



Jack Ogaja



# **Higher Order Horizontal Discretization of Euler-Equations in a Non-Hydrostatic NWP and RCM Model**



**Cuvillier Verlag Göttingen**  
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HIGHER ORDER HORIZONTAL DISCRETIZATION OF EULER-EQUATIONS IN  
A NON-HYDROSTATIC NWP AND RCM MODEL



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## ABSTRACT

For the first time, efficient fourth order accurate horizontal discretization of the Euler equations has been implemented in an operational NWP and RCM model implicitly conserving the kinetic energy of the rotational flow, which allowed simulation of regional climate over a European domain with COSMO model without horizontal numerical filter. The first spatial scheme implemented is a central difference fourth order scheme which is a natural extension of the COSMO spatial schemes by consistent discretization of linear and nonlinear terms of the model's Euler equations to fourth order accuracy. The second, is a symmetric fourth order accurate scheme previously used for turbulence studies.

It has been shown analytically that none of the previously implemented numerical schemes (COSMO schemes) is higher order (higher than second order) accurate. In contrast, the new schemes have been shown to be fourth order accurate. It is further shown that the symmetric scheme conserves rotational part of the momentum and kinetic energy *a priori*. The ageostrophic divergent part is discretized fourth order accurate. Analysis of the numerical errors of the schemes show that the new schemes exhibit significantly improved accuracy in terms of amplitude error, and slight improvement in terms of phase errors in comparison with the COSMO schemes. Linear stability analysis of the new schemes reveal improved linear stability which allows significantly larger time steps compared to the COSMO schemes. Theoretical analysis further reveal significant decrease in alias error of the symmetric schemes, which reduces significantly the non-linear instability of the schemes compared to other schemes. Together with conservation of the kinetic energy, it allows stable simulations without numerical diffusion.

A two-dimensional numerical idealized test has been used to reveal the second order and fourth order convergence properties of the schemes. Instability attributed to inconsistent discretization of non-linear and linear terms of the model's Euler equations has been demonstrated using the same idealized test case and a real case study. The model's numerical accuracy, stability and simulation quality using different schemes have been further assessed by analysis of a series of regional climate simulations over European domain using ERA-interim re-analysis boundary data. The results show a significant improvement of the models stability and effective resolution. The results also reveal significant effects of the new schemes on the coupling between the model's resolved dynamics and the sub-grid scale physical parameterizations. This has particularly enhanced the bias in the model's simulated convective precipitation.



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