JOHN RYAN

Clifford Algebras in Analysis and Related Topics

STUDIES IN ADVANCED MATHEMATICS

Studies in Advanced Mathematics

Clifford Algebras in Analysis and Related Topics

Studies in Advanced Mathematics

Series Editors

STEVEN G. KRANTZ
Washington University in St. Louis

Editorial Board

R. Michael Beals
Rutgers University

Dennis de Turck
University of Pennsylvania

Ronald DeVore
University of South Carolina

L. Craig Evans
University of California at Berkeley

Gerald B. Folland
University of Washington

William Helton
University of California at San Diego

Norberto Salinas
University of Kansas

Michael E. Taylor
University of North Carolina

Titles Included in the Series

Real Analysis and Foundations, *Steven G. Krantz*
CR Manifolds and the Tangential Cauchy-Riemann Complex, *Albert Boggess*
Elementary Introduction to the Theory of Pseudodifferential Operators, *Xavier Saint Raymond*
Fast Fourier Transforms, *James S. Walker*
Measure Theory and Fine Properties of Functions, *L.Craig Evans and Ronald Gariepy*
Partial Differential Equations and Complex Analysis, *Steven G. Krantz*
The Cauchy Transform Theory, Potential Theory, and Conformal Mapping, *Steven R. Bell*
Several Complex Variables and the Geometry of Real Hypersurfaces, *John P. D'Angelo*
An Introduction to Operator Algebra, *Kehe Zhu*
Modern Differential Geometry of Curves and Surfaces, *Alfred Gray*
Wavelets: Mathematics and Applications, *John Benedetto and Michael W. Frazier*
Vibration and Damping in Distributed Systems
  Volume I: Analysis, Estimation, Attenuation, and Design, *Goong Chen and Jianxin Zhou*
  Volume II: WKB and Wave Methods, Visualization and Experimentation, *Goong Chen and Jianxin Zhou*
A Guide to Distribution Theory and Fourier Transforms, *Robert Strichartz*
Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem, *Peter B. Gilkey*
Course in Abstract Harmonic Analysis, *Gerald B. Folland*
Dynamical Systems: Stability, Symbolic Dynamics, and Chaos, *Clark Robinson*
Fourier Analysis and Partial Differential Equations, *José Garcia-Cuerva, Eugenio Hernández, Fernando Soria, and José-Luis Torrea*

JOHN RYAN
University of Arkansas

Clifford Algebras in Analysis and Related Topics

CRC PRESS
Boca Raton New York London Tokyo

Library of Congress Cataloging-in-Publication Data

Clifford algebras in analysis and related topics / edited by John Ryan
p. cm. – (Studies in advanced mathematics)
"Based on a conference held in Fayetteville, Arkansas during the weekend, April 8-10th, 1993" – Introd.
Includes bibliographical references.
ISBN 0-8493-8481-8
1. Clifford algebras. 2. Mathematical analysis. I. Ryan, John, 1955- II. Series.
QA199.C56 1996
515′.1 – dc20 95-23667
CIP

This book contains information obtained from authentic and highly regarded sources. Reprinted material is quoted with permission, and sources are indicated. A wide variety of references are listed. Reasonable efforts have been made to publish reliable data and information, but the author and the publisher cannot assume responsibility for the validity of all materials or for the consequences of their use.

Neither this book nor any part of it may be reproduced or transmitted by any means, electronic or mechanical, including photocopying, microfilming, and recording, or by any information storage or retrieval system, without prior permission in writing from the publisher.

All rights reserved. Authorization to photocopy items for internal or personal use, or the personal or internal use of specific clients, may be granted by CRC Press, Inc., provided that $.50 per page photocopied is paid directly to Copyright Clearance Center, 27 Congress Street, Salem, MA 01970 USA. The fee code for users of the Transactional Reporting Service is ISBN 0-8493-8481-8/96/$0.00+$.50. The fee is subject to change without notice. For organizations that have been granted a photocopy license by the CCC, a separate system of payment has been arranged.

CRC Press, Inc.'s consent does not extend to copying for general distribution, for promotion, for creating new works, or for resale. Specific permission must be obtained in writing from CRC Press for such copying.

Direct all inquiries to CRC Press, Inc., 2000 Corporate Blvd., N.W., Boca Raton, Florida 33431.

©1996 by CRC Press, Inc.

No claim to original U.S. Government works.
International Standard Book Number 0-8493-8481-8
Library of Congress Number 95-23667
Printed in the United States of America 1 2 3 4 5 6 7 8 9 0
Printed on acid-free paper

Contents

1 Acknowledgment 1

2 Conference Participants 3

3 Introduction
John Ryan 7

4 Problem Book
H. Blaine Lawson 17

5 Clifford Algebras, Fourier Theory, Singular Integrals, and Harmonic Functions on Lipschitz Domains
Alan McIntosh 33

- 5.1 Introduction . . . 33
- 5.2 Lecture 1 . . . 34
  - 5.2.1 (A) Spectral theory of $\xi \in \mathbb{R}^m$ . . . 36
  - 5.2.2 (B) Spectral theory of $\zeta \in \mathbb{C}^m$ . . . 37
- 5.3 Lecture 2 . . . 40
  - 5.3.1 (C) Spectral theory of commuting matrices . . . 40
  - 5.3.2 (D) Spectral theory of the Dirac operator $\mathbf{D}$ . . . 43
- 5.4 Lecture 3 . . . 50
  - 5.4.1 (E) Monogenic functions . . . 50
  - 5.4.2 (F) Singular convolution integrals on Lipschitz surfaces 56
- 5.5 Lecture 4 . . . 63
  - 5.5.1 (G) Fourier transforms . . . 63
  - 5.5.2 (H) Correspondence between monogenic functions and holomorphic functions of several complex variables under Fourier transforms . . . 67
- 5.6 Lecture 5 . . . 71
  - 5.6.1 (J) Algebras of singular integrals on Lipschitz surfaces, and functional calculi of Dirac operators on these surfaces . . . 71
  - 5.6.2 (K) Boundary value problems for harmonic functions 75
- 5.7 Additional Material . . . 78
  - 5.7.1 (L) Singular integrals on the boundary of a strongly Lipschitz domain . . . 78
  - 5.7.2 (M) More about singular convolution integrals on Lipschitz surfaces . . . 80

6 Clifford Algebras and H_∞ Functional Calculi of Commuting Operators
Chun Li and Alan McIntosh **89**
6.1 Introduction 89
6.2 The H_∞ functional calculus of a single operator 89
6.3 Fourier transform between holomorphic functions and monogenic functions 92
6.4 H_∞ functional calculi of n–tuples of commuting operators . 97

7 Hypercomplex Variable Techniques in Harmonic Analysis
Marius Mitrea **103**
7.1 Introduction 103
7.2 Clifford Algebra Rudiments 104
7.3 Elements of Clifford Analysis 106
7.4 Non-Homogeneous Dirac Operators 108
7.5 Clifford Algebra-Valued Singular Integral Operators 109
7.6 Hardy Spaces on Lipschitz Domains 112
7.7 Rellich Type Formulas for Monogenic Functions 120
7.8 A Burkholder-Gundy-Silverstein Theorem for Monogenic Functions in Lipschitz Domains 121
7.9 Some Applications to the Theory of Harmonic Functions in Lipschitz Domains 123

8 Some Applications of Conformal Covariance in Clifford Analysis
John Ryan **129**
8.1 Introduction 129
8.2 Preliminaries 131
8.3 Conformal Invariance of Cells of Harmonicity 134
8.4 $V(n)$ and Real Clifford Analysis 137
8.5 $V(\mathbf{C}^n)$ and Complex Clifford Analysis 145
8.6 The Bergman Kernel and Harmonic Measure 149
8.7 More on Conformal Covariance in Complex Clifford Analysis 153

9 Singular Integrals with Monogenic Kernels on the m-Torus and their Lipschitz Perturbations
Tao Qian **157**
9.1 Introduction 157
9.2 Preliminary 159
9.3 Fourier Transforms 161
9.4 Singular Integral Operators and Fourier Multiplier Operators on Periodic Lipschitz Surfaces 167

10 Scattering Theory for Orthogonal Wavelets
Palle E.T. Jorgensen **173**
10.1 Introduction 173
10.2 Preliminaries 174
10.3 Scattering Theory 177
10.4 Wavelets 181
10.5 Multiresolutions 184
10.6 The Integral Translation Group 188
10.7 The Besicovich Spaces 192

11 Acoustic Scattering, Galerkin Estimates and Clifford Algebras
Björn Jawerth and Marius Mitrea **199**
11.1 Introduction 199
11.2 Definitions and Preliminary Results 202
11.3 Inverting Quaternionic-Valued Boundary Operators 206
11.4 Finite Element Spaces 208
11.5 Local Galerkin Estimates 211

12 Clifford Algebras, Hardy Spaces, and Compensated Compactness
Zhijian Wu **217**
12.1 Introduction 217
12.2 Preliminaries 219
12.3 Components of the product of two monogenic functions . . 222
12.4 Proof of the main theorems 229
12.5 Applications in compensated compactness 236

13 Frame Decompositions of Form-Valued Hardy Spaces
J.E. Gilbert, J.A. Hogan, and J.D. Lakey **239**
13.1 Introduction 239
13.2 Dilatation Frames 241
13.3 The deRham complex and Hardy spaces of forms 246
13.4 Projections on $H^p_{\rho_r}$ 250
13.5 Lusin's representation for $H^p_{\rho_r}$ 251
13.6 Discrete frames: the case of one-forms 255
13.7 Concluding remarks and problems 256

14 Applications of Clifford Analysis to Inverse Scattering for the Linear Hierarchy in Several Space Dimensions
Daniel B. Dix **261**
14.1 Introduction 261
14.2 Review of Some Clifford Analysis 262

14.3 Relation between Complex and Clifford Analysis in Dimension One . . . 265
14.4 Review of the Linear Hierarchy in One Space Dimension . . 272
14.5 The Linear Hierarchy in Several Space Dimensions 278

15 On Riemann-Hilbert Problems for Nonhomogeneous Dirac Equations in a Half Space of $\mathbf{R}^m (m \geq 2)$
Zhenyuan Xu and Chiping Zhou **285**
15.1 Introduction . . . 285
15.2 Preliminaries . . . 286
15.3 Riemann-Hilbert Problems for Nonhomogeneous Dirac Equations . . . 289

16 Regularity and Approximation Results for the Maxwell Problem on C^1 and Lipschitz Domains
Marius Mitrea, Rodolfo H. Torres, and Grant V. Welland **297**
16.1 Introduction . . . 297
16.2 The vector Helmholtz equation . . . 300
16.3 The singular integral operator M . . . 301
16.4 Sobolev-Besov space regularity results . . . 302
16.5 Some approximation results . . . 304

17 Continuity of Calderón-Zygmund Type Operators on the Predual of a Morrey Space
Josefina Alvarez **309**
17.1 Introduction . . . 309
17.2 Definition of the Space $\mathbf{L}^{p,\varphi}$ . . . 310
17.3 The Space $\mathbf{L}^{p,\varphi}$ as the Dual of the Atomic Space $\mathbf{H}^{p,\varphi}$ . . . 311
17.4 A Notion of a Molecule in the Space $\mathbf{H}^{p,\varphi}$ . . . 312
17.5 Continuity on the Spaces $\mathbf{H}^{p,\varphi}$ and $\mathbf{L}^{p,\varphi}$ of Calderón-Zygmund Type Operators . . . 315

18 Neumann Type Problems for the Dirac Operator
Chiping Zhou **321**
18.1 Introduction . . . 321
18.2 Neumann Problems for Monogenic Functions . . . 323
18.3 Neumann Problems for Nonhomogeneous Dirac Equations . 328

19 The Hyperholomorphic Bergman Projector and its Properties
E. Ramírez de Arellano, M. Shapiro, and N. Vasilevski **333**
19.1 Introduction . . . 333
19.2 Preliminaries . . . 334

19.3 The T-operator and the singular integral operator over a domain 337
19.4 Bergman hyperholomorphic spaces and Bergman hyperholomorphic projector 338
19.5 Some algebras generated by the Bergman projector 341

20 Multivector Solutions to the Hyperholomorphic Massive Dirac Equation
William M. Pezzaglia, Jr. **345**
20.1 Introduction 345
20.2 Algebraic Notation 346
20.2.1 Classical Galilean Space and The Pauli Algebra . . . 347
20.2.2 Minkowski Spacetime and Majorana Algebra 347
20.2.3 Automorphisms and Conservation Laws 348
20.3 Functional Solutions of the Massive Dirac Equation 350
20.3.1 Relativistic Quantum Wave Equations 350
20.3.2 Meta-Monogenic Functions 351
20.3.3 Multivectorial Hilbert Space 352
20.4 Integral Meta-Monogenic Solutions 354
20.4.1 The Propagating Kernel 354
20.4.2 Green Function 356
20.4.3 Path Integral Formulation (PIF) 357
20.5 Summary 358

21 Möbius Transformations, Vahlen Matrices, and their Factorization
Pertti Lounesto **361**
21.1 Similarities and transversions 361
21.2 The counter-example of J. Maks 362
21.3 Recent mistakes in factorization 363
21.4 How to factor Vahlen matrices? 364

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

1

Acknowledgment

This contributed volume is derived from a conference held in Arkansas.

It is my pleasure to acknowledge the help of the Department of Mathematical Sciences, the Graduate School, and the J. William Fulbright College of Arts and Sciences of the University of Arkansas, Fayetteville, for their generous funding of the conference on which this contributed volume is based. During a part of the time this volume was in preparation the editor was supported by a grant from the Arkansas Science and Technology Authority. I am grateful to A.S.T.A. for their support.

Thanks are also due to the other members of the University of Arkansas Spring Lecture Series Committee, for their help and support in a year of preparation for the meeting. Specifically, thanks are due to Itrel Monroe, chairman of the committee, and John Duncan, Mihalis Maliakas, and William Summers. A special thanks is due to Janet Miller for her tremendous secretarial support both before and during the meeting.

I am very grateful to William Pezzaglia, who advertised the conference by placing abstracts of the talks and various announcements in the Clifford Algebra e-mail Bulletin Board, clf_alg@stars.sfsu.edu, and to Jaime Keller for publishing a report on the conference in the journal *Advances in Applied Clifford Algebra.*

I would also like to thank Steven Krantz for his strong encouragement and constant support of this project. I am also indebted to Wayne Yuhasz, and Nora Konopka of CRC Press and Lori Pickert of Archetype for all their help.

All papers appearing in this volume were refereed. For their editorial assistance I am truly grateful to Pascal Auscher, Tao Qian, Chun Li, Alan McIntosh, Marius Mitrea, Palle Jorgensen, Grant Welland, Zhijian Wu, Stephen Semmes, Daniel Luecking, John Akeroyd, John Duncan, Itrel Monroe, and Mihalis Maliakas.

For their patient assistance with my limitations with TeX and e-mail, I am indebted to my colleagues Dan Luecking and Bob Miller.

0-8493-8481-8/96/$0.00+$.50
© 1996 by CRC Press

I am indebted to Brenda Barrett for the typing of all of my own contributions to this project.

Thanks are also due to John Duncan, Gary Tee, A. Gsponer and Andreas Sparschuh for their assistance with some historical details appearing in the introduction.

A special thanks is due to Cherie Moore of Washington University, Saint Louis, for the gargantuan task of placing this volume in camera-ready form.

Also, my heartfelt thanks go to my wife Christine, and my daughters Tammy and Danielle, for their constant support throughout the nearly two years this project has taken to come to fruition.

John Ryan, Arkansas, September, 1994.

Conference Participants

John Adams Department of Physics, San Francisco State University, San Francisco, California 94132.
E-mail: adams@stars.sfsu.edu

Josefina Alvarez Department of Mathematical Sciences, New Mexico State University, Las Cruces, New Mexico 88003-0001.
E-mail: jalvarez@nmsu.edu

Enrique Ramírez de Arellano Departamento de Matemáticas, CINVESTAV del IPN, Mexico, D.F., MEXICO.
E-mail: eramirez@math.cinvestav.mx

Pascal Auscher Departément de Mathématiques, Université de Rennes I, Campus de Beaulieu, 35042 Rennes Cedex, FRANCE.
E-mail: auscher@univ-rennes1.fr

Ruediger Belch Rutherford College, The University of Canterbury, Kent CT2 7NX, UNITED KINGDOM.
E-mail: rb5@ukc.ac.uk

Don Burkhardt Department of Physics, University of Georgia, Athens, Georgia 30602.

Guy David Mathématique Bât 425, Université de Paris-Sud, 9140 Orsay Cedex, FRANCE.
E-mail: gdavid@anh.matups.fr

Daniel Dix Department of Mathematics, University of South Carolina at Columbia, Columbia, South Carolina 29208.
E-mail: dix@bigcheese.math.scarolina.edu

Matt Enjalran Department of Physics, University of Massachusetts at Amherst, Amherst, Massachusetts 01003.
E-mail: menjalran@phast.umass.edu

John Gilbert Department of Mathematics, University of Texas at Austin, Austin, Texas 78712.
E-mail: gilbert@math.utexas.edu

Angela Grant Department of Mathematics, Lincoln University, 820 Chestnut, Jefferson City, Missouri 65102.

Bjorn Jawerth Department of Mathematics, University of South Carolina at Columbia, Columbia, South Carolina 29208.
E-mail: bj@loki.math.scarolina.edu

0-8493-8481-8/96/$0.00+$.50
© 1996 by CRC Press

Palle E.T. Jorgensen Department of Mathematics, University of Iowa, Iowa City, Iowa 52242-1466.
E-mail: jorgen@math.uiowa.edu

Carlos Kenig Department of Mathematics, University of Chicago, 5734 University Avenue, Chicago, Illinois 60637.
E-mail: cek@math.uchicago.edu

Frank Kippig Fachbereich Mathematik, Bergakademie Freiberg, Bernhard-von-Cotta-Staße 2, 0-9200 Freiberg, Saxony, GERMANY.
E-mail: kippig @mathe.ba-freiberg.d400.de

Joe Lakey Department of Mathematics, University of Texas at Austin, Austin, Texas 78712.
E-mail: lakey@math.utexas.edu

Chun Li School of Mathematics, Physics, Computing, and Electronics, Macquarie University, North Ryde, New South Wales 2109, AUSTRALIA.
E-mail: chun@macadam.mpce.mq.edu.au

Pertti Lounesto Institute of Mathematics, Helsinki University of Technology, SF-02150 Espoo, Helsinki, FINLAND.
E-mail: lounesto@dopey.hut.fi

Alan McIntosh School of Mathematics, Physics, Computing, and Electronics, Macquarie University, North Ryde, New South Wales 2109, AUSTRALIA.
E-mail: alan@macadam.mpce.mq.edu.au

Gary Miller Department of Mathematics, University of New Brunswick, Fredericton, New Brunswick E3B 5A3, CANADA.
E-mail: gary@math.unb.ca

Marius Mitrea Department of Mathematics, University of Minnesota at Minneapolis, Minnesota 55455.
E-mail: mitrea@math.umn.edu

Robert E. Mullins Department of Mathematics and Computer Science, William Wehr Physics Building, Marquette University, Milwaukee, Wisconsin 53233.
E-mail: bobm@mscs.mu.edu

Tao Qian Department of Mathematics, Statistics, and Computing Science, University of New England, Armidale, New South Wales 2351, AUSTRALIA.
E-mail: tao@neumann.une.edu.au

William M. Pezzaglia, Jr. Department of Physics, California State University at Hayward, Hayward, California 94542.
E-mail: bpezzag@mcs.csuhayward.edu

John Ryan Department of Mathematical Sciences, University of Arkansas at Fayetteville, Fayetteville, Arkansas 72701.
E-mail: jryan@comp.uark.edu

Stephen Semmes Department of Mathematics, Rice University, Houston, Texas 77251.
E-mail: semmes@math.rice.edu

Michael Shapiro Departamento de Matemáticas, ESFM del IPN, Edificio 9, Unidad ALM, 07300, Mexico, D.F., MEXICO.
E-mail: mshapiro@godel.esfm.ipn.mx

Zhongwei Shen Department of Mathematics, Purdue University, West Lafayette, Indiana 47907-1395.
E-mail: shenz@math.purdue.edu

Don Small Department of Mathematics, University of New Brunswick, Fredericton, New Brunswick E3B 5A3, CANADA.
E-mail: don@math.unb.edu

Karel Stroethoff Department of Mathematics, University of Montana, Missoula, Montana 59812.
E-mail: ma_kms@selway.umt.edu

Terrance Tao Department of Mathematics, Princeton University, Princeton, New Jersey 08544.
E-mail: tao@math.princeton.edu

Toma Tonev Department of Mathematics, University of Montana, Missoula, Montana 59812.
E-mail: ma_tt@lewis.umt.edu

Rodolfo Torres Department of Mathematics, University of Michigan at Ann Arbor, Ann Arbor, Michigan 48109.
E-mail: rodolfo.torres@math.lsa.umich.edu

Nikolai Vasilevski Departamento de Matemáticas, CINVESTAV del IPN, Mexico, D.F., MEXICO.
E-mail: nvasilev@math.cinvestav.mx

Greg Verchota Department of Mathematics, Syracuse University, Syracuse, New York 13244.
E-mail: verchota@math.syr.edu

Grant Welland Department of Mathematics, University of Missouri at Saint Louis, 8001 Natural Bridge Road, Saint Louis, Missouri 63121.
E-mail: welland@eads.umsl.edu

Zhijian Wu Department of Mathematics, University of Alabama at Tuscaloosa, Tuscaloosa, Alabama 35487.
E-mail: zwu@mathdept.as.ua.edu

Zhenyuan Xu Department of Mathematics, Ryerson Polytechnic University, 350 Victoria Street, Toronto, Ontario M5B 2K3, CANADA.
E-mail: zxu@acs.ryerson.ca

Keith Yale Department of Mathematics, University of Montana, Missoula, Montana 59812.
E-mail: ma_iky@relway.umt.edu

Chiping Zhou Department of Mathematics, University of Hawaii at Honolulu Community College, 874 Dillingham Boulevard, Honolulu, Hawaii 96817.
E-mail: chiping@pulua.hcc.hawaii.edu

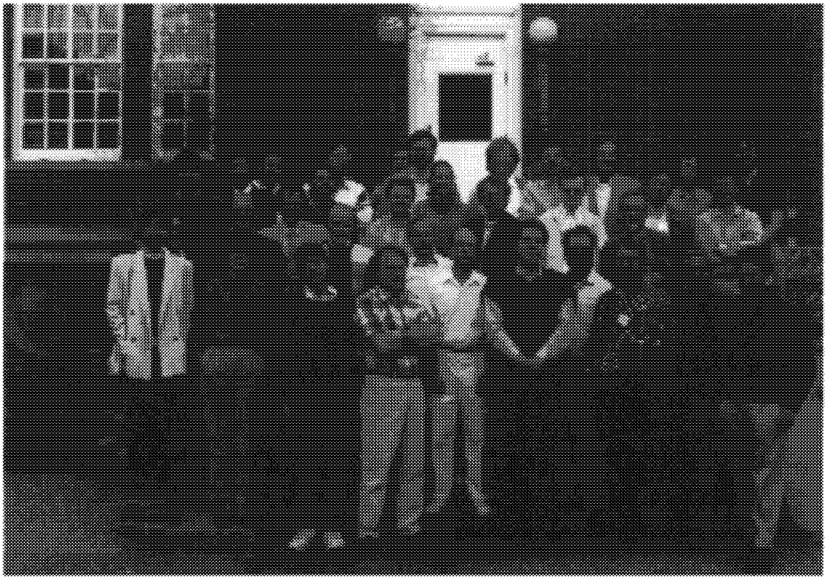


FIGURE 2.1

'CLIFFORD ALGEBRAS IN ANALYSIS'
April 8–10 1993
UNIVERSITY OF ARKANSAS, FAYETTEVILLE

Front Row (Left to Right):
Chun Li (Sydney, Australia), Xhenyun Xu (Toronto, Canada), Terrance Tao (Princeton), Garry Miller (New Brunswick, Canada), Don Small (New Brunswick, Canada), Pertti Lounesto (Helsinki, Finland), Ruediger Belch (Kent, UK), Naoki Kimura (Arkansas), Alan McIntosh (Sydney, Australia).
Second Row:
Debora Gayle (San Francisco), Enrique Ramirez De Arellano (Mexico City, Mexico), Palle Jorgensen (Iowa), Don Burkhardt (Georgia), Toma Tonev (Montana), William M. Pezzaglia, Jr. (San Francisco).
Third Row:
Pascal Auscher (Rennes, France), Chiping Zhou (Hawaii), John Adams (San Francisco), John Gilbert (Texas), Josefina Alvarez (New Mexico), Keith Yale (Montana), Frank Kippig (Freiberg, Germany), Michael Shapiro (Mexico City, Mexico), Tao Qian (Armidale, Australia), Xiangsheng Xu (Arkansas).
Back Row:
Serge Tabachnikov (Arkansas), Zhijian Wu (Alabama), Guy David (Paris, France), Karel Stroethoff (Montana), John Ryan (Arkansas), Nikolai Vasilevski (Mexico City, Mexico), Itrel Monroe (Arkansas), Zhongwei Shen (Indiana), Daniel Dix (South Carolina).
Photographer: Matthew Enjalran (Amherst, MA)

Location: OLD POST OFFICE (Fayetteville, AR)
Plaque of Senator William Fulbright on left side

3

Introduction

John Ryan

Clifford analysis started as an attempt to generalize one-variable complex analysis to higher dimensions using Clifford algebras generated from Euclidean space. More recently, deep and unexpected links to classical harmonic analysis, several complex variables, and representation theory have been discovered. In the early stages the subject was developed exclusively in three and four dimensions using the quaternionic division algebra, which is an example of a Clifford algebra. Later it was realized that results obtained in the quaternionic setting, particularly the generalization of Cauchy's integral formula, did not exclusively rely on the division algebra property of the quaternions, but that it is sufficient for an algebra to contain a vector subspace where all non-zero vectors are invertible in the algebra. In the Clifford algebra setting, this invertibility corresponds to the usual Kelvin inversion of vectors in Euclidean space. This fact is not too surprising, given that Clifford algebras are specifically designed to help describe the geometric properties of quadratic forms on vectors spaces, see for instance, [2]. For some time, it has been understood by most people working with Clifford analysis that most results so far obtained in quaternionic analysis more or less automatically extend to all finite dimensions using Clifford algebras.

In fact, Clifford algebras are remarkably simple algebras to set up. Loosely speaking, one would like to consider R^n as a subspace of some algebra, so that under the algebra multiplication we have that $x^2 = -\|x\|^2$ for each vector x in R^n. If no other constraints are introduced, the minimal algebra satisfying this requirement is an example of a universal Clifford algebra. When $n = 1$, we obtain the complex number system. When $n = 2$, we obtain the quaternions. For $n > 2$, the algebra is no longer a division algebra. However, each non-zero vector x in R^n is invertible in the algebra, with multiplicative inverse $x^{-1} = \frac{-x}{\|x\|^2}$. This inverse corresponds, up to a sign, to the Kelvin inverse of the vector x.

Clifford algebras were introduced in the nineteenth century by mathe-

0-8493-8481-8/96/$0.00+$.50
© 1996 by CRC Press

maticians and mathematical physicists in various attempts to provide a good foundation to geometric calculus in Euclidean space. An historical account of this development and Clifford's role here is given in the introduction of [10] and some papers therein, e.g., [26]. Clifford was a student of Maxwell's. He is one of the youngest scientists ever to be elected a Fellow of the Royal Society, and he was Professor of Applied Mathematics at University College, London. He died of tuberculosis at the age of thirty-three in 1879. His interest in the algebra that bears his name arose, in part, from his attempts to place Maxwell's work on electromagnetism in a more mathematically rigorous setting. His paper [11] describing these algebras appeared in the *American Journal of Mathematics* one year before his death.

The subject of Clifford analysis has been discovered and independently rediscovered about ten times in the last century. The earliest known work on the subject is by A.C. Dixon [16]. Alfred Cardew Dixon (1865–1936) was Professor of Mathematics at Queen's University, Belfast, from 1901 to 1930. He was a Fellow of the Royal Society, and President of the London Mathematical Society from 1931 to 1933. Later, C. Lanczos described the rudiments of quaternionic analysis in his doctoral thesis [31]. In the 1920's, Felix Klein independently rediscovered the area, [30]. In the 1930's and 1940's, the Swiss mathematician Rudolph Fueter and his students published about fifteen papers on the subject, e.g., [18]. Most of these papers appeared in the journal *Commentarii Mathematici Helvetici.* An excellent summary of this work is given in a paper of A. Sudbery [56], and a more detailed account is given in some lecture notes of E. Bareiss [4]. Surprisingly, many topics covered by Fueter and his collaborators have not been touched upon in more recent books on the subject, though Gursey and Tze [23] have used some of these results in their study of Yang Mills field equations.

At much the same time as Fueter's work appeared, Moisil and Theodorescu [39] worked on closely related results. This appears to have been the start of a period of research by Romanian mathematicians into aspects of Clifford analysis and related topics which spanned a period of over thirty years; see for instance, [27,41] and references therein. It was also during the 1930's that possible links to mathematical physics were first noted. In particular, the differential operator arising in the generalized Cauchy-Riemann equations, and the "conjugate" of this operator combine to give the Laplacian in Euclidean space. This is in complete analogy to the fact that in one-variable complex analysis, the operators $\frac{d}{dz}$ and $\frac{d}{d\bar{z}}$ combine to give the Laplacian in two-dimensional space. Earlier, Dirac [17] had used a matrix representation of a Clifford algebra to introduce a factorization of the wave operator, or d'Alambertian, in terms of two first-order differential operators. For this reason, the differential operator arising in the generalized Cauchy-Riemann equations of Clifford analysis is often referred to as the Dirac operator. This operator corresponds to the $d+\delta$ operator acting on differential forms over R^n, where d is deRham's exterior derivative and δ its adjoint. However, it should be pointed out that the alternating algebra does not possess the correct algebraic structure to admit a meaningful Cauchy integral formula, akin to the one from Clifford analysis. Essentially, the alternating algebra generated from R^n does not incorporate Kelvin inversion.

It was also at this time that it was first noted that the analysis so far developed in the quaternion setting generalized to the Clifford algebra setting; see for instance, a paper of Haefeli [24]. Some further work in this direction was developed in the 1950's by Sce, [46].

In a four-year period in the late 1960's and early 1970's independent papers by Richard Delanghe [14], T.E. Littlewood and C.D. Gay [35], David Hestenes [25], and Viorel Iftimie [27] were published. Each of these papers illustrated how many aspects of one-variable complex analysis extend to Euclidean space using Clifford algebras. Here, fundamental, but implicit, use is made of Kelvin inversion to set up Cauchy's integral formula, Laurent and Taylor series, etc. In particular, in [27] Iftimie sets up basic results on Cauchy transforms over domains in R^n, and establishes Plemelj formulae for Hölder continuous functions defined over compact Liapunov surfaces in Euclidean space. Using the Plemelj formulae, he is able to show that the square of the singular Cauchy transform over such a surface is, when acting on Hölder continuous functions, the identity map. This is in complete analogy to the case in complex analysis. Consequently, the stage was set for applying Clifford analysis to study boundary value problems. More recently, these results have been extended to L^p-spaces over the boundaries of Lipschitz domains in R^n; see for instance [33,34].

Clifford analysis provides an extremely rich framework for generalizing many results from one-variable complex analysis. A review of the basic results of Clifford analysis is given in [21,Ch.4]. One subtle difference is that the generalized analytic functions, which are often called monogenic functions, are defined on domains in R^n, and usually take values in the Clifford algebra generated from that space or some spinor subspace of the algebra. One apparent limitation to the theory is that the pointwise multiplication of two monogenic functions is, in general, not a monogenic function. This follows from the noncommutativity of the algebra. Though it should be pointed out that in [51], a very natural product is introduced which reduces in the two-dimensional setting to the usual product. Other basic properties of one-variable complex analysis do not hold in the Clifford analysis setting, e.g., the Riemann mapping theorem.

The term "Clifford analysis" was first coined in the late 1970's, when the editor of this volume used it as a title of a manuscript. The manuscript was referenced by Sommen in [50], and most of the main results for this manuscript appeared in [42]. Later, the term was used by Brackx, Delanghe, and Sommen, [8], for the title of the first book in the area.

During the 1970's and 1980's, research into Clifford analysis started to become significantly less sporadic and isolated. Richard Delanghe began to build a research group at Ghent State University, Belgium, which has become the largest group currently working in the area. In particular, Frank Sommen [51] independently rediscovered a result of Littlewood and Gay [35] showing that real analytic functions defined on domains in R^{n-1} have Cauchy-Kowalweski extensions to monogenic functions defined in some neighborhood in R^n. Although this result is extremely simple, as are many basic results in Clifford analysis, it has a basic impact of linking problems in real analysis, in R^{n-1}, to function theory over domains in one higher dimension. In particular, in [52] Sommen uses this idea to link up Clifford analysis in R^n with the Fourier transform over R^{n-1}. It is in this work and in his later work on plane wave decompositions, [53], that it is realized that this analysis requires both the use of complex numbers and their generalization, the real Clifford algebras. In particular, both algebras are used fundamentally to set up projection operators to describe the decomposition of special classes of functions defined on R^{n-1} into classes of monogenic functions defined on upper- and lower-half-space in R^n. For this reason, it becomes necessary to introduce complex Clifford algebras. For mathematicians, this effectively, and efficiently, dispenses with objections raised by some physicists to the use of complex Clifford algebras; see for instance, remarks made in [26].

The projection operators mentioned in the previous paragraph are, in fact, Fourier transforms of the Plemelj formulae/operators for upper- and lower-half-space. Moreover, the singular Cauchy transform over R^{n-1} is the vector sum over R^{n-1} of the Riesz transforms described by Stein and Weiss in [54,55]. So, this singular Cauchy transform can be seen as a generalization of the Hilbert transform over the line. It follows that the work of Stein and Weiss, [54,55], on H^p-spaces in R^n using conjugate harmonic functions fits perfectly into the context of Clifford analysis. This point is well described in [20,Ch.2]. In fact, conjugate harmonic functions are vector-valued harmonic functions whose derivatives are symmetric, and have vanishing trace. Such a system of equations is called a Riesz system, and is a special case of the Cauchy-Riemann equations arising in Clifford analysis.

It was during the mid-1980's that R. Coifman had the idea that many hard problems in classical harmonic analysis could either be simplified or solved using Clifford analysis. This arose in the context of the Coifman-McIntosh-Meyer theorem, [13], which establishes the L^2-boundedness of the double-layer potential operator over Lipschitz graphs in R^n. This was a landmark result in classical harmonic analysis which was cited at the time in a report to the American Mathematical Society listing three recent dramatic examples of progress in theoretical mathematics. The original proof for the case $n = 2$ uses the complex number system, but in higher dimensions the Calderón rotation method is used. Coifman suggested that the two-dimensional proof can be mimicked in higher dimensions using Clifford algebras and Dirac operators, giving rise to a more natural proof. This was carried out for Lipschitz graphs with small Lipschitz constant by Margaret Murray, [40]. The argument was completed for all Lipschitz constants by Alan McIntosh [36]. A key idea here is that the double-layer potential operator over a sufficiently smooth surface is the real, or scalar, part of the singular Cauchy transform over the surface in R^n. This idea and these results had the impact of opening up the field to a much broader spectrum of mathematical interests.

Originally, the L^2-boundedness of the double-layer potential operator was worked out over Liapunov surfaces. So, the surface is C^1 with a Hölder continuous derivative. This added smoothness gives sufficient cancellation for one to deduce that the operator is weakly singular. It follows that the operators $\frac{1}{2}I \pm DLP$ are Fredholm, where I is the identity and DLP is the double-layer potential operator. Some more work reveals that these operators are injective, and so they are invertible. Consequently, it becomes an easy matter to use invertibility to produce solutions to the interior and exterior Dirichlet problems for such domains.

When one replaces Liapunov surfaces by Lipschitz surfaces, one no longer has the cancellation property mentioned in the previous paragraph. So the Fredholm operator theory is no longer available, and one needs to find different techniques. The first step in solving the Dirichlet problem over Lipschitz surfaces is to establish the L^2-boundedness of the double-layer potential over such surfaces. Several proofs of this result have now appeared, and some of them make use of Clifford analysis. One main advantage of the Clifford algebra-based proofs is that they bring to light the functional calculus of Dirac operators over Lipschitz surfaces, and unify much of the existing theory.

Also, in the 1980's, Ahlfors rediscovered results of Vahlen [58] and showed [1] that Möbius transformations in R^n could be described using a group of 2×2 matrices with values in a Clifford algebra. This inspired some authors to use Vahlen matrices to find analogues of Schwarzian derivatives on R^n [9,44].

The analogue of the Vahlen group over Minkowski space is the Lie group $SU(2,2)$. This group is used, [29], to describe the conformal covariance of the Dirac operator and its iterates over Minkowski space. This too, can be placed in the context of Clifford algebras. These ideas, together with ideas described in a paper of Imaeda's [28] and many other references given in this introduction, have inspired the editor of this volume to study intertwining operators for conformally covariant operators over Euclidean space and $\mathbf{C}^n$, and to study Clifford analysis over very general types of cells of harmonicity in $\mathbf{C}^n$, together with links with several complex variables, see for instance, [43,45], and references therein.

The introduction of Vahlen matrices has inspired some authors in the early 1990's to develop Clifford analysis over hyperbolic space. Also, following ideas presented in [29] and mentioned in the previous paragraph, it would seem desirable to see further work done on the links between Clifford analysis and twistor theory. Work in this direction has been initiated in the last chapter of [15]. It would also seem likely that work previously done on automorphic forms and involving the use of Lie groups such as $SU(n,n)$, $Sp(n,R)$, and $Sp(n,\mathbf{C})$ could also be developed using Vahlen matrices over Minkowski-type spaces. Hopefully, some new and interesting results are awaiting discovery here. In addition, it would be nice to see closer ties developed between Clifford analysis and the study of Dirac operators over general spin manifolds, and to see links with the Atiyah-Singer index theorem as developed in [5,6,20,32].

The area of Clifford analysis which has seen the most rapid growth in recent years has been the one inspired by Coifman on applications to classical harmonic analysis and the theory of singular integrals. Besides the references that we have cited so far, there is also the work of Auscher and Tchamitchian [3], Gaudry, Long, and Qian [19], and Mitrea [38], where Clifford algebra-valued Haar wavelets/martingales are used to deduce the L^2-boundedness of the double-layer potential operator over Lipschitz surfaces in R^n. This extends to Euclidean space a proof due to Coifman, Jones, and Semmes, of the same result in the complex plane using complex-valued Haar wavelets; see [12].

A very good summary of these results, together with the Clifford $T(b)$ theorem is given in the Master's thesis of Terrance Tao [57]. Further very interesting results involving Clifford analysis within singular integral theory have been developed in recent times by Stephen Semmes [47,48,49]. Besides these recent developments on singular integrals and their applications to boundary value problems, Gürlebeck and Sprössig [22] have also considered related problems over Liapunov surfaces. Their approach also considers the use of colocation methods and other numerical techniques.

It should be pointed out that this review of the development of Clifford analysis, though intended to be fairly thorough, is by no means complete. Firstly, it is almost certain that there are still some long-forgotten papers in the area which will eventually be rediscovered. This seems inevitable, as such papers keep turning up with a fair regularity. Also, constraints of space and time prevented us from pointing out some further interesting developments and works in this area, or related areas.

This volume is based on a conference held in Fayetteville, Arkansas during the Easter weekend, April 8–10th, 1993. The conference was entitled "Clifford Algebras in Analysis", and the principal speaker was Alan McIntosh, of Macquarie University, Australia. Though there have been three other conferences on Clifford algebras and their applications in mathematical physics, [7,10,37], including one which took place one month after this one, this is the first conference, together with the proceedings, which deals almost exclusively with the impact of Clifford analysis on harmonic analysis. We were fortunate to be able to gather a highly-distinguished group of researchers in classical harmonic analysis and Clifford analysis for the meeting. It is hoped that the meeting and this volume will help set the pace for future research in this fascinating and growing area of mathematics. To this end, we have included a selection of open problems provided by many researchers with interests in this area. The idea for such a list came from a similar problem book in function theory developed by Walter Hayman and David Brannan. It is hoped that the problem book produced here will be added to with the passage of time, and will be addressed in future publications and conferences.

Bibliography

[1] L. V. Ahlfors, *Möbius transformations in R^n expressed through* 2×2 *matrices of Clifford numbers*, Complex Variables **5**, (1986), 215–224.

[2] M. F. Atiyah, R. Bott, and A. Shapiro, *Clifford modules*, Topology **3**, (1965), 3–38.

[3] P. Auscher and Ph. Tchamitchian, *Bases d'ondelettes sur les courbes corde-arc, noyou de Cauchy et Espaces de Hardy associeés*, Revista Matematica Iberoamericana **5**, (1989), 139–170.

[4] E. Bareiss, *Functions of a hypercomplex variable*, lecture notes based on talks given by Rudolph Fueter, University of Zürich, 1948–1949.

[5] N. Berline, E. Getzler, and M. Vergne, *Heat Kernels and Dirac Operators*, Springer-Verlag, Heidelberg, 1992.

[6] B. Booss-Bavnbeck and K. Wojciechowski, *Elliptic Boundary Problems for Dirac Operators*, Birkhauser, Basel, 1993.

[7] F. Brackx, R. Delanghe, and H. Serras, eds., *Clifford Algebras and their Applications in Mathematical Physics*, Deinze 1993, Kluwer, Dordrecht, 1993.

[8] F. Brackx, R. Delanghe, and F. Sommen, *Clifford Analysis*, Pitman, London, 1982.

[9] T. K. Carne, *The Schwarzian derivative for conformal maps*, J. Reine Angew. Math. **408**, (1990), 10–33.

[10] J. S. R. Chisholm and A.K. Common, eds., *Clifford Algebras and their Applications in Mathematical Physics*, Riedel, Dordrecht, 1986.

[11] W. K. Clifford, *Applications of Grassmann's extensive algebra*, Amer. J.of Math. **1**, (1978), 350–358.

[12] R. Coifman, P. Jones, and S. Semmes, *Two elementary proofs of the L^2-boundedness of Cauchy integrals on Lipschitz curves*, J. of the A.M.S. **2**, (1989), 553–564.

[13] R. Coifman, A. McIntosh, and Y. Meyer, *L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes*, Annals of Mathematics **116**, (1982), 361–387.

[14] R. Delanghe, *On regular-analytic functions with values in a Clifford algebra*, Math. Ann. **185**, (1970), 91–111.

[15] R. Delanghe, F. Sommen, and V. Soucek, *Clifford Algebra and Spinor-Valued Functions : A Function Theory for the Dirac Operator*, Kluwer, Dordrecht, 1992.

[16] A. C. Dixon, *On the Newtonian potential*, Quarterly Journal of Mathematics **35**, (1904), 283–296.

[17] P. A. M. Dirac, *The quantum theory of the electron, I*, Proceedings of the Royal Society A **117**, (1928), 610–624.

[18] R. Fueter, *Die Funktionentheorie der Differentialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit Vier Reallen Variablen*, Commentarii Mathematici Helvetici **7**, (1934–1935), 307–330.

[19] G. Gaudry, R. Long, and T. Qian, *A martingale proof of L^2-boundedness of Clifford-valued singular integrals*, Annali di Matematica, Pura Appl. **165**, (1993), 369–394.

[20] J. Gilbert and M. Murray, *Clifford Algebras and Dirac Operators in Harmonic Analysis*, Cambridge University Press, Cambridge, 1991.

[21] R. P. Gilbert and J. Buchanan, *First-Order Elliptic Systems : A Function Theoretic Approach*, Academic Press, San Diego, 1983.

[22] K. Gürlebeck and W. Sprössig, *Quaternionic Analysis and Elliptic Boundary Value Problems*, Birkhäuser Verlag, Basel, 1990.

[23] F. Gursey and H. C. Tze, *Complex and quaternionic analyticity in chiral and gauge theories, 1*, Annals of Physics **128**, (1980), 29–130.

[24] H. G. Haefeli, *Hypercomplexe Differentiale*, Commentarii Mathematici Helvetici **20**, (1947), 382–420.

[25] D. Hestenes, *Multivector functions*, J. Math. Anal. Appl. **24**, (1968), 467–473.

[26] D. Hestenes, *A unified language for mathematics and physics*, Clifford Algebras and Their Applications in Mathematical Physics, Reidel, Dordrecht, 1–23, 1986.

[27] V. Iftimie, *Fonctions hypercomplexes*, Bull. Math. de la Soc. Sci. Math. de la R.S. de Roumanie **9**, (1965) 279–332.

[28] K. Imaeda, *A new formulation of classical electrodynamics*, Nuovo Cimento **32B**, (1976), 138–162.

[29] H. P. Jakobsen and M. Vergne, *Wave and Dirac operators, and representations of the conformal group*, J. Funct. Anal. **24**, (1977), 52–106.

[30] F. Klein, *Vorlesungen über nichteuklidische Geometrie*, Springer, Berlin, 1928.

[31] C. Lanczos, *Die Funktionentheoretischen Beziehungen Der Maxwellschen Aethergleichungen*, Doctoral Thesis, Budapest, 1919.

[32] H. B. Lawson, Jr. and M.-L. Michelson, *Spin Geometry*, Princeton University Press, Princeton, NJ, 1989.

[33] C. Li, A. McIntosh, and S. Semmes, *Convolution singular integrals on Lipschitz surfaces*, J. of the A.M.S. **5**, (1992), 455–481.

[34] C. Li, A. McIntosh, and T. Qian, *Clifford algebras, Fourier transforms, and singular convolution operators on Lipschitz surfaces*, Revista Mathematica Iberoamericana, to appear.

[35] T. E. Littlewood and C. D. Gay, *Analytic spinor fields*, Proc. Roy. Soc. **A313**, (1969), 491–507.

[36] A. McIntosh, *Clifford algebras and the higher-dimensional Cauchy integral*, Approximation Theory and Function Spaces, Banach Center Publications **22**, (1989), 253–267.

[37] A. Micali, R. Boudet, and J. Helmstetter, eds., *Clifford Algebras and their Applications in Mathematical Physics*, Kluwer, Dordrecht, 1992.

[38] M. Mitrea, *Singular Integrals, Hardy Spaces, and Clifford Wavelets*, Lecture Notes in Mathematics **No.1575**, (1994), Springer-Verlag, Heidelberg.

[39] Gr. C. Moisil and N. Theodorescu, *Fonctions holomorphes dans l'espace*, Mathematica (Cluj) **5**, (1931), 142–159.

[40] M. Murray, *The Cauchy integral, Calderón commutation, and conjugation of singular integrals in R^m*, Trans. of the A.M.S. **298**, (1985), 497–518.

[41] D. Pascali, *Representations of quaternionic areolar polynomials in tridimensional space*, Stud. Cerc. Mat. **18**, (1966), 239–242

[42] J. Ryan, *Clifford analysis with generalized elliptic and quasi-elliptic functions*, Appl. Anal. **13**, (1982), 151–171.

[43] J. Ryan, *Cells of harmonicity and generalized Cauchy integral formulae*, Proc. of the London Math. Soc. **60**, (1990), 295–318.

[44] J. Ryan, *Generalized Schwarzian derivatives for generalized fractional linear transformations*, Ann. Polon. Math. **LVII**, (1992), 29–44.

[45] J. Ryan, *Some applications of conformal covariance in Clifford analysis*, this volume.

[46] M. Sce, *Sulle Serie de Potenzi Nei Moduli Quadratic*, Lincei Rend. Sci. Fis. Mat. et Nat. **23**, (1957), 220–225.

[47] S. Semmes, *Differentiable function theory on hypersurfaces in R^n (without bounds on their smoothness)*, Indiana Univ. Math. J. **39**, (1990), 985–1004.

[48] S. Semmes, *Analysis vs. geometry on a class of rectifiable hypersurfaces in R^n*, Indiana Univ. Math. J. **39**, (1990), 1005–1035.

[49] S. Semmes, *Chord-arc surfaces with small constant, I*, Adv. in Math. **85**, (1991), 198–223.

[50] F. Sommen, *Spherical monogenics and analytic functionals on the unit sphere*, Tokyo J. Math. **4**, (1981), 427–456.

[51] F. Sommen, *A product and an exponential function in hypercomplex function theory*, Appl. Anal. **12**, (1981), 13–26.

[52] F. Sommen, *Microfunctions with values in a Clifford algebra, II*, Scientific Papers of the College of Arts and Sciences, University of Tokyo **36**, (1986), 15–37.

[53] F. Sommen, *Plane wave decomposition of monogenic functions, Ann. Polon. Math.*, **49**, (1988), 101–114.

[54] E. M. Stein and G. Weiss, *On the theory of harmonic functions of several variables, I: The theory of H^p-spaces*, Acta Math, **103**, (1960), 25–62.

[55] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, NJ, 1971.

[56] A. Sudbery, *Quaternionic analysis*, Math. Proc. of the Cambridge Philosophical Soc., **85**, (1979), 199–225.

[57] T. Tao, *Convolution operators generated by right-monogenic and harmonic kernels*, Master's Thesis, Flinders University, Australia, 1992.

[58] K. Th. Vahlen, *Über Bewegungen und Complexe Zahlen*, Math. Ann. **55**, (1902), 585–593.

4

Problem Book

H. Blaine Lawson (State University of New York at Stonybrook)

Is there a reproducing kernel of Cauchy type for solutions to the Dirac equation over (spin) manifolds with negative curvature?

John Ryan (University of Arkansas, Fayetteville)

Suppose M is a smooth, real, n-dimensional, compact manifold in $\mathbf{C}^n$, satisfying

(i) $M \cap N(\underline{z}) = \{z\}$

(ii) $TM_z \cap N(\underline{z}) = \{\underline{z}\}$

for each $\underline{z} \in M$ ($N(\underline{z}) = \{\underline{z'} \in \mathbf{C}^n : (z - z')^2 = 0\}$—the null cone in $\mathbf{C}^n$).

Suppose Ω is a domain in $\mathbf{C}^n$ with $M \subseteq \Omega$ and $h : \Omega \to \mathbf{C}$ is a complex harmonic function so that $h(z)$ is holomorphic and

$$\sum_{j=1}^{n} \frac{\partial^2 h}{\partial z_j^2} = 0.$$

Is it true that for each such M we have that $h|_M$ satisfies a maximum principle?

Pascal Auscher (Université de Rennes, France)

It is known that the Clifford Haar-type b-wavelets for the $T(b)$ theorem can be defined when $b(x)$ is not only *accretive*, i.e.,

$$b(x) = b_0(x)e_0 - 1\sum_{i=1}^{n} b_i(x)e_i, b_0(x) \geq \delta_0 > 0, \qquad A.0$$

but *pseudo-accretive*, i.e.,

$$\left|\frac{1}{|Q|}\int_{\phi} b(x)dx\right| \geq \delta_0 > 0 \qquad A.1$$

0-8493-8481-8/96/$0.00+$.50
© 1996 by CRC Press

and even *para-accretive*: $\exists\delta_0 > 0$, $\delta_1 < 1$, for each cube Q, there exists a sub-cube R such that

$$\left|\frac{1}{|R|}\int_R b(x)dx\right| \geq \delta_0 > 0 \quad\text{and}\quad |R| > \delta_1|Q|. \qquad A.2$$

Smooth Clifford b-wavelets can be obtained under conditon (A.0) with arbitrary high regularity.

Smooth Clifford b-wavelets can also be obtained under condition (A.1), but the regularity seems to be related to the smallness of δ_0: the smaller δ_0, the smaller r (r is then the Hölder regularity of the b-wavelets).

Problem 1: What is the exact relation between regularity of the b-wavelets and δ_0 of condition (A.1)?

Problem 2: Construct smooth Clifford b-wavelets under para-accretivity condition (A.2).

T. Tao (Princeton University)

Let Ω be an open subset of $\mathbf{R}^m$, and let u be a scalar-valued bounded harmonic function on Ω. Does there always exist a bi-monogenic function f on Ω such that $[f]_0 = u$ (i.e., the scalar part of f is u) when

(a) Ω is a sphere, or a rectangular box? (Answer: Yes, explicit condition possible)

(b) Ω is a bounded star-like region?

(c) Any generalizations (e.g., Ω with null m^{th} homotopy group)?

Also, can f always be chosen so that $Range(f) \subseteq span(e_0, e_1, \ldots, e_m)$?

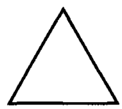
(Editor's comment: When Ω is star-shaped, one can construct a left-monogenic function whose real part is u. See, for instance: A. Sudbery Quaternionic Analysis, Math. Proc. of the Cambridge Philosophical Society **86** *(1979) 199–225, or J. Ryan Complexified Clifford analysis, Complex Variables* **1** *(1982) 151–171. Also, where Ω is a Lipschitz domain the answer is yes. See M. Mitrea 'Clifford algebras and boundary estimates for harmonic functions' Clifford Algebra and their Applications in Mathematical Physics, ed by F. Brackx, R. Delanghe and H. Serras, Kluwer, 1993.)*

Palle E.T. Jorgensen (University of Iowa)

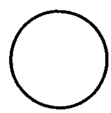
Let $\Omega \subset \mathbf{R}^n$, $n > 1$, be open and bounded, and let $D = \sum_1^n E_j \frac{\partial}{\partial x_j}$ be the corresponding Dirac/Clifford operator, acting on vector functions which are C^∞ and compactly supported in Ω. Then D is a symmetric Hilbert space operator with dense domain and corresponding adjoint D^*. Give a geometric description of the domain of D^*. In Jorgensen's talk, a selfadjoint extension, D_A, was described, $D \subset D_A \subset D^*$. Find the spectrum of D_A, and relate it directly to the geometry of Ω.

Let the operator D_A be defined as above, and associated with some $\Omega \subset \mathbf{R}^n$. Suppose $n = 2$, and Ω is one of the following:

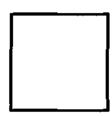
Ex. 1.



Ex. 2.



Ex. 3.



Then answer the problems for these special cases. For Ex. 3, $\left\{\frac{1}{r-1}\frac{\partial}{\partial x_j}\right\}$, $j = 1, 2$ (two symmetric operators in $L^2(\Omega)$), have commuting selfadjoint extension operators in the scalar space $L^2(\Omega)$. Relate the joint spectrum of these to the spectrum of D_A.

Nikolai Vasilevski and Michael Shapiro
(ESFM del I.P.N., Mexico City, Mexico)

Let Ω be a domain in $\mathbf{H}$, the quaternions. Also, let

$$(Df)(x) = \sum_{k=0}^{3} \frac{\partial f}{\partial t} + i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$(D^{\psi} f)(x) = \sum_{k=0}^{3} \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k$$

for $f : \Omega \to \mathbf{H}$. Introduce the Bergman kernel function of the domain Ω

$$B(x,\xi) = \left(D_\xi \bar{D}_x g\right)(x,\xi) = D_x \cdot D_\xi g(x,\xi),$$

where $g(x,\xi)$ is the classical Green function of Ω, and the Bergman operator

$$(Bf)(x) = \int_\Omega B(x,\xi) f(\xi) d\xi : L_2(\Omega) \to L_2(\Omega).$$

Let $X_0(x)$ be a characteristic function at a subdomain Ω_0 of Ω.

Problem: Calculate $\Delta = sp\,(B - X_0(x)I)^2$.

This problem is an essential step for describing various algebras generated by Bergman operators. It is known that $\Delta \subset [0,1]$, and *most probably*: $\Delta = [0,1]$.

Each $f : \Omega \to \mathbf{H}$ can be represented in the "complex" form

$$f = f^{(1)} + f^{(2)} j,$$

where

$$f^{(k)} : \Omega \subset \mathbf{R}^4 = \mathbf{C}^2 \to \mathbf{C}.$$

We have

$$D = 2\left(\frac{\partial}{\partial \bar{z}_2} + k \frac{\partial}{\partial \bar{z}_2}\right),$$

and thus the set of hyperholomorphic functions contains (but does not coincide with) the set of all holomorphic (in the sense of complex analysis) mappings $f = (f^{(1)}, f^{(2)}) : \Omega \subset \mathbf{C}^2 \to \mathbf{C}$.

Problem: Find connections between the Bergman function $B(x,\xi)$ of a domain Ω and the two-dimensional complex analysis Bergman function of a domain Ω.

Zhijian Wu (University of Alabama, Tuscaloosa)

Let

$$A = \left\{ f : f(x,y) \text{ } monogenic \text{ } in \text{ } \mathbf{R}^{n+1}_+ : \frac{1}{4} \int\int_{\mathbf{R}^{n+1}_+} |f(x,y)|^2 dx dy < \infty \right\}.$$

A is a subspace of $L^2(\mathbf{R}^{n+1}_+)$. One can find the orthonormal projection $P : L^2(\mathbf{R}^{n+1}_+) \to A$. P can be expressed as an integral operator.

Question: Can we express $I - P$ as an integral operator?

Example: If $n = 1$, the answer for this question is Yes. In fact, if $\varphi \in C_0^\infty(\mathbf{R}_+^2)$,

$$(I - P)(\varphi)(w) = \int_{\mathbf{R}_+^2} \frac{\bar{\partial}\varphi(z) Im\, z}{(w - z)(w - \bar{z})} dxdy.$$

John Ryan (University of Arkansas, Fayetteville), and Tao Qian (University of New England, Australia)

The surfaces of star-shaped domains have global parametrizations that may allow us to solve some boundary value problems first on these domains, and then transfer the solutions to more general domains. The following question then arises: What are the domains in $\mathbf{R}^n$, $n \geq 3$, that are the ranges under conformal mappings of the star-shaped ones? When $n = 2$, simply connected domains are the ranges under conformal mappings of the star-shaped ones. In the higher-dimensional case, the only conformal mappings are the elements of the Möbius transform group, and they are not monogenic functions.

Tao Qian (University of New England, Australia)

Singular integral theory with holomorphic (monogenic) kernels has been developed on Lipschitz surfaces and Lipschitz perturbations of the n-torus (see the expository papers of A. McIntosh and T. Qian in this collection).

What is the analogue on n-dimensional solid balls? What is the analogue on n-dimensional complex balls?

Pertti Lounesto (Helsinki University of Technology, Finland)

The Maxwell equations can be condensed into one equation by Clifford bivectors (at least in an isotropic and homogeneous media). The Maxwell equations are also conformally covariant (as photons are massless). Furthermore, the solutions of sourceless Maxwell equations are monogenic. Under a Möbius transformation $x \to g(x) = (ax+b)/(cx+d)$, a monogenic function $F(x)$ is transformed as follows

$$F(x) \to G(x) = \frac{\widetilde{cx+d}}{|cx+d|^n} F(g(x)),$$

where $G(x)$ is also monogenic. However, for a bivector field $F(x)$, the transformed field $G(x)$ is not, in general, a bivector. Is it possible to transform the electromagnetic bivector field $F(x)$ under Möbius transformations so that the transformed field is also a bivector?

Klaus Gürlebeck (Technical University of Chemnitz, Germany)

It is known in complex function theory that under certain conditions a continuous function r defined on a closed curve Γ can be factorized in the following form,

$$r(t) = r_-(t)t^\kappa r_+(t)\forall t \in \Gamma, \tag{1}$$

where r_- allows a holomorphic extension into the exterior domain and r_+ has a holomorphic extension into the interior domain with boundary Γ. The value κ is an integer. If r is a rational function, one can show in a constructive way that there exists a factorization (1) with rational functions r_- and r_+.

Problem: Is it possible to find a similar factorization (explicitly?), also for functions defined on the boundary Γ of a bounded domain G, in R^n, with values in (real) Clifford algebras? The most interesting case for applications is the case of quaternionic-valued functions.

Wolfgang Sprößig (Bergakademie Freiberg Technische Universität, Freiberg, Germany)

Suppose ω is a closed, bounded rectangle in R^2. Then $\omega \times R \subseteq R^3$ is called a channel-domain. For a channel-domain with density φ we have the electric field

$$\underline{E} = \sum_{i=1}^{3} E_i e_i,$$

the magnetic field

$$\underline{H} = \sum_{i=1}^{3} H_i e_i,$$

and the electric conductivity κ, dielectric constant ε, and permeability μ. We consider in the channel the solution of the following stationary Maxwell equations:

$$div\, \varepsilon \underline{E} = 0 \qquad div\, \mu \underline{H} = 0 \qquad rot\, \underline{E} = 0 \qquad rot\, \underline{H} = \kappa \underline{E}.$$

Prove the existence, uniqueness, and regularity of the solution, if the normal components of the solution is given on the boundary of the channel; so that $\underline{s} \cdot \underline{H} = g$ for g belonging to a suitable function space.

Pertti Lounesto (Helsinki University of Technology, Finland)

Does the Clifford algebra version of the Bott periodicity theorem dictate results in Clifford analysis which vary from dimension to dimension? For instance, are there results in three dimensions which do not hold in seven dimensions, even though there might be similar results in $11 = 8+3$ dimensions?

Marius Mitrea (University of South Carolina)

Recall the generalized Hardy spaces $\mathcal{H}^p(\Omega)$ discussed by Kenig in [Ke] for Ω a (special) Lipschitz domain in the complex plane. For $1 < p < \infty$, they also have a natural (and in many respects, satisfactory) extension to higher dimensions within the Clifford algebra framework (cf., e.g., [Mi]). In this setting, establish a Riesz boundary behavior theory for the end-point case $p = 1$.

References:

[Ke] C.E. Kenig, *Weighted H^p spaces on Lipschitz domains*, Am. J. Math. **102**(1980), 129–163.

[Mi] M. Mitrea, *Clifford Wavelets, Singular Integrals, and Hardy Spaces*, Lecture Notes in Math., 1575, Springer-Verlag (1994).

Josefina Alvarez (New Mexico State University)

Let T be a Calderón-Zygmund operator in the sense of R. Coifman and Y. Meyer. That is to say, assume that the distribution kernel $k(x,y)$ of T satisfies the pointwise condition

$$|k(x,y) - k(x,z)| \leq C\frac{|y-z|^\delta}{|x-z|^{n+\delta}}$$

if $2|y-z| < |x-z|$, for some $0 < \delta \leq 1$. Concerning the continuity of the operator T on Hardy spaces, the following results are known.

(i) T maps continuously H^p into L^p for $\dfrac{n}{n+\delta} < p \leq 1$, and this result is optimal.

(ii) T maps continuously $H^{\frac{n}{n+\delta}}$ into $L^{\frac{n}{n+\delta},\infty}$.

(iii) T maps continuously $H^{p,\infty}$ into $L^{p,\infty}$, for $\dfrac{n}{n+\delta} < p \leq 1$.

Problem: Is the result in (iii) optimal? If not, what continuity result can be formulated for $p = \dfrac{n}{n+\delta}$?

Zhenyuan Xu (Ryerson Polytechnical University, Canada)

It is well known that the index plays a very important rôle in the study of boundary value problems in complex analysis. For instance, consider the

Riemann-Hilbert problem

$$\frac{\partial}{\partial \bar{z}} w = 0, \quad in\ \Omega,$$

$$R_e\left[\lambda\left(\bar{z}\right) w(z)\right] = \gamma(z), \quad on\ \Gamma = \partial\Omega,$$

where Ω is a unit disk with center at the origin. The index is defined by

$$\kappa = \frac{1}{2\pi} \Delta_{\Gamma} \arg \lambda(z).$$

Then, for $\kappa \geq 0$, the Riemann-Hilbert problem is solvable for any Hölder continuous functions $\lambda(z)$ and $\gamma(z)$. Moreover, the solution linearly depends on $2\kappa + 1$ real arbitrary constants. For $\kappa < 0$, the Riemann-Hilbert problem is not solvable, except when $\lambda(z)$ and $\gamma(z)$ satisfy $-2\kappa - 1$ consistent conditions. Is there an analogue of the index for Clifford-valued functions in R^m?

Daniel B. Dix (University of South Carolina)

I will employ the notation described in my paper.

(1) Consider a homogeneous "scalar" Dirac equation $\mathcal{D}m = T(x, v)m$, where $v(k', t)$ is a $\mathbf{C}_n$-valued function of $k' \in \mathbf{R}^n$ and $t \in \mathbf{R}$. $T(x, v)$ is a linear operator, depending parametrically on x and v, which acts on $m(x, t, k)$, and yields a distribution on $\mathbf{R}^{n+1}$, parameterized by x and t, which is an appropriate right-hand side for the inhomogeneous Dirac equation. Suppose this equation has a solution with asymptotic behavior

$$m(x,t,k)\ e_0 + \sum_{h=0}^{\infty} \left[\sum_{(l) \in n^h / S_h} W_{(l)}(k) q_{h,(l)}(x,t) \right], \quad \|k\| \to \infty, |k_0| > \varepsilon.$$

The coefficients $Q_{h,(l)}(x, t)$ of this asymptotic expansion satisfy a complicated coupled system of nonlinear evolution equations. Can simple examples of the operator $T(x, v)$ and a linear evolution of $v(k', t)$ be found such that this infinite coupled system of evolution equations effectively reduces to a coupled system involving only finitely many of the coefficients and their partial derivatives in the x variables? If so, then this would be a truly multidimensional example of a nonlinear system of partial differential equations solvable by a Clifford inverse scattering method. In such an example, how would one define the forward scattering transform?

(2) In complex analysis, we have the notion of the sheaf of holomorphic functions on a complex manifold. In particular, we can discuss holomorphic functions defined on a neighborhood of ∞ on the Riemann

sphere. Is there some analogue of this in Clifford analysis? In particular, how would one make sense out of the "sheaf of monogenic functions" on $S^{n+1} = \mathbf{R}^{n+1} \cup \{\infty\}$? Is there a monogenic analogue of (some substantial portion of) the theory of Riemann surfaces?

Stephen Semmes (Rice University, Houston, Texas)

Generalizations of Complex Analysis to Higher Codimensions

There are two particularly prominent methods for generalizing classical complex analysis in the plane to $\mathbf{R}^n$ for $n > 2$. The first is to use a Cauchy-Riemann system, like the classical Riesz system (of vector fields which are curl- and divergence-free) or Clifford holomorphicity. These are first-order linear elliptic systems of partial differential equations. The second main approach is to look at quasi-regular mappings, which are (roughly speaking) maps for which the maximal stretching of the differential at any point is bounded by a constant multiplied by the minimal stretching at that point. These two approaches are very different in style—the first is better suited for linear analysis, while the second is more geometric in focus—but they are also at opposite extremes in terms of dimensions. This point is illustrated by the following observation. Let $f(x)$ be a Clifford-holomorphic function on some domain in $\mathbf{R}^n$. For each point x_0 in the domain, the differential of f at x_0 is controlled by its restriction (as a linear mapping) to any hyperplane through the origin in $\mathbf{R}^n$. This follows from the definition of Clifford holomorphicity, which provides a formula for the derivative in any given direction v in terms of the derivatives in the remaining $n - 1$ directions. By contrast, if $f(x)$ is quasi-regular, then the differential of f at x_0 is controlled by its restriction to any line through the origin, by definition of quasi-regularity.

I like to think of Clifford analysis (and other Riesz systems) as being "codimension-1 complex analysis" on $\mathbf{R}^n$. This is also related to the usual integration formulas in Clifford analysis (like Cauchy's theorem) which involve integrals over hypersurfaces. Similarly, quasi-regular mappings define a kind of codimension-$(n-1)$ complex analysis.

Problem: Find interesting kinds of "codimension-d" complex analysis on $\mathbf{R}^n$ for other choices of d.

It is not clear exactly what this means, but there are some basic principles. By definition, "complex analysis" should deal with a class of functions or mappings which are distinguished by a condition on their first derivatives. In codimension-d complex analysis, the differential of a "holomorphic" object should be controlled (somehow) by its restriction to any codimension-d-plane through the origin. One might hope for nice integral formulae, but for codimension-d submanifolds. There should be some interesting interplay between "holomorphic" objects on the complement of a d-dimensional submanifold and their boundary behavior. These principles should be viewed more as illustrative than definitive, and I am certainly not saying that there is at most one reasonable codimension-d complex analysis.

When $d > 1$, I don't believe that there is a nice codimension-d complex analysis that is based on a first-order linear system and which has roughly the same analytic features as for Riesz systems and Clifford analysis when $d = 1$. I envision two types of theories, one which is nonlinear, not so algebraic, and more brutally geometric, and another which is linear, has interesting integral formulae, and which probably uses differential forms more seriously and is more degenerate analytically when $d > 1$ than when $d = 1$. Of course, we see part of this dichotomy already in the contrast between Clifford analysis and quasi-regular mappings in $\mathbf{R}^n$ when $n > 2$.

One reason for raising the issure of codimension-d complex analysis is that I would like to have a nice higher-codimension version of some of the results in [DS], [S1], [S2], and [S3]. In particular, in [S1] and [S2] there are some integration-by-parts computations which are used to good advantage, and I would like to have suitable extensions of these arguments to higher codimensions. In these extensions, there should be integral formulae with topological content, just as the Cauchy formula in codimension-1 contains the information of when a point lies in a given domain or its exterior. In the higher-codimension case, the analogous topological issue is the linking number of pairs of spheres (and other submanifolds). There are, of course, classical integral formulae for computing linking numbers (see [F], especially p.79ff), but I have never managed to use them to obtain interesting analytic information, as occurs in codimension-1 in [S1] and [S2].

I am not convinced that singular integral operators will have such an important role in higher-codimension (linear) complex analysis as in codimension-1. I am more optimistic about approximations to the identity and square functions estimates. In codimension-1 complex analysis, there are some interesting approximations to the identity on hypersurfaces which are built out of the Cauchy kernel and which contain interesting geometric information about the surface (see [S3], especially (1.0) on p.1010). One can imagine analogous objects in higher codimensions, using differential forms and Clifford algebras. These higher-dimensional analogues could be associated to $(d-1)$-spheres which link a given codimension-d submanifold (on which we are to have the approximation to the identity), just as in the codimension-1 case the approximation to the identity is defined using pairs of points which lie in different components of the complement of the hypersurface (i.e., linking 0-spheres). Unfortunately, I have not succeeded in producing anything that I can work with analytically. See [DS, Sect.8, Ch.3, Part III] for different remarks on the same general topic.

References

[DS] G. David and S. Semmes, "Analysis of and on Uniformly Rectifiable Sets", *Mathematical Surveys and Monographs* **38**(1993), Am. Math. Soc.

[F] H. Flanders, *Differential Forms, with Applications to the Physical Sciences*, Academic Press, 1963.

[S1] S. Semmes, A criterion for the boundedness of singular integrals on hypersurfaces, *Trans. Am. Math. Soc.* **311**(1989), pp.501–513.

[S2] S. Semmes, Differentiable function theory on hypersurfaces in $\mathbf{R}^n$ (without bounds on their smoothness), *Ind. Math. J.* **39**(1990), 985–1004.

[S3] S. Semmes, Analysis vs. geometry on a class of rectifiable hypersurfaces in $\mathbf{R}^n$, *Ind. Math. J.* **39**(1990), pp.1005–1035.

Rodolfo H. Torres (University of Michigan at Ann Arbor) and Grant Welland (University of Missouri at Saint Louis)

Let D be a bounded Lipschitz domain in R^3, and let N be the unit outward normal to the boundary of the domain. Let k be a complex number, and let $\Phi(X) = -\dfrac{e^{ik|X|}}{4\pi|X|}$ be the fundamental solution for the Helmholtz operator $\Delta + k^2 I$.

Problem: Find the spectrum in $L^2_T(\partial D)$ (square integrable vectors fields on ∂D) of the operator M given by

$$MF(P) = p.v. \int_{\partial D} N(P) \times curl\,(\Phi(P-Q)F(Q))\, d\sigma(Q).$$

This operator arises in the study of boundary value problems for Maxwell equations using the method of layer potentials. The knowledge of the spectrum will be important in solving some transmission problems. For transmission problems in Lipschitz domains, see, for example,

(1) L. Escauriaza, E. Fabes, and G. Verchota, "On a regularity theorem for weak solutions to transmission problems with internal Lipschitz boundaries", *Proc. Am. Math. Soc.* **115**(1992), pp.1069–1076.

(2) M. Mitrea, R. Torres, and G. Welland, "Regularity and approximation results for the Maxwell problem in C^1 and Lipschitz domains", in this proceedings.

(3) R. Torres, "A transmission problem in the scattering of electromagnetic waves by a penetrable object", preprint.