CRC REVIVALS

Handbook of Mathematical Sciences

Sixth Edition

Edited by
William H. Beyer

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CRC Handbook of Mathematical Sciences

6th Edition

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PREFACE

Numerical tables of mathematical and statistical functions are in continual demand by professional scientists, by those in the teaching profession, and by students of mathematics and related sciences. The *CRC Handbook of Mathematical Sciences,* published by CRC Press, Inc., contains the most up-to-date, authoritative, logically arranged, and ready-usable collection of reference material available. Prior to the preparation of this 6th Edition of the *Handbook*, the contents of predecessor editions were carefully examined to determine if certain tables and/or reference materials should be altered, expanded, or deleted. The net result is that this 6th Edition has been prepared to provide an adequately broad spectrum of traditional and modern mathematical sciences data necessary for today's scientific needs, even in light of today's computer technology.

The same successful format which characterized the 5th Edition has been retained. Material is presented in a multisectional format, with each section containing a valuable collection of fundamental reference material — both expository and tabular in form. The format is such that existing sections can be expanded or reduced as necessary, and new sections can be developed as warranted. The 6th Edition has been vastly improved by the addition of material on numerical solutions of nonlinear equations, statistical tests of hypotheses, statistical confidence intervals, and analysis of variance tables. Omissions include tables involving squares and square roots, cube and cube roots; logarithms of the binomial coefficients; reciprocals, circumferences and areas of circles; natural trigonometric functions for angles in radius; and financial tables. It is hoped that these changes will prove to be beneficial to the users of the *Handbook*.

The editor gratefully acknowledges the services of Paul Gottehrer, Editor, for the handling of the detail work which is so essential in the final production of this edition.

All errors called to our attention have, to the best of the Editor's knowledge, been corrected. As in the past CRC Press, Inc., and the Editor invite and welcome regular input from the many users of this handbook. Since the inception, some 20 years ago, of *The CRC Handbook of Mathematical Sciences* (formerly called *The CRC Handbook of Mathematics*), users of the *Handbook* have forwarded data, advice, guidance, and constructive criticisms. This information has provided a most effective means for keeping the editions of the *Handbook* updated, accurate, and abreast of the times.

William H. Beyer, Editor

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HOWARD EVES, PH.D.
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Trigonometry

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The Laplace Transform

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GREEK ALPHABET

Greek letter	Greek name	English equivalent	Greek letter	Greek name	English equivalent
Α α	Alpha	a	Ν ν	Nu	n
Β β	Beta	b	Ξ ξ	Xi	x
Γ γ	Gamma	g	Ο ο	Omicron	ŏ
Δ δ	Delta	d	Π π	Pi	p
Ε ε	Epsilon	ĕ	Ρ ρ	Rho	r
Ζ ζ	Zeta	z	Σ σ ς	Sigma	s
Η η	Eta	ē	Τ τ	Tau	t
Θ θ ϑ	Theta	th	Υ υ	Upsilon	u
Ι ι	Iota	i	Φ φ ϕ	Phi	ph
Κ κ	Kappa	k	Χ χ	Chi	ch
Λ λ	Lambda	l	Ψ ψ	Psi	ps
Μ μ	Mu	m	Ω ω	Omega	ō

THE NUMBER OF EACH DAY OF THE YEAR

Day of Mo.	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.	Day of Mo.
1	1	32	60	91	121	152	182	213	244	274	305	335	1
2	2	33	61	92	122	153	183	214	245	275	306	336	2
3	3	34	62	93	123	154	184	215	246	276	307	337	3
4	4	35	63	94	124	155	185	216	247	277	308	338	4
5	5	36	64	95	125	156	186	217	248	278	309	339	5
6	6	37	65	96	126	157	187	218	249	279	310	340	6
7	7	38	66	97	127	158	188	219	250	280	311	341	7
8	8	39	67	98	128	159	189	220	251	281	312	342	8
9	9	40	68	99	129	160	190	221	252	282	313	343	9
10	10	41	69	100	130	161	191	222	253	283	314	344	10
11	11	42	70	101	131	162	192	223	254	284	315	345	11
12	12	43	71	102	132	163	193	224	255	285	316	346	12
13	13	44	72	103	133	164	194	225	256	286	317	347	13
14	14	45	73	104	134	165	195	226	257	287	318	348	14
15	15	46	74	105	135	166	196	227	258	288	319	349	15
16	16	47	75	106	136	167	197	228	259	289	320	350	16
17	17	48	76	107	137	168	198	229	260	290	321	351	17
18	18	49	77	108	138	169	199	230	261	291	322	352	18
19	19	50	78	109	139	170	200	231	262	292	323	353	19
20	20	51	79	110	140	171	201	232	263	293	324	354	20
21	21	52	80	111	141	172	202	233	264	294	325	355	21
22	22	53	81	112	142	173	203	234	265	295	326	356	22
23	23	54	82	113	143	174	204	235	266	296	327	357	23
24	24	55	83	114	144	175	205	236	267	297	328	358	24
25	25	56	84	115	145	176	206	237	268	298	329	359	25
26	26	57	85	116	146	177	207	238	269	299	330	360	26
27	27	58	86	117	147	178	208	239	270	300	331	361	27
28	28	59	87	118	148	179	209	240	271	301	332	362	28
29	29	*	88	119	149	180	210	241	272	302	333	363	29
30	30		89	120	150	181	211	242	273	303	334	364	30
31	31		90		151		212	243		304		365	31

* In leap years, after February 28, add 1 to the tabulated number.

I. CONSTANTS AND CONVERSION FACTORS

SI SYSTEM OF MEASUREMENT

SI, which is the abbreviation of the French words "Systeme Internationale d'Unites," is the accepted abbreviation for the International Metric System, which has seven base units, as shown below.

UNITS FOR A SYSTEM OF MEASURES AS USED INTERNATIONALLY

Quantity measured	Unit	Abbreviation
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	degree Kelvin	K
Luminous intensity	candela	cd
Amount of substance	mole	mol

Supplementary and Derived Units From Base Units as used Internationally

Supplementary Units

Plane angle	radian	rad	
Solid angle	steradian	sr	

Derived Units

Area	square meter	m^2	
Volume	cubic meter	m^3	
Frequency	hertz	Hz	(s^{-1})
Density	kilogram per cubic meter	kg/m^3	
Velocity	meter per second	m/s	
Angular velocity	radian per second	rad/s	
Acceleration	meter per second squared	m/s^2	
Angular acceleration	radian per second squared	rad/s^2	
Force	newton	N	$(kg\cdot m/s^2)$
Pressure	newton per sq meter	N/m^2	
Kinematic viscosity	sq meter per second	m^2/s	
Dynamic viscosity	newton-second per sq meter	$N\cdot s/m^2$	
Work, energy, quantity of heat	joule	J	$(N\cdot m)$
Power	watt	W	(J/s)
Electric charge	coulomb	C	$(A\cdot s)$
Voltage, potential difference, electromotive force	volt	V	(W/A)
Electric field strength	volt per meter	V/m	
Electric resistance	ohm	Ω	(V/A)
Electric capacitance	farad	F	$(A\cdot s/V)$
Magnetic flux	weber	Wb	$(V\cdot s)$
Inductance	henry	H	$(V\cdot s/A)$
Magnetic flux density	tesla	T	(Wb/m^2)
Magnetic field strength	ampere per meter	A/m	
Magnetomotive force	ampere	A	
Luminous flux	lumen	lm	$(cd\cdot sr)$
Luminance	candela per sq meter	cd/m^2	
Illumination	lux	lx	(lm/m^2)

RECOMMENDED UNIT PREFIXES

Multiples and submultiples	Prefixes	Symbols
10^{18}	exa	E
10^{15}	peca	P
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{2}	hecto	h
10	deka	da
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ (greek mu)
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a

DEFINED VALUES AND EQUIVALENTS

Meter .... (m)	1 650 763.73 wave lengths in vacuo of the unperturbed transition $2p_{10} - 5d_5$ in ^{86}Kr
Kilogram .... (kg)	mass of the international kilogram at Sèvres, France
Second .... (s)	1/31 556 925.974 7 of the tropical year at 12^h ET, 0 January 1900
Degree Kelvin .... (°K)	defined in the thermodynamic scale by assigning 273.16°K to the triple point of water (freezing point, 273.15°K = 0°C)
Unified atomic mass unit .... (u)	1/12 the mass of an atom of the ^{12}C nuclide
Mole .... (mol)	amount of substance containing the same number of atoms as 12 g of pure ^{12}C
Standard acceleration of free fall .... (g_n)	9.806 65 M s^{-2}, 980.665 cm s^{-2}
Normal atmospheric pressure .... (atm)	101 325 N m^{-2}, 1 013 250 dyn cm^{-2}
Thermochemical calorie .... (cal_{th})	4.1840 J, 4.1840 × 10^7 erg
International Steam Table calorie .... (cal_{IT})	4.1868 J, 4.1868 × 10^7 erg
Liter .... (l)	0.001 000 028 m^3, 1 000.028 cm^3 (recommended by CIPM, 1950)
	0.001 m^3, 1000 cm^3 (recommended by GCWM 1964)
Inch .... (in)	0.0254 m, 2.54 cm
Pound (avdp) .... (lb)	0.453 592 37 kg, 453.592 37 g

CONVERSION FACTORS

Conversion Factors – Metric to English

To obtain	Multiply	By
Inches	Centimeters	0.3937007874
Feet	Meters	3.280839895
Yards	Meters	1.093613298
Miles	Kilometers	0.6213711922
Ounces	Grams	$3.527396195 \times 10^{-2}$
Pounds	Kilograms	2.204622622
Gallons	Liters	0.2641720524
Fluid ounces	Milliliters (cc)	$3.381402270 \times 10^{-2}$
Square inches	Square centimeters	0.1550003100
Square feet	Square meters	10.76391042
Square yards	Square meters	1.195990046
Cubic inches	Milliliters (cc)	$6.102374409 \times 10^{-2}$
Cubic feet	Cubic meters	35.31466672
Cubic yards	Cubic meters	1.307950619

Conversion Factors – English to Metric*

To obtain	Multiply	By
Microns	Mils	**25.4**
Centimeters	Inches	**2.54**
Meters	Feet	**0.3048**
Meters	Yards	**0.9144**
Kilometers	Miles	**1.609344**
Grams	Ounces	28.34952313
Kilograms	Pounds	**0.45359237**
Liters	Gallons	**3.785411784**
Milliliters (cc)	Fluid ounces	29.57352956
Square centimeters	Square inches	**6.4516**
Square meters	Square feet	**0.09290304**
Square meters	Square yards	**0.83612736**
Milliliters (cc)	Cubic inches	**16.387064**
Cubic meters	Cubic feet	$2.831684659 \times 10^{-2}$
Cubic meters	Cubic yards	0.764554858

Conversion Factors – General*

To obtain	Multiply	By
Atmospheres	Feet of water @ 4°C	2.950×10^{-5}
Atmospheres	Inches of mercury @ 0°C	3.342×10^{-2}
Atmospheres	Pounds per square inch	6.804×10^{-2}
BTU	Foot-pounds	1.285×10^{-3}
BTU	Joules	9.480×10^{-4}
Cords	Cubic feet	**128**

*Boldface numbers are exact; others are given to ten significant figures where so indicated by the multiplier factor.

Conversion Factors – General (Continued)

To obtain	Multiply	By
Degree (angle)	Radians	57.2958
Ergs	Foot-pounds	1.356×10^{7}
Feet	Miles	**5280**
Feet of water @ 4°C	Atmospheres	33.90
Foot-pounds	Horsepower-hours	1.98×10^{6}
Foot-pounds	Kilowatt-hours	2.655×10^{6}
Food-pounds per min	Horsepower	3.3×10^{4}
Horsepower	Foot-pounds per sec	1.818×10^{-3}
Inches of mercury @ 0°C	Pounds per square inch	2.036
Joules	BTU	1054.8
Joules	Foot-pounds	1.35582
Kilowatts	BTU per min	1.758×10^{-2}
Kilowatts	Foot-pounds per min	2.26×10^{-5}
Kilowatts	Horsepower	0.745712
Knots	Miles per hour	0.86897624
Miles	Feet	1.894×10^{-4}
Nautical miles	Miles	0.86897624
Radians	Degrees	1.745×10^{-2}
Square feet	Acres	**43560**
Watts	BTU per min	17.5796

Temperature Factors

$$°F = 9/5\ (°C) + 32$$

Fahrenheit temperature = 1.8 (temperature in kelvins) −459.67

$$°C = 5/9\ [(°F) - 32]$$

Celsius temperature = temperature in kelvins −273.15

Fahrenheit temperature = 1.8 (Celsius temperature) +32

CONVERSION FACTORS

U. S. AND METRIC UNITS

Each unit in bold face type is followed by its equivalent in other units of the same quantity.

Acre—0.0015625 square mile (statute); 4.3560×10^{4} square feet; 0.40468564 hectare.
Bushel—(U.S.)—1.244456 cubic feet; 2150.42 cubic inches; 0.035239 cubic meter; 35.23808 liters.
Centimeter—0.0328084 foot; 0.393701 inch.
Circular Mil—7.853982×10^{-7} square inches; 5.067075×10^{-6} square centimeters.
Cubic Centimeter—0.061024 cubic inch; 0.270512 dram (U. S. fluid); 16.230664 minims (U. S.); 0.999972 milliliter.
Cubic Foot—0.803564 bushel (U. S.); 7.480520 gallons (U. S. liquid); 0.028317 cubic meter; 28.31605 liters.
Cubic Inch—16.387064 cubic centimeters.
Cubic Meter—35.314667 cubic feet; 264.17205 gallons (U. S. liquid).
Foot—0.3048 meter.
Gallon (U. S. liquid)—0.1336816 cubic foot; 0.832675 gallon (British); 231 cubic inches; 0.0037854 cubic meter; 3.785306 liters.
Grain—0.06479891 gram.
Gram—0.00220462 pound (avoirdupois); 0.035274 ounce (avoirdupois); 15.432358 grains.
Hectare—2.471054 acres; 1.07639×10^{5} square feet.
Inch—2.54 centimeters.
Kilogram—2.204623 pounds (avoirdupois).
Kilometer—0.621371 mile (statute).
Liter—0.264179 gallon (U. S. liquid); 0.0353157 cubic foot; 1.056718 quarts (U. S. liquid).
Meter—1.093613 yards; 3.280840 feet; 39.37008 inches.
Mile (statute)—1.609344 kilometers.
Ounce (U. S. fluid)—1.804688 cubic inches; 29.573730 cubic centimeters.
Ounce (avoirdupois)—28.349523 grams.
Ounce (apothecary or troy)—31.103486 grams.
Pint (U. S. liquid)—0.473163 liter; 473.17647 cubic centimeters.
Pound (avoirdupois)—0.453592 kilogram; 453.59237 grams.
Pound (apothecary or troy)—0.3732417 kilogram, 373.24172 grams.
Quart (U. S. dry)—1.10119 liters.
Quart (liquid)—0.946326 liter.
Radian—57.295779 degrees.
Rod—5.0292 meters.
Square Centimeter—0.155000 square inch.
Square Foot—0.09290304 square meter.
Square Inch—645.16 square millimeters.
Square Meter—10.763910 square feet.
Square Yard—0.836127 square meter.
Ton (short)—907.18474 kilograms.
Yard—0.9144 meter.

See Index for extensive Conversion Factors.

METRIC CONVERSION TABLE

Inches	Centimeters	Centimeters	Inches
1	2.54	1	0.393701
2	5.0	2	0.787402
3	7.62	3	1.181103
4	10.16	4	1.574804
5	12.70	5	1.968505
6	15.24	6	2.362206
7	17.78	7	2.755907
8	20.32	8	3.149608
9	22.86	9	3.543309

Feet	Meters	Meters	Feet
1	0.3048	1	3.280840
2	0.6096	2	6.561680
3	0.9144	3	9.842520
4	1.2192	4	13.123360
5	1.5240	5	16.404200
6	1.8288	6	19685040
7	2.1336	7	22.965880
8	2.4384	8	26.246720
9	2.7432	9	29.527560

Yards	Meters	Meters	Yards
1	0.9144	1	1.0936133
2	1.8288	2	2.1872266
3	2.7432	3	3.2808399
4	3.6576	4	4.3744532
5	4.5720	5	5.4680665
6	5.4864	6	6.5616798
7	6.4008	7	7.6552931
8	7.3152	8	8.7489064
9	8.2296	9	9.8425197

Miles (statute)	Kilometers	Kilometers	Miles
1	1.609344	1	0.6213712
2	3.218688	2	1.2427424
3	4.828032	3	1.8641136
4	6.437376	4	2.4854848
5	8.046720	5	3.1068560
6	9.656064	6	3.7282272
7	11.265408	7	4.3495984
8	12.874752	8	4.9709696
9	14.484096	9	5.5923408

Square inches	Square centimeters	Square centimeters	Square inches
1	6.45	1	0.155
2	12.90	2	0.310
3	19.36	3	0.465
4	25.81	4	0.620
5	32.26	5	0.775
6	38.71	6	0.930
7	45.16	7	1.085
8	51.61	8	1.240
9	58.06	9	1.395

METRIC CONVERSION TABLE (continued)

Square feet	Square meters	Square meters	Square feet
1	0.0929	1	10.76
2	0.1858	2	21.53
3	0.2787	3	32.29
4	0.3716	4	43.06
5	0.4645	5	53.82
6	0.5574	6	64.58
7	0.6503	7	75.35
8	0.7432	8	86.11
9	0.8361	9	96.88

Cubic inches	Cubic centimeters	Cubic centimeters	Cubic Inches
1	16.39	1	0.0610
2	32.77	2	0.1221
3	49.16	3	0.1831
4	65.55	4	0.2441
5	81.94	5	0.3051
6	98.32	6	0.3661
7	114.71	7	0.4272
8	131.10	8	0.4882
9	147.48	9	0.5492

Cubic feet	Cubic meters	Cubic meters	Cubic feet
1	0.0283	1	35.3
2	0.0566	2	70.6
3	0.0850	3	105.9
4	0.1133	4	141.3
5	0.1416	5	176.6
6	0.1699	6	211.9
7	0.1982	7	247.2
8	0.2265	8	282.5
9	0.2549	9	317.8

Cubic feet	Liters	Liters	Cubic feet
1	28.32	1	0.0353
2	56.63	2	0.0706
3	84.95	3	0.1060
4	113.26	4	0.1413
5	141.58	5	0.1766
6	169.90	6	0.2119
7	198.21	7	0.2472
8	226.53	8	0.2825
9	254.84	9	0.3178

U.S. gallons	Liters	Liters	U.S. gallons
1	3.785306	1	0.264179
2	7.570612	2	0.528358
3	11.355918	3	0.792537
4	15.141224	4	1.056716
5	18.926530	5	1.320895
6	22.711836	6	1.585074
7	26.497142	7	1.849253
8	30.282448	8	2.113432
9	34.067754	9	2.377611

METRIC CONVERSION TABLE (continued)

British or Imperial gallons	Liters	Liters	British or Imperial gallons
1	4.546	1	0.220
2	9.092	2	0.440
3	13.638	3	0.660
4	18.184	4	0.880
5	22.730	5	1.100
6	27.276	6	1.320
7	31.822	7	1.540
8	36.368	8	1.760
9	40.914	9	1.980

Pounds (av)	Kilograms	Kilograms	Pounds (av)
1	0.45359237	1	2.2046226
2	0.90718474	2	4.4092452
3	1.36077711	3	6.6138678
4	1.81436948	4	8.8184904
5	2.26796185	5	11.0231130
6	2.72155422	6	13.2277356
7	3.17514659	7	15.4323582
8	3.62873896	8	17.6369808
9	4.08233133	9	19.8416034

Ounces Avoirdupois	Grams	Grams	Ounces Avoirdupois
1	28.350	1	0.035274
2	56.699	2	0.070548
3	85.049	3	0.10582
4	113.40	4	0.14110
5	141.75	5	0.17637
6	170.10	6	0.21164
7	198.45	7	0.24692
8	226.80	8	0.28219
9	255.15	9	0.31747

Pounds per foot	Kilograms per meter	Kilograms per meter	Pounds per foot
1	1.4882	1	0.6720
2	2.9763	2	1.3439
3	4.4645	3	2.0159
4	5.9527	4	2.6879
5	7.4408	5	3.3598
6	8.9290	6	4.0318
7	10.4171	7	4.7038
8	11.9053	8	5.3758
9	13.3935	9	6.0477

Pounds per square inch	Kilograms per square centimeter	Kilograms per square centimeter	Pounds per square inch
1	0.0703	1	14.22
2	0.1406	2	28.45
3	0.2109	3	42.67
4	0.2812	4	56.89
5	0.3515	5	71.12
6	0.4218	6	85.34
7	0.4922	7	99.96
8	0.5625	8	113.79
9	0.6328	9	128.01

METRIC CONVERSION TABLE (continued)

Pounds per square inch	Kilonewtons per square meter	Kilonewtons per square meter	Pounds per square inch
1	6.895	1	0.145
2	13.790	2	0.290
3	20.684	3	0.435
4	27.579	4	0.580
5	34.474	5	0.725
6	41.369	6	0.870
7	48.263	7	1.015
8	55.158	8	1.160
9	62.053	9	1.305

Pounds per square foot	Kilograms per square meter	Kilograms per square meter	Pounds per square foot
1	4.88	1	0.2048
2	9.77	2	0.4096
3	14.65	3	0.6144
4	19.53	4	0.8193
5	24.41	5	1.0241
6	29.30	6	1.2289
7	34.18	7	1.4337
8	39.06	8	1.6385
9	43.94	9	1.8433

Pound feet	Kilogram meters	Kilogram meters	Pound feet
1	0.138	1	7.23
2	0.277	2	14.47
3	0.415	3	21.70
4	0.553	4	28.93
5	0.691	5	36.17
6	0.830	6	43.40
7	0.968	7	50.63
8	1.106	8	57.87
9	1.244	9	65.10

Foot pounds force	Joules	Joules	Foot pounds force
1	1.356	1	0.7376
2	2.712	2	1.4751
3	4.068	3	2.2127
4	5.423	4	2.9502
5	6.779	5	3.6878
6	8.135	6	4.4254
7	9.491	7	5.1629
8	10.847	8	5.9005
9	12.202	9	6.6381

British thermal units	Kilojoules	Kilojoules	British thermal units
1	105.51	1	94.78
2	211.01	2	189.56
3	316.52	3	284.35
4	422.02	4	379.13
5	527.53	5	473.91
6	633.03	6	568.69
7	738.54	7	663.47
8	844.04	8	758.25
9	949.55	9	853.04

METRIC CONVERSION TABLE (continued)

Horsepower	Kilowatts	Kilowatts	Horsepower
1	0.746	1	1.341
2	1.491	2	2.682
3	2.237	3	4.023
4	2.983	4	5.364
5	3.729	5	6.705
6	4.474	6	8.046
7	5.220	7	9.387
8	5.966	8	10.728
9	6.711	9	12.069

Pounds force	Newtons	Newtons	Pounds force
1	4.448	1	0.22481
2	8.896	2	0.44962
3	13.345	3	0.67443
4	17.793	4	0.89924
5	22.241	5	1.12404
6	26.689	6	1.34885
7	31.138	7	1.57366
8	35.586	8	1.79847
9	40.034	9	2.02328

CONVERSION FACTORS

L. P. Buseth

To convert from	To	Multiply by
Abampere	Ampere	*10*
Abcoulomb	Coulomb	*10*
Abfarad	Farad	1×10^{9}
Abhenry	Henry	1×10^{-9}
Abmho	Siemens (mho)	1×10^{9}
Abohm	Ohm	1×10^{-9}
Abvolt	Volt	1×10^{-8}
Acre	Hectare	0.40468564
	Square foot	*43560*
	Square kilometer	4.046856×10^{-3}
	Square meter	4046.85642
	Square mile	1.5625×10^{-3} (1/640)
	Square yard	*4840*
Acre (U.S. Survey)	Square meter	4046.872610
Acre-foot	Cubic meter	1233.482
	Cubic yard	1613.333
Acre-inch	Cubic foot	*3630*
	Cubic meter	102.7902
	Gallon (Brit.)	22610.67
	Gallon (U.S.)	27154.29
Ampere (int., mean)	Ampere	0.99985
Ampere (int., U.S.)	Ampere	0.999835
Ampere/square centimeter	Ampere/square inch	*6.4516*
Ampere/square inch	Ampere/square centimeter	0.1550003
Ampere-hour	Coulomb	*3600*
Ampere(-turn)	Gilbert	1.256637
Ångström	Nanometer	*0.1*
Apostilb	Candela/square meter	0.3183099 ($1/\pi$)
Are	Square foot	1076.391
	Square meter	*100*
Astronomical unit	Kilometer	1.4959787×10^{8}
Atmosphere	Atmosphere (tech.)	1.033227
	Bar	*1.01325*
	Foot of H_2O (conv.)	33.89854
	Inch of Hg (conv.)	29.92126
	Kilogram-force/square centimeter	1.033227
	Kilopascal	*101.325*
	Meter of H_2O (conv.)	10.33227
	Millibar	*1013.25*
	Millimeter of Hg (conv.)	760
	Newton/square centimeter	*10.1325*
	Pascal (N/square meter)	1.01325×10^{5}
	Pound-force/square foot	2116.22
	Pound-force/square inch	14.69595
	Ton-force (long)/square foot	0.944740
	Ton-force (short)/square foot	1.058108
	Ton-force (long)/square inch	6.56069×10^{-3}
	Ton-force (short)/square inch	7.34797×10^{-3}
	Torr	*760*
Atmosphere (tech.)	Atmosphere	0.967841
	Bar	*0.980665*
	Foot of H_2O (conv.)	32.8084
	Inch of Hg (conv.)	28.9590
	Kilogram-force/square centimeter	*1*
	Kilopascal	*98.0665*
	Meter of H_2O (conv.)	*10*
	Millibar	*980.665*
	Millimeter of Hg (conv.)	735.559
	Newton/square centimeter	9.80665
	Pascal (N/m^2)	98066.5
	Pound-force/square inch	14.22334
Bag (Brit.)	Gallon (Brit.)	*24*
Bar	Atmosphere	0.9869233
	Atmosphere (tech.)	1.019716
	Dyne/square centimeter	1×10^{6}
	Foot of H_2O (conv.)	33.4553
	Inch of Hg (conv.)	29.5300
	Kilogram-force/square centimeter	1.019716
	Kilopascal	*100*
	Meter of H_2O (conv.)	10.19716
	Millibar	*1000*
	Millimeter of Hg (conv.)	750.062
	Newton/square centimeter	*10*
	Pascal (N/m^2)	1×10^{5}
	Pound-force/square foot	2088.54
	Pound-force/square inch	14.50377
	Ton-force (long)/square foot	0.932385
	Ton-force (short)/square foot	1.04427
	Ton-force (long)/square inch	6.47490×10^{-3}
	Ton-force (short)/square inch	7.25189×10^{-3}
	Torr	750.062
Barleycorn (Brit.)	Inch	0.333333 (1/3)
Barn	Square meter	1×10^{-28}
Barrel (Brit., beer)	Gallon (Brit.)	*36*
	Liter	163.6592
Barrel (Brit., wine)	Gallon (Brit.)	*31.5*
	Liter	143.2018
Barrel (petroleum)	Cubic foot	5.614583
	Cubic meter	0.1589873
	Gallon (Brit.)	34.97232
	Gallon (U.S.)	*42*
	Liter	158.9873
Barrel (U.S., dry)	Bushel (U.S.)	3.281219
	Cubic foot	4.083333
	Cubic inch	*7056*
	Cubic meter	0.1156271
	Liter	115.6271
	Pint (U.S., dry)	209.998
	Quart (U.S., dry)	104.9990
Barrel (U.S., cranb.)	Cubic inch	*5826*
	Liter	95.4710
Barrel (U.S., liquid)	Cubic foot	*4.2109375*
	Cubic inch	*7276.5*
	Cubic meter	0.1192405
	Gallon (Brit.)	26.22925
	Gallon (U.S.)	*31.5*
	Liter	119.2405
Barye	Bar	1×10^{-6}
	Dyne/square centimeter	*1*
Becquerel	Curie	2.702703×10^{11}
Biot	Ampere	*10*
Board foot	Cubic foot	0.083333 (1/12)
Bolt (cloth)	Foot	*120*
Btu	Calorie	251.996
	Cubic foot-atmosphere	*0.367717*
	Foot-poundal	25036.9
	Foot-pound-force	778.169
	Horsepower-hour	3.93015×10^{-4}
	Horsepower-hour (metric)	3.98466×10^{-4}
	Joule	1055.056
	Kilocalorie	0.251996
	Kilogram-force-meter	107.586
	Kilowatt-hour	2.93071×10^{-4}
	Liter-atmosphere	10.4126
	Watt-hour	0.293071
Btu (39 °F, 4 °C)	Joule	1059.67
Btu (60 °F, 15.6 °C)	Joule	1054.68
Btu (mean)	Joule	1055.87
Btu (thermochemical)	Joule	1054.350
Btu/cubic foot	Joule/cubic meter	37258.9

CONVERSION FACTORS (continued)

To convert from	To	Multiply by
	kilocalorie/cubic meter	8.89915
Btu/°F	Calorie/°C	453.592
	Joule/°C	1899.10
Btu/hour	Btu/minute	0.0166667 (*1/60*)
	Btu/second	2.77778×10^{-4}
	Calorie/second	0.0699988
	Foot-pound-force/second	0.216158
	Horsepower	3.93015×10^{-4}
	Watt	0.293071
Btu/(hour × square foot)	Watt/square meter	3.15459
Btu/(hour × square foot × °F)	Calorie/second × square meter × °C)	1.35623
	Watt/(square meter × °C)	5.67826
Btu/(hour × square foot × °F/foot)	Watt/(meter × °C)	1.73073
Btu/(hour × square foot × °F/inch)	Watt/(meter × °C)	0.144228
Btu/minute	Calorie/second	4.19993
	Horsepower	0.0235809
	Watt	17.5843
Btu/(minute × square foot)	Watt/square meter	189.273
Btu/pound	Calorie/gram	0.555556
	Joule/kilogram	*2326*
	Kilocalorie/kilogram	0.555556
	Watt-hour/kilogram	0.646111
Btu/(pound × °F)	Calorie/(gram × °C)	*1*
	Joule/(kilogram × °C)	*4186.8*
Btu/second	Horsepower	1.41485
	Kilowatt	1.055056
Btu/(second × square foot)	Kilowatt/square meter	11.3565
Btu/(second × square foot × °F)	Kilowatt/(square meter × °C)	20.4417
Btu/(second × square foot × °F/foot)	Kilowatt/(meter × °C)	6.23064
Btu/(second × square foot × °F /inch)	Watt/(meter × °C)	519.220
Btu/square foot	Joule/square meter	11356.5
	watt-hour/square meter	3.15459
Bucket (Brit.)	Gallon (Brit.)	*4*
Bushel (Brit.)	Bushel (U.S.)	1.032057
	Gallon (Brit.)	*8*
	Liter	*36.36872*
Bushel (U.S.)	Barrel (U.S., dry)	0.3047647
	Bushel (Brit.)	0.9689390
	Cubic foot	1.244456
	Cubic inch	*2150.42*
	Gallon (Brit.)	7.751512
	Gallon (U.S., liquid)	*9.309177*
	Liter	35.23907
	Peck (U.S.)	*4*
	Pint (U.S., dry)	*64*
	Quart (U.S., dry)	*32*
Butt (Brit.)	Gallon (Brit.)	*108* or *126*
Cable length (int.)	Foot	607.6115
	Meter	*185.2*
	Mile (nautical)	*0.1*
Cable length (U.S.)	Foot	*720*
	Meter	*219.456*
	Mile (nautical)	0.1184968
	Mile (statute)	0.1363636
Caliber	Inch	*0.01*
	Millimeter	*0.254*
Calorie	Btu	3.96832×10^{-3}
	Cubic foot-atmosphere	1.45922×10^{-3}
	Foot-poundal	99.3543
	Foot-pound-force	3.08803
	Horsepower-hour	1.55961×10^{-6}
	Horsepower-hour (metric)	1.58124×10^{-6}
	Joule	*4.1868*
	Kilocalorie	*0.001*
	Kilogram-force-meter	0.426935
	Kilowatt-hour	1.163×10^{-6}
	Liter-atmosphere	0.0413205
	Watt-hour	1.163×10^{-3}
Calorie (15°C)	Joule	4.1855
Calorie (20°C)	Joule	4.18190
Calorie (mean)	Joule	4.19002
Calorie (thermochem.)	Joule	*4.184*
Calorie/°C	Btu/°F	2.20462×10^{-3}
	Joule/°F	*2.326*
Calorie/gram	Btu/pound	*1.8*
	Joule/kilogram	*4186.8*
Calorie/(gram × °C)	Btu/(pound × °F)	*1*
	Joule/(kilogram × °C)	*4186.8*
Calorie/minute	Watt	*0.06978*
Calorie/(minute × square centimeter)	Watt/square meter	*697.8*
Calorie/second	Watt	*4.1868*
Calorie/(second × square centimeter)	Kilowatt/square meter	*41.868*
Calorie/(second × square centimeter × °C)	Kilowatt/(square meter × °C)	*41.868*
Calorie/(second × square centimeter × °C/centimeter)	Watt/(meter × °C)	*418.68*
Calorie/square centimter	Kilojoule/square meter	41.868
Candela	Hefner unit	1.11
	Lumen/steradian	*1*
Candela/square centimeter	Candela/square foot	*929.0304*
	Candela/square inch	*6.4516*
	Lambert	3.141593 (π)
Candela/square foot	Candela/square inch	6.944444×10^{-3} (*1/144*)
	Candela/square meter	10.76391
	Foot-lambert	3.141593 (π)
	Lambert	3.381582×10^{-3}
Candela/square inch	Candela/square centimeter	0.1550003
	Candela/square foot	*144*
	Foot-lambert	452.3893
	Lambert	0.4869478
Candela/square meter	Candela/square foot	*0.09290304*
	Lambert	3.141593×10^{-4}
Carat (metric)	Gram	*0.2*
Cental	Kilogram	*45.359237*
	Pound	*100*
°C heat unit (chu)	Btu	*1.8*
	Calorie	453.592
	Joule	1899.10
Centiliter	Cubic centimeter	*10*
	Cubic inch	0.6102374
	Drachm (Brit., fliud)	2.815606
	Dram (U.S., fluid)	2.705122

CONVERSION FACTORS (continued)

To convert from	To	Multiply by
	Ounce (Brit., fluid)	0.3519508
	Ounce (U.S., fluid)	0.3381402
Centimeter	Foot	0.03280840
	Inch	0.3937008
	Micrometer	*10000*
	Mil	393.7008
	Millimeter	*10*
	Yard	0.01093613
Centimeter of Hg (conv.)	Atmosphere	0.0131579
	Millibar	13.3322
	Millimeter of H_2O (conv.)	*135.951*
	Pascal	1333.22
	Pound-force/square inch	0.193368
Centimeter of H_2O (conv.)	Atmosphere	9.67841×10^{-4}
	Millibar	*0.980665*
	Millimeter of Hg (conv.)	0.735559
	Kilogram-force/square centimeer	*0.001*
	Pascal	*98.0665*
	Pound-force/square inch	0.0142233
Centimeter/second	Foot/minute	1.968504
	Foot/second	0.03280840
	Kilometer/hour	*0.036*
	Meter/minute	*0.6*
	Mile/hour	0.02236936
Centimeter/square second	Foot/square second	0.03280840
	Kilometer/(hour × second)	*0.036*
	Meter/square second	*0.01*
	Mile/(hour × second)	0.02236936
Centipoise	Pascal-second	*0.001*
Centistokes	Square meter/second	1×10^{-6}
Chain (Gunter's)	Foot	*66*
Chain (Ramsden's)	Foot	*100*
Circular inch	Circular mil	1×10^{6}
	Square centimeter	5.067075
	Square inch	0.7853982
Circular mil	Square inch	7.853982×10^{-7}
	Square micrometer	506.7075
	Square mil	0.7853982
Circular millimeter	Square millimeter	0.7853982
Circumference	Degree	*360*
	Gon (grade)	*400*
	Radian	6.283185 (2π)
Clo	(°C × square meter)/watt	0.2003712
Cord	Cord-foot	*8*
	Cubic foot	*128*
Cord-foot	Cord	*0.125 (1/8)*
	Cubic foot	*16*
Coulomb	Ampere-second	*1*
Cubic centimeter	Cubic foot	3.531467×10^{-5}
	Cubic inch	0.06102374
	Cubic meter	1×10^{-6}
	Cubic millimeter	*1000*
	Cubic yard	1.307951×10^{-6}
	Drachm (Brit., fluid)	0.2815606
	Dram (U.S., fluid)	0.2705122
	Gallon (Brit.)	2.199692×10^{-4}
	Gallon (U.S.)	2.641721×10^{-4}
	Gill (Brit.)	7.039016×10^{-3}
	Gill (U.S.)	8.453506×10^{-3}
	Liter	*0.001*
	Milliliter	*1*
	Minim (Brit.)	16.89364
	Minim (U.S.)	16.23073
	Ounce (Brit., fluid)	0.03519508
	Ounce (U.S., fluid)	0.03381402
	Pint (Brit.)	1.759754×10^{-3}
	Pint (U.S., dry)	1.816166×10^{-3}

To convert from	To	Multiply by
	Pint (U.S., liquid)	2.113376×10^{-3}
	Quart (Brit.)	8.798770×10^{-4}
	Quart (U.S., dry)	9.080830×10^{-4}
	Quart (U.S., liquid)	1.056688×10^{-3}
Cubic centimeter/gram	Cubic foot/pound	0.0160185
Cubic centimeter/second	Cubic foot/minute	2.118880×10^{-3}
	Liter/hour	*3.6*
Cubic centimeter-atmosphere	Joule	*0.101325*
	Watt-hour	2.814583×10^{-5}
Cubic decimeter	Cubic centimeter	*1000*
	Cubic foot	0.03531467
	Cubic inch	61.02374
	Cubic meter	*0.001*
	Liter	*1*
Cubic foot	Acre-foot	2.295684×10^{-5}
	Board foot	*12*
	Bushel (Brit.)	0.7786044
	Bushel (U.S.)	0.8035640
	Cord	7.8125×10^{-3} *(1/128)*
	Cord-foot	*0.0625 (1/16)*
	Cubic centimeter	28316.847
	Cubic inch	*1728*
	Cubic meter	0.028316847
	Cubic yard	0.03703704 (*1/27*)
	Gallon (Brit.)	6.228835
	Gallon (U.S.)	7.480519
	Liter	28.316847
	Pint (Brit.)	49.83068
	Pint (U.S., dry)	51.42809
	Pint (U.S., liquid)	59.84416
	Quart (Brit.)	24.91534
	Quart (U.S., dry)	25.71405
	Quart (U.S., liquid)	29.92208
Cubic foot/hour	Cubic centimeter/second	7.865791
	Liter/minute	0.4719474
Cubic foot/minute	Cubic centimeter/second	471.9474
	Gallon (Brit.)/second	0.1038139
	Gallon (U.S.)/second	0.1246753
Cubic foot/pound	Cubic meter/kilogram	0.06242796
Cubic foot/second	Cubic meter/hour	101.9406
	Cubic yard/minute	2.222222
	Gallon (Brit.)/minute	373.7301
	Gallon (U.S.)/minute	448.8312
	Liter/minute	1699.011
Cubic foot-atmosphere	Btu	2.71948
	Calorie	685.298
	Foot-pound-force	2116.22
	Joule	2869.205
	Kilogram-force-meter	292.577
	Liter-atmosphere	28.31685
	Watt-hour	0.7970012
Cubic foot (pound-force/ square inch)	Btu	0.7970012
	Calorie	
	Joule	0.185050
	Watt-hour	46.6317
		195.238
		0.0542327
Cubic inch	Board foot	6.944444×10^{-3} *(1/144)*
	Bushel (Brit.)	4.505813×10^{-4}
	Bushel (U.S.)	4.650254×10^{-4}
	Cubic centimeter	*16.387064*
	Cubic foot	5.787037×10^{-4} *(1/1728)*
	Cubic meter	1.6387064×10^{-5}
	Cubic yard	2.143347×10^{-5}
	Drachm (Brit., fluid)	4.613952
	Dram (U.S., fluid)	4.432900
	Gallon (Brit.)	3.604650×10^{-3}
	Gallon (U.S.)	4.329004×10^{-3} *(1/231)*
	Liter	*0.016387064*
	Milliliter	*16.387064*
	Ounce (Brit., fluid)	0.5767440

CONVERSION FACTORS (continued)

To convert from	To	Multiply by
	Ounce (U.S., fluid)	0.5541126
	Pint (Brit.)	0.02883720
	Pint (U.S., dry)	0.02976163
	Pint (U.S., liquid)	0.03463203
	Quart (Brit.)	0.01441860
	Quart (U.S., dry)	0.01488081
	Quart (U.S., liquid)	0.01731602
Cubic inch/minute	Cubic centimeter/second	0.2731177
Cubic kilometer	Cubic mile	0.2399128
Cubic meter	Barrel (petroleum)	6.289811
	Barrel (U.S., dry)	8.648490
	Barrel (U.S., liquid)	8.386414
	Bushel (U.S.)	28.37759
	Cubic centimeter	1×10^{6}
	Cubic decimeter	1000
	Cubic foot	35.31467
	Cubic inch	61023.74
	Cubic yard	1.307951
	Gallon (Brit.)	219.9692
	Gallon (U.S.)	264.1721
	Liter	1000
	Pint (Brit.)	1759.754
	Pint (U.S., dry)	1816.166
	Pint (U.S., liquid)	2113.376
	Quart (Brit.)	879.8770
	Quart (U.S., dry)	908.0830
	Quart (U.S., liquid)	1056.688
	Register ton	0.3531467
Cubic meter/kilogram	Cubic foot/pound	16.01846
Cubic mile	Cubic kilometer	4.168182
Cubic millimeter	Cubic centimeter	0.001
	Cubic inch	6.102374×10^{-5}
	Minim (Brit.)	0.01689364
	Minim (U.S.)	0.01623073
Cubic yard	Bushel (Brit.)	21.02232
	Bushel (U.S.)	21.69623
	Cubic foot	27
	Cubic inch	46656
	Cubic meter	0.76455486
	Gallon (Brit.)	168.1786
	Gallon (U.S.)	201.9740
	Liter	764.5549
Cubic yard/minute	Cubic foot/second	0.45
	Gallon (Brit.)/second	2.802976
	Gallon (U.S.)/second	3.366234
	Liter/second	12.74258
Cubit	Inch	18
Cup (metric)	Milliliter	200
Cup (U.S.)	Milliliter	236.588
	Ounce (U.S. fluid)	8
Curie	Becquerel	3.7×10^{10}
Darcy	Square meter	9.869233×10^{-13}
Day (mean solar)	Hour	24
	Minute	1440
	Second	86400
Day (sidereal)	Second	86164.09
Decibel	Neper	0.115129255
Degree (angular)	Circumference	2.777778×10^{-3} (1/360)
	Gon (grade)	1.111111
	Minute (angular)	60
	Quadrant	0.01111111 (1/90)
	Radian	0.01745329
	Second (angular)	3600
Degree/foot	Radian/meter	0.05726146
Degree/inch	Radian/meter	0.6871375
Degree/second	revolution/minute	0.1666667 (1/6)
°C (temp. interval)	°Fahrenheit	1.8
	°Rankine	1.8
	Kelvin	1
(°C × hour)/kilocalorie	°C/watt	0.859845
(°C × hour × square meter)/kilocalorie	(°C × square meter)/watt	0.859845
°F (temp. interval)	°Celsius	0.5555556 (5/9)
	°Rankine	1
	Kelvin	0.5555556 (5/9)
(°F × hour)/Btu	°C/watt	1.89563
(°F × hour × square foot)/Btu	(°C × square meter)/watt	0.176110
(°F/inch × hour × square foot)/Btu	(°C × meter)/watt	6.93347
Denier	Tex	0.111111 (1/9)
Drachm (Brit. fluid)	Dram (U.S., fluid)	0.9607599
	Milliliter	3.551633
	Minim (Brit.)	60
	Ounce (Brit. fluid)	0.125 (1/8)
Dram (apoth. or troy)	Dram (avoirdupois)	2.1942857
	Grain	60
	Gram	3.8879346
	Ounce (apoth. or troy)	0.125 (1/8)
	Pennyweight	2.5
	Scruple	3
Dram (avoirdupois)	Grain	27.34375
	Gram	1.7718452
	Ounce (avoirdupois)	0.0625 (1/16)
Dram (U.S., fluid)	Cubic centimeter	3.696691
	Cubic inch	0.2255859
	Drachm (Brit. fluid)	1.040843
	Gallon (U.S.)	9.765625×10^{-4} (1/1024)
	Gill (U.S.)	0.03125 (1/32)
	Milliliter	3.696691
	Minim (U.S.)	60
	Ounce (U.S., fluid)	0.125 (1/8)
	Pint (U.S., liquid)	7.8125×10^{-3} (1/128)
	Quart (U.S., liquid)	3.90625×10^{-3} (1/256)
Dyne	Kilogram-force	1.019716×10^{-6}
	Newton	1×10^{-5}
	Poundal	7.233014×10^{-5}
	Pound-force	2.248089×10^{-6}
Dyne/centimeter	Newton/meter	0.001
Dyne/square centimeter	Bar	1×10^{-6}
	Kilogram-force/square centimeter	1.019716×10^{-6}
	Millimeter of Hg (conv.)	7.50062×10^{-4}
	Millimeter of H_2O (conv.)	0.01019716
	Pascal (N/square meter)	0.1
	Pound-force/square inch	1.450377×10^{-5}
Dyne-centimeter	Erg	1
	Foot-poundal	2.37304×10^{-6}
	Foot-pound-force	7.37562×10^{-8}
	Joule	1×10^{-7}
	Kilogram-force-meter	1.019716×10^{-8}
	Newton-meter	1×10^{-7}
Dyne-second/square centimeter	Poise	1
	Pascal-second	0.1
Electronvolt	Erg	1.60219×10^{-12}
	Joule	1.60219×10^{-19}
Ell	Inch	45
Erg	Dyne-centimeter	1
	Joule	1×10^{-7}
	Watt-hour	2.777778×10^{-11}
Erg/(square centimeter × second)	Watt/square meter	0.001
Farad (int. mean)	Farad	0.999510
Farad (int. U.S.)	Farad	0.999505
Fathom	Foot	6
Fermi	Meter	1×10^{-15}
Firkin (Brit.)	Gallon (Brit.)	9
Firkin (U.S.)	Gallon (U.S.)	9
Foot	Centimeter	30.48
	Foot (U.S. Survey)	0.999998
	Inch	12
	Meter	0.3048
	Millimeter	304.8
	Mile (nautical)	1.645788×10^{-4}
	Mile (statute)	1.893939×10^{-4}

CONVERSION FACTORS (continued)

To convert from	To	Multiply by
	Yard	0.333333 (1/3)
Foot (U.S. Survey)	Foot	1.000002
	Meter	0.30480060960
Foot of H_2O (conv.)	Atmosphere	0.0294998
	Bar	0.0298907
	Inch of Hg (conv.)	0.882671
	Kilogram-force/square centimeter	0.03048
	Millimeter of Hg (conv.)	22.4198
	Pascal (N/square meter)	2989.07
	Pound-force/square inch	0.433527
Foot/°F	Meter/°C	0.54864
Foot/hour	Meter/second	8.466667×10^{-5}
Foot/minute	Kilometer/hour	0.018288
	Knot	9.87473×10^{-3}
	Meter/second	5.08×10^{-3}
	Mile/hour	0.01136364 (1/88)
Foot/second	Kilometer/hour	1.09728
	Knot	0.5924838
	Meter/minute	18.288
	Meter/second	0.3048
	Mile/hour	0.6818182
Foot/square second	Kilometer/(hour × second)	1.09728
	Meter/square second	0.3048
	Mile/(hour × second)	0.6818182
Foot to the fourth power	Meter to the fourth power	8.630975×10^{-3}
Foot-candle	Lumen/square foot	1
	Lumen/square meter	10.76391
	Lux	10.76391
Foot-lambert	Candela/square centimeter	3.426259×10^{-4}
	Candela/square foot	0.3183099 (1/π)
	Candela/square meter	3.426259
	Lambert	1.076391×10^{-3}
	Meter-lambert	10.76391
Foot-poundal	Btu	3.99411×10^{-5}
	Calorie	0.0100650
	Foot-pound-force	0.0310810
	Joule	0.0421401
	Kilogram-force-meter	4.29710×10^{-3}
	Liter-atmosphere	4.15891×10^{-4}
	Watt-hour	1.17056×10^{-5}
Foot-pound-force	Btu	1.28507×10^{-3}
	Calorie	0.323832
	Cubic foot-atmosphere	4.72541×10^{-4}
	Foot-poundal	32.1740
	Horsepower-hour	5.05051×10^{-7}
	Horsepower-hour (metric)	5.12055×10^{-7}
	Joule	1.35582
	Kilogram-force-meter	0.138255
	Liter-atmosphere	0.0133809
	Newton-meter	1.35582
	Watt-hour	3.76616×10^{-4}
Foot-pound-force/hour	Watt	3.76616×10^{-4}
Foot-pound-force/minute	Horsepower	3.03030×10^{-5}
	Horsepower (metric)	3.07233×10^{-5}
	Watt	0.0225970
Foot-pound-force/second	Horsepower	1.81818×10^{-3} (1/550)
	Horsepower (metric)	1.84340×10^{-3}
	Watt	1.355818
Franklin	Coulomb	3.335641×10^{-10}
Furlong	Foot	660
	Meter	201.168
	Mile (statute)	0.125 (1/8)
	Yard	220
Gal	Centimeter/square second	1
	Meter/square second	0.01
Gallon (Brit.)	Bushel (Brit.)	0.125 (1/8)
	Cubic centimeter	4546.09
	Cubic foot	0.1605437
	Cubic inch	277.4194
	Cubic yard	5.946061×10^{-3}
	Drachm (Brit., fluid)	1280
	Gallon (U.S.)	1.200950
	Gill (Brit.)	32
	Liter	4.54609
	Minim (Brit.)	76800
	Ounce (Brit., fluid)	160
	Peck (Brit.)	0.5
	Pint (Brit.)	8
	Quart (Brit.)	4
Gallon (U.S., dry)	Bushel (U.S.)	0.125 (1/8)
	Cubic inch	268.8025
	Liter	4.404884
Gallon (U.S., liquid)	Barrel (petroleum)	0.02380952 (1/42)
	Cubic centimeter	3785.412
	Cubic foot	0.13368056
	Cubic inch	231
	Cubic yard	4.951132×10^{-3}
	Dram (U.S., fluid)	1024
	Gallon (Brit.)	0.8326742
	Gill (U.S.)	32
	Liter	3.785412
	Minim (U.S.)	61440
	Ounce (U.S., fluid)	128
	Pint (U.S., liquid)	8
	Quart (U.S., liquid)	4
Gallon (Brit.)/minute	Cubic foot/hour	9.632619
	Cubic foot/second	2.675728×10^{-3}
	Cubic meter/hour	0.2727654
	Liter/second	0.07576817
Gallon (U.S.)/minute	Cubic foot/hour	8.020834
	Cubic foot/second	2.228009×10^{-3}
	Cubic meter/hour	0.2271247
	Liter/second	0.06309020
Gamma	Tesla	1×10^{-9}
Gauss	Tesla	1×10^{-4}
	Weber/square meter	1×10^{-4}
Geepound	Slug	1
Gigawatt-hour	Kilowatt-hour	1×10^{6}
Gilbert	Ampere	0.7957747
Gill (Brit.)	Cubic centimeter	142.0653
	Cubic inch	8.669357
	Gallon (Brit.)	0.03125 (1/32)
	Gill (U.S.)	1.200950
	Milliliter	142.0653
	Ounce (Brit. fluid)	5
	Pint (Brit.)	0.25 (1/4)
	Quart (Brit.)	0.125 (1/8)
Gill (U.S.)	Cubic centimeter	118.2941
	Cubic inch	7.21875
	Gallon (U.S.)	0.03125 (1/32)
	Gill (Brit.)	0.8326742
	Milliliter	118.2941
	Ounce (U.S., fluid)	4
	Pint (U.S., liquid)	0.25 (1/4)
	Quart (U.S., liquid)	0.125 (1/8)
Gon (grade)	Circumference	0.002 5 (1/400)
	Degree (angular)	0.9
	Minute (angular)	54
	Radian	0.01570796
	Second (angular)	3240
Grain	Carat (metric)	0.32399455
	Dram	0.03657143
	Milligram	64.79891
	Ounce (avoirdupois)	2.285714×10^{-3}
	Ounce (troy)	2.083333×10^{-3} (1/480)
	Pennyweight	0.04166667 (1/24)
	Pound	1.428571×10^{-4} (1/7000)
	Scruple	0.05 (1/20)
Grain/cubic foot	Milligram/liter	2.288352

CONVERSION FACTORS (continued)

To convert from	To	Multiply by
Grain/gallon (Brit.)	Milligram/liter	14.25377
Grain/gallon (U.S.)	Milligram/liter	17.11806
	Pound/million gallons	142.8571
Gram	Carat (metric)	5
	Dram	0.56438339
	Grain	15.432358
	Kilogram	0.001
	Milligram	1000
	Ounce (avoirdupois)	0.035273962
	Ounce (troy)	0.032150747
	Pennyweight	0.64301493
	Pound	2.2046226×10^{-3}
	Scruple	0.77161792
	Ton (metric)	1×10^{-6}
Gram/(centimeter × second)	Poise	1
Gram/cubic centimeter	Kilogram/cubic decimeter	1
	Kilogram/cubic meter	1000
	Kilogram/liter	1
	Pound/cubic foot	62.42796
	Pound/cubic inch	0.03612729
	Pound/gallon (Brit.)	10.02241
	Pound/gallon (U.S.)	8.345404
Gram/cubic meter	Grain/cubic foot	0.4369957
Gram/liter	Grain/gallon (Brit.)	70.15689
	Grain/gallon (U.S.)	58.41783
	Gram/cubic centimeter	0.001
	Kilogram/cubic meter	1
	Pound/cubic foot	0.0624280
	Pound/gallon (Brit.)	0.0100224
	Pound/gallon (U.S.)	8.34540×10^{-3}
Gram/meter	Ounce/yard	0.03225451
Gram/milliliter	Gram/cubic centimeter	1
Gram/square meter	Ounce/square foot	0.3277058
	Ounce/square yard	0.02949352
Gram/ton (long)	Gram/ton (metric)	0.9842065
	Gram/ton (short)	0.8928571
	Milligram/kilogram	0.9842065
Gram/ton (metric)	Gram/ton (long)	1.016047
	Gram/ton (short)	0.9071847
	Milligram/kilogram	1
Gram/ton (short)	Gram/ton (long)	1.12
	Gram/ton (metric)	1.102311
	Milligram/kilogram	1.102311
Gram-force	Dyne	980.665
	Newton	9.80665×10^{-3}
Gram-force/square centimeter	Pascal	98.0665
Gram-force-centimeter	Erg	980.665
	Joule	9.80665×10^{-5}
Gray	Joule/kilogram	1
Hand	Inch	4
Hectare	Acre	2.471054
	Are	100
	Square foot	1.076391×10^{5}
	Square kilometer	0.01
	Square meter	10000
	Square mile	3.861022×10^{-3}
	Square yard	11959.90
Hectogram	Kilogram	0.1
Hectoliter	Cubic meter	0.1
Hefner unit	Candela	0.903
Henry (int. mean)	Henry	1.00049
Henry (int. U.S.)	Henry	1.000495
Hogshead (U.S.)	Gallon (U.S.)	63
Horsepower	Btu/hour	2544.43
	Btu/minute	42.4072
	Btu/second	0.706787
	Foot-pound-force/hour	1.98×10^{6}
	Foot-pound-force/minute	33000
	Foot-pound-force/second	550
	Horsepower (metric)	1.01387
	Joule/second	745.700
	Kilocalorie/hour	641.186
	Kilocalorie/minute	10.6864
	Kilocalorie/second	0.178107
	Kilogram-force-meter/second	76.0402
	Kilowatt	0.745700
Horsepower (boiler)	Kilowatt	9.80950
Horsepower (electric)	Kilowatt	0.746
Horsepower (metric)	Foot-pound-force/second	542.476
	Horsepower	0.986320
	Kilocalorie/hour	632.415
	Kilocalorie/minute	10.54025
	Kilocalorie/second	0.175671
	Kilogram-force-meter/second	75
	Kilowatt	0.735499
Horsepower (water)	Kilowatt	0.746043
Horsepower-hour	Btu	2544.43
	Foot-pound-force	1.98×10^{6}
	Horsepower-hour (metric)	1.01387
	Joule	2.68452×10^{6}
	Kilocalorie	641.186
	Kilogram-force-meter	2.73745×10^{5}
	Kilowatt-hour	0.745700
	Megajoule	2.68452
Horsepower-hour (metric)	Horsepower-hour	0.986320
	Joule	2.64780×10^{6}
	Kilocalorie	632.415
	Kilogram-force-meter	2.7×10^{5}
	Kilowatt-hour	0.735499
	Megajoule	2.64780
Hour (mean solar)	Day	0.04166667 (1/24)
	Minute	60
	Second	3600
	Week	5.952381×10^{-3} (1/168)
Hundredweight (long)	Hundredweight (short)	1.12
	Kilogram	50.80234544
	Pound	112
	Ton (long)	0.05
	Ton (metric)	0.050802345
	Ton (short)	0.056
Hundredweight (short)	Hundredweight (long)	0.89285714
	Kilogram	45.359237
	Pound	100
	Ton (long)	0.044642857
	Ton (metric)	0.045359237
	Ton (short)	0.05
Inch	Centimeter	2.54
	Foot	0.08333333 (1/12)
	Mil	1000
	Millimeter	25.4
	Yard	0.02777778 (1/36)
Inch of Hg (conv.)	Atmosphere	0.0334211
	Foot of H_2O (conv.)	1.132925
	Inch of H_2O (conv.)	13.5951
	Kilogram-force/square centimeter	0.0345316
	Millibar	33.8639
	Millimeter of H_2O (conv.)	345.316
	Pascal	3386.39
	Pound-force/square inch	0.491154
Inch of H_2O (conv.)	Inch of Hg (conv.)	0.0735559
	Kilogram-force/square centimeter	2.54×10^{-3}
	Millibar	2.49089
	Millimeter of Hg (conv.)	1.86832
	Pascal	249.089
	Pound-force/square inch	0.0361273

CONVERSION FACTORS (continued)

To convert from	To	Multiply by
Inch/°F	Millimeter/°C	*45.72*
Inch/hour	Millimeter/minute	0.4233333
	Millimeter/second	7.055556×10^{-3}
	Foot/minute	1.388889×10^{-3}
Inch/minute	Foot/hour	*5*
	Meter/hour	*1.524*
	Millimeter/second	0.4233333
Inch/second	Foot/hour	*300*
	Meter/minute	1.524
Inch to the fourth power	Meter to the fourth power	4.162314×10^{-7}
Joule	Btu	9.47817×10^{-4}
	Calorie	0.238846
	Centrigrade heat unit	5.26565
	Cubic foot-atmosphere	3.48529×10^{-4}
	Cubic foot-pound-force/square inch	5.12196×10^{-3}
	Erg	1×10^{7}
	Foot-poundal	23.7304
	Foot-pound-force	0.737562
	Horsepower-hour	3.72506×10^{-7}
	Horsepower-hour (metric)	3.77673×10^{-7}
	Kilogram-force-meter	0.101972
	Liter-atmosphere	9.86923×10^{-3}
	Newton-meter	*1*
	Watt-hour	2.777778×10^{-4} (*1/3600*)
	Watt-second	*1*
Joule/°C	Btu/°F	5.26565×10^{-4}
Joule/gram	Btu/pound	0.429923
	Kilocalorie/kilogram	0.238846
Joule/(gram × °C)	Btu/(pound × °F)	0.238846
	Kilocalorie/(kilogram × °C)	0.238846
Joule/hour	Watt	2.777778×10^{-4} (*1/3600*)
Joule/minute	Watt	0.01666667 (*1/60*)
Joule/second	Watt	*1*
Kelvin (temp. interval)	°Celsius	*1*
	°Fahrenheit	*1.8*
	°Rankine	*1.8*
Kilderkin (Brit.)	Gallon (Brit.)	*18*
Kilocalorie	Btu	3.96832
	Calorie	*1000*
	Joule	*4186.8*
Kilocalorie/cubic meter	Btu/cubic foot	0.112370
	Kilojoule/cubic meter	*4.1868*
Kilocalorie/hour	Watt	*1.163*
Kilocalorie/(hour × square meter)	Watt/square meter	*1.163*
Kilocalorie/(hour × square meter × °C)	Watt/(square meter × °C)	*1.163*
Kilocalorie/(hour × square meter × °C/centimeter)	Watt/(meter × °C)	*0.01163*
Kilocalorie/kilogram	Btu/pound	*1.8*
	Joule/gram	*4.1868*
Kilocalorie/(kilogram × °C)	Btu/(pound × °F)	*1*
	Kilojoule/(kg × °C)	*4.1868*
Kilocalorie/minute	Foot-pound-force/second	51.4671
	Horsepower	0.0935765
	Horsepower (metric)	0.0948744
	Watt	*69.78*
Kilocalorie/second	Kilowatt	*4.1868*
Kilogram	Grain	15432.358
	Gram	*1000*
	Hundredweight (long)	0.019684131
	Hundredweight (short)	0.022046226
	Ounce (avoirdupois)	35.273962
	Ounce (troy)	32.150747
	Pound	2.2046226
	Ton (long)	9.8420653×10^{-4}
	Ton (metric)	*0.001*
	Ton (short)	1.1023113×10^{-3}
Kilogram/cubic meter	Gram/cubic centimeter	*0.001*
	Gram/liter	*1*
	Pound/cubic foot	0.06242796
	Pound/cubic inch	3.612729×10^{-5}
Kilogram/meter	Gram/centimeter	*10*
	Pound/foot	0.6719690
	Pound/inch	0.05599741
Kilogram-force	Dyne	9.80665×10^{5}
	Newton	*9.80665*
	Pound-force	2.20462
	Poundal	70.9316
Kilogram-force/square centimeter	Atmosphere	0.967841
	Atmosphere (technical)	*1*
	Bar	*0.980665*
	Foot of H_2O (conv.)	32.8084
	Inch of Hg (conv.)	28.9590
	Kilogram-force/square millimeter	*0.01*
	Meter of H_2O (conv.)	*10*
	Millimeter of Hg (conv.)	735.559
	Newton/square millimeter	*0.0980665*
	Pascal (N/square meter)	*98066.5*
	Pound-force/square foot	2048.16
	Pound-force/square inch	14.22334
	Ton-force (long)/square foot	0.914358
	Ton-force (short)/square foot	1.02408
	Ton-force (long)/square inch	6.34971×10^{-3}
	Ton-force (short)/square inch	7.11167×10^{-3}
Kilogram-force/square meter	Pascal	*9.80665*
Kilogram-force/square millimeter	Newton/square millimeter	*9.80665*
	Megapascal	*9.80665*
	Pound-force/square inch	1422.334
Kilogram-force-meter	Btu	9.29491×10^{-3}
	Calorie	2.34228
	Cubic foot-atmosphere	3.41790×10^{-3}
	Erg	9.80665×10^{7}
	Foot-poundal	232.715
	Foot-pound-force	7.23301
	Horsepower-hour	3.65304×10^{-6}
	Horsepower-hour (metric)	3.70370×10^{-6}
	Joule	*9.80665*
	Liter-atmosphere	0.0967841
	Newton-meter	*9.80665*
	Watt-hour	2.72407×10^{-3}
Kilometer	Astronomical unit	6.68459×10^{-9}
	Foot	3280.840
	Light year	1.05702×10^{-13}
	Mile (nautical)	0.5399568
	Mile (statute)	0.6213712
	Yard	1093.613
Kilometer/hour	Foot/minute	54.68066
	Foot/second	0.9113444
	Inch/second	10.93613
	Knot	0.5399568
	Meter/minute	16.66667
	Meter/second	0.2777778
	Mile/hour	0.6213712
Kilometer/(hour × second)	Centimeter/square second	27.77778
	Foot/square second	0.9113444
	Meter/square second	0.2777778
	Mile/(hour × second)	0.6213712

CONVERSION FACTORS (continued)

To convert from	To	Multiply by
Kilopascal	Pound-force/square foot	20.8854
	Pound-force/square inch	0.1450377
Kilopond	Kilogram-force	*1*
	Newton	*9.80665*
Kilowatt	Btu/hour	3412.14
	Btu/minute	56.8690
	Btu/second	0.947817
	Foot-pound-force/hour	2.65522×10^{6}
	Foot-pound-force/minute	44253.7
	Foot-pound-force/second	737.562
	Horsepower	1.34102
	Horsepower (metric)	1.35962
	Joule/hour	3.6×10^{6}
	Joule/minute	*60000*
	Joule/second	*1000*
	Kilocalorie/hour	859.845
	Kilocalorie/minute	14.3308
	Kilocalorie/second	0.238846
	Kilogram-force-meter/hour	3.67098×10^{5}
	Kilogram-force-meter/minute	6118.30
	Kilogram-force-meter/second	101.972
Kilowatt-hour	Btu	3412.14
	Foot-pound-force	2.65522×10^{6}
	Horsepower-hour	1.34102
	Horsepower-hour (metric)	1.35962
	Joule	3.6×10^{6}
	Kilocalorie	859.845
	Kilogram-force-meter	3.67098×10^{5}
	Megajoule	*3.6*
Kilowatt-hour/pound	Btu/pound	3412.14
	Joule/gram	7936.641
	Kilocalorie/kilogram	1895.63
Kilowatt-hour/kilogram	Btu/pound	1547.72
Kip	Pound-force	*1000*
Kip/square inch	Newton/square millimeter	6.89476
	Megapascal	6.89476
Knot	Foot/minute	101.2686
	Foot/second	1.687810
	Kilometer/hour	*1.852*
	Meter/minute	30.86667
	Meter/second	0.5144444
	Mile (nautical)/hour	*1*
	Mile (statute)/hour	1.150779
Lambert	Candela/square centimeter	0.3183099 ($1/\pi$)
	Candela/square foot	295.7196
	Candela/square inch	2.053608
	Candela/square meter	3183.099
	Foot-lambert	*929.0304*
Langley	Joule/square meter	*41840*
Last (Brit.)	Gallon (Brit.)	*640*
League (nautical)	Mile (nautical)	*3*
League (statute)	Mile (statute)	*3*
Light year	Astronomical unit	63239.7
	Kilometer	9.46053×10^{12}
	Mile	5.87850×10^{12}
	Parsec	0.306595
Line	Inch	*0.1* or 0.083333 (*1/12*)
	Millimeter	*2.54* or 2.116667
Line	Weber	1×10^{-8}
Link	Chain	*0.01*
Liter	Bushel (Brit.)	0.027496156
	Bushel (U.S.)	0.02837759
	Cubic centimeter	*1000*
	Cubic decimeter	*1*
	Cubic foot	0.03531467
	Cubic inch	61.02374
	Cubic meter	*0.001*
	Cubic yard	1.307951×10^{-3}
	Drachm (Brit., fluid)	281.5606
	Dram (U.S., fluid)	270.5122
	Gallon (Brit.)	0.21996925
	Gallon (U.S.)	0.26417205
	Gill (Brit.)	7.039016
	Gill (U.S.)	8.453506
	Milliliter	1000
	Minim (Brit.)	16893.64
	Minim (U.S.)	16230.73
	Ounce (Brit., fluid)	35.19508
	Ounce (U.S., fluid)	33.81402
	Pint (Brit.)	1.759754
	Pint (U.S., dry)	1.816166
	Pint (U.S., liquid)	2.113376
	Quart (Brit.)	0.8798770
	Quart (U.S., dry)	0.9080830
	Quart (U.S., liquid)	1.056688
Liter (1901—1964)	Cubic decimeter	1.000028
Liter/minute	Cubic foot/hour	2.118880
	Cubic foot/second	5.885778×10^{-4}
	Gallon (Brit.)/hour	13.19815
	Gallon (Brit.)/second	3.666154×10^{-3}
	Gallon (U.S.)/hour	15.85032
	Gallon (U.S.)/second	4.402868×10^{-3}
Liter/second	Cubic foot/hour	127.1328
	Cubic foot/minute	2.118880
	Gallon (Brit.)/hour	791.8893
	Gallon (Brit.)/minute	13.19815
	Gallon (U.S.)/hour	951.0194
	Gallon (U.S.)/minute	15.85032
Liter-atmosphere	Btu	0.0960376
	Calorie	24.2011
	Cubic foot-atmosphere	0.0353147
	Cubic foot-pound-force/square inch	0.518983
	Foot-poundal	2404.48
	Foot-pound-force	74.7335
	Horsepower-hour	3.77442×10^{-5}
	Horsepower-hour (metric)	3.82677×10^{-5}
	Joule	*101.325*
	Kilogram-force-meter	10.3323
	Watt-hour	0.0281458
Liter-bar	Joule	*100*
Lumen/square centimeter	Lux	*10000*
	Phot	*1*
Lumen/square foot	Lux	10.76391
Lumen/square meter	Lumen/square foot	*0.09290304*
	Lux	*1*
Lux	Lumen/square meter	*1*
	Phot	1×10^{-4}
Maxwell	Weber	1×10^{-8}
Megajoule	Kilowatt-hour	0.2777778
Megapascal	Bar	*10*
	Newton/square millimeter	*1*
Megohm	Ohm	1×10^{6}
Meter	Ångström	1×10^{10}
	Fathom	0.5468066
	Foot	3.2808399
	Foot (U.S. Survey)	3.2808333
	Inch	39.37007874
	Micrometer	1×10^{6}
	Mile (nautical)	5.399568×10^{-4}
	Mile (statute)	6.213712×10^{-4}
	Nanometer	1×10^{9}
	Yard	1.093613298
Meter/hour	Foot/minute	0.05468066
	Foot/second	9.113444×10^{-4}
	Millimeter/minute	16.66667
	Millimeter/second	0.2777778
Meter/minute	Foot/second	0.05468066
	Kilometer/hour	*0.06*

CONVERSION FACTORS (continued)

To convert from	To	Multiply by
	Knot	0.03239741
	Mile (statute)/hour	0.03728227
	Millimeter/second	16.66667
Meter/second	Foot/minute	196.8504
	Kilometer/hour	*3.6*
	Kilometer/minute	*0.06*
	Knot	1.943844
	Mile (statute)/hour	2.236936
	Mile (statute)/minute	0.03728227
Meter/square second	Foot/square second	3.280840
	Kikometer/(hour × second)	3.6
	Mile/(hour × second)	2.236936
Meter-candle	Lux	*1*
Mho (ohm^{-1})	Siemens	*1*
Microfarad	Farad	1×10^{-6}
Microgram	Grain	1.5432358×10^{-5}
	Gram	1×10^{-6}
Micrometer	Ångström	*10000*
	Mil	0.03937008
	Millimeter	*0.001*
	Nanometer	*1000*
Micron	Micrometer	*1*
Mil	Inch	*0.001*
	Micrometer	*25.4*
	Millimeter	*0.0254*
Mile (nautical)	Foot	6076.1155
	Kikometer	*1.852*
	Mile (statute)	1.150779
	Yard	2025.372
Mile (statute)	Chain (Gunter's)	*80*
	Chain (Ramsden's)	*52.8*
	Foot	*5280*
	Furlong	*8*
	Inch	*63360*
	Kilometer	*1.609344*
	Light year	1.70111×10^{-13}
	Meter	*1609.344*
	Mile (nautical)	0.86897624
	Parsec	5.21552×10^{-14}
	Rod	*320*
	Yard	*1760*
Mile (U.S. Survey)	Meter	1609.3472187
Mile/gallon (Brit.)	Kikometer/liter	0.354006
Mile/gallon (U.S.)	Kilometer/liter	0.425144
Mile/hour	Foot/minute	*88*
	Foot/second	1.466667
	Kilometer/hour	*1.609344*
	Knot	0.8689762
	Meter/minute	*26.8224*
	Meter/second	*0.44704*
Mile/(hour × minute)	Centimeter/square second	0.7450667
Mile/(hour × second)	Centimeter/square second	*44.704*
Mile/minute	Foot/second	*88*
	Kilometer/hour	96.56064
	Knot	52.13857
	Meter/second	*26.8224*
Millibar	Pascal	*100*
Milligram	Carat (metric)	0.005
	Dram	5.6438339×10^{-4}
	Grain	0.015432358
	Ounce (avoirdupois)	3.5273962×10^{-5}
	Ounce (troy)	3.2150747×10^{-5}
	Pennyweight	6.4301493×10^{-4}
	Pound	2.2046226×10^{-6}
	Scruple	7.7161792×10^{-4}
Milligram/assay ton (Brit.)	Milligram/kilogram	30.612245
	Ounce (troy)/ton (long)	*1*
Milligram/assay ton (U.S.)	Milligram/kilogram	34.285714
	Ounce(troy)/ton (short)	*1*
Milligram/kilogram	Gram/ton (metric)	*1*
	Pound/ton (short)	*0.002*
Milligram/liter	Grain/gallon (Brit.)	0.07015689
	Grain/gallon (U.S.)	0.05841783
	Gram/cubic meter	*1*
	Pound/cubic foot	6.242796×10^{-5}
Milligram/cubic meter	Grain/cubic foot	4.369957×10^{-4}
Milligram-force	Dyne	*0.980665*
	Newton	9.80665×10^{-6}
Milligram-force/ centimeter	Dyne/centimeter	*0.980665*
	Newton/meter	9.80665×10^{-4}
Milligram-force/inch	Dyne/centimeter	0.386089
	Newton/meter	3.86089×10^{-4}
Milliliter	Cubic centimeter	*1*
Millimeter	Ångström	1×10^{7}
	Inch	0.03937008
	Micrometer	*1000*
Millimeter of Hg (conv.)	Atmosphere	1.315789×10^{-3}
	Dyne/square centimeter	1333.224
	foot of H_2O (conv.)	0.0446033
	Gram-force/square centimeter	*1.35951*
	Millibar	1.333224
	Millimeter of H_2O (conv.)	*13.5951*
	Pascal	133.3224
	Pound-force/square foot	2.78450
	Pound-force/square inch	0.0193368
	Torr	1
Millimeter of H_2O (conv.)	Atmosphere	9.67841×10^{-5}
	Gram-force/square centimeter	*0.1*
	Millibar	*0.0980665*
	Millimeter of Hg (conv.)	0.0735559
	Pascal	*9.80665*
	Pound-force/square inch	1.42233×10^{-3}
Millimicron	Nanometer	*1*
Minim (Brit.)	Drachm (Brit., fluid)	0.01666667 (*1/60*)
	Milliliter	0.05919388
	Minim (U.S.)	0.9607599
	Ounce (Brit., fluid)	2.083333×10^{-3} (*1/480*)
Minim (U.S.)	Dram (U.S., fluid)	0.01666667 (*1/60*)
	Milliliter	0.06161152
	Minim (Brit.)	1.040843
	Ounce (U.S., fluid)	2.083333×10^{-3} (*1/480*)
Minute	Day	6.944444×10^{-4} (*1/1440*)
	Hour	0.01666667 (*1/60*)
	Second	*60*
	Week	9.920635×10^{-5}
Minute (angular)	Circumference	4.629630×10^{-5}
	Degree (angular)	0.01666667 (*1/60*)
	Gon (grade)	0.01851852 (*1/54*)
	Quadrant	1.851852×10^{-4}
	Radian	2.908882×10^{-4}
	Second (angular)	*60*
Month (mean of 4-year period)	Day	*30.4375*
	Hour	*730.5*
	Minute	*43830*
	Second	2.6298×10^{6}
	Week	4.348214
Nail (Brit).	Inch	*2.25*
Nanometer	Ångström	*10*
	Micrometer	*0.001*
	Mil	3.937008×10^{-5}
Neper	Decibel	8.685890
Newton	Dyne	1×10^{5}
	Kilogram-force	0.1019716
	Poundal	7.23301

CONVERSION FACTORS (continued)

To convert from	To	Multiply by
	Pound-force	0.224809
Newton/square centimeter	Newton/square millimeter	0.01
	Pascal	10000
Newton/square meter	Pascal	1
Newton/square millimeter	Kilogram-force/square millimeter	0.1019716
	Megapascal	1
	Ton-force (metric)/square meter	101.9716
Newton-meter	Foot-pound-force	0.737562
	Joule	1
	Kilogram-force-meter	0.1019716
	Watt-hour	2.777778×10^{-4}
	Watt-second	1
Nit	Candela/square meter	1
Noggin (Brit.)	Gill (Brit.)	1
Nox	Lux	0.001
Oersted	Ampere/meter	79.57747
Ohm (int. mean)	Ohm	1.00049
Ohm (int. U.S.)	Ohm	1.000495
Ohm/foot	Ohm/meter	3.280840
Ohm-centimeter	Ohm-meter	0.01
Ohm-circular mil/foot	Ohm-meter	1.662426×10^{-9}
Ohm-meter	Ohm-square millimeter/meter	1×10^{6}
Ohm-square millimeter/meter	Ohm-meter	1×10^{-6}
Ounce (avoirdupois)	Dram	16
	Grain	437.5
	Gram	28.349523
	Ounce (apot. or troy)	0.91145833
	Pennyweight	18.229167
	Pound	0.0625 (1/16)
	Scruple	21.875
Ounce (troy or ap.)	Grain	480
	Gram	31.1034768
	Ounce (avoirdupois)	1.0971429
	Penneyweight	20
	Pound (avoirdupois)	0.068571429
	Scruple	24
Ounce (Brit. fluid)	Cubic centimeter	28.41306
	Cubic inch	1.733871
	Drachm (Brit., fluid)	8
	Dram (U.S., fluid)	7.686079
	Gallon (Brit.)	6.25×10^{-3} (1/160)
	Gill (Brit.)	0.2
	Milliliter	28.41306
	Minim (Brit.)	480
	Ounce (U.S., fluid)	0.9607599
	Pint (Brit.)	0.05
	Quart (Brit.)	0.025 (1/40)
Ounce (U.S., fluid)	Cubic centimeter	29.57353
	Cubic inch	1.8046875
	Dram (U.S., fluid)	8
	Gallon (U.S.)	7.8125×10^{-3} (1/128)
	Gill (U.S.)	0.25
	Milliliter	29.57353
	Minim (U.S.)	480
	Ounce (Brit., fluid)	1.040843
	Pint (U.S., liquid)	0.0625 (1/16)
	Quart (U.S., liquid)	0.03125 (1/32)
Ounce (avoirdupois)/cubic foot	Kilogram/cubic meter	1.001154
Ounce (avoirdupois)/cubic inch	Kilogram/cubic meter	1729.994
Ounce (avoirdupois)/gallon (Brit.)	Kilogram/cubic meter	6.236023
Ounce (avoirdupois)/gallon (U.S.)	Kilogram/cubic meter	7.489152
Ounce (avoirdupois)/square foot	Gram/square meter	305.1517
Ounce (avoirdupois)/square yard	Gram/square meter	33.90575
Ounce (avoirdupois)/ton(long)	Gram/ton (metric)	27.90179
	Milligram/kilogram	27.90179
Ounce (avoirdupois)/ton(short)	Gram/ton (metric)	31.25
	Milligram/kilogram	31.25
Ounce (avoirdupois)/yard	Gram/meter	31.00342
Ounce-force (avoirdupois)	Newton	0.2780139
Ounce-force (avoirdupois)/square inch	Pascal	430.922
Ounce-force (avoirdupois)-inch	Newton-meter	7.06155×10^{-3}
Pace	Foot	2.5
Palm	Inch	3
Parsec	Astronomical unit	2.06265×10^{5}
	Kilometer	3.0857×10^{13}
	Light year	3.26164
	Mile (statute)	1.91735×10^{13}
Part per million	Gram/ton (metric)	1
	Milligram/kilogram	1
	Milliliter/cubic meter	1
	Ounce(avoirdupois)/ton (long)	0.03584
	Ounce (avoirdupois)/ton(short)	0.032
	Ounce(troy)/ton(long)	0.03266667
	Ounce(troy)/ton(short)	0.02916667
Pascal	Atmosphere	9.869233×10^{-6}
	Bar	1×10^{-5}
	Dyne/square centimeter	10
	Foot of H_2O (conv.)	3.34552×10^{-4}
	Inch of Hg (conv.)	2.95300×10^{-4}
	Inch of H_2O (conv.)	4.01463×10^{-3}
	Kilogram-force/square centimeter	1.01972×10^{-5}
	Millibar	0.01
	Millimeter of Hg (conv.)	7.50062×10^{-3}
	Millimeter of H_2O (conv.)	0.101972
	Newton/square meter	1
	Newton/square millimeter	1×10^{-6}
	Poundal/square foot	0.671969
	Pound-force/square foot	0.0208854
	Pound-force/square inch	1.45038×10^{-4}
	Torr	7.50062×10^{-3}
Pascal-second	Poise	10
Peck (Brit.)	Gallon (Brit.)	2
Peck (U.S.)	Bushel (U.S.)	0.25
	Quart (U.S., dry)	8
Pennyweight	Dram	0.87771429
	Grain	24
	Gram	1.55517384
	Ounce (avoirdupois)	0.054857143
	Ounce (apoth. or troy)	0.05
	Pound	3.4285714×10^{-3}
Perch	Foot	16.5
Phot	Lux	10000
Pica (printer's)	Point (printer's)	12
Picofarad	Farad	1×10^{-12}
Pint (Brit.)	Cubic centimeter	568.26125
	Cubic inch	34.67743
	Gallon (Brit.)	0.125 (1/8)
	Gill (Brit.)	4
	Liter	0.56826125
	Milliliter	568.26125
	Ounce (Brit., fluid)	20
	Pint (U.S., dry)	1.032057
	Pint (U.S. liquid)	1.200950
	Quart (Brit.)	0.5
Pint (U.S., dry)	Bushel (U.S.)	0.015625 (1/64)
	Cubic centimeter	550.6105
	Cubic inch	33.6003125
	Liter	0.5506105
	Milliliter	550.6105

CONVERSION FACTORS (continued)

To convert from	To	Multiply by
	Peck (U.S.)	0.0625 (1/16)
	Pint (Brit.)	0.9689390
	Quart (U.S. dry)	0.5
Pint (U.S., liquid)	Cubic centimeter	473.1765
	Cubic inch	28.875
	Gallon (U.S.)	0.125 (1/8)
	Gill (U.S.)	4
	Liter	0.4731765
	Milliliter	473.1765
	Ounce (U.S., fluid)	16
	Pint (Brit.)	0.8326742
	Quart (U.S., liquid)	0.5
Point (printer's, Didot)	Millimeter	0.3760650
Point (printer's, U.S.)	Inch	0.013837
	Millimeter	0.3514598
Poise	Dyne-second/square centimeter	1
	Gram/(centimeter × second)	1
	Pascal-second	0.1
Pole (Brit.)	Foot	16.5
Pond	Gram-force	1
Pottle (Brit.)	Gallon (Brit.)	0.5
Pound (avoirdupois)	Dram	256
	Grain	7000
	Gram	453.59237
	Hundredweight (long)	8.9285714×10^{-3}
	Hundredweight (short)	0.01
	Kilogram	0.45359237
	Ounce (avoirdupois)	16
	Ounce (troy)	14.583333
	Pennyweight	291.66667
	Pound (troy)	1.2152778
	Scruple	350
	Stone (Brit.)	0.07142857 (1/14)
	Ton (long)	4.4642857×10^{-4}
	Ton (metric)	4.5359237×10^{-4}
	Ton (short)	5×10^{-4} (1/2000)
Pound (troy)	Dram (troy)	96
	Grain	5760
	Gram	373.2417216
	Ounce (troy)	12
	Pennyweight	240
	Pound (avoirdupois)	0.82285714
	Scruple	288
Pound/acre	Kilogram/hectare	1.120851
Pound/cubic foot	Gram/liter	16.01846
	Kilogram/cubic meter	16.01846
	Pound/cubic inch	5.787037×10^{-4}
Pound/cubic inch	Gram/cubic centimeter	27.679905
	Pound/cubic foot	1728
Pound/cubic yard	Kilogram/cubic meter	0.5932764
Pound/foot	Kilogram/meter	1.488164
Pound/(foot × hour)	Pascal-second	4.133789×10^{-4}
Pound/(foot × second)	Pascal-second	1.488164
Pound/gallon (Brit.)	Gram/cubic centimeter	0.09977637
	Gram/liter	99.77637
	Kilogram/cubic meter	99.77637
	Pound/cubic foot	6.228835
	Ton(long)/cubic yard	0.07507968
Pound/gallon (U.S.)	Gram/cubic centimeter	0.1198264
	Gram/liter	119.8264
	Kilogram/cubic meter	119.8264
	Pound/cubic foot	7.480519
	Ton(short)/cubic yard	0.1009870
Pound/hour	Gram/minute	7.559873
	Gram/second	0.1259979
	Kilogram/day	10.88622
Pound/horsepower-hour	Kilogram/megajoule	0.1689659
	Kilogram/kilowatt-hour	0.6082774
Pound/inch	Kilogram/meter	17.85797
Pound/minute	Gram/second	7.559873
	Kilogram/hour	27.21554
Pound/second	Kilogram/hour	1632.932
	Kilogram/minute	27.21554

To convert from	To	Multiply by
Pound/square foot	Kilogram/square meter	4.882428
Poundal	Gram-force	14.0981
	Newton	0.1382550
	Pound-force	0.0310810
Poundal/square foot	Pascal	1.488164
Poundal-foot	Newton-meter	0.0421401
Poundal-second/square foot	Pascal-second	1.488164
Pound-force	Kilogram-force	0.453592
	Newton	4.44822
	Poundal	32.1740
Pound-force/foot	Newton/meter	14.5939
Pound-force/inch	Newton/meter	175.127
Pound-force/square foot	Atmosphere	4.72541×10^{-4}
	Bar	4.78803×10^{-4}
	Foot of H_2O (conv.)	0.0160185
	Gram-force/square centimeter	0.488243
	Inch of Hg (conv.)	0.0141390
	Millimeter of Hg (conv.)	0.359131
	Millimeter of H_2O (conv.)	4.88243
	Pascal	47.8803
	Pound-force/square inch	6.944444 × 10 (1/144)
Pound-force/square inch	Atmosphere	0.0680460
	Bar	0.0689476
	Foot of H_2O (conv.)	2.30666
	Inch of Hg (conv.)	2.03602
	Kilogram-force/square centimeter	0.0703070
	Meter of H_2O (conv.)	0.703070
	Millibar	68.9476
	Millimeter of Hg (conv.)	51.7149
	Pascal	6894.76
	Pound-force/square foot	144
Pound-force-foot	Newton-meter	1.35582
Pound-force-foot/inch	Newton-meter/meter	53.3787
Pound-force-inch	Newton-meter	0.112985
Pound-force-inch/inch	Newton-meter/ meter	4.44822
Pound-force-second/square foot	Pascal-second	47.8803
Pound-force-second/square inch	Pascal-second	6894.76
Psi	Pound-force/square inch	1
Puncheon (Brit.)	Gallon (Brit.)	70
Quadrant	Degree (angular)	90
	Gon (grade)	100
	Minute (angular)	5400
	Radian	1.570796 ($\pi/2$)
Quart (Brit.)	Cubic centimeter	1136.5225
	Cubic foot	0.04013591
	Cubic inch	69.35486
	Gallon (Brit.)	0.25 (1/4)
	Gill (Brit.)	8
	Liter	1.1365225
	Ounce (Brit., fluid)	40
	Pint (Brit.)	2
	Quart (U.S., dry)	1.032057
	Quart (U.S., liquid)	1.200950
Quart (U.S., dry)	Bushel (U.S.)	0.03125 (1/32)
	Cubic centimeter	1101.221
	Cubic foot	0.03888925
	Cubic inch	67.200625
	Liter	1.101221
	Peck (U.S.)	0.125 (1/8)
	Pint (U.S., dry)	2
	Quart (U.S., liquid)	1.163647
Quart (U.S., liquid)	Cubic centimeter	946.35295
	Cubic foot	0.03342014
	Cubic inch	57.75
	Dram (U.S., fluid)	256
	Gallon (U.S.)	0.25 (1/4)

CONVERSION FACTORS (continued)

To convert from	To	Multiply by
	Gill (U.S.)	8
	Liter	0.94635295
	Ounce (U.S., fluid)	32
	Pint (U.S., liquid)	2
	Quart (Brit.)	0.8326742
	Quart (U.S., dry)	0.8593670
Quarter (Brit., cap.)	Gallon (Brit.)	64
Quarter (Brit., mass)	Pound	28
Quarter (U.S., long)	Pound	560
Quarter (U.S., short)	Pound	500
Quintal	Kilogram	100
Rad	Gray	0.01
	Joule/kilogram	0.01
Radian	Circumference	0.1591549 (1/2 π)
	Degree (angular)	57.295780
	Gon (grade)	63.66198
	Minute (angular)	3437.747
	Quadrant	0.6366198 (2/π)
	Revolution	0.1591549
	Second (angular)	2.062648×10^{5}
Radian/centimeter	Degree/millimeter	5.729578
	Degree/foot	1746.375
	Degree/inch	145.5313
Radian/second	Revolution/minute	9.549297
Radian/square second	Revolution/square minute	572.9578
Register ton	Cubic foot	100
	Cubic meter	2.831685
Rem	Sievert	0.01
Revolution	Degree (angular)	360
	Gon (Grade)	400
	Radian	6.283185 (2 π)
Revolution/minute	Degree/second	6
Reyn	Pascal-second	6894.76
Rhe	1/pascal-second	10
Right angle	Degree	90
	Gon (grade)	100
Rod	Foot	16.5
Roentgen	Coulomb/kilogram	2.58×10^{-4}
Rood (Brit.)	Acre	0.25 (1/4)
	Square meter	1011.7141
Rope (Brit.)	Foot	20
Scruple	Dram (apoth. or troy)	0.3333333 (1/3)
	Grain	20
	Gram	1.2959782
	Ounce (avoirdupois)	0.045714286
	Ounce (apoth. or troy)	0.04166667 (1/24)
	Pennyweight	0.83333333 (10/12)
	Pound	2.857143×10^{-3} (1/350)
Scruple (Brit. fluid)	Minim (Brit.)	20
Seam (Brit.)	Gallon (Brit.)	64
Second (angular)	Degree	2.777778×10^{-4} (1/3600)
	Gon (grade)	3.086420×10^{-4} (1/3240)
	Minute (angular)	0.01666667 (1/60)
	Radian	4.848137×10^{-6}
Shake	Second	1×10^{-8}
Siemens	Mho (ohm^{-1})	1
Slug	Geepound	1
	Kilogram	14.5939
	Pound	32.1740
Slug/cubic foot	Kilogram/cubic meter	515.379
Slug/(foot × second)	Pascal-second	47.8803
Span	Inch	9
Sphere	Steradian	12.56637 (4 π)
Square centimeter	Circular mil	1.973525×10^{5}
	Circular millimeter	127.3240
	Square foot	1.076391×10^{-3}
	Square inch	0.1550003
	Square meter	1×10^{-4}
	Square millimeter	100
	Square yard	1.195990×10^{-4}
Square chain(Gunter's)	Acre	0.1
	Square foot	4356
	Square meter	404.6856
Square chain (Ramsden's)	Square foot	10000
Square chain(U.S. Survey)	Square meter	404.687261
Square degree	Steradian	3.046174×10^{-4}
Square foot	Acre	2.295684×10^{-5}
	Square centimeter	929.0304
	Square chain (Gunter's)	2.295684×10^{-4}
	Square chain (Ramsden's)	1×10^{-4}
	Square inch	144
	Square link (Gunter's)	2.295684
	Square meter	0.09290304
	Square mile	3.587006×10^{-8}
	Square rod	3.673095×10^{-3}
	Square yard	0.1111111 (1/9)
Square foot (U.S. Survey)	Square meter	0.092903412
Square foot/hour	Square meter/second	2.58064×10^{-5}
Square inch	Circular mil	1.273240×10^{6}
	Circular millimeter	821.4432
	Square centimeter	6.4516
	Square foot	6.944444×10^{-3} (1/144)
	Square millimeter	645.16
Square inch/second	Square foot/minute	0.4166667
	Square meter/hour	2.322576
Square kilometer	Acre	247.1054
	Hectare	100
	Square foot	1.076391×10^{7}
	Square meter	1×10^{6}
	Square mile	0.38610216
	Square yard	1.195990×10^{6}
Square link(Gunter's)	Square foot	0.4356
Square link(Ramsden's)	Square foot	1
Square meter	Acre	2.471054×10^{-4}
	Are	0.01
	Hectare	1×10^{-4}
	Square centimeter	10000
	Square chain (Gunter's)	2.471054×10^{-3}
	Square foot	10.76391
	Square inch	1550.003
	Square kilometer	1×10^{-6}
	Square link (Gunter's)	24.71054
	Square mile	3.861022×10^{-7}
	Square yard	1.195990
Square mil	Circular mil	1.273240
	Square inch	1×10^{-6}
	Square micrometer	645.16
	Square millimeter	6.4516×10^{-4}
Square mile	Acre	640
	Square chain (Gunter's)	6400
	Square foot	2.78784×10^{7}
	Square kilometer	2.589988110
	Square meter	2.589988×10^{6}
	Square rod	1.024×10^{5}
	Square yard	3.0976×10^{6}
	Township	0.02777778 (1/36)
Square mile (U.S. Survey)	Square kilometer	2.589998470
Square millimeter	Circular mil	1973.525
	Circular millimeter	1.273240
	Square centimeter	0.01
	Square inch	1.550003×10^{-3}
	Square mil	1550.003
Square rod	Acre	0.00625 (1/160)
	Square foot	272.25
	Square meter	25.29285
Square yard	Acre	2.066116×10^{-4}
	Square foot	9
	Square inch	1296
	Square meter	0.83612736
	Square mile	3.228306×10^{-7}

CONVERSION FACTORS (continued)

To convert from	To	Multiply by
Standard (Petrograd)	Cubic foot	*165*
Statampere	Ampere	3.335641×10^{-10}
Statcoulomb	Coulomb	3.335641×10^{-10}
Statfarad	Farad	1.112650×10^{-12}
Stathenry	Henry	8.987552×10^{11}
Statmho	Siemens	1.112650×10^{-12}
Statohm	Ohm	8.987552×10^{11}
Statvolt	Volt	299.7925
Steradian	Sphere	0.07957747 ($1/4\pi$)
	Spherical right angle	0.6366198 ($2/\pi$)
	Square degree	3282.806
Stere	Cubic meter	*1*
Stilb	Candela/square centimeter	*1*
Stokes	Square meter/second	1×10^{-4}
Stone	Pound	*14*
Tablespoon (metric)	Milliliter	*15*
Tablespoon (U.S.)	Milliliter	14.79
Teaspoon (metric)	Milliliter	*5*
Teaspoon (U.S.)	Milliliter	4.93
Terawatt-hour	Kilowatt-hour	1×10^{9}
Tesla	Weber/square meter	*1*
Tex	Denier	*9*
	Gram/kilometer	*1*
Therm	Btu	1×10^{5}
Thou	Mil	*1*
Ton (assay, (Brit.)	Gram	32.66667
Ton (assay, U.S.)	Gram	29.16667
Ton (long)	Hundredweight (long)	*20*
	Hundredweight (short)	*22.4*
	Kilogram	*1016.0469088*
	Pound	*2240*
	Ton (metric)	1.016047
	Ton (short)	*1.12*
Ton (metric)	Hundredweight (long)	19.684131
	Hundredweight (short)	22.046226
	Kilogram	*1000*
	Pound	2204.6226
	Ton (long)	0.98420653
	Ton (short)	1.1023113
Ton (short)	Hundredweight (long)	17.857143
	Hundredweight (short)	*20*
	Kilogram	*907.18474*
	Pound	*2000*
	Ton (long)	0.89285714
	Ton (metric)	*0.90718474*
Ton(long)/cubic yard	Kilogram/cubic meter	1328.939
Ton (metric)/cubic meter	Gram/cubic centimeter	*1*
	Kilogram/cubic decimeter	*1*
Ton(short)/cubic yard	Kilogram/cubic meter	1186.553
Ton-force (long)	Newton	9964.02
Ton-force (metric)	Newton	*9806.65*
Ton-force (short)	Newton	8896.44
Ton-force(long)/square foot	Atmosphere	1.05849
	Bar	1.07252
	Kilogram-force/square centimeter	1.09366
	Newton/square millimeter	0.107252
	Pascal	1.07252×10^{5}
	Pound-force/square inch	15.5556
Ton-force(long)/square inch	Atmosphere	152.423
	Bar	154.443
	Kilogram-force/square centimeter	157.488
	Newton/square millimeter	15.4443
	Pascal	1.54443×10^{7}
	Pound-force/square inch	*2240*
Ton-force(metric)/square meter	Atmosphere	0.0967841
	Bar	*0.0980665*
	Kilogram-force/square centimeter	*0.1*
	Newton/square millimeter	9.80665×10^{-3}
	Pascal	*9806.65*
	Pound-force/square inch	1.42233
Ton-force (short)/square foot	Atmosphere	0.945083
	Bar	0.957605
	Kilogram-force/square centimeter	0.976486
	Newton/square millimeter	0.0957605
	Pascal	9.57605×10^{4}
	Pound-force/square inch	13.8889
Ton-force (short)/square inch	Atmosphere	136.092
	Bar	137.895
	Kilogram-force/square centimeter	140.614
	Newton/square millimeter	13.7895
	Pascal	1.37895×10^{7}
	Pound-force/square inch	*2000*
Tonne	Kilogram	*1000*
Torr	Millibar	1.333224
	Millimeter of Hg (conv.)	1
	Pascal	133.3224
Township (U.S.)	Square kilometer	93.23957
	Square mile	*36*
Unit pole	Weber	1.256637×10^{-7}
Volt (int. mean)	Volt	1.00034
Volt (int. U.S.)	Volt	1.000330
Volt/inch	Volt/meter	39.37008
Volt-second	Weber	*1*
Watt	Btu/hour	3.41214
	Btu/minute	0.0568690
	Calorie/minute	14.3308
	Calorie/second	0.238846
	Erg/second	1×10^{7}
	Foot-pound-force/minute	44.2537
	Foot-pound-force/second	0.737562
	Horsepower	1.34102×10^{-3}
	Horsepower (metric)	1.35962×10^{-3}
	Joule/second	*1*
	Kilocalorie/hour	0.859845
	Kilogram-force-meter/second	0.101972
Watt (int. mean)	Watt	1.00019
Watt (int. U.S.)	Watt	1.000165
Watt/square inch	Btu/(hour × square foot)	491.348
	Kilocalorie/(hour × square meter)	1332.76
	Watt/square meter	1550.003
Watt/square meter	Kilocalorie/(hour × square meter)	0.859845
Watt-hour	Btu	3.41214
	Calorie	859.845
	Foot-pound-force	2655.22
	Horsepower-hour	1.34102×10^{-3}
	Horsepower-hour (metric)	1.35962×10^{-3}
	Joule	*3600*
	Kilogram-force-meter	367.098
	Liter-atmosphere	35.5292
Watt-second	Erg	1×10^{7}
	Joule	*1*
	Newton-meter	*1*
Weber	Maxwell	1×10^{8}
Weber/square meter	Gauss	*10000*

CONVERSION FACTORS (continued)

To convert from	To	Multiply by
Week	Day	*7*
	Hour	*168*
	Minute	*10080*
	Month	0.2299795
	Second	6.048×10^{5}
X-unit	Meter	1.00202×10^{-13}
Yard	Centimeter	*91.44*
	Fathom	*0.5*
	Foot	*3*
	Inch	*36*
	Meter	*0.9144*
	Mile	5.681818×10^{-4}
Year (calendar, mean of 4-year period)	Day	*365.25*
	Hour	*8766*
	Minute	5.2596×10^{5}
	Second	3.15576×10^{7}
	Week	52.17857
Year (leap)	Day	*366*
Year (normal calendar)	Day	*365*
	Hour	*8760*
	Minute	5.256×10^{5}
	Second	3.1536×10^{7}
	Week	52.14286
Year (sidereal)	Day	365.25636
	Second	3.155815×10^{7}
	Year (tropical)	1.0000388
Year (tropical)	Day	365.24220
	Second	3.1556926×10^{7}
	Year (sidereal)	0.9999612

DECIMAL EQUIVALENTS OF COMMON FRACTIONS

		1/64	= 0.015 625		11/32	22/64	= 0.343 75			43/64	= 0.671 875
	1/32	2/64	= .031 25			23/64	= .359 375	11/16	22/32	44/64	= .687 5
		3/64	= .046 875	3/8	12/32	24/64	= .375			45/64	= .703 125
1/16	2/32	4/64	= .062 5			25/64	= .390 625		23/32	46/64	= .718 75
		5/64	= .078 125		13/32	26/64	= .406 25			47/64	= .734 375
	3/32	6/64	= .093 75			27/64	= .421 875	3/4	24/32	48/64	= .75
		7/64	= .109 375	7/16	14/32	28/64	= .437 5			49/64	= .765 625
1/8	4/32	8/64	= .125			29/64	= .453 125		25/32	50/64	= .781 25
		9/64	= .140 625		15/32	30/64	= .468 75			51/64	= .796 875
	5/32	10/64	= .156 25			31/64	= .484 375	13/16	26/32	52/64	= .812 5
		11/64	= .171 875	1/2	16/32	32/64	= .50			53/64	= .828 125
3/16	6/32	12/64	= .187 5			33/64	= .515 625		27/32	54/64	= .843 75
		13/64	= .203 125		17/32	34/64	= .531 25			55/64	= .859 375
	7/32	14/64	= .218 75			35/64	= .546 875	7/8	28/32	56/64	= .875
		15/64	= .234 375	9/16	18/32	36/64	= .562 5			57/64	= .890 625
1/4	8/32	16/64	= .25			37/64	= .578 125		29/32	58/64	= .906 25
		17/64	= .265 625		19/32	38/64	= .593 75			59/64	= .921 875
	9/32	18/64	= .281 25			39/64	= .609 375	15/16	30/32	60/64	= .937 5
		19/64	= .296 875	5/8	20/32	40/64	= .625			61/64	= .953 125
5/16	10/32	20/64	= .312 5			41/64	= .640 625		31/32	62/64	= .968 75
		21/64	= .328 125		21/32	42/64	= .656 25			63/64	= .984 375

FUNDAMENTAL PHYSICAL CONSTANTS

DR. E. RICHARD COHEN

The following table contains data which are a tentative revision of the 1963 values of the fundamental physical constants.

It has become increasingly clear in the last several years that the 1963 analysis of the fundamental physical constants by Cohen and DuMond must be revised and that the values recommended at the time are in error by as much as 100 ppm. The strongest evidence for this revision came from the measurement in 1967 of macroscopic phase coherence in superconductors by Langenberg, Parker and Taylor at the University of Pennsylvania. Their measured value of the quantum of magnetic flux ($h/2e$), measured with an accuracy of 4 ppm, was inconsistent with the 1963 recommendation by 10 times that amount. This verified the growing evidence from spectroscopic and microwave data that the value of the fine structure constant needed a revision of 20 ppm.

It is therefore clear that a complete revision of the 1963 recommendation is necessary. Such a revision will of course include experimental data in addition to that on the fine structure constant, including careful attention to the electrical standards maintained by each national standards laboratory as recalibrated with respect to BIPM in 1968, effective January 1, 1969.

The following table of numerical values of the physical constants is intended as a general indication of the extent of the revision required in the 1963 values. Because of the tentative nature, and since the full effect of experimental correlations between data have not been calculated, no errors are quoted for these values. The numerical values, although tentative, and not representing a full reassessment of the available data are believed to be more reliable than the 1963 values. A more recent discussion of the status of the physical constants as of approximately January, 1969, is given by B. N. Taylor, W. H. Parker, and D. N. Langenberg in Reviews of Modern Physics, July 1969.

FUNDAMENTAL PHYSICAL CONSTANTS

Constant	Symbol	Old value	New value		*Correction ppm
Speed of light in vacuum	c	2.997925_1	2.997925	$\times 10^{8}$ ms^{-1}	0
Gravitational constant	G	6.670_5	6.670	10^{-11} Nm^{2}kg^{-2}	0
Elementary charge	e	1.60210_2	1.6022	10^{-19}C	+60
		4.80298_7	4.8032	10^{-10} esu	+60
Avogadro constant	N_A	6.02252_9	6.0222	10^{26} kmole^{-1}	−60
Mass unit	u	1.66043_2	1.66053	10^{-27} kg	+60
Electron rest mass	m_e	9.10908_{13}	9.1096	10^{-31} kg	+60
		5.48597_3	5.48593	10^{-4} u	0
Proton rest mass	m_p	1.67252_3	1.67262	10^{-27} kg	+60
		1.00727663_8	1.00727661	u	0
Neutron rest mass	m_n	1.67482_3	1.67492	10^{-27} kg	+60
		1.0086654_4	1.0086652	u	0
Faraday constant	F	9.64870_5	9.6487	10^{4}C mole^{-1}	0
		2.89261_{25}	2.8926	10^{14} esu	0
Planck constant	h	6.62559_{16}	6.6262	10^{-34} Js	+100
	$h/2\pi$	1.054494_{25}	1.05459	10^{-34} Js	+100
Fine-structure constant	α	7.29720_3	7.29735	10^{-3}	+20
$2\pi e^2/hc$	$1/\alpha$	137.0388_6	137.0360		−20
Charge-to-mass ratio	e/m_e	1.758796_6	1.75880	10^{11}C kg^{-1}	0
for electron		5.27274_2	5.27276	10^{17} esu	0
Quantum of magnetic flux	hc/e	4.13556_4	4.13571	10^{-11} Wb	+40
		1.379474_{13}	1.37952	10^{-17} esu	+40
Rydberg constant	R_∞	1.0973731_1	1.0973731	10^{7} m^{-1}	0
Bohr radius	a_0	5.29167_2	5.29177	10^{-11} m	+20
Compton wavelength of	$\lambda_c = h/m_ec$	2.42621_2	2.42631	10^{-12} m	+40
electron	$\lambda_c/2\pi$	3.86144_3	3.86159	10^{-13} m	+40
Gyromagnetic ratio of	γ	2.675192_7	2.67519	10^{8} rad s^{-1}T^{-1}	0
proton	$\gamma/2\pi$	4.25770_1	4.2577	10^{7} Hz T^{-1}	0
(Uncorrected for	γ'	2.675123_7	†2.67512	10^{8} s^{-1}T^{-1}	0
diamagnetism H_2O)	$\gamma'/2\pi$	4.25759_1	†4.257586	10^{7} Hz T^{-1}	0
Bohr magneton	μ_B	9.2732_2	9.2741	10^{-24} J T^{-1}	+100
Nuclear magneton	μ_N	5.05050_{13}	5.0510	10^{-27} J T^{-1}	+100
Proton Moment	μ_p	1.41049_4	1.4106	10^{-26} J T^{-1}	+80
	μ_p/μ_N	2.79276_2	2.79278		0
(Uncorrected for diamagnetism in H_2O sample)		2.79268_2	2.79271		0
Gas constant	R_0	8.31434_{35}	8.3143	J deg^{-1} mole^{-1}	0
Boltzmann constant	k	1.38054_6	1.3806	10^{-23} J deg^{-1}	+60
First radiation constant ($2\pi hc^2$)	c_1	3.74150_9	3.7418	10^{-16} W m^2	+80
Second radiation constant (hc/k)	c_2	1.43879_6	1.4388	10^{-2} m deg	0
Stephan-Boltzmann constant	σ	5.6697_{10}	5.6696	10^{-8} W m^{-2} deg^{-4}	−20

* This column gives the correction resulting only from the increase of 20 ppm in the value of the fine structure constant, not the total change from 1963 to the tentative new value.

† The value for the gyromagnetic ratio of the proton has been recommended by the Comité International des Poids et Mesures in their meeting of 14–17 October 1968 for international metrological usage. This value is based on the 1969 BIPM scales of resistance and electromotive force which are in agreement, as exactly as is possible, with the (absolute) definitions of electrical units adopted by the Conférence Generale des Poids et Mesures.

MISCELLANEOUS CONSTANTS

PHYSICAL CONSTANTS

Equatorial radius of the earth = 6378.388 km = 3963.34 miles (statute).

Polar radius of the earth, 6356.912 km = 3949.99 miles (statute).

1 degree of latitude at 40° = 69 miles.

1 international nautical mile = 1.15078 miles (statute) = 1852 m = 6076.115 ft.

Mean density of the earth = 5.522 g/cm^3 = 344.7 lb/ft^3.

Constant of gravitation, (6.673 ± 0.003) X 10^3 cm^3 gm^{-1} s^{-2}.

Acceleration due to gravity at sea level, latitude 45° = 980.6194cm/s^2 = 32,1726 ft/sec^2

Length of seconds pendulum at sea level, latitude 45° = 99.3575 cm = 39.1171 in.

1 knot (international) = 101.269 ft/min = 1.6878 ft/sec = 1.1508 miles (statute)/hr.

1 micron = 10^{-4} cm.

1 angstrom = 10^{-8} cm.

Mass of hydrogen atom = (1.67339 ± 0.0031) X 10^{-24} g.

Density of mercury at 0°C = 13.5955 g/ml.

Density of water at 3.98°C = 1.000000 g/ml.

Density, maximum, of water, at 3.98°C = 0.999973 g/cm^3.

Density of dry air at 0°C, 760 mm = 1.2929 g/liter.

Velocity of sound in dry air at 0°C = 331.36 m/s - 1087.1 ft/sec.

Velocity of light in vacuum = (2.997925 ± 0.000002) X 10^{10} cm/s.

Heat of fusion of water 0°C = 79.71 cal/g.

Heat of vaporization of water 100°C = 539.55 cal/g.

Electrochemical equivalent of silver 0.001118 g/sec international amp.

Absolute wave length of red cadmium light in air at 15°C, 760 mm pressure = 6438.4696 A.

Wave length of orange-red line of krypton 86 = 6057.802 A.

π CONSTANTS

π = 3.14159 26535 89793 23846 26433 83279 50288 41971 69399 37511

$1/\pi$ = 0.31830 98861 83790 67153 77675 26745 02872 40689 19291 48091

π^2 = 9.86960 44010 89358 61883 44909 99876 15113 53136 99407 24079

$\log_e \pi$ = 1.14472 98858 49400 17414 34273 51353 05871 16472 94812 91531

$\log_{10} \pi$ = 0.49714 98726 94133 85435 12682 88290 89887 36516 78324 38044

$\log_{10} \sqrt{2\pi}$ = 0.39908 99341 79057 52478 25035 91507 69595 02099 34102 92128

CONSTANTS INVOLVING e

e = 2.71828 18284 59045 23536 02874 71352 66249 77572 47093 69996

$1/e$ = 0.36787 94411 71442 32159 55237 70161 46086 74458 11131 03177

e^2 = 7.38905 60989 30650 22723 04274 60575 00781 31803 15570 55185

$M = \log_{10} e$ = 0.43429 44819 03251 82765 11289 18916 60508 22943 97005 80367

$1/M = \log_e 10$ = 2.30258 50929 94045 68401 79914 54684 36420 76011 01488 62877

$\log_{10} M$ = 9.63778 43113 00536 78912 29674 98645 − 10

π^e AND e^π CONSTANTS

π^e = 22.45915 77183 61045 47342 71522

e^π = 23.14069 26327 79269 00572 90864

$e^{-\pi}$ = 0.04321 39182 63772 24977 44177

$e^{\frac{1}{2}\pi}$ = 4.81047 73809 65351 65547 30357

$i^i = e^{-\frac{1}{2}\pi}$ = 0.20787 95763 50761 90854 69556

NUMERICAL CONSTANTS

$\sqrt{2}$ = 1.41421 35623 73095 04880 16887 24209 69807 85696 71875 37695

$\sqrt[3]{2}$ = 1.25992 10498 94873 16476 72106 07278 22835 05702 51464 70151

$\log_e 2$ = 0.69314 71805 59945 30941 72321 21458 17656 80755 00134 36026

$\log_{10} 2$ = 0.30102 99956 63981 19521 37388 94724 49302 67681 89881 46211

$\sqrt{3}$ = 1.73205 08075 68877 29352 74463 41505 87236 69428 05253 81039

$\sqrt[3]{3}$ = 1.44224 95703 07408 38232 16383 10780 10958 83918 69253 49935

$\log_e 3$ = 1.09861 22886 68109 69139 52452 36922 52570 46474 90557 82275

$\log_{10} 3$ = 0.47712 12547 19662 43729 50279 03255 11530 92001 28864 19070

OTHER CONSTANTS

Euler's Constant γ = 0.57721 56649 01532 86061

$\log_e \gamma$ = −0.54953 93129 81644 82234

Golden Ratio ϕ = 1.61803 39887 49894 84820 45868 34365 63811 77203 09180

NUMBERS CONTAINING π

	Number	Logarithm		Number	Logarithm
π	3.1415 927	0.4971 499	$2\pi^2$	19.7392 088	1.2953 297
2π	6.2831 853	0.7981 799	$\pi/180$	0.0174 533	8.2418 774 − 10
3π	9.4247 780	0.9742 711	$180/\pi$	57.2957 795	1.7581 226
4π	12.5663 706	1.0992 099	$4\pi^2$	39.4784 176	1.5963 597
8π	25.1327 412	1.4002 399	$1/\pi^2$	0.1013 212	9.0057 003 − 10
$\pi/2$	1.5707 963	0.1961 199	$1/(2\pi^2)$	0.0506 606	8.7046 703 − 10
$\pi/3$	1.0471 976	0.0200 286	$1/(4\pi^2)$	0.0253 303	8.4036 403 − 10
$\pi/4$	0.7853 982	9.8950 899 − 10	$\sqrt{\pi}$	1.7724 539	0.2485 749
$\pi/6$	0.5235 988	9.7189 986 − 10	$\frac{\sqrt{\pi}}{2}$	0.8862 269	9.9475 449 − 10
$\pi/8$	0.3926 991	9.5940 599 − 10	$\frac{\sqrt{\pi}}{4}$	0.4431 135	9.6465 149 − 10
$2\pi/3$	2.0943 951	0.3210 586	$\sqrt{\frac{\pi}{2}}$	1.2533 141	0.0980 599
$4\pi/3$	4.1887 902	0.6220 886	$\sqrt{\frac{2}{\pi}}$	0.7978 846	9.9019 401 − 10
$1/\pi$	0.3183 099	9.5028 501 − 10	π^3	31.0062 767	1.4914 496
$2/\pi$	0.6366 198	9.8038 801 − 10	$\sqrt[3]{\pi}$	1.4645 919	0.1657 166
$4/\pi$	1.2732 395	0.1049 101	$1/\sqrt[3]{\pi}$	0.6827 841	9.8342 834 − 10
$1/(2\pi)$	0.1591 549	9.2018 201 − 10	$\sqrt[3]{\pi^2}$	2.1450 294	0.3314 332
$1/(4\pi)$	0.0795 775	8.9007 901 − 10	$1/\sqrt{\pi}$	0.5641 896	9.7514 251 − 10
$1/(6\pi)$	0.0530 516	8.7246 989 − 10	$1/\sqrt{2\pi}$	0.3989 423	9.6009 101 − 10
$1/(8\pi)$	0.0397 887	8.5997 601 − 10	$2/\sqrt{\pi}$	1.1283 792	0.0524 551
π^2	9.8696 044	0.9942 997			

MULTIPLES OF $\frac{\pi}{2}$

n	$n\frac{\pi}{2}$	n	$n\frac{\pi}{2}$	n	$n\frac{\pi}{2}$	n	$n\frac{\pi}{2}$
1	1.57079 63268	26	40.84070 44967	51	80.11061 26665	76	119.38502 08364
2	3.14159 26536	27	42.41150 08235	52	81.68140 89933	77	120.95131 71632
3	4.71238 89804	28	43.98229 71503	53	83.25220 53201	78	122.52211 34900
4	6.28318 53072	29	45.55309 34771	54	84.82300 16469	79	124.09290 98168
5	7.85398 16340	30	47.12388 98038	55	86.39379 79737	80	125.66370 61436
6	9.42477 79608	31	48.69468 61306	56	87.96459 43005	81	127.23450 24704
7	10.99557 42876	32	50.26548 24574	57	89.53539 06273	82	128.80529 87972
8	12.56637 06144	33	51.83627 87842	58	91.10618 69541	83	130.37609 51240
9	14.13716 69412	34	53.40707 51110	59	92.67698 32809	84	131.94689 14508
10	15.70796 32679	35	54.97787 14378	60	94.24777 96077	85	133.51768 77776
11	17.27875 95947	36	56.54866 77646	61	95.81857 59345	86	135.08848 41044
12	18.84955 59215	37	58.11946 40914	62	97.38937 22613	87	136.65928 04312
13	20.42035 22483	38	59.69026 04182	63	98.96016 85881	88	138.23007 67580
14	21.99114 85751	39	61.26105 67450	64	100.53096 49149	89	139.80087 30847
15	23.56194 49019	40	62.83185 30718	65	102.10176 12417	90	141.37166 94115
16	25.13274 12287	41	64.40264 93986	66	103.67255 75685	91	142.94246 57383
17	26.70533 75555	42	65.97344 57254	67	105.24335 38953	92	144.51326 20651
18	28.27433 38823	43	67.54424 20522	68	106.81415 02221	93	146.08405 83919
19	29.84513 02091	44	69.11503 83790	69	108.38494 65488	94	147.65485 47187
20	31.41592 65359	45	70.68583 47058	70	109.95574 28765	95	149.22565 10455
21	32.98672 28627	46	72.25663 10326	71	111.52653 92024	96	150.79644 73723
22	34.55751 91895	47	73.82742 73594	72	113.09733 55292	97	152.36724 36991
23	36.12831 55163	48	75.39822 36862	73	114.66813 18560	98	153.93804 00259
24	37.69911 18431	49	76.96902 00129	74	116.23892 81828	99	155.50883 63527
25	39.26990 81699	50	78.53981 63397	75	117.80972 45096	100	157.07963 26795

II. ALGEBRA

FACTORS AND EXPANSIONS

$$(a \pm b)^2 = a^2 \pm 2ab + b^2.$$
$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3.$$
$$(a \pm b)^4 = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4.$$
$$a^2 - b^2 = (a - b)(a + b).$$
$$a^2 + b^2 = (a + b\sqrt{-1})(a - b\sqrt{-1}).$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$
$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$
$$a^4 + b^4 = (a^2 + ab\sqrt{2} + b^2)(a^2 - ab\sqrt{2} + b^2).$$
$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \ldots + b^{n-1}).$$
$$a^n - b^n = (a + b)(a^{n-1} - a^{n-2}b + \ldots - b^{n-1}),$$
for even values of n.
$$a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + \ldots + b^{n-1}),$$
for odd values of n.
$$a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2).$$
$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$
$$(a + b + c)^3 = a^3 + b^3 + c^3 + 3a^2(b + c) + 3b^2(a + c) + 3c^2(a + b) + 6abc.$$
$$(a + b + c + d + \ldots)^2 = a^2 + b^2 + c^2 + d^2 + \ldots + 2a(b + c + d + \ldots) + 2b(c + d + \ldots) + 2c(d + \ldots) + \ldots$$

See also under Series.

POWERS AND ROOTS

$a^x \times a^y = a^{(x+y)}$. $\quad a^0 = 1$ [if $a \neq 0$] $\quad (ab)^x = a^x b^x$.

$\dfrac{a^x}{a^y} = a^{(x-y)}$. $\quad a^{-x} = \dfrac{1}{a^x}$. $\quad \left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$.

$(a^x)^y = a^{xy}$. $\quad a^{\frac{1}{x}} = \sqrt[x]{a}$. $\quad \sqrt[x]{ab} = \sqrt[x]{a}\,\sqrt[x]{b}$.

$\sqrt[x]{\sqrt[y]{a}} = \sqrt[xy]{a}$. $\quad a^{\frac{x}{y}} = \sqrt[y]{a^x}$. $\quad \sqrt[x]{\dfrac{a}{b}} = \dfrac{\sqrt[x]{a}}{\sqrt[x]{b}}$.

PROPORTION

If $\dfrac{a}{b} = \dfrac{c}{d}$, then $\dfrac{a+b}{b} = \dfrac{c+d}{d}$,

$$\frac{a-b}{b} = \frac{c-d}{d}, \qquad \frac{a-b}{a+b} = \frac{c-d}{c+d}.$$

*ARITHMETIC PROGRESSION

An arithmetic progression is a sequence of numbers such that each number differs from the previous number by a constant amount, called the *common difference.*

If a_1 is the first term; a_n the nth term; d the common difference; n the number of terms; and s_n the sum of n terms—

$$a_n = a_1 + (n - 1)d, \quad s_n = \frac{n}{2}[a_1 + a_n].$$

$$s_n = \frac{n}{2}[2a_1 + (n - 1)d].$$

The arithmetic mean between a and b is given by $\dfrac{a + b}{2}$.

*GEOMETRIC PROGRESSION

A geometric progression is a sequence of numbers such that each number bears a constant ratio, called the *common ratio*, to the previous number.

If a_1 is the first term; a_n the nth term; r the common ratio; n the number of terms; and s_n the sum of n terms

$$a_n = a_1 r^{n-1}; \; s_n = a_1 \frac{1 - r^n}{1 - r}$$

$$= a_1 \frac{r^n - 1}{r - 1}, \quad r \neq 1.$$

$$= \frac{a_1 - ra_n}{1 - r}$$

$$= \frac{ra_n - a_1}{r - 1}$$

If $|r| < 1$, then the sum of an infinite geometrical progression converges to the limiting value

$$\frac{a_1}{1 - r}, \quad \left[s_\infty = \lim_{n \to \infty} \frac{a_1(1 - r^n)}{1 - r} = \frac{a_1}{1 - r}\right]$$

The geometric mean between a and b is given by $\sqrt{ab}$.

*It is customary to represent a_n by l in a finite progression and refer to it as the last term.

HARMONIC PROGRESSION

A sequence of numbers whose reciprocals form an arithmetic progression is called an harmonic progression. Thus

$$\frac{1}{a_1}, \frac{1}{a_1 + d}, \frac{1}{a_1 + 2d}, \ldots, \frac{1}{a_1 + (n - 1)d}, \ldots,$$

where

$$\frac{1}{a_n} = \frac{1}{a_1 + (n - 1)d}$$

forms an harmonic progression. The harmonic mean between a and b is given by $\dfrac{2ab}{a + b}$.

If A, G, H respectively represent the arithmetic mean, geometric mean, and harmonic mean between a and b, then $G^2 = AH$.

QUADRATIC EQUATIONS

Any quadratic equation may be reduced to the form,—

$$ax^2 + bx + c = 0.$$

Then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If a, b, and c are real then:
If $b^2 - 4ac$ is positive, the roots are real and unequal;
If $b^2 - 4ac$ is zero, the roots are real and equal;
If $b^2 - 4ac$ is negative, the roots are imaginary and unequal.

CUBIC EQUATIONS

A cubic equation, $y^3 + py^2 + qy + r = 0$ may be reduced to the form,—

$$x^3 + ax + b = 0$$

by substituting for y the value, $x - \frac{p}{3}$. Here

$$a = \tfrac{1}{3}(3q - p^2) \text{ and } b = \tfrac{1}{27}(2p^3 - 9pq + 27r).$$

For solution let,—

$$A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}, \qquad B = \sqrt[3]{+\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}},$$

then the values of x will be given by,

$$x = A + B, \quad -\frac{A+B}{2} + \frac{A-B}{2}\sqrt{-3}, \quad -\frac{A+B}{2} - \frac{A-B}{2}\sqrt{-3}.$$

If p, q, r are real, then:

If $\frac{b^2}{4} + \frac{a^3}{27} > 0$, there will be one real root and two conjugate complex roots;

If $\frac{b^2}{4} + \frac{a^3}{27} = 0$, there will be three real roots of which at least two are equal;

If $\frac{b^2}{4} + \frac{a^3}{27} < 0$, there will be three real and unequal roots.

Trigonometric Solution of the Cubic Equation

The form $x^3 + ax + b = 0$ with $ab \neq 0$ can always be solved by transforming it to the trigonometric identity

$$4\cos^3\theta - 3\cos\theta - \cos(3\theta) \equiv 0.$$

Let $x = m\cos\theta$, then

$$x^3 + ax + b \equiv m^3\cos^3\theta + am\cos\theta + b \equiv 4\cos^3\theta - 3\cos\theta - \cos(3\theta) \equiv 0.$$

Hence

$$\frac{4}{m^3} = -\frac{3}{am} = \frac{-\cos(3\theta)}{b},$$

from which follows that

$$m = 2\sqrt{-\frac{a}{3}}, \quad \cos(3\theta) = \frac{3b}{am}.$$

Any solution θ_1 which satisfies $\cos(3\theta) = \dfrac{3b}{am}$, will also have the solutions

$$\theta_1 + \frac{2\pi}{3} \quad \text{and} \quad \theta_1 + \frac{4\pi}{3}.$$

The roots of the cubic $x^3 + ax + b = 0$ are

$$2\sqrt{-\frac{a}{3}}\cos\theta_1, \quad 2\sqrt{-\frac{a}{3}}\cos\left(\theta_1 + \frac{2\pi}{3}\right), \quad 2\sqrt{-\frac{a}{3}}\cos\left(\theta_1 + \frac{4\pi}{3}\right).$$

Example where hyperbolic functions are necessary for solution with latter procedure

The roots of the equation $x^3 - x + 2 = 0$ may be found as follows:

Here

$$a = -1, \quad b = 2, \quad m = 2\sqrt{\tfrac{1}{3}} = 1.155$$

$$\cos(3\theta) = \frac{6}{-1.155} = -5.196$$

$$\cos(3\theta) = -\cos(3\theta - \pi) = -\cosh[i(3\theta - \pi)] = -5.196.$$

Using hyperbolic function tables for $\cosh[i(3\theta - \pi)] = 5.196$, it is found that

$$i(3\theta - \pi) = 2.332.$$

Thus

$$3\theta - \pi = -i(2.332).$$

$$3\theta = \pi - i(2.332)$$

$$\theta_1 = \frac{\pi}{3} - i(0.777)$$

$$\theta_1 + \frac{2\pi}{3} = \pi - i(0.777)$$

$$\theta_1 + \frac{4\pi}{3} = \frac{5\pi}{3} - i(0.777)$$

$$\begin{aligned}\cos\theta_1 &= \cos\left[\frac{\pi}{3} - i(0.777)\right]\\ &= \left(\cos\frac{\pi}{3}\right)[\cos i(0.777)] + \left(\sin\frac{\pi}{3}\right)[\sin i(0.777)]\\ &= \left(\cos\frac{\pi}{3}\right)(\cosh 0.777) + i\left(\sin\frac{\pi}{3}\right)(\sinh 0.777)\\ &= (0.5)(1.317) + i(0.866)(0.858) = 0.659 + i(0.743).\end{aligned}$$

Note that

$$\cos\mu = \cosh(i\mu) \quad \text{and} \quad \sin\mu = -i\sinh(i\mu).$$

Similarly

$$\begin{aligned}\cos\left(\theta_1 + \frac{2\pi}{3}\right) &= \cos[\pi - i(0.777)]\\ &= (\cos\pi)(\cosh 0.777) + i(\sin\pi)(\sinh 0.777)\\ &= -1.317,\end{aligned}$$

and

$$\begin{aligned}\cos\left(\theta_1 + \frac{4\pi}{3}\right) &= \cos\left[\frac{5\pi}{3} - i(0.777)\right]\\ &= \left(\cos\frac{5\pi}{3}\right)(\cosh 0.777) + i\left(\sin\frac{5\pi}{3}\right)(\sinh 0.777)\\ &= (0.5)(1.317) - i(0.866)(0.858) = 0.659 - i(0.743).\end{aligned}$$

The required roots are

$$\begin{aligned}1.155[0.659 + i(0.743)] &= 0.760 + i(0.858)\\ (1.155)(-1.317) &= -1.520\\ (1.155)[0.659 - i(0.743)] &= 0.760 - i(0.858).\end{aligned}$$

QUARTIC OR BIQUADRATIC EQUATION

A quartic equation,

$$x^4 + ax^3 + bx^2 + cx + d = 0,$$

has the *resolvent cubic equation*

$$y^3 - by^2 + (ac - 4d)y - a^2d + 4bd - c^2 = 0.$$

Let y be any root of this equation, and

$$R = \sqrt{\frac{a^2}{4} - b + y}.$$

If $R \neq 0$, then let

$$D = \sqrt{\frac{3a^2}{4} - R^2 - 2b + \frac{4ab - 8c - a^3}{4R}}$$

and

$$E = \sqrt{\frac{3a^2}{4} - R^2 - 2b - \frac{4ab - 8c - a^3}{4R}}$$

If $R = 0$, then let

$$D = \sqrt{\frac{3a^2}{4} - 2b + 2\sqrt{y^2 - 4d}}$$

and

$$E = \sqrt{\frac{3a^2}{4} - 2b - 2\sqrt{y^2 - 4d}}.$$

Then the four roots of the original equation are given by

$$x = -\frac{a}{4} + \frac{R}{2} \pm \frac{D}{2}$$

and

$$x = -\frac{a}{4} - \frac{R}{2} \pm \frac{E}{2}.$$

EQUATION $x^n = c$

Using DeMoivre's theorem:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta;\ i = \sqrt{-1},$$

the equation $x^n = c$ has n roots given by

$$x = \sqrt[n]{c}\left(\cos\frac{2m\pi}{n} + i\sin\frac{2m\pi}{n}\right) \text{if } c > 0,$$

or

$$x = \sqrt[n]{-c}\left(\cos\frac{(2m+1)\pi}{n} + i\sin\frac{(2m+1)\pi}{n}\right) \text{ if } c < 0,$$

where m takes the n values $0,1,2, \ldots (n-1)$ giving n roots.

PARTIAL FRACTIONS

This section applies only to rational algebraic fractions with numerator of lower degree than the denominator. Improper fractions can be reduced to proper fractions by long division.

Every fraction may be expressed as the sum of component fractions whose denominators are factors of the denominator of the original fraction.

Let $N(x)$ = numerator, a polynomial of the form

$$n_0 + n_1x + n_2x^2 + \cdots + n_ix^i$$

I. *Non-repeated Linear Factors*

$$\frac{N(x)}{(x-a)G(x)} = \frac{A}{x-a} + \frac{F(x)}{G(x)}$$

$$A = \left[\frac{N(x)}{G(x)}\right]_{x=a}$$

$F(x)$ determined by methods discussed in the following sections.

Example:

$$\frac{x^2+3}{x(x-2)(x^2+2x+4)} = \frac{A}{x} + \frac{B}{x-2} + \frac{F(x)}{x^2+2x+4}$$

$$A = \left[\frac{x^2+3}{(x-2)(x^2+2x+4)}\right]_{x=0} = -\frac{3}{8}$$

$$B = \left[\frac{x^2+3}{x(x^2+2x+4)}\right]_{x=2} = \frac{4+3}{2(4+4+4)} = \frac{7}{24}$$

II. *Repeated Linear Factors*

$$\frac{N(x)}{x^mG(x)} = \frac{A_0}{x^m} + \frac{A_1}{x^{m-1}} + \cdots + \frac{A_{m-1}}{x} + \frac{F(x)}{G(x)}$$

$$F(x) = f_0 + f_1x + f_2x^2 + \cdots, \quad G(x) = g_0 + g_1x + g_2x^2 + \cdots$$

$$A_0 = \frac{n_0}{g_0}, \quad A_1 = \frac{n_1 - A_0g_1}{g_0}, \quad A_2 = \frac{n_2 - A_0g_2 - A_1g_1}{g_0}$$

General term:

$$A_k = \frac{1}{g_0}\left[n_k - \sum_{i=0}^{k-1} A_ig_{k-i}\right]$$

$$*m = 1\begin{cases} f_0 = n_1 - A_0g_1 \\ f_1 = n_2 - A_0g_2 \\ f_j = n_{j+1} - A_0g_{j+1} \end{cases}$$

$$m = 2\begin{cases} f_0 = n_2 - A_0g_2 - A_1g_1 \\ f_1 = n_3 - A_0g_3 - A_1g_2 \\ f_j = n_{j+2} - [A_0g_{j+2} + A_1g_{j+1}] \end{cases}$$

$$m = 3\begin{cases} f_0 = n_3 - A_0 g_3 - A_1 g_2 - A_2 g_1 \\ f_1 = n_4 - A_0 g_4 - A_1 g_3 - A_2 g_2 \\ f_j = n_{j+3} - [A_0 g_{j+3} + A_1 g_{j+2} + A_2 g_{j+1}] \end{cases}$$

$$\text{any } m\text{: } f_j = n_{m+j} - \sum_{i=0}^{m-1} A_i g_{m+j-i}$$

Example:

$$\frac{x^2 + 1}{x^3(x^2 - 3x + 6)} = \frac{A_0}{x^3} + \frac{A_1}{x^2} + \frac{A_2}{x} + \frac{f_1 x + f_0}{x^2 - 3x + 6}$$

$$A_0 = \frac{1}{6}, \quad A_1 = \frac{0 - (\frac{1}{6})(-3)}{6} = \frac{1}{12},$$

$$A_2 = \frac{1 - (\frac{1}{6})(1) - (\frac{1}{12})(-3)}{6} = \frac{13}{72},$$

$$m = 3\begin{cases} f_0 = 0 - \frac{1}{6}(0) + \frac{1}{12}(1) - \frac{13}{72}(-3) = \frac{11}{24} \\ f_1 = 0 - \frac{1}{6}(0) - \frac{1}{12}(0) - \frac{13}{72}(1) = -\frac{13}{72} \end{cases}$$

*Note: If $G(x)$ contains linear factors, $F(x)$ may be determined by previous section I.

III. *Repeated Linear Factors*

$$\frac{N(x)}{(x - a)^m G(x)} = \frac{A_0}{(x - a)^m} + \frac{A_1}{(x - a)^{m-1}} + \cdots + \frac{A_{m-1}}{(x - a)} + \frac{F(x)}{G(x)}$$

Change to form $\dfrac{N'(y)}{y^m G'(y)}$ by substitution of $x = y + a$. Resolve into partial fractions in terms of y as described in Section II. Then express in terms of x by substitution $y = x - a$.

Example:

$$\frac{x - 3}{(x - 2)^2(x^2 + x + 1)}$$

Let $x - 2 = y, \quad x = y + 2$

$$\frac{(y + 2) - 3}{y^2[(y + 2)^2 + (y + 2) + 1]} = \frac{y - 1}{y^2(y^2 + 5y + 7)} = \frac{A_0}{y^2} + \frac{A_1}{y} + \frac{f_1 y + f_0}{y^2 + 5y + 7}$$

$$A_0 = -\frac{1}{7}, \quad A_1 = \frac{1 - (-\frac{1}{7})(5)}{7} = \frac{12}{49},$$

$$m = 2\begin{cases} f_0 = 0 - (-\frac{1}{7})(1) - (\frac{12}{49})(5) = -\frac{53}{49} \\ f_1 = 0 - (-\frac{1}{7})(0) - (\frac{12}{49})(1) = -\frac{12}{49} \end{cases}$$

$$\therefore \frac{y - 1}{y^2(y^2 + 5y + 7)} = \frac{-\frac{1}{7}}{y^2} + \frac{\frac{12}{49}}{y} + \frac{-\frac{12}{49}y - \frac{53}{49}}{y^2 + 5y + 7}$$

Let $y = x - 2$, then

$$\frac{x - 3}{(x - 2)^2(x^2 + x + 1)} = \frac{-\frac{1}{7}}{(x - 2)^2} + \frac{\frac{12}{35}}{(x - 2)} + \frac{-\frac{12}{49}(x - 2) - \frac{53}{49}}{x^2 + x + 1}$$

$$= -\frac{1}{7(x - 2)^2} + \frac{12}{35(x - 2)} + \frac{-12x - 29}{49(x^2 + x + 1)}$$

IV. *Repeated Linear Factors*

Alternative method of determining coefficients:

$$\frac{N(x)}{(x-a)^m G(x)} = \frac{A_0}{(x-a)^m} + \cdots + \frac{A_k}{(x-a)^{m-k}} + \cdots + \frac{A_{m-1}}{x-a} + \frac{F(x)}{G(x)}$$

$$A_k = \frac{1}{k!}\left\{D_x^k\left[\frac{N(x)}{G(x)}\right]\right\}_{x=a}$$

where D_x^k is the differentiating operator, and the derivative of zero order is defined as:

$$D_x^0 u = u.$$

V. *Factors of Higher Degree*

Factors of higher degree have the corresponding numerators indicated.

$$\frac{N(x)}{(x^2 + h_1 x + h_0)G(x)} = \frac{a_1 x + a_0}{x^2 + h_1 x + h_0} + \frac{F(x)}{G(x)}$$

$$\frac{N(x)}{(x^2 + h_1 x + h_0)^2 G(x)} = \frac{a_1 x + a_0}{(x^2 + h_1 x + h_0)^2} + \frac{b_1 x + b_0}{(x^2 + h_1 x + h_0)} + \frac{F(x)}{G(x)}$$

$$\frac{N(x)}{(x^3 + h_2 x^2 + h_1 x + h_0)G(x)} = \frac{a_2 x^2 + a_1 x + a_0}{x^3 + h_2 x^2 + h_1 x + h_0} + \frac{F(x)}{G(x)}$$

etc.

Problems of this type are determined first by solving for the coefficients due to linear factors as shown above, and then determining the remaining coefficients by the general methods given below.

VI. *General Methods for Evaluating Coefficients*

1. $$\frac{N(x)}{D(x)} = \frac{N(x)}{G(x)H(x)L(x)} = \frac{A(x)}{G(x)} + \frac{B(x)}{H(x)} + \frac{C(x)}{L(x)} + \cdots$$

Multiply both sides of equation by $D(x)$ to clear fractions. Then collect terms, equate like powers of x, and solve the resulting simultaneous equations for the unknown coefficients.

2. Clear fractions as above. Then let x assume certain convenient values ($x = 1, 0, -1, \ldots$). Solve the resulting equations for the unknown coefficients.

3. $$\frac{N(x)}{G(x)H(x)} = \frac{A(x)}{G(x)} + \frac{B(x)}{H(x)}$$

Then

$$\frac{N(x)}{G(x)H(x)} - \frac{A(x)}{G(x)} = \frac{B(x)}{H(x)}$$

If $A(x)$ can be determined, such as by Method I, then $B(x)$ can be found as above.

BASIC CONCEPTS IN ALGEBRA

DR. W. E. DESKINS

I. ALGEBRA OF SETS

1. Intuitively a set is a collection of objects called the elements of the set. Set and set membership are generally accepted as basic, undefined terms used to define and construct mathematical systems.

   The notation $a \in A$ indicates that a is an element of the set A. The notation $a \notin A$ means that a is not a member of A.

   A set is sometimes specified by listing its elements within a set of braces: $\{a\}$ is the set containing only the element a.
2. Set A is a subset of set B provided $a \in A$ implies $a \in B$. This is denoted by $A \subseteq B$. Every set has as a subset the empty or null set, denoted by ϕ, which has no elements.
3. Set A equals set B, written $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$. A is a proper subset of B, sometimes indicated by $A \subset B$, if and only if $A \subseteq B$ and $A \neq B$; then B has at least one element which does not belong to A.
4. The Cartesian product of sets A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. A subset R of $A \times A$ is a binary relation on A, and this is an equivalence relation on A provided (i) $(a, a) \in R$ for every $a \in A$, (ii) $(a, b) \in R$ implies $(b, a) \in R$, and (iii) $(a, b) \in R$ and $(b, c) \in R$ imply $(a, c) \in R$. Ordinary equality of numbers, equality of sets, and congruence of plane figures are examples of equivalence relations.
5. A subset F of $A \times B$ is a *function* from A to B provided each element of A appears exactly once as the first element of a pair in F. A function F from A to B is *onto* provided each element of B appears at least once as the second element of a pair in F. It is *one-to-one* provided each element of B appears at most once as the second element of a pair in F. A function from $A \times A$ to A is a *binary operation* on A. Addition and multiplication of ordinary numbers are examples of binary operations.
6. If consideration is restricted to elements and subsets of a particular set I, then I is the universal set.
7. Common binary operations on subsets of I are: $A \cup B$, the union or join of sets A and B, is the set of all elements of I which belong to either A or B or both A and B.

   $A \cap B$, the intersection or meet of sets A and B, is the set of all elements of I which belong to both A and B.

   $A \setminus B$, the difference of sets A and B, is the set of elements of I which belong to A but not B.

   The difference $I \setminus A$ is denoted by A' and called the complement of A (relative to I). Except in dealing with the concept of complementation the use of a universal set is not essential to the above ideas.
8. Some theorems basic to the Algebra of Sets:

   Let A, B, and C be arbitrary subsets of a universal set I.

   (a) (Commutativity) $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

   (b) (Associativity) $(A \cup B) \cup C = A \cup (B \cup C)$ and
   $(A \cap B) \cap C = A \cap (B \cap C)$.

(c) (Distributivity) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
(d) (Idempotency) $A \cup A = A \cap A = A$.
(e) Properties of I and ϕ: $A \cap I = A \cup \phi = A$,
$A \cup I = I$, and
$A \cap \phi = \phi$.
(f) $(A \cap B) \cup (A \backslash B) = A$.
(g) $(A \backslash B) \cup B = A \cup B$.
(h) $A \subseteq A \cup B$.
(i) $A \cap B \subseteq A$.
(j) $A \cup B = A$ if and only if $B \subseteq A$.
(k) $A \cap B = A$ if and only if $A \subseteq B$.
(m) $A \backslash B = A \backslash (A \cap B)$.
(n) (DeMorgan's Theorem) $(A \backslash B) \cap (A \backslash C) = A \backslash (B \cup C)$ and
$(A \backslash B) \cup (A \backslash C) = A \backslash (B \cap C)$.
(o) $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$.
(p) $A \cup A' = I$ and $A \cap A' = \phi$.

9. A mathematical system S is a set $S = \{E, O, A\}$ where E is a nonempty set of elements, O is a set of relations and operations on E, and A is a set of axioms, postulates, or assumptions concerning the elements of E and O.
10. The Algebra of Sets provides an example of a mathematical system called a Boolean Algebra (or Boolean Ring) which is defined as:
 Set E of elements $a, b, c, \ldots$;
 Set O of 2 binary operations $\oplus$ and $\otimes$; (Here $a \oplus b$ denotes the image of (a, b) under the binary operation.)
 Set A of axioms for all a, b, c of E:

A_1. The binary operations are commutative; i.e.,

$$a \oplus b = b \oplus a \quad \text{and} \quad a \otimes b = b \otimes a.$$

A_2. Each binary operation is distributive over the other; i.e.,

$$a \oplus (b \otimes c) = (a \oplus b) \otimes (a \oplus c) \quad \text{and} \quad a \otimes (b \oplus c) = (a \otimes b) + (a \otimes c).$$

A_3. There exist elements e and z in E such that for each $a \in E$, $a \oplus z = a$ and $a \otimes e = a$.

A_4. For each $a \in E$ there exists an element $a' \in E$ such that $a \otimes a' = e$ and $a \oplus a' = z$.

In the algebra of subsets of a (universal) set I, ϕ plays the role of z, I that of e, $\cup$ that of $\oplus$, and $\cap$ that of $\otimes$.

11. A Boolean Algebra has the Principle of Duality: If the interchanges of $\left\{\begin{matrix}\oplus \text{ and } \otimes \\ e \text{ and } z\end{matrix}\right\}$ are made in a correct statement, then the result is also a correct statement.
12. In addition to the Algrebra of Sets which is a Boolean Algebra, other representations of Boolean Algebra that are interesting of themselves and valuable for their applications are:
 (a) The Algebra of Symbolic Logic
 (b) The Algebra of Switching Currents

Algebra of Sets	*Binary Operator*	*Symbolic Logic*	*Binary Operator*	*Switching Circuits*	*Binary Operator*
Union of 2 sets	$\cup$	Disjunction of 2 propositions	$\vee$	2 switches in $\parallel$	$+$
Intersection of 2 sets	$\cap$	Conjunction of 2 propositions	$\wedge$	2 switches in series	$\times$
Complement of a set A		Negation of a proposition T, F	$\sim$	on-off, or 1, 0	1 or $^{-}$

Both in symbolic logic and in switching circuits, we can consider "0" and "1" as the elements, in the former representing "False" and "True"; in the latter, "Off" and "On", satisfying the following "rules":

$$\left.\begin{array}{l} 0 + 0 = 0 \\ 1 + 1 = 1 \end{array}\right\} \text{i.e. } a + a = a$$

$$\left.\begin{array}{l} 0 \times 0 = 0 \\ 1 \times 1 = 1 \end{array}\right\} \text{i.e. } a \times a = a$$

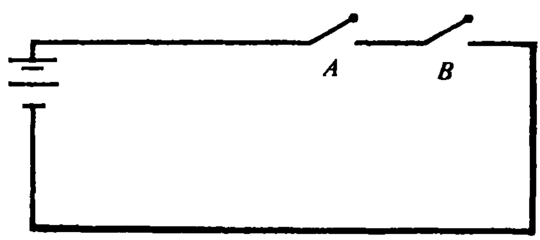
$$0' = 1$$

$$1' = 0$$

$$\left.\begin{array}{l} 0 + 0' = 0 + 1 = 1 \\ 1 + 1' = 1 + 0 = 1 \end{array}\right\} \text{i.e. } a + a' = 1$$

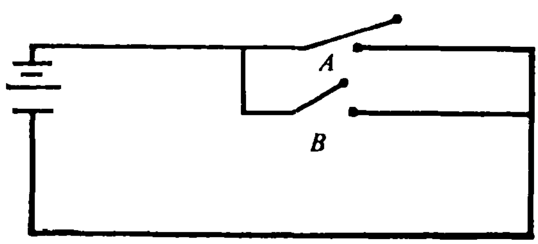
$$\left.\begin{array}{l} 0 \times 0' = 0 \times 1 = 0 \\ 1 \times 1' = 1 \times 0 = 0 \end{array}\right\} a \times a' = 0$$

In switching circuits:



Series Circuit

$A \times B$



Parallel Circuit

$A + B$

here, "0" represents an open circuit:  and "1"

$A = 0$

a closed circuit: 

$A = 1$

In the Algebra of Symbolic Logic, we use *Truth Tables* to define the operations $\wedge$, $\vee$, $\sim$ as follows:

					Other Operators Used	
p	q	$p \wedge q$	$p \vee q$	$\sim p$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	T	T	F	T	T
T	F	F	T	F	F	F
F	T	F	T	T	T	F
F	F	F	F	T	T	T

13. In order to re-emphasize the use of switching circuits and their relation to truth tables the following is included. Conventionally a "1" represents "True" and a "0" represents "False." The switching circuit symbols are $-$, $\cdot$, $+$, $\rightarrow$, $\equiv$ representing "Not," "And," "Or," "Implies," "Equivalent" respectively and their Truth Table Definitions are

p	q	$p \cdot q$	$p + q$	$-p$	$p \rightarrow q$	$p \equiv q$
0	0	0	0	1	1	1
0	1	0	1	1	1	0
1	0	0	1	0	0	0
1	1	1	1	0	1	1

The comparison with the Algebra of Symbolic Logic being obvious. The "rules" for these circuits are as follows:

$$0 + 0 = 0$$
$$1 + 1 = 1$$
$$1 + 0 = 0 + 1 = 1$$
$$0 \cdot 0 = 0$$
$$1 \cdot 1 = 1$$
$$0 \cdot 1 = 1 \cdot 0 = 0$$
$$\bar{0} = 1$$
$$\bar{1} = 0$$

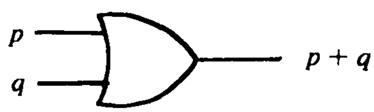
Mechanical switches or relays are represented by

——— p ——— or ——— $\bar{p}$ ———

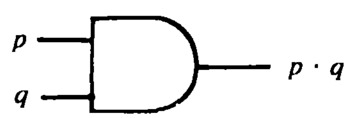
the former indicating that the circuit is closed, i.e. the switch is made, when $p = 1$ and the latter indicating the converse namely that the circuit is closed when $\bar{p} = 1$ or, what amounts to the same thing, when $p = 0$.

Electronic switches or gates are represented by more complex symbols—four in all, three of which are independent and can stand alone

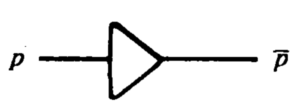
"Or"



"And"



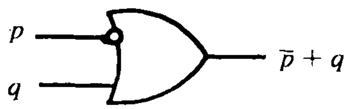
"Not"



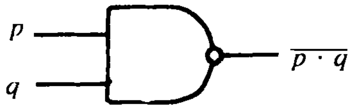
and one which represents the negation of an input or an output and is used with one of the above

"Not" 

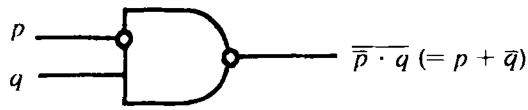
An example of its use on an input line is



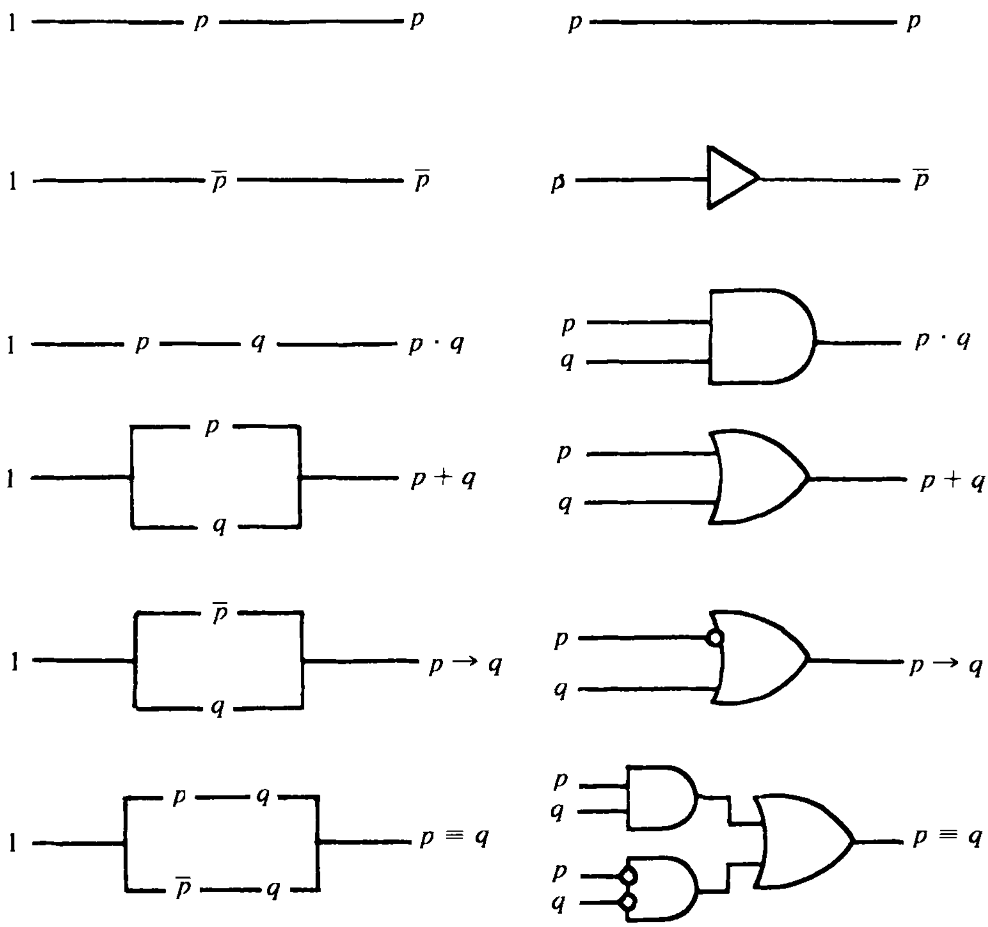
or on an output is



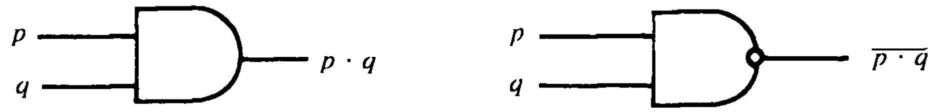
and on both



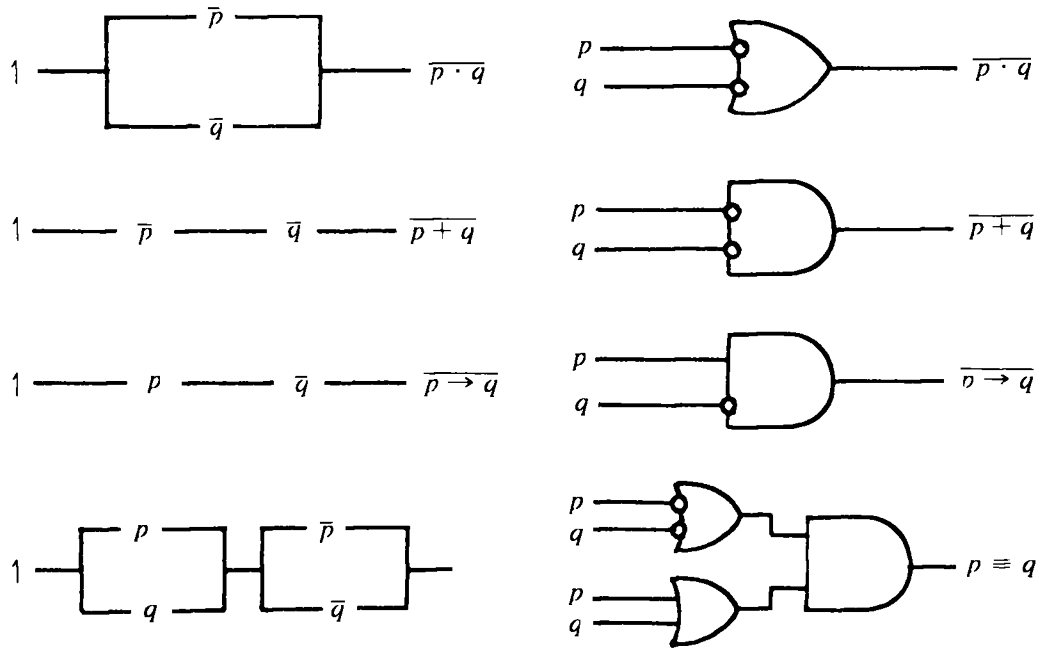
The basic functions obtained from the two types of switching circuits are



All the above electronic circuits can be negated by simply adding a negating circle to the output as for example in



Alternative circuits however, which are direct analogues of their relay switching counterparts, are



The operation + is sometimes referred to as the "Inclusive Or" and $\not\equiv$ as the "Exclusive Or", the former having the value "True" when both the inputs are "True"—see the truth table. Note that $p \not\equiv q$ is a shorthand for $\overline{p \equiv q}$.

14. Two sets A and B are equivalent (have same cardinal) if and only if there exists a one-to-one correspondence between the elements of the two sets. This is an equivalence relation on the collection of subsets of set I.

A set is infinite if and only if it is equivalent with a proper subset of itself.

A set is called countably (denumerably) infinite if it is equivalent with the set of all positive integers. The set of all rational numbers is countably infinite but the set of all real numbers is noncountably infinite. The cardinal of the set of all rational numbers is denoted by (aleph null); the cardinal of the set of reals is denoted by (aleph).

II. ABSTRACT ALGEBRAIC SYSTEMS

1. *Semigroup.* A semigroup is a system $\{S, \theta, A\}$; S is a nonempty set $\{a, b, c, \ldots\}$, θ consists of one binary operation on S, denoted by $*$, and A consists of the axiom

A_1. Associativity: $a * (b * c) = (a * b) * c$ for all $a, b \circ c \in S$.

Basic Theorem. (*Generalized Associativity*). If $a_1, a_2, \ldots, a_n$ are elements of S then all associations of the n elements yield the same "product". (For example,

$$a * ((b * c) * d) = a * (b * (c * d)) = (a * b) * (c * d), \text{etc.})$$

2. *Group*. A group is a system $\{G, \theta, A\}$; G is a nonempty set $\{a, b, c, \ldots\}$, θ consists of one binary operation denoted by $\circ$, and A consists of the axioms:
   - A_1. Associativity: $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in G$.
   - A_2. Identity Element: G contains an element e having the property, $a \circ e = e \circ a = a$ for every $a \in G$.
   - A_3. Inverse Element: For each $a \in G$ there is an element $a' \in G$ with the property, $a \circ a' = a' \circ a = e$.

   If the following additional axiom belongs to A,
   - A_4. Commutativity: $a \circ b = b \circ a$ for all $a, b \in G$. Then the group is called Abelian (after Niels Henrik Abel). Some basic theorems:
     - (a) The element e (Axiom A_2) is unique. Then e is *the* identity element of G.
     - (b) The element a' (Axiom A_3) is unique for each $a \in G$. Then a' is *the* inverse of a in G.
     - (c) The equation $a \circ x = b$ has a unique solution in G, viz., $x = a' \circ b$.
     - (d) $(a')' = a$ and $(a \circ b)' = b' \circ a'$.
     - (e) $a \circ b = a \circ c$ if and only if $b = c$.

       If a nonempty subset H of G satisfies the two conditions:
       - H_1. $a \circ b \in H$ whenever $a, b \in H$. (Closure)
       - H_2. $a \in H$ if and only if $a' \in H$.

     then H is a *subgroup* of G.

     (Lagrange). If G is a finite set then the number of elements in H divides the number of elements in G.

     Example of group. Let G be the set of all one-to-one functions from a nonempty S onto itself. For any $f, g \in G$, define the function $f \circ g$ as the function which maps s onto $f(g(s))$, for each $s \in S$. Relative to this binary operation G is a group, the *symmetric group* of all permutations on S.

     Each group is essentially a subgroup of the symmetric group of some set S.

3. *Ring*. A ring is a system $\{R, \theta, A\}$; R is a nonempty set $\{a, b, c, \ldots\}$, θ consists of two binary operations denoted by $+$ and $\times$, and A consists of the axioms:
   - A_0. Relative to addition (i.e., $+$) R is an Abelian group in which the identity element is denoted by z and the inverse of a is denoted by $-a$.
   - M_0. Relative to multiplication (i.e., $\times$) R is a semigroup.
   - D_1. Left distributive: $a \times (b + c) = (a \times b) + (a \times c)$, all $a, b, c \in R$.
   - D_2. Right distributive: $(b + c)a = (b \times a) + (c \times a)$, all $a, b, c \in R$.

EXAMPLE 1. The set of all integers (whole numbers) and ordinary addition and multiplication.

EXAMPLE 2. The set of all real functions continuous on the interval $0 \leq y \leq 1$, with addition and multiplication defined by $(f + g)(y) = f(y) + g(y)$, sum of real numbers, and $(f \times g)(y) = f(y) \times g(y)$, product of real numbers.

Special types of rings have been studied extensively.

3.1 *Integral Domain*. An integral domain is a ring R in which multiplication $(\times)$ satisfies the additional assumptions:
   - M_1. Commutativity: $a \times b = b \times a$ for all a and b in R.
   - M_2. Multiplicative identity: R contains an element $e \neq z$ with the property $a \times e = e \times a = a$ for all a in R.
   - M_3. Cancellation: $a \times b = a \times c$ if and only if $b = c$.

   An element u of integral domain R is a *unit* provided R contains v such that $u \times v = e$.

An element p of integral domain R is a *prime* (irreducible element) provided $p = a \times b$ implies that exactly one of the elements a or b is a unit.

The elements of integral domain R which differ from z and are neither units nor primes are *composites*.

In some integral domains (such as the ring of integers) each composite can be factored uniquely (up to unit factors) as the product of a finite set of primes. However in the integral domain of all entire functions this is not true.

3.2 *Field.* A field is an integral domain in which every element except z is a unit. In other words, the non-z elements form an Abelian group relative to multiplication ($\times$).

EXAMPLE 1. The rational field consisting of ordinary fractions, addition, and multiplication.

EXAMPLE 2. The set of all real numbers $a + b\sqrt{2}$, a and b rational. Then

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2} \text{ and}$$
$$(a + b\sqrt{2}) \times (c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2}.$$

Besides these well-known examples there exist finite fields (sometimes called Galois fields).

EXAMPLE 3. Let p be a prime integer. Denote by $GF(p)$ the p integers $0, 1, \ldots, p - 1$. Define addition($\oplus$) of two of these elements a and b as the remainder of $a + b$ (ordinary addition) after division by p. (Thus $1 \oplus (p - 1) = 0$.)

Define $a \otimes b$, the product, to be the remainder of ab (ordinary multiplication) after division by p. (Thus, when $p = 3$, $2 \otimes 2 = 1$.) The resulting system $\{GF(p), \oplus, \otimes\}$ is the (modular) field of integers modulo p.

3.3 *Skew Field or Division Ring.* A skew field is a ring in which the non-z elements form a group relative to multiplication ($\times$).

The classical example of a skew field is the ring of real *quaternions*, first described by W. R. Hamilton. A quaternion is expressible in the form $ae + bi + cj + dk$ where a, b, c, and d are real numbers and e, i, j, and k are elements which commute with all real numbers and multiply as follows:

$$e \times e = e, \quad e \times i = i \times e = i, \quad e \times j = j \times e = j, \quad e \times k = k \times e = k;$$
$$i \times i = -e, \quad i \times j = k, \quad j \times i = -k, \quad i \times k = -j, \quad k \times i = j,$$
$$j \times j = -e, \quad j \times k = i, \quad k \times j = -i, \quad k \times k = -e.$$

These elements distribute over addition. e is generally identified with and written as the real number 1.

3.4 *Matric Ring.* The matric ring $M_n(R)$ over the ring R, where n is a positive integer, consists of all doubly-ordered sets of n^2 elements of R, written as an array

$$\begin{pmatrix} a_{1,1} & a_{1,2} \cdots a_{1,n} \\ a_{2,1} & a_{2,2} \cdots a_{2,n} \\ \vdots & \\ a_{n,1} & \cdot\cdot \quad a_{n,n} \end{pmatrix} = (a_{i,j})$$

with addition and multiplication defined as follows:

$(a_{i,j}) + (b_{i,j}) = (a_{i,j} + b_{i,j})$
$(a_{i,j}) \times (b_{i,j}) = (c_{i,j})$
where

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k}\, b_{k,j}, \quad i = 1, \ldots, n \quad \text{and} \quad j = 1, \ldots, n.$$

If $n > 1$, then multiplication is noncommutative in general; i.e., $(a_{i,j}) \times (b_{i,j})$ can differ from $(b_{i,j}) \times (a_{i,j})$. Moreover, the product of two nonzero matrices can be the zero matrix (which consists of only the element z in all n^2 positions).

A similar useful method for forming a new ring from a known ring utilizes sequences.

3.5 *Power Series and Polynomial Ring.* Let R be a ring in which multiplication $(\times)$ is commutative. The set $PS(R)$ of all sequences $(a_0, a_1, \ldots)$ with $a_i \in R$ is the power series ring of R, with addition and multiplication defined as

$(a_0, a_1, \ldots) \oplus (b_0, b_1, \ldots) = (a_0 + b_0, \;\; a_1 + b_1, \ldots)$ and
$(a_0, a_1, \ldots) \oplus (b_0, b_1, \ldots) = (c_0, c_1, \ldots)$

where

$$c_0 = a_0 \times b_0, \quad c_1 = a_0 \times b_1 + a_1 \times b_0, \ldots, \text{and,}$$

generally,

$$c_n = a_0 \times b_n + a_1 \times b_{n-1} + \cdots + a_n \times b_0.$$

The subset $P(R)$ of $PS(R)$ consisting of those sequences $(a_0, a_1, \ldots)$ in which at most only finitely many of the a_i differ from z, form a ring relative to the addition and multiplication just defined. This ring $\{P(R), \oplus, \otimes\}$, is the polynomial ring of R.

Some theorems for rings, fields, etc.

(a) In a ring R, if $a = b$ and $c = d$, then $a + c = b + d$ and $a \times c = b \times d$.
(b) In a ring R, $-(-a) = a$; $(-a) \times b = a \times (-b) = -(a \times b)$; and $(-a) \times (-b) = a \times b$, for all $a, b \in R$.
(c) In a ring R, $a \times z = z \times a = z$, for all $a \in R$.
(d) In a ring R the equation $a + x = b$ has a unique solution, viz., $x = -a + b$.
(f) In a field, skew field, or integral domain, $a \times b = z$ if and only if a and/or b equals z.
(g) A finite integral domain is a field.
(h) The polynomial ring of an integral domain is also an integral domain.
(i) The power series ring of an integral domain is also an integral domain.
(j) A ring is a field provided it is both an integral domain and a skew field.
(k) If R is a (skew) field, then the equation $a \times y = b$, $a \neq z$, has a unique solution $y = a' \times b$.
(l) The polynomial ring and the power series ring of a field are unique factorization domains.

4. *Vector Space.* A vector space $V(F)$ over a field F consists of a nonempty set V (the vectors), a binary operation $(\oplus)$ on V, a function (called *scalar multiplication*) from the product set $F \times V$ onto V with the image of (a, ν) denoted by $a \circ \nu$, and the following axioms:

A_0. Relative to addition $(\oplus)$ V is an Abelian group in which the identity element (vector) is denoted by z and the inverse of ν is denoted by $-\nu$.

M_1. $a \circ (b \circ \nu) = (ab) \circ \nu$ for all $a, b \in F$ and $\nu \in F$. (Here ab denotes the product of a and b in F.)

M_2. $1 \circ \nu = \nu$ for all $\nu \in V$. (Here 1 denotes the multiplicative indentity element of F.)

D_1. $a \circ (\mu \oplus \nu) = (a \circ \mu) \oplus (a \circ \nu)$ for all $a \in F$, $\mu, \nu \in V$.

D_2. $(a + b) \circ \nu = (a \circ \nu) \oplus (b \circ \nu)$ for all $a, b \in F$, $\nu \in V$. (Here $+$ denotes addition in the field F.)

The elements of F are referred to as *scalars*.

EXAMPLE 1. The polynomial ring $P(F)$ of a field F is a vector space over F. In this example scalar multiplication is a special case of the multiplication defined for $P(F)$.

EXAMPLE 2. Denote by $C_n(F)$ the set of all n-tuples, $(a_1, a_2, \ldots, a_n)$, n a positive integer, with all $a_i \in F$. Define

$$(a_1, \ldots, a_n) \oplus (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n) \quad \text{and}$$
$$c \circ (a_1, \ldots, a_n) = (c \times a_1, \ldots, c \times a_n),$$

where $+$ and $\times$ denote the addition and multiplication, respectively, of the field F. Relative to $\oplus$ and $\circ$, $C_n(F)$ is a vector space, called the *n-dimensional coordinate space* over F.

A vector space $V(F)$ is *n-dimensional* over F provided V contains n elements ν_1, ν_2, $\ldots$, ν_n such that each element $\nu \in V$ is uniquely expressible in the form

$$\nu = a_1 \circ \nu_1 \oplus a_2 \circ \nu_2 \oplus \cdots \oplus a_n \circ \nu_n$$

for some $a_1, a_2, \ldots, a_n \in F$.

Two vector spaces $V(F)$ and $W(F)$ over the field of scalars F are *isomorphic* provided there is a one-to-one correspondence between the elements of V and the elements of W which is preserved under the arithmetic of the two spaces.

Basic Theorem. An n-dimensional vector space $V(F)$ is isomorphic with the coordinate space $C_n(F)$ (of Example 2, above).

MATRICES AND DETERMINANTS

DR. R. E. BARGMANN

1. GENERAL DEFINITIONS

1.1. A matrix is an array of numbers, consisting of m rows and n columns. It is usually denoted by a bold-face capital letter, e.g.,

$$\mathbf{A} \qquad \mathbf{\Sigma} \qquad \mathbf{M}$$

1.2. The (i,j) element of a matrix is the element occurring in row i and column j. It is usually denoted by a lower-case letter with subscripts, e.g.,

$$a_{ij} \qquad \sigma_{ij} \qquad m_{ij}$$

Exceptions to this convention will be stated where required.

1.3. A matrix is called rectangular if m (number of rows) $\neq n$ (number of columns).

1.4. A matrix is called square if $m = n$.

1.5a. In the transpose of a matrix $\mathbf{A}$, denoted by $\mathbf{A}'$, the element in the j'th row and i'th column of $\mathbf{A}$ is equal to the element in the i'th row and j'th column of $\mathbf{A}'$. Formally $(\mathbf{A}')_{ij} = (\mathbf{A})_{ji}$ where the symbol $(\mathbf{A}')_{ij}$ denotes the (i,j) element of $\mathbf{A}'$.

1.5b. The Hermitian conjugate of a matrix $\mathbf{A}$, denoted by $\mathbf{A}^{\mathrm{H}}$ or $\mathbf{A}^{\dagger}$ is obtained by transposing $\mathbf{A}$ and replacing each element by its conjugate complex. Hence if

$$a_{kl} = u_{kl} + iv_{kl}$$

then

$$(\mathbf{A}^{\mathrm{H}})_{kl} = u_{lk} - iv_{lk}$$

where typical elements have been denoted by (k,l) to avoid confusion with $i = \sqrt{-1}$.

1.6a. A square matrix is called symmetric if $\mathbf{A} = \mathbf{A}'$.

1.6b. A square matrix is called Hermitian if $\mathbf{A} = \mathbf{A}^{\mathrm{H}}$.

1.7. A matrix with m rows and 1 column is called a column vector and is usually denoted by bold faced, lower-case letters, e.g.,

$$\boldsymbol{\beta} \qquad \mathbf{x} \qquad \mathbf{a}$$

1.8. A matrix with one row and n columns is called a row vector and is usually denoted by a primed, bold faced, lower-case letter, e.g.,

$$\mathbf{a}' \qquad \mathbf{c}' \qquad \boldsymbol{\mu}'$$

1.9. A matrix with one row and one column is called a scalar, and is usually denoted by a lower-case letter, occasionally italicized.

1.10. The diagonal extending from upper left (NW) to lower right (SE) is called the principal diagonal of a square matrix.

1.11a. A matrix with all elements above the principal diagonal equal to zero is called a lower triangular matrix.

Example

$$\mathbf{T} = \begin{bmatrix} t_{11} & 0 & 0 \\ t_{21} & t_{22} & 0 \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \text{ is lower triangular}$$

1.11b. The transpose of a lower triangular matrix is called an upper triangular matrix.

1.12. A square matrix with all off-diagonal elements equal to zero is called a diagonal matrix, denoted by the letter **D** with subscript indicating the typical element in the principal diagonal.

Example

$$\mathbf{D}_a = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \text{ is diagonal}$$

2. ADDITION, SUBTRACTION, AND MULTIPLICATION

2.1. Two matrices **A** and **B** can be added (subtracted) if the number of rows (columns) in **A** equals the number of rows (columns) in **B**.

$$\mathbf{A} \pm \mathbf{B} = \mathbf{C}$$

implies

$$a_{ij} \pm b_{ij} = c_{ij} \qquad i = 1, 2, \ldots m$$
$$j = 1, 2, \ldots n$$

2.2. Multiplication of a matrix or vector by a scalar implies multiplication of each element by the scalar. If

$$\mathbf{B} = \gamma \mathbf{A}$$

then

$$b_{ij} = \gamma a_{ij}$$

for all elements.

2.3a. Two matrices, **A** and **B**, can be multiplied if the number of columns in **A** equals the number of rows in **B**.

2.3b. Let **A** be of order $(m \times n)$ (have m rows and n columns) and **B** of order $(n \times p)$. Then the product of two matrices $\mathbf{C} = \mathbf{AB}$, is a matrix of order $(m \times p)$ with elements

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

This states that c_{ij} is the scalar product of the i'th row vector of **A** and the j'th column vector of **B**.

Example

$$\begin{bmatrix} 3 & 4 & 2 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -4 \\ 0 & -1 & 2 \\ 6 & -3 & 9 \end{bmatrix} = \begin{bmatrix} 15 & -16 & 14 \\ -4 & -4 & -11 \end{bmatrix}$$

e.g.,

$$c_{23} = [2 \quad 3 \quad -1] \begin{bmatrix} -4 \\ 2 \\ 9 \end{bmatrix}$$

$$= 2 \times (-4) + 3 \times 2 + (-1) \times 9 = -11$$

2.3c. In general, matrix multiplication is not commutative

$$\mathbf{AB} \neq \mathbf{BA}$$

2.3d. Matrix multiplication is associative

$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$$

2.3e. The distributive law for multiplication and addition holds as in the case of scalars,

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$$
$$\mathbf{C}(\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB}$$

2.4. In some applications, the term-by-term product of two matrices **A** and **B** of identical order is defined as

$$\mathbf{C} = \mathbf{A} * \mathbf{B}$$

where

$$c_{ij} = a_{ij}b_{ij}$$

2.5. $(\mathbf{ABC})' = \mathbf{C}'\mathbf{B}'\mathbf{A}'$

2.6. $(\mathbf{ABC})^H = \mathbf{C}^H\mathbf{B}^H\mathbf{A}^H$

2.7. If both **A** and **B** are symmetric, then $(\mathbf{AB})' = \mathbf{BA}$. Note that the product of two symmetric matrices is generally not symmetric.

3. RECOGNITION RULES AND SPECIAL FORMS

3.1. A column (row) vector with all elements equal to zero is called a null vector, and usually denoted by the symbol **0**.

3.2. A null matrix has all elements equal to zero.

3.3a. A diagonal matrix with all elements equal to one in the principal diagonal is called the identity matrix **I**.

3.3b. $\gamma\mathbf{I}$, i.e., a diagonal matrix with all diagonal elements equal to a constant γ, is called a scalar matrix.

3.4. A matrix which has only one element equal to one and all others equal to zero is called an elementary matrix $(\mathbf{EL})_{ij}$.

Example

$$(\mathbf{EL})_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The order of the matrix is usually implicit.

3.5a. The symbol **j** is reserved for a column vector with all elements equal to 1.

3.5b. The symbol **j**′ is reserved for a row vector with all elements equal to 1.

3.6. An expression ending with a column vector is a column vector.

Example

$$\mathbf{ABx} = \mathbf{y}$$

(It is assumed that rule 2.3a is satisfied, else matrix multiplication would not be defined.)

3.7. An expression beginning with a row vector is a row vector.

Example

$$\mathbf{y}'(\mathbf{A} + \mathbf{BC}) = \mathbf{d}'$$

3.8. An expression beginning with a row vector and ending with a column vector, is a scalar.

Example

$$\mathbf{a}'\mathbf{B}\mathbf{c} = \gamma$$

3.9a. If $\mathbf{Q}$ is a square matrix, the scalar $\mathbf{x}'\mathbf{Q}\mathbf{x}$ is called a quadratic form. If $\mathbf{Q}$ is non-symmetric, one can always find a symmetric matrix $\mathbf{Q}^*$ such that

$$\mathbf{x}'\mathbf{Q}\mathbf{x} = \mathbf{x}'\mathbf{Q}^*\mathbf{x}$$

where

$$(\mathbf{Q}^*)_{ij} = \tfrac{1}{2}(q_{ij} + q_{ji})$$

3.9b. If $\mathbf{Q}$ is a square matrix the scalar $\mathbf{x}^H\mathbf{Q}\mathbf{x}$ is called a Hermitian form.

3.10. A scalar $\mathbf{x}'\mathbf{Q}\mathbf{y}$ is called a bilinear form.

3.11. The scalar $\mathbf{x}'\mathbf{x} = \Sigma x_i^2$, i.e., the sum of squares of all elements of $\mathbf{x}$.

3.12. The scalar $\mathbf{x}'\mathbf{y} = \Sigma x_i y_i$, i.e., the sum of products of elements in $\mathbf{x}$ by those in $\mathbf{y}$. $\mathbf{x}$ and $\mathbf{y}$ have the same number of elements.

3.13. The scalar $\mathbf{x}'\mathbf{D}_w\mathbf{x} = \Sigma w_i x_i^2$ is called a weighted sum of squares.

3.14. The scalar $\mathbf{x}'\mathbf{D}_w\mathbf{y} = \Sigma w_i x_i y_i$ is called a weighted sum of products.

3.15a. The vector $\mathbf{A}\mathbf{j}$ is a column vector whose elements are the row sums of $\mathbf{A}$.

3.15b. The vector $\mathbf{j}'\mathbf{A}$ is a row vector whose elements are the column sums of $\mathbf{A}$.

3.15c. The scalar $\mathbf{j}'\mathbf{A}\mathbf{j}$ is the sum of all elements in $\mathbf{A}$. Schematically

$$\begin{array}{c|c} \mathbf{A} & \mathbf{A}\mathbf{j} \\ \hline \mathbf{j}'\mathbf{A} & \mathbf{j}'\mathbf{A}\mathbf{j} \end{array}$$

3.16a. If $\mathbf{B} = \mathbf{D}_w\mathbf{A}$; then $b_{ij} = w_i a_{ij}$.

3.16b. If $\mathbf{B} = \mathbf{A}\mathbf{D}_w$; then $b_{ij} = a_{ij} w_j$.

3.17. Interchanging summation and matrix notation:

If

$$\mathbf{A}\mathbf{B}\mathbf{C}\mathbf{D} = \mathbf{E}$$

then

$$e_{ij} = \sum_k \sum_l \sum_m a_{ik} b_{kl} c_{lm} d_{mj}$$

The second subscript of an element must coincide with the first of the next one. Reordering and transposing may be required.

Example

If

$$e_{ij} = \sum_k \sum_l \sum_m a_{kl} b_{ki} c_{jm} d_{ml}$$

$$= \sum_k \sum_l \sum_m b_{ki} a_{kl} d_{ml} c_{jm}$$

Then

$$\mathbf{E} = \mathbf{B}'\mathbf{A}\mathbf{D}'\mathbf{C}'$$

3.18a. $\mathbf{A}'\mathbf{A}$ is a symmetric matrix whose (i,j) element is the scalar product of the i'th column vector and the j'th column vector of $\mathbf{A}$.

3.18b. $\mathbf{A}\mathbf{A}'$ is a symmetric matrix whose (i,j) element is the scalar product of the i'th row vector and the j'th row vector of $\mathbf{A}$.

4. DETERMINANTS

4.1a. A determinant $|\mathbf{A}|$ or det(**A**) is a scalar function of a square matrix defined in such a way that

$$|\mathbf{A}|\;|\mathbf{B}| = |\mathbf{AB}|$$

and

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

4.1b. $|\mathbf{A}| = |\mathbf{A}'|$

4.2.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

4.3.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{vmatrix} = \sum (-1)^{\delta} a_{1i_1} a_{2i_2} \cdots a_{ni_n}$$

where the sum is over all permutations

$$i_1 \neq i_2 \neq \cdots i_n$$

and δ denotes the number of exchanges necessary to bring the sequence $(i_1, i_2, \ldots i_n)$ back into the natural order $(1, 2, \ldots n)$.

4.4. If two rows (columns) in a matrix are exchanged, the determinant will change its sign.

4.5. A determinant does not change its value if a linear combination of other rows (columns) is added to any given row (column).

Example

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

where

$$b_{2i} = a_{2i} + \gamma_1 a_{1i} + \gamma_3 a_{3i} + \gamma_4 a_{4i}$$
$$i = 1, 2, 3, 4$$

$\gamma_1, \gamma_3, \gamma_4$ arbitrary.

4.6. If the i'th row (column) equals (a constant times) the j'th row (column) of a matrix, its determinant is equal to zero, $(i \neq j)$.

4.7. If, in a matrix **A**, each element of a row (column) is multiplied by a constant γ, the determinant is multiplied by γ.

4.8. $|\gamma\mathbf{A}| = \gamma^n |\mathbf{A}|$ assuming that **A** is of order $(n \times n)$.

4.9. The cofactor of a square matrix **A**, $\mathrm{cof}_{ij}(\mathbf{A})$ is the determinant of a matrix obtained by striking the i'th row and j'th column of **A** and choosing positive (negative) sign if $i + j$ is even (odd).

Example

$$\text{cof}_{23}\begin{bmatrix} 2 & 4 & 3 \\ 6 & 1 & 5 \\ -2 & 1 & 3 \end{bmatrix} = - \begin{vmatrix} 2 & 4 \\ -2 & 1 \end{vmatrix}$$

$$= -(2 + 8) = -10$$

4.10. (Laplace Development)

$$|\mathbf{A}| = a_{i1}\text{cof}_{i1}(\mathbf{A}) + a_{i2}\text{cof}_{i2}(\mathbf{A}) + \cdots + a_{in}\text{cof}_{in}(\mathbf{A})$$

$$= a_{1j}\text{cof}_{1j}(\mathbf{A}) + a_{2j}\text{cof}_{2j}(\mathbf{A}) + \cdots + a_{nj}\text{cof}_{nj}(\mathbf{A})$$

for any row i or any column j.

4.11. Numerical Evaluation of the determinant of a symmetric matrix.

Note: If $\mathbf{A}$ is non-symmetric, form $\mathbf{A}'\mathbf{A}$ or $\mathbf{A}\mathbf{A}'$ by rule 3.18, obtain its determinant, and take the square root.

("Forward Doolittle Scheme", "left side")

Let

$$p_{11} = a_{11}, \quad p_{12} = a_{12} = a_{21}, \ldots p_{1n} = a_{1n}$$

$$\begin{array}{ccccc}
p_{11} & p_{12} & p_{13} & \cdots & p_{1n} \\
1 & u_{12} & u_{13} & \cdots & u_{1n} \\
\hline
 & a_{22} & a_{23} & \cdots & a_{2n} \\
 & p_{22} & p_{23} & \cdots & p_{2n} \\
 & 1 & u_{23} & \cdots & u_{2n} \\
\hline
 & & a_{33} & \cdots & a_{3n} \\
 & & p_{33} & \cdots & p_{3n} \\
 & & 1 & \cdots & u_{3n} \\
\hline
 & & \cdot & \cdots & \cdot \\
 & & & & a_{nn} \\
 & & & & p_{nn} \\
 & & & & 1
\end{array}$$

$$\begin{aligned}
u_{1i} &= p_{1i}/p_{11} & i &= 1, 2, \ldots n \\
p_{2i} &= a_{2i} - u_{12}p_{1i} & i &= 2, 3, \ldots n \\
u_{2i} &= p_{2i}/p_{22} & & \\
p_{3i} &= a_{3i} - u_{13}p_{1i} - u_{23}p_{2i} & i &= 3, 4, \ldots n \\
u_{3i} &= p_{3i}/p_{33} & & \\
p_{ki} &= a_{ki} - u_{1k}p_{1i} - u_{2k}p_{2i} - \cdots - u_{k-1,k}p_{k-1,i} & i &= k, k+1, \ldots n \\
 & & k &= 2, 3, \ldots n \\
u_{ki} &= p_{ki}/p_{kk} & &
\end{aligned}$$

If, at some stage, $p_{kk} = 0$, reordering of rows and columns may be required. If the matrix is positive-definite (see 8.16) (always true for $\mathbf{A}\mathbf{A}'$ or $\mathbf{A}'\mathbf{A}$, see rule 10.24), none of the

p_{kk} will be zero. The p_{ii} are called pivots. Then

$$|\mathbf{A}| = \prod_{i=1}^{n} p_{ii}$$

Further, if **A** is partitioned

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}'_{12} & \mathbf{A}_{22} \end{bmatrix}$$

where $\mathbf{A}_{11}$ is of order $(k \times k)$, then

$$|\mathbf{A}_{11}| = \prod_{i=1}^{k} p_{ii}$$

(Numerical Examples: see 6.14.)

5. SINGULARITY AND RANK

5.1. A matrix **A** is called singular if there exists a vector $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{Ax} = \mathbf{0}$ or $\mathbf{A'x} = \mathbf{0}$. Note $\mathbf{x} \neq \mathbf{0}$ if a single element of **x** is unequal 0. If a matrix is not singular, it is called non-singular.

5.2. If a matrix $\mathbf{A}_1$ can be formed by selection of r rows and columns of **A** such that $\mathbf{A}_1\mathbf{x} \neq \mathbf{0}$ or $\mathbf{A}_1'\mathbf{x} \neq \mathbf{0}$ for every $\mathbf{x} \neq \mathbf{0}$, and if addition of an $(r + 1)st$ row and column would produce a singular matrix, r is called the rank of **A**.

Example

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 7 \\ 3 & 7 & 13 \\ 1 & 1 & -1 \end{bmatrix}$$

Note that

$$[1, \quad 1, \quad -1] \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 7 \\ 3 & 7 & 13 \end{bmatrix} = [0 \quad 0 \quad 0]$$

and

$$[1, \quad -1, \quad -1] \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 7 \\ 1 & 1 & -1 \end{bmatrix} = [0 \quad 0 \quad 0]$$

but

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$[x_1 \quad x_2] \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \neq [0, \quad 0]$$

for any arbitrary

$$[x_1, \quad x_2] \neq [0, \quad 0].$$

Hence the matrix has rank 2.

5.3. If $\mathbf{A}$ has rank r and if $\mathbf{A}_1$ is a non-singular submatrix consisting of r rows and columns of $\mathbf{A}$, then $\mathbf{A}_1$ is called a basis of $\mathbf{A}$.

5.4a. The determinant of a square singular matrix is 0.

5.4b. The determinant of a non-singular matrix is $\neq 0$.

5.5. $\text{rank}\,(\mathbf{AB}) \leq \min\,[\text{rank}\,(\mathbf{A}), \text{rank}\,(\mathbf{B})]$.

5.6. $\text{rank}\,(\mathbf{AA}') = \text{rank}\,(\mathbf{A}'\mathbf{A}) = \text{rank}\,(\mathbf{A})$.

5.7. $|\mathbf{A}'\mathbf{A}| = |\mathbf{AA}'| = |\mathbf{A}|^2$ if $\mathbf{A}$ is square.

5.8. $|\mathbf{A}'\mathbf{A}| = |\mathbf{AA}'| \geq 0$ for every $\mathbf{A}$ with real elements.

6. INVERSION

(Regular Case, non-singular matrices)

6.1. If $\mathbf{A}$ is square and non-singular ($|\mathbf{A}| \neq 0$) there exists a unique matrix $\mathbf{A}^{-1}$ such that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

6.2. $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$ (provided that all inverses exist).

6.3. $(\mathbf{A}^{-1})' = (\mathbf{A}')^{-1}$

6.4. $\mathbf{Ax} = \mathbf{b}$ is a system of linear equations. If $\mathbf{A}$ is square and non-singular, there exists a unique solution

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

6.5. $(\gamma\mathbf{A})^{-1} = (1/\gamma)\,\mathbf{A}^{-1}$

6.6. $|\mathbf{A}^{-1}| = 1/|\mathbf{A}|$

6.7. $\mathbf{D}_w^{-1} = \mathbf{D}_{1/w}$ where $\mathbf{D}$ is a diagonal matrix.

6.8. If

$$\mathbf{A} = \mathbf{B} + \mathbf{uv}'$$

then

$$\mathbf{A}^{-1} = \mathbf{B}^{-1} - \lambda\mathbf{yz}'$$

where

$$\mathbf{y} = \mathbf{B}^{-1}\mathbf{u}, \quad \mathbf{z}' = \mathbf{v}'\mathbf{B}^{-1},$$

and

$$\lambda = 1/(1 + \mathbf{z}'\mathbf{u})$$

Example 6.8.1

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 4 & 5 \\ 3 & 9 & 12 & 15 \\ 2 & 4 & 11 & 10 \\ 1 & 2 & 4 & 10 \end{bmatrix}$$

This matrix can be written as

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} [1 \quad 2 \quad 4 \quad 5] = \mathbf{B} + \mathbf{uv}'$$

$$\mathbf{B}^{-1} = \begin{bmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/5 \end{bmatrix}$$

$$\mathbf{y} = \mathbf{B}^{-1}\mathbf{u} = \begin{bmatrix} 1/3 \\ 1 \\ 2/3 \\ 1/5 \end{bmatrix}$$

$$\mathbf{z}' = \mathbf{v}'\mathbf{B}^{-1} = [1/3 \quad 2/3 \quad 4/3 \quad 1]$$

$$\mathbf{z}'\mathbf{u} = 1/3 \times 1 + 2/3 \times 3 + 4/3 \times 2 + 1 \times 1 = 6$$

$$\lambda = 1/7$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/5 \end{bmatrix} - (1/7) \begin{bmatrix} 1/3 \\ 1 \\ 2/3 \\ 1/5 \end{bmatrix} [1/3 \quad 2/3 \quad 4/3 \quad 1]$$

$$= (1/315) \begin{bmatrix} 100 & -10 & -20 & -15 \\ -15 & 75 & -60 & -45 \\ -10 & -20 & 65 & -30 \\ -3 & -6 & -12 & 54 \end{bmatrix}$$

(This rule is especially useful if all off-diagonal elements are equal, then $\mathbf{u} = k\mathbf{j}$ and $\mathbf{v}' = \mathbf{j}'$ and $\mathbf{B}$ is diagonal.)

6.9. Let $\mathbf{B}$ (elements b_{ij}) have a known inverse, $\mathbf{B}^{-1}$ (elements b^{ij}). Let $\mathbf{A} = \mathbf{B}$ except for one element $a_{rs} = b_{rs} + k$. Then the elements of $\mathbf{A}^{-1}$ are

$$a^{ij} = b^{ij} - \frac{kb^{ir}b^{sj}}{1 + kb^{sr}}.$$

6.10. (Partitioning)

Let

$$\mathbf{A} = \begin{array}{c c} & \begin{array}{cc} (p) & (q) \end{array} \\ \begin{array}{c} (p) \\ (q) \end{array} & \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{bmatrix} \end{array}$$

(letters in parentheses denote order of the submatrices)

Let $\mathbf{B}^{-1}$ and $\mathbf{E}^{-1}$ exist. Then

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{U} \end{bmatrix}$$

where

$$\mathbf{X} = (\mathbf{B} - \mathbf{C}\mathbf{E}^{-1}\mathbf{D})^{-1}$$
$$\mathbf{U} = (\mathbf{E} - \mathbf{D}\mathbf{B}^{-1}\mathbf{C})^{-1}$$
$$\mathbf{Y} = -\mathbf{B}^{-1}\mathbf{C}\mathbf{U}$$
$$\mathbf{Z} = -\mathbf{E}^{-1}\mathbf{D}\mathbf{X}$$

6.11. (Partitioning of Determinants)

Let

$$|\mathbf{A}| = \begin{vmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{vmatrix} \qquad \text{(same structure as in 6.10)}$$

Then

$$|\mathbf{A}| = |\mathbf{E}|\,|(\mathbf{B} - \mathbf{C}\mathbf{E}^{-1}\mathbf{D})| = |\mathbf{B}|\,|(\mathbf{E} - \mathbf{D}\mathbf{B}^{-1}\mathbf{C})|$$

6.12. Let

$$\mathbf{A} = \mathbf{B} + \mathbf{UV}$$

where $\mathbf{B}(n \times n)$ has an inverse

U is of order $(n \times k)$, with k usually very small

V is of order $(k \times n)$

(the special case for $k = 1$ is treated in 6.8).

Then

$$\mathbf{A}^{-1} = \mathbf{B}^{-1} - \mathbf{Y}\boldsymbol{\Lambda}\mathbf{Z}$$

where

$$\mathbf{Y} = \mathbf{B}^{-1}\mathbf{U}(n \times k)$$
$$\mathbf{Z} = \mathbf{V}\mathbf{B}^{-1}(k \times n)$$

and

$$\boldsymbol{\Lambda}(k \times k) = [\mathbf{I} + \mathbf{ZU}]^{-1}$$

6.13. Let a_{ij} denote the elements of **A** and a^{ij} those of $\mathbf{A}^{-1}$. Then

$$a^{ij} = \mathrm{cof}_{ji}(\mathbf{A})/|\mathbf{A}|$$

where cof is the determinant defined in 4.9.

6.14. "Doolittle" Method of inverting symmetric matrices (see also 4.11). Let

$$p_{11} = a_{11}, \quad p_{12} = a_{12} = a_{21}, \ldots p_{1n} = a_{1n} = a_{n1}$$

Forward Solution

$$\begin{array}{ccccc|ccccc}
p_{11} & p_{12} & p_{13} & \cdots & p_{1n} & 1 & & & & \\
1 & u_{12} & u_{13} & \cdots & u_{1n} & u_{1\mathrm{I}} & & & & \\
\hline
 & a_{22} & a_{23} & \cdots & a_{2n} & 0 & 1 & & & \\
 & p_{22} & p_{23} & \cdots & p_{2n} & p_{2\mathrm{I}} & p_{2\mathrm{II}} & & & \\
 & 1 & u_{23} & \cdots & u_{2n} & u_{2\mathrm{I}} & u_{2\mathrm{II}} & & & \\
\cline{2-7}
 & & a_{33} & \cdots & a_{3n} & 0 & 0 & 1 & & \\
 & & p_{33} & \cdots & p_{3n} & p_{3\mathrm{I}} & p_{3\mathrm{II}} & p_{3\mathrm{III}} & & \\
 & & 1 & \cdots & u_{3n} & u_{3\mathrm{I}} & u_{3\mathrm{II}} & u_{3\mathrm{III}} & & \\
\cline{3-8}
 & & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & & \\
 & & & & a_{nn} & 0 & 0 & 0 & \cdots & 1 \\
 & & & & p_{nn} & p_{n\mathrm{I}} & p_{n\mathrm{II}} & p_{n\mathrm{III}} & \cdots & p_{nN} \\
 & & & & 1 & u_{n\mathrm{I}} & u_{n\mathrm{II}} & u_{n\mathrm{III}} & \cdots & u_{nN}
\end{array}$$

$$
\begin{array}{lll}
u_{1i} = p_{1i}/p_{11} & & i = 1, 2, \ldots n, \mathrm{I} \\
p_{2i} = a_{2i} - u_{12}p_{1i} & & i = 2, 3, \ldots n, \mathrm{I}, \mathrm{II} \\
u_{2i} = p_{2i}/p_{22} & & \\
p_{3i} = a_{3i} - u_{13}p_{1i} - u_{23}p_{2i} & & i = 3, 4, \ldots n, \mathrm{I}, \mathrm{II}, \mathrm{III} \\
u_{3i} = p_{3i}/p_{33} & & \\
p_{ki} = a_{ki} - u_{1k}p_{1i} - u_{2k}p_{2i} - \cdots - u_{k-1,k}p_{k-1,i} & & i = k, k+1, \ldots n, \mathrm{I}, \mathrm{II}, \ldots K \\
 & & k = 2, 3, \ldots n \\
u_{ki} = p_{ki}/p_{kk} & &
\end{array}
$$

Backward Solution

(j refers to Arabic, J refers to Roman numerals)

The elements of $\mathbf{A}^{-1}$ are a^{ij}

$$
\begin{array}{ll}
a^{nj} = u_{nJ} & j = 1, 2, \ldots n; \\
 & J = \mathrm{I}, \mathrm{II}, \ldots N \\
a^{n-1j} = u_{n-1,J} - u_{n-1,n}a^{nj} & j = 1, 2, \ldots (n-1); \\
 & J = \mathrm{I}, \mathrm{II}, \ldots (N-1) \\
a^{n-2,j} = u_{n-2,J} - u_{n-2,n}a^{nj} - u_{n-2,n-1}a^{n-1,j} & j = 1, 2, \ldots (n-2); \\
 & J = \mathrm{I}, \mathrm{II}, \ldots (N-2) \\
a^{n-k,j} = u_{n-k,J} - u_{n-k,n}a^{nj} - u_{n-k,n-1}a^{n-1,j} - \cdots - u_{n-k,n-k+1}a^{n-k+1,j} & j = 1, 2, \ldots (n-k); \\
 & J = \mathrm{I}, \mathrm{II}, \ldots (N-k); \\
 & k = 1, 2, \ldots (n-1),
\end{array}
$$

and $a^{ji} = a^{ij}$.

Numerical Example 6.14.1.

Invert the Matrix

$$
\begin{bmatrix} 25 & 30 & -10 \\ 30 & 40 & -6 \\ -10 & -6 & 17 \end{bmatrix}
$$

a_1	25	30	−10	1		
u_1	1	1.2	−0.4	0.04		
	a_2	40	−6	0	1	
	p_2	4	6	−1.2	1	
	u_2	1	1.5	−0.3	0.25	
		a_3	17	0	0	1
		p_3	4	2.2	−1.5	1
		u_3	1	0.55	−0.375	0.25
	1.61	−1.125	0.55			
	−1.125	0.8125	−0.375			
	0.55	−0.375	0.25			

Enter row a_1.
Elements in u_1 = Elements in a_1 divided by $a_{11}(=25)$.
Enter row a_2.

$$p_{22} = 40 - 1.2 \times 30 = 4$$
$$p_{23} = -6 - 1.2 \times (-10) = 6$$
$$p_{21} = 0 - 1.2 \times 1 = -1.2$$
$$p_{2II} = 1$$

Elements in u_2 = Elements in p_2 divided by $p_{22}(=4)$.
Enter row a_3.

$$p_{33} = 17 - (-0.4) \times (-10) - 1.5 \times 6 = 4$$
$$p_{31} = 0 - (-0.4) \times 1 - 1.5 \times (-1.2) = 2.2$$
$$p_{3II} = 0 - 1.5 \times 1 = -1.5$$
$$p_{3III} = 1$$

Elements in u_3 = Elements in p_3 divided by $p_{33}(=4)$.

Copy the right-hand side of the last (third) u – row as the last column below the double line.

$$a^{21} = -0.3 - 1.5 \times 0.55 = -1.125$$
$$a^{22} = 0.25 - 1.5 \times (-0.375) = 0.8125$$
$$a^{23} = 0 - 1.5 \times 0.25 = -0.375 \quad \text{(check against } a^{32}\text{)}.$$

These are entered in the next to last (second) column below.

$$a^{11} = 0.04 - (-0.4) \times 0.55 - 1.2 \times (-1.125) = 1.61$$
$$a^{12} = 0 - (-0.4) \times (-0.375) - 1.2 \times 0.8125 = -1.125 \quad \text{(check against } a^{21}\text{)}$$
$$a^{13} = 0 - (-0.4) \times (0.25) - 1.2 \times (-0.375) = 0.55 \quad \text{(check against } a^{31}\text{)}.$$

6.15. A matrix is called orthogonal if $\mathbf{A}' = \mathbf{A}^{-1}$ (or $\mathbf{AA}' = \mathbf{I}$).

7. TRACES

7.1. If $\mathbf{A}$ is a square matrix then the trace of $\mathbf{A}$ is $tr\,\mathbf{A} = \sum_i a_{ii}$, i.e., the sum of the diagonal elements.

7.2. If $\mathbf{A}$ is of order $(m \times k)$ and $\mathbf{B}$ of order $(k \times m)$ then $tr(\mathbf{AB}) = tr(\mathbf{BA})$.

7.3. If $\mathbf{A}$ is of order $(m \times k)$, $\mathbf{B}$ of order $(k \times r)$ and $\mathbf{C}$ of order $(r \times m)$, then

$$tr(\mathbf{ABC}) = tr(\mathbf{BCA}) = tr(\mathbf{CAB}).$$

7.3a. If $\mathbf{b}$ is a column vector and $\mathbf{c}'$ a row vector, then

$$tr(\mathbf{Abc}') = tr(\mathbf{bc}'\mathbf{A}) = \mathbf{c}'\mathbf{Ab}$$

since the trace of a scalar is the scalar.

7.4. $tr(\mathbf{A} + \gamma\mathbf{B}) = tr\mathbf{A} + \gamma tr\mathbf{B}$; where γ is a scalar.

7.5. $tr(\mathbf{EL})_{ij}\mathbf{A} = tr\mathbf{A}(\mathbf{EL})_{ij} = a_{ji}$; where $(\mathbf{EL})_{ij}$ is an elementary matrix as defined in 3.4.

7.6. $tr(\mathbf{EL})_{ij}\mathbf{A}(\mathbf{EL})_{rs}\mathbf{B} = a_{jr}b_{si}$

(These rules are useful in matrix differentiation)

7.7. The trace of the second order of a square matrix $\mathbf{A}$ is the sum of the determinants of all $\binom{n}{2}$ matrices of order (2×2) which can be formed by intersecting rows i and j with columns i and j.

$$tr_2\mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ \cdots + \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$+ \cdots + \begin{vmatrix} a_{22} & a_{2n} \\ a_{n2} & a_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} a_{n-1,n-1} & a_{n-1,n} \\ a_{n,n-1} & a_{nn} \end{vmatrix}$$

7.8. The trace of the k'th order of a square matrix is the sum of the determinants of all $\binom{n}{k}$ matrices of order $(k \times k)$ which can be formed by intersecting any k rows of $\mathbf{A}$ with the same k columns.

$$tr_k\mathbf{A} = \sum \begin{vmatrix} a_{i_1 i_1} & a_{i_1 i_2} & \cdots & a_{i_1 i_k} \\ a_{i_2 i_1} & a_{i_2 i_2} & \cdots & a_{i_2 i_k} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i_k i_1} & a_{i_k i_2} & \cdots & a_{i_k i_k} \end{vmatrix}$$

where the sum extends over all combinations of n elements taken k at a time in order

$$i_1 < i_2 < \cdots < i_k.$$

7.9. Rules 7.2 and 7.3 (cyclic exchange) are valid for trace of k'th order.

7.10. $tr_n\mathbf{A} = |\mathbf{A}|$ if $\mathbf{A}$ is of order $(n \times n)$.

8. CHARACTERISTIC ROOTS AND VECTORS

8.1. If $\mathbf{A}$ is a square matrix of order $(n \times n)$, then $|\mathbf{A} - \lambda\mathbf{I}| = 0$ is called the characteristic equation of the matrix $\mathbf{A}$. It is a polynomial of the n'th degree in λ.

8.2. The n roots of the characteristic equation (not necessarily distinct) are called the characteristic roots of $\mathbf{A}$

$$ch(\mathbf{A}) = \lambda_1, \lambda_2, \ldots \lambda_n$$

8.3. The characteristic equation of $\mathbf{A}$ can be obtained by the relation

$$\lambda^n - (tr\mathbf{A})\lambda^{n-1} + (tr_2\mathbf{A})\lambda^{n-2} - (tr_3\mathbf{A})\lambda^{n-3} \cdots - (-1)^n(tr_{n-1}\mathbf{A})\lambda + (-1)^n|\mathbf{A}| = 0$$

where tr_k is defined in 7.8.

Example 8.3.1

$$\mathbf{A} = \begin{bmatrix} 25 & 30 & -10 \\ 30 & 40 & -6 \\ -10 & -6 & 17 \end{bmatrix}$$

$tr\mathbf{A} = 25 + 40 + 17 = 82$

$tr_2\mathbf{A} = (25 \times 40 - 30 \times 30) + (25 \times 17 - 10 \times 10) + (40 \times 17 - 6 \times 6) = 1069$

$tr_3\mathbf{A} = |\mathbf{A}| = 25 \times 4 \times 4 = 400$

(cf. 6.14 and procedure stated in 4.11)

Hence

$$\lambda^3 - 82\lambda^2 + 1069\lambda - 400 = 0$$

The solutions (by Newton iteration) are

$$\lambda_1 = 65.86108$$
$$\lambda_2 = 15.75339$$
$$\lambda_3 = 0.38553$$

These are the characteristic roots of **A**.

8.4. $ch(\mathbf{A} + \gamma\mathbf{I}) = \gamma + ch(\mathbf{A})$

8.5. $ch(\mathbf{AB}) = ch(\mathbf{BA})$

(except that **AB** or **BA** may have additional roots equal to zero).

8.6. $ch(\mathbf{A}^{-1}) = 1/ch(\mathbf{A})$

8.7. If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the roots of **A** then

$$\sum_i \lambda_i = tr\mathbf{A}$$

$$\sum_{i<j} \lambda_i\lambda_j = tr_2\mathbf{A}$$

$$\sum_{i<j<k} \lambda_i\lambda_j\lambda_k = tr_3\mathbf{A}$$

$$\prod_i \lambda_i = |\mathbf{A}|$$

8.8. If $\mathbf{x}'$ denotes the radius vector (running coordinates $[x, y, z]$) and if a matrix **Q** is positive-definite, then

$$(\mathbf{x}' - \mathbf{x}_0')\mathbf{Q}^{-1}(\mathbf{x} - \mathbf{x}_0) = 1$$

is the equation of an ellipsoid with center at $[x_0, y_0, z_0] = \mathbf{x}_0'$ and semi-axes equal to the square roots of the characteristic roots of **Q**.

8.9. The characteristic roots of a triangular (or diagonal) matrix are the diagonal elements of the matrix.

8.10. If **A** is a real matrix with positive roots, then

$$ch_{\min}(\mathbf{AA}') \le [ch_{\min}(\mathbf{A})]^2 \le [ch_{\max}(\mathbf{A})]^2 \le ch_{\max}(\mathbf{AA}')$$

where $ch_{\min}$ denotes the smallest and $ch_{\max}$ the largest root.

8.11. The ratio of two quadratic forms (**B** non-singular)

$$u = \frac{\mathbf{x}'\mathbf{Ax}}{\mathbf{x}'\mathbf{Bx}}$$

attains stationary values at the roots of $\mathbf{B}^{-1}\mathbf{A}$. In particular

$$u_{\max} = ch_{\max}(\mathbf{B}^{-1}\mathbf{A}) \qquad \text{and} \qquad u_{\min} = ch_{\min}(\mathbf{B}^{-1}\mathbf{A})$$

8.12. The equation system

$$\mathbf{Ax} = \lambda\mathbf{x}$$

permits non-zero solutions only if λ is one of the characteristic roots of **A**. Such a solution **x** is called a characteristic vector.

8.13. If **x** is a solution to 8.12, so is $\gamma\mathbf{x}$ for an arbitrary scalar γ.

8.14. A solution **x** which has unit length ($\mathbf{x}'\mathbf{x} = 1$) is called the eigenvector associated with the characteristic root λ of **A**. The vector is frequently denoted by **e**.

8.15. A real symmetric matrix has real roots.

8.16. A matrix **A** is called positive-definite (abbreviated p.d.) if the quadratic form $\mathbf{x}'\mathbf{Ax} > 0$ for every $\mathbf{x} \neq \mathbf{0}$.

8.17. A matrix **A** is called positive-semidefinite (abbreviated p.s.d.) if the quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x} > 0$ and/or $\mathbf{x}'\mathbf{A}\mathbf{x} = 0$ for some $\mathbf{x} \neq \mathbf{0}$.

8.18. A positive-definite real symmetric matrix has only positive characteristic roots.

8.19. If a real symmetric matrix is positive-semidefinite, it has no negative roots. The number of non-zero roots equals the rank of the matrix.

8.20. If all roots of a real symmetric matrix are distinct, the associated eigenvectors are distinct.

8.21. The matrix of eigenvectors

$$\mathbf{E} = [\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n]$$

of a real symmetric matrix is (or can be chosen to be) orthogonal.

8.22. $\mathbf{AE} = \mathbf{ED}_\lambda$.

8.23. For a real symmetric matrix, $\mathbf{A} = \mathbf{ED}_\lambda\mathbf{E}'$ (decomposition into matrices of unit rank)

$$\mathbf{E}'\mathbf{AE} = \mathbf{D}_\lambda$$

where $\mathbf{D}_\lambda$ denotes the diagonal matrix of characteristic roots ordered in the same way as the eigenvector columns in **E**.

8.24. If $f(\lambda)$ is a polynomial in λ, then

$$f(\mathbf{A}) = \mathbf{ED}_{f(\lambda)}\mathbf{E}^{-1}$$

where λ are the characteristic roots of **A** and **E** is the matrix of associated eigenvectors. If **A** is symmetric, $\mathbf{E}^{-1} = \mathbf{E}'$.

Example 8.24.1

Consider the matrix in 8.3 (and 6.14).

$$\mathbf{A} = \begin{bmatrix} 25 & 30 & -10 \\ 30 & 40 & -6 \\ -10 & -6 & 17 \end{bmatrix}$$

The characteristic roots were found in Example 8.3.1,

$$\lambda_1 = 65.86108 \qquad \lambda_2 = 15.75339 \qquad \lambda_3 = 0.38553.$$

To find some **x** such that $\mathbf{Ax} = \lambda_1\mathbf{x}$, we arbitrarily set the first element of **x** equal to 1. Using only the first two rows of **A** we solve the equation system

$$25 + 30x_2 - 10x_3 = 65.86108$$

$$30 + 40x_2 - 6x_3 = 65.86108x_2$$

which yields $x_2 = 1.24294$ and $x_3 = -0.35729$. Substitution of these values into the third equation

$$-10 - 6x_2 + 17x_3 = 65.86108x_3$$

yields zero to five decimal places, indicating the accuracy of the first characteristic root. To reduce to unit length the characteristic vector

$$[1 \quad 1.24294 \quad -0.35729]$$

we divide each element by

$$\sqrt{1 + 1.24294^2 + 0.35729^2}$$

and thus obtain the first eigenvector

$$[0.61170 \quad 0.76030 \quad -0.21855]$$

This, written as a column vector, is $\mathbf{e}_1$. Repeating the same process for the second and third eigenvectors we obtain

$$\mathbf{e}_2 = \begin{bmatrix} -0.08659 \\ 0.33896 \\ 0.93681 \end{bmatrix} \qquad \mathbf{e}_3 = \begin{bmatrix} 0.78634 \\ -0.55412 \\ 0.27318 \end{bmatrix}$$

The three vectors can be placed into the eigenvector matrix **E**, which is easily seen to be orthogonal.

9. CONDITIONAL INVERSES

9.1. Any matrix **A** (singular or non-singular, rectangular or square) has some conditional or generalized inverse $\mathbf{A}^{(-1)}$ defined by the relation

$$\mathbf{A}\mathbf{A}^{(-1)}\mathbf{A} = \mathbf{A}.$$

9.2. If (and only if) **A** is square and non-singular, $\mathbf{A}^{(-1)}$ is unique and equals $\mathbf{A}^{-1}$. Otherwise there will be infinitely many matrices $\mathbf{A}^{(-1)}$ which satisfy the defining relation 9.1.

9.3a. If **A** is rectangular ($n \times m$) of rank m, with $m < n$, then $\mathbf{A}^{(-1)}$ is of order ($m \times n$) and $\mathbf{A}^{(-1)}\mathbf{A} = \mathbf{I}(m \times m)$. Then $\mathbf{A}^{(-1)}$ is called an inverse from the left. $\mathbf{A}\mathbf{A}^{(-1)} \neq \mathbf{I}$ in this case.

9.3b. If **A** is rectangular ($n \times m$) of rank n, with $m > n$, then $\mathbf{A}^{(-1)}$ is of order ($m \times n$) and $\mathbf{A}\mathbf{A}^{(-1)} = \mathbf{I}(n \times n)$. Then $\mathbf{A}^{(-1)}$ is called an inverse to the right. In this case,

$$\mathbf{A}^{(-1)}\mathbf{A} \neq \mathbf{I}.$$

9.3c. For a square, singular matrix, $\mathbf{A}\mathbf{A}^{(-1)} \neq \mathbf{I}$ and $\mathbf{A}^{(-1)}\mathbf{A} \neq \mathbf{I}$.

Example 9.3.1

$$\mathbf{A} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

The row vector [1/3 0 0] is an inverse from the left. The row vector

$$[x \quad y \quad (1 - 3x - 2y)]$$

is a conditional inverse of the above matrix **A** for any values of x and y. It is called the generalized inverse of **A**.

Example 9.3.2

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 7 & 9 \end{bmatrix}$$

A conditional inverse is

$$\mathbf{A}^{(-1)} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here it was obtained by inversion of the basis (the 2×2 matrix in the upper left-hand corner) and replacement of the other elements by zeros.

9.4. A square matrix $\mathbf{A}$ is called idempotent if $\mathbf{AA} = \mathbf{A}^2 = \mathbf{A}$.

9.5. $\mathbf{AA}^{(-1)}$ and $\mathbf{A}^{(-1)}\mathbf{A}$ are idempotent.

9.6. All characteristic roots of idempotent matrices are either zero or one.

9.7. A system of linear equations (m equations in n unknowns)

$$\mathbf{Ax} = \mathbf{b}$$

is called consistent if there exists some solution $\mathbf{x}$ which satisfies the equation system.

Example 9.7.1
The system

$$\begin{aligned} x + y &= 2 \\ 2x + 2y &= 4 \qquad \text{is consistent.} \end{aligned}$$

Example 9.7.2
The system

$$\begin{aligned} x + y &= 2 \\ 2x + 2y &= 5 \qquad \text{is inconsistent,} \end{aligned}$$

for no pair of values (x, y) will satisfy this system.

9.8. If, in a system of equations (rectangular or square)

$$\mathbf{Ax} = \mathbf{b}$$

$\mathbf{AA}^{(-1)}\mathbf{b} = \mathbf{b}$ for some conditional inverse $\mathbf{A}^{(-1)}$, then $\mathbf{AA}^{(-1)}\mathbf{b} = \mathbf{b}$ for every conditional inverse of $\mathbf{A}$, and $\mathbf{Ax} = \mathbf{b}$ is consistent. Conversely, if $\mathbf{AA}^{(-1)}\mathbf{b} \neq \mathbf{b}$ for some conditional inverse $\mathbf{A}^{(-1)}$ then $\mathbf{AA}^{(-1)}\mathbf{b} \neq \mathbf{b}$ for every conditional inverse of $\mathbf{A}$, and $\mathbf{Ax} = \mathbf{b}$ is inconsistent.

9.9. If $\mathbf{Ax} = \mathbf{b}$ is consistent, then $\mathbf{x} = \mathbf{A}^{(-1)}\mathbf{b}$ is a solution (generally a different one for each $\mathbf{A}^{(-1)}$).

9.10. Let $\mathbf{y}$ ($p \times 1$) be a set of linear functions of the solutions $\mathbf{x}$ ($n \times 1$) of a consistent system of equations $\mathbf{Ax} = \mathbf{b}$, given by the relation $\mathbf{y} = \mathbf{Cx}$. Then $\mathbf{y} = \mathbf{Cx}$ is called unique if the same values of $\mathbf{y}$ will result regardless which solution $\mathbf{x}$ is used.

Example 9.10.1

$$\begin{aligned} 3x + 4y + 5z &= 22 \\ x + y + z &= 6 \end{aligned}$$

is a consistent system. One solution would be

$$x = 3 \quad y = 2 \quad z = 1$$

Another solution is

$$x = 2 \quad y = 4 \quad z = 0$$

The linear function

$$[7 \quad 9 \quad 11] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = u$$

($7x + 9y + 11z = u$) will have the same value (50) regardless which of the two (or any other) solutions is substituted. Thus u is unique.

9.11. Let $\mathbf{Ax} = \mathbf{b}$ be a consistent system of equations. For $\mathbf{Cx} = \mathbf{y}$ to be a unique linear combination of the solution $\mathbf{x}$, it is necessary and sufficient that $\mathbf{CA}^{(-1)}\mathbf{A} = \mathbf{C}$. If this relation holds for some $\mathbf{A}^{(-1)}$ it will hold for every conditional inverse of $\mathbf{A}$. If it is violated for some $\mathbf{A}^{(-1)}$ it will be violated for every $\mathbf{A}^{(-1)}$, and y will be non-unique.

9.12. Let $\mathbf{A}$ be of rank r and select r rows and r columns which form a basis of $\mathbf{A}$. Then a conditional inverse of $\mathbf{A}$ can be obtained as follows: Invert the $(r \times r)$ matrix, place the inverse (without transposing) into the r rows corresponding to the column numbers and the r columns corresponding to the row numbers of the basis, and place zero into all remaining elements. Thus, if $\mathbf{A}$ is of order (5×4) and rank 3, and if rows 1, 2, 4 and columns 2, 3, 4 are selected as a basis, $\mathbf{A}^{(-1)}$, of order (4×5) will contain the inverse elements of the basis in rows 2, 3, 4 and columns 1, 2, 4, and zeros elsewhere. (See example 9.3.2)

9.13. If $\mathbf{A}$ is a square, singular matrix of order $(n \times n)$ and rank r, let $\mathbf{M}$ be a matrix of order $[n \times (n - r)]$ and $\mathbf{K}$ another matrix of order $[(n - r) \times n]$ chosen in such a way that $\mathbf{A} + \mathbf{MK}$ is non-singular. Then $(\mathbf{A} + \mathbf{MK})^{-1}$ is a conditional inverse of $\mathbf{A}$.

Example 9.13.1

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

is of order (4×4) and rank 3. Take $\mathbf{M} = \mathbf{j}$ (column vector of ones) and $\mathbf{K} = \mathbf{j}'$ (row vector of ones). Then $\mathbf{A} + \mathbf{MK} = \mathbf{A} + \mathbf{jj}' = 4\mathbf{I}$. Hence $(1/4)\ \mathbf{I}$ is a conditional inverse of $\mathbf{A}$.

9.14. The "Doolittle" method (see 6.14) can be employed to obtain a conditional inverse of a symmetric matrix. If, at any stage, the leading element of the p-row is zero, that cycle is disregarded.

Example 9.14.1

Invert, conditionally, the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & -2 & 4 \\ 2 & 17 & 11 & 6 \\ -2 & 11 & 10 & 1 \\ 4 & 6 & 1 & 30 \end{bmatrix}$$

4	2	−2	4	1			
1	.5	−.5	1	.25			
	17	11	6	0	1		
	16	12	4	−.5	1		
	1	.75	.25	−.03125	.0625		
		10	1	0	0	1	
		0					
			30	0	0	0	1
			25	−.875	−.25	0	1
			1	−.035	−.01	0	.04

$$\begin{bmatrix} .29625 & -.0225 & 0 & -.035 \\ -.0225 & .065 & 0 & -.01 \\ 0 & 0 & 0 & 0 \\ -.035 & -.01 & 0 & .04 \end{bmatrix} = \mathbf{A}^{(-1)}$$

10. MATRIX DIFFERENTIATION

10.1a. If the elements of a matrix $\mathbf{Y}$ ($m \times n$) are functions of a scalar, x, the expression

$$\partial \mathbf{Y}/\partial x$$

denotes a matrix of order ($m \times n$) with elements $\partial y_{ij}/\partial x$.

10.1b. If the elements of a column (row) vector $\mathbf{y}$ ($\mathbf{y}'$) are functions of a scalar, x, the expression

$$\partial \mathbf{y}/\partial x \quad (\partial \mathbf{y}'/\partial x)$$

denotes a column (row) vector with elements $\partial y_i/\partial x$.

10.2a. If y is a scalar function of $m \times n$ variables, x_{ij}, arranged into a matrix $\mathbf{X}$, the expression

$$\partial y/\partial \mathbf{X}$$

denotes a matrix with elements $\partial y/\partial x_{ij}$.

(Note: Partial differentiation is performed with respect to the element in row i and column j of $\mathbf{X}$. If the same x-variable occurs in another place as, e.g., in a symmetric matrix, differentiation with respect to the distinct (repeated) variable is performed in two stages.)

Example 10.2.1

If $y = \mathbf{j}'\mathbf{X}\mathbf{j}$ (sum of all elements of a square matrix), $\partial y/\partial \mathbf{X}$ is a matrix of ones. If $\mathbf{X}$ is symmetric, one can introduce a new notation $x_{ij} = x_{ji} = z_{ij}$. Then

$$\begin{aligned} \partial y/\partial z_{ij} &= (\partial y/\partial x_{ij})(\partial x_{ij}/\partial z_{ij}) \\ &\quad + (\partial y/\partial x_{ji})(\partial x_{ji}/\partial z_{ij}) \\ &= 1 + 1 = 2 \quad (\text{if } i \neq j) \\ &= 1 \qquad\qquad (\text{if } i = j). \end{aligned}$$

10.2b. If y is a scalar function of n variables, x_i, arranged into a column (row) vector $\mathbf{x}$ ($\mathbf{x}'$), the expression

$$\partial y/\partial \mathbf{x} \qquad (\partial y/\partial \mathbf{x}')$$

denotes a column (row) vector with elements $\partial y/\partial x_i$.

10.3. If $\mathbf{y}$ is a column vector with m elements, each a function of n variables, x_i, arranged into a row vector $\mathbf{x}'$, the expression $\partial \mathbf{y}/\partial \mathbf{x}'$ denotes a matrix with m rows and n columns, with elements $\partial y_i/\partial x_j$.

10.4. $\partial \mathbf{Y}/\partial y_{ij} = (\mathbf{EL})_{ij}$ (see definition of (EL) in 3.4).

10.5. $\partial \mathbf{UV}/\partial x = (\partial \mathbf{U}/\partial x)\mathbf{V} + \mathbf{U}(\partial \mathbf{V}/\partial x)$.

10.6. $\partial \mathbf{AY}/\partial x = \mathbf{A}(\partial \mathbf{Y}/\partial x)$ (if elements of $\mathbf{A}$ are not functions of x).

10.7. $\partial \mathbf{Y}'/\partial y_{ij} = (\mathbf{EL})_{ji}$

10.8. $\partial \mathbf{A}'\mathbf{YA}/\partial x = \mathbf{A}'(\partial \mathbf{Y}/\partial x)\mathbf{A}$

10.9. $\partial \mathbf{Y}'\mathbf{AY}/\partial x = (\partial \mathbf{Y}'/\partial x)\mathbf{AY} + \mathbf{Y}'\mathbf{A}(\partial \mathbf{Y}/\partial x)$

10.10. $\partial \mathbf{a}'\mathbf{x}/\partial \mathbf{x} = \mathbf{a}$

10.11. $\partial \mathbf{x}'\mathbf{x}/\partial \mathbf{x} = 2\mathbf{x}$

10.12. $\partial \mathbf{x}'\mathbf{A}\mathbf{x}/\partial \mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{A}'\mathbf{x}$
10.13. (Chain Rule No. 1) $\quad \partial \mathbf{y}/\partial \mathbf{x}' = (\partial \mathbf{y}/\partial \mathbf{z}')(\partial \mathbf{z}/\partial \mathbf{x}')$
10.14. $\partial \mathbf{A}\mathbf{x}/\partial \mathbf{x}' = \mathbf{A}$
10.15. $\partial tr\mathbf{X}/\partial \mathbf{X} = \mathbf{I}$
10.16. $\partial tr\mathbf{A}\mathbf{X}/\partial \mathbf{X} = \partial tr\mathbf{X}\mathbf{A}/\partial \mathbf{X} = \mathbf{A}'$
10.17. $\partial tr\mathbf{A}\mathbf{X}\mathbf{B}/\partial \mathbf{X} = \mathbf{A}'\mathbf{B}'$
10.18. $\partial tr\mathbf{X}'\mathbf{A}\mathbf{X}/\partial \mathbf{X} = \mathbf{A}\mathbf{X} + \mathbf{A}'\mathbf{X}$
10.19. $\partial \log |\mathbf{X}| /\partial \mathbf{X} = (\mathbf{X}')^{-1}$ $\quad$ (log to base e).
10.20. $\partial \mathbf{Y}^{-1}/\partial x = -\mathbf{Y}^{-1}(\partial \mathbf{Y}/\partial x)\mathbf{Y}^{-1}$
10.21. (Chain Rule No. 2)

$$\partial y/\partial x = tr(\partial y/\partial \mathbf{Z})(\partial \mathbf{Z}'/\partial x)$$

where y and x are scalars. The scalar y is a function of $m \times n$ variables z_{ij}, and each of the z_{ij} is a function of x.

Example 10.21.1
Obtain $\log |\mathbf{R} - \mathbf{F}\mathbf{F}'| /\partial \mathbf{F}$, where $\mathbf{R}$ is symmetric.
By Chain Rule No. 2:

$\partial \log |\mathbf{R} - \mathbf{F}\mathbf{F}'| /\partial f_{ij}$

$= tr[\partial \log |\mathbf{R} - \mathbf{F}\mathbf{F}'| /\partial(\mathbf{R} - \mathbf{F}\mathbf{F}')][\partial(\mathbf{R} - \mathbf{F}\mathbf{F}')/\partial f_{ij}]$ (since $\mathbf{R}$ and $\mathbf{F}\mathbf{F}'$ are symmetric)
$= tr(\mathbf{R} - \mathbf{F}\mathbf{F}')^{-1}[\partial(\mathbf{R} - \mathbf{F}\mathbf{F}')/\partial f_{ij}]$ (by 10.19)
$= tr(\mathbf{R} - \mathbf{F}\mathbf{F}')^{-1}[-(\partial \mathbf{F}/\partial f_{ij})\mathbf{F}' - \mathbf{F}(\partial \mathbf{F}'/\partial f_{ij})]$ (by 10.5)
$= tr(\mathbf{R} - \mathbf{F}\mathbf{F}')^{-1}[-(\mathbf{EL})_{ij}\mathbf{F}' - \mathbf{F}(\mathbf{EL})_{ji}]$ (by 10.4 and 10.7)
$= -tr(\mathbf{R} - \mathbf{F}\mathbf{F}')^{-1}(\mathbf{EL})_{ij}\mathbf{F}' - tr(\mathbf{R} - \mathbf{F}\mathbf{F}')^{-1}\mathbf{F}(\mathbf{EL})_{ji}$
$= -tr(\mathbf{EL})_{ij}\mathbf{F}'(\mathbf{R} - \mathbf{F}\mathbf{F}')^{-1} - tr(\mathbf{EL})_{ji}(\mathbf{R} - \mathbf{F}\mathbf{F}')^{-1}\mathbf{F}$ (by 7.3)
$= -[\mathbf{F}'(\mathbf{R} - \mathbf{F}\mathbf{F}')^{-1}]_{ji} - [(\mathbf{R} - \mathbf{F}\mathbf{F}')^{-1}\mathbf{F}]_{ij}$,

where $[\,]_{ij}$ denotes the (i,j) element of the matrix in brackets (by 7.5),

$= -[(\mathbf{R} - \mathbf{F}\mathbf{F}')^{-1}\mathbf{F}]_{ij} - [(\mathbf{R} - \mathbf{F}\mathbf{F}')^{-1}\mathbf{F}]_{ij}$ (since $\mathbf{R} - \mathbf{F}\mathbf{F}'$ is symmetric)
$= -2[(\mathbf{R} - \mathbf{F}\mathbf{F}')^{-1}\mathbf{F}]_{ij}$.

Hence, by definition 10.2a

$$\partial \log |\mathbf{R} - \mathbf{F}\mathbf{F}'| /\partial \mathbf{F} = -2(\mathbf{R} - \mathbf{F}\mathbf{F}')^{-1}\mathbf{F}.$$

10.22. $|\partial \mathbf{y}/\partial \mathbf{x}'| = J(\mathbf{y}; \mathbf{x})$ is called the Jacobian or Functional Determinant used in variable transformation of multiple integrals. Formally, if $\mathbf{y}$ is a column vector with m elements, each a function of m variables x_i arranged into a row vector $\mathbf{x}'$,

$$dx_1\, dx_2 \ldots dx_m = |\partial \mathbf{y}/\partial \mathbf{x}'|^{-1} dy_1\, dy_2 \ldots dy_m.$$

10.23. For a scalar y (a function of m variables x_i) to attain a stationary value, it is necessary that

$$\partial y/\partial \mathbf{x} = \mathbf{0}.$$

10.24. For a stationary value to be a minimum (maximum) it is necessary that

$$\partial(\partial y/\partial \mathbf{x})/\partial \mathbf{x}' \qquad (-\partial(\partial y/\partial \mathbf{x})/\partial \mathbf{x}')$$

be a positive-definite matrix for the value of $\mathbf{x}$ satisfying 10.23.

Example 10.24.1
Find the values of $\boldsymbol{\beta}$ which minimize $u = \mathbf{x}'\mathbf{x}$ (the sum of squares of x_i) where $\mathbf{x} = \mathbf{y} - \mathbf{A}\boldsymbol{\beta}$ (with $\mathbf{y}$ and $\mathbf{A}$ known and fixed).

$$\begin{aligned} \partial u/\partial \boldsymbol{\beta}' &= (\partial u/\partial \mathbf{x}')(\partial \mathbf{x}/\partial \boldsymbol{\beta}') \quad \text{(by Chain Rule No. 1)} \\ &= -2\mathbf{x}'\mathbf{A} \quad \text{(by 10.11 and 10.14)} \end{aligned}$$

Hence

$$\begin{aligned} \partial u/\partial \boldsymbol{\beta} &= -2\mathbf{A}'\mathbf{x} \\ &= -2\mathbf{A}'(\mathbf{y} - \mathbf{A}\boldsymbol{\beta}). \end{aligned}$$

Hence, for a stationary value, by 10.23, it is necessary that

$$\mathbf{A}'\mathbf{A}\hat{\boldsymbol{\beta}} = \mathbf{A}'\mathbf{y}$$

where $\hat{\boldsymbol{\beta}}$ denotes the values which make u stationary. Now,

$$\partial(\partial u/\partial \boldsymbol{\beta})/\partial \boldsymbol{\beta}' = 2\partial(\mathbf{A}'\mathbf{A}\boldsymbol{\beta})/\partial \boldsymbol{\beta}' = 2\mathbf{A}'\mathbf{A}.$$

If $\mathbf{A}$ has real elements, and if $\mathbf{A}'\mathbf{A}$ is non-singular, then it is positive-definite (since, given an arbitrary real $\mathbf{x} \neq \mathbf{0}$, $\mathbf{x}'\mathbf{A}'\mathbf{A}\mathbf{x} = \mathbf{z}'\mathbf{z}$, with $\mathbf{z} = \mathbf{A}\mathbf{x}$; thus this is a sum of squares). Hence $\hat{\boldsymbol{\beta}}$ minimizes u.

10.25. (Generalized Newton Iteration)

Let $\mathbf{x}_0'$ be an initial estimate (m elements) of the roots of the m equations

$$\mathbf{f}(\mathbf{x}') = \mathbf{0}$$

where the m elements of the column vector $\mathbf{f}$ are each functions of $x_1, x_2, \ldots x_m$. Then an improved root is

$$\mathbf{x}_1 = \mathbf{x}_0 - \mathbf{Q}_0^{-1}\mathbf{f}(\mathbf{x}_0'),$$

where $\mathbf{Q}_0$ is the matrix of derivatives $\partial\mathbf{f}/\partial\mathbf{x}'$ evaluated at $\mathbf{x} = \mathbf{x}_0$. The usual procedure consists of evaluating $\mathbf{f}(\mathbf{x}_0')$, then solving $\mathbf{Q}_0\mathbf{u} = \mathbf{f}(\mathbf{x}_0')$ for $\mathbf{u}$. Then $\mathbf{x}_1 = \mathbf{x}_0 - \mathbf{u}$.

Example 10.25.1

Solve

$$\begin{aligned} f_1(x,y) &= x^3 - x^2y + y^2 - 3.526 = 0 \\ f_2(x,y) &= x^3 + y^3 - 14.911 = 0 \end{aligned}$$

$$\mathbf{Q} = \begin{bmatrix} 3x^2 - 2xy & 2y - x^2 \\ 3x^2 & 3y^2 \end{bmatrix}$$

Take $x_0 = 1, \quad y_0 = 2$

$$\begin{aligned} f_1(x_0,y_0) &= -0.526 \\ f_2(x_0,y_0) &= -5.911 \end{aligned}$$

$$\mathbf{Q}_0 = \begin{bmatrix} -1 & 3 \\ 3 & 12 \end{bmatrix}$$

$$\begin{aligned} -u + 3v &= -0.526 \\ 3u + 12v &= -5.911 \end{aligned}$$

$$\text{yields } u = -0.55, \quad v = -0.36.$$

Then,

$$\begin{aligned} x_1 &= x_0 - u = 1.55 \\ y_1 &= y_0 - v = 2.36 \\ f_1(x_1,y_1) &= 0.0976 \\ f_2(x_1,y_1) &= 1.9572 \end{aligned}$$

$$\mathbf{Q}_1 = \begin{bmatrix} -0.1085 & 2.3175 \\ 7.2075 & 16.7088 \end{bmatrix}$$

$$-0.1085u + 2.3175v = 0.0976$$
$$7.2075u + 16.7088v = 1.9572$$
$$\text{yields } u = 0.157, \quad v = 0.049.$$

Then

$$x_2 = x_1 - u = 1.393$$
$$y_2 = y_1 - v = 2.311$$
$$f_1(x_2, y_2) = 0.03337$$
$$f_2(x_2, y_2) = 0.13443$$

$$\mathbf{Q}_2 = \begin{bmatrix} -0.61710 & 2.68155 \\ 5.82135 & 16.02216 \end{bmatrix}$$

$$-0.61710u + 2.68155v = 0.03337$$
$$5.82135u + 16.02216v = 0.13443$$
$$\text{yields } u = -0.0068, \quad v = 0.0109.$$

Then,

$$x_3 = x_2 - u = 1.3998$$
$$y_3 = y_2 - v = 2.3001$$

(The exact roots are $x = 1.4$ and $y = 2.3$).

11. STATISTICAL MATRIX FORMS

11.1. Let E denote the expectation operator, and let $\mathbf{y}$ be a set of p random variables. Then

$$E(\mathbf{y}) = \boldsymbol{\mu}$$

states that $E(y_i) = \mu_i \quad (i = 1, 2, \ldots p)$.

11.2. Let var denote variance. Then

$$\mathrm{var}(\mathbf{y}) = \boldsymbol{\Sigma}$$

denotes a $p \times p$ symmetric matrix whose elements are $\mathrm{cov}(y_i, y_j)$, and whose diagonal elements are $\mathrm{var}(y_i)$, where cov denotes covariance.

11.3. $E(\mathbf{AY} + \mathbf{b}) = \mathbf{A}E(\mathbf{y}) + \mathbf{b} = \mathbf{A}\boldsymbol{\mu} + \mathbf{b}$

11.4. $\mathrm{var}(\mathbf{Ay} + \mathbf{b}) = \mathbf{A}\,\mathrm{var}(\mathbf{y})\,\mathbf{A}' = \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'$

11.5. $\mathrm{cov}(\mathbf{y}, \mathbf{z}')$ denotes a matrix with elements $\mathrm{cov}(y_i, z_j)$.
$\mathrm{cov}(\mathbf{z}, \mathbf{y}') = [\mathrm{cov}(\mathbf{y}, \mathbf{z}')]'$.

11.6. $\mathrm{cov}(\mathbf{Ay} + \mathbf{b}, \mathbf{z}'\mathbf{C} + \mathbf{d}') = \mathbf{A}\,\mathrm{cov}(\mathbf{y}, \mathbf{z}')\,\mathbf{C}$

11.7. $\mathrm{var}(\mathbf{y}) = E(\mathbf{yy}') - E(\mathbf{y})E(\mathbf{y}')$

11.8. $\mathrm{cov}(\mathbf{y}, \mathbf{z}') = E(\mathbf{yz}') - E(\mathbf{y})E(\mathbf{z}')$

11.9. (Expected "sum of squares")

$$E(\mathbf{y}'\mathbf{Qy}) = tr[\mathbf{Q}\,\mathrm{var}\,(\mathbf{y})] + E(\mathbf{y}')\mathbf{Q}E(\mathbf{y}).$$

11.10. If a matrix $\mathbf{Q}$ is symmetric and positive-definite, one can find a lower triangular matrix $\mathbf{T}$ (with positive diagonal terms, for uniqueness) such that $\mathbf{TT}' = \mathbf{Q}$. The matrices $\mathbf{T}$ and $\mathbf{T}^{-1}$ can be obtained from the Doolittle pattern (6.14) (Gauss elimination or square-root method) as follows: In each cycle, divide the p-row (left and right hand side) by $\sqrt{p_{ii}}$ (instead of p_{ii} for the u-row). Thus obtain rows designated as t-rows. The left-hand side (Arabic subscripts) is $\mathbf{T}'$, and the right-hand side (Roman subscripts) is $\mathbf{T}^{-1}$

11.11. If a coordinate system $\mathbf{x}$ is oblique, and if the cosines between reference vectors (scalar products of basis vectors of unit length) are stated in a symmetric matrix $\mathbf{Q}$, then $\mathbf{T}^{-1}\mathbf{x} = \mathbf{y}$ is an orthogonal system, where T is obtained from Q by 11.10.

11.12. The likelihood function of a sample of size n from a multivariate normal distribution (p responses), with common variance-covariance matrix $\Sigma(p \times p)$, and with means or main effects replaced by maximum-likelihood or least-squares estimates, can be written as

$$\log L = -\frac{np}{2}\log 2\pi - \frac{n}{2}\log|\Sigma| - \frac{n}{2}\, tr\, \Sigma^{-1}\mathbf{S}$$

where $\Sigma(p \times p)$ is the common variance-covariance matrix, and $\mathbf{S}$ is its maximum-likelihood estimate (matrix of sums of squares and products due to error, divided by sample size n).

11.13. If Σ has a structure under a model or null hypothesis, and if elements of Σ are to be estimated, by maximum-likelihood, two cases can be distinguished: (11.14) Σ^{-1} has the same structure (intraclass correlation, mixed model, compound symmetry, factor analysis). (11.15) Σ^{-1} has a different structure (autocorrelation, Simplex structure).

11.14. If the structure of Σ and Σ^{-1} is identical, and if u and v are elements (or functions of elements) of Σ^{-1} then estimates of Σ can be obtained from the relations (usually requiring Newton iteration, see 10.25):

$$\partial \log L/\partial u = \frac{n}{2}\, tr\, \mathbf{A}(\Sigma - \mathbf{S})$$

where $\mathbf{A} = \partial\Sigma^{-1}/\partial u$, is frequently an elementary matrix (see 3.4 and, especially, the rules 7.5 and 7.6).

$$\partial^2 \log L/\partial u \partial v = \frac{n}{2}\, tr(\partial\mathbf{A}/\partial v)(\Sigma - \mathbf{S}) + \frac{n}{2}\, tr\, \mathbf{A}\Sigma^{-1}\mathbf{B}\Sigma^{-1}$$

where $\mathbf{B} = \partial\Sigma^{-1}/\partial v$. These rules are useful to obtain Newton iterations and asymptotic variance-covariance matrices of the estimates.

11.15. If the structures of Σ and Σ^{-1} are different, then an estimate of Σ can be obtained from the relations

$$\partial \log L/\partial x = -\frac{n}{2}\, tr\, \mathbf{A}(\Sigma^{-1} - \mathbf{Q}),$$

where

$$\mathbf{Q} = \Sigma^{-1}\mathbf{S}\Sigma^{-1}$$

and

$$\mathbf{A} = \partial\Sigma/\partial x \qquad \text{(see comments in 11.14).}$$

$$\partial^2 \log L/\partial x \partial y = -\frac{n}{2}\, tr(\partial\mathbf{A}/\partial y)(\Sigma^{-1} - \mathbf{Q}) + \frac{n}{2}\, tr\, \mathbf{A}\Sigma^{-1}\mathbf{B}(\Sigma^{-1} - \mathbf{Q}) - \frac{n}{2}\, tr\, \mathbf{AQB}\Sigma^{-1},$$

where

$$\mathbf{B} = \partial\Sigma/\partial y$$

x and y are elements (or functions of elements) of Σ. The comments of 11.14 apply, but the iterative procedure is considerably more complex.

Suggestions for further reading:

A. S. Householder, *The Theory of Matrices in Numerical Analysis*, Blaisdell, 1964.
S. R. Searle, *Matrix Algebra for the Biological Sciences*, Wiley, 1966.
P. H. Schonemann, Matrix differentiation of traces and determinants, *Psychometrika*, 1966.

III. COMBINATORIAL ANALYSIS

POWERS OF NUMBERS

The larger numbers are expressed exponentially to at least seven significant figures. The approximate value written as a whole number may be obtained by shifting the decimal point to the right by the number of places indicated in the exponent of 10 shown at the head of each group of values. For example: the approximate value of 33^8 is found in the table as 14.064086×10^{11}. Written as a whole number it is 1 406 408 600 000.

n	n^4	n^5	n^6	n^7	n^8	n^9
1	1	1	1	1	1	1
2	16	32	64	128	256	512
3	81	243	729	2187	6561	19683
4	256	1024	4096	16384	65536	262144
5	625	3125	15625	78125	390625	1953125
6	1296	7776	46656	279936	1679616	10077696
7	2401	16807	117649	823543	5764801	40353607
8	4096	32768	262144	2097152	16777216	134217728
9	6561	59049	531441	4782969	43046721	387420489
					$\times 10^8$	$\times 10^9$
10	10000	100000	1000000	10000000	1.000000	1.000000
11	14641	161051	1771561	19487171	2.143589	2.357948
12	20736	248832	2985984	35831808	4.299817	5.159780
13	28561	371293	4826809	62748517	8.157307	10.604499
14	38416	537824	7529536	105413504	14.757891	20.661047
15	50625	759375	11390625	170859375	25.628906	38.443359
16	65536	1048576	16777216	268435456	42.949673	68.719477
17	83521	1419857	24137569	410338673	69.757574	118.587876
18	104976	1889568	34012224	612220032	110.199606	198.359290
19	130321	2476099	47045881	893871739	169.835630	322.687698
				$\times 10^9$	$\times 10^{10}$	$\times 10^{11}$
20	160000	3200000	64000000	1.280000	2.560000	5.120000
21	194481	4084101	85766121	1.801089	3.782286	7.942800
22	234256	5153632	113379904	2.494358	5.487587	12.072692
23	279841	6436343	148035889	3.404825	7.831099	18.011527
24	331776	7962624	191102976	4.586471	11.007531	26.418075
25	390625	9765625	244140625	6.103516	15.258789	38.146973
26	456976	11881376	308915776	8.031810	20.882706	54.295037
27	531441	14348907	387420489	10.460353	28.242954	76.255975
28	614656	17210368	481890304	13.492929	37.780200	105.784560
29	707281	20511149	594823321	17.249876	50.024641	145.071460
			$\times 10^8$	$\times 10^{10}$	$\times 10^{11}$	$\times 10^{13}$
30	810000	24300000	7.290000	2.187000	6.561000	1.968300
31	923521	28629151	8.875037	2.751261	8.528910	2.643962
32	1048576	33554432	10.737418	3.435974	10.995116	3.518437
33	1185921	39135393	12.914680	4.261844	14.064086	4.641148
34	1336336	45435424	15.448044	5.252335	17.857939	6.071699
35	1500625	52521875	18.382656	6.433930	22.518754	7.881564
36	1679616	60466176	21.767823	7.836416	28.211099	10.155996
37	1874161	69343957	25.657264	9.493188	35.124795	12.996174
38	2085136	79235168	30.109364	11.441558	43.477921	16.521610
39	2313441	90224199	35.187438	13.723101	53.520093	20.872836
			$\times 10^9$	$\times 10^{10}$	$\times 10^{12}$	$\times 10^{14}$
40	2560000	102400000	4.096000	16.384000	6.553600	2.621440
41	2825761	115856201	4.750104	19.475427	7.984925	3.273819
42	3111696	130691232	5.489032	23.053933	9.682652	4.066714
43	3418801	147008443	6.321363	27.181861	11.688200	5.025926
44	3748096	164916224	7.256314	31.927781	14.048224	6.181218
45	4100625	184528125	8.303766	37.366945	16.815125	7.566806
46	4477456	205962976	9.474297	43.581766	20.047612	9.221902
47	4879681	229345007	10.779215	50.662312	23.811287	11.191305
48	5308416	254803968	12.230590	58.706834	28.179280	13.526055
49	5764801	282475249	13.841287	67.822307	33.232931	16.284136
50	6250000	312500000	15.625000	78.125000	39.062500	19.531250

POWERS OF NUMBERS (Continued)

n	n^4	n^5	n^6	n^7	n^8	n^9
			$\times 10^9$	$\times 10^{11}$	$\times 10^{13}$	$\times 10^{14}$
50	6250000	312500000	15.625000	7.812500	3.906250	19.531250
51	6765201	345025251	17.596288	8.974107	4.576794	23.341652
52	7311616	380204032	19.770610	10.280717	5.345973	27.799059
53	7890481	418195493	22.164361	11.747111	6.225969	32.997636
54	8503056	459165024	24.794911	13.389252	7.230196	39.043059
55	9150625	503284375	27.680641	15.224352	8.373394	46.053666
56	9834496	550731776	30.840979	17.270948	9.671731	54.161694
57	10556001	601692057	34.296447	19.548975	11.142916	63.514620
58	11316496	656356768	38.068693	22.079842	12.806308	74.276587
59	12117361	714924299	42.180534	24.886515	14.683044	86.629958
		$\times 10^8$	$\times 10^{10}$	$\times 10^{11}$	$\times 10^{13}$	$\times 10^{16}$
60	12960000	7.776000	4.665600	27.993600	16.796160	1.007770
61	13845841	8.445963	5.152037	31.427428	19.170731	1.169415
62	14776336	9.161328	5.680024	35.216146	21.834011	1.353709
63	15752961	9.924365	6.252350	39.389806	24.815578	1.563381
64	16777216	10.737418	6.871948	43.980465	28.147498	1.801440
65	17850625	11.602906	7.541889	49.022279	31.864481	2.071191
66	18974736	12.523326	8.265395	54.551607	36.004061	2.376268
67	20151121	13.501251	9.045838	60.607116	40.606768	2.720653
68	21381376	14.539336	9.886748	67.229888	45.716324	3.108710
69	22667121	15.640313	10.791816	74.463533	51.379837	3.545209
		$\times 10^8$	$\times 10^{10}$	$\times 10^{12}$	$\times 10^{14}$	$\times 10^{16}$
70	24010000	16.807000	11.764900	8.235430	5.764801	4.035361
71	25411681	18.042294	12.810028	9.095120	6.457535	4.584850
72	26873856	19.349176	13.931407	10.030613	7.222041	5.199870
73	28398241	20.730716	15.133423	11.047399	8.064601	5.887159
74	29986576	22.190066	16.420649	12.151280	8.991947	6.654041
75	31640625	23.730469	17.797852	13.348389	10.011292	7.508469
76	33362176	25.355254	19.269993	14.645195	11.130348	8.459064
77	35153041	27.067842	20.842238	16.048523	12.357363	9.515169
78	37015056	28.871744	22.519960	17.565569	13.701144	10.686892
79	38950081	30.770564	24.308746	19.203909	15.171088	11.985160
		$\times 10^8$	$\times 10^{10}$	$\times 10^{12}$	$\times 10^{14}$	$\times 10^{16}$
80	40960000	32.768000	26.214400	20.971520	16.777216	13.421773
81	43046721	34.867844	28.242954	22.876792	18.530202	15.009464
82	45212176	37.073984	30.400667	24.928547	20.441409	16.761955
83	47458321	39.390406	32.694037	27.136051	22.522922	18.694026
84	49787136	41.821194	35.129803	29.509035	24.787589	20.821575
85	52200625	44.370531	37.714952	32.057709	27.249053	23.161695
86	54700816	47.042702	40.456724	34.792782	29.921793	25.732742
87	57289761	49.842092	43.362620	37.725479	32.821167	28.554415
88	59969536	52.773192	46.440409	40.867560	35.963452	31.647838
89	62742241	55.840594	49.698129	44.231335	39.365888	35.035640
		$\times 10^9$	$\times 10^{11}$	$\times 10^{13}$	$\times 10^{15}$	$\times 10^{17}$
90	65610000	5.904900	5.314410	4.782969	4.304672	3.874205
91	68574961	6.240321	5.678693	5.167610	4.702525	4.279298
92	71639296	6.590815	6.063550	5.578466	5.132189	4.721614
93	74805201	6.956884	6.469902	6.017009	5.595818	5.204111
94	78074896	7.339040	6.898698	6.484776	6.095689	5.729948
95	81450625	7.737809	7.350919	6.983373	6.634204	6.302494
96	84934656	8.153727	7.827578	7.514475	7.213896	6.925340
97	88529281	8.587340	8.329720	8.079828	7.837434	7.602311
98	92236816	9.039208	8.858424	8.681255	8.507630	8.337478
99	96059601	9.509900	9.414801	9.320653	9.227447	9.135172
100	100000000	10.000000	10.000000	10.000000	10.000000	10.000000

POSITIVE POWERS OF TWO

n	2^n	n	2^n
1	2	51	22517 99813 68524 8
2	4	52	45035 99627 37049 6
3	8	53	90071 99254 74099 2
4	16	54	18014 39850 94819 84
5	32	55	36028 79701 89639 68
6	64	56	72057 59403 79279 36
7	128	57	14411 51880 75855 872
8	256	58	28823 03761 51711 744
9	512	59	57646 07523 03423 488
10	1024	60	11529 21504 60684 6976
11	2048	61	23058 43009 21369 3952
12	4096	62	46116 86018 42738 7904
13	8192	63	92233 72036 85477 5808
14	16384	64	18446 74407 37095 51616
15	32768	65	36893 48814 74191 03232
16	65536	66	73786 97629 48382 06464
17	13107 2	67	14757 39525 89676 41292 8
18	26214 4	68	29514 79051 79352 82585 6
19	52428 8	69	59029 58103 58705 65171 2
20	10485 76	70	11805 91620 71741 13034 24
21	20971 52	71	23611 83241 43482 26068 48
22	41943 04	72	47223 66482 86964 52136 96
23	83886 08	73	94447 32965 73929 04273 92
24	16777 216	74	18889 46593 14785 80854 784
25	33554 432	75	37778 93186 29571 61709 568
26	67108 864	76	75557 86372 59143 23419 136
27	13421 7728	77	15111 57274 51828 64683 8272
28	26843 5456	78	30223 14549 03657 29367 6544
29	53687 0912	79	60446 29098 07314 58735 3088
30	10737 41824	80	12089 25819 61462 91747 06176
31	21474 83648	81	24178 51639 22925 83494 12352
32	42949 67296	82	48357 03278 45851 66988 24704
33	85899 34592	83	96714 06556 91703 33976 49408
34	17179 86918 4	84	19342 81311 38340 66795 29881 6
35	34359 73836 8	85	38685 62622 76681 33590 59763 2
36	68719 47673 6	86	77371 25245 53362 67181 19526 4
37	13743 89534 72	87	15474 25049 10672 53436 23905 28
38	27487 79069 44	88	30948 50098 21345 06872 47810 56
39	54975 58138 88	89	61897 00196 42690 13744 95621 12
40	10995 11627 776	90	12379 40039 28538 02748 99124 224
41	21990 23255 552	91	24758 80078 57076 05497 98248 448
42	43980 46511 104	92	49517 60157 14152 10995 96496 896
43	87960 93022 208	93	99035 20314 28304 21991 92993 792
44	17592 18604 4416	94	19807 04062 85660 84398 38598 7584
45	35184 37208 8832	95	39614 08125 71321 68796 77197 5168
46	70368 74417 7664	96	79228 16251 42643 37593 54395 0336
47	14073 74883 55328	97	15845 63250 28528 67518 70879 00672
48	28147 49767 10656	98	31691 26500 57057 35037 41758 01344
49	56294 99534 21312	99	63382 53001 14114 70074 83516 02688
50	11258 99906 84262 4	100	12676 50600 22822 94014 96703 20537 6
		101	25353 01200 45645 88029 93406 41075 2

NEGATIVE POWERS OF TWO

n	2^{-n}
0	1.0
1	0.5
2	0.25
3	0.125
4	0.0625
5	0.03125
6	0.01562 5
7	0.00781 25
8	0.00390 625
9	0.00195 3125
10	0.00097 65625
11	0.00048 82812 5
12	0.00024 41406 25
13	0.00012 20703 125
14	0.00006 10351 5625
15	0.00003 05175 78125
16	0.00001 52587 89062 5
17	0.00000 76293 94531 25
18	0.00000 38146 97265 625
19	0.00000 19073 48632 8125
20	0.00000 09536 74316 40625
21	0.00000 04768 37158 20312 5
22	0.00000 02384 18579 10156 25
23	0.00000 01192 09289 55078 125
24	0.00000 00596 04644 77539 0625
25	0.00000 00298 02322 38769 53125
26	0.00000 00149 01161 19384 76562 5
27	0.00000 00074 50580 59692 38281 25
28	0.00000 00037 25290 29846 19140 625
29	0.00000 00018 62645 14923 09570 3125
30	0.00000 00009 31322 57461 54785 15625
31	0.00000 00004 65661 28730 77392 57812 5
32	0.00000 00002 32830 64365 38696 28906 25
33	0.00000 00001 16415 32182 69348 14453 125
34	0.00000 00000 58207 66091 34674 07226 5625
35	0.00000 00000 29103 83045 67337 03613 28125
36	0.00000 00000 14551 91522 83668 51806 64062 5
37	0.00000 00000 07275 95761 41834 25903 32031 25
38	0.00000 00000 03637 97880 70917 12951 66015 625
39	0.00000 00000 01818 98940 35458 56475 83007 8125
40	0.00000 00000 00909 49470 17729 28237 91503 90625
41	0.00000 00000 00454 74735 08864 64118 95751 95312 5
42	0.00000 00000 00227 37367 54432 32059 47875 97656 25
43	0.00000 00000 00113 68683 77216 16029 73937 98828 125
44	0.00000 00000 00056 84341 88608 08014 86968 99414 0625
45	0.00000 00000 00028 42170 94304 04007 43484 49707 03125
46	0.00000 00000 00014 21085 47152 02003 71742 24853 51562 5
47	0.00000 00000 00007 10542 73576 01001 85871 12426 75781 25
48	0.00000 00000 00003 55271 36788 00500 92935 56213 37890 625
49	0.00000 00000 00001 77635 68394 00250 46467 78106 68945 3125
50	0.00000 00000 00000 88817 84197 00125 23233 89053 34472 65625

SUMS OF POWERS OF INTEGERS, $\sum_{k=1}^{n} k^m$

$(m = 1, 2, 3, 4);\ 1 \leq n \leq 40$

n	Σk	Σk^2	Σk^3	Σk^4
1	1	1	1	1
2	3	5	9	17
3	6	14	36	98
4	10	30	100	354
5	15	55	225	979
6	21	91	441	2275
7	28	140	784	4676
8	36	204	1296	8772
9	45	285	2025	15333
10	55	385	3025	25333
11	66	506	4356	39974
12	78	650	6084	60710
13	91	819	8281	89271
14	105	1015	11025	127687
15	120	1240	14400	178312
16	136	1496	18496	243848
17	153	1785	23409	327369
18	171	2109	29241	432345
19	190	2470	36100	562666
20	210	2870	44100	722666
21	231	3311	53361	917147
22	253	3795	64009	1151403
23	276	4324	76176	1431244
24	300	4900	90000	1763020
25	325	5525	105625	2153645
26	351	6201	123201	2610621
27	378	6930	142884	3142062
28	406	7714	164836	3756718
29	435	8555	189225	4463999
30	465	9455	216225	5273999
31	496	10416	246016	6197520
32	528	11440	278784	7246096
33	561	12529	314721	8432017
34	595	13685	354025	9768353
35	630	14910	396900	11268978
36	666	16206	443556	12948594
37	703	17575	494209	14822755
38	741	19019	549081	16907891
39	780	20540	608400	19221332
40	820	22140	672400	21781332

SUMS OF POWERS OF THE FIRST n INTEGERS*

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{k=1}^{n} k^4 = \frac{n}{30}(n+1)(2n+1)(3n^2+3n-1).$$

$$\sum_{k=1}^{n} k^5 = \frac{n^2}{12}(n+1)^2(2n^2+2n-1).$$

$$\sum_{k=1}^{n} k^6 = \frac{n}{42}(n+1)(2n+1)(3n^4+6n^3-3n+1).$$

$$\sum_{k=1}^{n} k^7 = \frac{n^2}{24}(n+1)^2(3n^4+6n^3-n^2-4n+2).$$

$$\sum_{k=1}^{n} k^8 = \frac{n}{90}(n+1)(2n+1)(5n^6+15n^5+5n^4-15n^3-n^2+9n-3).$$

$$\sum_{k=1}^{n} k^9 = \frac{n^2}{20}(n+1)^2(2n^6+6n^5+n^4-8n^3+n^2+6n-3).$$

$$\sum_{k=1}^{n} k^{10} = \frac{n}{66}(n+1)(2n+1)(3n^8+12n^7+8n^6-18n^5 - 10n^4+24n^3+2n^2-15n+5).$$

Note that

$\sum_{k=1}^{n} k^p = 1^p + 2^p + 3^p + \cdots + n^p$ is a function of n which can be conveniently generated by use of the *following proposition*

If
$$\sum_{k=1}^{n} k^p = a_1 n^{p+1} + a_2 n^p + a_3 n^{p-1} + \cdots + a_{p+1} n$$

then

$$\sum_{k=1}^{n} k^{p+1} = \frac{p+1}{p+2} a_1 n^{p+2} + \frac{p+1}{p+1} a_2 n^{p+1} + \frac{p+1}{p} a_3 n^p + \cdots + \frac{p+1}{2} a_{p+1} n^2 + \left[1 - (p+1)\sum_{k=1}^{p+1} \frac{a_k}{(p+3-k)}\right] n$$

Example *Since* $\sum_{k=1}^{n} k = \frac{1}{2}n^2 + \frac{1}{2}n$, then

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \text{ and from this result}$$

$$\sum_{k=1}^{n} k^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \quad \text{etc.}$$

This proposition is extracted from a paper written by Michael A. Budin and Arnold J. Cantor entitled "Simplified Computation of Sums of Powers of Integers."