A Primer for the Monte Carlo Method

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publishing history

This book was published in Russian in 1968, 1972, and 1978. While it is a popular book, it is referred to in rigorous applied papers; teachers also use it as a textbook. With this in mind, the author largely revised the book and published its fourth edition in Russian in 1985.

In English, the book was first published in 1974 by Chicago University Press without the author's permission (the USSR joined the Universal Copyright Convention only in 1973). The second English publication was by Mir Publishers, in 1975 and 1985 (translations of the second and third Russian editions).

The fourth edition of the book was translated only into German (1991, Deutscher Verlag de Wissenschaften).

abstract

The Monte Carlo method is a numerical method of solving mathematical problems by random sampling. As a universal numerical technique, the Monte Carlo method could only have emerged with the appearance of computers. The field of application of the method is expanding with each new computer generation.

This book contains the main schemes of the Monte Carlo method and various examples of how the method can be used in queuing theory, quality and reliability estimations, neutron transport, astrophysics, and numerical analysis.

The principal goal of the book is to show researchers, engineers, and designers in various areas (science, technology, industry, medicine, economics, agriculture, trade, etc.) that they may encounter problems in their respective fields that can be solved by the Monte Carlo method.

The reader is assumed to have only a basic knowledge of elementary calculus. Section 2 presents the concept of random variables in a simple way, which is quite enough for understanding the simplest procedures and applications of the Monte Carlo method.

The fourth revised and enlarged Russian edition (1985; German trans. 1991) can be used as a university textbook for students-nonmathematicians.

preface

The principal goal of this book is to suggest to specialists in various areas that there are problems in their fields that can be solved by the Monte Carlo method.

Many years ago I agreed to deliver two lectures on the Monte Carlo method, at the Department of Computer Technology of the Public University in Moscow. Shortly before the first lecture, I discovered, to my horror, that most of the audience was unfamiliar with probability theory. It was too late to retreat: more than two hundred listeners were eagerly waiting. Accordingly, I hurriedly inserted in the lecture a supplementary part that surveyed the basic concepts of probability. This book's discussion of random variables in Chapter 1 is an outgrowth of that part, and I feel that I must say a few words about it.

Everyone has heard, and most have even used, the words "probability" and "random variable." The intuitive idea of probability (considered as frequency) more or less corresponds to the true meaning of the term. But the layman's notion of a random variable is rather different from the mathematical definition. Therefore, the concept of probability is assumed to be understood, and only the more complicated concept of the random variable is clarified in the first chapter. This explanation cannot replace a course in probability theory: the presentation here is simplified, and no proofs are given. But it does give the reader enough acquaintance with random variables for an understanding of Monte Carlo techniques.

The problems considered in Chapter 2 are fairly simple and have been selected from diverse fields. Of course, they cannot encompass all the areas in which the method can be applied. For example, not a word in this book is devoted to medicine, although the method enables us to calculate radiation doses in X-ray therapy (see Computation of Neutron Transmission Through a Plate in Chapter 2). If we have a program for computing the absorption of radiation in various body tissues, we can select the dosage and direction of irradiation that most efficiently ensures that no harm is done to healthy tissues.

The Russian version of this book is popular, and is often used as a textbook for students-nonmathematicians. To provide greater mathematical depth, the fourth Russian edition includes a new Chapter 3 that is more advanced than the material presented in the preceding editions (which assumed that the reader had only basic knowledge of elementary calculus). The present edition also contains additional information on different techniques for modeling random variables, an approach to quasi-Monte Carlo methods, and a modern program for generating pseudorandom numbers on personal computers.

Finally, I am grateful to Dr. E. Gelbard (Argonne National Laboratory) for encouragement in the writing.

I. Sobol'
Moscow, 1993

introduction

general idea of the method

The Monte Carlo method is a numerical method of solving mathematical problems by the simulation of random variables.

The Origin of the Monte Carlo Method

The generally accepted birth date of the Monte Carlo method is 1949, when an article entitled "The Monte Carlo method" by Metropolis and Ulam[1] appeared. The American mathematicians John von Neumann and Stanislav Ulam are considered its main originators. In the Soviet Union, the first papers on the Monte Carlo method were published in 1955 and 1956 by V. V. Chavchanidze, Yu. A. Shreider and V. S. Vladimirov.

Curiously enough, the theoretical foundation of the method had been known long before the von Neumann-Ulam article was published. Furthermore, well before 1949 certain problems in statistics were sometimes solved by means of random sampling — that is, in fact, by the Monte Carlo method. However, because simulation of random variables by hand is a laborious process, use of the Monte Carlo method as a universal numerical technique became practical only with the advent of computers.

As for the name "Monte Carlo," it is derived from that city in the Principality of Monaco famous for its ... casinos. The point is that one of the simplest mechanical devices for generating random numbers is the roulette wheel. We will discuss it in Chapter 2 under Generating Random Variables on a Computer. But it appears worthwhile to answer here one frequently asked question: "Does the Monte Carlo method help one win at roulette?" The answer is *No*; it is not even an attempt to do so.

Example: the "Hit-or-Miss" Method

We begin with a simple example. Suppose that we need to compute the area of a plane figure S. This may be a completely arbitrary figure with a curvilinear boundary; it may be defined graphically or analytically, and be either connected or consisting of several parts. Let S be the region drawn in Figure 1, and let us assume that it is contained completely within a unit square.

Choose at random N points in the square and designate the number of points that happen to fall inside S by N'. It is geometrically obvious that the area of S is approximately equal to the ratio N'/N. The greater the N, the greater the accuracy of this estimate.

The number of points selected in Figure 1 is $N = 40$. Of these, $N' = 12$ points appeared inside S. The ratio $N'/N = 12/40 = 0.30$, while the true area of S is 0.35.

In practice, the Monte Carlo method is not used for calculating the area of a plane figure. There are other methods (quadrature formulas) for this, that, though they are more complicated, provide much greater accuracy.

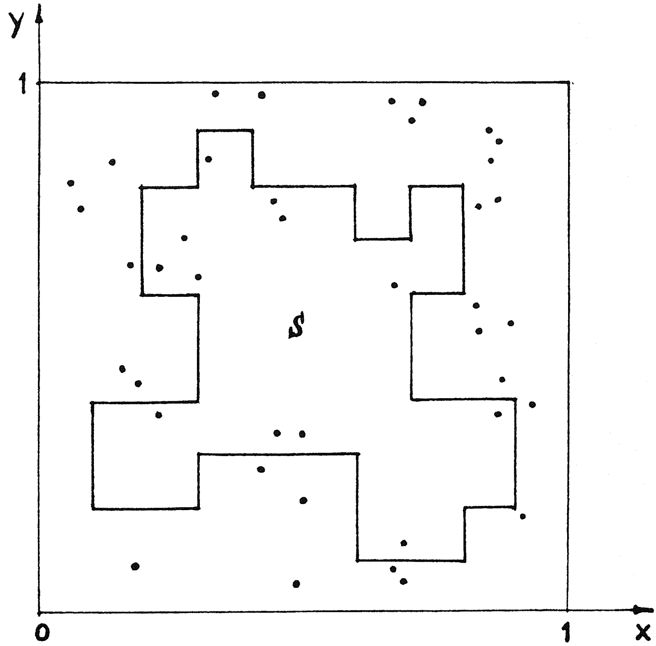

Fig. 1. N random points in the square. Of these, N' points are inside S. The area of S is approximately N'/N.

However, the hit-or-miss method shown in our example permits us to estimate, just as simply, the "multidimensional volume" of a body in a multidimensional space; in such a case the Monte Carlo method is often the only numerical method useful in solving the problem.

Two Distinctive Features of the Monte Carlo Method

One advantageous feature of the Monte Carlo method is the simple structure of the computation algorithm. As a rule, a program is written to carry out one random trial (in our previous "hit-or-miss" example one has to check whether a selected random point inside the square also lies within S). This trial is repeated N times, each trial being independent of the rest, and then the results of all trials are averaged. Therefore, the Monte Carlo method is sometimes called the method of statistical trials.

A second feature of the method is that, as a rule, the error of calculations is proportional to $\sqrt{D/N}$, where D is some constant, and N is the number of trials. Hence, it is clear that to decrease the error by a factor of 10 (in other words, to obtain another decimal digit in the result), it is necessary to increase N (and thus the amount of work) by a factor of 100.

Obtaining high accuracy in this way is clearly impossible. Consequently, it is usually said that the Monte Carlo method is primarily useful for solving those problems that require moderate accuracy, e.g., 5 to 10%. However, any particular problem can be solved by different versions of the Monte Carlo method having different values of D. The accuracy of the result can be significantly improved by an ingenious choice of a computation method having a considerably smaller value of D.

The pluralized term "Monte Carlo methods" is frequently used, emphasizing that the same problem can be solved by simulating different random variables.

Problems that are Solvable by the Monte Carlo Method

To understand what kinds of problems are solvable by the Monte Carlo method, it is important to note that the method enables simulation of any process whose development is influenced by random factors. Second, for many mathematical problems involving no chance, the method enables us to artificially construct a probabilistic model (or several such models), making possible the solution of the problems. In fact, this was done in our earlier "hit-or-miss" example.