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Stochastically Forced Compressible Fluid Flows

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Notation

$\aleph_0$	Cardinality of the set of integers
c, C	Generic constants which differ from line to line
$a \lesssim b$	$a \leq cb$
B^c	Complement of a set B
$\mathcal{O}$	A domain – an open connected subset of $\mathbb{R}^N$
$\lvert\mathcal{O}\rvert$	Lebesgue measure of the domain $\mathcal{O}$
$\mathfrak{L}$	1-dimensional Lebesgue measure
$\mathfrak{L}^N$	N-dimensional Lebesgue measure
$\mathfrak{B}$	Borel σ-algebra
$\mathbb{T}^N$	N-dimensional flat torus $([-1,1]\lvert_{\{-1,1\}})^N$
$\mathbb{T}_L^{N+1}$	Space-time $(N+1)$-dimensional torus $([-L,L]\lvert_{\{-L,L\}}) \times \mathbb{T}^N$
$\mathbb{R}^{n\times N}$	Space of $n \times N$ matrices over $\mathbb{R}$
$\mathbb{A} : \mathbb{B}$	Scalar product $\sum_{ij} A_{ij}B_{ij}$ between two matrices $\mathbb{A}$, $\mathbb{B}$
$\mathbb{I}$	Identity matrix $(\delta_{ij})_{i,j=1}^N$ in $\mathbb{R}^{N\times N}$
B_b	Bounded Borel measurable functions
C	Continuous functions
C_c	Continuous functions with compact support
C_0	Continuous functions vanishing at infinity
C_b	Bounded continuous functions
C^α	α-Hölder continuous functions
C^k	k-times continuously differentiable functions
C_c^k	C^k-functions with compact support
$C^{k,\alpha}$	k-times continuously differentiable functions with α-Hölder continuous derivatives
C^∞	∞-times continuously differentiable functions
$C_c^\infty / \mathcal{D}$	C^∞-functions with compact support
$\mathcal{D}'$	Dual of C_c^∞
C_{div}^∞	C^∞-functions with vanishing divergence
$\mathcal{D}'_{\mathrm{div}}$	Dual of C_{div}^∞
L^p	Lebesgue space of p-integrable functions
L_{loc}^p	Lebesgue space of locally p-integrable functions
L_{div}^p	L^p-functions with vanishing divergence
p'	Dual exponent of p: $p' = p/(p-1)$
$W^{k,p}$	Sobolev functions with differentiability k and integrability p
$W_{\mathrm{div}}^{k,p}$	$W^{k,p}$-functions with vanishing divergence
$W^{-k,p}$	Dual space of $W^{k,p'}$
$(e_{\mathbf{m}})_{\mathbf{m}\in\mathbb{Z}^N}$	Trigonometric polynomials on $\mathbb{T}^N$
$\mathcal{M}_b$	Bounded signed measures
$\mathcal{M}_b^+$	Non-negative bounded measures

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$\mathcal{M}_R^+$	Non-negative Radon measures
Δ^{-1}	Solution operator to the Laplace equation
$\mathcal{P}_H\mathbf{v}$	Helmholtz projection $\mathbf{v} - \nabla\Delta^{-1}\operatorname{div}\mathbf{v}$ of a function $\mathbf{v} : \mathbb{R}^N \to \mathbb{R}^N$ ($\mathbb{T}^N \to \mathbb{T}^N$)
$\mathcal{Q}\mathbf{v}$	Gradient part $\nabla\Delta^{-1}\operatorname{div}\mathbf{v}$ of a function $\mathbf{v} : \mathbb{R}^N \to \mathbb{R}^N$ ($\mathbb{T}^N \to \mathbb{T}^N$)
X^*	Dual space of X
$\Vert\cdot\Vert_X$	Norm on X
$\langle\cdot,\cdot\rangle_X$	Inner product on X
$\langle\cdot,\cdot\rangle_{X^*,X}$	Duality pairing between X^* and X
$\rightharpoonup$	Weak convergence
$\overset{*}{\rightharpoonup}$	Weak-$*$ convergence
$\overset{d}{\to}$	Convergence in law
$L^p(0,T;X)$	Bochner space of X-valued p-integrable functions
$L^p_{\mathrm{loc}}(0,\infty;X)$	Bochner space of X-valued locally p-integrable functions
$C([0,T];X)$	Continuous functions with values in X
$C_{\mathrm{loc}}([0,\infty);X)$	Locally continuous functions with values in X
$C^\alpha([0,T];X)$	α-Hölder continuous functions with values in X
$C_w([0,T];X)$	Weakly continuous functions with values in X
$W^{k,p}(0,T;X)$	k-times weakly differentiable functions with values in X and integrability p
$(\Omega,\mathfrak{F},\mathbb{P})$	Probability space with sample space Ω, σ-algebra $\mathfrak{F}$, and probability measure $\mathbb{P}$
$(\mathfrak{F}_t)_{t\geq0}$	Filtration
$(\sigma_t[\mathbf{U}])_{t\geq0}$	Canonical filtration/history of a stochastic process/random distribution $\mathbf{U}$
$(\Omega,\mathfrak{F},(\mathfrak{F}_t)_{t\geq0},\mathbb{P})$	Filtered probability space with filtration $(\mathfrak{F}_t)_{t\geq0}$
$([0,1],\overline{\mathfrak{B}([0,1])},\mathfrak{L})$	Standard probability space
$\mathbb{E}$	Expectation
$\mathbb{E}[\cdot\vert\mathfrak{F}]$	Conditional expectation given $\mathfrak{F}$
$\mathcal{L}[\cdot]$	Law of a random variable
$\mathcal{L}_X[\cdot]$	Law of a random variable on the space X
$\overset{d}{\sim}$	Equality in law
$L^p_{\mathrm{prog}}(\Omega\times[0,T];X)$	L^p-integrable progressively measurable X-valued random variable
$L(\mathfrak{U},H)$	Continuous linear operators from $\mathfrak{U}\to H$
$L_2(\mathfrak{U},H)$	Hilbert–Schmidt operators from $\mathfrak{U}\to H$
$(e_k)_{k\in\mathbb{N}}$	Complete orthonormal system in $\mathfrak{U}$
$W=\sum_{k=1}^\infty e_kW_k$	Cylindrical Wiener process in $\mathfrak{U}$
$\langle\langle\mathbf{U}\rangle\rangle$	Quadratic variation of the stochastic process $\mathbf{U}$
$\langle\langle\mathbf{U},\mathbf{V}\rangle\rangle$	Cross variation of stochastic processes $\mathbf{U}$ and $\mathbf{V}$

Contents

Part I: **Preliminary results**

1 Elements of functional analysis

We will exclusively use functions $v = v(t,x)$ with the time $t \in I$ and the space variable $x \in \mathcal{O} \subset \mathbb{R}^N$, where I is an interval and $\mathcal{O}$ denotes a *domain* – an open connected subset of $\mathbb{R}^N$. Sometimes, it will be convenient to separate the time and space variables and consider $v = v(t,\cdot)$ as a mapping ranging in a suitable topological space X of functions depending on the x-variable. To avoid problems related to the presence of a kinematic boundary in the equation of fluid mechanics, we mostly focus on functions that are space periodic, meaning the spatial domain $\mathcal{O}$ is identified with the flat torus $\mathbb{T}^N$, given by

$$\mathbb{T}^N = \left([-1,1]|_{\{-1,1\}}\right)^N.$$

The length of period 2 is taken only for the sake of convenience. All results stated in this book have been obtained for a general torus given by

$$\Pi_{i=1}^N [a_i, b_i]|_{\{a_i,b_i\}}.$$

If not otherwise stated, all functions (or vector-valued functions) are real-valued.

1.1 Continuous functions, measures

For a topological space X, the symbol $C(X)$ denotes the space of continuous functions on X, $C_c(X)$ is the space of all continuous functions compactly supported in X, and $C_b(X)$ is the space of all bounded continuous functions on X.

If K is compact, $C(K)$ is a Banach space with the norm

$$\|v\|_{C(K)} = \sup_{y\in K} |v(y)|, \quad v \in C(K).$$

For $X \subset \mathbb{R}^N$ or $X \subset \mathbb{R}$ we simply write $\|\cdot\|_{C_x}$ and $\|\cdot\|_{C_t}$. Similarly, for functions $v : K \to Y$ ranging in a metric space Y with metric d_Y, we define a metric on $C(K;Y)$ as

$$d_{C(K;Y)}[v,w] = \sup_{y\in K} d_Y[v(y), w(y)], \quad v, w \in C(K;Y).$$

If there is no danger of confusion, we write $C(K)$ instead of $C(K;\mathbb{R}^M)$.

The following result is known as the *Arzelà–Ascoli theorem*; see Kelley [Kel55, Chapter 7, Theorem 17].

Theorem 1.1.1. *Let $K \subset \mathbb{R}^N$ be compact and Y a compact topological metric space endowed with a metric d_Y. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of functions in $C(K;Y)$ that is equi-*

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continuous, meaning that, for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$d_Y[v_n(y), v_n(z)] \leq \varepsilon \text{ provided } |y - z| < \delta \text{ independently of } n \in \mathbb{N}.$$

Then $(v_n)_{n\in\mathbb{N}}$ is precompact in $C(K;Y)$, that is, there exist a subsequence (not relabeled) and a function $v \in C(K;Y)$ such that

$$\sup_{y\in K} d_Y[v_n(y), v(y)] \to 0 \quad \text{as } n \to \infty.$$

Next we recall the *Stone–Weierstrass theorem*; see Cullen [Cul68].

Theorem 1.1.2. *Suppose K is a compact Hausdorff space and $\mathcal{A}$ is a subalgebra of $C(K;\mathbb{R})$ which contains a non-zero constant function. Then $\mathcal{A}$ is dense in $C(K;\mathbb{R})$ if and only if it separates points.*

Remark 1.1.3. A set of continuous functions $\mathcal{A}$ on K separates points if, for $x, y \in K$, $x \neq y$, there is $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. Note that a topological space is Hausdorff if, for any two points $x \neq y$, there are open sets U_x, U_y, $x \in U_x$, $y \in U_y$, $U_x \cap U_y = \emptyset$. In particular, any topological space in which $C(X;\mathbb{R})$ separates points is Hausdorff and the "if" part of Theorem 1.1.2 holds without the explicit requirement K to be Hausdorff.

A function vanishes at infinity if, for any $\varepsilon > 0$, there is a compact $K_\varepsilon \subset X$ such that $|f(x)| < \varepsilon$ for $x \notin K_\varepsilon$. The space of continuous functions vanishing at infinity is denoted as $C_0(X;\mathbb{R})$. There is an extension of the Stone–Weierstrass theorem to locally compact spaces; see de Branges [dB59].

Theorem 1.1.4. *Suppose K is a locally compact topological space and $\mathcal{A}$ is a subalgebra of $C_0(K;\mathbb{R})$ that separates points such that, for any $x \in X$, there is $f \in \mathcal{A}$ such that $f(x) \neq 0$. Then $\mathcal{A}$ is dense in $C_0(X;\mathbb{R})$.*

Let $\mathcal{M}^+(X)$ denote the set of all non-negative measures on X, meaning all non-negative σ-additive set-functions defined on a σ-field of measurable subsets of X. The following is the *Riesz representation theorem*; see Rudin [Rud87, Chapter 2, Theorem 2.14].

Theorem 1.1.5. *Let X be a locally compact Hausdorff metric space. Let f be a non-negative linear functional defined on the space $C_c(X)$.*

Then there exists a σ-algebra of measurable sets containing all Borel sets and a unique non-negative measure $\mu_f \in \mathcal{M}^+(X)$ such that

$$\langle f, g\rangle = \int_X g \, \mathrm{d}\mu_f \quad \text{for any } g \in C_c(X).$$

Moreover, the measure μ_f enjoys the following properties:
- *We have $\mu_f[K] < \infty$ for any compact $K \subset X$.*
- *We have*

$$\mu_f[E] = \sup\{\mu_f[K] \mid K \subset E,\ K \text{ compact}\}$$

for any open set $E \subset X$.
- *We have*

$$\mu_f[V] = \inf\{\mu(E) \mid V \subset E,\ E \text{ open}\}$$

for any Borel set V.
- *If E is μ_f-measurable, $\mu_f(E) = 0$, and $A \subset E$, then A is μ_f-measurable.*

1.2 Topological spaces

The topological spaces we deal with, besides admitting a σ-field of Borel sets, will satisfy certain *separation* properties. Possibly the weakest assumption in this sense is that a topological space X is completely regular (Tikhonov space), X is Hausdorff, and $C(X)$ separates points from closed sets: for any $x \in X$ and a closed set $F \subset X$ with $x \notin F$, there is $f \in C(X)$ such that $f(x) = 1$, $f|_F = 0$. The topology on a completely regular space is the coarsest topology making all functions from $C(X)$ or $C_b(X)$ continuous. Every subspace of a completely regular space is completely regular. In particular, if Y is completely regular and $X \hookrightarrow Y$ is a continuous injection, then X is completely regular. Any metric space is completely regular. In this book we deal almost exclusively with *topological vector spaces*, where the algebraic operations of addition and multiplication by a scalar are continuous. In particular, any Hausdorff topological vector space is Tikhonov. Topological vector spaces admit a uniform structure. Specifically, any neighborhood $\mathcal{U}(x)$ of a point x can be written as $x + \mathcal{U}$, where $\mathcal{U}$ is a neighborhood of zero. The uniform structure is necessary for a proper definition of some stochastic concepts like convergence in probability.

Most statements in the theory of stochastic PDEs use *Polish spaces*.

Definition 1.2.1. A topological space is *Polish* if the topology on X is separable and completely metrizable.

Later (see Definition 2.1.3) we introduce a larger class of *sub-Polish* spaces. These are, roughly speaking, topological spaces that admit a continuous injection into a Polish space.

The symbol $\mathcal{M}_R^+(X)$ denotes the set of non-negative Radon measures on X, meaning non-negative Borel measures μ such that

$$\mu[E] = \sup\{\mu[K] \mid K \subset E, K \text{ compact}\} \quad \text{for any open set } E \subset X.$$

Proposition 1.2.2. *If X is Polish, then every finite Borel measure is a Radon measure.*

For the proof see, e.g., Bogachev [Bog07].

1.3 Differentiable functions, distributions

The symbol

$$\partial_{y_i} g(y) := \frac{\partial g}{\partial_{y_i}}, \quad y = [y_1, \dots, y_N]$$

stands for the *partial derivative* of a function g defined on an open neighborhood of a point $y \in \mathbb{R}^N$.

The space of functions having k continuous derivatives are denoted C^k. If K is a compact set, then $C^k(K)$ is the space of functions from $C^k(\mathbb{R}^N)$ restricted to K. $C^{k,\nu}(\mathcal{O})$, $\nu \in (0,1)$, is the subspace of $C^k(\mathcal{O})$-functions having their kth derivatives ν-Hölder continuous in $\mathcal{O} \subset \mathbb{R}^N$. $C^{k,1}(\mathcal{O})$ is a subspace of $C^k(\mathcal{O})$ of functions whose kth derivatives are Lipschitz on $\mathcal{O}$. For a bounded domain $\mathcal{O}$, the spaces $C^k(\overline{\mathcal{O}})$ and $C^{k,\nu}(\overline{\mathcal{O}})$, $\nu \in (0,1]$, are Banach spaces with norms

$$\|u\|_{C^k_x} = \max_{|\alpha| \le k} \sup_{x \in \mathcal{O}} |\partial^\alpha u(x)|$$

and

$$\|u\|_{C^{k,\nu}_x} = \|u\|_{C^k_x} + \max_{|\alpha|=k} \sup_{(x,y)\in\mathcal{O}^2, x \neq y} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\nu},$$

where $\partial^\alpha u$ stands for the partial derivative $\partial^{\alpha_1}_{x_1} \dots \partial^{\alpha_N}_{x_N} u$ of order $|\alpha| = \sum_{i=1}^N \alpha_i$. The spaces $C^{k,\nu}(\overline{\mathcal{O}}; \mathbb{R}^M)$ are defined in a similar way. However, for notational simplicity the target space $\mathbb{R}^M$ will not be explicitly mentioned. Finally, we set $C^\infty = \bigcap_{k=0}^\infty C^k$.

The symbol $C^k_c(\mathcal{O})$, $k \in \{0, 1, \dots, \infty\}$, denotes the vector space of functions belonging to $C^k(\mathcal{O})$ and having compact support in $\mathcal{O}$. If $\mathcal{O} \subset \mathbb{R}^N$ is an open set, the symbol $\mathcal{D}(\mathcal{O})$ will be used alternatively for the space $C^\infty_c(\mathcal{O})$ endowed with the topology induced by the convergence

$$\varphi_n \to \varphi \quad \text{in } \mathcal{D}(\mathcal{O}),$$

if there is $K \subset \mathcal{O}$, a compact such that $\mathrm{supp}[\varphi_n] \subset K$ for any $k = 0, 1, \dots$ and

$$\varphi_n \to \varphi \quad \text{in } C^k(K). \tag{1.1}$$

The dual space $\mathcal{D}'(\mathcal{O})$ is the space of *distributions* on $\mathcal{O}$. Similarly, we define $\mathcal{D}'(\mathcal{O}; \mathbb{R}^M)$. Continuity of a linear form belonging to $\mathcal{D}'(\mathcal{O})$ is understood with respect to the convergence introduced in (1.1). We also consider the space of periodic distributions $\mathcal{D}'(\mathbb{T}^N)$ defined on the flat torus $\mathbb{T}^N$.

A differential operator ∂^α of order $|\alpha|$ can be identified with a distribution

$$\langle \partial^\alpha v, \varphi \rangle = (-1)^{|\alpha|} \langle v, \partial^\alpha \varphi \rangle = (-1)^{|\alpha|} \int_{\mathcal{O}} v \partial^\alpha \varphi \, \mathrm{d}y, \quad \varphi \in \mathcal{D}(\mathcal{O}),$$

where the most right identity makes sense whenever v is a locally integrable function.

1.4 Integrable functions

Let $\mathcal{O}$ be a measurable subset of $\mathbb{R}^N$ and X a separable Banach space with norm $\|\cdot\|_X$. The *Lebesgue space* $L^p(\mathcal{O}; X)$ is the space of Bochner measurable functions v ranging in the Banach space X such that the norm

$$\|v\|^p_{L^p_x X} = \int_{\mathcal{O}} \|v(y)\|^p_X \, \mathrm{d}y \text{ is finite}, \quad 1 \le p < \infty.$$

Similarly, $v \in L^\infty(\mathcal{O}; X)$ if v is Bochner measurable and

$$\|v\|_{L^\infty_x X} = \operatorname*{ess\,sup}_{y \in \mathcal{O}} \|v(y)\|_X < \infty.$$

The symbol $L^p_{\mathrm{loc}}(\mathcal{O}; X)$ denotes the vector space of locally L^p-integrable functions, meaning

$$v \in L^p_{\mathrm{loc}}(\mathcal{O}; X) \quad \text{if } v \in L^p(K; X) \text{ for any compact set } K \text{ in } \mathcal{O}.$$

We will omit the target space and write $L^p(\mathcal{O})$ instead of $L^p(\mathcal{O}; X)$ whenever no confusion arises.

The dual spaces to the L^p spaces are characterized in the following theorem; see Gajewski et al. [GGZ75, Chapter IV, Theorem 1.14, Remark 1.9], Edwards [Edw94], and Pedregal [Ped97, Chapter 6, Theorem 6.14].

Theorem 1.4.1. (1) *Let $\mathcal{O} \subset \mathbb{R}^N$ be a measurable set, X a Banach space that is reflexive and separable, and $1 \le p < \infty$. Then any continuous linear form $\xi \in [L^p(\mathcal{O}; X)]^*$ admits a unique representation $w_\xi \in L^{p'}(\mathcal{O}; X^*)$,*

$$\langle \xi, v \rangle_{L^{p'}(\mathcal{O}; X^*); L^p(\mathcal{O}; X)} = \int_{\mathcal{O}} \langle w_\xi(y), v(y) \rangle_{X^*; X} \, \mathrm{d}y \quad \textit{for all } v \in L^p(\mathcal{O}; X),$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Moreover, the norm on the dual space is given by

$$\|\xi\|_{[L^p_x X]^*} = \|w_\xi\|_{L^{p'}_x X^*}.$$

Accordingly, the spaces $L^p(\mathcal{O};X)$ *are reflexive for* $1 < p < \infty$ *as soon as* X *is reflexive and separable. Identifying* ξ *with* w_ξ*, we obtain the Riesz representation theorem*

$$[L^p(\mathcal{O};X)]^* = L^{p'}(\mathcal{O};X^*), \quad \|\xi\|_{[L^p_x X]^*} = \|\xi\|_{L^{p'}_x X^*}, \quad 1 \le p < \infty.$$

(2) *If the Banach space* X *is merely separable, we have*

$$[L^p(\mathcal{O};X)]^* = L^{p'}_{w^*}(\mathcal{O};X^*) \quad \textit{for } 1 \le p < \infty,$$

where

$$\begin{aligned} L^{p'}_{w^*}(\mathcal{O};X^*) := \{\xi : \mathcal{O} \to X^* \mid y \in \mathcal{O} \mapsto \langle \xi(y), v\rangle_{X^*;X} \textit{ measurable } \forall v \in X, \\ y \mapsto \|\xi(y)\|_{X^*} \in L^{p'}(\mathcal{O})\}. \end{aligned}$$

For L^p-spaces we also report *Hölder's inequality*

$$\|uv\|_{L^r_x} \le \|u\|_{L^p_x}\|v\|_{L^q_x}, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q},$$

for any $u \in L^p(\mathcal{O})$, $v \in L^q(\mathcal{O})$, $\mathcal{O} \subset \mathbb{R}^N$, and the *interpolation inequality*

$$\|v\|_{L^r_x} \le \|v\|^{\lambda}_{L^p_x}\|v\|^{(1-\lambda)}_{L^q_x}, \quad \frac{1}{r} = \frac{\lambda}{p} + \frac{1-\lambda}{q}, \quad p < r < q,\ \lambda \in (0,1),$$

for any $v \in L^p \cap L^q(\mathcal{O})$, $\mathcal{O} \subset \mathbb{R}^N$; see Adams [Ada75, Chapter 2].

Finally, we recall the celebrated and frequently used *Gronwall's lemma*; see Carroll [Car13].

Lemma 1.4.2. *Let* $a \in L^1(0,T)$, $a \ge 0$, $\beta \in L^1(0,T)$, $b_0 \in \mathbb{R}$, *and*

$$b(\tau) = b_0 + \int_0^\tau \beta(t)\,\mathrm{d}t$$

be given. Let $r \in L^\infty(0,T)$ *satisfy*

$$r(\tau) \le b(\tau) + \int_0^\tau a(t)r(t)\,\mathrm{d}t \quad \textit{for a.a. } \tau \in [0,T].$$

Then

$$r(\tau) \le b_0 \exp\left(\int_0^\tau a(t)\,\mathrm{d}t\right) + \int_0^\tau \beta(t)\exp\left(\int_t^\tau a(s)\,\mathrm{d}s\right)\mathrm{d}t$$

for a.a. $\tau \in [0,T]$.

1.5 Compactness and convergence of integrable functions

Let X be a Banach space, B_X the closed unit ball in X, and B_{X^*} the closed unit ball in the dual space X^*. Then we have:

(1) B_X is weakly compact only if X is reflexive. This is Kakutani's theorem; see Theorem III.6 in Brezis [Bre83].
(2) B_{X^*} is weakly-$*$-compact. This is the Banach–Alaoglu theorem; see Theorem III.15 in Brezis [Bre83].
(3) If X is separable, then B_{X^*} is sequentially weakly-$*$-compact; see Theorem III.25 in Brezis [Bre83].
(4) A non-empty subset of a Banach space X is weakly relatively compact only if it is sequentially weakly relatively compact. This is the Eberlein–Shmuliyan–Grothendieck theorem; see Paragraph 24 in Kothe [KK83].

In view of the above results we get:
- Any bounded sequence in $L^p(\mathcal{O})$, where $1 < p < \infty$ and $\mathcal{O} \subset \mathbb{R}^N$ is a domain, is relatively weakly compact.
- Any bounded sequence in $L^\infty(\mathcal{O})$, where $\mathcal{O} \subset \mathbb{R}^N$ is a domain, is relatively weakly-$*$ compact.

The situation for L^1, which is neither reflexive nor dual of a Banach space, is clarified in the following theorem; see Ekeland–Temam [ET99, Chapter 8, Theorem 1.3] and Pedregal [Ped97, Lemma 6.4].

Theorem 1.5.1. *Let $\mathcal{V} \subset L^1(\mathcal{O})$, where $\mathcal{O} \subset \mathbb{R}^N$ is a bounded measurable set.*

Then the following statements are equivalent:
- *any sequence $(v_n)_{n\in\mathbb{N}} \subset \mathcal{V}$ contains a subsequence weakly converging in $L^1(\mathcal{O})$;*
- *for any $\varepsilon > 0$, there exists $k > 0$ such that*
$$\int_{\{|v|\geq k\}} |v(y)|\,\mathrm{d}y \leq \varepsilon \quad \text{for all } v \in \mathcal{V};$$
- *for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $v \in \mathcal{V}$,*
$$\int_M |v(y)|\,\mathrm{d}y < \varepsilon,$$
for any measurable set $M \subset \mathcal{O}$, such that
$$|M| < \delta;$$
- *there exists a non-negative function $\Phi \in C([0,\infty))$*
$$\lim_{z\to\infty} \frac{\Phi(z)}{z} = \infty,$$
such that
$$\sup_{v\in\mathcal{V}} \int_{\mathcal{O}} \Phi(|v(y)|)\,\mathrm{d}y \leq c.$$

1.6 Sobolev spaces

There is a vast amount of literature devoted to the study of Sobolev spaces. We restrict ourselves to listing some standard results. The reader may consult the monographs by Adams [Ada75], Kufner et al. [KJF77], Maz'ya [Maz13], or Ziemer [Zie89] for more information.

The *Sobolev spaces* $W^{k,p}(\mathcal{O})$, $1 \le p \le \infty$, with k being a positive integer, are the spaces of functions having all distributional derivatives up to order k in $L^p(\mathcal{O})$. The norm in $W^{k,p}(\mathcal{O})$ is defined as

$$\|\mathbf{v}\|_{W_x^{k,p}} = \begin{cases} (\sum_{|\alpha| \le k} \|\partial^\alpha v\|_{L_x^p}^p)^{1/p} & \text{if } 1 \le p < \infty, \\ \max_{|\alpha| \le k}\{\|\partial^\alpha v\|_{L_x^\infty}\} & \text{if } p = \infty, \end{cases}$$

where the symbol ∂^α stands for any partial derivative of order $|\alpha|$.

If $1 \le p < \infty$, then $W^{k,p}(\mathcal{O})$ is separable and the space $C^k(\overline{\mathcal{O}})$ is its dense subspace (if $\mathcal{O}$ has a Lipschitz boundary).

The space $W^{1,\infty}(\mathcal{O})$, where $\mathcal{O}$ is a bounded Lipschitz domain, is isometrically isomorphic to the space $C^{0,1}(\overline{\mathcal{O}})$ of Lipschitz functions on $\overline{\mathcal{O}}$.

The symbol $W_0^{k,p}(\mathcal{O})$ denotes the completion of $C_c^\infty(\mathcal{O})$ with respect to the norm $\|\cdot\|_{W_x^{k,p}}$. In what follows, we identify $W^{0,p}(\mathcal{O}) = W_0^{0,p}(\mathcal{O})$ with $L^p(\mathcal{O})$.

The differentiability of a composition of a Sobolev function with a Lipschitz function is clarified in the following result; see Ziemer [Zie89, Section 2.1].

Lemma 1.6.1. *If $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function and $f \circ v \in L^p(\mathcal{O})$ for some $v \in W^{1,p}(\mathcal{O})$, then $f \circ v \in W^{1,p}(\mathcal{O})$ and*

$$\partial_{x_j}[f \circ v](x) = f'(v(x))\partial_{x_j} v(x) \quad \textit{for a.a. } x \in \mathcal{O}.$$

Duals to Sobolev spaces are characterized in the following theorem; see Adams [Ada75, Theorem 3.8] and Maz'ya [Maz13, Section 1.1.14].

Theorem 1.6.2. *Let $\mathcal{O} \subset \mathbb{R}^N$ be a domain and let $1 \le p < \infty$. Then the dual space $[W_0^{k,p}(\mathcal{O})]^*$ is a proper subspace of the space of distributions $\mathcal{D}'(\mathcal{O})$. Moreover, any linear form $f \in [W_0^{k,p}(\mathcal{O})]^*$ admits a representation*

$$\langle f, v\rangle_{[W_0^{k,p}]^*; W_0^{k,p}} = \sum_{|\alpha| \le k} \int_{\mathcal{O}} (-1)^{|\alpha|} w_\alpha \partial^\alpha v \, \mathrm{d}x,$$
$$\textit{where } w_\alpha \in L^{p'}(\mathcal{O}), \ \frac{1}{p} + \frac{1}{p'} = 1. \tag{1.2}$$

The norm of f in the dual space is given by

$$\|f\|_{[W_0^{k,p}(Q)]^*} = \begin{cases} \inf\{(\sum_{|\alpha| \le k} \|w_\alpha\|_{L_x^{p'}}^{p'})^{1/p'} \mid w_\alpha \textit{ satisfy } (1.2)\} & \textit{if } 1 < p < \infty, \\ \inf\{\max_{|\alpha| \le k}\{\|w_\alpha\|_{L_x^\infty}\} \mid w_\alpha \textit{ satisfy } (1.2)\} & \textit{if } p = 1. \end{cases}$$

The infimum is attained in both cases.

The dual space of the Sobolev space $W_0^{k,p}(\mathcal{O})$ is denoted as $W^{-k,p'}(\mathcal{O})$. The dual space of the Sobolev space $W^{k,p}(\mathcal{O})$ admits formally the same representation equation (1.2). However, it cannot be identified as a space of distributions on $\mathcal{O}$.

The important result is the Rellich–Kondrachov embedding theorem for Sobolev spaces; see Ziemer [Zie89, Theorem 2.5.1, Remark 2.5.2].

Theorem 1.6.3. *Let $\mathcal{O} \subset \mathbb{R}^N$ be a bounded Lipschitz domain.*

(i) *Then, if $kp < N$ and $p \geq 1$, the space $W^{k,p}(\mathcal{O})$ is continuously embedded in $L^q(\mathcal{O})$ for any*

$$1 \leq q \leq p^* = \frac{Np}{N-kp}.$$

Moreover, the embedding is compact if $k > 0$ and $q < p^*$.

(ii) *If $kp = N$, the space $W^{k,p}(\mathcal{O})$ is compactly embedded in $L^q(\mathcal{O})$ for any $q \in [1,\infty)$.*

(iii) *If $kp > N$, then $W^{k,p}(\mathcal{O})$ is continuously embedded in $C^{k-[N/p]-1,\nu}(\overline{\mathcal{O}})$, where $[\cdot]$ denotes the integer part and*

$$\nu = \begin{cases} [\frac{N}{p}] + 1 - \frac{N}{p} & \textit{if } \frac{N}{p} \notin Z, \\ \textit{arbitrary positive number in } (0,1) & \textit{if } \frac{N}{p} \in Z. \end{cases}$$

Moreover, the embedding is compact if $0 < \nu < [\frac{N}{p}] + 1 - \frac{N}{p}$.

As a straightforward corollary, we get the following dual result.

Theorem 1.6.4. *Let $\mathcal{O} \subset \mathbb{R}^N$ be a bounded domain. Let $k > 0$ and $q < \infty$ satisfy*

$$\begin{aligned} &q > \frac{p^*}{p^*-1}, \quad \textit{where } p^* = \frac{Np}{N-kp} \textit{ if } kp < N, \\ &q > 1 \quad \textit{for } kp = N, \end{aligned}$$

or

$$q \geq 1 \quad \textit{if } kp > N.$$

Then the space $L^q(\mathcal{O})$ is compactly embedded into the space $W^{-k,p'}(\mathcal{O})$, $1/p + 1/p' = 1$.

Remark 1.6.5. We have formulated this section on real-valued functions for the ease of presentation. However, all results extend in a straightforward manner to the case of vectorial functions ranging in $\mathbb{R}^M$ with $M \geq 2$.

1.7 Sobolev spaces of periodic functions

We focus on space periodic functions defined on the flat torus $\mathbb{T}^N$. Although all spaces we shall deal with are real, it is convenient to introduce the complex trigonometric polynomials

$$e_{\mathbf{m}}(x) = \exp(\mathrm{i}\mathbf{m} \cdot \pi x), \quad \mathbf{m} = [m_1, \dots, m_N] \in \mathbb{Z}^N.$$

The space $\mathcal{D}'(\mathbb{T}^N)$ is defined as the space of continuous linear forms on $\mathcal{D}(\mathbb{T}^N) = C_c^\infty(\mathbb{T}^N) = C^\infty(\mathbb{T}^N)$. The vector-valued form $\mathcal{D}'(\mathbb{T}^N; \mathbb{R}^M)$ may be defined analogously. Each distribution $v \in \mathcal{D}'(\mathbb{T}^N)$ can be identified with the infinite sequence of its *Fourier coefficients*, as described by

$$a_{\mathbf{m}}[v] = \frac{1}{(2\pi)^N} \langle v, \overline{e}_{\mathbf{m}} \rangle, \quad \text{formally } v \approx \sum_{\mathbf{m} \in \mathbb{Z}^N} a_{\mathbf{m}}[v] e_{\mathbf{m}},$$

where $\overline{e}_{\mathbf{m}}$ is the complex conjugate.

1.7.1 Hilbertian structure

The *Sobolev spaces* $W^{k,2}(\mathbb{T}^N)$ of periodic functions having derivatives up to the order k in $L^2(\mathbb{T}^N)$ can be characterized as $v \in \mathcal{D}'(\mathbb{T}^N)$ such that

$$\|v\|^2_{W^{k,2}(\mathbb{T}^N)} = \sum_{\mathbf{m} \in \mathbb{Z}^N} (|\mathbf{m}| + 1)^{2k} a^2_{\mathbf{m}}[v] < \infty. \tag{1.3}$$

The definition can be used even for a general exponent $k \in \mathbb{R}$. In particular, we have $(W^{k,2}(\mathbb{T}^N))^* = W^{-k,2}(\mathbb{T}^N)$ for any $k \in \mathbb{R}$. This identification corresponds to the Gelfand triple

$$W^{k,2}(\mathbb{T}^N) \hookrightarrow L^2(\mathbb{T}^N) \approx (L^2(\mathbb{T}^N))^* \hookrightarrow W^{-k,2}(\mathbb{T}^N), \quad k \geq 0,$$

where L^2 has been identified with its dual via Riesz isometry.

The spaces $W^{k,2}$ are separable Hilbert spaces endowed with the scalar product. We have

$$\langle v, w \rangle = \sum_{\mathbf{m} \in \mathbb{Z}^N} (|\mathbf{m}| + 1)^{2k} a_{\mathbf{m}}[v] \overline{a}_{\mathbf{m}}[w].$$

In accordance with Theorem 1.6.3 and Theorem 1.6.4, we have the compact embedding

$$W^{k,2}(\mathbb{T}^N) \overset{c}{\hookrightarrow} C(\mathbb{T}^N) \quad \text{whenever } k > \frac{N}{2},$$

whence we have

$$L^1(\mathbb{T}^N) \overset{c}{\hookrightarrow} W^{-k,2}(\mathbb{T}^N), \quad k > \frac{N}{2}. \tag{1.4}$$

We may also consider time-dependent (periodic) functions defined on the $(N+1)$-dimensional torus

$$\mathbb{T}_L^{N+1} = [-L, L]|_{\{-L,L\}} \times \mathbb{T}^N.$$

We summarize:

- The spaces $W^{k,2}(\mathbb{T}^N)$, $W^{k,2}(\mathbb{T}_L^{N+1})$ are separable Hilbert spaces, in particular Polish spaces.
- If $X \hookrightarrow W^{k,2}(\mathbb{T}^N)$ or $X \hookrightarrow W^{k,2}(\mathbb{T}_L^{N+1})$ with continuous embedding, then X is a completely regular Hausdorff space (Tikhonov space). Moreover, X admits a countable family of continuous functions separating points, namely

$$f_{\mathbf{m}}[v] = a_{\mathbf{m}}[v], \quad \mathbf{m} \in \mathbb{Z}^N.$$

1.7.2 L^p-structure

We start with the presentation of a combination of DeLeeuw's theorem on Fourier multipliers on $\mathbb{T}^N$ (see Stein [Ste70, Chapter 7, Theorem 3.8]) and the Hörmander–Mikhlin theorem (see Stein [Ste70, Chapter 4, Theorem 3]).

Theorem 1.7.1. *Let* $\mathbf{M} \in L^\infty(\mathbb{R}^N)$ *possess classical derivatives up to order* $[N/2]+1$ *in* $\mathbb{R}^N \setminus \{0\}$ *such that*

$$|\partial_\alpha \mathbf{M}(\xi)| \le c_\alpha |\xi|^{-|\alpha|}, \quad |\xi| \neq 0, \ |\alpha| \le [N/2]+1.$$

Then the operator $\mathcal{L}$*, since we know*

$$\mathcal{L}[v] = \sum_{|\mathbf{m}| \in \mathbb{Z}^N} \mathbf{M}(\mathbf{m}) a_{\mathbf{m}}[v] e_{\mathbf{m}},$$

is bounded on $L^p(\mathbb{T}^N)$*,* $1 < p < \infty$*.*

Consider the projection operator

$$\Pi_{\mathbf{M}} : W^{k,2}(\mathbb{T}^N) \to L^2(\mathbb{T}^N) \quad \text{defined as } \Pi_{\mathbf{M}}[v] = \sum_{|m_i| \le M_i,\ i=1,\dots,N} a_{\mathbf{m}}[v] e_{\mathbf{m}}.$$

In accordance with Theorem 1.7.1, $\Pi_{\mathbf{M}}$ is bounded as an operator on $L^p(\mathbb{T}^N)$, $1 < p < \infty$. Moreover (see Weisz [Wei12, Theorem 4.1]), we have

$$\|\Pi_{\mathbf{M}}[v]\|_{L^p_x} \le c_p \|v\|_{L^p_x}, \quad \text{and} \quad \Pi_{\mathbf{M}}[v] \to v \quad \text{in } L^p(\mathbb{T}^N) \text{ as } \min_i\{M_i\} \to \infty. \tag{1.5}$$

1.7.3 Regularization by convolution kernels

Let $\theta_\delta^x \in C^\infty(\mathbb{T}^N)$ be a family of regularizing kernels. More specifically,

$$\theta_\delta^x(x) = \frac{1}{\delta^N}\theta\left(\frac{x}{\delta}\right), \quad \theta \in C_c^\infty((-1,1)^N), \quad \theta(x) = \theta(|x|), \quad \int_{\mathbb{T}^N} \theta(x) = 1. \tag{1.6}$$

For $v \in \mathcal{D}'(\mathbb{T}^N)$, we define its regularization $[v]_{x,\delta}$ as the convolution

$$[v]_{x,\delta}(x) = v * \theta_\delta^x \equiv \langle v, \theta_\delta^x(x - \cdot)\rangle.$$

The following results can be found in Amann [Ama95, Chapter III.4] or Brezis [Bre83, Chapter IV.4]:

- If $v \in L^1(\mathbb{T}^N)$, then we have $[v]_{x,\delta} \in C^\infty(\mathbb{T}^N)$.
- If $v \in L^p(\mathbb{T}^N)$, $1 \le p < \infty$, then

$$\|[v]_{x,\delta}\|_{L_x^p} \le \|v\|_{L_x^p}$$

  and

$$[v]_{x,\delta} \to v \quad \text{in } L^p(\mathbb{T}^N) \text{ as } \delta \to 0.$$

- If $v \in L^\infty(\mathbb{T}^N)$, then

$$\|[v]_{x,\delta}\|_{L_x^\infty} \le \|v\|_{L_x^\infty}.$$

- If $v \in L^1(\mathbb{T}^N)$, then

$$[v]_{x,\delta}(x) \to v(x) \quad \text{whenever } x \text{ is a Lebesgue point of } v.$$

  In particular,

$$[v]_{x,\delta} \to v \quad \text{a.e. in } \mathbb{T}^N.$$

We recall that, for $v \in L^1(\mathcal{O}; X)$, the Lebesgue points $x \in \mathcal{O}$ are characterized by the property

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} \|v(y) - v(x)\|_X \,\mathrm{d}y \to 0 \quad \text{as } r \to 0,$$

where $B_r(x) \subset \mathcal{O}$ is a ball with radius r, centered at x.

The above concept may be extended to a larger class of generalized functions as long as the operation of convolution with a smooth kernel is well-defined, notably to the space of distributions; see Section 2.2.2.

1.8 Bochner spaces

In this section we present supplementary material for Bochner spaces. They can be seen as particular cases of the vector-valued functions introduced in Sections 1.1 and 1.4, where $\mathcal{O} = (0,T)$. These spaces are of crucial importance for time-dependent PDEs. Sometimes, it will be convenient to consider functions from Bochner spaces (depending on space and time) as space-time distributions in $\mathcal{D}'(\mathbb{R} \times \mathbb{T}^N)$ or even $\mathcal{D}'([-L,L]|_{\{-L,L\}} \times \mathbb{T}^N)$, defined on the space-time torus

$$\mathbb{T}_L^{N+1} = [-L,L]|_{\{-L,L\}} \times \mathbb{T}^N,$$

extending them conveniently outside the interval I. Similarly to (1.3) we have the embedding

$$L^1(0,T;L^1(\mathbb{T}^N)) \overset{c}{\hookrightarrow} W^{-k,2}(\mathbb{T}_L^{N+1}), \quad k > \frac{N+1}{2},\ L \geq T. \tag{1.7}$$

1.8.1 Time regularity

Let X be a separable Banach space. For $u \in L^1(0,T;X)$ we consider the distribution

$$C_c^\infty((0,T)) \to X, \quad \phi \mapsto \int_0^T u(t)\phi'(t)\,\mathrm{d}t.$$

Let Y be a Banach space with $X \hookrightarrow Y$ continuously. If there is $v \in L^1(0,T;Y)$ such that

$$\int_0^T u(t)\phi'(t)\,\mathrm{d}t = -\int_0^T v(t)\phi(t)\,\mathrm{d}t \quad \text{for all } \phi \in C_c^\infty((0,T)),$$

then we say that v is the weak derivative of u in Y and write $v = \partial_t u$. The space $W^{1,p}(0,T;X)$ consists of those functions from $L^p(0,T;X)$ having weak derivatives in $L^p(0,T;X)$. It is a Banach space with the norm

$$\|u\|_{W^{1,p}(0,T;X)}^p := \|u\|_{L^p(0,T;X)}^p + \|\partial_t u\|_{L^p(0,T;X)}^p.$$

Obviously this can be iterated to define the spaces $W^{k,p}(0,T;X)$, $k \in \mathbb{N}$.

In order to study the time regularity of functions from Bochner spaces, we recall the concept of continuity introduced in Section 1.1.

Definition 1.8.1. Let X be a Banach space with norm $\|\cdot\|_X$, $T > 0$ and $\alpha \in (0,1]$. Then:

– $C([0,T];X)$ denotes the set of functions $u : [0,T] \to X$ being continuous with respect to the norm topology, i.e.,

$$u(t_k) \to u(t_0) \quad \text{in } X,$$

for any sequence $(t_k)_{k\in\mathbb{N}} \subset [0,T]$ with $t_k \to t_0$.

– $C_w([0,T];X)$ denotes the set of functions $u : [0,T] \to X$ being continuous with respect to the weak topology, i.e.,

$$u(t_k) \rightharpoonup u(t_0) \quad \text{in } X,$$

for any sequence $(t_k)_{k\in\mathbb{N}} \subset [0,T]$ with $t_k \to t_0$. Equivalently, we may say that u belongs to $C_w([0,T];X)$ if the scalar functions $t \mapsto \langle x^*, u(t,\cdot)\rangle$ belong to $C([0,T])$, for any $x^* \in X^*$.

– $C^\alpha([0,T];X)$ denotes the set of functions $u : [0,T] \to X$ being α-Hölder continuous with respect to the norm topology, i.e.,

$$\sup_{t,s\in[0,T];t\neq s} \frac{\|u(t)-u(s)\|_X}{|t-s|^\alpha} < \infty.$$

Obviously, we have the inclusions

$$C^\alpha([0,T];X) \subset C([0,T];X) \subset C_w([0,T];X),$$

for any $\alpha \in (0,1]$.

We introduce *convergence* in $C_w([0,T];X)$ by stating

$$v_n \to v \text{ in } C_w([0,T];X) \quad \text{if } \sup_{t\in[0,T]} |\langle x^*, v_n - v\rangle_{X^*,X}| \to 0 \ \forall x^* \in X^*.$$

If the space X is separable and reflexive, then the unit ball $B_X \subset X$ is a metrizable compact set and the above convergence generates a metric topology on $C_w([0,T];B_X)$ in the sense specified in Section 1.1.

1.8.2 Compact embeddings

The following theorem shows how to obtain compactness in Bochner spaces. The original version was developed by Aubin and Lions (see Aubin [Aub63], Lions [Lio69, Section 1.5], or the survey paper by Simon [Sim86]).

Theorem 1.8.2. *Let (V,X,Y) be a triple of separable and reflexive Banach spaces such that the embedding $V \hookrightarrow X$ is compact and the embedding $X \hookrightarrow Y$ is continuous. Then the embedding*

$$\{u \in L^p(0,T;V) : \ \partial_t u \in L^p(0,T;Y)\} \hookrightarrow L^p(0,T;X)$$

is compact for $1 < p < \infty$.

In the context of stochastic PDEs we will be confronted with functions having only fractional derivatives in time. We define for $p \in (1,\infty)$ and $\alpha \in (0,1)$ the norm

$$\|u\|^p_{W^{\alpha,p}(0,T;X)} := \|u\|^p_{L^p(0,T;X)} + \int_0^T \int_0^T \frac{\|u(\sigma_1) - u(\sigma_2)\|^p_X}{|\sigma_1 - \sigma_2|^{1+\alpha p}} \,\mathrm{d}\sigma_1 \,\mathrm{d}\sigma_2.$$

The space $W^{\alpha,p}(0,T;X)$ is now defined as the subspace of $L^p(0,T;X)$ consisting of those functions having finite $W^{\alpha,p}(0,T;X)$-norm. It can be shown that this is a complete space and we have $W^{1,p}(0,T;X) \subset W^{\alpha,p}(0,T;X) \subset L^p(0,T;X)$. The following variant of Theorem 1.8.2 holds (see Flandoli–Gątarek [FG95, Theorem 2.1]).

Theorem 1.8.3. *Let (V,X,Y) be a triple of separable and reflexive Banach spaces such that the embedding $V \hookrightarrow X$ is compact and the embedding $X \hookrightarrow Y$ is continuous. Then the embedding*

$$L^p(0,T;V) \cap W^{\alpha,p}(0,T;Y) \hookrightarrow L^p(0,T;X)$$

is compact for $1 < p < \infty$ and $0 < \alpha < 1$.

Using the continuous embedding $C^\alpha([0,T],Y) \hookrightarrow W^{\alpha,p}(0,T;Y)$, we obtain the following.

Corollary 1.8.4. *Let (V,X,Y) be a triple of separable and reflexive Banach spaces such that the embedding $V \hookrightarrow X$ is compact and the embedding $X \hookrightarrow Y$ is continuous. Then the embedding*

$$L^p(0,T;V) \cap C^\alpha([0,T];Y) \hookrightarrow L^p(0,T;X)$$

is compact for $1 < p < \infty$ and $0 < \alpha < 1$.

We will use Corollary 1.8.4 at various occasions in order to obtain compactness for stochastic PDEs. Typically, solutions are Hölder continuous in a negative Sobolev space, so we have $Y = W^{-\ell,2}(\mathbb{T}^N)$ for some $\ell \in \mathbb{N}$. On the other hand, these functions also belong to $L^p(0,T;L^p(\mathbb{T}^N))$ (or $L^p(0,T;W^{1,p}(\mathbb{T}^N))$) for some $p \in (1,\infty)$. This means we have $V = L^p(\mathbb{T}^N)$ (or $V = W^{1,p}(\mathbb{T}^N)$). Corollary 1.8.4 applies with $X = W^{-1,p}(\mathbb{T}^N)$ (or $X = L^p(\mathbb{T}^N)$).

In view of the applications to compressible Navier–Stokes equations, we have to deal with weakly continuous functions. The following result is appropriate to handle this situation.

Theorem 1.8.5. *Let $\alpha \geq 0$, $1 < p < \infty$, and $\ell \in \mathbb{R}$. Then*

$$L^\infty(0,T;L^p(\mathbb{T}^N)) \cap C^\alpha([0,T];W^{\ell,2}(\mathbb{T}^N)) \hookrightarrow C_w([0,T];L^p(\mathbb{T}^N)).$$

If $\alpha > 0$, then the embedding is sequentially compact, meaning any sequence

$$(v_n)_{n\in\mathbb{N}} \quad \text{bounded in } L^\infty(0,T;L^p(\mathbb{T}^N)) \cap C^\alpha([0,T];W^{\ell,2}(\mathbb{T}^N))$$

contains a subsequence $(v_{n_k})_{k\in\mathbb{N}}$ such that

$$v_{n_k} \to v \quad \text{in } C_w([0,T];L^p(\mathbb{T}^N)).$$

Proof. First we have to show that

$$\langle x^*, v(t,\cdot)\rangle \in C([0,T]) \quad \text{for any } x^* \in L^{p'}(\mathbb{T}^N),$$

whenever $v \in L^\infty(0,T;L^p(\mathbb{T}^N)) \cap C^\alpha([0,T];W^{\ell,2}(\mathbb{T}^N))$ and p' is the conjugate exponent of p. As the norm in L^p is weakly lower semi-continuous, we deduce $v(t,\cdot) \in B(r)$ for *any* t, where $B(r)$ is a ball in $L^p(\mathbb{T}^N)$ of suitable radius $r > 0$. The collection of trigonometric polynomials $(e_{\mathbf{m}})_{\mathbf{m}\in\mathbb{Z}^N}$ defined in Section 1.7 generates a basis in $W^{\ell,2}(\mathbb{T}^N)$ for any ℓ, and their finite linear combinations are dense in $L^q(\mathbb{T}^N)$ for any $1 \le q < \infty$, in particular, in $L^{p'}(\mathbb{T}^N)$. Consequently,

$$\begin{aligned}
&\left|\langle x^*, v(t,\cdot)\rangle - \langle x^*, v(s,\cdot)\rangle\right| \\
&\le \left|\left\langle \sum_{|\mathbf{m}|\le M} \beta_{\mathbf{m}} e_{\mathbf{m}}, v(t,\cdot) - v(s,\cdot)\right\rangle\right| + \left|\left\langle x^* - \sum_{|\mathbf{m}|\le M} \beta_{\mathbf{m}} e_{\mathbf{m}}, v(t,\cdot) - v(s,\cdot)\right\rangle\right| \\
&\le \left|\left\langle \sum_{|\mathbf{m}|\le M} \beta_{\mathbf{m}} e_{\mathbf{m}}, v(t,\cdot) - v(s,\cdot)\right\rangle\right| + r\left\| x^* - \sum_{|\mathbf{m}|\le M} \beta_{\mathbf{m}} e_{\mathbf{m}}\right\|_{L_x^{p'}} \\
&\le c(M,\ell)\|v\|_{C_t^\alpha W_x^{\ell,2}} |t-s|^\alpha + r\left\| x^* - \sum_{|\mathbf{m}|\le M} \beta_{\mathbf{m}} e_{\mathbf{m}}\right\|_{L_x^{p'}},
\end{aligned} \tag{1.8}$$

where the last term can be made small uniformly for all $s,t \in [0,T]$ by taking suitable $\beta_{\mathbf{m}}$ and M large enough.

If $\alpha > 0$ we may apply the abstract Arzelà–Ascoli theorem (Theorem 1.1.1). The ball $B(r)$ is indeed weakly sequentially compact and the desired equi-continuity of the sequence $(v_n)_{n\in\mathbb{N}}$ follows easily from (1.8). □

1.8.3 Regularization by convolution kernels

This section is dedicated to the regularization of time-dependent functions. In order to avoid problems related to progressive measurability (which typically arise in our applications to stochastic PDEs) we regularize functions backwards in time. Consequently, it is convenient to extend them appropriately for $t \le 0$. For $v \in L^1(-1,T;X)$, where X is a Banach space, we consider the time regularization

$$[v]_{t,\delta}(t) = v * \theta_\delta^t(\cdot - \delta) = \int_{-\infty}^{\infty} \theta_\delta^t(t - \delta - s)v(s)\,\mathrm{d}s.$$

Here, the regularizing kernel is a function of t satisfying (1.6) for $N = 1$.

Referring again to Amann [Ama95, Chapter III.4] and Brezis [Bre83, Chapter IV.4], we have:
- If $v \in L^1(-1,T;X)$, then we have $[v]_{t,\delta} \in C^\infty((-1,T);X)$.
- If $v \in L^p(-1,T;X)$, $1 \le p < \infty$, then

$$\|[v]_{t,\delta}\|_{L_t^p X} \le \|v\|_{L_t^p X}$$

and

$$[v]_{t,\delta} \to v \quad \text{in } L^p(0,T;X) \text{ as } \delta \to 0.$$

- If $v \in L^\infty(-1,T;X)$, then

$$\|[v]_{t,\delta}\|_{L_t^\infty X} \le \|v\|_{L_t^\infty X}.$$

- If $v \in L^1(-1,T;X)$, then

$$[v]_{t,\delta}(t) \to v(t) \quad \text{in } X \text{ whenever } t \text{ is a Lebesgue point of } v.$$

In particular,

$$[v]_{t,\delta} \to v \quad \text{in } X \text{ as } \delta \to 0 \text{ a.e. in } (0,T).$$

2 Elements of stochastic analysis

We introduce the basic stochastic framework used in this book. We present only a selection of the principal concepts and ideas of stochastic analysis, as the reader is expected to be familiar with the basic notions of probability theory. Part of the results presented in this chapter can be found in the literature. The classical and widely used monographs include for instance Karatzas–Shreve [KS91] and Da Prato–Zabczyk [DPZ92] and we invite the reader to consult these textbooks for further details. In addition, we include a number of original results needed for the study of the compressible Navier–Stokes system later on.

To be more precise, in Section 2.2 we introduce the notion of random distributions (see Definition 2.2.1). It is a generalization of stochastic processes which allows one to treat random elements in the weakest possible topology, namely, the weak-$*$ topology of the space of space-time distributions $\mathcal{D}'(I \times \mathbb{T}^N)$, where $I \subset \mathbb{R}$. For the sake of simplicity, the results will be stated only for $I = \mathbb{R}$, with obvious modifications for a general interval I. In the subsequent sections we show how the classical theory of Itô's stochastic integration and its applications to stochastic PDEs can be formulated in the context of random distributions. We believe that this new perspective is interesting in its own right and will prove useful also for researchers working on other models in fluid dynamics or other fields.

2.1 Random variables and stochastic processes

Throughout the book $(\Omega, \mathfrak{F}, \mathbb{P})$ denotes a complete *probability space* with a σ-field $\mathfrak{F}$ and a probability measure $\mathbb{P}$. The probability space $([0,1], \overline{\mathfrak{B}([0,1])}, \mathfrak{L})$, where $\mathfrak{L}$ denotes the Lebesgue measure, is called *standard*. Here, $\overline{\mathfrak{B}}$ denotes the completion $\mathfrak{B}$ and $\mathfrak{L}$ denotes the one-dimenisonal Lebesgues measure. A *filtration* is a non-decreasing family of sub-σ-fields of $\mathfrak{F}$, that is, $\mathfrak{F}_t \subset \mathfrak{F}$ for all $t \geq 0$ and $\mathfrak{F}_s \subset \mathfrak{F}_t$ whenever $s \leq t$. We say that the filtration $(\mathfrak{F}_t)_{t\geq 0}$ satisfies the *usual conditions*, provided it is complete and right-continuous. In other words,

$$\{N \in \mathfrak{F};\ \mathbb{P}(N) = 0\} \subset \mathfrak{F}_0, \quad \mathfrak{F}_t = \mathfrak{F}_{t+} := \bigcap_{s>t} \mathfrak{F}_s \quad \text{for all } t \geq 0.$$

The multiple $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$ is then called a *stochastic basis* or a *filtered probability space*.

We proceed with basic definitions concerning random variables.

Definition 2.1.1. Let $(X, \mathcal{A})$ be a measurable space. An X-valued *random variable* is a measurable mapping $\mathbf{U} : (\Omega, \mathfrak{F}) \to (X, \mathcal{A})$. We denote by $\sigma(\mathbf{U})$ the smallest σ-field with respect to which $\mathbf{U}$ is measurable. More precisely,

$$\sigma(\mathbf{U}) := \{\{\omega \in \Omega;\ \mathbf{U}(\omega) \in A\};\ A \in \mathcal{A}\}$$

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and $\sigma(\mathbf{U}) \subset \mathfrak{F}$. In addition, we denote by $\mathcal{L}[\mathbf{U}]$ or also $\mathcal{L}_X[\mathbf{U}]$ the law of $\mathbf{U}$ on X, that is, $\mathcal{L}[\mathbf{U}]$ is the pushforward probability measure on $(X, \mathcal{A})$ given by

$$\mathcal{L}[\mathbf{U}](A) = \mathbb{P}(\mathbf{U} \in A), \quad A \in \mathcal{A}.$$

Definition 2.1.2. Let $(X, \mathcal{A})$ be a measurable space. We say that two X-valued random variables $\mathbf{U}$ and $\mathbf{V}$ are *equal in law*, if $\mathcal{L}[\mathbf{U}]$ and $\mathcal{L}[\mathbf{V}]$ coincide.

We stress that the assumptions on the state space X will vary in the sequel. Most of the notions presented below require a topology on X and therefore we assume that X is a topological space equipped with a Borel σ-field. In addition, it is convenient that the topology on X is completely determined by the family of continuous functions. Specifically, we consider *Tikhonov spaces*, meaning completely regular and Hausdorff. As a matter of fact, we deal almost exclusively with *topological vector spaces*, in particular with the class of locally convex topological vector spaces. These are vector spaces equipped with a topology that renders the vector addition as well as the scalar multiplication continuous and, in addition, the topology is generated by a family of seminorms $(p_\gamma)_{\gamma\in\Gamma}$.

Many concepts in the theory of stochastic processes require a certain *uniformity* of the topology. Simplifications occur in the case of *Polish spaces*, that is, separable spaces that are completely metrizable. This is also the common setting found in the literature. However, the delicate structure of the compressible Navier–Stokes system studied in the main body of this book naturally leads to spaces which are generally not metrizable, such as Banach spaces equipped with weak topology. Hence we will formulate the basic notions on probability theory in a wider generality. In particular, all spaces we shall deal with will admit a countable family of bounded continuous functions that separates points. Given such a family of continuous functions $(g_n)_{n\in\mathbb{N}}$ on X, we define an embedding

$$\jmath : X \to [-1,1]^{\aleph_0}, \quad \jmath(x) = \big(g_n(x)\big)_{n\in\mathbb{N}}.$$

Here, we have tacitly assumed that all functions g_n range in $(-1,1)$. Note that $[-1,1]^{\aleph_0}$ is a compact Polish space. This motivates the following definition.

Definition 2.1.3 (sub-Polish space)**.** Let (X, τ) be a topological space such that there exists a countable family

$$\{g_n : X \to (-1,1);\, n \in \mathbb{N}\}$$

of continuous functions that separate points of X. Then (X, τ) is called a *sub-Polish* space.

The following characterization of equality in law will be frequently used in the sequel.

Lemma 2.1.4. *Let X be a Tikhonov topological space equipped with the Borel σ-field. Let $\mathbf{U}$ and $\mathbf{V}$ be X-valued random variables. Then $\mathcal{L}[\mathbf{U}] = \mathcal{L}[\mathbf{V}]$, provided*

$$\langle \mathcal{L}[\mathbf{U}], f \rangle = \langle \mathcal{L}[\mathbf{V}], f \rangle,$$

or, equivalently,

$$\mathbb{E}[f(\mathbf{U})] = \mathbb{E}[f(\mathbf{V})]$$

holds true for all $f \in C_b(X)$.

Although several function spaces we use in the book, notably the space of distributions, are not first countable, the convergence results are usually stated in terms of *sequences* rather than nets. The sequential language seems more adequate for describing the asymptotic behavior of stochastic processes and several results are simply only true for sequences. We proceed with various notions on the convergence of random variables. First, we introduce the almost sure convergence which corresponds to the almost everywhere convergence known from measure theory.

Definition 2.1.5. Let X be a topological space equipped with the Borel σ-field and let $\mathbf{U}$ and $\mathbf{U}_n$, $n \in \mathbb{N}$, be X-valued random variables on $(\Omega, \mathfrak{F}, \mathbb{P})$. We say that $\mathbf{U}_n$ converges to $\mathbf{U}$ *almost surely*, provided

$$\mathbb{P}\Big(\omega \in \Omega;\ \lim_{n\to\infty} \mathbf{U}_n(\omega) = \mathbf{U}(\omega)\Big) = 1.$$

In other words, there exists a set of full probability $\Omega^* \subset \Omega$ such that, for every $\omega \in \Omega^*$, the following statement holds: if $\mathcal{U} \subset X$ is an open neighborhood of $\mathbf{U}(\omega)$, then there exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, we have $\mathbf{U}_n(\omega) \in \mathcal{U}$.

Next, we define the probabilistic analogue of convergence in measure. To this end, we restrict ourselves to the case of topological vector spaces. Recall that, if X is a topological vector space, the topology on X is uniform. This means that any neighborhood $\mathcal{U}(x)$ of a point x takes the form $x + \mathcal{U} = \{x + y;\ y \in \mathcal{U}\}$, where $\mathcal{U}$ is a neighborhood of 0.

Definition 2.1.6. Let X be a topological vector space. Assume that $\mathbf{U}$ and $\mathbf{U}_n$, $n \in \mathbb{N}$, are X-valued random variables on $(\Omega, \mathfrak{F}, \mathbb{P})$. We say that $\mathbf{U}_n$ converges to $\mathbf{U}$ *in probability* if, for every $\mathcal{U} \subset X$ which is an open neighborhood of 0, we have

$$\lim_{n\to\infty} \mathbb{P}\big(\omega \in \Omega;\ \mathbf{U}_n(\omega) \notin \mathbf{U}(\omega) + \mathcal{U}\big) = 0. \tag{2.1}$$

Remark 2.1.7. As pointed out, the definition extends easily to *uniform spaces* but we do not need this generality here.