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Introduction

> We are just an advanced breed of monkeys on a minor planet of a very average star.
> But we can understand the Universe. That makes us something very special.
>
> |Stephen Hawking|

Modern mathematical cosmology was constructed between 1907 and 1915 by Albert Einstein, in which he used his gravity model to understand the dynamics of the universe. This model was built using his general theory of relativity (also known as general relativity), which was constructed in 1916 using Riemannian geometry. Although his model stated that the universe is expanding, observations did not support this prediction until 1922. In 1922, Alexander Friedman used the modified equations of general relativity to obtain the same result as Einstein of an expanding universe. Since there was no observational evidence of cosmic expansion, Einstein modified the field equations of general relativity by adding a term called the cosmological constant. The cosmological constant provides a repulsion to compensate the gravity attraction and to stop expansion, leading to a static model. In 1929, observational evidence changed the fate of the Einstein's general relativity model: Edwin Hubble's research on the red shift of distant galaxies confirmed the prediction that the universe is expanding. As a result, Einstein considered the cosmological constant as his biggest blunder. Following Hubble's discovery, cosmologists started to construct expanding universe models in the context of general relativity, in which the consequences of different assumptions about the distribution of matter in the universe are investigated. Therefore, the initially simple cosmological models have been replaced by more complex models taking into account nonlinearity and dissipation.

The modern cosmological models are based on the Friedman–Lemaitre family of models, which are built from the Robertson–Walker (1934) spatially homogeneous and isotropic geometries. Although there is observational evidence supporting these models on the largest scales, at smaller scales they do not provide a good description. The questions are [19]: On what scales is the geometry of the universe nearly Friedman–Robertson–Walker (FRW)? Why is it FRW? How did the universe come to have such an improbable geometry?

The answer to these questions can be found in inflation theory [22, 31]. According to this theory, the quantum fluctuations in the very early universe formed the seeds of inhomogeneities that could then grow. To examine these questions one needs to consider the family of cosmological solutions in the full state space of solutions, allowing one to see how realistic models are, related to each other and to higher symmetry models including, in particular, the Friedman–Lemaitre models.

Here we discuss general techniques for examining the FRW-type family of models and their generalizations, which could be useful in describing the universe at large scales. First of all, in FRW-type approaches the universe is characterized by cosmic-scale parameters, which are functions of the global time variable. From this point of

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view the isotropic universe is a dynamical system with one degree of freedom. But at smaller scales the anisotropy of the universe could be important, which is why one can consider a more general situation, with three different scale parameters depending on one global time. In this case, which is in the class of the so-called Bianchi family of universes, we have a dynamical system with three degrees of freedom and the FRW universe appears as the only symmetric reduction valid for the isotropic case. The general anisotropic case can be described by the Riccati equation and this equation admits transformation to the time dependent damped harmonic oscillator. This is why such models are called oscillatory models of the universe.

A time dependent metric also leads to the problem of the complexity of the physics in a time dependent background. Brandenberger [9] showed that inflationary metrics also imply time dependent frequency for the gravitational wave modes. This allows us to extend the canonical quantization method for nonunitary time evolution to include the quantization formalism for a parametric oscillator. [29] studied a harmonic oscillator with a time dependent frequency and a constant mass in an expanding universe. In the inflating case, the FRW metrics produce a damped harmonic oscillator equation for the partial waves of the field $h_{\mu\nu}$ [21]. [1] discussed the canonical quantization of nonunitary time evolution in an inflating universe. They considered gravitational wave modes in the FRW metrics in a de Sitter phase, then applied the quantization method to the damped oscillator mentioned above. Following this, the doubling of the $h_{\mu\nu}$ partial waves, which was called double universe, was shown by [3].

If damped oscillatory models are very important at small scales, the natural question is what happens upon quantization of these models, when one cannot neglect quantum fluctuations. One of the first approaches to quantize the damped harmonic oscillator is to start with the classical equation of motion, then find the Lagrangian and then the Hamiltonian, which will lead to the Hamilton equations of motion, and finally to quantize them by the canonical formalism method. This approach is called the (Bateman)–Caldirola–Kanai model, which derives quantum mechanics from a dissipative Hamiltonian. This Hamiltonian was actually proposed earlier by Bateman, but in a classical context [5]. This approach has the attractive quality of providing an exact solution, in essence because the classical equation of motion has an exact solution and formal quantization merely has the effect of converting the classical variables into operators. A second approach uses an interaction Hamiltonian and applies perturbation theory. One is a rather simple system (the undamped harmonic oscillator) that we construct, but an environment of the damped harmonic oscillator also exists. These, in fact, close the system, which creates a realistic or artificial embedding within a larger system that preserves energy. This way, Hamiltonians that describe a total, conserved energy can be obtained. An example of this line of thought is the so-called doubling the degrees of freedom approach. In fact, this idea can also be traced back to a Hamiltonian that was coined by Bateman, the so-called dual Hamiltonian [5]. The idea is that the damped oscillator is coupled to its time reversed image oscillator, which absorbs the energy lost so that the energy of the whole system is conserved or closed. In fact, since the phase space of the whole system describes the damped harmonic oscillator and its image, the degrees of freedom are effectively doubled. Another way of looking at this is that adding a time reversed oscillator restores the breaking of the time reversal symmetry. Difficulties arose during earlier attempts to elaborate this idea, such as time evolution leading out of the Hilbert space of states, but later a satisfactory quantization could be achieved within the framework of quantum field theory [8, 14]. The doubling of degrees of freedom approach has the conceptual disadvantage that the environment to which the damped harmonic oscillator is coupled is artificial. However, the word 'artificial' is only used for well-known systems. Since the structure of universe is not well defined, this approach has the advantage of showing the main form of dissipation as a system.

Apart from these approaches for a damped harmonic oscillator with a constant frequency and damping coefficient, the general form of a time dependent Hamiltonian that describes a classical forced oscillator with a time dependent damping coefficient and frequency were studied by [23]. This kind of system was also considered by other studies [27, 33]. Moreover, Kim demonstrated that canonical transformations in classical mechanics correspond to unitary transformations in quantum mechanics [43]. Additionally, Kim and Lee studied time dependent harmonic and anharmonic oscillators and found the exact Fock space and density operator for a time dependent anharmonic oscillator [28].

The goal of the first part of this book is to study the dissipative geometry of universe models in general relativity in the following contexts.

In Chapter 1, the fundamental definitions of general relativity, such as Christoffel symbols, Riemann tensor, Ricci tensor and Ricci scalar (Section 1.1) as well as the definitions of the Einstein field equations both in the presence and in the absence of matter, and the definitions of the cosmological constant, are given. We also discuss one of the most important tensors of general relativity: the energy momentum tensor, which tells us the energy-like aspects of the system. In Section 1.2, we discuss the universe as a dynamical system based on time dependent and scale factor dependent metrics. In this framework, the solution of the Einstein field equations and the derivation of the equation of state are given. These equations are particularly important since they can help us understand the universe as a dynamical system.

In Chapter 2, the construction of the universe models begins with the idea that the universe on large scales is isotropic and homogeneous. As a result, the Friedman universe models are considered, including four basic group of models: static, empty, non-empty with zero cosmological constant, and non-empty models with nonzero cosmological constant (Section 2.1). In Section 2.2, Milne's model and its fundamental properties are discussed, and Milne's model and the Friedman models are compared.

In Chapter 3, anisotropic and homogeneous universe models are investigated in terms of different density and pressure functions. In Section 3.1, the general solution of the field equations is obtained with respect to the anisotropic and homogeneous metrics. In the subsequent subsections, the particular solution of the field equations in the radiation dominated model is given.

In Chapter 4, the linearization of the Einstein equations are given, which produces gravitational waves on the Minkowski background and from the Fourier expansion of the field. The Fourier component of the field satisfying the harmonic oscillator equation with constant frequency (Section 4.1) is obtained. Following this, in Section 4.2, the linearization of the same equation on the de Sitter background produces damped harmonic oscillator systems with respect to the Bateman approach; the double universe models can be formed with respect to this approach. In Chapter 5, applying the factorization procedure to Friedman equations, bosonic and fermionic models of FRW universe are described. In Chapter 6, we consider an oscillatory universe models with time dependent gravitational and cosmological constants.

The second part of this book is devoted to study of the variational formulation of time dependent harmonic oscillators.

The Lagrangian and the Hamiltonian descriptions are crucial to understand the damped oscillator in quantum and classical theory. Hence, the background of these descriptions is given in Chapter 7. In Section 7.1, we give the definitions of the generalized coordinates and the velocities. In Section 7.2, a formulation for the study of a mechanical system, which is called least action principle, is discussed. In Section 7.3, the Hamiltonian and Hamilton's equations, the Poisson brackets and the properties of the Poisson brackets are discussed. In Section 8.1, the solution of damped harmonic oscillator is considered for three different cases: overdamping, critical damping and underdamping. An extension of analytical mechanics to include dissipation, is discussed in Section 8.2. In Section 8.3, we give the definition of the Bateman dual description, and using this approach we investigate the Lagrangian and the Hamiltonian functions for doublet damped oscillator systems. In Section 8.4, the time dependent Hamiltonian with time dependent mass satisfying the standard damped harmonic oscillator equation, which is called the Caldirola–Kanai Hamiltonian, is given. In Section 8.5, the Caldirola–Kanai Hamiltonian with a constant damping coefficient and frequency is quantized.

In Chapter 9, the two different formulations of damped oscillator with time dependent damping and frequency are related to the self-adjoint extension of the Sturm Liouville problem (Section 9.1) are given. In Section 9.2, the particular representations for the time dependent frequencies and the damping coefficient functions, related with different special functions are discussed. In Chapter 10, the Riccati representation of the special functions as oscillator-type problems is considered and some particular cases are given. In Chapter 11, quantization of damped oscillators with time dependent damping and frequency is considered. Exact quantum solution of this problem in Gaussian form is constructed in terms of the Riccati equation and the classical damped parametric oscillator, studies in previous chapters.

Part I: **Dissipative geometry and general relativity theory**

1 Pseudo-Riemannian geometry and general relativity

The two theories established by Albert Einstein – special relativity and general relativity in 1905 and 1915 respectively – are the modern theories of space and time. These theories changed our view of Newton's concepts of absolute time.

In both special and general relativity theories, the notions of separate vectors in space and time are abandoned, and the notions of spacetime and four-dimensional quantities are introduced. In this four-dimensional spacetime, the separation between two events is given by the spacetime interval, also called the metric, ds^2. In special relativity, the spacetime interval in four dimensions is given by the Minkowski (four-dimensional, flat) metric,

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \ , \tag{1.1}$$

where $\eta_{\mu\nu}$ is the metric tensor,

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{1.2}$$

and x^μ represent coordinates in the Minkowski space,

$$x^\mu = (ct, x, y, z) \ . \tag{1.3}$$

As can be seen, the Minkowski spacetime metric/interval is similar to the Euclidean space. For example, in Euclidean space, the infinitesimal spatial distance between two points is simply $ds^2 = dx^2 + dy^2 + dz^2$. The main difference is that while all the space coordinate contributions are positive, the time coordinate appears with negative sign in the Minkowski metric.

Additionally, events in general relativity occur in four-dimensional, curved spacetime rather than in flat Minkowski spacetime. Curved spacetime is defined by pseudo-Riemannian geometry, where the separation between two events like in the Riemannian spacetime is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \ , \tag{1.4}$$

in which $g_{\mu\nu}$ is called the Riemannian metric tensor. In contrast to the Minkowski metric $\eta_{\mu\nu}$, the pseudo-Riemannian metric is coordinate dependent, $g_{\mu\nu}(x)$. The metric components transform as a tensor

$$g_{\mu\nu} = \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial x^\beta}{\partial x^\nu} g_{\alpha\beta} \ , \tag{1.5}$$

where the partial derivatives $\frac{\partial x^\alpha}{\partial x^\mu}$ and $\frac{\partial x^\beta}{\partial x^\nu}$ form transformation matrices of the basis vectors.

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In the next section, the definitions and formulations of general relativity theory will be given in a general framework.

1.1 Curvature of spacetime and Einstein field equations

Suppose a general coordinate system with general basis e_α and a general four-vector V, then this four-vector can be represented as

$$V = V^\alpha e_\alpha \,, \tag{1.6}$$

in which V^α are the vector components. The first derivative of this vector in terms of general coordinates becomes

$$\frac{\partial V}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta} e_\alpha + V^\alpha \frac{\partial e_\alpha}{\partial x^\beta} . \tag{1.7}$$

Here, the partial derivatives of the basis vectors are

$$\frac{\partial e_\alpha}{\partial x^\beta} = \Gamma^\mu_{\alpha\beta} e_\mu \,, \tag{1.8}$$

where $\Gamma^\mu_{\alpha\beta}$ stands for the Christoffel symbols. If we substitute the derivative of the basis vectors (1.8) into the first derivative of the four-vector, we obtain

$$\frac{\partial V}{\partial x^\beta} = \underbrace{\left(\frac{\partial V^\alpha}{\partial x^\beta} + V^\mu \Gamma^\alpha_{\mu\beta} \right)}_{components} \underbrace{e_\alpha}_{basis} \,. \tag{1.9}$$

The components of the first derivative of the four-vector give the covariant derivative

$$V^\alpha_{;\beta} = V^\alpha{}_{,\beta} + V^\mu \Gamma^\alpha_{\mu\beta} \,. \tag{1.10}$$

Note that covariant derivative of a vector is a tensor. As is shown, the covariant derivative specifies a derivative of a four-vector along tangent vectors in curved spacetime. Here the connection, or the Christoffel symbol $\Gamma^\alpha_{\mu\beta}$, holds some important properties:
(1) it is symmetric: $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$
(2) it is torsion-free (no twist of a moving frame), indicating that the metric is covariantly constant $g_{\mu\nu;\beta} = 0$.

The torsion-free property of the Christoffel symbols is particularly important. Similarly, the partial derivative of the metric tensor in special relativity is zero. However, in the arbitrary coordinate system of the pseudo-Riemannian geometry, the partial derivative of the metric tensor will not give zero, since the metric components are coordinate dependent. In the latter case, connections can be thought of as inertial forces.

Computationally it is very difficult to obtain the Christoffel symbols. However, for symmetric connection compatible with the metric, it is much easier to calculate the