The Newman Lectures on Mathematics

John Newman
Vincent Battaglia

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$\frac{\partial^2 \Phi}{\partial x^2} = LC \frac{\partial^2 \Phi}{\partial t^2}$$

The Newman Lectures on Mathematics

The Newman Lectures on Mathematics

John Newman
Vincent Battaglia

Pan Stanford Publishing

Published by

Pan Stanford Publishing Pte. Ltd.
Penthouse Level, Suntec Tower 3
8 Temasek Boulevard
Singapore 038988

Email: editorial@panstanford.com
Web: www.panstanford.com

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

The Newman Lectures on Mathematics

ISBN 978-981-4774-25-3 (Hardcover)
ISBN 978-1-315-10885-8 (eBook)

Contents

Introduction and Philosophical Remarks ix

1. Differentiation of Integrals **1**

2. Linear, First-Order Differential Equations **3**

3. Linear Systems **5**

4. Linearization of Nonlinear Problems **7**

5. Reduction of Order **13**

6. Linear, Second-Order Differential Equations **17**

7. Euler's Equation and Equations with Constant Coefficients **21**

7.1 Linear Equations with Constant Coefficients 21

7.2 Equidimensional Equations 23

8. Series Solutions and Singular Points **25**

9. Legendre's Equation and Special Functions **31**

10. Laplace Transformation **37**

11. The Sturm–Liouville System **47**

12. Numerical Methods for Ordinary Differential Equations **51**

13. Vector Calculus **57**

14. Classification and Examples of Partial Differential Equations **69**

14.1 Elliptic Equations 71

14.2 Parabolic Equations 73

14.3 Hyperbolic Equations 73

15. Steady Heat Conduction in a Rectangle **77**

16. Coordinate Transformations **89**

17. Disk Electrode in an Insulating Plane **95**

18. Suspension of Charged Drops **103**

19. Transient Temperature Distribution in a Slab **107**
19.1 Solution by Separation of Variables. Series Solutions and Superposition 108
19.2 Similarity Solution for Short Times or Thick Slabs 109
19.3 Solution by the Method of Laplace Transforms 111

20. Inversion of Laplace Transforms by the Method of Residues **115**

21. Similarity Transformations **119**

22. Superposition Integrals and Integral Equations **125**
22.1 Duhamel's Theorem 127
22.2 Integral Equations 128
22.3 Catalytic Reaction at a Surface 130
22.4 Superposition Integral from Laplace's Equation 131
22.5 Wealth of Superposition Integrals 133

23. Migration in Rapid Double-Layer Charging **141**

24. Fourier Transforms **147**
24.1 Finite Fourier Transform 147
24.2 Extension to the Fourier Transform 149
24.3 More General Fourier Transforms 151
24.4 Diffusion from a Finite Region 152
24.5 The Convolution Integral 154
24.6 Finishing the Diffusion Problem 155
24.7 Formal Connection with Laplace Transforms 157

25. Conformal Mapping **163**
25.1 A Coordinate Transformation 163
25.2 Analytic Functions 167
25.3 Conformal Mapping 169

25.4 Preserving Laplace's Equation 169
25.5 The Schwarz–Christoffel Transformation 172
25.6 Translating the Flux Densities 176
25.7 Transformations for a Photoelectrochemical Cell 177

26. Calculus of Variations **181**
26.1 The Euler–Lagrange Formula 182
26.2 Application to the Shortest Distance 183
26.3 Optimization with a Constraint 184
26.4 Optimization with End Points Not Fixed 185
26.5 Other Examples 188

Further Readings **199**

Index: Mathematics 201

Index: Physics 205

Introduction and Philosophical Remarks

This book covers vector calculus and ordinary and partial differential equations, both with Laplace transforms. However, the treatment of ordinary differential equations is somewhat abstract because many ordinary differential equations of interest arise in the course of solving partial differential equations. Occasionally, a problem is reduced to an integral equation or the utility of numerical or perturbation methods is indicated. Singular perturbations have been detailed further in Volume 3, *The Newman Lectures on Transport Phenomena*, of this book series.

The present book emphasizes that readers should be able to analyze the problems they encounter in chemical engineering courses. It exposes them to methods of mathematical thinking and presents selected examples that illustrate the techniques that are useful in a typical evening by the fire with a pen and a pad of paper. The book does not focus on the rigorous proof of theorems such as existence and uniqueness of solutions. However, it does give importance to formal manipulations because trivial errors consume a lot of time, which can otherwise be devoted to useful activities. It suggests the readers to follow these three important steps for problem solving:

1. Formulate the problem in mathematical terms. This may be done with varying degrees of completeness or detail, but one should always make sure that the important features of the physical situation are adequately described. For problems of common types, this part may be easy, but some problems are new and require careful consideration.
2. Work through to obtain a solution using the mathematical tools at your disposal. You may need to introduce additional approximations or assumptions in order to get an answer.
3. Contemplate the physical meaning of the results. You should be able to explain qualitatively why the results behave as they do. Be on the lookout for physical absurdities or impossibilities. These may result from an incorrect formulation of the problem

with neglect of important factors or from approximations or even errors introduced during the solution. It should be possible to check, at least *a posteriori*, the validity of an approximation. On the other hand, one may want to live with the consequences of an approximation, recognizing the limitations of the solution in certain regions.

Chapter 1

Differentiation of Integrals

In the solution to differential equations, either in abstract terms or in specific instances, one frequently arrives at an integral of the general form

$$I(x) = \int_{L_1(x)}^{L_2(x)} F(x,\xi)d\xi \,. \qquad (1.1)$$

This is so because the differential equation is considered solved if the expression for the unknown can be reduced to such an integral, even if the integral cannot be evaluated in the closed form.

In order to verify such a solution or to derive its properties, it may be necessary to differentiate it. Hence, you should verify that its derivative is

$$\frac{dI}{dx} = \frac{dL_2}{dx}F[x,L_2(x)] - \frac{dL_1}{dx}F[x,L_1(x)] + \int_{L_1(x)}^{L_2(x)} \frac{\partial F}{\partial x}d\xi \,. \qquad (1.2)$$

This is called the Leibniz rule.

Problems

1.1 Verify Eq. 1.2.

1.2 Differentiate $\int_{y/2\sqrt{Dt}}^{\infty} e^{-\xi^2} d\xi$ with respect to t.

The Newman Lectures on Mathematics
John Newman and Vincent Battaglia

ISBN 978-981-4774-25-3 (Hardcover), 978-1-315-10885-8 (eBook)
www.panstanford.com

1.3 Differentiate the integral of Problem 1.2 with respect to y.

1.4 Can the integral in Eq. 1.2 be evaluated directly, since it is an integral of a derivative?

Chapter 2

Linear, First-Order Differential Equations

The general form of a linear, first-order differential equation is

$$\frac{dy}{dx} + a(x)y = f(x). \tag{2.1}$$

This can be solved by means of an integrating factor F,

$$F = \exp\left\{\int_{L_1}^{x} a\,dx\right\}, \tag{2.2}$$

where the lower limit L_1 can be selected to make F as simple as possible. For example, if $a = x$, the selection $L_1 = 0$ gives $F = e^{x^2/2}$. If $a = 1/x$, the selection $L_1 = 1$ gives $F = x$.

Multiplication of Eq. 2.1 by F yields

$$F\frac{dy}{dx} + Fay = \frac{dFy}{dx} = Ff\,. \tag{2.3}$$

The form 2 of the integrating factor F is easy to remember since it converts the left side of Eq. 2.3 to a perfect differential. To verify this, one needs to make use of Eq. 1.2 for the differentiation of an integral.

Integration of Eq. 2.3 gives

$$Fy = \int_{L_2}^{x} Ff dx + A\,, \tag{2.4}$$

where A is a constant of integration and where again L_2 can be selected so as to yield the simplest expression. Thus, the general solution for y is

The Newman Lectures on Mathematics
John Newman and Vincent Battaglia

ISBN 978-981-4774-25-3 (Hardcover), 978-1-315-10885-8 (eBook)
www.panstanford.com

$$y = e^{-\int_{L_1}^{x} a\,dx} \int_{L_2}^{x} f e^{\int_{L_1}^{x} a\,dx} dx + A e^{-\int_{L_1}^{x} a\,dx}. \tag{2.5}$$

This general solution contains one arbitrary constant A, as is appropriate for a first-order equation. Here one recognizes the first term in Eq. 2.5 as a particular solution to Eq. 2.1 and the second term as the general solution to the corresponding homogeneous equation, as discussed in the next section on linear systems.

As an example, the general solution to the equation

$$\frac{dp}{dx} + 2xp = 1 \tag{2.6}$$

is

$$p = e^{-x^2} \int_0^x e^{x^2} dx + A e^{-x^2}. \tag{2.7}$$

One should notice that here we have not been careful to distinguish the dummy variable of integration x from the independent variable x. If both appeared in the integrand, one would have to distinguish the two. For example, Eq. 2.7 should, more properly, be written as

$$p = e^{-x^2} \int_0^x e^{\xi^2} d\xi + A e^{-x^2} = \int_0^x e^{\xi^2 - x^2} d\xi + A e^{-x^2}. \tag{2.8}$$

Problems

2.1 Show that Eq. 2.5 is a solution to Eq. 2.1.

2.2 Write the general solution to Eq. 2.6 with $L_1 = 1$ and $L_2 = 2$ and show that the result is equivalent to Eq. 2.7.

Chapter 3

Linear Systems

A linear problem is to determine y from the equation

$$\mathscr{L}\{y\} = F, \tag{3.1}$$

where F is given and $\mathscr{L}$ is a linear operator. A linear operator has the following properties:

1. $\mathscr{L}\{ay\} = a\,\mathscr{L}\{y\}$, where a is a scalar constant. (From this, it follows that $\mathscr{L}\{0\} = 0$.)
2. $\mathscr{L}\{y + z\} = \mathscr{L}\{y\} + \mathscr{L}\{z\}$.

For example, differentiation is a linear operation.

Because linear problems are more tractable than nonlinear problems, considerable attention is devoted in applied mathematics to their solution. This relative simplicity is related to the property of superposition of solutions. A linear problem is homogeneous if $F = 0$:

$$\mathscr{L}\{y\} = 0. \tag{3.2}$$

This is also said to be the homogeneous equation corresponding to Eq. 3.1. It follows from the properties of a linear operator that if y and z are each a solution to a linear, homogeneous problem, then $Ay + Bz$ is also a solution, where A and B are arbitrary constants. For example, $y = x$ and $y = 1$ are both solutions to the equation

$$\frac{d^2y}{dx^2} = 0. \tag{3.3}$$

The Newman Lectures on Mathematics
John Newman and Vincent Battaglia

ISBN 978-981-4774-25-3 (Hardcover), 978-1-315-10885-8 (eBook)
www.panstanford.com

Hence, the general solution to this equation is

$$y = Ax + B. \tag{3.4}$$

We know this to be the general solutions because two independent constants are appropriate to a second-order equation. The principle of superposition can then be applied to the linear problem 1 because the general solution to that problem can be expressed as

$$y = y_\mathrm{p} + y_\mathrm{h}, \tag{3.5}$$

where y_p is any particular solution to Eq. 3.1 and y_h is the general solution to the corresponding homogeneous Eq. 3.2. Thus, the original problem can be decomposed into two simpler problems. This explains the terminology used to describe the solution given by Eq. 2.5. The concept of superposition of solutions to linear problems will be used repeatedly in this book.

Problems

3.1 Is $\mathscr{L}$ a linear operator if it is defined as

a. $\mathscr{L}\{y\} = y^2$?
b. $\mathscr{L}\{y\} = y + 2$?
c. $\mathscr{L}\{y\} = dy/dx$?
d. $\mathscr{L}\{y\} = a_1(x)y$?
e. $\mathscr{L}\{y\} = (d/dx)[a_1(x)y]$?

3.2 If $\mathscr{L}$ is a linear operator, which of the following problems are linear problems?

a. $\mathscr{L}\{y\} = 3y$
b. $\mathscr{L}\{y\} = 2x + b$
c. $\mathscr{L}\{y\} = e^y$

Chapter 4

Linearization of Nonlinear Problems

The relative ease of treating linear problems frequently leads one to introduce approximations that produce a linear problem. There is something of an art in the formulation of mathematical models since the model is useless if it is intractable and equally useless if it fails to describe the salient features of the physical system. One recommended procedure would be to formulate a detailed and relatively precise model into which one can subsequently introduce approximations of a mathematical nature. It is usually possible to use the approximate solution thereby obtained to assess the validity of the approximations that have been made. Thus, the contemplation of an approximate solution is an important part of analysis.

If an approximate solution is slightly in error or is significantly in error, but only in a restricted domain, it may be possible to make a correction. This leads to the very important perturbation methods, which give a sound mathematical basis to many approximate solutions. One can state, as a general principle, that if the nature of the approximations is well understood, it should always be possible to use the approximate solution as a basis for a perturbation expansion.

While perturbation methods are strictly outside the scope of this book, we shall, from time to time, draw attention to cases where an approximate solution might be examined in detail and a perturbation expansion would be appropriate.

The Newman Lectures on Mathematics
John Newman and Vincent Battaglia

ISBN 978-981-4774-25-3 (Hardcover), 978-1-315-10885-8 (eBook)
www.panstanford.com

Linearization of nonlinear problems finds widespread use. In the examination of the stability of Poiseuille flow in a tube, it is only necessary to examine whether a small, arbitrary disturbance superposed on the basic, steady velocity profile will grow or decay in time or distance down the tube. In process dynamics and control, the response of a system at a given steady state to minor fluctuations in input variables or external conditions can be analyzed by linearization. In electronic amplification, vacuum tubes and transistors can be treated as linear elements, and the useful range of the equipment is thus defined.

Linearization is also used widely in the numerical solution to nonlinear problems where, by iteration or successive approximations, it is frequently possible to obtain the desired solution to the original problem. A simple example is the Newton–Raphson method for determining the root of a function $y(x) = 0$. This is illustrated graphically in Fig. 4.1. For an initial value x_0, one calculates y_0 and the derivative dy/dx. By linearization, one next calculates a second approximation:

$$y = y_0 + \frac{dy}{dx}(x_1 - x_0) = 0. \tag{4.1}$$

$$x_1 = x_0 - y_0/(dy/dx). \tag{4.2}$$

When this method works, it converges very rapidly. Many successive approximation methods are generalizations of this concept (see, for example, Chapter 12).

Let us illustrate how a particular problem might be linearized to yield useful results. An experimental flow loop is sketched in Fig. 4.2. The hydraulic characteristics of the experimental apparatus are approximated by the resistance of an orifice. The problem is that the pump introduces pulsations in the flow rate. A closed air cavity has been installed between the pump and the experimental apparatus in order to damp these pulsations. On an intuitive basis, we might anticipate that low-frequency pulsations would be little damped, while high-frequency pulsations might be effectively eliminated. How should the air cavity be designed without the solution to a complicated nonlinear problem? We shall neglect inertial effects here, although they can alter significantly the damping characteristics. We shall also neglect the hydrostatic head in the air cavity and take $p_c = p_1$ (see Fig. 4.2).

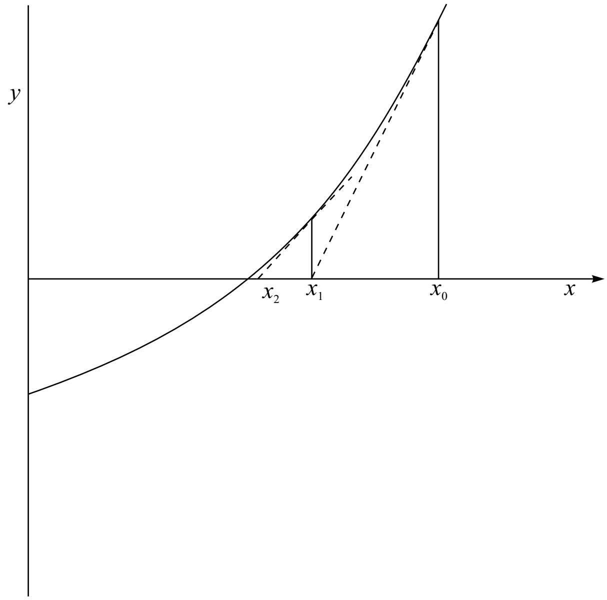

Figure 4.1 Use of the Newton–Raphson method to find *x* such that *y*(*x*) = 0.

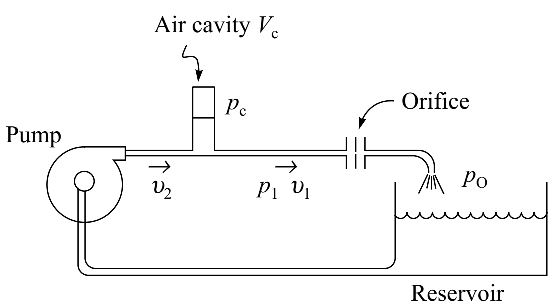

Figure 4.2 The use of an air cavity to damp pulsations in a flow loop.

Let the various flow quantities be represented as sums of their average values and oscillating parts, presumed to be small:

$$\left.\begin{aligned} v_1 &= v + v_1' \\ v_2 &= v + v_2' \\ p_c &= p + p' \\ V_c &= V + V' \end{aligned}\right\}. \tag{4.3}$$

The resistance of the orifice is described by the equation

$$p_c - p_0 = \frac{\rho}{2C^2}\left(\frac{A_T^2}{A_0^2} - 1\right)\upsilon_1 |\upsilon_1|, \tag{4.4}$$

where A_0 is the area of the orifice opening, A_T is the cross-sectional area of the tubing, and C is the orifice coefficient. The pressure in the air cavity is described by an equation for adiabatic expansion

$$p_c V_c^{\gamma} = \text{constant}, \tag{4.5}$$

where γ is the polytropic coefficient of the gas. The material balance on the liquid is

$$\frac{dV_c}{dt} = A_T(\upsilon_1 - \upsilon_2). \tag{4.6}$$

With Eq. 4.3, Eq. 4.4 for the orifice becomes

$$p - p_0 + p' = \frac{\rho}{2C^2}\left(\frac{A_T^2}{A_0^2} - 1\right)\left(\upsilon^2 + 2\upsilon\upsilon_1' + (\upsilon_1')^2\right). \tag{4.7}$$

The square of the small term υ_1' is to be neglected in the linearization. Then the steady components of Eq. 4.7 can be equated

$$p - p_0 = \frac{\rho}{2C^2}\left(\frac{A_T^2}{A_0^2} - 1\right)\upsilon^2, \tag{4.8}$$

and the nonsteady components can be equated

$$p' = \frac{\rho}{2C^2}\left(\frac{A_T^2}{A_0^2} - 1\right)2\upsilon\upsilon_1'. \tag{4.9}$$

Equation 4.5 can be linearized by differentiation

$$V_c^{\gamma}\frac{dp'}{dt} + \gamma p_c V_c^{\gamma-1}\frac{dV'}{dt} = 0. \tag{4.10}$$

When the squares of the small, oscillating terms are neglected, this becomes

$$\frac{dV'}{dt} = -\frac{V}{\gamma p}\frac{dp'}{dt}. \tag{4.11}$$

Equation 4.6 is already linear and can be written as

$$\frac{dV'}{dt} = A_T(\upsilon_1' - \upsilon_2'). \tag{4.12}$$

Combination of Eqs. 4.9, 4.11, and 4.12 gives

$$\frac{dV'}{dt} = A_T(v_1' - v_2') = -\frac{V}{\gamma p}\frac{dp'}{dt} = -\frac{V}{\gamma p}\frac{\rho}{C^2}\left(\frac{A_T^2}{A_0^2} - 1\right)v\frac{dv_1'}{dt} \quad (4.13)$$

or

$$\tau\frac{dv_1'}{dt} + v_1' = v_2', \quad (4.14)$$

where

$$\tau = \frac{V\rho v}{\gamma p A_T C^2}\left(\frac{A_T^2}{A_0^2} - 1\right). \quad (4.15)$$

Suppose now that the pump output can be represented as

$$v_2' = A \sin \omega t. \quad (4.16)$$

After transients have decayed, the oscillating velocity through the orifice will be

$$v_1' = B \sin(\omega t + \phi). \quad (4.17)$$

The ratio of the amplitude of the oscillation of the velocity in the orifice to that in the velocity of the pump output will be

$$\frac{|v_1'|}{|v_2'|} = \frac{B}{A} = \frac{1}{\sqrt{1 + \omega^2\tau^2}}. \quad (4.18)$$

We see from these results that high-frequency oscillations will be more strongly damped than low-frequency oscillations. Furthermore, better damping occurs for a large volume of the air cavity and a high resistance of the orifice and will also depend on the flow velocity.