Yanpei Liu

Topological Theory of Graphs

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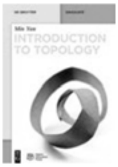

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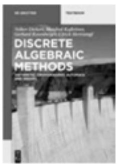

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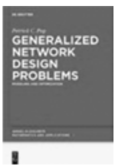

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Topological Theory of Graphs

DE GRUYTER

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ISBN 978-3-11-047669-9
e-ISBN (PDF) 978-3-11-047949-2
e-ISBN (EPUB) 978-3-11-047922-5
Set-ISBN 978-3-11-047950-8

Library of Congress Cataloging-in-Publication Data
A CIP catalog record for this book has been applied for at the Library of Congress.

Bibliographic information published by the Deutsche Nationalbibliothek
The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available on the Internet at http://dnb.dnb.de.

Typesetting: Integra Software Services Pvt. Ltd.
Printing and binding: CPI books GmbH, Leck
Cover image: ktsimage/iStock/thinkstock
♾ Printed on acid-free paper
Printed in Germany

www.degruyter.com

Preface to DG Edition

The De Gruyter (DG) edition is based on my previous monograph *Topological Theory on Graphs*, published by the University of Science and Technology of China (USTC) Press in 2008, with updates and improvements from two main sources.

One is that the new developments of four ways, with five pairs of theorems, enable us to get criteria, for determining the embeddability of a graph, on a surface (orientable and non-orientable) of genus arbitrarily given, as shown in Liu [229, 231, 232, 236]. Before them, only few results were known for surfaces of small genera ($\neq 0$), such as Abrams and Slilaty [1], Archdeacon [11], Archdeacon and Huneke [12], Archdeacon and Siran [14], Glover and Huneke [100], Glover et al. [101].

Notably, for the specific case of genus zero, one of the five pairs leads to the criteria for planarity, as given in the pair of Theorems 4.2.5 and 4.3.2 relating to a pair of homology and cohomology. They have a number of corollaries, including the three theorems obtained by Lefschetz [171], MacLane [261] and Whitney [392] for the planarity of a graph at a time. This causes item 4.5.10, in the Notes Section 4.5 of Chapter 4. Subsequently, item 4.5.9 is completed in the core theoretical stage of the three stages: theoretization, efficientization and intelligentization, involving with my research.

The other is the progresses in applications and usages of joint tree model described in Section 9.1.

Each Notes section in chapters is accompanied by at least one new item. I would like to mention the following:

Notes Section 1.5 in Chapter 1. Item 1.5.5 was provided for reminding readers of the universality of vector space, sketched in Section 1.4 as an abstract linear space, motivated from the background in Liu [237]. Item 1.5.6 is for accessing the efficiency of theoretical results in this book, or polynomial complexity as shown in Cook [54] and Karp [159].

Notes Section 2.6 in Chapter 2. Item 2.6.6 was presented for perspective developments available in the theory of polyhedra, shown in Liu [230, 233, 234, 238], with relevant references as complimentary for the reader.

Notes Section 3.7 in Chapter 3. Formula (3.7.1) was put into item 3.7.3 to show that the topological classification of surfaces can be done, via only the three types of transformations. Item 3.7.9 was provided to enhance readers' understanding, intrinsically from topology.

Notes Section 4.5 in Chapter 4. Further to item 4.5.10 mentioned earlier, item 4.5.11 illustrates one of the new approaches, to investigate the structure of cycle spaces in a graph, via an example.

Notes Section 5.6 in Chapter 5. Item 5.6.7 was suggested to generalize the polyhedral form, from Jordan curve axiom to the surface closed curve axiom, for recognizing whether, or not, a graph can be embedded onto a surface of given genus not zero.

Notes Section 6.5 in Chapter 6. Item 6.5.12 shows that the relationship among graphs, polyhedra, embedding and maps can be clarified via symmetries.

Notes Section 7.5 in Chapter 7. Item 7.5.6 enables us to go in a new way, for classifying knots, or links, by observing the relationship between embeddings of general networks with binary weight of edges on surfaces and knots, or links, via the correspondence between a 4-regular graph and a pair of two general graphs mutually dual.

Notes Section 8.6 in Chapter 8. Item 8.6.10 presents a theoretical framework, inspired from the pair of homology and cohomology in Sections 4.2 and 4.3, to detect a type of homology on a graph, as the dual of cohomology in Theorem 8.2.1, and to establish a new pair of criteria for the embeddability of a graph, on a surface of genus arbitrarily given, via the polyhedral theory, described in Liu [227].

Notes Section 9.5 in Chapter 9. Items 9.5.5–9.5.10 reflect a series of progresses, for determining the up-embeddability on surfaces, handle (orientable genus) and crosscap (momorientable genus) polynomials, maximum genus, genus (minimum!), average genus, etc., of graphs, or digraphs, based on the joint tree model described in Section 9.1.

Notes Section 10.5 in Chapter 10. Recent result on the planarity of a graph, by a single forbidden configuration, was mentioned in item 10.5.5, as shown in Ref. [238].

Notes Sections 11.5 and 12.5 in, respectively, in Chapters 11 and 12. Both items 11.4.9 and 12.5.8 indicate the reason why the minors are not available as a forbidden configuration, for the properties considered with the inheritness.

Notes Section 13.8 in Chapter 13. Both items 13.8.8 and 13.8.9 reflect new progresses on minimality and maximality on graphs with surfaces, based on joint tree model.

Notes Section 14.6 in Chapter 14. Item 14.6.6 shows a new approach, hopefully to recognize whether, or not, a regular matroid is graphic, or cographic, to strengthen and expand Theorems 14.4.1 and 14.5.1.

In Notes Section 15.6 in Chapter 15. Item 15.6.8 provides a theoretical framework to characterize whether, or not, two knots, or two links, are in the same class of pan-polynomial equivalence on the basis of Section 15.3.

In addition, Theorem 13.5.1 was improved and revised. Proofs of certain conclusions are completed and concise, or more accurate, such as in Lemma 3.1.2, Theorem 5.4.1, Lemma 5.5.1, Theorem 13.5.1, etc.

Last but not least, I would like to take this opportunity to express my sincere thanks to Rongxia Hao, Erling Wei and Liangxia Wan for their careful reading of the manuscript with corrections on grammar. Some of researches were partially supported by NNSFC under Grant No. 11371052.

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Beijing
September 2016

Preface to USTC Edition

The subject of this book reflects new developments established mainly by the author himself in company with a few cooperators, most of them being his former and present graduate students in the foundation, as mentioned in Liu [216, 217]. The central idea is to extract suitable parts of a topological object such as a graph which is not necessarily to be with symmetry, as linear spaces which are all with symmetry for exploiting global properties in construction of the objects. This is a way of combinatorizations and further algebraications of an object via relationship among their subspaces.

Graphs are dealt with three vector spaces over GF(2) generated by 0 (dimensional)-cells, 1 (dimensional)-cells and 2 (dimensional)-cells, with the finite field of order 2. The first two spaces were known from, e.g., Lefschetz [172] by taking 0-cells and 1-cells as, respectively, vertices and edges. Of course, a graph is only a 1-complex without two cells.

Since the 1950s, in Wu [402] and Tutte [335, 346], the chain groups generated by 0-cells and 1-cells over, respectively, GF(2) and the real field were independently used for describing a graph. And they both, after ten years, adopted non-adjacent pair of edges as a 2-cell for which the cohomology on a graph was allowed to be established.

Their results especially in Wu [402–406] enabled the author to create a number of types of planarity auxiliary graphs induced from the graph considered for the study of the efficiency of theorems in Liu [192, 193, 202, 205, 225] as one approach. Another approach can be seen in Liu [206–208, 226].

More interestingly, two decades after Liu [192], in Archdeacon and Siran [14], a theta graph (network) was used for characterizing the planarity of a given graph. The theta graph can be seen to be a type of planarity auxiliary graph (network) because planarity auxiliary graphs are subgraphs of the theta graph. However, in virtue of the order of theta network upper bounded by a exponential function of the size of given graph and that of planarity auxiliary network by a quadratic polynomial of the size of given graph, theorems deduced from a theta network are all without efficiency while those from planarity auxiliary graphs are all with efficiency.

The effects of planarity auxiliary graphs are reflected in Chapters 8, 10, 11, 12 and 13 with a number of extensions.

On the other hand, in Liu [214] a graph was dealt with a set of polyhedra via double covering the edge set by travels under certain condition so that travels were treated as 2-cells. These enable us to discover homology and another type of cohomology for showing the sufficiency of Eulerian necessary condition in this circumstance. Further, all the results for the planarity of a graph in Whitney [392] on the duality, MacLane [261, 262] on a circuit basis and Lefschetz [171] on a circuit double covering have a universal view in this way. In fact, our polyhedra are all on such surfaces, i.e., 2-dimensional compact manifolds without boundary. If a boundary is allowed on a surface, the Eulerian necessary condition is not always sufficient in general. Some people used to miss the boundary condition.

The effects of this theoretical thinking are reflected in Chapters 4, 5, 7 and 14.

Because of the clarification of the joint tree model of a polyhedron in Liu [218, 219] by the present author recently on the basis of Liu [195, 196], we are allowed to write a brief description on the theories of surfaces and polyhedra in Chapters 2 and 3 and related topics in Chapters 6, 9 and 15.

Although quotient embeddings (current graph and its dual, voltage graph) were quite active in constructing an embedding of a graph on a surface with its genus minimum in a period of decades, this book has no space for them. One reason is that some writers such as White [382], Ringel [288] and Liu [216, 217], etc., have already mentioned them. Another reason is that only graphs with higher symmetry are suitable for quotient embeddings, or for employing the covering space method, whence this book is for general graphs without such a limitation of symmetry.

In spite of refinements and simplifications for known results, this book still contains a number of new results, for example Section 5.2, the sufficiency in the proof of Theorem 5.2.1, Sections 9.4, 11.3 and 11.4, 13.1 and 13.2, 13.4 and 13.5 etc., only to name a few. Researches were partially supported by the NNSF in China under Grant No. 60373030 and No. 10571013.

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Beijing
December 2007

Contents

1 Preliminaries

Throughout, for the sake of brevity, the usual logical conventions are adopted: disjunction, conjunction, negation, implication, equivalence, universal quantification and existential quantification denoted, respectively, by the familiar symbols:$\vee$, $\wedge$, $\neg$, $\Rightarrow$, $\Leftrightarrow$, $\forall$ and $\exists$. And, $x.y$ is for the section y in Chapter x.

In the context, ($i.j.k$) refers to item k of section j in Chapter i.

1.1 Sets and relations

A *set* is a collection of objects with some common property, which might be numbers, points, symbols, letters or whatever even sets except itself to avoid paradoxes. The objects are said to be *elements* of the set. We always denote elements by italic lower case letters and sets by upper case letter. The statement "x is (is not) an element of M" is written as $x \in M(x \notin M)$. A set is often characterized by a property. For example,

$$M = \{x \mid x \leq 4, \text{positive integer}\} = \{1, 2, 3, 4\}.$$

The *cardinality* of a set M (or the number of elements of M if finite) is denoted by $\mid M \mid$.

Let A, B be two sets. If $(\forall a)\ (a \in A \Rightarrow a \in B)$, then A is said to be a *subset* of B which is denoted by $A \subseteq B$. Further, we may define the three main operations: union, intersection and subtraction, respectively, as $A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$, $A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$ and $A \setminus B = \{x \mid (x \in A) \wedge (x \notin B)\}$.

If $B \subseteq A$, then $A \setminus B = A - B$ is denoted by $\overline{B}(A)$, which is said to be the *complement* of B in A. If all the sets are considered as subsets of Ω, then the complement of A in Ω is simply denoted by $\overline{A}$. The *empty* denoted by $\emptyset$ is the set without element. For those operations on subsets of Ω mentioned above, we have the following laws:

Idempotent law $\forall A \subseteq \Omega, A \cap A = A \cup A = A$.

Commutative law $\forall A, B \subseteq \Omega, A \cup B = B \cup A; A \cap B = B \cap A$.

Associative law $\forall A, B, C \subseteq \Omega, A \cup (B \cup C) = (A \cup B) \cup C; A \cap (B \cap C) = (A \cap B) \cap C$.

Absorption law $\forall A, B \subseteq \Omega, A \cap (A \cup B) = A \cup (A \cap B) = A$.

Distributive law $\forall A, B, C \subseteq \Omega, A \cup (B \cap C) = (A \cup B) \cap (A \cup C); A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Universal bound law $\forall A \subseteq \Omega, \emptyset \cap A = \emptyset,\ \emptyset \cup A = A; \Omega \cap A = A, \Omega \cup A = \Omega$.

Unary complement law $\forall A \subseteq \Omega, A \cap \overline{A} = \emptyset; A \cup \overline{A} = \Omega$.

The unary complement law is also called the *excluded middle law* in logic.

From the laws described earlier, we may obtain a large number of important results. Here, only a few are listed for usage in this context.

DOI 10.1515/9783110479492-001

Theorem 1.1.1. $\forall A \subseteq \Omega$,

$$\begin{cases} (\forall X \subseteq \Omega)\Big((A \cap X = A) \vee (A \cup X = X)\Big) \\ \qquad\qquad \Rightarrow A = \emptyset; \\ (\forall X \subseteq \Omega)\Big((A \cap X = X) \vee (A \cup X = A)\Big) \\ \qquad\qquad \Rightarrow A = \Omega. \end{cases} \tag{1.1.1}$$

Theorem 1.1.2. $\forall A, B \subseteq \Omega$,

$$A \cap B = A \Leftrightarrow A \cup B = B. \tag{1.1.2}$$

Theorem 1.1.3. $\forall A, B, C \subseteq \Omega$,

$$(A \cap B = A \cap C) \wedge (A \cup B = A \cup C) \Leftrightarrow B = C. \tag{1.1.3}$$

Theorem 1.1.4. $\forall A \subseteq \Omega$,

$$\overline{\overline{A}} = A. \tag{1.1.4}$$

Theorem 1.1.5. $\forall A, B \subseteq \Omega$,

$$\overline{A \cup B} = \overline{A} \cap \overline{B}; \overline{A \cap B} = \overline{A} \cup \overline{B}. \tag{1.1.5}$$

From those described above, it is seen that $\overline{\emptyset} = \Omega$ and $\overline{\Omega} = \emptyset$. Further, the symmetry (or duality) that any proposition related to $\cup, \cap, \emptyset, \Omega$ can be transformed into another by interchanging $\cup$ and $\cap$, $\emptyset$ and Ω.

For $A, B \subseteq \Omega$, an *injection* (or 1-to-1 correspondence) between A and B is a mapping $\alpha : A \to B$, such that $\forall a, b \in A,\ a \neq b \Rightarrow \alpha(a) \neq \alpha(b)$. A *surjection* between A and B is a mapping $\beta : A \to B$, such that $(\forall b \in B)(\exists a \in A)(\beta(a) = b)$. If a mapping is both an injection and a surjection, then it is called a *bijection*. Two sets are said to be *isomorphic* if there is a bijection between them. Two isomorphic sets A and B, or write $A \sim B$, are always treated as the same. Of course, for finite sets, it is trivial to justify if two sets are isomorphic by the fact: $\forall A, B \subseteq \Omega,\ A \sim B \Leftrightarrow | A |=| B |$.

For a set M, let $M \times M = \{\prec x, y \succ | \ \forall x, y \in M\}$ which is said to be the *Cartesian product* of M. Here, $\prec x, y \succ \neq \prec y, x \succ$ in general.

A *binary relation* R on M is a subset of $M \times M$. The adjective “binary” of the relation will often be omitted in the context. If the relation R holds for $x, y \in M$, then we write $\prec x, y \succ \in R$, or xRy. An *order*, denoted by $\preceq$, is a relation R which satisfies the following three laws:

Reflective law $\forall x \in M, xRx$.

Antisymmetry law $\forall x, y \in M, xRy \wedge yRx \Rightarrow x = y$.

Transitive law $\forall x, y, z \in M, xRy \wedge yRz \Rightarrow xRz$.

The set M with the order $\preceq$ is said to be a *poset* (or partial order set) denoted by $(M, \preceq)$.

Theorem 1.1.6. *In a poset* $(M, \preceq)$, $\forall x_1, x_2, \ldots, x_n \in M$,

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_1 \Rightarrow x_1 = x_2 = \cdots = x_n. \tag{1.1.6}$$

The theorem is sometimes called the *anti-circularity law*. If a relation only satisfies Reflective law and Transitive law but not Anti-symmetry law, then it is called the *quasi-order*, which is denoted by $\bullet \prec$. A set M with $\bullet \prec$ is said to be a *quoset* denoted by $(M, \bullet \prec)$.

Theorem 1.1.7. *Any subset S of a quoset $(M, \bullet \prec)$ is itself a quoset with the restriction of the quasi-order to S.*

If a quasi-order R on M satisfies the symmetry law described below, then it is called an *equivalent relation*, or simply an *equivalence* denoted by $\sim$.

Symmetry law $\forall x, y \in M, xRy \Rightarrow yRx$.

For the equivalence $\sim$ on M, we are allowed to define the set $x(M) = \{y \mid \forall y \in M, y \sim x\}$, which is said to be the *equivalent class* for $x \in M$. The set that consists of all the equivalent classes is called the *quotient set* of $(M, \sim)$ denoted by $M/\sim$. In a quoset $(M, \bullet\prec)$, let $\sim_{\bullet\prec}$ be defined by

$$\forall x, y \in M, x \sim_{\bullet\prec} y \Leftrightarrow (x \bullet \prec y) \wedge (y \bullet \prec x). \tag{1.1.7}$$

Then, it is seen that $\sim_{\bullet\prec}$ is an equivalence on M and that $(M/\sim_{\bullet\prec}, \bullet \prec)$ is also a quoset.

Theorem 1.1.8. *A quoset $(M, \bullet \prec)$ is a poset if, and only if, $M/\sim_{\bullet\prec} = M$, or say, it satisfies the anti-circularity law.*

In a poset $(M, \preceq)$, we define the *strict inclusion* , denoted by $\prec$, of the order by the *anti-reflective law*: $\neg x \in M, x \prec x$ and the *transitive law*: $(x \prec y) \wedge (y \prec z) \Rightarrow x \prec z$ while noticing that $x \preceq y \Leftrightarrow (x \prec y) \vee (x = y)$. If an order $\preceq$ on M satisfies the alternative law described below, then it is called a *total order*, or a *linear order*.

Alternative law $\forall x, y \in M, x \npreceq y \Rightarrow y \preceq x$.

A set with a total order is said to be a *chain*. The *length* of a chain with n elements is defined to be $n - 1$. From Theorem 1.1.7 and the definitions, we may have

Theorem 1.1.9. *Any subset of a poset is a poset and any subset of a chain is a chain.*

The *converse* of a relation R on M is, by definition, the relation $R^* : \forall x, y \in M, xR^*y \Leftrightarrow yRx$. It is obvious from inspection of the three laws for order to have

Theorem 1.1.10 (Duality principle). *The converse of any order is itself an order.*

In a poset $(M, \preceq)$, there may have an element $a : \forall x \in M, a \preceq x$. Because of Antisymmetry law, such an element, if it exists, is a unique one which is called the *least element* denoted by O. In a dual case, the *greatest element*, if it exists, is denoted by I. The elements O and I, when they exist, are called *universal bounds* of the poset.

Theorem 1.1.11. *A chain has the universal bounds if it is finite.*

In a poset $(M, \preceq)$, an element $a \in M : \forall x \in M, x \preceq a \Rightarrow x = a$ is called a *minimal element*. Dually, a *maximal element* is defined as $a \in M : \forall x \in M, a \preceq x \Rightarrow a = x$.

Theorem 1.1.12. *Any finite non-empty poset* $(M, \preceq)$ *has minimal and maximal elements.*

A mapping $\tau : M \to N$ from a poset $(M, \preceq)$ to a poset $(N, \preceq)$ is called *order preserving*, or *isotone* if it satisfies

$$\forall x, y \in M, \ x \preceq y \Leftrightarrow \tau(x) \preceq \tau(y). \tag{1.1.8}$$

Further, if an isotone $\tau : M \to N$ satisfies

$$\forall x, y \in M, \tau(x) \preceq \tau(y) \Rightarrow x \preceq y, \tag{1.1.9}$$

then it is called an *isomorphism*. Two posets $(M, \preceq)$ and $(N, \preceq)$ are said to be *isomorphic*, that is $(M, \preceq) \cong (N, \preceq)$, if there is an isomorphism between them. All isomorphic posets are treated as the same. However, it is not trivial as for sets to justify if two posets are isomorphic in general.

An *upper bound* of a subset X of a poset $(M, \preceq)$ is an element $a : \forall x \in X, x \preceq a$. The *least upper bound* (or l.u.b.) is an upper bound $b : a \preceq b \Rightarrow a = b$, where a is another upper bound of X. Dually, a *lower bound* and the *greatest lower bound* (g.l.b.). The *length* of a poset is the l.u.b. of the lengths of chains in the poset. A *lattice* is a poset if any two x and y of whose elements has a g.l.b. or *meet* denoted by $x \wedge y$ and an l.u.b. or *join* denoted by $x \vee y$. A lattice $L = (M, \preceq; \vee, \wedge)$ is *complete* if each of its subset X has an l.u.b. and a g.l.b.. Moreover, we have known that all finite-length lattices are complete.

Let 2^Ω be the set that consists of all subsets of Ω. From Section 1.1, we may see that $(2^\Omega, \subseteq; \cup, \cap)$ is a lattice. In fact, we have

Theorem 1.1.13. *A poset is a lattice if, and only if, it satisfies the idempotent, commutative, associative and absorption laws.*

Two lattices $(M, \preceq; \vee, \wedge)$ and $(N, \preceq; \vee, \wedge)$ are *isomorphic* if there is an isomorphism τ between $(M, \preceq)$ and $(M, \preceq)$ such that $\forall x, y \in M$,

$$(\tau(x \vee y) = \tau(x) \vee \tau(y)) \wedge (\tau(x \wedge y) = \tau(x) \wedge \tau(y)). \tag{1.1.10}$$

Of course, it is non-trivial as well to justify if two lattices are isomorphic in general.

1.2 Partitions and permutations

A *partition* of a set X is such a set of subsets of X that any two subsets are without common element and the union of all the subsets is X.

Theorem 1.2.1. *A partition $P(X)$ of a set X determines an equivalence on X such that the subsets in $P(X)$ are the equivalent classes.*

Let $P(X) = \{p_1, p_2, \ldots, p_{k_1}\}$ and $Q(X) = \{q_1, q_2, \ldots, q_{k_2}\}$ be two partitions of X. If for any q_j, $1 \le j \le k_1$, there exists a p_i, $1 \le i \le k_2$ such that $q_j \subset p_i$, then $Q(X)$ is called a *refinement* of $P(X)$ and $P(X)$, an *enlargement* of $Q(X)$ except only for $P(X) = Q(X)$. The partition of X with each subset of a single element, or only one subset which is X in its own right is, respectively, called the 0-*partition*, or 1-*partition* and denoted by $0(X)$, or $1(X)$.

Theorem 1.2.2. *For a set X and its partition $P(X)$, the 0-partition $0(X)$ (or 1-partition $1(X)$) can be obtained by refinements (or enlargements) for at most $O(\log |X|)$ times in the worst case.*

Proof. In the worst case, it suffices to consider $P(X) = 1(X)$(or $0(X)$) and only one more subset produced in a refinement. Because of

$$1 + 2 + 2^2 + \cdots + 2^{\log |X|} = \frac{2^{1+\log |X|} - 1}{2 - 1} = O(|X|), \tag{1.2.1}$$

the times of refinements (or enlargements) needed for getting $0(X)$ (or $1(X)$) is $O(\log |X|)$. The theorem is obtained. □

For two partitions $P = \{p_1, p_2, \dots, p_s\}$ and $Q = \{q_1, q_2, \dots, q_t\}$ of a set X, the *family intersection* of P and Q is defined to be

$$P \cap Q = \bigcup_{i=1}^{s} \{p_i \cap q_1, p_i \cap q_2, \dots, p_i \cap q_t\}. \tag{1.2.2}$$

Actually, $\{p_i \cap q_1, p_i \cap q_2, \dots, p_i \cap q_t\}$ for $i = 1, 2, \dots, l$ are partitions of p_i.

Theorem 1.2.3. *The family intersection satisfies the commutative and associate laws. And further, $P \cap Q$ is a refinement of both P and Q.*

A *permutation* of a set X is a bijection of X to itself. Because elements in a set are no distinction, they are allowed to be distinguished by natural numbers as $X = \{x_1, x_2, \dots\}$, or simply $X = \{1, 2, \dots\}$. So, a permutation of set $L = \{1, 2, \dots, l\}$ can be expressed as

$$\begin{pmatrix} 1 & 2 & 3 & \dots & l \\ i_1 & i_2 & i_3 & \dots & i_l \end{pmatrix}. \tag{1.2.3}$$

If $i_j = j$ for all $1 \le j \le l$, the permutation is called the *identity*. From Theorem 1.1.4, the identity is unique.

Theorem 1.2.4. *Let π be a permutation of set $L = \{1, 2, \dots, l\}$, then for any $i \in L$ there is an integer $n \ge 0$ such that $p^n i = i$.*

Proof. By contradiction. If there is no such an integer, by the 1–to–1 property it is a contradiction to the finiteness of l. □

On the basis of this theorem, the set $X_i = \{i, \pi i, \pi^2 i, \dots, \pi^{n-1} i\}$ is called the *orbit* of i. Because any element in X_i has the same orbit as i, it can also be called an orbit of π, denoted by $\mathrm{Orb}_\pi\{i\}$, or simply $\{i\}_\pi$. Because any two orbits of a permutation are either same or disjoint, all orbits form a partition of L.

An orbit with the order in its own right is called a *cyclic permutation*, or in brief, a *cycle*. The cycle corresponding to $\mathrm{Orb}_\pi\{i\}$ is denoted by $\mathrm{Orb}_\pi(i)$, or simply $(i)_\pi$. Because of the disjointness among orbits, by considering that the composite of disjoint cycles satisfies the commutative law and the associate law, a permutation can always be expressed as a product of cycles. The *order* of a cycle is one greater than its *length*, i.e., the number of elements in the cycle. A cycle of order 1 is called a *fixed point* of the permutation. All the fixed points in a permutation are always omitted in its cyclic expression.

As an example,

$$\begin{aligned}\begin{pmatrix}1\ 2\ 3\ 4\ 5\ 6\ 7\\ 2\ 5\ 1\ 6\ 3\ 4\ 7\end{pmatrix} &= (1,2,5,3)(4,6)(7)\\ &= (1,2,5,3)(4,6).\end{aligned}$$

However, the product of two cycles with a common element is not commutative in general. For example, $P_1 = (1,3,2)$ and $P_2 = (1,2,4)$,

$$P_1P_2 = (2,4,3) \neq (1,3,4) = P_2P_1.$$

Because $C^{sk} = 1$ on the order k of a cycle C and any positive integer s, it can be seen from Theorem 1.2.4 that if permutation $\pi = C_1C_2 \cdots C_n$, where C_i, $1 \le i \le n$, are all the disjoint cycles of order n_i, then π has its order $[l_1, l_2, \ldots, l_n]$, the least common multiple $(\mathrm{lcm}\{l_1, l_2, \ldots, l_n\} = [l_1, l_2, \ldots, l_n])$ of $l_1, l_2, \ldots$, and l_n.

Theorem 1.2.5. *The unique inverse of a cycle $C = (c_1, c_2, \ldots, c_n)$ is the cycle $C^{-1} = (c_n, c_{n-1}, \ldots, c_1)$.* ■

Let $\Sigma_{|L|}$ be the set of all permutations on L. The cardinality of L is also called the *degree* of permutations. For two permutations π and σ in $\Sigma_{|L|}$, if there is a permutation $\rho \in \Sigma_{|L|}$ such that $\pi = \rho\sigma\rho^{-1}$, then π and σ are *conjugates* for ρ.

Let $\gamma = (x_1, x_2, \ldots, x_r)$ be a cycle and τ, another permutation in $\Sigma_{|L|}$. For $y \in L$, if $x = \tau^{-1}y$ is not in γ, then $\tau\gamma\tau^{-1}y = \tau x = \tau(\tau^{-1}y) = y$. Otherwise, if $x = \tau^{-1}y = x_i (1 \le i \le r)$, then $\tau\gamma\tau^{-1}y = \tau x_i = x_{i+1}$. This implies that

$$\begin{aligned}\tau\gamma\tau^{-1} &= \tau(x_1, x_2, \ldots, x_r)\tau^{-1}\\ &= (\tau x_1, \tau x_2, \ldots, \tau x_r).\end{aligned} \tag{1.2.4}$$

For $\pi \in \Sigma_{|L|}$, let $c(\pi)$ be the number of cycles in its cyclic partition and l_i, the number of cycles of length i, $1 \le i \le c(\pi)$. The *cyclic type* of permutation π is defined to be the decreased sequence of l_i, $1 \le i \le c(\pi)$.

Theorem 1.2.6. *Two permutations are conjugate if, and only if, they have a same cyclic type.*

Proof. The necessity is obvious because of eq. (1.2.4) for cyclic partition representation of permutations. Conversely, for any two permutations with a same cyclic type, assume with one cycle each without generality as $\pi = (x_1, x_2, \ldots, x_r)$ and $\sigma = (y_1, y_2, \ldots, y_r)$, it is seen from eq. (1.2.4) that let

$$\tau = \begin{pmatrix}x_1\ x_2\ \ldots\ x_r\\ y_1\ y_2\ \ldots\ y_r\end{pmatrix},$$

then $\tau\pi\tau^{-1} = \sigma$. Therefore, π and σ are conjugates. □

Two particular cases should be mentioned for conjugate pair $\{\pi, \sigma\}$ of permutations. One is for $\pi = \sigma$ and the other, $\pi = \sigma^{-1}$. The former is called *self-conjugate* and the later, *inverse conjugate*. If $\pi = (x_1, x_2, \dots, x_r)$ and $\sigma = (y_1, y_2, \dots, y_r)$, then the self-conjugate is only for $\tau = 1_r$, the identity of degree r and the inverse conjugate is for

$$\tau = (x_1, x_r)(x_2, x_{r-1}) \dots (x_{\lfloor r/2 \rfloor}, x_{\lfloor r/2 \rfloor+1}).$$

Let $D = \langle a_1, a_2, \dots, a_d \rangle$ be a set with linear order $a_1 < a_2 < \cdots < a_d$. An ordered pair $\langle a_i, a_j \rangle$ is called an *inversion* if $1 \leq j < i \leq d$. Let $\mathrm{sgn}(\pi)$ denote the total number of inversions in the sequence $\prec x_1, x_2, \dots, x_d \succ$ with linear order $x_1 \prec x_2 \prec \dots \prec x_d$ for

$$\pi = \begin{pmatrix} d_1 & d_2 & \dots & d_r \\ x_1 & x_2 & \dots & x_r \end{pmatrix}.$$

The permutation π is said to be *even* or *odd* accordingly as $\sin(\pi)$ is even or odd. The mapping $(-1)^{\mathrm{sgn}(\pi)}$ from a permutation π to $\{1, -1\}$ is called the *parity* of π. A cycle of length 2 is called *transposition*. A transposition (x_i, x_j), assume $x_i < x_j$ and $i < j$ without loss of generality, is always an odd permutation because of odd number of inversions as $\langle x_j, x_i \rangle$ with pairs (x_j, x_k) and (x_k, x_i) for $i < k < j$.

By observing that a cycle

$$\begin{aligned}(a_1, a_2, \dots, a_l) = (a_1, a_l)(a_1, a_{l-1}) \cdots \\ (a_1, a_3)(a_1, a_2),\end{aligned} \tag{1.2.5}$$

any permutation can be represented by a composite of transpositions.

Because for $1 \leq j < k < l$,

$$(a_j, a_{k+1}) = (a_k, a_{k+1})(a_j, a_k)(a_k, a_{k+1}), \tag{1.2.6}$$

the transposition representations of a permutation may have different numbers of transpositions. A transposition in form as a_i, a_{i+1}, $1 \leq i < l$, is said to be *adjacent*.

Theorem 1.2.7. *Any permutation π of degree at least 2 has an adjacent transposition representation of the same congruent number of transpositions modulo 2 as $\mathrm{sgn}(\pi)$.*

Proof. First, we show the existence of such a representation. By virtue of eqs. (1.2.5) and (1.2.6), an adjacent transposition representation can be found. Then, by considering that a transposition and the two sides of eq. (1.2.6) have all an odd number of inversions, such a representation has its total number of inversions the congruent number of transpositions as $\mathrm{sgn}(\pi)$. □

Theorem 1.2.8. *For any two permutations π and σ,*

$$\operatorname{sgn}(\pi\sigma) = \operatorname{sgn}(\pi) + \operatorname{sgn}(\sigma) \pmod 2). \tag{1.2.7}$$

Proof. Since each transposition involves odd number of inversions, from Theorem 1.2.7, expression (1.2.7) holds. □

By virtue of eq. (1.2.6), we have

$$(-1)^{\operatorname{sgn}(\pi\sigma)} = (-1)^{\operatorname{sgn}(\pi)}(-1)^{\operatorname{sgn}(\sigma)}. \tag{1.2.8}$$

i.e., the parity of composite of two permutations is the product of their parities.

Theorem 1.2.9. *All transposition representations of a permutation have the same parity of the permutation.*

Proof. A direct conclusion of Theorem 1.2.8 in the case that one of π and σ is the identity. □

1.3 Graphs and networks

A *graph* denoted by $G = (V, E)$ is a set V, the *vertex set* whose elements are called *vertices*, with a binary relation $E \subseteq V * V = \{(u, v) \mid \forall u, v \in V, u \neq v\}$. Here, $(u, v) = (v, u)$. E is said to be an *edge set* whose elements are called *edges*. Occasionally, (u, u) and repetition of an element in E are allowed to be called a *loop* and a *multi-edge*, respectively. $|V|$ is the *order* of G, which is denoted by ν, and $|E|$, the *size* denoted by ϵ. Of course, only *finite* graphs, which are those of finite order, are considered without specific explanation in this book. The graph whose edge set is $V * V$ is called a *complete graph* denoted by K_ν, or simply K when without confusion. If a graph $H = (V(H), E(H))$ satisfies $V(H) \subseteq V$ and $E(H) \subseteq E$, then it is called a *subgraph* of G denoted by $H \subseteq G$. It is easily seen that all graphs are subgraphs of a complete graph and that the *empty graph* denoted by $\emptyset$ as well is a subgraph of any graph. A graph without an edge is an *isolated graph* and the graph with a single vertex, *trivial graph*.

Theorem 1.3.1. $\forall V_1 \subseteq V_2, E_1 \subseteq E_2$,

$$(V_1, E_1) = G_1 \subseteq G_2 = (V_2, E_2) \Leftrightarrow E_1 \subseteq V_1 * V_1. \tag{1.3.1}$$

Similarly to the case for sets in Section 1.1, we can define the operations: *union* and *intersection* as follows: $\forall G_1 = (V_1, E_1), G_2 = (V_1, E_2) \subseteq K$,

$$G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2); \tag{1.3.2}$$

$$G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2). \tag{1.3.3}$$

It is easily shown that $(2^K, \subseteq)$, 2^K is the set of all subgraphs of K, is a poset with the idempotent, commutative, associative and absorption laws for $\cup$ and $\cap$ defined earlier. Therefore, from Theorem 1.1.13, $(2^K, \subseteq; \cup, \cap)$ is a lattice.

For an edge $e = (u, v) \in E$, u and v are said to be *adjacent*, or simply write "u adj v", and e is said to be *incident with* u or v, or write "e ind u" or "e ind v". Conversely, u or v is said to be *incident to* e, or write "u ind e" or "v ind e" as well. An edge can be considered to consist of two *semi-edges*: $[u, v)$ and $(u, v]$. The *valency* of vertex v, denoted by $\rho(v)$, is the number of semi-edges incident with v. A vertex is *odd* if $\rho(v) = 1 \pmod 2$; otherwise, *even*. A vertex of valency k is said to be *k-valent* for $k \geq 0$. A 0-valent vertex is called an *isolated* vertex. An *articulate* vertex is 1-valent.

Theorem 1.3.2. *In a graph, the number of odd vertices is even.*

A subgraph H of G is called a *vertex-induced* subgraph denoted by $H = G[V(H)]$ if $E(H) = \{(u, v) \mid \forall u, v \in V(H), (u, v) \in E\}$. If a subgraph H of G satisfies that $V(H) = \{v \mid \exists e \in E(H), v \text{ ind } e\}$, then it is called an *edge-induced* subgraph denoted by $H = G[E(H)]$. We may see that $\forall H \subseteq G$,

$$H = G[V(H)] \Leftrightarrow \forall u, v \in V(H),\ \neg e = (u, v) \in E \setminus E(H)$$

and

$$H = G[E(H)] \Leftrightarrow \neg v \in V(H),\ \rho_H(v) = 0.$$

Let $2^{[G;v]}$ and $2^{[G;e]}$ be the sets of all vertex- and edge-induced subgraphs of G, respectively. It is easily shown from inspection of the three laws for partial order in Section 1.1 that both $(2^{[G;v]}, \subseteq)$ and $(2^{[G;e]}, \subseteq)$ are posets. Further, both $(2^{[G;v]}, \subseteq)$ and $(2^{[G;e]}, \subseteq)$ are lattices, although the union and the intersection of induced subgraphs are not closed on them in general.

A *trail* between two vertices u and v in G denoted by $T_{rl}(u, v)$ is a sequence of edges $e_1, e_2, \ldots, e_l$, such that $e_i = (v_i, v_{i+1}), i = 1, 2, \ldots, l,\ u = v_1, v = v_{l+1}$. Here, l is called the *length*. When $u = v$, the trail $T_{rl}(u, v)$ is called a *travel* denoted by $T_{rl}(u)$, or simply T_{rl}. If all the edges in $T_{rl}(u, v)$ are distinct, then the trail is called a *walk*, denoted by $T_r(u, v)$. When $u = v$, the walk $T_r(u, v)$ is called a *tour*, denoted by $T_r(u)$, or simply T_r. If the edge-induced subgraph $H = G[E(T_r(u, v))]$ satisfies that $(\rho_H(u) = \rho_H(v) = 1) \wedge (\rho_H(v_i) = 2, i = 1, 2, \ldots, l-1)$, then the walk is called a *path*, denoted by $P(u, v)$. When $u = v$, the path

$P(u, v)$ is a *circuit* denoted by $C(u)$, or C. Of course, walks and paths can be both seen as edge-induced subgraphs. Two vertices are said to be *connected* if there is a path between them. If all pairs of vertices in G are connected, then G is a *connected graph*. It is easy to check by the reflective, symmetry and transitive laws in Section 1.1 that the connectedness between two vertices is an equivalence on the vertex set, which is denoted by $\sim_c$.

Theorem 1.3.3. *A graph $G = (V, E)$ is connected if, and only if, $| V/\sim_c| = 1$.*

Let $\sigma =| V/ \sim_c|$, which is called the *number of components* of G. For a vertex v, we define $G - v = (V \setminus \{v\}, E \setminus E_v)$, where $E_v = \{e \mid \forall e \in E, e \ \text{ind} \ v\}$. A vertex v is called a *cut-vertex* if $\sigma(G - v) > \sigma$. Similarly, a *cut-edge* $e : \sigma(G - e) > \sigma, \ G - e = (V, E \setminus \{e\})$. A *tree* is such a graph that it is connected and is of least size. We may show that all trees of order v are of the same size, which is $v - 1$. A graph whose components are all trees is called a *forest*.

Theorem 1.3.4. *A graph of order v is a tree if, and only if, its size is $v - 1$ and all its edges are cut edges.*

A graph that has neither isolated vertex nor cut vertex is called a *block*, or a *non-separable* one. It is obvious from inspection of O1, Õ2 and O3 in Section 1.2 that the statement "two edges are on the same circuit" defines an equivalence denoted by $\sim_b$ on the edge set of a graph.

Theorem 1.3.5. *A graph without isolated vertex is non-separable if, and only if, $|E/\sim_b|= 1$.*

A subgraph H of G is said to be of *spanning* if $V(H) = V$. A spanning circuit is called a *Hamiltonian* circuit and a spanning tour on which each edge of the graph occurs, a *Eulerian* tour in the graph. If a graph has a Hamiltonian circuit, or a Eulerian tour, then it is a *Hamiltonian*, or *a Eulerian graph*, respectively.

Theorem 1.3.6. *A connected graph is Eulerian if, and only if, all the valencies of its vertices are even.*

For a graph G, if $V = A + B$ (i.e., $A \cup B$ provided $A \cap B = \emptyset$) and both $G[A]$ and $G[B]$ are isolated graphs, then G is called a *bipartite graph* denoted by $G = (A, B; E)$. If $E = \{e = (u, v) \mid \forall(u \in A)(v \in B)\}$, then the bipartite graph $(A, B; E)$ is called a *complete* one denoted by $K_{\alpha,\beta}$, where $\alpha =| A |$ and $\beta =| B |$.

Theorem 1.3.7. *A graph is bipartite if, and only if, it is without a circuit of odd length.*

If any pair of elements in a subset of V or E is not adjacent, then the subset is said to be *independent*. An independent subset of E for a graph $G = (V, E)$ is also called a *matching*. If a matching induces a spanning subgraph of G, then it is said to be *perfect*. For $a \in V$, let $N_a = \{v \mid \forall v \in V, v \text{ adj } a\}$ and for $A \subseteq V$, let

$$N(A) = \bigcup_{a \in A} N_a \backslash A.$$

Theorem 1.3.8. *A bipartite graph $G = (X, Y; E)$ has a perfect matching if, and only if, $\forall A \subseteq X$ and $\forall A \subseteq Y, \mid N(A) \mid \geq \mid A \mid$.*

It is known that any graph can be realized as a subset of 3-Euclidean space such that the edges are represented by sections of curves (in fact, straight segments here) any of whose pairs is without common point except only for the end points of the sections, which represent the common end of the corresponding edges. Such a representation of a graph is called an *embedding* in the space. However, not all graphs have an embedding in the plane, or 2-Euclidean space. If a graph has an embedding in the plane, then it is said to be *planar*.

A *bisection* is an operation of transforming $G = (V, E)$ into a graph $(V + \{w\}, (E \setminus \{(u, v)\}) + \{(u, w), (w, v)\})$. If a graph can be obtained from another one by a series of bisections and/or the inverses, then the two graphs are said to be *homeomorphic*.

Theorem 1.3.9. *A graph is planar if, and only if, it has no subgraph homeomorphic to K_5 or $K_{3,3}$.*

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be *isomorphic* if there is a bijection $\tau : V_1 \to V_2$ such that

$$\forall u, v \in V_1,\ (u, v) \in E_1 \Leftrightarrow (\tau(u), \tau(v)) \in E_2. \tag{1.3.4}$$

The bijection τ defined by eq. (1.3.4) is called an *isomorphism* between G_1 and G_2. An *automorphism* of G is an isomorphism between G and itself. It would be the most difficult problem among those are mentioned to justify if two graphs are isomorphic in general.

Similarly, a *digraph* (or a directed graph) denoted by $D = (V, A)$ is a set V, which is also called the *vertex set*, with a binary relation $A \subseteq V \times V = \{\prec u, v \succ \mid \forall u \in V, \forall v \in V\}$, which is called the *arc set*. All the above discussions have analogues in the directed case. Particularly, a poset $P = (M; \preceq)$ can be represented by a digraph $Dos = (M, Aos)$, where $\prec x, y \succ \in Aos \Leftrightarrow (x \preceq y) \wedge (\neg z, x \prec z \prec y)$, or say x is *covered* by y for $x, y \in M$. If a graph of order v is associated with an injection (almost in any case, a bijection) from its vertex set to (onto) the integer set $(\{1, 2, \ldots, v\})$, then it is said to be *labelled*. The injection is called the *labelling*. The image of a vertex under the labelling is called its *label*. Of course, an isomorphism between labelled (directed) graphs has to be considered with the labels on vertices (directions on edges).

A *network* N is such a graph $G = (V, E)$ with a real function $w(e) \in \mathbf{R}, e \in E$ on E, and hence write $N = (G; w)$. Usually, a network N is denoted by the graph G itself if no confusion occurs.

Finite recursion principle On a finite set A, choose $a_0 \in A$ as the initial element at the 0th step. Assume a_i is chosen at the ith, $i \geq 0$, step with a given rule. If not all elements available from a_i are already chosen, choose one of them as a_{i+1} at the $i+1$st step by the rule, then a chosen element will be encountered in finite steps unless all available elements of A have been chosen.

Finite restrict recursion principle On a finite set A, choose $a_0 \in A$ as the initial element at the 0th step. Assume a_i is chosen at the ith, $i \geq 0$, step with a given rule. If a_0 is not available from a_i, choose one of elements available from a_i as a_{i+1} at the $i+1$st step by the rule, then a_0 will be encountered in finite steps unless all available elements of A are chosen.

The two principles above are very useful in finite sets, graphs and networks, even in a wide range of combinatorial optimizations.

Let $N = (G; w)$ be a network where $G = (V, E)$ and $w(e) = -w(e) \in Z_n = \{0, 1, \ldots, n-1\}$, i.e., mod n, $n \geq 1$, integer group. For example, $Z_1 = \{0\}$, $Z_2 = B = \{0, 1\}$, and so on. Suppose $x_v = -x_v \in Z_n$, $v \in V$, are variables. Let us discuss the system of equations

$$x_u + x_v = w(e) \pmod n, \ e = (u, v) \in E \tag{1.3.5}$$

on Z_n.

Theorem 1.3.10. *System of equations (1.3.5) has a solution on Z_n if, and only if, there is no circuit C such that*

$$\sum_{e \in C} w(e) \neq 0 \pmod n \tag{1.3.6}$$

on N.

Proof. Necessity. Assume C is a circuit satisfying eq. (1.3.6) on N. Because the restricted part of eq. (1.3.5) on C has no solution, the whole system of equations (1.3.5) has to have no solution either. Therefore, N has no such circuit. This is a contradiction to the assumption

Sufficiency. Let $x_0 = a \in Z_n$, start from $v_0 \in V$. Assume $v_i \in V$ and $x_i = a_i$ at step i. Choose one of $e_i = (v_i, v_{i+1}) \in E$ without used (otherwise, backward 1 step as the step i). Choose v_{i+1} with $a_{i+1} = a_i + w(e_i)$ at step $i+1$. If a circuit such as $\{e_0, e_1, \ldots, e_l\}$, $e_j = (v_j, v_{j+1}), 0 \leq j \leq l, v_{l+1} = v_0$, occurs within a permutation of indices, then from eq. (1.3.6)

$$
\begin{aligned}
a_{l+1} &= a_l + w(e_l)\\
&= a_{l-1} + w(e_{l-1}) + w(e_l)\\
&\dots\\
&= a_0 + \sum_{j=0}^{l} w(e_j) = a_0.
\end{aligned}
$$

Because the system of equations obtained by deleting all the equations for all the edges on the circuit from eq. (1.3.5) is equivalent to the original system of equations (1.3.5). By virtue of the finite recursion principle a solution of eq. (1.3.5) can always be extracted. □

When $n = 2$, this theorem has a variety of applications. In Ref. [194], where Theorem 1.3.7 is a special case, some applications can be seen. Further, its extension on a non-Abelian group can also be done while the system of equations are not yet linear but quadratic.

A graph is said to be *even* if the valency of each vertex is even.

Theorem 1.3.11. *A graph is even if, and only if, its edges set has a cycle partition.*

Proof. Since what is obtained from an even graph by deleting all the edges on a cycle is still an even graph, based on the finite recursion principle, the theorem is done. □

Let $G = (V, E)$ be a graph where $V = \P(X)$, and $E = \{Bx | x \in X\}$ where $\P(X)$ is a partition on

$$B(X) = \bigcup_{x \in X} Bx$$

and $Bx = \{x(0), x(1)\}$ for a set X. Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if, and only if, there exists a bijection $\iota\colon X_1 \to X_2$ such that the diagrams

$$
\begin{tikzcd}
X_1 \arrow[r, "\iota"] \arrow[d, "\sigma_1"'] & X_2 \arrow[d, "\sigma_2"] \\
X_1 \arrow[r, "\iota"'] & X_2
\end{tikzcd}
\tag{1.3.7}
$$

for $\sigma_i = B_i, \P_i$, $i = 1, 2$, are commutative. Let $\mathrm{Aut}(G)$ be the automorphism group of G.

On the other hand, a *semi-arc isomorphism* between two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is defined to be such a bijection $\tau\colon B_1(X_1) \to B_2(X_2)$ that

$$\begin{tikzcd} B_1(X_1) \arrow[r, "\tau"] \arrow[d, "\sigma_1"'] & B_2(X_2) \arrow[d, "\sigma_2"] \\ B_1(X_1) \arrow[r, "\tau"'] & B_2(X_2) \end{tikzcd} \tag{1.3.8}$$

for $\sigma_i = B_i, \P_i$, $i = 1, 2$, are commutative. Let $\mathrm{Aut}_{1/2}(G)$ be the semi-arc automorphism group of G.

Theorem 1.3.12. *If* $\mathrm{Aut(G)}$ *and* $\mathrm{Aut}_{1/2}(\mathrm{G})$ *are, respectively, the automorphism and semi-arc automorphism groups of graph G, then*

$$\mathrm{Aut}_{1/2}(\mathrm{G}) = \mathrm{Aut(G)} \times S_2^l, \tag{1.3.9}$$

where l is the number of self-loops on G and S_2 is the symmetric group of degree 2.

Proof. Since each automorphism of G just induces two semi-arc isomorphisms of G for a self-loop, the theorem is true. □

1.4 Groups and spaces

If a *group* denoted by $\Gamma = (X, \lozenge)$ is a set X with a binary operation $\gamma{:}X \times X \to X$, it would be better to write $x\lozenge y$ for $\prec x, y \succ \gamma$ referring to "$\lozenge$" as the operation, such that the laws $\Gamma1$, $\Gamma2$ and $\Gamma3$ described below are satisfied.

Γ1 (Associative law) $\forall x, y, z \in X$, $(x\lozenge y)\lozenge z = x\lozenge(y\lozenge z)$.

Γ2 (Identity law) $(\exists 1_\Gamma$ (or simply 1) $\in X)(\forall x \in X, x\lozenge 1_\Gamma = x)$.

Γ3 (Inverse law) $(\forall x \in X)(\exists y \in X, x\lozenge y = 1_\Gamma)$.

The element 1 in Γ2 is called a *right identity* and the element y in Γ3 a *right inverse* of x. We may also define a *left identity* and a *left inverse* of an element. However, it is easily shown that they are all unique and the left one equals to the right. So we are allowed to call 1 the *identity* and x^{-1} the *inverse* of x.

The *order* of a group $\Gamma = (X, \lozenge)$ is defined to be $|\,\Gamma\,| = |X|$. We can see that $(1, \lozenge)$ is a group, which is called the *trivial group* or the *identity group*. In this book, a group $\Gamma = (X, \lozenge)$ is always written as $\Gamma = X$ without specific indication. If a group Γ satisfies the condition Γ4 below, then it is said to be *Abelian*.

Γ4 (Commutative law) $\forall x, y \in \Gamma$, $x\lozenge y = y\lozenge x$.

There are two commonly used ways of writing the group operation of Γ. One is the *additive* notation by writing $x\lozenge y$ as a "sum" $x + y$ with the identity 0_Γ (or 0) and the inverse $-x$ of x especially for Abelian groups. The other is the *multiplicative* notation by using a "product" $x \bullet y$ (or xy), 1_Γ (or 1) and x^{-1} as $x \Diamond y$, the identity and the inverse of x, respectively, for general groups.

Let $\Gamma = (X, \bullet)$ be a group and let $Y \subseteq X$. If $\Lambda = (Y, \bullet)$ is a group, then Λ is called a *subgroup* of Γ, denoted by $\Lambda \subseteq \Gamma$. Of course, the identity group is a subgroup of any group and a group is a subgroup of itself.

Theorem 1.4.1. $\forall Y,\ \emptyset \neq Y \subseteq X, \Lambda = (Y, \bullet) \subseteq \Gamma = (X, \bullet) \Leftrightarrow (\forall x, y \in Y)(xy^{-1} \in Y)$.

Let $\Gamma_i = (X_i, \bullet) \subseteq \Gamma = (X, \bullet), i \in I$. It is easily seen that $\cap_{i\in I}\Gamma_i = (\cap_{i\in I}X_i, \bullet) \subseteq \Gamma$, which is called the *intersection*. For an $S \subseteq X$, the intersection, denoted by $\langle S \rangle$, of all subgroups that contains S is called the subgroup *generated* by S in Γ. The subgroup $\langle \cup_{i\in I}X_i \rangle$ denoted by $\cup_{i\in I}\Gamma_i$ is called the *join* of subgroups $\Gamma_i,\ i \in I$. Let $\mathbf{\Gamma}$ be the set that consists of all subgroups of Γ. Then, it is obvious from inspection of the laws O1–O3 in Section 1.1 and Theorem 1.1.13 that $(\mathbf{\Gamma}, \subseteq; \cup, \cap)$ is a lattice, more precisely, a complete lattice because any subset of $\mathbf{\Gamma}$ has the l.u.b., which is the intersection, and the g.l.b., which is the join, in $\mathbf{\Gamma}$.

A subgroup Λ of a group Γ is said to be *normal*, or write $\Lambda \lhd \Gamma$, if it satisfies one of the following three equivalent conditions:

$$\begin{aligned} \forall x \in \Gamma, x\Lambda = \Lambda x &\Leftrightarrow \forall x \in \Gamma, x^{-1}\Lambda x = \Lambda \\ &\Leftrightarrow \forall x \in \Gamma, \forall y \in \Lambda, x^{-1}yx \in \Lambda. \end{aligned} \tag{1.4.1}$$

It is easily seen that any subgroup of an Abelian group is normal. However, in general, there exist subgroups that are not normal for a non-Abelian group. One may also see that the set of all normal subgroups of a group forms a complete lattice with the inclusion as the order and with the intersection and the join as the two operations.

Because it can be shown that the relation, denoted by $\sim_N$:

$$x \sim_N y \ \Leftrightarrow\ \exists h \in N, x = hy, \tag{1.4.2}$$

provides an equivalence on the set X of the group $\Gamma(X, \bullet)$ for $N \lhd \Gamma$. We are allowed to define the *quotient* (or *factor*) group of N in Γ to be

$$\Gamma/N \ = \ (X/{\sim_N}, \bullet), \tag{1.4.3}$$

where $(Nx)(Ny) = N(xy)$. The order of Γ/N is called the *index* of N in Γ.

Let Γ and Λ be two groups. A function $\alpha : \ \Gamma \to \Lambda$ is called a *homomorphism* from Γ to Λ if

$$\forall x, y \in \Gamma,\ \alpha(xy) \ = \ \alpha(x)\alpha(y). \tag{1.4.4}$$

Because $o : \Gamma \to 1_\Lambda$ is a homomorphism, which is called *zero homomorphism*, the set $\mathrm{Hom}(\Gamma, \Lambda)$ of all homomorphisms from Γ to Λ is always non-empty. A homomorphism from Γ to Γ itself is said to be an *endomorphism* of Γ. The identity function $\iota : \Gamma \to \Gamma$ is an endomorphism of Γ.

For a homomorphism α from Γ to Λ, let

$$\begin{cases} \mathrm{Im}\,\alpha = \alpha(\Gamma) = \{\alpha(x) \mid \forall x \in X\}; \\ \mathrm{Ker}\,\alpha = \{x \mid \forall x \in X, \alpha(x) = 1_\Lambda\}, \end{cases} \tag{1.4.5}$$

which is said to be the *image*, the *kernel* of α, respectively. It is easy to check by Theorem 1.4.1 that $\mathrm{Im}\,\alpha \subseteq \Lambda$ and $\mathrm{Ker}\,\alpha \lhd \Gamma$. If a homomorphism α from Γ to Λ satisfies $\mathrm{Ker}\,\alpha = 1_\Gamma$, then α is called a *monomorphism*. If a homomorphism α from Γ to Λ has $\mathrm{Im}\,\alpha = \Lambda$, then α is called an *epimorphism*. A homomorphism that is both a monomorphism and an epimorphism is said to be an *isomorphism*. Two groups Γ and Λ are said to be *isomorphic*, or written as $\Gamma \cong \Lambda$, when there is an isomorphism between them. An isomorphism from Γ to Γ itself is called an *automorphism* of Γ. It can be easily shown from inspection of the laws $\Gamma1$–$\Gamma3$ that the set of all automorphisms of Γ is a group, which is called the *automorphism group* of Γ, denoted by $\mathrm{Aut}\,\Gamma$.

Theorem 1.4.2 (First isomorphism law). $\forall \alpha \in \mathrm{Hom}(\Gamma, \Lambda)$,

$$\Gamma/\mathrm{Ker}\,\alpha \cong \mathrm{Im}\,\alpha.$$

Based on Theorem 1.4.2, we are allowed to call $\Gamma/\mathrm{Ker}\,\alpha$ the *coimage* of α. If $N \lhd \Gamma$, then the mapping $\phi : x \mapsto Nx$ is an epimorphism from Γ to Γ/N with $\mathrm{Ker}\,\phi = N$. We call ϕ the *canonical homomorphism*.

For two groups $\Lambda = (Y, \bullet) \subseteq (X, \bullet) = \Gamma$, let $\Gamma\Lambda = (XY, \bullet)$, where $XY = \{xy \mid \forall x \in X, \forall y \in Y\}$. One may see that $\forall \Lambda \subseteq \Gamma, N \lhd \Gamma \Rightarrow \Lambda \cap N \lhd \Lambda$.

Theorem 1.4.3 (Second isomorphism law). $\forall \Lambda \subseteq \Gamma, \forall N \lhd \Gamma$,

$$\Lambda/N \cap \Lambda \cong N\Lambda/N.$$

Let N and Q be two normal subgroups of a group Γ and let $N \subseteq Q$. Then, it is known that $Q/N \lhd \Gamma/N$.

Theorem 1.4.4 (Third isomorphism law). $\forall N, Q \lhd \Gamma$,

$$N \subseteq Q \Rightarrow (\Gamma/N)/(Q/N) \cong \Gamma/Q.$$

Let Φ be a group, S a non-empty set and $\sigma : S \to \Phi$, a function. Then, Φ, or precisely (Φ, σ), is said to be *free* on S if for each function $\alpha : S \to \Gamma$, there is a unique homomorphism $\beta : \Phi \to \Gamma$ such that $\alpha = \beta\sigma$. A group which is free on some set is called a *free group*. From the definition it can be derived that σ is injective and that Im σ generates Φ. In fact, it can be shown that for any non-empty set S there exists a group Φ and a function $\sigma : S \to \Phi$ such that Φ is free on S and $\Phi = \langle \mathrm{Im}\, \sigma \rangle$.

Theorem 1.4.5. *If Φ_1 is free on S_1 and Φ_2 is free on S_2, then $\Phi_1 \cong \Phi_2 \Leftrightarrow \mid S_1 \mid = \mid S_2 \mid$.*

This theorem allows us to define the *rank* of a free group as the cardinality of any set on which it is free. Further, we have known that any group is an image of a free group. Such an image is called a presentation of the group. More precisely, a *free presentation* of a group Γ is an epimorphism $\pi : \Phi \to \Gamma$, Φ is a free group. From Theorem 1.4.2, we have $\Phi/\mathrm{Ker}\ \pi \cong \Gamma$. The elements of Ker$\pi$ are called *relators* of the presentation. Therefore, any group can be characterized by generators and relaters. Although a presentation of a group is known, to justify if two groups are isomorphic in general is still not easy because a group may have different kinds of presentations.

A *space* (or precisely a *vector space* or *linear space*) over $\boldsymbol{F}$ denoted by $(\mathscr{X}, \boldsymbol{F}; +, \bullet)$ (or simply write $\mathscr{X}$) is an Abelian group $(\mathscr{X}, +)$, or $\mathscr{X}$ as well, associated with a field $(\boldsymbol{F}, +, \bullet)$, or simply $\boldsymbol{F}$, and two binary operations: "+", called the *sum* and "•", the *scalar product*, satisfying the following four axioms: Vects.1–4. The sum is with the same symbol as the addition on the group $\mathscr{X}$ and the addition on the field $\boldsymbol{F}$. The scalar product $a \bullet A$, or simply aA, is defined for $a \in \boldsymbol{F}$ and $A \in \mathscr{X}$ and is with the same symbol as the multiplication on $\boldsymbol{F}$. Members of $\mathscr{X}$ are called *vectors*, and those of $\boldsymbol{F}$, *scalars*.

Vect.1 $\forall a \in \boldsymbol{F}, \forall A, B \in \mathscr{X}, a(A + B) = aA + aB.$

Vect.2 $\forall a, b \in \boldsymbol{F}, \forall A \in \mathscr{X}, (a + b)A = aA + bA.$

Vect.3 $\forall a, b \in \boldsymbol{F}, \forall A \in \mathscr{X}, (ab)A = a(bA).$

Vect.4 $\forall A \in \mathscr{X}, 1A = A.$

It seems that the only notational distinction we have to make between vectors and scalars is to denote the zero elements of $\mathscr{X}$ and $\boldsymbol{F}$ by $0_{\mathscr{X}}$ and $0_{\boldsymbol{F}}$, respectively. However, since it is easily shown, from the axioms Vects.1–4, that $\forall A \in \mathscr{X}, 0_{\boldsymbol{F}}A = 0_{\mathscr{X}}$ and that $\forall a \in \boldsymbol{F}$, $a0_{\mathscr{X}} = 0_{\mathscr{X}}$, the distinction will almost always be dropped and $0_{\boldsymbol{F}}$, $0_{\mathscr{X}}$ be written simply 0.

A subset $\mathscr{Y} \subseteq \mathscr{X}$ of a space $\mathscr{X}$ over $\boldsymbol{F}$ is said to be a *subspace* , denoted by $\mathscr{Y} \subseteq_{\mathrm{vect}} \mathscr{X}$ (or simply $\mathscr{Y} \subseteq \mathscr{X}$ without confusion), of $\mathscr{X}$ if $\mathscr{Y}$ is a space over $\boldsymbol{F}$ in its own right, but with respect to the same operations as $\mathscr{X}$. The zero vector 0 belongs to any space and itself is a space called the *zero space* or *trivial space* denoted by 0 as well. Any non-zero vector of order 2 with 0 here forms a subspace, which is denoted by $\mathscr{I}$.

Theorem 1.4.6. $\forall \mathscr{Y} \subseteq \mathscr{X}, \mathscr{Y} \subseteq_{\mathrm{vect}} \mathscr{X} \Leftrightarrow$

$$(\forall A, B \in \mathscr{Y}, A + B \in \mathscr{Y}) \wedge (\forall a \in \boldsymbol{F},\ \forall A \in \mathscr{Y}, aA \in \mathscr{Y}).$$

Proof. The necessity is straight forward. Conversely, because $\mathscr{Y} \subseteq \mathscr{X}$, from the last statement, Vects.2–4 hold and from the first statement, Vect.1 holds for $\mathscr{Y}$. The sufficiency is obtained. □

Apparently, for spaces we are also allowed to introduce the two operations: $\cap$, the intersection, and $\cup$, the join, as described before for groups and find that $(2^{\mathscr{X}}, \subseteq; \cup, \cap)$ forms a lattice, of course, a complete one.

In what follows, we are only concerned with the field $\boldsymbol{F} = \mathrm{GF}(2)$, the finite field of two elements for spaces. In this case, the space is called a *binary space*. For any $A \in \mathscr{X}$, we always have $A + A = 0$, the zero vector. That is of characteristic 2. Suppose $\mathscr{X} = 2^X$ is the free Abelian group $\langle x | \forall x \in X \rangle$ generated by all the elements of X. Then, a vector is also a subset of X. We always employ the same symbol to denote a vector of $\mathscr{X}$ and a subset in X. Let $A \in \mathscr{X}$, then

$$A = \sum_{x \in X} A_x x = \sum_{x \in A} x, \tag{1.4.6}$$

where A_x is said to be the *coefficient*, or *component* of A on x. Of course, $A_x = 1$, if $x \in A$; 0, otherwise.

On the space $\mathscr{X}$, we define an *inner product* denoted by (A, B) for $A, B \in \mathscr{X}$ as

$$(A, B) = \sum_{x \in X} A_x B_x. \tag{1.4.7}$$

By this notation, we have the relation:

$$A_x = (A, x), \forall x \in X. \tag{1.4.8}$$

If for $A, B \in \mathscr{X}$, $(A, B) = 0$, then A and B are said to be *orthogonal* denoted by $A \perp B$ or $B \perp A$ from the symmetry: $(A, B) = (B, A)$. Here, one may see

$$\forall A, B \in \mathscr{X}, (A, B) = 0 \Leftrightarrow |A \cap B| = 0 \pmod 2. \tag{1.4.9}$$

If $(A, A) = 0$, then vector A is said to be *even*. Let $\mathscr{A}(\mathscr{X})$ be the set of all even vectors in $\mathscr{X}$. It can be seen from inspection of axioms Vects.1–4 that $\mathscr{A}(\mathscr{X})$ is a subspace of $\mathscr{X}$ and is called the *alternating* (or *symplectic*) space on X.

Further, we may also see that for $A \in \mathscr{X}$ given,

$$A = 0 \Leftrightarrow \forall B \in \mathscr{X}, (A, B) = 0. \tag{1.4.10}$$

Or, in other words, the inner product is *non-degenerate*.

If a vector A satisfies that $\forall B \in \mathscr{B}, (A, B) = 0$, then it is said to be *orthogonal* to $\mathscr{B}$ and denoted by $A \perp \mathscr{B}$.

Let $\mathscr{A}$ and $\mathscr{B}$ be two subspaces of $\mathscr{X}$. If

$$\mathscr{A} = \{A | \ \forall A \in \mathscr{X}, A \perp \mathscr{B}\}, \tag{1.4.11}$$

then $\mathscr{A}$ is said to be the *orthogonal space* of $\mathscr{B}$ in $\mathscr{X}$ and is denoted by $\mathscr{A} = \mathscr{B}^{\perp}$. Moreover, from the symmetry of the inner product, we have

$$(\mathscr{B}^{\perp})^{\perp} = \mathscr{B}. \tag{1.4.12}$$

In Chapters 4 and 8, we shall see a number of spaces related to graphs. Almost all results for them can be extended to general spaces over GF(2), the finite field of two elements.

1.5 Notes

1.5.1 This book is in principle designed to be self-contained in the background presented in this chapter. One might still like to read more materials related to topology. References can be chosen such as Alexandroff [5], Giblin [97], Greenberg [107], Massey [269], Stillwell [321], Agoston [2], or Lefschetz [173], Williams [398].

1.5.2 Permutations are established from partitions on a set. Such an idea enables us to observe embeddings, or super maps of a graph as permutations, from the graph as a partition. A description in a certain detail can be seen in Liu [218, 219, 224, 234]. Most books on basic algebra involve permutations such as Jacobson [157], Gilbert [98], particularly Dixon and Mortimer [73].

1.5.3 A graph turns out a partition of the ground set from a set by a binary group sticking on from Liu [218]. Although a great number of books on graphs have appeared in literature as Bellman et al. [24], Berge [25], Biggs [31], Bondy and Murty [35], Capobianco and Molluzzo [39], Chan [41], Chen [43], Fiorini and Wilson [86], Golumbic [105], Harary [122], Iri [156], Kaufman [162], Kuo [165], Lovasz [252], Tutte [347, 350], Zykov [427], et al. Only a few, more or less, directly related to this book are listed as Ore [273], Tutte [349], Ringel [286], White [382], Lefschetz [172], Wu [404] and Liu [216, 217], particularly more popular book: Gross and Tucker [108].

1.5.4 Those mentioned in Section 1.4 are all extracted from Liu [216]. One might also like to read more about general groups such as MacLane and Birkhoff [263], and Robinson [292].

1.5.5 One might see that the vector space can be generalized to an abstract linear space from, for an example, Theorem 1.2.1 in Liu [237]. To read more about linear space (advanced) is referred to Roman [293].

1.5.6 By considering theoretical efficiency, basic knowledge of data structure, algorithm and complexity should be known. The reader is suggested to read from Aho et al. [3], Pralts [277], Garey and Johnson [95] if necessary.

2 Polyhedra

2.1 Polygon double covers

A *polygon*, denoted by $(a, b, c, \dots)$, is a finite set of letters in a cyclic order. In general, such a polygon can be represented by the infinite face of a connected plane graph conformed with convex polygons and articulate edges, or the inner face of a regular polygon. Hence, the letters in a polygon are allowed with repetition of each letter at most twice (with the same power or different powers: 1 always omitted and -1) in the first case. For a letter a, a^{-1} is called the *inverse* of a. The inverse satisfies the following two rules:

Inverse rule 1 For a letter a, $(a^{-1})^{-1} = a$.

Inverse rule 2 For two letters a and b, $(ab)^{-1} = b^{-1}a^{-1}$, or $(a, b)^{-1} = (b^{-1}, a^{-1})$.

Two polygons A_1 and A_2 are dealt with the same if one becomes the other by one of the following alternatives:

No.diff.gon1 For $a \in A_1$, A_2 is different from A_1 only in interchanging the positions of the two occurrences of a, if any.

No.diff.gon2 For $a, b \in A_1$, A_2 is different from A_1 only in interchange between a and b.

Let polygon $A = (a_1, a_2, \dots, a_l)$, then polygons $(a_2, a_3, \dots, a_1), \dots, (a_l, a_1, \dots, a_{l-1})$ are, respectively, called *cyclic left shift* of A in $1, 2, \dots, l-1$ bits.

No.diff.gon3 A_2 is any of all the cyclic left shifts of A_1.

The polygon $(a_l, \dots, a_2, a_1)$ is called a *reversion*, denoted by $(a_1, a_2, \dots, a_l)^{\mathrm{rv}}$, of polygon $(a_1, a_2, \dots, a_l)$.

No.diff.gon4 $A_2 = (A_1)^{\mathrm{rv}}$.

The polygon $(a_1^{-1}, a_2^{-1}, \dots, a_l^{-1})$ is called a *conversion*, denoted by $(a_1, a_2, \dots, a_l)^{\mathrm{cv}}$, of polygon $(a_1, a_2, \dots, a_l)$.

No.diff.gon5 $A_2 = (A_1)^{\mathrm{cv}}$.

An *inversion* of polygon $A = (a_1, a_2, \dots, a_l)$ is defined to be $A^{\mathrm{iv}} = (a_l^{-1}, \dots, a_2^{-1}, a_1^{-1})$.

Proposition 2.1.1. *For any polygon A,*

$$A^{iv} = (A^{rv})^{cv} = (A^{cv})^{rv}. \tag{2.1.1}$$

Proof. Easy to check by the definitions. □

On the basis of this proposition, it is from the inverse rule 2 seen that $A^{\mathrm{iv}} = A^{-1}$.

DOI 10.1515/9783110479492-002

If a set of polygons has each letter occuring exactly twice, then it is called a *double cover* on the set of all letters in the polygons.

A *polyhedron* P is a set $\mathscr{C} = \{C_i|1 \le i \le k\}$, $k \ge 1$, of polygons which forms a double cover on a set A of letters, where C_i is called a *face* of P such that no proper subset of $\mathscr{C}$ is a double cover of a subset of A.

This is the combinatorial representation of Heffter's in Heffter [133] (1891, and more than half a century later, Edmonds' in Edmonds [83] as dual case) for a polyhedron.

Let $P = \{C_i|1 \le i \le k\}$ be a polyhedron and $X = X_P$, the set of all letters in P. An element (or letter) in X is called an *edge* of P. The property that the two occurrences of a letter with the same or different directions in a polyhedron is called the *status* of the edge. By sticking the group B of two elements on X, each edge consists of two *semi-edges* as $\{x^+, x^-\}$, or written as $\{+x, -x\}$, $\{x, x^{-1}\}$, or $\{x, \bar{x}\}$ for certain convenience. Each semi-edge $+x$, or $-x$, is compounded with its copy marked by a prime, i.e. $+x'$ or $-x'$ (x', or $x^{-1'}$) respectively. Then, an edge is further considered as

$$\{+x, +x', -x, -x'\} \text{ or simply, } \{x, x', x^{-1}, x^{-1'}\}$$

and hence $\{+x, +x'\}$ or $\{-x, -x'\}$ as well is now a semi-edge. The set

$$\mathscr{X}(P) = \sum_{x \in P} (\{x^+, x^{-1}\} + \{x+, x^{-1}\}') \tag{2.1.2}$$

is called a *ground set* of P. An element of the ground set is also called a *quarter* (of an edge).

Attention 2.1.1.

(1) For $x \in X$ and $x \in \mathscr{X}_P$, x has different meanings. The former is a letter and the latter, a quarter of an edge.

(2) For $x \in \mathscr{X}_P$, both $'$ and $^{-1}$ are seen as permutations on the ground set, i.e.,

$$' = \prod_{x \in X + X^{-1}} (x, x') \quad \text{and} \quad ^{-1} = \prod_{x \in X + X'} (x, x^{-1}), \tag{2.1.3}$$

where $X' = \{x' | \forall x \in X\}$ and $X^{-1} = \{x^{-1} | \forall x \in X\}$ for $X \subseteq \mathscr{X}_P$.

(3) For $x, y \in \mathscr{X}_P$, $(xy)' = y'x'$, $(xy)^{-1} = y^{-1}x^{-1}$, and $x'^{-1} = x^{-1'}$.

A face A in polyhedron P is seen in companion with A^{-1} on its ground set.

Proposition 2.1.2. *Let P be a polyhedron with its face set $\mathscr{A}$. Then, P is determined by the permutation π_P on its ground set as*

$$\pi_P = \prod_{A \in \mathscr{A}} (A)(A^{-1}), \tag{2.1.4}$$

in which two occurrences of a letter with the same power are distinguished by one with a prime.

Proof. By observing that all cycles appearing in eq. (2.1.4) form a partition, in view of Section 1.2 the conclusion is seen. □

Let $\sigma = {}'$ and $\delta = {}^{-1}$ be the permutations shown in eq. (2.1.3) on the ground set $\mathcal{X}_P$, i.e. for $x \in \mathcal{X}_P$,

$$\sigma(x) = \begin{cases} y', & \text{when } x = y; \\ y, & \text{when } x = y' \end{cases} \tag{2.1.5}$$

and for $x \in \mathcal{X}_P$,

$$\delta(x) = \begin{cases} y^{-1}, & \text{when } x = y; \\ y, & \text{when } x = y^{-1} \end{cases} \tag{2.1.6}$$

Then, $\pi_P^* = \pi_P \sigma \delta$ is a permutation on $\mathcal{X}_P$ as well.

Lemma 2.1.1. *On $\mathcal{X}_P$, $\delta \pi_P = \pi_P^{-1} \delta$.*

Proof. By virtue of $\pi_P \delta x = \pi_P x^{-1} = (\pi_P^{-1} x)^{-1} = \delta(\pi_P^{-1} x) = (\delta \pi_P) x$, by the arbitrariness of $x \in \mathcal{X}_P$ the lemma is obtained. □

Lemma 2.1.2. *On $\mathcal{X}_P$, $\sigma \pi_P^* = \pi_P^{*-1} \sigma$.*

Proof. By considering that

$$\begin{aligned} \sigma \pi_P^* &= \sigma \pi_P \sigma \delta = \sigma(\pi_P \delta)\sigma \text{ (by Lemma 2.1.1)} \\ &= \sigma(\delta \pi_P^{-1})\sigma = (\sigma^{-1} \delta^{-1} \pi_P^{-1})\sigma = \pi_P^{*-1} \sigma, \end{aligned}$$

the lemma is done. □

Lemma 2.1.3. *For $x \in \mathcal{X}_P$, two orbits $(x)_{\pi_P^*}$ and $(x')_{\pi_P^*}$ are disjoint and conjugate.*

Proof. By virtue of Lemma 2.1.2, the two orbits have the same type. From Theorem 1.2.6, they are conjugate. □

Theorem 2.1.1. *Permutation π_P^* on $\mathcal{X}_P$ determines a polyhedron.*

Proof. On the basis of Lemma 2.1.3, each pair of the conjugate orbits determine a polygon when the prime is omitted. Then, the set of all such polygons form a polyhedron. □

The polyhedron P^* obtained by omitting the power -1 and then replacing the prime by -1 from the permutation π_P^* shown in this theorem is called a *dual* of P. A face of the dual P^* is defined to be a *vertex* of P.

For a polyhedron P determined by permutation π_P on the ground set $\mathscr{X}_P$, the transposition

$$(x^{-1}, \pi_P x) = (\delta x, \pi_P x)$$

is called an *angle*. Two semi-edges incident with the same angle is said to be *V-adjacent*. Then, an equivalence called *V-adjacence* by appending the transitive law on the V-adjacent relation is obtained on the set of all semi-edges.

Theorem 2.1.2. *A set of semi-edges of a polyhedron forms a vertex if, and only if, it is an equivalent class under V-equivalence.*

Proof. In fact, a conjugate pair of cycles on π_P^* determines a equivalent class under V-equivalence. This is the theorem. □

Example 2.1.1. Only one polygon $(ae^{-1}b^{-1}cdefdb^{-1}afc^{-1})$ forms a polyhedron named by P. The permutation that determines P is

$$\begin{aligned}\pi_P = {} & (ae^{-1}b^{-1}cde'fdb'^{-1}a'f'c'^{-1})\\ & (a^{-1}c'f'^{-1}a'^{-1}b'd^{-1}f^{-1}e'^{-1}d^{-1}c^{-1}be).\end{aligned}$$

Then,

$$\begin{aligned}\pi_P^* = {} & (ab'c)(a'c'b)(df^{-1}c'^{-1})(d'c^{-1}f'^{-1})\\ & (a^{-1}f'e'^{-1})(a'^{-1}e^{-1}f)(b^{-1}d'^{-1}e')(b'^{-1}e'd^{-1}).\end{aligned}$$

By omitting the power -1 and then replacing the prime by -1 on π_P^*, we have

$$P^* = (ab^{-1}c)(dfc^{-1})(af^{-1}e^{-1})(bd^{-1}e^{-1}).$$

Theorem 2.1.3. *For two polyhedra P and Q, P is a dual of Q if, and only if, Q is a dual of P. Or in other words, $P^{**} = P$.*

Proof. By observing that

$$\pi_P^{**} = (\pi_P\sigma\delta)\delta\sigma = \pi_P(\sigma\delta\delta\sigma) = \pi_P(\sigma\sigma) = P,$$

the theorem is done from Theorem 2.1.1. □

2.2 Supports and skeletons

A *support* of polyhedron $P = \{C_i | 1 \le i \le k\}$ is the network formed by graph $U = (V_U, E_U)$ with a weight w on E_U where $V_U = \{C_i | 1 \le i \le k\}$, $(C_i, C_j) \in E_U$ if, and only if, C_i and C_j, $1 \le i, j \le k$, have a common letter, and