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Graziano Crasta and Ilaria Fragalà

1 Geometric issues in PDE problems related to the infinity Laplace operator

Abstract: We review some recent results related to the homogeneous Dirichlet problem for the infinity Laplace equation with a constant source in a bounded domain. We characterize the geometry of domains for which an overdetermined problem admits a viscosity solutions. An essential tool is a regularity result for viscosity solutions in convex domains, obtained by the convex envelope method introduced by Alvarez, Lasry, and Lions.

Keywords: Overdetermined problems, infinity Laplacian

AMS Classification: Primary 49K20, Secondary 49K30, 35J70, 35N25.

1.1 Introduction

Our primary interest in partial differential equation (PDE) problems for the infinity Laplacian operator raised from the following overdetermined problem:

$$\begin{cases} -\Delta_\infty u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ |\nabla u| = c & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

whose study was firstly proposed in [6].

Let us recall that the infinity Laplacian is the strongly nonlinear and highly degenerated differential operator defined for smooth functions u by

$$\Delta_\infty u := \nabla^2 u \nabla u \cdot \nabla u\,.$$

It was firstly discovered by Aronsson in the sixties in connection with the so-called absolutely minimizing Lipschitz extensions and later in the nineties a fundamental advance concerning the existence and uniqueness of solutions came by Jensen. In the last decade, also due to their connection with tug-of-war games, boundary value problems involving the infinity Laplace operator have received a great impulse thanks to the contribution of several authors; without any attempt of completeness, let us quote the papers [2–4, 15, 16, 23–25, 27], where the reader may find further related references.

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On the other hand, starting from the fundamental paper by Serrin [26], overdetermined problems of the type (1.1) have been studied for many operators (the basic examples being the Laplace and p-Laplace operator; see for instance [5, 14, 18, 19, 26]), not including the infinite Laplacian operator. In all these cases it is known that, if the overdetermined problem (1.1) admits a solution, then Ω is a ball.

An intriguing discovery is that this is not the case for the infinity Laplacian, unless more regularity (and topological) assumptions are required on the domain Ω.

Motivated by the aim of characterizing the shape of domains where problem (1.1) admits a solution, we were led to study a number of geometrical and regularity matters, going from the concavity properties of the unique solution to the Dirichlet problem given by the first two equations in (1.1), to the study of sets with positive reach and empty interior in $\mathbb{R}^n$.

In this chapter, we review our achievements on these topics to this day. Our choice is in favor of an intuitive presentation: though the results are rigorously stated, they are introduced in an informal way, enlightening the main ideas and avoiding all technicalities. In this spirit, we invoke more than once heuristic arguments, and we limit ourselves to sketch the proofs, referring for all details to the original papers.

The outline of the chapter is as follows.

In Section 1.2, we recall some basic facts concerning existence, uniqueness, and regularity for the homogeneous Dirichlet problem with a constant source term.

In Section 1.3, we deal with a simplified version of problem (1.1) where solutions are searched in the family of functions having prescribed level lines, and precisely the same level lines as the distance function from $\partial\Omega$. Studying the problem in this setting leads to introduce a class of domains, that we call "stadium-like," for which the cut locus agrees with the set of maximal distance from the boundary.

In Section 1.4, we present the geometric results we obtained for stadium-like domains, which rely on a new classification of closed sets with positive reach and empty interior. These results are essentially two-dimensional.

In Section 1.5, we deal with problem (1.1) in its general and quite challenging formulation.

To pursue our attempt of showing that the field is extremely rich, and many relevant questions remain unsolved, we conclude the chapter with a short section of open problems.

1.2 On the Dirichlet problem

In this section, we briefly discuss the Dirichlet problem for the infinity Laplace equation with a constant source term:

$$\begin{cases} -\Delta_\infty u = 1 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.1}$$

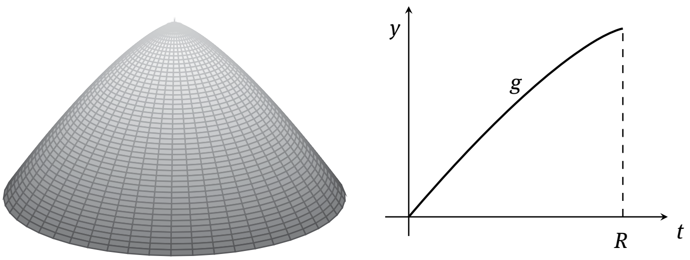

Fig. 1: Radial solution of the Dirichlet problem (2.1).

We begin with a basic example in order to get a feeling with the problem and motivate the use of viscosity solutions.

Example 2.1. Let $\Omega = B_R(0)$ be the ball of radius R centered at the origin. Let us look for a radial solution to problem (2.1) of the form $u(x) = g(R-|x|)$, where $g\colon [0, R] \to \mathbb{R}$ is a continuous function, of class C^2 in the interval $(0, R)$. The Dirichlet boundary condition gives $g(0) = 0$. On the other hand, if we want u to be differentiable at $x = 0$ (which is *a posteriori* justified by Theorem 2.2 stated hereafter) we have to require that $g'(R) = 0$. Hence, we have to solve the following one-dimensional boundary value problem for the function g:

$$-\Delta_\infty u(x) = -g''(R-|x|)\,[g'(R-|x|)]^2 = 1, \qquad g(0) = 0, \quad g'(R) = 0\,.$$

We easily obtain

$$g(t) = c_0[R^{4/3} - (R-t)^{4/3}], \qquad t \in [0, R] \quad (c_0 = 3^{4/3}/4)$$

(see Figure 1). The function $u(x) = g(R-|x|)$ is of class $C^{1,1/3}(B_R) \cap C^2(B_R \setminus \{0\})$. This shows that there are no radial solutions of class $C^2(B_R)$.

We shall turn back to the lackness of classical (i.e., C^2) solutions for the Dirichlet problem (2.1) in arbitrary domains in Section 1.5.

By the moment, we limit ourselves to consider the above example as a heuristic explanation why solutions to problem (2.1) cannot be expected to be classical. Moreover, we also observe that the notion of weak solutions is ruled out, because the equation is fully nonlinear and cannot be written in the divergence form. In fact, the right notion of solution to problem (2.1) is one of the viscosity solution. We shortly recall it below, for the benefit of the reader, referring to [8] for more details.

A *viscosity subsolution* to the equation $-\Delta_\infty u - 1 = 0$ is a function $u \in C(\Omega)$ which, for every $x_0 \in \Omega$, satisfies

$$-\Delta_\infty \varphi(x_0) - 1 \le 0 \quad \text{whenever } \varphi \in C^2(\Omega) \text{ and } u - \varphi \text{ has a local maximum at } x_0\,, \tag{2.2}$$

or equivalently

$$-\langle Xp, p\rangle - 1 \le 0 \quad \forall (p, X) \in J^{2,+}_\Omega u(x_0)\,. \tag{2.3}$$

Here the second-order superjet $J^{2,+}_{\Omega}u(x_0)$ of a function $u \in C(\overline{\Omega})$ at a point $x_0 \in \Omega$ denotes the set of pairs $(p, A) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}_{\mathrm{sym}}$ such that

$$u(y) \leq u(x_0) + \langle p, y - x_0 \rangle + \frac{1}{2} \langle A(y - x_0), y - x_0 \rangle + o\left(|y - x_0|^2\right)$$

as $y \to x_0$, $y \in \Omega$.

Similarly, a *viscosity supersolution* to the equation $-\Delta_\infty u - 1 = 0$ is a function $u \in C(\Omega)$ which, for every $x_0 \in \Omega$, satisfies

$$-\Delta_\infty \varphi(x_0) - 1 \geq 0 \quad \text{whenever } \varphi \in C^2(\Omega) \text{ and } u - \varphi \text{ has a local minimum at } x_0, \tag{2.4}$$

or equivalently

$$-\langle Xp, p \rangle - 1 \geq 0 \quad \forall (p, X) \in J^{2,-}_{\Omega}u(x_0) \tag{2.5}$$

(the second-order subjet $J^{2,-}_{\Omega}u(x_0)$ is defined analogously to the superjet with the inequality reversed).

Finally, a *viscosity solution* to problem (2.1) is a function $u \in C(\overline{\Omega})$ such that $u = 0$ on $\partial\Omega$ and u is a viscosity solution to $-\Delta_\infty u = 1$ in Ω, meaning it is both a viscosity subsolution and a viscosity supersolution on Ω, according to the above definition.

We are now in a position to recall the basic known facts concerning existence, uniqueness, and regularity for viscosity solutions to problem (2.1).

Theorem 2.2 (Basic properties of viscosity solutions to (2.1)). *The Dirichlet problem* (2.1) *admits a unique viscosity solution u. Moreover, u is differentiable at every point of* Ω.

Both existence and uniqueness of viscosity solution have been obtained by Lu and Wang in [24], by adapting the nowadays standard approach for viscosity solutions of nondegenerate second-order fully nonlinear equations. In particular, existence is obtained by Perron's method, while uniqueness is a consequence of the following comparison principle.

Theorem 2.3 (Comparison principle). *Let $u, v \in C(\overline{\Omega})$ be, respectively, viscosity sub- and supersolutions of $-\Delta_\infty u = 1$ in Ω. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.*

The fact that the unique solution u to (2.1) is differentiable everywhere has been recently proved by Lindgren [23], by adapting the method of Evans and Smart [16] for infinity harmonic functions.

1.3 On the overdetermined problem: the simple (web) case

In this section, we consider a simplified version of the overdetermined problem (1.1) and introduce a class of domains where such a simplified version turns out to admit a solution.

To follow an intuitive approach, let us present a heuristic argument. Assume for a moment that u is a smooth solution to (1.1), and consider the *gradient flow* associated with u, i.e., the flow generated by the ordinary differential equation

$$\dot{x}(t) = \nabla u(x(t)) .$$

Solutions of this differential equation will be called *characteristics*. If $x(t)$, $t \in [0, T)$, is a characteristic, and $\varphi(t) := u(x(t))$ denotes the restriction of u along this solution, we have

$$\begin{aligned}\dot{\varphi}(t) &= |\nabla u(x(t))|^2 , \\ \ddot{\varphi}(t) &= 2 \left\langle D^2 u(x)\nabla u(x),\ \nabla u(x)\right\rangle = 2\Delta_\infty u(x) = -2\end{aligned}$$

i.e., $\varphi(t) = \varphi(0) + \dot{\varphi}(0)\, t - t^2$. Moreover, if $x(0) = y \in \partial\Omega$, from the conditions $u(y) = 0$ and $|\nabla u(y)| = c$ we can determine explicitly φ as

$$\varphi(t) = \sqrt{c}\, t - t^2 . \tag{3.1}$$

On the other hand, from this information we cannot reconstruct the expression of the solution u, because in general we do not know the geometry of characteristics, which clearly depends on the solution itself!

However, there is a special case when this geometry is explicitly known, namely when the function u belongs to the following class:

Definition 3.1 (Web functions). We say that u is a *web function* if it only depends on the distance d from the boundary of $\partial\Omega$, that is it can be written for some function w as $u(x) = w(d(x))$.

As we are going to realize immediately, when dealing with problem (1.1) within the class of web functions, there are two subsets of $\overline{\Omega}$ related with the geometry of d which turn out to play a crucial role. We introduce them below:

Definition 3.2 (Cut locus and high ridge). The *cut locus* $\overline{\Sigma}(\Omega)$ of Ω is the closure in $\overline{\Omega}$ of the set $\Sigma(\Omega)$ of points of non differentiability of d. The *high ridge* $\mathrm{M}(\Omega)$ of Ω is the set where d achieves its maximum over $\overline{\Omega}$ (called the inradius ρ_Ω of the set Ω).

Figure 2 shows the cut locus and the high ridge when Ω is a rectangle.

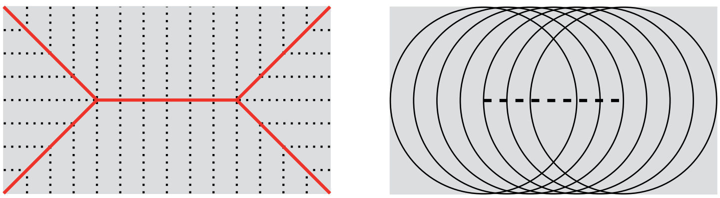

Fig. 2: Cut locus (solid), characteristics (dotted), high ridge (dashed).

Observe now that, for a generic domain Ω, if u is a web function, ∇u is parallel to ∇d, and hence the characteristics of u are line segments normal to the boundary. More precisely, a characteristic is a line segment which starts at a point of the boundary, is normal to the boundary itself, and reaches a point of the cut locus (for instance, some characteristics of a web function on a rectangle are the dotted line segments in Figure 2).

Moreover, if u is written as $w(d)$, we have $|\nabla u(y)| = w'(0)$ for every $y \in \partial\Omega$, so that the condition $|\nabla u| = c$ on $\partial\Omega$ is automatically satisfied, with $c = w'(0)$. Thus, asking that the unique viscosity solution to problem (2.1) is a web function we immediately obtain a solution to the overdetermined problem (1.1).

By arguing as in Example 2.1, namely solving a one-dimensional boundary value problem for the function w, we obtain

$$w(t) = c_0(R^{4/3} - (R-t)^{4/3}) \qquad c_0 := \frac{3^{4/3}}{4}, \quad R := \frac{c^3}{3}.$$

If we now impose that u is differentiable, then we find that all characteristics must have the same length R, and that this length R must coincide with the inradius ρ_Ω.

In other words, the requirement that all characteristics must have the same length is equivalent to ask a precise geometric condition on Ω, which is the coincidence between cut locus and high ridge. Accordingly, we set the following:

Definition 3.3 (Stadium-like domains). A set $\Omega \subset \mathbb{R}^n$ is said to be a *stadium-like domain* if $\mathrm{M}(\Omega) = \overline{\Sigma}(\Omega)$.

Clearly, the rectangle is not a stadium-like domain. Some examples of stadium-like domains are represented in Figure 3.

The heuristic arguments presented above can be made rigorous and yield the following result. It has been proved in [6] in the regular case (for C^1 solutions and C^2 domains) and in [10] in the general case (with no regularity assumption on u and Ω).

Theorem 3.4 (Web-viscosity solutions). *The unique viscosity solution to problem* (2.1) *is a web function if and only if Ω is a stadium-like domain. In this case, the web-viscosity solution is given by*

$$u(x) = \psi_\Omega(x) := g(d(x)) = c_0\left[\rho_\Omega^{4/3} - (\rho_\Omega - d(x))^{4/3}\right]. \tag{3.2}$$

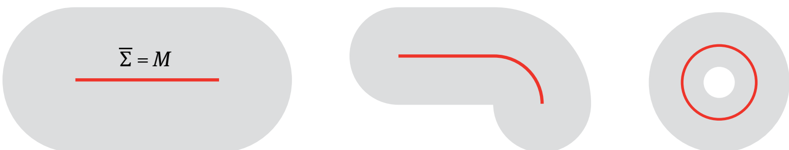

Fig. 3: Stadium-like domains.

1.4 On stadium-like domains

In view of Theorem 3.4, a natural question is whether and how is it possible to characterize the geometry of stadium-like domains. A complete classification of them has been given in [9] in dimension $n = 2$; a similar statement in higher dimensions has been proved until now only under the convexity assumption. To prepare our results, we have to recall the fundamental notion of set of positive reach introduced by Federer in [17].

Definition 4.1 (Set of positive reach). Let $S \subset \mathbb{R}^n$ be a nonempty closed set, and let d_S denote the distance function from S. We say that S is a set of *positive reach* if there exists $r_S > 0$ (called radius of proximal smoothness) such that every point of the tubular neighborhood

$$\{x \in \mathbb{R}^n : \ 0 < d_S(x) < r_S\} \tag{4.1}$$

has a unique projection on S.

Federer himself proved that S has positive reach if and only if S is *proximally* C^1, which means that the distance function d_S is of class C^1 in a tubular neighborhood of the form (4.1). (If this is the case, it can be proved that d_S is of class $C^{1,1}$ in such tubular neighborhood.)

In [9, Theorem 2], we have obtained the following complete characterization of planar sets with positive reach and empty interior:

Theorem 4.2 (Characterization of planar proximally C^1 sets with empty interior). *Let $S \subset \mathbb{R}^2$ be closed, proximally C^1, with empty interior, and connected. Then S is either a singleton, or a one-dimensional manifold of class $C^{1,1}$.*

Sketch of the proof. The proof is of marked geometric stamp, and here we limit ourselves to give a rough idea of it. It consists basically in performing a careful analysis of the so-called contact set. Namely, we fix a point $p \in S$ and a positive r smaller than the radius of proximal smoothness, and study the contact set of p into S_r, which is defined as the set $C_r(p)$ where the circumference of radius r centered at p meets the boundary of the tubular neighborhood $\{d_S(x) < r\}$. The main issue in the proof amounts to show that $C_r(p)$ consists either of two antipodal points, or of a semicircumference. Once one has this geometric characterization of the contact set, it is rather easy to deduce that S is locally the graph of a Lipschitz function g. Finally, the fact that it is of class $C^{1,1}$ comes from the fact that g is both semiconcave and semiconvex. □

We explicitly note that a one-dimensional connected manifold can be with boundary (two points) or without boundary (a closed curve), see Figure 4.

It is interesting to observe that, as soon as we require d_S to be of class C^2 in a tubular neighborhood of S, then the second case in Figure 4 (manifold with boundary) cannot happen. More precisely, let us set the following:

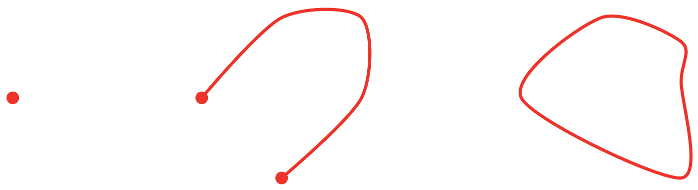

Fig. 4: Planar proximally C^1 sets with empty interior.

Definition 4.3 (Proximally C^k sets). We say that a nonempty closed subset S of $\mathbb{R}^n$ is *proximally* C^k if there exists $r_S > 0$ such that d_S is of class C^k in a tubular neighborhood of S of the form (4.1).

Then, we have (see [9, Theorem 3]):

Theorem 4.4 (Characterization of proximally C^2 sets with empty interior). *Let $S \subset \mathbb{R}^2$ be closed, proximally C^2, with empty interior, and connected. Then S is either a singleton, or a one-dimensional manifold of class C^2 without a boundary.*

The above statement can be generalized to the case when, with the analogous meaning as in Definition 4.3, the set S is proximally $C^{k,\alpha}$, for some $k \geq 2$ and $\alpha \in [0, 1]$, or proximally C^∞, or proximally C^ω. Accordingly, S turns out to be a manifold, respectively, of class $C^{k,\alpha}$, C^∞, or C^ω (cf. [9, Remark 4 (iii) and Remark 23]).

A direct consequence of Theorems 4.2 and 4.4 is the following characterization of stadium-like domains. To understand it, one has to think of S as playing the role of the set $\mathrm{M}(\Omega) = \overline{\Sigma}(\Omega)$, which is a nonempty closed set with empty interior (notice in fact that the high ridge $\mathrm{M}(\Omega)$ cannot have interior points, since otherwise there would be points where $\nabla d = 0$). Accordingly, the set Ω has to be thought as a tubular neighborhood of S.

Theorem 4.5 (Characterization of planar domains with $\mathrm{M} = \overline{\Sigma}$). *Let $\Omega \subset \mathbb{R}^2$ be an open bounded connected set with $\mathrm{M}(\Omega) = \overline{\Sigma}(\Omega)$. Then Ω is either a disk or a parallel neighborhood of a one-dimensional $C^{1,1}$ manifold.*

If in addition Ω is C^2, then Ω is either a disk or a parallel neighborhood of a one-dimensional C^2 manifold with no boundary.

If Ω is also simply connected, then Ω is a disk.

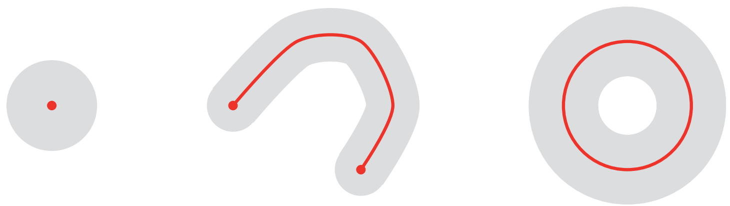

Fig. 5: Stadium-like domains.

The three possibilities are shown in Figure 5.

In [9, Theorem 12], we also proved a partial extension for convex sets in higher dimension.

Theorem 4.6 (Extension to higher dimensions). *Let $\Omega \subset \mathbb{R}^n$ be an open bounded convex set. If $\mathrm{M}(\Omega) = \overline{\Sigma}(\Omega)$ and Ω is C^2, then Ω is a ball.*

Now our Theorem 3.4 can be rephrased in the following much more “visual” way:

Theorem 4.7 (Web-viscosity solutions). *The unique viscosity solution to problem* (2.1) *is a web function if and only the shape of Ω can be characterized as in Theorem 4.5 (in dimension $n = 2$) and 4.6 (in any dimension provided Ω is assumed to be convex).*

1.5 On the overdetermined problem: the general (non-web) case

Up to now, we have characterized the geometry of sets for which the overdetermined problem (1.1) admits a solution in the class of web functions. (We stress once more that, in this class of functions, the overdetermined problem (1.1) is equivalent to the Dirichlet problem (2.1), since the condition $|\nabla u|$ constant on $\partial\Omega$ is automatically satisfied.)

In this section, we are going to consider what happens in the general case, i.e., without the restriction to web functions. Recalling the heuristic argument given at the beginning of Section 1.3, we see that we have to face with a number of additional difficulties. In particular, the following two main problems emerge.

- Since u is unknown and, *a priori*, its level lines do not have any specific form, the geometry of the trajectories of the gradient flow is unknown.
- Even worse, we do not know if the gradient flow is well posed. Namely, in general, we only know that ∇u is locally bounded, and it is never locally Lipschitz, as we shall see in Theorem 5.4 that u never belongs to $C^{1,1}(\Omega)$. This means that we cannot use the standard Cauchy–Lipschitz theory for ordinary differential equations for the gradient flow $\dot{x} = \nabla u(x)$. Moreover, even if we were able to prove an intermediate regularity result between local boundedness and local Lipschitzianity for ∇u (e.g., that it is locally in BV or in some Sobolev space), we could not even apply the Ambrosio–Di Perna–Lions theory of regular Lagrange flows, because we do not have a lower bound for the measure $\operatorname{div} \nabla u$.

Our approach is motivated by the above remarks, and in particular it stems from the will of recovering the well-posedness of the gradient flow. In this respect it is well known that, in order to have at least forward well posedness, it is enough u to be *locally semiconcave*. By definition, this means that there exists a constant $C \geq 0$ such that

$$u(x+h) + u(x-h) - 2u(x) \leq C|h|^2 \qquad \forall [x-h, x+h] \subset \Omega\,,$$

where $[x - h, x + h]$ denotes the segment in $\mathbb{R}^n$ joining the two points $x - h$ and $x + h$.

In fact, the forward uniqueness of solutions follows from the property

$$\langle \nabla u(y) - \nabla u(x),\ y - x\rangle \le C|y - x|^2 ,$$

which is the analogous, for differentiable semiconcave functions, of the monotonicity of the gradient of a (differentiable) concave function. Now, if $x(t)$ and $y(t)$ are two solutions of the gradient flow defined in a common interval $[0, \tau)$, setting $w(t) := |y(t) - x(t)|^2/2$ we obtain

$$\dot{w}(t) = \langle \nabla u(y(t)) - \nabla u(x(t)),\ y(t) - x(t)\rangle \le 2\,C\,w(t) .$$

Hence, if $w(t_0) = 0$ for some $t_0 \in [0, \tau)$, i.e., if $x(t_0) = y(t_0)$, then by Gronwall's inequality we obtain that $w(t) = 0$ for every $t \in [t_0, \tau)$, i.e., $x(t) = y(t)$ for every $t \in [t_0, \tau)$.

For a review on semiconcave functions, we refer to [7].

In this perspective, our first step will be to set up a regularity result for u, proving that u is locally semiconcave. Unfortunately, we are not able to obtain such a result in full generality, but we have to restrict to convex domains without corners. More precisely, we are going to assume that

$$\Omega \text{ is convex and satisfies an interior sphere condition.} \tag{$H\Omega$}$$

Theorem 5.1 (Power concavity and semiconcavity of solutions). *Assume* $(H\Omega)$ *and let* u *be the viscosity solution to the Dirichlet problem* (2.1). *Then,* $u^{3/4}$ *is concave in* Ω. *In particular,* u *is locally semiconcave in* Ω.

Sketch of the proof. Let us outline the strategy we adopt in order to prove that the function $w := -u^{3/4}$ is convex in Ω. For the detailed proof, we refer to [11, Theorem 1].

We first observe that w is well defined (since $u > 0$ in Ω), and it is the unique viscosity solution of the Dirichlet problem

$$\begin{cases} -\Delta_\infty w - \frac{1}{w}\left[\frac{1}{3}|\nabla w|^4 + \left(\frac{3}{4}\right)^3\right] = 0 & \text{in } \Omega , \\ w = 0 & \text{on } \partial\Omega . \end{cases} \tag{5.1}$$

At first sight, the equation satisfied by w looks more complicate than the original one for u. On the other hand, thanks to the structure of such equation (we refer in particular to the factor $1/w$ in front of the brackets), we are enabled to adapt the convex envelop method developed by Alvarez et al. (see [1]). It consists essentially in the following steps.

(i) Prove that the convex envelope w_{**} of w is a viscosity supersolution to (5.1). This is the most challenging task, where the structure of the equation intervenes. The detailed proof can be found in [11]. The main ingredients are:
   - the representation of the convex envelope as

$$w_{**}(x) = \inf_{k \le n+1} \left\{ \sum_{i=1}^k \lambda_i w(x_i) \,:\, x = \sum_{i=1}^k \lambda_i x_i,\, x_i \in \overline{\Omega},\, \lambda_i > 0,\, \sum_{i=1}^k \lambda_i = 1 \right\} ;$$

- the fact that, since the normal derivative of w with respect to the external normal is $+\infty$ at every boundary point of Ω, in our case the points x_i in the formula above cannot lie on the boundary of Ω;
- Proposition 1 in [1];
- the concavity of the map $Q \mapsto 1/\mathrm{tr}((p \otimes p)Q^{-1})$.

(ii) By Step (i) and the comparison principle (that for Equation (5.1) has been proved in [24, Theorem 3]), it follows that $w_{**} \geq w$ in Ω.

(iii) By definition of convex envelope, it is immediate that $w_{**} \leq w$ in Ω.

By combining Steps (ii) and (iii), we conclude that w coincides with its convex envelope, so that $w = -u^{3/4}$ is a convex function. From this power-concavity property of u, it is straightforward to conclude that u is locally semiconcave in Ω. ☐

Since u is locally semiconcave and differentiable everywhere, we obtain at once the following regularity property (see [7, Prop. 3.3.4]).

Corollary 5.2 (C^1-regularity of solutions). *Assume* ($H\Omega$) *and let u be the viscosity solution to the Dirichlet problem* (2.1). *Then, u is continuously differentiable in* Ω.

Let us now turn back to the overdetermined boundary value problem (1.1), in the light of the regularity results obtained so far for the solution u to problem (2.1) in Ω. In order not to face with boundary regularity matters for u at the boundary of Ω (for which however some results are available in the literature, see [20, 21, 28]), in the following we will assume that u is C^1 *up to* the boundary, namely that

$$\exists\, \delta > 0: \ u \text{ is of class } C^1 \text{ on } \{x \in \overline{\Omega}: \ d(x) < \delta\}\,. \tag{Hu}$$

As a consequence of Corollary 5.2 and assumption (Hu), for every initial point $x_0 \in \overline{\Omega}$ the Cauchy problem

$$\begin{cases} \dot{x} = \nabla u(x)\,, \\ x(0) = x_0 \end{cases}$$

turns out to admit a unique forward solution $\mathbf{X}(\cdot, x_0)$, defined on some maximal interval $[0, T(x_0))$. Moreover, we can prove that $t \mapsto \mathbf{X}(t, x_0)$ reaches in finite time a maximum point of u and then stops there.

Characteristics are now back at our disposal! So, let us resume the heuristic approach started in Section 1.3, consisting in studying the solution along such curves. Assume for a moment that the solution u of the Dirichlet problem (2.1) is smooth enough (let's say C^2), and consider the P-function

$$P(x) := \tfrac{1}{4}\, |\nabla u(x)|^4 + u(x)\,.$$

If $x(\cdot) = \mathbf{X}(\cdot, y)$ is a characteristic, then

$$\frac{\mathrm{d}}{\mathrm{d}t} P(x(t)) = |\nabla u(x)|^2 \left\langle D^2 u(x) \nabla u(x),\ \nabla u(x) \right\rangle + |\nabla u(x)|^2 = 0\,,$$

so that the P-function is constant along the gradient flow.

If, in addition, we require the overdetermined condition $|\nabla u| = c$ on $\partial\Omega$ to hold, we have that $P(y) = c^4/4$ at every point $y \in \partial\Omega$. From this information, it follows that the P-function is constant along the set spanned by the gradient flow, i.e., on the whole Ω. In turn, the constancy of P over Ω allows us to characterize the expression of u and the shape of Ω exactly in the same way as was done in Section 1.3 in the web setting. Indeed, the following result holds.

Theorem 5.3 (P-function). *Under the assumptions* $(H\Omega)$–(Hu), *let u be the unique solution to problem* (2.1). *If $P(x) = \lambda \le c_0\rho_\Omega^{4/3}$ for a.e. $x \in \Omega$, then u is the web function defined in* (3.2) *and Ω is a stadium-like domain (for which the conclusions of Theorem 4.7 hold).*

Sketch of the proof. The function ψ_Ω in (3.2) is the unique viscosity solution of the Hamilton–Jacobi equation

$$H(u, \nabla u) := \tfrac{1}{4}|\nabla u|^4 + u - \lambda = 0 .$$

On the other hand, $u \in C^1(\Omega)$ is a classical solution of the same equation (since P is continuous and so $P = \lambda$ in Ω). Therefore, $u = \psi_\Omega$. In particular, since u is a web function, the conclusions of Theorem 4.7 hold. □

Unfortunately, in general, u is not regular enough to prove that P is constant a.e. in Ω. Actually, the heuristic argument leading to the constancy of P can be made rigorous only provided u is at least of class $C^{1,1}$, and this kind of regularity never occurs. More precisely, the optimal expected regularity is $C^{1,1/3}$ according to the result below, which is obtained essentially by dealing with ODE's along the gradient flow of u, and, in particular, exploiting the expression of u along characteristics given by equation (3.1).

Theorem 5.4 (Regularity threshold). *If the unique solution u to problem* (2.1) *is of class $C^{1,1}(A \setminus K)$, where $K := \operatorname{argmax}_{\overline{\Omega}}(u)$ and A is a neighborhood of K, then for any $\alpha > 1/3$ it cannot occur that u is of class $C^{1,\alpha}(A)$.*

Nevertheless, not everything is lost. Still by exploiting characteristics, we can argue to obtain, in place of the constancy of the P-function, some useful upper and lower bounds for it.

Theorem 5.5 (P-function inequalities). *Under the assumptions* $(H\Omega)$–(Hu), *let u be the unique solution to problem* (2.1). *Then,*

$$\min_{\partial\Omega} \frac{|\nabla u|^4}{4} \le P(x) \le \max_{\overline{\Omega}} u \qquad \forall x \in \overline{\Omega} .$$

Sketch of the proof. Observe that, if $y \in \partial\Omega$ then $P(\mathbf{X}(0, y)) = P(y) = |\nabla u(y)|^4/4$; on the other hand, for t large enough, $\mathbf{X}(t, y)$ is a maximum point of u, so that $P(\mathbf{X}(t, y)) = \max u$. Then to prove the statement, it is enough to show that P is nondecreasing along the gradient flow. To this end, in order to obtain a bit more of regularity, we consider

Fig. 6: Domains considered in Theorem 5.6.

the supremal convolutions

$$u^\varepsilon(x) = \sup_y \left\{ u(y) - \frac{|x-y|^2}{2\varepsilon} \right\}.$$

By the local semiconcavity of u, these convolutions are of class $C^{1,1}$. Moreover, thanks to the so-called magic properties of their superjets, they turn out to be subsolutions of the PDE. Hence the corresponding approximated P-functions

$$P_\varepsilon := \frac{|\nabla u^\varepsilon|^4}{4} + u^\varepsilon$$

are nondecreasing along the gradient flow of u^ε. Finally by passing to the limit as $\varepsilon \to 0^+$ we get the desired monotonicity property for P. □

The bounds for the P-function obtained in Theorem 5.5 do not give us enough information to deduce a complete characterization of domains where the overdetermined problem (1.1) admits a solution. However, they are quite helpful to get at least a partial target. Namely we can prove the following result, showing that the same conclusions of Theorem 4.7 continue to hold without asking the solution to be a web function, provided some *a priori* geometric restrictions on Ω are imposed.

Theorem 5.6 (Serrin-type theorem for Δ_∞). *Assume* $(H\Omega)$–(Hu). *Further assume that there exists an inner ball B of radius ρ_Ω which meets $\partial\Omega$ at two diametral points (see Figure 6 left). If there exists a solution u to the overdetermined problem* (1.1), *then u is the web function defined in* (3.2), *and Ω is a stadium-like domain (for which the conclusions of Theorem 4.7 hold).*

Sketch of the proof. Let $p, q \in \partial\Omega$ be the two diametral points belonging to $\partial B \cap \partial\Omega$, and let D be a stadium-like domain D that contains Ω and is tangent to Ω at p and q (see Figure 6, right). Let u_B and u_D denote, respectively, the solutions of the Dirichlet problem (2.1) in B and D. By comparison, we have

$$u_B \le u \le u_D \qquad \text{in } \overline{B}.$$

In particular, this implies that $u = u_B = u_D$ on the segment $[p, q]$ and that $\nabla u = \nabla u_B = \nabla u_D$ at p and q, so that $|\nabla u_B| = |\nabla u_D| = c$ at these two points. In turn, this gives

$\max u_D = c^4/4$ and hence, by Theorem 5.5, we obtain

$$\frac{c^4}{4} = \min_{\partial\Omega} \frac{|\nabla u|^4}{4} \le P(x) \le \max_{\overline{\Omega}} u \le \frac{c^4}{4}\,.$$

Now the conclusion follows from Theorem 5.3. □

1.6 Open problems

We list below some open questions related to the results reviewed above, which are in our opinion interesting challenges for further research.

- *Problem 1.* Provide a complete characterization of stadium-like domains in higher dimensions (i.e., remove the convexity assumption in Theorem 4.6).
- *Problem 2.* Provide a general version of Serrin theorem for Δ_∞ (i.e., remove the geometric restrictions on Ω in Theorem 5.6).
- *Problem 3.* Prove that the solution to the Dirichlet problem (2.1) is actually of class $C^{1,1/3}(\Omega)$ (i.e., show that the regularity threshold of Theorem 5.4 is achieved).
- *Problem 4.* To some extent surprisingly, the geometric condition $\overline{\Sigma}(\Omega) = M(\Omega)$ appears independently in the paper [29], where it is shown that on stadium-like domains the infinity Laplacian admits a unique ground state. (An infinity ground state is, roughly speaking, the limit as $p \to +\infty$ of a sequence of solutions to the Euler–Lagrange equation for the nonlinear Rayleigh quotient associated with the p-Laplacian). As recently shown in [22], the uniqueness of an infinity ground state is false, in general, and the geometric characterization of domains where it is true is a completely open problem. It would be interesting to understand whether ∞-ground states are unique in all convex domains or just in stadium-like ones.

Note added in proof. Recently, some of the results presented in this chapter have been generalized to the case of the normalized infinity Laplace operator, see [12]. Moreover, we address to the forthcoming paper [13] for some developments on Problem 2.

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2 Solution of free boundary problems in the presence of geometric uncertainties

Abstract: This chapter is concerned with solving Bernoulli's exterior free boundary problem in the case of an interior boundary which is random. We model this random free boundary problem such that the expectation and the variance of the sought domain can be defined. In order to numerically approximate the expectation and the variance, we propose a sampling method like the (quasi-) Monte Carlo quadrature. The free boundary is determined for each sample by the trial method which is a fixed-point-like iteration. Extensive numerical results are given in order to illustrate the model.

Keywords: Bernoulli's exterior free boundary problem, random boundary

2.1 Introduction

Let $T \subset \mathbb{R}^n$ denote a bounded domain with boundary $\partial T = \Gamma$. Inside the domain T, we assume the existence of a simply connected subdomain $S \subset T$ with boundary $\partial S = \Sigma$. The resulting annular domain $T \setminus \overline{S}$ is denoted by D. The topological situation is visualized in Figure 1.

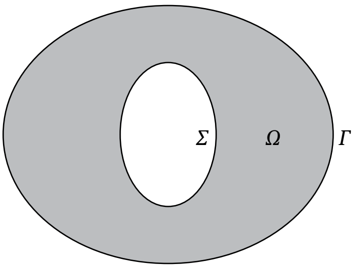

Fig. 1: The domain D and its boundaries Γ and Σ.

We consider the following overdetermined boundary value problem in the annular domain D:

$$\begin{aligned} \Delta u &= 0 && \text{in } D\,, \\ \|\nabla u\| &= f && \text{on } \Gamma\,, \\ u &= 0 && \text{on } \Gamma\,, \\ u &= 1 && \text{on } \Sigma\,, \end{aligned} \tag{1.1}$$

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where $f > 0$ is a given constant. We like to stress that the non-negativity of the Dirichlet data implies that u is positive in D. Hence, there holds the identity

$$\|\nabla u\| \equiv -\frac{\partial u}{\partial \mathbf{n}} \quad \text{on } \Gamma \tag{1.2}$$

since u admits homogeneous Dirichlet data on Γ.

We arrive at Bernoulli's exterior free boundary problem if the boundary Γ is unknown. In other words, we seek a domain D with a fixed boundary Σ and unknown boundary Γ such that the overdetermined boundary value problem (1.1) is solvable. This problem has many applications in engineering sciences such as fluid mechanics, see [10], or electromagnetics, see [6, 7] and references therein. In the present form, it models, for example, the growth of anodes in electrochemical processes. For the existence and uniqueness of solutions, we refer the reader to, e.g., [3, 4, 17]; see also [9] for the related interior free boundary problem. Results concerning the geometric form of the solutions can be found in [1] and references therein.

In this chapter, we try to model and solve the free boundary problem (1.1) in the case that the interior boundary is uncertain, i.e., if $\Sigma = \Sigma(\omega)$ with an additional parameter $\omega \in \Omega$. This model is of practical interest in order to treat, for example, tolerances in fabrication processes or if the interior boundary is only known by measurements which typically contain errors. We are thus looking for a tuple $(D(\omega), u(\omega))$ such that it holds

$$\begin{aligned} \Delta u(\omega) &= 0 && \text{in } D(\omega)\,, \\ \|\nabla u(\omega)\| &= f && \text{on } \Gamma(\omega)\,, \\ u(\omega) &= 0 && \text{on } \Gamma(\omega)\,, \\ u(\omega) &= 1 && \text{on } \Sigma(\omega)\,. \end{aligned} \tag{1.3}$$

The questions to be answered in the following are:

- How to model the random domain $D(\omega)$? What is the associated expectation and the variance?
- Do the expectation and the variance exist and are they finite?
- What is the expectation and the variance of the potential $u(\omega)$ if the domain $D(\omega)$ is uncertain?
- How to compute the solution to the random free boundary problem numerically?

For the sake of simplicity, we restrict our consideration to the two-dimensional situation. Nevertheless, the extension to higher dimensions is straightforward and is left to the reader.

The rest of this chapter is organized as follows. Section 2.2 is dedicated to answering the first two questions. We start by defining appropriate function spaces to define the stochastic model. Afterward, we define the random inner boundary and the resulting random outer boundary. Especially, we provide a theorem which guarantees the well posedness of the random free boundary problem under consideration. Moreover, we introduce here the expectation and the variance of the domain's boundaries. Finally, we give an analytic example which shows that the solution of the free boundary problem depends nonlinearly on the stochastic parameter. In Section 2.3, we answer the latter two questions. We propose the use of boundary integral equations for the solution of the underlying boundary value problem. This significantly decreases the effort for the numerical solution. In particular, we can describe the related potential of the free boundary problem in terms of Green's representation formula. This also allows us to define its expectation and its variance. For the numerical approximation of the free boundary, we apply a trial method in combination with a Nyström discretization of the boundary integral equations. Section 2.4 is then devoted to the numerical examples. We will present here four different examples in order to illustrate different aspects of the proposed approach. We especially show that there is a clear difference between the expected free boundary and the free boundary which belongs to the expected interior boundary. As an important result, it follows thus that one cannot ignore random influences in numerical simulations. Finally, in Section 2.5, we state some concluding remarks.

2.2 Modelling uncertain domains

2.2.1 Notation

In the sequel, let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a complete and separable probability space with σ-algebra $\mathcal{F}$ and probability measure $\mathbb{P}$. Here, complete means that $\mathcal{F}$ contains all $\mathbb{P}$-null sets. In the sequel, for a given Banach space X, the Bochner space $L^p_{\mathbb{P}}(\Omega; X)$, $1 \le p \le \infty$, consists of all equivalence classes of strongly measurable functions $v\colon \Omega \to X$ whose norm

$$\|v\|_{L^p_{\mathbb{P}}(\Omega;X)} := \begin{cases} \left(\int_\Omega \|v(\cdot, \omega)\|_X^p \,\mathrm{d}\mathbb{P}(\omega) \right)^{1/p}, & p < \infty \\ \operatorname*{ess\,sup}_{\omega\in\Omega} \|v(\cdot, \omega)\|_X, & p = \infty \end{cases}$$

is finite. If $p = 2$ and X is a separable Hilbert space, then the Bochner space is isomorphic to the tensor product space $L^2_{\mathbb{P}}(\Omega) \otimes X$. Note that, for notational convenience, we will always write $v(\phi, \omega)$ instead of $(v(\omega))(\phi)$ if $v \in L^p_{\mathbb{P}}(\Omega; X)$. For more details on Bochner spaces, we refer the reader to [14].

2.2.2 Random interior boundary

Throughout the chapter, the domain $D(\omega)$ will be identified by its boundaries $\Sigma(\omega)$ and $\Gamma(\omega)$. Indeed, we assume that $\Sigma(\omega)$ is $\mathbb{P}$-almost surely starlike. This enables us to parameterize this random boundary in accordance with

$$\Sigma(\omega) = \{\mathbf{x} = \boldsymbol{\sigma}(\phi, \omega) \in \mathbb{R}^2 \colon \boldsymbol{\sigma}(\phi, \omega) = q(\phi, \omega)\mathbf{e}_r(\phi), \phi \in I\} .$$

Here, $\mathbf{e}_r(\phi) := [\cos(\phi), \sin(\phi)]^\intercal$ is the radial direction and $I := [0, 2\pi]$ is the parameter interval. The radial function $q(\phi, \omega) \geq \underline{c} > 0$ has to be in the Bochner space $L^2(\Omega; C^2_{\mathrm{per}}(I))$, where $C^2_{\mathrm{per}}(I)$ denotes the Banach space of periodic, twice continuously differentiable functions, i.e.,

$$C^2_{\mathrm{per}}(I) := \{f \in C(I) \colon f^{(i)}(0) = f^{(i)}(2\pi), i = 0, 1, 2\} ,$$

equipped with the norm

$$\|f\|_{C^2_{\mathrm{per}}(I)} := \sum_{i=0}^{2} \max_{x \in I} \left|f^{(i)}(x)\right| .$$

For our purposes, we assume that $q(\phi, \omega)$ is described by its expectation

$$\mathbb{E}[q](\phi) = \int_\Omega q(\phi, \omega) \, \mathrm{d}\mathbb{P}(\omega)$$

and its covariance

$$\mathrm{Cov}[q](\phi, \phi') = \mathbb{E}[q(\phi, \omega)q(\phi', \omega)] = \int_\Omega q(\phi, \omega)q(\phi', \omega) \, \mathrm{d}\mathbb{P}(\omega) .$$

Then, $q(\phi, \omega)$ can be represented by the so-called *Karhunen–Loève expansion*, cf. [16],

$$q(\phi, \omega) = \mathbb{E}[q](\phi) + \sum_{k=1}^{N} q_k(\phi) Y_k(\omega) .$$

Herein, the functions $\{q_k(\phi)\}_k$ are scaled versions of the eigenfunctions of the Hilbert–Schmidt operator associated with $\mathrm{Cov}[q](\phi, \phi')$. Common approaches to numerically recover the Karhunen–Loève expansion from these quantities are, e.g., given in [13] and the references therein. By construction, the random variables $\{Y_k(\omega)\}_k$ in the Karhunen–Loève expansion are uncorrelated. For our modelling, we shall also require that they are independent, which is a common assumption. Moreover, we suppose that they are identically distributed with $\operatorname{img} Y_k(\omega) = [-1, 1]$. Note that it holds

$$\mathbb{V}[q](\phi) = \int_\Omega \{q(\phi, \omega) - \mathbb{E}[q](\phi)\}^2 \, \mathrm{d}\mathbb{P}(\omega) = \sum_{k=1}^{N} \left(q_k(\phi)\right)^2 .$$

2.2.3 Random exterior boundary

If the interior boundary $\Sigma(\omega)$ is starlike, then also the exterior boundary $\Gamma(\omega)$ is starlike. In particular, it also follows that the free boundary $\Gamma(\omega)$ is C^∞-smooth, see [2] for details. Hence, the exterior boundary can likewise be represented via its parameterization:

$$\Gamma(\omega) = \{\mathbf{x} = \boldsymbol{\gamma}(\phi, \omega) \in \mathbb{R}^2 : \boldsymbol{\gamma}(\phi, \omega) = r(\phi, \omega)\mathbf{e}_r(\phi), \phi \in I\} . \tag{2.1}$$

The following theorem guarantees us the well posedness of the problem under consideration, cf. [4, 17]. It shows that it holds $r(\phi, \omega) \in L^\infty_{\mathbb{P}}(\Omega, C^2_{\text{per}}(I))$ if $q(\phi, \omega)$ is almost surely bounded and thus that $\boldsymbol{\gamma}(\phi, \omega)$ is well defined.

Theorem 2.1. *Assume that $q(\phi, \omega)$ is uniformly bounded almost surely, i.e.,*

$$q(\phi, \omega) \le \underline{R} \quad \text{for all } \phi \in I \text{ and } \mathbb{P}\text{-almost every } \omega \in \Omega . \tag{2.2}$$

Then, there exists a unique solution $(D(\omega), u(\omega))$ to (1.3) for almost every $\omega \in \Omega$. Especially, with some constant $\overline{R} > \underline{R}$, the radial function $r(\phi, \omega)$ of the associated free boundary (2.1) satisfies

$$q(\phi, \omega) < r(\phi, \omega) \le \overline{R} \quad \text{for all } \phi \in I \text{ and } \mathbb{P}\text{-almost every } \omega \in \Omega .$$

Proof. In view of (2.2), it follows that

$$\Sigma(\omega) \subset B_{\underline{R}}(\mathbf{0}) := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| < \underline{R}\}$$

for almost every $\omega \in \Omega$. Hence, for fixed $\omega \in \Omega$, [17, Theorem 1] guarantees the unique solvability of (1.3). In particular, there exists a constant $\overline{R} > \underline{R}$ such that $\Gamma(\omega) \subset B_{\overline{R}}(\mathbf{0})$ whenever $\Sigma(\omega) \subset B_{\underline{R}}(\mathbf{0})$. Therefore, the claim follows since $q(\phi, \omega)$ is supposed to be uniformly bounded in $\omega \in \Omega$. □

2.2.4 Expectation and variance of the domain

Having the parameterizations $\boldsymbol{\sigma}(\omega)$ and $\boldsymbol{\gamma}(\omega)$ at hand, we can obtain the expectation and the variance of the domain $D(\omega)$.

Theorem 2.2. *The expectation of the domain $D(\omega)$ is given via the expectations of its boundaries' parameterizations in accordance with*

$$\mathbb{E}[\partial D(\omega)] = \mathbb{E}[\Sigma(\omega)] \cup \mathbb{E}[\Gamma(\omega)] ,$$

where

$$\begin{aligned}\mathbb{E}[\Sigma(\omega)] &= \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbb{E}[q(\phi, \omega)]\mathbf{e}_r(\phi), \phi \in I\} ,\\ \mathbb{E}[\Gamma(\omega)] &= \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbb{E}[r(\phi, \omega)]\mathbf{e}_r(\phi), \phi \in I\} .\end{aligned}$$

Proof. For the proof, we introduce the global parameterization $\boldsymbol{\delta} : [0, 4\pi) \to \partial D(\omega)$ given by

$$\boldsymbol{\delta}(\phi, \omega) = \begin{cases} \boldsymbol{\sigma}(\phi, \omega), & \phi \in [0, 2\pi) ,\\ \boldsymbol{\gamma}(\phi - 2\pi, \omega), & \phi \in [2\pi, 4\pi) . \end{cases} \tag{2.3}$$

Then, it holds per definition that

$$\mathbb{E}[\partial D(\omega)] = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbb{E}[\boldsymbol{\delta}(\phi, \omega)], \phi \in [0, 4\pi)\} .$$

Therefore, the expected boundary $\mathbb{E}[\partial D(\omega)]$ consists of all points $\mathbf{x} \in \mathbb{R}^2$ with

$$\mathbf{x} = \begin{cases} \mathbb{E}[\boldsymbol{\sigma}(\phi, \omega)], & \phi \in [0, 2\pi) , \\ \mathbb{E}[\boldsymbol{\gamma}(\phi - 2\pi, \omega)], & \phi \in [2\pi, 4\pi) . \end{cases}$$

This is equivalent to

$$\mathbf{x} = \begin{cases} \mathbb{E}[q(\phi, \omega)]\mathbf{e}_r(\phi), & \phi \in [0, 2\pi) , \\ \mathbb{E}[r(\phi - 2\pi, \omega)]\mathbf{e}_r(\phi - 2\pi), & \phi \in [2\pi, 4\pi) , \end{cases}$$

which immediately implies the assertion. □

The variance of the domain $D(\omega)$ is obtained in a similar way as the expectation. In particular, it suffices to take only the radial part of the variance into account due to the star shapedness.

Theorem 2.3. *The variance of the domain $D(\omega)$ in the radial direction is given via the variances of its boundaries parameterizations in accordance with*

$$\mathbb{V}[\partial D(\omega)] = \mathbb{V}[\Sigma(\omega)] \cup \mathbb{V}[\Gamma(\omega)]$$

where

$$\begin{aligned} \mathbb{V}[\Sigma(\omega)] &= \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbb{V}[q(\phi, \omega)]\mathbf{e}_r(\phi), \phi \in I\} , \\ \mathbb{V}[\Gamma(\omega)] &= \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbb{V}[r(\phi, \omega)]\mathbf{e}_r(\phi), \phi \in I\} . \end{aligned}$$

Proof. We shall again employ the global parameterization $\boldsymbol{\delta}(\phi, \omega)$ from (2.3). For the sake of notational convenience, we denote its centered version by

$$\boldsymbol{\delta}_0(\phi, \omega) := \boldsymbol{\delta}(\phi, \omega) - \mathbb{E}[\boldsymbol{\delta}(\phi, \omega)] ,$$

and likewise for $\boldsymbol{\sigma}(\phi, \omega)$ and $\boldsymbol{\gamma}(\phi, \omega)$.

The variance of $D(\omega)$ can be determined as the trace of the covariance

$$\mathrm{Cov}[\partial D(\omega)] = \{\mathbf{X} \in \mathbb{R}^{2\times 2} : \mathbf{X} = \mathbb{E}[\boldsymbol{\delta}_0(\phi, \omega)\boldsymbol{\delta}_0(\phi', \omega)^\intercal], \phi \in [0, 4\pi)\} .$$

From this representation, one concludes that $\mathrm{Cov}[\partial D(\omega)]$ consists of all (2×2) matrices $\mathbf{X}$ with

$$\mathbf{X} = \begin{cases} \mathbb{E}[\boldsymbol{\sigma}_0(\phi, \omega)\boldsymbol{\sigma}_0(\phi', \omega)^\intercal], & \phi, \phi' \in [0, 2\pi) , \\ \mathbb{E}[\boldsymbol{\sigma}_0(\phi, \omega)\boldsymbol{\gamma}_0(\phi' - 2\pi, \omega)^\intercal], & \phi \in [0, 2\pi), \phi' \in [2\pi, 4\pi) , \\ \mathbb{E}[\boldsymbol{\gamma}_0(\phi - 2\pi, \omega)\boldsymbol{\sigma}_0(\phi', \omega)^\intercal], & \phi \in [2\pi, 4\pi), \phi' \in [0, 2\pi) , \\ \mathbb{E}[\boldsymbol{\gamma}_0(\phi - 2\pi, \omega)\boldsymbol{\gamma}_0(\phi' - 2\pi, \omega)^\intercal], & \phi, \phi' \in [2\pi, 4\pi) . \end{cases}$$

The situation $\phi = \phi'$ can only appear in the first or last case. These can be reformulated with $\phi, \phi' \in [0, 2\pi)$ as

$$\begin{aligned}\operatorname{Cov}[\boldsymbol{\sigma}, \boldsymbol{\sigma}](\phi, \phi') &= \mathbb{E}[\boldsymbol{\sigma}_0(\phi, \omega)\boldsymbol{\sigma}_0(\phi', \omega)^\intercal] \\ &= \mathbb{E}\big[\big(q(\phi, \omega) - \mathbb{E}[q](\phi)\big)\big(q(\phi', \omega) - \mathbb{E}[q](\phi)\big)\big]\mathbf{e}_r(\phi)\mathbf{e}_r(\phi')^\intercal\end{aligned}$$

and likewise as

$$\begin{aligned}\operatorname{Cov}[\boldsymbol{\gamma}, \boldsymbol{\gamma}](\phi, \phi') &= \mathbb{E}[\boldsymbol{\gamma}_0(\phi, \omega)\boldsymbol{\gamma}_0(\phi', \omega)^\intercal] \\ &= \mathbb{E}\big[\big(r(\phi, \omega) - \mathbb{E}[r](\phi)\big)\big(r(\phi', \omega) - \mathbb{E}[r](\phi)\big)\big]\mathbf{e}_r(\phi)\mathbf{e}_r(\phi')^\intercal .\end{aligned}$$

By setting $\phi = \phi'$, we arrive at

$$\operatorname{Cov}[\boldsymbol{\sigma}, \boldsymbol{\sigma}](\phi, \phi) = \mathbb{V}[q](\phi)\mathbf{e}_r(\phi)\mathbf{e}_r(\phi)^\intercal \text{ and } \operatorname{Cov}[\boldsymbol{\gamma}, \boldsymbol{\gamma}](\phi, \phi) = \mathbb{V}[q](\phi)\mathbf{e}_r(\phi)\mathbf{e}_r(\phi)^\intercal .$$

To get the radial part of the variances, we multiply the last expression by the radial direction $\mathbf{e}_r$ which yields the desired assertion. □

Consequently, in view of having $\mathbb{E}[q(\phi, \omega)]$ and $\mathbb{V}[q(\phi, \omega)]$ at hand, we need just to compute the expectation $\mathbb{E}[r(\phi, \omega)]$ and the variance $\mathbb{V}[r(\phi, \omega)]$ to obtain the expectation and the variance of the random domain $D(\omega)$.

2.2.5 Stochastic quadrature method

For numerical simulation, we aim at approximating $\mathbb{E}[r(\phi, \omega)]$ and $\mathbb{V}[r(\phi, \omega)]$ with the aid of a (quasi-) Monte Carlo quadrature. To that end, we first parameterize the stochastic influences in $q(\phi, \omega)$ by considering the parameter domain $\square := [-1, 1]^N$ and setting

$$q(\phi, \mathbf{y}) = \mathbb{E}[q](\phi) + \sum_{k=1}^{N} q_k(\phi)y_k \quad \text{for } \mathbf{y} = [y_1, \ldots, y_N]^\intercal \in \square .$$

Especially, we have $q(\phi, \mathbf{y}) \in L^\infty(\square; C^2_{\text{per}}(I))$ if $q(\phi, \omega) \in L^\infty(\Omega; C^2_{\text{per}}(I))$. Here, the space $L^\infty(\square; C^2_{\text{per}}(I))$ is equipped with the pushforward measure $\mathbb{P}_{\mathbf{Y}}$, where $\mathbf{Y} = [Y_1, \ldots, Y_N]^\intercal$. This measure is of product structure due to the independence of the random variables. If the measure $\mathbb{P}_{\mathbf{Y}}$ is absolutely continuous with respect to the Lebesgue measure, then there exists a density $\rho(\mathbf{y})$, which is also of product structure, such that there holds

$$\mathbb{E}[q](\phi) = \int_\Omega q(\phi, \omega)\,\mathrm{d}\mathbb{P}(\omega) = \int_\square q(\phi, \mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} .$$

In complete analogy, we have for the variance

$$\mathbb{V}[q](\phi) = \int_\Omega \big(q(\phi, \omega)\big)^2\,\mathrm{d}\mathbb{P}(\omega) - \big(\mathbb{E}[q](\phi)\big)^2 = \int_\square \big(q(\phi, \mathbf{y})\big)^2\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} - \big(\mathbb{E}[q](\phi)\big)^2 .$$

Now, if

$$F\colon L^\infty(\Omega; C^2_{\mathrm{per}}(I)) \to L^\infty(\Omega; C^2_{\mathrm{per}}(I)), \quad q(\phi,\omega) \mapsto r(\phi,\omega) \tag{2.4}$$

denotes the solution map, the expectation and the variance of $r(\phi,\omega)$ are given according to

$$\mathbb{E}[r](\phi) = \mathbb{E}[F(q)](\phi) \quad \text{and} \quad \mathbb{V}[r](\phi) = \mathbb{V}[F(q)](\phi)\,.$$

In view of this representation, we can apply a (quasi-) Monte Carlo quadrature in order to approximate the desired quantities.

The Monte Carlo quadrature and the quasi-Monte Carlo quadrature approximate the integral of a sufficiently smooth function f over $\square$ by a weighted sum according to

$$\int_\square f(\mathbf{y})\,\mathrm{d}\mathbf{y} \approx \frac{1}{M}\sum_{i=1}^M f(\mathbf{y}_i)\,.$$

Herein, the sample points $\{\mathbf{y}_1,\dots,\mathbf{y}_M\}$ are either chosen randomly with respect to the uniform distribution, which results in the Monte Carlo quadrature, or according to a deterministic low-discrepancy sequence, which results in the quasi-Monte Carlo quadrature. The Monte Carlo quadrature can be shown to converge, in the mean square sense, with a dimension-independent rate of $M^{-1/2}$. The quasi-Monte Carlo quadrature based, for example, on Halton points, cf. [11], converges instead with the rate $M^{\delta-1}$ for arbitrary $\delta > 0$. Although, for the quasi-Monte Carlo quadrature, the integrand has to provide bounded first-order mixed derivatives. For more details on this topic, see [5] and the references therein.

In our particular problem under consideration, the expectation $\mathbb{E}[r](\phi)$ and the variance $\mathbb{V}[r](\phi)$ are finally computed in accordance with

$$\mathbb{E}[r](\phi) = \mathbb{E}[F(q)](\phi) \approx \frac{1}{M}\sum_{i=1}^M F(q(\phi,\mathbf{y}_i))\rho(\mathbf{y}_i)$$

and

$$\mathbb{V}[r](\phi) = \mathbb{V}[F(q)](\phi) \approx \frac{1}{M}\sum_{i=1}^M \Big(F(q(\phi,\mathbf{y}_i))\Big)^2\rho(\mathbf{y}_i) - \left(\frac{1}{M}\sum_{i=1}^M F(q(\phi,\mathbf{y}_i))\rho(\mathbf{y}_i)\right)^2\,.$$

2.2.6 Analytical example

The calculations can be performed analytically if the interior boundary $\Sigma(\omega)$ is a circle around the origin with radius $q(\omega)$. Then, due to symmetry, also the free boundary $\Gamma(\omega)$ will be a circle around the origin with unknown radius $r(\omega)$. We shall thus focus on this particular situation in order to verify that the radius $r(\omega)$ depends nonlinearly on the stochastic input $q(\omega)$. Hence, on the associated expected domain $\mathbb{E}[D(\omega)]$, the overdetermined boundary value problem (1.1) has, in general, no solution.

Using polar coordinates and making the ansatz $|u(\rho, \phi)| = y(\rho)$, the solution with respect to the prescribed Dirichlet boundary condition of (1.1) has to satisfy

$$y'' + \frac{y'}{\rho} = 0, \quad y(q(\omega)) = 1, \quad y(r(\omega)) = 0\,.$$

The solution to this boundary value problem is given by

$$y(\rho) = \frac{\log\left(\frac{\rho}{r(\omega)}\right)}{\log\left(\frac{q(\omega)}{r(\omega)}\right)}\,.$$

The desired Neumann boundary condition at the free boundary $r(\omega)$ yields the equation

$$-y'(r(\omega)) = \frac{1}{r(\omega)\log\left(\frac{r(\omega)}{q(\omega)}\right)} \stackrel{!}{=} f\,,$$

which can be solved by means of Lambert's W-function:

$$r(\omega) = \frac{1}{fW\left(\frac{1}{fq(\omega)}\right)}\,. \tag{2.5}$$

Thus, the free boundary $r(\omega)$ depends nonlinearly on $q(\omega)$ since it generally holds

$$\mathbb{E}[r(\omega)] = \mathbb{E}\left[\frac{1}{fW\left(\frac{1}{fq(\omega)}\right)}\right] \neq \frac{1}{fW\left(\frac{1}{f\mathbb{E}[q]}\right)}\,. \tag{2.6}$$

Notice that the right-hand side would be the (unique) solution of the free boundary problem in the case of the interior circle of radius $\mathbb{E}[q(\omega)]$. Thus, indeed the overdetermined boundary value problem (1.1) will, in general, not be fulfilled on the expected domain $\mathbb{E}[D(\omega)]$.

2.3 Computing free boundaries

2.3.1 Trial method

For computing the expected domain $\mathbb{E}[D(\omega)]$ and its variance $\mathbb{V}[D(\omega)]$, we have to be able to determine the free boundary $\Gamma(\omega)$ for each specific realization of the fixed boundary $\Sigma(\omega)$. This will be done by the so-called trial method, which is a fixed point type iterative scheme. For the sake of simplicity in representation, we omit the stochastic variable ω in this section, i.e., we assume that $\omega \in \Omega$ is fixed.

The trial method for the solution of the free boundary problem (1.1) requires an update rule. Suppose that the current boundary in the k-th iteration is Γ_k and let the current state u_k satisfy

$$\begin{aligned} \Delta u_k &= 0 && \text{in } D_k\,,\\ u_k &= 1 && \text{on } \Sigma\,,\\ -\frac{\partial u_k}{\partial \mathbf{n}} &= f && \text{on } \Gamma_k\,. \end{aligned} \tag{3.1}$$

The new boundary Γ_{k+1} is now determined by moving the old boundary into the radial direction, which is expressed by the update rule

$$\boldsymbol{\gamma}_{k+1} = \boldsymbol{\gamma}_k + \delta r_k \mathbf{e}_r .$$

The update function $\delta r_k \in C^2_{\text{per}}([0, 2\pi])$ is chosen such that the desired homogeneous Dirichlet boundary condition is approximately satisfied at the new boundary Γ_{k+1}, i.e.,

$$0 \stackrel{!}{=} u_k \circ \boldsymbol{\gamma}_{k+1} \approx u_k \circ \boldsymbol{\gamma}_k + \left(\frac{\partial u_k}{\partial \mathbf{e}_r} \circ \boldsymbol{\gamma}_k\right) \delta r_k \quad \text{on } [0, 2\pi] , \tag{3.2}$$

where u_k is assumed to be smoothly extended into the exterior of the domain D_k. We decompose the derivative of u_k in the direction $\mathbf{e}_r$ into its normal and tangential components

$$\frac{\partial u_k}{\partial \mathbf{e}_r} = \frac{\partial u_k}{\partial \mathbf{n}} \langle \mathbf{e}_r, \mathbf{n} \rangle + \frac{\partial u_k}{\partial \mathbf{t}} \langle \mathbf{e}_r, \mathbf{t} \rangle \quad \text{on } \Gamma_k \tag{3.3}$$

to arrive finally at the following iterative scheme (cf. [9, 12, 18]):

(1) Choose an initial guess Γ_0 of the free boundary.

(2a) Solve the boundary value problem with the Neumann boundary condition on the free boundary Γ_k.

(2b) Update the free boundary Γ_k such that the Dirichlet boundary condition is approximately satisfied at the new boundary Γ_{k+1}:

$$\delta r_k = -\frac{u_k}{\frac{\partial u_k}{\partial \mathbf{e}_r}} = -\frac{u_k}{f \langle \mathbf{n}, \mathbf{e}_r \rangle + \frac{\partial u_k}{\partial \mathbf{t}} \langle \mathbf{t}, \mathbf{e}_r \rangle} . \tag{3.4}$$

(3) Repeat step 2 until the process becomes stationary up to a specified accuracy.

Notice that the update equation (3.4) is always solvable at least in a neighborhood of the optimum free boundary Γ since there it holds $-\partial u / \partial \mathbf{e}_r = f \langle \mathbf{e}_r, \mathbf{n} \rangle > 0$ due to $\partial u_k / \partial \mathbf{t} = 0$, $f > 0$ and $\langle \mathbf{e}_r, \mathbf{n} \rangle > 0$ for starlike domains.

2.3.2 Discretizing the free boundary

For the numerical computations, we discretize the radial function r_k associated with the boundary Γ_k by a trigonometric polynomial according to

$$r_k(\phi) = \frac{a_0}{2} + \sum_{i=1}^{n-1} \{a_i \cos(i\phi) + b_i \sin(i\phi)\} + \frac{a_n}{2} \cos(n\phi) . \tag{3.5}$$

This obviously ensures that r_k is always an element of $C^2_{\text{per}}(I)$. To determine the update function δr_k, represented likewise by a trigonometric polynomial, we insert the $m \geq 2n$ equidistantly distributed points $\phi_i = 2\pi i / m$ into the update equation (3.4):

$$\delta r_k = -\frac{u_k}{f \langle \mathbf{n}, \mathbf{e}_r \rangle + \frac{\partial u_k}{\partial \mathbf{t}} \langle \mathbf{t}, \mathbf{e}_r \rangle} \quad \text{in all the points } \phi_1, \ldots, \phi_m .$$

This is a discrete least-squares problem which can simply be solved by the normal equations. In view of the orthogonality of the Fourier basis, this means just a truncation of the trigonometric polynomial.

2.3.3 Boundary integral equations

Our approach to determine the solution u_k of the state equation (3.1) relies on the reformulation as a boundary integral equation by using Green's fundamental solution

$$G(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \log \|\mathbf{x} - \mathbf{y}\|_2 .$$

Namely, the solution $u_k(\mathbf{x})$ of (3.1) is given in each point $\mathbf{x} \in D$ by Green's representation formula

$$u_k(\mathbf{x}) = \int\limits_{\Gamma_k \cup \Sigma} \left\{ G(\mathbf{x}, \mathbf{y}) \frac{\partial u_k}{\partial \mathbf{n}}(\mathbf{y}) - \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} u_k(\mathbf{y}) \right\} \mathrm{d}\sigma_{\mathbf{y}} . \tag{3.6}$$

Using the jump properties of the layer potentials, we obtain the direct boundary integral formulation of the problem

$$\frac{1}{2} u_k(\mathbf{x}) = \int\limits_{\Gamma_k \cup \Sigma} G(\mathbf{x}, \mathbf{y}) \frac{\partial u_k}{\partial \mathbf{n}}(\mathbf{y}) \, \mathrm{d}\sigma_{\mathbf{y}} - \int\limits_{\Gamma_k \cup \Sigma} \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}_{\mathbf{y}}} u_k(\mathbf{y}) \, \mathrm{d}\sigma_{\mathbf{y}} , \tag{3.7}$$

where $\mathbf{x} \in \Gamma_k \cup \Sigma$. If we label the boundaries by $A, B \in \{\Gamma_k, \Sigma\}$, then (3.7) includes the single-layer operator

$$\mathcal{V}\colon C(A) \to C(B), \quad (\mathcal{V}_{AB}\rho)(\mathbf{x}) = -\frac{1}{2\pi} \int\limits_A \log \|\mathbf{x} - \mathbf{y}\|_2 \, \rho(\mathbf{y}) \, \mathrm{d}\sigma_{\mathbf{y}} \tag{3.8}$$

and the double-layer operator

$$\mathcal{K}\colon C(A) \to C(B), \quad (\mathcal{K}_{AB}\rho)(\mathbf{x}) = \frac{1}{2\pi} \int\limits_A \frac{\langle \mathbf{x} - \mathbf{y}, \mathbf{n}_{\mathbf{y}} \rangle}{\|\mathbf{x} - \mathbf{y}\|_2^2} \rho(\mathbf{y}) \, \mathrm{d}\sigma_{\mathbf{y}} \tag{3.9}$$

with the densities $\rho \in C(A)$ being the Cauchy data of u on A. Equation (3.7) in combination with (3.8) and (3.9) indicates the Neumann-to-Dirichlet map, which for problem (3.1) induces the following system of integral equations:

$$\begin{bmatrix} \frac{1}{2} I + \mathcal{K}_{\Gamma\Gamma} & -\mathcal{V}_{\Sigma\Gamma} \\ \mathcal{K}_{\Gamma\Sigma} & -\mathcal{V}_{\Sigma\Sigma} \end{bmatrix} \begin{bmatrix} u_k|_\Gamma \\ \frac{\partial u_k}{\partial \mathbf{n}}|_\Sigma \end{bmatrix} = \begin{bmatrix} \mathcal{V}_{\Gamma\Gamma} & -\mathcal{K}_{\Sigma\Gamma} \\ \mathcal{V}_{\Gamma\Sigma} & -(\frac{1}{2} I + \mathcal{K}_{\Sigma\Sigma}) \end{bmatrix} \begin{bmatrix} -f \\ 1 \end{bmatrix} . \tag{3.10}$$

The boundary integral operator on the left-hand side of this coupled system of the boundary integral equation is continuous and satisfies a Gårding inequality with respect to the product Sobolev space $L^2(\Gamma) \times H^{-1/2}(\Sigma)$ provided that $\mathrm{diam}(\Omega) < 1$.

Since its injectivity follows from potential theory, this system of integral equations is uniquely solvable according to the Riesz–Schauder theory.

The next step to the solution of the boundary value problem is the numerical approximation of the integral operators included in (3.10) which first requires the parameterization of the integral equations. To that end, we insert the parameterizations $\boldsymbol{\sigma}$ and $\boldsymbol{\gamma}_k$ of the boundaries Σ and Γ_k, respectively. For the approximation of the unknown Cauchy data, we use the collocation method based on trigonometric polynomials. Applying the trapezoidal rule for the numerical quadrature and the regularization technique along the lines of [15] to deal with the singular integrals, we arrive at an exponentially convergent Nyström method provided that the data and the boundaries and thus the solution are arbitrarily smooth.

2.3.4 Expectation and variance of the potential

We shall comment on the expectation and the variance of the potential. To that end, we consider a specific sample $\omega \in \Omega$ and assume that the associated free boundary $\Gamma(\omega)$ is known. Then, with the aid of the parameterizations

$$\boldsymbol{\sigma}(\omega)\colon [0, 2\pi] \to \Sigma(\omega) \quad \text{and} \quad \boldsymbol{\gamma}(\omega)\colon [0, 2\pi] \to \Gamma(\omega)\,,$$

we arrive, in view of (3.6), for $\mathbf{x} \in D(\omega)$ at the potential representation

$$u(\mathbf{x}, \omega) = \sum_{A \in \{\Sigma(\omega), \Gamma(\omega)\}} \int_0^{2\pi} \{k_A^{\mathcal{V}}(\mathbf{x}, \phi, \omega)\rho_A^{\mathcal{V}}(\phi, \omega) - k_A^{\mathcal{K}}(\mathbf{x}, \phi, \omega)\rho_A^{\mathcal{K}}(\phi, \omega)\}\,\mathrm{d}\phi\,, \tag{3.11}$$

where

$$\begin{aligned} k_{\Sigma(\omega)}^{\mathcal{V}}(\mathbf{x}, \phi, \omega) &= G(\mathbf{x}, \boldsymbol{\sigma}(\phi, \omega))\|\boldsymbol{\sigma}'(\phi, \omega)\|_2\,, \\ k_{\Gamma(\omega)}^{\mathcal{V}}(\mathbf{x}, \phi, \omega) &= G(\mathbf{x}, \boldsymbol{\gamma}(\phi, \omega))\|\boldsymbol{\gamma}'(\phi, \omega)\|_2\,, \end{aligned}$$

and

$$\begin{aligned} k_{\Sigma(\omega)}^{\mathcal{K}}(\mathbf{x}, \phi, \omega) &= \frac{\partial G(\mathbf{x}, \boldsymbol{\sigma}(\phi, \omega))}{\partial \mathbf{n}_{\mathbf{y}}}\|\boldsymbol{\sigma}'(\phi, \omega)\|_2\,, \\ k_{\Gamma(\omega)}^{\mathcal{K}}(\mathbf{x}, \phi, \omega) &= \frac{\partial G(\mathbf{x}, \boldsymbol{\gamma}(\phi, \omega))}{\partial \mathbf{n}_{\mathbf{y}}}\|\boldsymbol{\gamma}'(\phi, \omega)\|_2\,. \end{aligned}$$

Moreover, the related densities are given according to

$$\begin{aligned} \rho_{\Sigma(\omega)}^{\mathcal{V}}(\phi, \omega) &= \frac{\partial u}{\partial \mathbf{n}}(\boldsymbol{\sigma}(\phi, \omega)), \quad & \rho_{\Gamma(\omega)}^{\mathcal{V}}(\phi, \omega) &= \frac{\partial u}{\partial \mathbf{n}}(\boldsymbol{\gamma}(\phi, \omega))\,, \\ \rho_{\Sigma(\omega)}^{\mathcal{K}}(\phi, \omega) &= u(\boldsymbol{\sigma}(\phi, \omega)), & \rho_{\Gamma(\omega)}^{\mathcal{K}}(\phi, \omega) &= u(\boldsymbol{\gamma}(\phi, \omega))\,. \end{aligned}$$

These densities coincide with the Cauchy data of the potential $u(\omega)$ on the boundary $\partial D(\omega)$.