly for the magnetostatic case given by (2.7) with $f = 0$ and

$b = 0$); our aim is simply to show here the advantage of the finite element methods we are going to introduce.

The magnetostatic problem has been considered since a long time, though very often in a simple topological situation, as it is probably the "most frequently encountered field problem in electrical engineering design" (see Chari et al. [28]).

A formulation in terms of a vector potential $\mathbf{A}$ such that $\operatorname{curl} \mathbf{A} = \boldsymbol{\mu}\mathbf{u}$ is quite classical, and has been analyzed by Coulomb [30], Barton and Cendes [12], Preis et al. [55] (see also the more recent point of view involving mimetic finite differences presented in Brezzi and Buffa [22], Lipnikov et al. [43]). Since the unknown is a vector field, the computational cost is higher than that needed to solve problem (2.7), that, as we will see in (7.7), in our formulation is essentially a scalar problem. Moreover, the magnetic vector potential approach presents two additional disadvantages: firstly, the right-hand sides f and b must be vanishing, or, if this not the case, one has the additional step given by the identification of a scalar function Ψ such that $\operatorname{div}(\boldsymbol{\mu} \operatorname{grad} \Psi) = f$ in Ω and $\boldsymbol{\mu} \operatorname{grad} \Psi \cdot \mathbf{n} = b$ on $\partial\Omega$; secondly, the vector potential $\mathbf{A}$ needs a gauge condition, thus another scalar equation (and unknown) has to be introduced. The method we devise in Section 6 for solving problem (2.1) has two steps: the first one has the aim of simply reducing the problem to the search of a suitable magnetic vector potential, and the second step can be performed without introducing a differential gauge, so that the overall scheme is cheap and efficient.

The remark concerning the computational cost also holds for many methods formulated in terms of the field $\mathbf{u}$: let us mention the mixed methods proposed by Kikuchi [42], Kanayama et al. [41], the least-squares approaches by Chang and Gunzburger [27], Bensow and Larson [14], Bochev et al. [18], the negative-norm least-squares schemes by Bramble and Pasciak [21], the weak Galerkin formulations by Wang and Wang [63], and the even more expensive two field-based methods by Rikabi et al. [58], Perugia [49] and Alotto and Perugia [10].

The co-volume method proposed by Nicolaides and Wu [48] is based on a system of two orthogonal grids like the classical Voronoi–Delaunay mesh pair, and for this reason this approach is not completely general, as some restrictions on the primal mesh and on the topological properties of the computational domain are needed.

Finally, the methods based on a magnetic scalar "potential" (the so-called reduced scalar potential) require the preliminary determination of a source field $\mathbf{H}_e$. Doing this by means of the Biot–Savart formula is not cheap from the computational point of view, and sometimes it induces cancellation errors (see Simkin and Trowbridge [62], Balac and Caloz [11]). In Mayergoyz et al. [45], it was suggested how to avoid this drawback by introducing an additional scalar potential, thus proposing a more expensive scheme (a complete analysis of this more complex formulation is in Bermudez et al. [15]). The method we propose in Section 7 for solving problem (2.7) presents two steps: the first one leads to a problem where the unknown is essentially a magnetic scalar "potential," but this is done without using the Biot–Savart formula, and in the end it turns out to be cheap and reliable.

Our methods in Section 6 and Section 7 are related to the so-called tree–cotree gauge used for the numerical approximation of magnetostatic and eddy current problems (see, e. g., Albanese and Rubinacci [1, 2], Ren and Razek [57], Manges and Cendes [44]); it could be seen as a rigorous mathematical version of that approach.

Before going on, a few remarks are in order. The techniques based on a tree–cotree decomposition of the nodes and the edges of the mesh can have some drawbacks, both for the construction of scalar or vector potentials and for the determination of a finite element basis. In fact, the stability of the methods depends on the choice of the tree (see Hiptmair [36]), and a clear theoretical result concerning the best selection for numerical approximation is not known. In this paper, as well as in our previous experience (see Alonso Rodríguez et al. [4], Alonso Rodríguez et al. [5]), choosing a breadth-first spanning tree has shown to be suitable and has lead to efficient numerical schemes. However, there are no rigorous results on this subject, and a deeper analysis, that would be quite interesting, could be the topic of a future research.

Let us introduce now some notation. In the following sections, we assume that $\Omega \subset \mathbb{R}^3$ is a polyhedral bounded domain with Lipschitz boundary $\partial\Omega$. We consider a tetrahedral triangulation $\mathcal{T}_h = (V, E, F, T)$ of $\overline{\Omega}$, denoting by V the set of vertices, E the set of edges, F the set of faces and T the set of tetrahedra of $\mathcal{T}_h$.

We will use these spaces of finite elements (see Monk [46, Section 5.6, Section 5.5, Section 5.4 and Section 5.7] for a complete presentation): the space L_h of continuous piecewise-linear elements, with dimension n_v, the number of vertices in $\mathcal{T}_h$; the space N_h of Nédélec edge elements of degree 1, with dimension n_e, the number of edges in $\mathcal{T}_h$; the space RT_h of Raviart–Thomas elements of degree 1, with dimension n_f, the number of faces in $\mathcal{T}_h$; the space PC_h of piecewise-constant elements, with dimension n_t, the number of tetrahedra in $\mathcal{T}_h$.

The following inclusions are well known:

$$L_h \subset H^1(\Omega) \quad , \quad N_h \subset H(\mathrm{curl};\Omega) \quad , \quad \mathrm{RT}_h \subset H(\mathrm{div};\Omega) \quad \mathrm{PC}_h \subset L^2(\Omega)\,.$$

Moreover, $\operatorname{grad} L_h \subset N_h$, $\operatorname{curl} N_h \subset \mathrm{RT}_h$ and $\operatorname{div} \mathrm{RT}_h \subset \mathrm{PC}_h$. The basis of L_h is denoted by $\{\psi_{h,1}, \dots, \psi_{h,n_v}\}$, with $\psi_{h,i}(v_j) = \delta_{i,j}$ for $1 \le i,j \le n_v$; the basis of N_h is denoted by $\{\mathbf{w}_{h,1}, \dots, \mathbf{w}_{h,n_e}\}$, with $\int_{e_j} \mathbf{w}_{h,i} \cdot \boldsymbol{\tau} = \delta_{i,j}$ for $1 \le i,j \le n_e$; the basis of RT_h is denoted by $\{\mathbf{r}_{h,1}, \dots, \mathbf{r}_{h,n_f}\}$, with $\int_{f_m} \mathbf{r}_{h,l} \cdot \boldsymbol{\nu} = \delta_{l,m}$ for $1 \le l,m \le n_f$.

Fixing a total ordering $v_1, \dots, v_{n_v}$ of the elements of V, an orientation on the elements of E and F is induced: if the end points of e_j are v_a and v_b for some $a, b \in \{1, \dots, n_v\}$ with $a < b$, then the oriented edge e_j will be denoted by $[v_a, v_b]$, with unit tangent vector $\boldsymbol{\tau} = \frac{v_b - v_a}{|v_b - v_a|}$; if the face f_m has vertices v_a, v_b and v_c with $a < b < c$, the oriented face f_m will be denoted by $[v_a, v_b, v_c]$ and its unit normal vector $\boldsymbol{\nu} = \frac{(v_b - v_a)\times(v_c - v_a)}{|(v_b - v_a)\times(v_c - v_a)|}$ is obtained by the right-hand rule.

We have already introduced the set of closed curves $\{\sigma_n\}_{n=1}^{g}$. We recall here that indeed they can be constructed as 1-cycles in $\mathcal{T}_h$, therefore, they are suitable for being employed in finite element approximation (see Hiptmair and Ostrowski [39]; see also Alonso Rodríguez et al. [4]).

6 Finite element approximation of problem (3.4)

We are ready now for the presentation of our finite element approximation procedure of problem (2.1). It can be performed in two steps. The first one, that is quite cheap, is finding a finite element potential $\mathbf{u}_h^\star \in \mathrm{RT}_h$ such that

$$\begin{cases} \operatorname{div} \mathbf{u}_h^\star = f_h & \text{in } \Omega \\ \displaystyle\int_{(\partial\Omega)_r} \mathbf{u}_h^\star \cdot \mathbf{n} = \alpha_r & \text{for each } r = 1, \dots, p\,, \end{cases} \tag{6.1}$$

where $f_h \in \mathrm{PC}_h$ is the piecewise-constant interpolant $I_h^{\mathrm{PC}} f$ of f. This can be done by means of a simple and efficient algorithm as shown in Alonso Rodríguez and Valli [9].

The second step concerns the numerical approximation of problem (3.4). Here, the main issue is to determine a finite element subspace of $\mathcal{W}_0$, and a suitable finite element basis. The natural choice is clearly

$$\mathcal{W}_{0,h} = \Big\{ \boldsymbol{\xi}_h \in \mathrm{RT}_h \mid \operatorname{div} \boldsymbol{\xi}_h = 0 \text{ in } \Omega, \\ \int_{(\partial\Omega)_r} \boldsymbol{\xi}_h \cdot \mathbf{n} = 0 \text{ for each } r = 1, \dots, p \Big\}\,. \tag{6.2}$$

For the ease of notation, let us set $n_Q = n_e - (n_v - 1)$. As proved in Alonso Rodríguez et al. [6], the dimension of $\mathcal{W}_{0,h}$ is equal to $n_Q - g$, and a basis is given by the curls of suitable Nédélec elements belonging to N_h.

To make clear this point, following Alonso Rodríguez et al. [6], some notation are necessary. As shown in Hiptmair and Ostrowski [39] (see also Alonso Rodríguez et al. [4]), it is possible to construct a set of 1-cycles $\{\sigma_n\}_{n=1}^g$, representing a basis of the first homology group $\mathcal{H}_1(\overline{\Omega}, \mathbb{Z})$, as a formal sum of edges in $\mathcal{T}_h$ with integer coefficients. More precisely, let us consider the graph given by the vertices and the edges of $\mathcal{T}_h$ on $\partial\Omega$. The number of connected components of this graph coincides with the number of connected components of $\partial\Omega$. For each $r = 0, 1, \dots, p$, let $S_{\partial\Omega}^r = (V_{\partial\Omega}^r, M_{\partial\Omega}^r)$ be a spanning tree of the corresponding connected component of the graph. Then consider the graph (V, E), given by all the vertices and edges of $\mathcal{T}_h$, and a spanning tree $S = (V, M)$ of this graph such that $M_{\partial\Omega}^r \subset M$ for each $r = 0, 1, \dots, p$. Let us order the edges in such a way that the edge e_l belongs to the cotree of S for $l = 1, \dots, n_Q$ and the edge e_{n_Q+i} belongs to the tree S for $i = 1, \dots, n_v - 1$. In particular, denote by e_q, $q = 1, \dots, 2g$, the set of edges of $\partial\Omega$, constructed by Hiptmair and Ostrowski [39], that have the following properties: they all belong to the cotree, and each one of them "closes" a 1-cycle γ_q that is a representative of a basis of the first homology group $\mathcal{H}_1(\partial\Omega, \mathbb{Z})$ (whose rank is indeed equal to $2g$). With this notation, we recall that the 1-cycles σ_n can be expressed as the formal sum

$$\sigma_n = \sum_{q=1}^{2g} A_{n,q}\gamma_q = \sum_{q=1}^{2g} A_{n,q}e_q + \sum_{i=n_Q+1}^{n_e} a_{n,i}e_i \,, \tag{6.3}$$

for suitable and explicitly computable integers $A_{n,q}$.

The idea that leads to the construction of the basis of $\mathcal{W}_{0,h}$ is now the following: first, consider the set

$$\{\operatorname{curl} \mathbf{w}_{h,l}\}_{l=2g+1}^{n_Q} \,,$$

Then look for g functions $\mathbf{z}_{h,\lambda} \in \mathrm{RT}_h$, $\lambda = 1, \dots, g$, of the form

$$\mathbf{z}_{h,\lambda} = \sum_{\upsilon=1}^{2g} c_\upsilon^{(\lambda)} \operatorname{curl} \mathbf{w}_{h,\upsilon} \,,$$

where the linearly independent vectors $\mathbf{c}^{(\lambda)} \in \mathbb{R}^{2g}$ are chosen in such a way that

$$\oint_{\sigma_n} \left(\sum_{\upsilon=1}^{2g} c_\upsilon^{(\lambda)} \mathbf{w}_{h,\upsilon} \right) \cdot d\mathbf{s} = 0$$

for $n = 1, \dots, g$. This can be done since σ_n is formed by the "closing" edges e_q, $q = 1, \dots, 2g$, and by edges belonging to the spanning tree, so that

$$\oint_{\sigma_n} \left(\sum_{\upsilon=1}^{2g} c_\upsilon^{(\lambda)} \mathbf{w}_{h,\upsilon} \right) \cdot d\mathbf{s} = \sum_{q=1}^{2g} A_{n,q} \int_{e_q} \left(\sum_{\upsilon=1}^{2g} c_\upsilon^{(\lambda)} \mathbf{w}_{h,\upsilon} \right) \cdot \boldsymbol{\tau} = \sum_{q=1}^{2g} A_{n,q} c_q^{(\lambda)} \,,$$

and the matrix $A \in \mathbb{Z}^{g\times 2g}$ with entries $A_{n,q}$ has rank g (see Hiptmair and Ostrowski [39], Alonso Rodríguez et al. [4, Section 6]). Thus we only have to determine a basis $\mathbf{c}^{(\lambda)} \in \mathbb{R}^{2g}$ of the kernel of A, $\lambda = 1, \dots, g$. An easy way for determining these vectors $\mathbf{c}^{(\lambda)}$ is presented in Alonso Rodríguez et al. [6].

Proposition 5. *The vector fields*

$$\{\operatorname{curl} \mathbf{w}_{h,l}\}_{l=2g+1}^{n_Q} \cup \left\{ \operatorname{curl} \left(\sum_{\upsilon=1}^{2g} c_\upsilon^{(\lambda)} \mathbf{w}_{h,\upsilon} \right) \right\}_{\lambda=1}^{g} \subset \mathcal{W}_{0,h}$$

are linearly independent and in particular they are a basis of $\mathcal{W}_{0,h}$.

Proof. The proof that these vector fields are linearly independent is in Alonso Rodríguez et al. [6, Proposition 2]. The second statement is then straightforward, as their number is $n_Q - g$, the dimension of $\mathcal{W}_{0,h}$. □

Let us denote this basis by $\{\operatorname{curl}\boldsymbol{\omega}_{h,l}\}_{l=g+1}^{n_Q}$, with

$$\boldsymbol{\omega}_{h,l} = \begin{cases} \mathbf{w}_{h,l} & \text{for } l = 2g+1,\dots,n_Q \\ \sum_{\upsilon=1}^{2g} c_\upsilon^{(l-g)}\mathbf{w}_{h,\upsilon} & \text{for } l = g+1,\dots,2g\,. \end{cases} \tag{6.4}$$

Proposition 6. *The vector fields $\{\boldsymbol{\omega}_{h,l}\}_{l=g+1}^{n_Q}$ are linearly independent.*

Proof. Suppose we have $\sum_{l=g+1}^{n_Q}\theta_l\boldsymbol{\omega}_{h,l} = \mathbf{0}$ for some θ_l. This can be rewritten as

$$\begin{aligned}\mathbf{0} &= \sum_{l=2g+1}^{n_Q}\theta_l\mathbf{w}_{h,l} + \sum_{l=g+1}^{2g}\theta_l\Big(\sum_{\upsilon=1}^{2g}c_\upsilon^{(l-g)}\mathbf{w}_{h,\upsilon}\Big)\\ &= \sum_{l=2g+1}^{n_Q}\theta_l\mathbf{w}_{h,l} + \sum_{\upsilon=1}^{2g}\Big(\sum_{l=g+1}^{2g}\theta_l c_\upsilon^{(l-g)}\Big)\mathbf{w}_{h,\upsilon}\,,\end{aligned}$$

thus $\theta_l = 0$ for $l = 2g+1,\dots,n_Q$ and $\sum_{l=g+1}^{2g}\theta_l c_\upsilon^{(l-g)} = 0$ for $\upsilon = 1,\dots 2g$, as $\{\mathbf{w}_{h,l}\}_{l=1}^{n_Q}$ are linearly independent. Since the vectors $\mathbf{c}^{(l-g)} \in \mathbb{R}^{2g}$, $l = g+1,\dots,2g$, are linearly independent, we also obtain $\theta_l = 0$ for $l = g+1,\dots,2g$, and the result follows. □

We are now in a position to formulate the finite element approximation of (3.4), that reads as follows:

$$\begin{aligned}\mathbf{W}_h \in \mathcal{W}_{0,h} : \int_\Omega \boldsymbol{\eta}\mathbf{W}_h\cdot\operatorname{curl}\mathbf{v}_h &= \int_\Omega \mathbf{J}\cdot\mathbf{v}_h - \int_\Omega \boldsymbol{\eta}\mathbf{u}_h^\star\cdot\operatorname{curl}\mathbf{v}_h\\ &\quad + \int_{\partial\Omega}\mathbf{a}\cdot\mathbf{v}_h \quad \forall\,\mathbf{v}_h \in N_h^\star\,,\end{aligned} \tag{6.5}$$

where

$$N_h^\star = \operatorname{span}\{\boldsymbol{\omega}_{h,l}\}_{l=g+1}^{n_Q}\,. \tag{6.6}$$

The corresponding algebraic problem is a square linear system of dimension $n_Q - g$, and it is uniquely solvable. In fact, we note that $\mathcal{W}_{0,h} = \operatorname{curl}N_h^\star$, hence we can choose $\mathbf{v}_h^\star \in N_h^\star$ such that $\operatorname{curl}\mathbf{v}_h^\star = \mathbf{W}_h$; from (6.5) we find at once $\mathbf{W}_h = \mathbf{0}$, provided that $\mathbf{J} = \mathbf{u}_h^\star = \mathbf{a} = \mathbf{0}$.

The convergence of this finite element scheme is easily shown by standard arguments. For the ease of reading, let us present the proof.

Theorem 3. *Let $\mathbf{W} \in \mathcal{W}_0$ and $\mathbf{W}_h \in \mathcal{W}_{0,h}$ be the solutions of problem* (3.4) *and* (6.5), *respectively. Set $\mathbf{u} = \mathbf{W} + \mathbf{u}^\star$ and $\mathbf{u}_h = \mathbf{W}_h + \mathbf{u}_h^\star$, where $\mathbf{u}^\star \in H(\mathrm{div};\Omega)$ and $\mathbf{u}_h^\star \in \mathrm{RT}_h$ are solutions to problem* (3.1) *and* (6.1), *respectively. Assume that* $\mathbf{u}$ *is regular enough, so* *that the interpolant $I_h^{\mathrm{RT}}\mathbf{u}$ is defined. Then the following error estimate holds:*

$$\|\mathbf{u}-\mathbf{u}_h\|_{H(\mathrm{div};\Omega)} \le c_0\big(\|\mathbf{u}-I_h^{\mathrm{RT}}\mathbf{u}\|_{L^2(\Omega)} + \|f - I_h^{\mathrm{PC}} f\|_{L^2(\Omega)}\big)\,. \tag{6.7}$$

Proof. Since $N_h^\star \subset H(\mathrm{curl};\Omega)$, we can choose $\mathbf{v} = \mathbf{v}_h \in N_h^\star$ in (3.4). By subtracting (6.5) from (3.4), we end up with

$$\int_\Omega \boldsymbol{\eta}[(\mathbf{W}+\mathbf{u}^\star) - (\mathbf{W}_h + \mathbf{u}_h^\star)] \cdot \mathrm{curl}\,\mathbf{v}_h = 0 \qquad \forall\, \mathbf{v}_h \in N_h^\star\,,$$

namely,

$$\int_\Omega \boldsymbol{\eta}(\mathbf{u}-\mathbf{u}_h)\cdot \mathrm{curl}\,\mathbf{v}_h = 0 \qquad \forall\, \mathbf{v}_h \in N_h^\star\,. \tag{6.8}$$

Then, recalling that $\mathcal{W}_{0,h} = \mathrm{curl}\,N_h^\star$, so that $\mathbf{W}_h = \mathrm{curl}\,\mathbf{v}_h^\star$ for a suitable $\mathbf{v}_h^\star \in N_h^\star$, using (6.8) we find

$$\begin{aligned} c_1\|\mathbf{u}-\mathbf{u}_h\|^2_{L^2(\Omega)} &\le \int_\Omega \boldsymbol{\eta}(\mathbf{u}-\mathbf{u}_h)\cdot(\mathbf{u}-\mathbf{u}_h)\\ &= \int_\Omega \boldsymbol{\eta}(\mathbf{u}-\mathbf{u}_h)\cdot(\mathbf{u}-\mathbf{W}_h-\mathbf{u}_h^\star)\\ &= \int_\Omega \boldsymbol{\eta}(\mathbf{u}-\mathbf{u}_h)\cdot(\mathbf{u}-\mathrm{curl}\,\mathbf{v}_h^\star-\mathbf{u}_h^\star)\\ &= \int_\Omega \boldsymbol{\eta}(\mathbf{u}-\mathbf{u}_h)\cdot(\mathbf{u}-\mathrm{curl}\,\mathbf{v}_h-\mathbf{u}_h^\star)\\ &\le c_2\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}\|\mathbf{u}-\boldsymbol{\xi}_h-\mathbf{u}_h^\star\|_{L^2(\Omega)} \quad \forall\,\boldsymbol{\xi}_h\in\mathcal{W}_{0,h}\,. \end{aligned}$$

We can choose $\boldsymbol{\xi}_h = (I_h^{\mathrm{RT}}\mathbf{u} - \mathbf{u}_h^\star) \in \mathcal{W}_{0,h}$; in fact, $\mathrm{div}(I_h^{\mathrm{RT}}\mathbf{u}) = I_h^{\mathrm{PC}}(\mathrm{div}\,\mathbf{u}) = I_h^{\mathrm{PC}} f = f_h$ and $\int_{(\partial\Omega)_r} I_h^{\mathrm{RT}}\mathbf{u}\cdot\mathbf{n} = \int_{(\partial\Omega)_r}\mathbf{u}\cdot\mathbf{n} = \alpha_r$ for each $r = 1,\dots,p$. Then it follows at once $\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)} \le \frac{c_2}{c_1}\|\mathbf{u}-I_h^{\mathrm{RT}}\mathbf{u}\|_{L^2(\Omega)}$. Moreover, $\mathrm{div}(\mathbf{u}-\mathbf{u}_h) = f - f_h = f - I_h^{\mathrm{PC}} f$, and the thesis is proved. □

A sufficient condition for defining the interpolant of $\mathbf{u}$ is that $\mathbf{u} \in (H^{\frac{1}{2}+\delta}(\Omega))^3$, $\delta > 0$ (see Monk [46, Lemma 5.15]). This is satisfied if, e. g., $\boldsymbol{\eta}$ is a scalar Lipschitz function in $\overline{\Omega}$ and $\mathbf{a} \in (H^\gamma(\partial\Omega))^3$, $\gamma > 0$ (see Alonso and Valli [3]). Moreover, if $\mathbf{u} \in (H^1(\Omega))^3$ and $f \in H^1(\Omega)$ we have $\|\mathbf{u}-\mathbf{u}_h\|_{H(\mathrm{div};\Omega)} = O(h)$.

6.1 The algebraic problem

The solution $\mathbf{W}_h \in \mathcal{W}_{0,h}$ is given by $\mathbf{W}_h = \sum_{l=g+1}^{n_Q} W_l\,\mathrm{curl}\,\boldsymbol{\omega}_{h,l}$. Hence the finite dimensional problem (6.5) can be rewritten as

$$\sum_{l=g+1}^{n_Q} W_l \int_\Omega \boldsymbol{\eta} \operatorname{curl} \boldsymbol{\omega}_{h,l} \cdot \operatorname{curl} \boldsymbol{\omega}_{h,m} = \int_\Omega \mathbf{J} \cdot \boldsymbol{\omega}_{h,m} - \int_\Omega \boldsymbol{\eta} \mathbf{u}_h^\star \cdot \operatorname{curl} \boldsymbol{\omega}_{h,m} + \int_{\partial\Omega} \mathbf{a} \cdot \boldsymbol{\omega}_{h,m}\,, \tag{6.9}$$

for each $m = g+1, \dots, n_Q$.

Theorem 4. *The matrix* $\mathbf{K}^\star$ *with entries*

$$K^\star_{ml} = \int_\Omega \boldsymbol{\eta} \operatorname{curl} \boldsymbol{\omega}_{h,l} \cdot \operatorname{curl} \boldsymbol{\omega}_{h,m}$$

is symmetric and positive definite.

Proof. It is enough to recall that the vector fields $\{\operatorname{curl} \boldsymbol{\omega}_{h,l}\}_{l=g+1}^{n_Q}$ are linearly independent (see Proposition 5). More precisely, they are a basis of $\mathcal{W}_{0,h}$, hence $\mathbf{K}^\star$ is the mass matrix in $\mathcal{W}_{0,h}$ with weight $\boldsymbol{\eta}$. □

7 Finite element approximation of problem (4.4)

Similar to the previous case, also the finite element approximation of problem (2.7) involves two steps. The first one is finding a finite element potential $\mathbf{u}_h^* \in N_h$ such that

$$\begin{cases} \operatorname{curl} \mathbf{u}_h^* = \mathbf{J}_h & \text{in } \Omega \\ \oint_{\sigma_n} \mathbf{u}_h^* \cdot d\mathbf{s} = \beta_n & \text{for each } n = 1, \dots, g\,, \end{cases} \tag{7.1}$$

where $\mathbf{J}_h \in \mathrm{RT}_h$ is the Raviart–Thomas interpolant $I_h^{\mathrm{RT}}\mathbf{J}$ of $\mathbf{J}$ (we therefore assume that $\mathbf{J}$ is so regular that its interpolant $I_h^{\mathrm{RT}}\mathbf{J}$ is defined; for instance, as already recalled, it is enough to assume $\mathbf{J} \in (H^{\frac{1}{2}+\delta}(\Omega))^3$, $\delta > 0$: see Monk [46, Lemma 5.15]). An efficient algorithm for computing $\mathbf{u}_h^*$, based on a tree–cotree decomposition of the mesh, is described in Alonso Rodríguez and Valli [9].

The second step is related to the numerical approximation of problem (4.4). It is quite easy to find a finite element subspace of $\mathcal{V}_0$ and a suitable finite element basis. The natural choice is clearly

$$\mathcal{V}_{0,h} = \Big\{ \mathbf{v}_h \in N_h \mid \operatorname{curl} \mathbf{v}_h = \mathbf{0} \text{ in } \Omega, \oint_{\sigma_n} \mathbf{v}_h \cdot d\mathbf{s} = 0 \text{ for each } n = 1, \dots, g \Big\}\,, \tag{7.2}$$

which can be rewritten as $\mathcal{V}_{0,h} = \operatorname{grad} L_h$. Since the dimension of this space is $n_v - 1$, a finite element basis is determined by taking $\operatorname{grad} \psi_{h,i}$, $i = 1, \dots, n_v - 1$, $\psi_{h,i}$ being the basis functions of the finite element space L_h.

The finite element approximation of (4.4) is easily obtained:

$$\begin{aligned}\mathbf{V}_h \in \mathcal{V}_{0,h} \ : \quad \int_\Omega \boldsymbol{\mu}\mathbf{V}_h \cdot \operatorname{grad}\phi_h &= -\int_\Omega f\phi_h - \int_\Omega \boldsymbol{\mu}\mathbf{u}_h^* \cdot \operatorname{grad}\phi_h \\ &\quad + \int_{\partial\Omega} b\phi_h \quad \forall\, \phi_h \in L_h^* \quad ,\end{aligned} \tag{7.3}$$

where

$$L_h^* = \operatorname{span}\{\psi_{h,i}\}_{i=1}^{n_v-1} = \{\phi_h \in L_h \,|\, \phi_h(v_{n_v}) = 0\}\,. \tag{7.4}$$

The corresponding algebraic problem is a square linear system of dimension $n_v - 1$, and it is uniquely solvable. In fact, since $\mathcal{V}_{0,h} = \operatorname{grad} L_h^*$, we can choose $\phi_h^* \in L_h^*$ such that $\operatorname{grad}\phi_h^* = \mathbf{V}_h$; from (7.3) we find at once $\mathbf{V}_h = \mathbf{0}$, provided that $f = b = 0$, $\mathbf{u}_h^* = \mathbf{0}$.

The convergence of this finite element scheme is easily proved by following the arguments previously presented.

Theorem 5. *Let $\mathbf{V} \in \mathcal{V}_0$ and $\mathbf{V}_h \in \mathcal{V}_{0,h}$ be the solutions of problem* (4.4) *and* (7.3)*, respectively. Set $\mathbf{u} = \mathbf{V} + \mathbf{u}^*$ and $\mathbf{u}_h = \mathbf{V}_h + \mathbf{u}_h^*$, where $\mathbf{u}^* \in H(\mathrm{curl};\Omega)$ and $\mathbf{u}_h^* \in N_h$ are solutions to problem* (4.1) *and* (7.1)*, respectively. Assume that $\mathbf{u}$ and $\mathbf{J}$ are regular enough, so that the interpolants $I_h^N\mathbf{u}$ and $I_h^{\mathrm{RT}}\mathbf{J}$ are defined. Then the following error estimate holds:*

$$\|\mathbf{u} - \mathbf{u}_h\|_{H(\mathrm{curl};\Omega)} \le c_0\big(\|\mathbf{u} - I_h^N\mathbf{u}\|_{L^2(\Omega)} + \|\mathbf{J} - I_h^{\mathrm{RT}}\mathbf{J}\|_{L^2(\Omega)}\big)\,. \tag{7.5}$$

Proof. Since $L_h^* \subset H^1(\Omega)$, we can choose $\phi = \phi_h \in L_h^*$ in (4.4). By subtracting (7.3) from (4.4), we end up with

$$\int_\Omega \boldsymbol{\mu}[(\mathbf{V} + \mathbf{u}^*) - (\mathbf{V}_h + \mathbf{u}_h^*)] \cdot \operatorname{grad}\phi_h = 0 \qquad \forall\, \phi_h \in L_h^*\,,$$

namely,

$$\int_\Omega \boldsymbol{\mu}(\mathbf{u} - \mathbf{u}_h) \cdot \operatorname{grad}\phi_h = 0 \qquad \forall\, \phi_h \in L_h^*\,. \tag{7.6}$$

Then, since $\mathcal{V}_{0,h} = \operatorname{grad} L_h^*$ and thus $\mathbf{V}_h = \operatorname{grad}\phi_h^*$ for a suitable $\phi_h^* \in L_h^*$, from (7.6) we find

$$\begin{aligned} c_1\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}^2 &\le \int_\Omega \boldsymbol{\mu}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{u}_h) \\ &= \int_\Omega \boldsymbol{\mu}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{V}_h - \mathbf{u}_h^*) \\ &= \int_\Omega \boldsymbol{\mu}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \operatorname{grad}\phi_h^* - \mathbf{u}_h^*) \end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \boldsymbol{\mu}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \operatorname{grad} \phi_h - \mathbf{u}_h^*) \\
&\leq c_2 \|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \|\mathbf{u} - \mathbf{v}_h - \mathbf{u}_h^*\|_{L^2(\Omega)} \quad \forall\, \mathbf{v}_h \in \mathcal{V}_{0,h} \,.
\end{aligned}
$$

We can choose $\mathbf{v}_h = (I_h^N \mathbf{u} - \mathbf{u}_h^*) \in \mathcal{V}_{0,h}$; in fact, $\operatorname{curl}(I_h^N \mathbf{u}) = I_h^{\mathrm{RT}}(\operatorname{curl} \mathbf{u}) = I_h^{\mathrm{RT}} \mathbf{J} = \mathbf{J}_h$ and $\oint_{\sigma_n} I_h^N \mathbf{u} \cdot d\mathbf{s} = \oint_{\sigma_n} \mathbf{u} \cdot d\mathbf{s} = \beta_n$ for each $n = 1, \ldots, g$. Then we find at once $\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \leq \frac{c_2}{c_1} \|\mathbf{u} - I_h^N \mathbf{u}\|_{L^2(\Omega)}$. Moreover, $\operatorname{curl}(\mathbf{u} - \mathbf{u}_h) = \mathbf{J} - \mathbf{J}_h = \mathbf{J} - I_h^{\mathrm{RT}} \mathbf{J}$, and the assertion follows. □

Sufficient conditions for defining the interpolants of $\mathbf{u}$ and $\mathbf{J} = \operatorname{curl} \mathbf{u}$ are that they both belong to $(H^{\frac{1}{2}+\delta}(\Omega))^3$, $\delta > 0$ (see Monk [46, Lemma 5.15 and Theorem 5.41]). This is for instance satisfied if $\boldsymbol{\mu}$ is a scalar Lipschitz function in $\overline{\Omega}$ and $b \in H^\gamma(\partial\Omega)$, $\gamma > 0$ (see Alonso and Valli [3]). Moreover, if $\mathbf{u} \in (H^1(\Omega))^3$ and $\mathbf{J} \in (H^1(\Omega))^3$ we have $\|\mathbf{u} - \mathbf{u}_h\|_{H(\operatorname{curl};\Omega)} = O(h)$.

7.1 The algebraic problem

The solution $\mathbf{V}_h \in \mathcal{V}_{0,h}$ is given by $\mathbf{V}_h = \sum_{i=1}^{n_v - 1} V_i \operatorname{grad} \psi_{h,i}$. Hence the finite dimensional problem (7.3) can be rewritten as

$$
\begin{aligned}
\sum_{i=1}^{n_v-1} V_i \int_\Omega \boldsymbol{\mu} \operatorname{grad} \psi_{h,i} \cdot \operatorname{grad} \psi_{h,j} = -\int_\Omega f \psi_{h,j} - \int_\Omega \boldsymbol{\mu} \mathbf{u}_h^* \cdot \operatorname{grad} \psi_{h,j} \\
+ \int_{\partial\Omega} b \psi_{h,j} \,,
\end{aligned}
\tag{7.7}
$$

for each $j = 1, \ldots, n_v - 1$.

We have at once the following.

Theorem 6. *The matrix* $\mathbf{K}^*$ *with entries*

$$
K_{ji}^* = \int_\Omega \boldsymbol{\eta} \operatorname{grad} \psi_{h,i} \cdot \operatorname{grad} \psi_{h,j}
$$

is symmetric and positive definite.

8 Numerical results

In this section, we present some numerical experiments with the aim of illustrating the effectiveness of the two proposed formulations and the behavior of their finite element approximation.

All the numerical computations have been performed by means of a MacBook Pro, with a processor 2.9 GHz Intel Core i7, 16 GB 2133 MHz RAM. We have used Netgen (see

[61]) to construct the meshes, and the package Pardiso (see [51, 50]) to solve the linear systems by means of a direct method (thus circumventing possible conditioning problems).

A peculiar point of our procedure is the choice of a suitable spanning tree of the graph given by the nodes and the edges of the mesh. As we have already noted, the stability of the method depends on this choice, in a way that is not completely clarified at the theoretical level. In our computations, we have systematically chosen a breadth-first spanning tree; this, together with the use of direct solvers for the algebraic systems, has always provided good numerical results. Breadth-first spanning trees have also shown to be an efficient choice in Alonso Rodríguez et al. [4], Alonso Rodríguez et al. [5].

We consider different test cases for each one of the two proposed formulations. For both formulations, the first test case is a problem with a known analytical solution. In this way, we can validate the code and illustrate the convergence properties of the finite element discretization. In the second test case, the data are very similar to those of the first test case, the difference only being a concentrated perturbation of the datum at the right-hand side of the divergence equation. We expect a solution that mainly differs from the solution of the first test case in a neighborhood of the support of the perturbation. For the problem in which the tangential component of the velocity is assigned, we present the computations for two different topological situations, in order to show that the approximation method is insensitive to the shape of the computational domain. In the third test case, the computational domain is similar to that of problem number 13 in the TEAM workshop (see [47]). The aim of this test case is to check the behaviour of the methods in a more realistic setting.

8.1 Numerical results for the problem with assigned tangential component on the boundary

Let us recall the system of equations that we consider:

$$\begin{cases} \operatorname{curl}(\boldsymbol{\eta}\mathbf{u}) = \mathbf{J} & \text{in } \Omega \\ \operatorname{div}\mathbf{u} = f & \text{in } \Omega \\ (\boldsymbol{\eta}\mathbf{u}) \times \mathbf{n} = \mathbf{a} & \text{on } \partial\Omega \\ \displaystyle\int_{(\partial\Omega)_r} \mathbf{u}\cdot\mathbf{n} = \alpha_r & \text{for each } r = 1,\dots,p\,. \end{cases}$$

For the sake of simplicity, in the sequel we will take $\boldsymbol{\eta}$ equal to the identity.

The data of the first test are such that the vector field $\mathbf{u} = [-x_1x_2, x_1x_2, 0]^T$ is the exact solution, hence in particular we have $\mathbf{J} = [0, 0, x_2 + x_1]^T$ and $f = x_1 - x_2$.

The computational domain Ω is a cylinder with a cavity. The cylinder has a vertical axis, height equal to $H = 100$, and the cross section given by the circle centered at the origin and of radius $R = 60$. The cavity is a similar cylinder but with height $h = 60$ and cross section of radius $r = 30$. The boundary of Ω has therefore two connected components. We include the Netgen file describing the geometry.

```
algebraic3d

solid cyl1 = cylinder(0,0,0;0,0,1; 60.)
         and plane( 0, 0, 50 ; 0, 0, 1 )
         and plane( 0, 0,-50 ; 0, 0, -1 );

solid cyl2 = cylinder(0,0,0;0,0,1; 30.)
         and plane( 0, 0, 30 ; 0, 0, 1 )
         and plane( 0, 0,-30 ; 0, 0, -1 );

solid cyl_in_cyl = cyl1 and not cyl2;

tlo cyl_in_cyl;
```

To check that the convergence rate is linear as expected, we solve the problem with five different meshes, described in Table 1.1.

Table 1.1: Description of the five meshes for the problem with assigned tangential component on the boundary (first test case, simply-connected domain).

	Elements	**Faces**	**Edges**	**Vertices**	**DOF**
Mesh 1	538	1246	886	180	707
Mesh 2	4304	9288	6048	1066	4983
Mesh 3	34 432	71 584	44 264	7114	37 151
Mesh 4	275 456	561 792	337 712	51 378	286 335
Mesh 5	2 203 648	4 450 816	2 636 256	389 090	2 247 167

The relative error is computed in the following way:

$$\mathrm{RE}(h) = \frac{\sqrt{\sum_{t\in T} |t|(u_{|t} - u_{h|t})^2}}{\sqrt{\sum_{t\in T} |t|(u_{|t})^2}}, \tag{8.1}$$

being T the set of tetrahedra of the mesh and $|t|$ the volume of the tetrahedron t.

The convergence rate is estimated comparing the error for two different meshes:

$$\text{Estimated Rate} = \frac{\log[\mathrm{RE}(h_1)/\mathrm{RE}(h_2)]}{\log(h_1/h_2)}. \tag{8.2}$$

Table 1.2: Relative error, mesh size, convergence rate and computational cost for the problem with assigned tangential component on the boundary (first test case, simply-connected domain).

	Relative error	h	Rate	CPU [ms]
Mesh 1	0.216	41.99		≈ 14
Mesh 2	0.131	31.68	1.657	≈ 62
Mesh 3	0.068	16.30	0.969	≈ 707
Mesh 4	0.034	8.16	0.998	≈ 11 161
Mesh 5	0.017	4.09	1.009	≈ 407 829

The results are reported in Table 1.2.

In the second test case, we consider a perturbed problem, namely, a problem with the same values of $\mathbf{J}$, $\mathbf{a}$, and α_r for each $r = 1, \ldots, p$, but with a new value for the divergence, given by $f_\epsilon = f + \epsilon$, where $\epsilon = 1000$ in the ball of radius 10 centered at the point $[45, 0, 0]^T$ and $\epsilon = 0$ otherwise. In Figure 1.1, one can compare the solutions of the first test case and of the second test case (namely, of the problem with a known analytical solution and of the perturbed problem). We are not showing the whole computational domain but only a cut along the plane $x_2 = 10$.

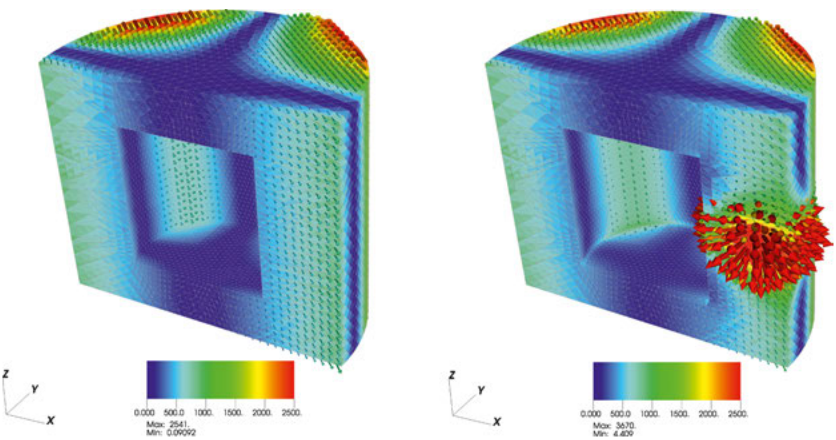

Figure 1.1: The solution $\mathbf{u}$ of the test problem in a simply-connected domain with a known analytical solution (left) and with a perturbed value for the divergence (right). In the figures, the domain is cut along the plane $x_2 = 10$.

In order to show the proposed method is also working for a domain with a more general topological shape, we have solved the problem for a toroidal domain with a concentric toroidal cavity. The connected components of the boundary are two and also the first Betti number of the computational domain is equal to two. More precisely, the computational domain Ω is the subtraction two domains: the larger one is the cylinder of height 2 with circular cross section of radius 1.2 minus the cylinder with the same height and cross section of radius 0.4; the cavity is the cylinder of height 1.6 with circular cross section of radius 1 minus the cylinder of the same height and cross section of radius 0.6. All the mentioned cylinders have their axis coincident with the x_3-axis.

For completeness, we include the Netgen file describing the geometry:

```
algebraic3d

solid cyl1a = cylinder(0,0,0;0,0,1; 1.2)
          and plane( 0, 0, 1 ; 0, 0, 1 )
          and plane( 0, 0,-1 ; 0, 0, -1 );

solid cyl1b = cylinder(0,0,0;0,0,1; 0.4)
          and plane( 0, 0, 1 ; 0, 0, 1 )
          and plane( 0, 0,-1 ; 0, 0, -1 );

solid cyl2a = cylinder(0,0,0;0,0,1; 1.)
          and plane( 0, 0, 0.8 ; 0, 0, 1 )
          and plane( 0, 0,-0.8 ; 0, 0, -1 );

solid cyl2b = cylinder(0,0,0;0,0,1; 0.6)
          and plane( 0, 0, 0.8 ; 0, 0, 1 )
          and plane( 0, 0,-0.8 ; 0, 0, -1 );

solid cyl1 = cyl1a and not cyl1b;
solid cyl2 = cyl2a and not cyl2b;

solid cyl_in_cyl = cyl1 and not cyl2;

tlo cyl_in_cyl;
```

The data of this test are such that the exact solution is $\mathbf{u} = [x_3x_1, x_3x_2, x_3^2]^T$, hence in particular we have $\mathbf{J} = [-x_2, x_1, 0]^T$ and $f = 4x_3$.

Again we have solved the problem with five different meshes, described in Table 1.3. The results are reported in Table 1.4.

In this case, the related perturbed problem has this form: we have kept the same values of $\mathbf{J}$, $\mathbf{a}$ and α_r for each $r = 1, \dots, p$, just modifying the datum at the right-hand side of the divergence equation, now given by $\widehat{f}_\epsilon = f + \epsilon$, with $\epsilon = -15$ in the ball centered at the point $[-0.8, 0, 0.9]^T$ and radius 0.1, $\epsilon = 15$ in the ball centered at the point $[0.8, 0, 0.9]^T$ and radius 0.1 and $\epsilon = 0$ otherwise.

Table 1.3: Description of the five meshes for the problem with assigned tangential component on the boundary (first test case, non-simply connected domain).

	Elements	Faces	Edges	Vertices	DOF
Mesh 1	1358	3169	2264	453	1810
Mesh 2	10 864	23 540	15 393	2717	12 675
Mesh 3	86 912	181 072	112 270	18 110	94 159
Mesh 4	695 296	1 419 584	854 668	130 380	724 287
Mesh 5	5 562 368	11 240 704	6 663 384	985 048	5 678 335

Table 1.4: Relative error, mesh size, convergence rate and computational cost for the problem with assigned tangential component on the boundary (first test case, non-simply connected domain).

	Relative error	h	Rate	CPU [ms]
Mesh 1	0.685	0.101		≈80
Mesh 2	0.498	0.060	1.634	≈170
Mesh 3	0.250	0.031	0.956	≈1931
Mesh 4	0.125	0.015	1.008	≈ 34 779
Mesh 5	0.063	0.008	1.012	≈2 481 110

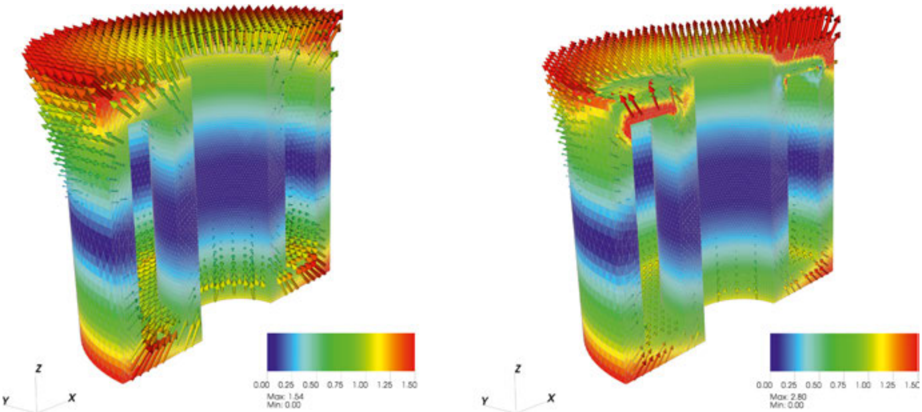

Figure 1.2: The solution **u** of the test problem in a non-simply connected domain with a known analytical solution (left) and with a perturbed value for the divergence (right). In the figures, only half of the domain is drawn.

In Figure 1.2, one can compare the solutions of the problem with a known analytical solution and of the perturbed problem. We are showing only half of the computational domain.

Let us note that we have not indeed constructed the basis described in Proposition 5, as we have used the set of generators $\{\mathrm{curl}\,\mathbf{w}_{h,l}\}_{l=1}^{n_Q}$, that in the case of a non-simply connected domain, are not linearly independent (the dimension of $\mathcal{W}_{0,h}$ is $n_Q - g$). In this case, the associated linear system is singular, but it is possible to find a solution in an efficient way (for instance, using the package Pardiso).

In the third test problem, the domain is the box $(-300\,300) \times (-300\,300) \times (-250\,250)$ (in mm), with three cavities corresponding to two channels and a plate (see Figure 1.3). The geometry is inspired to the problem number 13 in the TEAM workshop (see [47]). The thickness of the channels and the plate is $\delta = 3.2$ mm, the width $w = 50$ mm and the height $l = 126.4$ mm (so the plate is the hexaedron $(-1.6, 1.6) \times (-25, 25) \times (-63.2, 63.2)$). The distance between the plate and the channels is 0.5 mm, while the distance between the channels and the plane $x_2 = 0$ is 15 mm. The datum $\mathbf{J}$ is supported in a coil placed between the channels and the plate. More precisely, its support is the cylinder of height 100 mm with circular cross section centered at the origin and of radius 120 mm minus the analogous cylinder of the same height and cross section of radius 30 mm. Within the coil, we have $\mathbf{J} = [-x_2, x_1, 0]^T$, while $\mathbf{J}$ is zero outside the coil. All the other data, namely, f, $\mathbf{a}$ and α_r for $r = 1, \ldots, p$, are equal to zero.

The Netgen description of the geometry is the following.

```
algebraic3d

solid m1 = orthobrick(4.2,15,60;122.2,65,63.2);
solid m2 = orthobrick(4.2,15,-63.2;122.2,65,-60);
solid m3 = orthobrick(122.2,15,-63.2;125.4,65,63.2);

solid n1 = orthobrick(-122.2,-65,60;-4.2,-15,63.2);
solid n2 = orthobrick(-122.2,-65,-63.2;-4.2,-15,-60);
solid n3 = orthobrick(-125.4,-65,-63.2;-122.2,-15,63.2);

solid s = orthobrick(-1.6,-25,-63.2;1.6,25,63.2);

solid m    = m1 or m2 or m3;
solid n    = n1 or n2 or n3;
solid hole = m or n or s;

solid box = orthobrick(-300,-300,-250;300\,300\,250);

solid cyla = cylinder(0,0,0;0,0,1; 120.)
         and plane( 0, 0, 50 ; 0, 0, 1 )
         and plane( 0, 0,-50 ; 0, 0, -1 );

solid cylb = cylinder(0,0,0;0,0,1; 30.)
```

```
        and plane( 0, 0, 50 ; 0, 0, 1 )
        and plane( 0, 0,-50 ; 0, 0, -1 );

solid cyl = cyla and not cylb;

solid mat1 = box and not (hole or cyl);

tlo cyl;
tlo mat1;
```

In Figure 1.3, we show the computational domain and the datum **J**. A description of the used mesh is in Table 1.5. Figure 1.4 shows the solution **u** of the third test problem.

We also show in Figure 1.5 four level sets of the solution and in Figure 1.6 ten different level sets from $|\mathbf{u}| = 1000$ to $|\mathbf{u}| = 3000$.

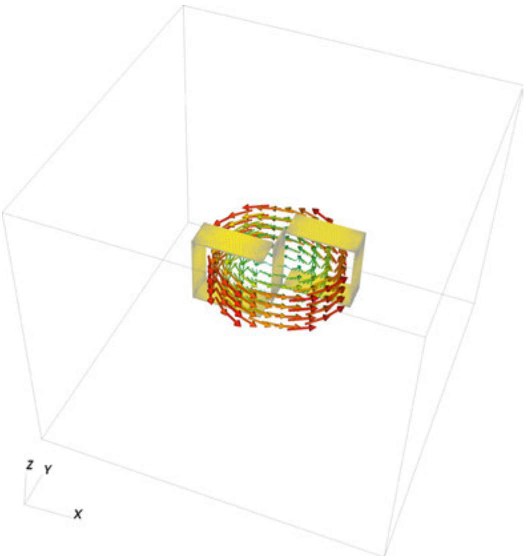

Figure 1.3: The computational domain and the datum of the third test problem.

Table 1.5: Description of the mesh for the third test problem.

Elements	Faces	Edges	Vertices	DOF
2 070 592	4 171 728	2 461 752	360 620	2 101 133

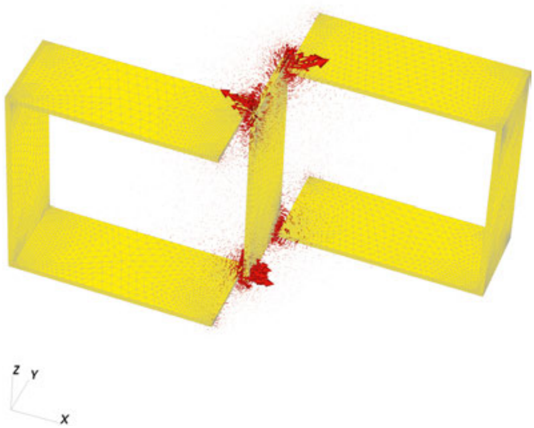

Figure 1.4: The solution $\mathbf{u}$ of the third test problem.

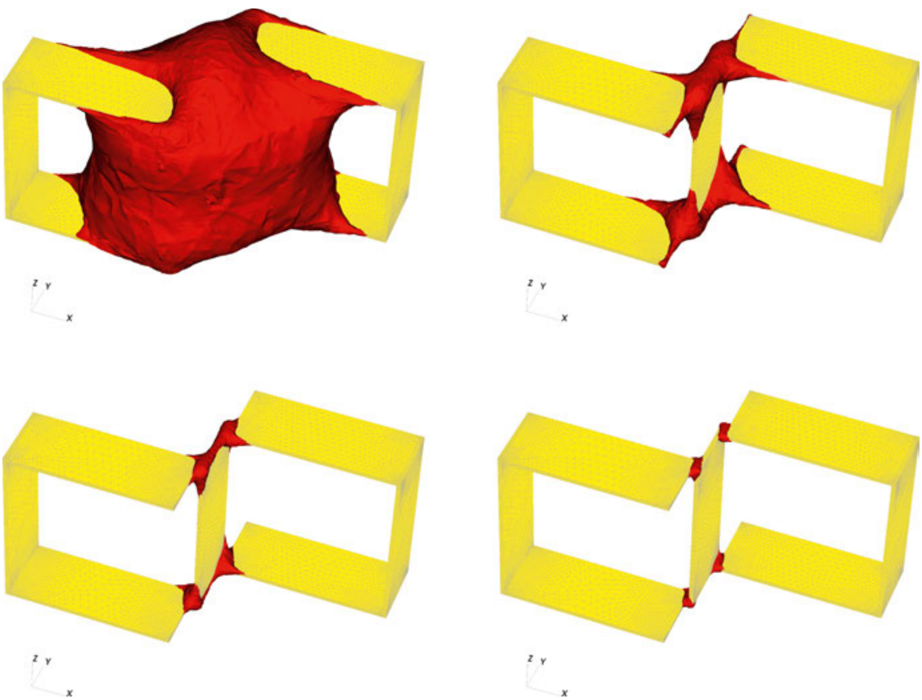

Figure 1.5: Four level sets of the solution $\mathbf{u}$ of the third test problem: $|\mathbf{u}| = 300$ (top-left), $|\mathbf{u}| = 1200$ (top-right), $|\mathbf{u}| = 2500$ (bottom-left), $|\mathbf{u}| = 5000$ (bottom-right).

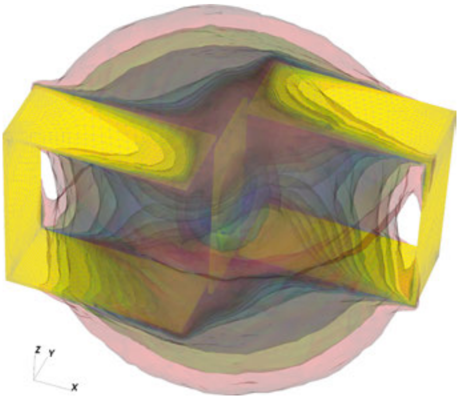

Figure 1.6: A single figure with ten level sets of the solution **u**, from $|\mathbf{u}| = 1000$ to $|\mathbf{u}| = 6000$.

8.2 Numerical results for the problem with assigned normal component on the boundary

We recall the system of equations:

$$\begin{cases} \operatorname{curl} \mathbf{u} = \mathbf{J} & \text{in } \Omega \\ \operatorname{div}(\boldsymbol{\mu}\mathbf{u}) = f & \text{in } \Omega \\ \boldsymbol{\mu}\mathbf{u} \cdot \mathbf{n} = b & \text{on } \partial\Omega \\ \oint\limits_{\sigma_n} \mathbf{u} \cdot d\mathbf{s} = \beta_n & \text{for each } n = 1, \dots, g\,, \end{cases}$$

and, for the sake of simplicity, in the sequel we will take $\boldsymbol{\mu}$ equal to the identity.

In the first and second test case, the computational domain is the toroidal domain with a concentric toroidal cavity that we have considered in the previous section. The data of the first test are again such that the exact solution is $\mathbf{u} = [x_3x_1, x_3x_2, x_3^2]^T$. In Table 1.6, we report the data of the meshes used for estimating the convergence rate, already presented in Table 1.3 but now including the number of degrees of freedom of this specific formulation.

Table 1.6: Description of the five meshes for the problem with assigned normal component on the boundary.

	Elements	Faces	Edges	Vertices	DOF
Mesh 1	1358	3169	2264	453	452
Mesh 2	10 864	23 540	15 393	2717	2716
Mesh 3	86 912	181 072	112 270	18 110	18 109
Mesh 4	695 296	1 419 584	854 668	130 380	130 379
Mesh 5	5 562 368	11 240 704	6 663 384	985 048	985 047