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# AN EXAMINATION OF LOGICAL POSITIVISM

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# PREFACE

I wish to express my gratitude to the members of the Sage School of Philosophy, especially Professors Burtt, Sabine, and Church, for their kindness in making many helpful suggestions and criticisms. I am also indebted to Dr. Rudolf Carnap for the explanation of several difficult points of logical syntax, and because he has allowed me to read an unpublished manuscript on the subject of meaning and verification. Finally, I wish to thank Professor Henry Bittermann, of Ohio State University, for reading the manuscript, and Mr. Manley Thompson, Jr., for reading the proof and preparing the indices.

J. R. W.

*July, 1936.*

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# AN EXAMINATION OF LOGICAL POSITIVISM

## INTRODUCTION

### I

The philosophy of the Viennese Circle has undergone so many radical changes since its formal organization in 1928 that a statement of its aims is likely to include too much, or to omit too much to be very informative. Nevertheless, the first statement of its official programme is sufficiently general to avoid misunderstanding.

In this official statement [1] the principal aims are set out as follows: first to provide a secure foundation for the sciences, and second to demonstrate the meaninglessness of all metaphysics. The method used to realize these aims is the logical analysis of all concepts and propositions. There have been other philosophical movements devoted to similar purposes, among which nineteenth century positivism and pragmatism may be mentioned. Likewise other philosophical movements have exclusively employed logical analysis to demonstrate their doctrines. The various contemporary realistic philosophies are specific instances. The unique characteristic of the philosophy of the Viennese Circle is the exclusive use of logical analysis to demonstrate positivistic theses.

The two most fundamental doctrines of Logical Positivism are (1) that propositions of existential import have an exclusively empirical reference, and (2) that this empirical reference can be conclusively shown by logical analysis. The empiristic doctrine is thus to be proved by a logical method. This calls for an account of logic consistent with

[1] "Wissenschaftliche Weltauffassung," *Der Wiener Kreis*, Wien, 1929, s. 15 ff.

such a thorough-going empiricism. At first sight this would seem to present a difficulty. Philosophical systems which employ logical methods almost exclusively would undoubtedly be expected to produce non-empirical results. If, however, logic is taken simply as a method of connecting meanings it is not difficult to reconcile logical methods with empirical results. If logical formulæ, in other words, assert nothing about the meanings of propositions, but simply show how such meanings are connected, then an empiricism based on a logical analysis of meanings is not inconsistent. This is what the Logical Positivists have attempted to do.

Their work then naturally falls into two parts: the foundations of a scientific method free from metaphysics, and the elimination of pseudo-concepts introduced by metaphysics into science and philosophy.

The distrust of metaphysics is almost as old as metaphysics itself. The rise of the great schools of ancient philosophy was followed by critical reactions in the form of scepticism. In the Middle Ages, although outspoken criticism of official metaphysics was prohibited, there is no doubt whatever of the widespread existence of radical anti-metaphysical movements in the schools. In modern times each great philosophic system has had an equally great critic. It is beyond the compass of this work to chronicle the history of these various attempts to demolish the great metaphysical systems. Nevertheless, some of the more prominent of them should be mentioned because they constitute the intellectual heritage of the contemporary form of positivism, with which this study is concerned.

In his criticial work Hume is the first great positivist. It was he no less than Kant who was responsible for the death-blow to deductive metaphysics. The discovery that the sphere of deductive reasoning is closed to statements about matters of fact (because deduction is no more than a complicated transformation occurring solely within the sphere of concepts) was known long before Hume wrote his great work. Hume's virtue was the thoroughgoing and relentless application of the discovery to all forms of abstract reasoning and, in particular, to metaphysics. Furthermore, the attempted reduction of statements about matters of fact to statements solely concerned with experience was the second great preparation that Hume made for the subsequent development of positivism. On the critical side, therefore, Hume is the positivist *par excellence*.[1] Nevertheless, he cannot be considered positivistic in the present sense of the term, since he seemed frequently to assume the existence of the trans-empirical world and to justify this assumption on the grounds of belief. It is not difficult to show that a thoroughgoing application of Hume's principles need not lead to scepticism. If scepticism is not the inevitable consequence of Hume's critical principles, then there is no *necessity* of introducing belief. On the other hand, the idea of belief as a method of description or explanation leads to as much metaphysics as before, so that there is no appreciable advance.[2] The use of psychological analysis, particularly in the case of belief, but elsewhere as well, is what principally distinguishes Hume from his positivistic descendants, just as the logical analyses of his works form the connecting link between them.[3] Many, if not all, of the principal doctrines of contemporary positivism derive from Hume. In almost all respects Hume is intellectually closer to the philosophy of the Viennese Circle than is the author of the *Cours de la Philosophie Positive*. The empiristic trend of Logical Positivism may safely be traced, I believe, to Hume. The logical foundation of Positivism must, nevertheless, be distinguished from the particular logical method employed to establish Hume's foundation. It is, therefore,

[1] Cf. "Wissenschaftliche Weltauffassung," op. cit., p. 17, apropos the two sources of the errors of metaphysics with Hume's *Enquiry concerning Human Understanding*, sec. xii, part iii.

[2] Hume realized this difficulty very clearly, but did not, it seems to me, take the obvious way out of it. See Appendix to his *Treatise of Human Nature*.

[3] Thus, for example, where Hume traced every idea to a corresponding impression or group of impressions, Logical Positivists reduce the meaning of every proposition to atomic facts.

necessary to look to Leibniz for the requisite methods. The structure of the propositions of logic has been thoroughly analysed only in recent times. Leibniz made the first great advance in this direction. He clearly distinguished between truths of reason and truths of fact, and he emphasized one fundamental property of logical truths. This was the doctrine that analytic propositions were unconditionally true because they possessed a certain formal property, namely the fact that the predicate of every analytic proposition can be shown to be contained in the subject. The division of propositions into truths of reason and truths of fact, together with the first approximation to an analysis of the former, places Leibniz among the sources of Logical Positivism. This, however, is not to say that Leibniz and the positivists have used this discovery for the same purposes, or even to say that the discovery is interpreted in the same way. The positivists are simply indebted to Leibniz for this discovery and for the use of logical analysis in philosophy.

At first sight it seems strange to include Immanuel Kant among the precursors of contemporary positivism. The point of departure of the Critical Philosophy is a reaction against the extravagances of empiricism as well as those of rationalism, the method of investigation adopted in the exposition of the system is far from empirical (being a composition of analytic and *a priori* synthetic reasoning, the former predominating [1] at times), and thus many of the results of the Critical Philosophy are completely in conflict with anything that might be called positivism. On the other hand, the presence of an *a priori* method in the system connects it with similar methods in contemporary positivism. These are details, however, and obscure the more intimate connection between positivism and the Critical system.

The real links are to be found in the common aims of the two philosophies. Both positivism and criticism desire to render the foundations of mathematics and natural

[1] Cf. Vaihinger, H., *Kommentar Zu Kant's Kritik*, pp. 412 *et seq.*

science absolutely secure and free from extraneous elements of a metaphysical character. Both positivism and criticism reject transempirical and deductive metaphysics. There is little doubt that the Kantian philosophy played its part in the development of positivism. Indeed, the famous Kantian refutation of the proofs for the existence of God is, in the main, in the exact spirit of contemporary positivistic thought.

Another aspect of the Kantian philosophy may have been of even greater influence than its anti-metaphysical direction. I refer to the phenomenalistic tendencies which characterized the second edition of the *Critique of Pure Reason*. Even though Kant was never a complete phenomenalist, the tendencies which guided the second edition were explicitly used by such positivistic thinkers as Avenarius and, in our own time, Schlick. From the logical point of view a strict adherence to phenomenalism on the terms of Kant's philosophy should lead to an abandonment of the transcendental object, on the one hand, and of the transcendental unity of apperception on the other. The phenomenalism thus produced is a close approximation to positivism in the contemporary sense of the word.

Hume, Leibniz, and Kant may, therefore, be regarded as the most influential precursors of positivistic thought. To these philosophers should be added those of the French Enlightenment. Generally speaking, the methods of the Enlightenment were almost as uncritically dogmatic as those of Continental rationalism and whatever of scholasticism remained. The general spirit of the Enlightenment was not genuinely anti-metaphysical and experimental, and is, therefore, not to be considered as a preparation for the positivism of the present day in any direct sense. This much, however, can be said. The reaction against the persistent elements of dogmatic theology and psychology that had characterized Continental thought probably paved the way for a more profound reaction against all metaphysics, including that of the Enlightenment itself, and so made a genuine appreciation of the experimental method possible. It was, then, not so much what the philosophers of the Enlightenment said or believed, but rather what they did, that provided the possibility of a positivistic philosophy.

The beginnings of positivism, as far as the philosophies which go by that name are concerned, are found in the first half of the nineteenth century. Three names stand out here as fundamentally important : Cournot, Germain, and Comte. The briefest outline of their essential methods and results will suffice to show what relationship these bear to the contemporary positivism of the Viennese Circle. From what immediately follows it will be clear that the connection is not as direct and intimate as might be supposed.

The Positive Philosophy of Comte has three distinctive marks, only one of which is retained in the contemporary doctrine of the Viennese Circle. It has, above all, a practical turn, that is the value of knowledge for human concerns is the principal criterion of what is truly scientific. Next in emphasis the historical approach to problems of thought is given primary importance. Finally the omnipotence of empirical method is asserted and defended with great force. Only this last-named characteristic constitutes a bond of connection with Logical Positivism. This connection, it must be admitted, is not very great.

Metaphysical assertions are regarded as hypotheses, somewhat less tangible than the hypotheses of theology. They have led to insoluble problems, and thus have outlived their period of utility. They are to be rejected on this ground. Such is the attitude toward metaphysics taken by the Positive Philosophy of Comte. The result is naturally a restriction of hypotheses to the empirically verifiable realm. Logical Positivism rejects metaphysics for quite another reason. Metaphysical assertions are not simply useless or indemonstrable ; they are nonsensical. There is no question of a significant assertion which cannot be verified. Utility does not concern the contemporary positivist of the Viennese Circle in any way. Hence the attitude toward metaphysics taken by Comte and that taken by the Viennese are by no means identical. There is, however, no great leap from the one to the other. "Do not concern yourself with assertions that are indemonstrable for such a concern is useless," leads quite easily to "Do not concern yourself with indemonstrable assertions, for such a concern is as senseless as the assertions which occasion it." There is a natural development from Comte to the Viennese Circle, but this development was not the one actually followed. With the exception of Mach and a very few others, most of the nineteenth century positivists regarded metaphysics as a set of significant but indemonstrable and useless assertions.

The contemporary form of positivism has little use for the historical analysis of thought forms which plays so great a role in Comte's system. For Comte something was to be learned about concepts from their historical setting. The Viennese Circle, however, prefers to treat all concepts on the same level. Herein lies another essential difference. The sole criterion of the value of concepts is, for the Viennese Circle, that of logical significance. Comte, on the other hand, seemed to give metaphysics a significant position in the development of thought.

The only direct connection between the older and the contemporary positivism is the insistence on empirical method as the sole source of truth. As we have observed, the reason for this faith differs in the two cases. The latter group alone claims a purely logical ground for its insistence on the primacy of empirical data.

The subsequent history of positivistic thought, so far as it concerns the present study, involves the development of empiricism in England and the scientific philosophers of Germany. John Stuart Mill and Herbert Spencer are usually reckoned as being in the direct line of descent from Comte. Of these two, Mill alone was a fairly consistent empiricist, but his psychologism in logical theory together with the tinge of realism which colours his ideas concerning matter are radical departures from Comte as well as off the direct historical path leading to Logical Positivism.

In order to take up the course of development of this school it is best to turn to the work of Mach, Avenarius, Hertz, and Popper-Lynkeus. (There are many others, but these are the most significant names.)

The work of these thinkers consisted largely in an attempt to relieve physics and psychology from metaphysics, Mach [1] in particular attempted to show how the absolute conceptions of physical reality were unverifiable and quite dispensable in physical inquiry. In this way the metaphysical idea of an objective world was vigorously attacked. At the same time Mach wanted to remove all metaphysics from psychology by showing that the psychological (or subjective) world as well as the objective physical world could be regarded as derivatives of the neutral elements of experience.

The chief difficulty with Mach's views seems to have been that his claim to neutrality in regard to metaphysical issues was not completely justified. There are, in fact, two aspects of his argument against metaphysics which are difficult, if not impossible, to disentangle. First there are the logical objections to the assumption of the existence of objects which, by definition, lie beyond the range of scientific investigation; for example, absolute space, material substance, and the ego. Certain purely logical and methodological objections against such concepts could be presented with great force without any alternative hypotheses for the explanation of the world. Mach sometimes argued in this simple logical way. But, in the second place, he suggested that all the metaphysical explanations of the concepts just mentioned could be replaced by the hypotheses of diverse arrangements of the neutral elements of experience. This somehow gave the impression that the logical objections to metaphysical ideas in science rested on the empirical hypothesis of "elements". This impression, rightly or

[1] *Beiträge zur Analyse der Empfindungen*, Jena, 1886.

wrongly conceived, was responsible in large part for the unfavourable reception of Mach's views. It certainly had less telling effect on metaphysics than if he had argued against metaphysical ideas on purely logical grounds.

Aside from this Mach is more closely allied with Logical Positivism than perhaps any previous thinker. If we try to discover the reason for the failure of Mach and his positivistic contemporaries to construct a consistent and absolutely neutral scientific philosophy, we must remember that the logical theory in the last quarter of the nineteenth century had not advanced to the point necessary to provide a logical method of antimetaphysical thought. There were, in the last decade of the century, several contributions to logic of the utmost importance (those of Pierce, Frege, Peano, Schroeder), but the influence of these contributions was not felt, and so it is not surprising that positivism did not make much headway among philosophers.

The contemporary positivistic thought of the Viennese Circle is a combination of the empiricism of the nineteenth century and the logical methods developed since that time. This combination makes it possible for positivists to adopt a genuinely neutral point of view and to avoid making decisions about matters which are beyond the scope of analysis. In this way their conclusions rest upon purely logical analyses and are not vitiated by unjustified pronouncements about empirical questions which philosophy cannot answer.

The empiricism of Hume, the logical methods of Leibniz, the critique of metaphysics by Hume and Kant, and finally the anti-metaphysical doctrines of Mach in respect of physics and psychology, prepared the way for Logical Positivism in so far as these systems contained the general methods and results. The specific logical technique has a somewhat different history, for the most part unrelated to the great philosophical systems.

Discounting the early investigations in symbolic logic, the first important studies were made by George Boole and

Augustus de Morgan in the first half of the nineteenth century. These investigations were followed by the development of the logic of relations by Charles Pierce and Ernst Schroeder. The logical foundation of mathematics was investigated about the same time by Giuseppe Peano and Gottlob Frege. The results of these labours were, I believe, the beginnings of a radically new method in philosophy.

Leibniz hoped for the time when two philosophers would "calculate" rather than discuss the outcome of a philosophical issue. The creation of a new method in logic seemed to realize his dream. The principal result of the work in symbolic logic in the nineteenth century was, in fact, applicable to many philosophical problems. I shall mention only a few of them here.

While many philosophers had clearly realized that propositional form and propositional inference frequently have relational structure (in contradistinction to the subject-predicate structure of the traditional form), the rigorous treatment of relational forms in terms of a logical calculus was not possible until Pierce's and Schroeder's work had been done. In so far as relational inference and the relational structure of propositions have any bearing on philosophical problems, this work was of fundamental significance.

Frege's treatment of the class-concept provided a theory of logic which seemed to demonstrate the nominalism of earlier empiristic philosophies. Now it is one thing to provide arguments of a philosophical or psychological nature for nominalism, and another to provide a demonstration of this doctrine in a rigorous mathematical way. Frege, and later Russell, attempted and, to some extent, succeeded in the latter. The elimination of abstract universals is an essential part of Frege's doctrine.

Finally, although the complete elucidation of the nature of analytic propositions was not accomplished until Wittgenstein's work on this subject was written, the formulation of logic and mathematics in symbolic notation made it possible to show (1) that all analytic reasoning falls into a system, and (2) that all analytic reasoning has some common element of structure. Leibniz and Kant had clearly explained the nature of analytic propositions of the subject-predicate form. Their explanation is correct for those propositions, but for propositions of relational form it has no significance. A complete explanation of the nature of an analytic proposition should apply to analytic propositions of whatever form. It was necessary to exhibit all these forms and the deductive principles which govern them before the essential analytic property common to all could be shown.

In general the development of symbolic logic provided a new method of investigation in philosophy and justified in part Hume's nominalistic empiricism and the Leibnizian distinction between truths of reason and truths of fact.

## II

The development of logic, especially the logic of relations, was responsible for the rise of present positivistic tendencies. The logical theories of Frege and Russell, culminating in the *Principia Mathematica*, are the principal source of the methodology of Logical Positivism. I shall, therefore, devote some space to Russell's doctrines as given in the *Principia Mathematica*.

The principal business of the *Principia Mathematica* is the deduction of logic and pure mathematics from a small number of premises all presumably of logical character. A premise is of logical character, according to Russell, if it is unconditionally true ; more exactly if it is always true, absolutely general, and contains no constants except logical ones. A logical constant is one of the following : "and," "or," "implies," "equivalence," "not both," "neither-nor," "not," "all," "there exists," etc.

Russell assumes certain ideas as undefined (not indefinable) within the limits of his system. These ideas are introduced in connection with the groups of axioms which determine their uses.

The calculus of propositions requires three undefined ideas, "proposition," "negation," "disjunction," symbolized by means of a kind of notation which is now in general use. The following dictionary will explain the essentials of this notation.

$p, q, r \ldots$ stand for propositions.

v is a binary operation involving two (or more) terms.

Thus "$p \vee q$" means "at least one of the two propositions is true".

$\sim$ is a unary operation involving one term or one complex of terms.

Thus $\sim p$ means "$p$ is false" and $\sim (p \vee q)$ means "$p$ or $q$ is false".

All the logical constants (excepting generalization and particularization) are definable in terms of "$\sim$" and "v". Thus $p$ and $q$, $p$ implies $q$, etc.:

$$p . q = df. \quad \sim ( \sim p \vee \sim q)$$

["$p$ and $q$ are true" is the same as saying "It is false that $p$ is false or $q$ is false."]

$$p \supset q = df. \ \sim p \vee q = \sim (p \, . \sim q)$$

["If $p$ then $q$" is the same as saying "Either $p$ is false or $q$ is true" which is the same as "It is false that $p$ is true and $q$ is false."] This may be a strange definition for implication but it serves the two purposes for which implication is used in logic and science. In logic implication is used in the proof of proposition as follows: If $A$ is a truth of logic, and if $A \supset B$, then $B$ is a truth of logic. It is necessary only that the truth of the antecedent depend on the truth of the consequent in order that the implication obtain.

In material inference we say if $A$ is factually true and if $A \supset B$ is factually true, then $B$ is factually true. This fails only when $A$ is true and $B$ is false. The *Principia* definition of "$\supset$" thus serves these purposes well.

$$p \equiv q = df. \ \ p \supset q . p \supset q$$

["$p$ and $q$ are equivalent" means "$p$ implies $q$, and $q$ implies $p$". Equivalent propositions are, in general, not identical. If $p = q$ then obviously $p \equiv q$, but if $p \equiv q$ we cannot generally infer $p = q$.]

With this dictionary we can state a set of axioms from which all the recognized principles of the logic of propositions may be deduced.

The Russell set of axioms (primitive propositions) is as follows :—

1. $p \vee p . \supset . p$ (tautology)
2. $p . \supset . p \vee q$ (addition)
3. $p \vee q . \supset . q \vee p$ (permutation)
4. $p \vee (q \vee r) . \supset . q \vee (p \vee r)$ (association)
5. $p \supset q : \supset . (p \vee r) \supset (q \vee r)$ (summation)

Some of these axioms are not independent, so we may use a set of three axioms.[1]

1. $p . \supset . \sim p \supset q$ which is the same as 2 above.
2. $\sim p \supset p . \supset . p$ which is the same as 1 above.
3. $p \supset q : \supset . q \supset r . \supset . p \supset r$ which is the same as 5 above.

For the purpose of proof, some nonformal axioms are required. It is not necessary to give them all here. The rules of inference and substitution suffice to show the character of such axioms.

Substitution. In any formula a symbol $p$ may be replaced by a symbol $q$ if the replacement is complete. E.g. $p . \supset . \sim p \supset q(\frac{p}{q}) = p . \supset . \sim p \supset p([``\frac{p}{q}"] =$ " replacing $q$ by $p$ ").

Inference. If $A$ is a truth of logic, and if $A \supset B$ is a truth of logic, then $B$ is a truth of logic. E.g. (1), (2), (3) are truths of logic and (1), (2), (3) $. \supset . p \supset p$ ; therefore $p \supset p$ is a truth of logic.

All logical proof for the logic of propositions proceeds by the application of such non-formal rules to the axioms. It is not necessary to give an example of such a proof because the procedure is self-explanatory.

It is clear that a system of axioms could be arranged in

[1] Lukasiewicz, J., Bernays, P., and Nicod, J., have proved this.

such a way as to generate all the consequent theorems mechanically. This is, of course, the ideal of procedure in logic.

The logic of functions of one or more variables ntroduces many new primitive ideas. A propositional form is often asserted to be true for all or some instances satisfying that form. Thus " all $x$ has $\phi$ " and " some $x$ has $\phi$ ". This involves the ideas of (1) a propositional function, (2) the quantifiers " $(\exists x)$ " . . ., " $(x)$ " . . ., (3) the apparent variable, and (4) the logical type.

A propositional function of one variable is a formula with one undetermined constituent such that an admissible determination of the constituent in question produces a proposition. Thus " $\phi\hat{x}$ ", when $a$ is inserted in the " $\hat{x}$ ", becomes " $\phi a$ ", which is a complete proposition.

In order to assert that all values of the " $\hat{x}$ " are true values of the function, the universal quantifier " $(x)$ . . . " must be introduced. Thus " $(x)\phi x$ " means " every $x$ has $\phi$ " where $\phi$ is any predicate of $x$. Similarly the assertion that at least one value satisfies $\phi\hat{x}$ involves the particular quantifier, " $(\exists x)$ . . ." Thus " $(\exists x)\phi x$ " means " at least one $x$ has $\phi$ ".

The " $x$ " in " $\phi x$ " is a real variable since the scope of $x$'s variation is not limited. However, in " $(x)$ . . ." or in " $(\exists x)$ . . ." the scope of $x$'s variation is fixed by the prefixes " $(x)$ . . ." or (" $\exists x)$ . . ." Here $x$ varies only apparently, and therefore is an " apparent variable ".

The use of the idea of a propositional function involves a limitation on the kind of values which the function can assume. Thus " man $(\hat{x})$ ", i.e. " $\hat{x}$ is a man " cannot assume any arbitrary kind of entity as a value. Three distinct cases arise ; (1) the value of a function may be a false proposition ; (2) the value of a function may be a true proposition ; (3) the function, when certain things are inserted as values, may become nonsense. In order to avoid case (3) certain rules must be introduced so as to limit the significance of propositional functions.

These rules constitute the theory of logical types. A logical type is the range of significance of a propositional function, i.e. the range of values of "$\phi\hat{x}$" for which "$\phi x \vee \sim \phi x$" obtains. The theory of types can be stated as follows: Under all circumstances it must be assumed to be non-significant:—

(1) for a function to be a value of itself or for a function not to be a value of itself. E.g. "$\phi(\phi)$" and "$\sim \phi(\phi)$" are both nonsense.

(2) for a function to be a value of another function, both of which have the same kind of object for values.

(3) for a function to be a value of a function of a higher order than itself when the degree of difference in order is greater than one.

(4) for a function to be a value of a function of lower order than itself.

It is possible to show that a vicious-circle paradox can be deduced from any violation of the typal rules. The most familiar of these paradoxes is the one about impredicable functions. If it is asserted that a function is not a value of itself, such functions as are not self-applicable may be defined as follows:—

Impr. $(F) . \equiv \sim F(F)$, i.e. "a function $F$ is impredicable" is equivalent to "$F$ is not $F$". Now either $F(F)$ or $\sim F(F)$ for every $F$ (by the law of excluded middle). Hence Impr. $(F)$ or $\sim$ Impr. $(F)$. Substitute Impr. for $F$, i.e. Impr. (Impr.). But *ex hypothesi* Impr. (Impr.) . $\equiv$ . $\sim$ Impr. (Impr.). This is a contradiction. There is only one way to eliminate such a contradiction, and that is to assume that both $F(F)$ and $\sim F(F)$ are nonsense. This limits the possible values of functions and thus the theory of types results.[1]

Now the theory of types has two parts, which are usually called the simple theory and the extended theory. The simple theory states that it is non-significant to suppose that a function can or cannot be a value of its own argument.

[1] Cf. Carnap, Rudolf, "Die Antinomien und die Unvollständigkeit der Mathematik," *Monatshefte für Mathematik und Physik*, Leipzig, 1934, 41. Band, 2 Heft, p. 264.

A hierarchy of types results from the theory. In order to understand this hierarchy it is necessary to introduce the notion of classes and relations or incomplete symbols.

In the *Principia Mathematica* propositional functions are treated separately according as they contain one or more than one argument-places.

A function of one argument, $\phi\hat{x}$, determines a range of values for which it is significant. Any function which determines the same range of values as another function is said to be formally equivalent to the latter function. Any set of formally equivalent functions forms a class and, conversely, if there is a class its members are values of a propositional function or of all formally equivalent functions. A class is the extension (i.e. the range of significant values) of a propositional function and of all the functions formally equivalent to this function.

Similar considerations obtain for functions of more than one argument, e.g. $\phi(\hat{x}, \hat{y})$. The problem here is more complicated. In the first case where functions of one argument were involved it was easy to define equivalent functions. Here, however, there is a greater complexity, since for the function $\phi\ (\hat{x}, \hat{y})$ there are many more propositions resulting from generalization. There are, in fact :—

$$(x)\ (y)\ \phi(x,y)\ ;\ (x)\ (\exists y)\ \phi(x,y)$$
$$(\exists x)\ (\exists y)\ \phi(x,y)\ ;\ (\exists y)\ (x)\ \phi(x,y).$$

If $y$ is a constant, i.e. if $y = a$, then there are more propositions, viz. :—

$$(x)\phi\ (x,a)\ ;\ (\exists x)\ \phi(x,a).$$

If $x$ is a constant, i.e. if $x = b$, then $(y)\ \phi(b,y)$ ; $(\exists y)$ $\phi(b,y)$.

There are thus eight different propositions formed by generalizing on the function $\phi(\hat{x}, \hat{y})$. Functions of three, four, etc., arguments would yield even more complicated kinds of generalization.

It is clear that functions of two arguments are formally equivalent if and only if they are satisfied by the same values.

A dyadic relation is the extension of all formally equivalent functions of two arguments.

A logical type is determined by a set of all classes which have the same kinds of members or all things which are members of the same kinds of classes.

Individuals form the first type, usually designated by "$\text{type}_0$". Classes of individuals form the second type. Classes of classes of individuals form the third type, and so forth. There is thus a hierarchy of types of the following structure :—

$$t_0 \quad t_1 \quad t_2 \quad t_3$$

The structure of any functions of whatever type must, therefore, be $t_n$ $(t_n - 1)$, i.e. functions of type $t_n$ must take arguments of the next lower type and only such arguments. There will be an infinity of types and in each such infinity there will be an infinity. This is clear for the following reason. If we begin with the type of individuals we can form classes, classes of classes, and so on without end. The dyadic relation of individuals, the classes of such relations, classes of such classes, and so on constitutes another infinity. In every case, then, there will be hierarchy of types containing infinitely many types and there will be an infinity of hierarchies of this kind.[1]

$$
\begin{array}{lll}
t_n & t_{nn} & t_{nnn} \ldots \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
 & & \uparrow \\
t_2 & t_{22} & t_{222} \ldots \\
 & & \uparrow \\
t_1 & t_{11} & t_{111} \ldots \text{[2]} \\
 & t_0 &
\end{array}
$$

[1] Russell and Whitehead disallow the possibility of infinitely many hierarchies and infinitely many types, but this would seem to be an arbitrary limitation imposed by their somewhat "realistic" interpretation of the axiom of infinity.

[2] This representation of the typal hierarchy is, I fear, over simplified. It conveys the general idea, however erroneous the details may be. The principal difficulty is that I have not indicated the place of heterogeneous relations, i.e. relations among terms of different type.

By means of certain conventions classes of heterogenous type may be constructed without contradiction.

We cannot have a class consisting of individuals and classes, but we can, under certain conditions, have a class of classes of individuals and relational couples. So much for the theory of types.

What are classes ? Russell recognized that the vicious-circle paradoxes arose from two related fallacies ; (1) the supposition that a function could be a value of its own argument ; (2) the supposition that a class (the extension of a function) was somehow a thing. These fallacies are eliminable by (1) the theory of types and (2) the related theory that classes are logical fictions. According to the second theory, a class symbol can only be defined in certain uses and is thus eliminable by application of the definitions. A class-symbol is therefore an incomplete symbol which does not directly represent *anything*. It is a notational convenience which seems necessary for the construction of logic and mathematics. The strict definition of an incomplete symbol may be given in the following form :—

> A symbol is incomplete : (1) if it does not represent any constituent of the fact symbolized by the proposition in which the symbol in question occurs ; and (2) if it is theoretically replaceable by symbols which do represent constituents of the fact without altering the meaning of the original proposition.

In the nominalistic terminology it could also be said that classes are mere words. In a theoretically perfect language class-symbols would be superfluous.

An unusual paradox thus arises from Russell's treatment of classes. As being incomplete symbols, classes are theoretically eliminable from any symbolic system *salve veritate*. Yet, as being the foundation of arithmetic, the theory of classes seems to be an indispensable part of mathematical logic. This is because cardinal numbers are defined as classes of similar (i.e. bi-uniquely correspondent) classes and because ordinal numbers are defined as classes of ordinally similar relations.

It might be supposed that a functional notation could be substituted for the class notation so that the use of incomplete symbols would be avoided. This, in fact, is not the case. The latter notation requires implicit definitions which involve incomplete symbols in the very same manner as the class theory.

This would seem to entail a distinction between two kinds of incomplete symbols, namely, those which are eliminable by application of definitions, and those which are not eliminable by this method. It is then open to logicians to pursue one of two possible methods of symbolic construction. On the one hand a system may be constructed in which no implicit definitions occur. On the other hand, an extension of the theory of definition may be devised so as to allow for implicitly defined terms.

An extension of the theory of types involves a further difficulty. According to Russell, functions have orders as well as types. A function of one argument, e.g. $\phi\hat{x}$, in which no function occurs as an apparent variable, is a first order function (also called "predicative functions" or "matrices"). Functions of the second order are those in which functions of the first order occur as apparent variables. Thus, if $\phi!\hat{x}$ [1] is a first order function, $\phi_2\hat{x}$ (e.g. $(\phi!)\phi!x$) is a second order function. For example, if "$(\phi!)\ \phi!x$" means "Napoleon had all the characteristics of a great general", then any one of these properties, $\phi!$, is a predicative function of Napoleon, but the property of possessing a set of properties is not predicative. In this particular case it is clear that the second order property is reducible to the first order properties, i.e. $\phi_2 x \supset \phi_1 x$, since the range of values of $\phi_1$ is manifestly contained in that of $\phi_2$. In general, however, it is not evident that $\phi_1 x \supset \phi_2 x$, $\phi_2 x \supset \phi_1 x$, or more generally still, it is not clear that $\phi_n x \supset \phi_{n+1} x$, $\phi_{n+1} x \supset \phi_n x$. An axiom is, therefore, required to insure the formal

[1] "$\phi!\hat{x}$" means "$\phi$ is a predicative function".

equivalence of higher and lower functions. The axiom must show that :—

$$(1)\ \phi_{n+1} \equiv \phi_n, \text{ and that}$$
$$(2)\ \phi_n \equiv \phi_1.$$

The need for this axiom is evident in two places in *Principia Mathematica.* The definition of identity of individuals reads : Two individuals are said to be identical if they possess all predicative properties in common. This depends on the axiom (2) which assures us that higher order properties are reducible to predicative properties, for individuals might otherwise share all predicative properties and yet differ in higher order properties. (The axiom (2) and the definition are equivalent to Leibniz's "identity of indiscernibles".) The axiom is also required in the foundation of the theory of real numbers. This phase of the problem will not be examined here.

There are still other axioms required for the foundation of mathematics. The first of this last group is the axiom of infinity which is equivalent to the assertion that there is an existent class for every inductive cardinal number, i.e. that there are infinitely many individuals.

The second axiom is required for the theory of infinite series. It states that, given a class of distinct and existent classes,[1] there exists a class composed of at least one member of each of the aforementioned classes. This is the multiplicative axiom.

The principal axioms of the *Principia* may be grouped as follows :—

A. Formal axioms of ungeneralized propositions :—

$$p \,.\supset.\, \sim p \supset q$$
$$\sim p \supset p \,.\supset.\, p$$
$$p \supset q :\supset. q \supset r \,.\supset.\, p \supset r.$$

B. Formal axioms of propositions involving apparent variables :—

[1] Distinct classes are classes without common members; existent classes are classes with at least one member.

$(x)\ \phi x\ .\supset .\ \phi u$ (Whatever holds of all, holds of any).

$\phi u\ .\supset .\ (\exists x)\ \phi x$ (Whatever holds for one holds for some).

C. Existence axioms.

1. Axiom of Reducibility: For every function of whatever order, there is a formally equivalent predicative function.

2. Axiom of Infinity: If A is an inductive cardinal number, then the cardinal successor of A exists.

3. Multiplicative axiom: For every class of existent classes, there is a class composed of at least one member of each of the aforementioned classes.

In addition to these formal axioms there are certain non-formal rules for manipulating the system. In particular, these rules are: The rule of substitution, the rule of inference, and the theory of types.

The existence axioms of the *Principia* are not formally certifiable, i.e. are not unconditional truths of logic, and hence may only be introduced as uncertified hypotheses. They seem to be required for the foundation of mathematics, but are, nevertheless, of a character quite unlike the other axioms of the *Principia* system.

The non-formal rules are also open to objection on the ground that they are not sufficiently segregated from the formal rules, and on the ground that they, too, require a kind of justification. A reason must be given, that is, for the fact that from truths of logic we can produce truths of logic by means of substitution and inference. And thus a formalization of these rules which are non-formal in respect of the *Principia* system would seem to be required. The presence of existence axioms and of unexplained rules of manipulation are recognized defects in *Principia Mathematica*.

For the purposes of this study the four most significant aspects of the logical system of *Principia Mathematica* are:—

1. The theory of types.
2. The theory of classes.

3. The theory of definition.
4. The theory of deduction.

Together these theories make possible the construction of a logical language with the following characteristics :—

1. Every sign in the language is either an undefined sign or an abbreviation for a group of signs which are themselves undefined or lead back to a group of undefined signs (theory of definition). A defined sign can always be replaced by the signs by means of which it is defined (substitution).

2. Transformations are allowed from one group of signs, which signify in a certain way, to another group of signs which signify the same things in a different way without altering the meanings (theory of definition and deduction).

3. Signs may be introduced which serve as notational conveniences but which do not, properly speaking, signify anything (theory of classes). It is thus possible to employ signs in a language without attributing an ontological reference to them.

4. The construction of complex signs from simple ones produces a hierarchy of signs which determines the place of every function or class (theory of types and orders). This makes it possible to determine the significance of signs by the mode of their construction and to avoid contradictions. A number of philosophical problems disappear by this method, and certain other problems can be solved without any trouble simply by indicating the method in which certain ideas are formed (i.e. the way certain sign-complexes are constructed).

5. All these aforementioned characteristics of the logical language serve to provide a means by which the meaning and verification of any proposition is determinable. In this way complex problems of meaning and truth are reducible to the simplest ones. All undefined signs can be listed in this language, and all complex defined signs are derivable, in various ways, from the simple undefined signs by quite explicit rules. The application of such rules solves

the problem of meaning and truth so far as complex signs are concerned. The meaning of the simple signs and truth of propositions constructed exclusively from such simple meanings are not determinable within the *Principia Mathematica* (or any similar system).

These are the reasons why Logical Positivism adopts a method of logical analysis for the foundation of science and the elimination of metaphysics. It should be observed that psychological analysis plays absolutely no rôle in this programme. The whole procedure involves simply the application of certain logical rules.

This stands in direct contrast to the methods not only of older positivistic thought but also of most philosophical investigation of the past and present. There is either a wholly psychological approach to problems (as in Pragmatism), or an admixture of logic and psychology without a clear distinction between the two methods (as in realism) in most discussions of philosophical issues.

The logical and mathematical theory of the *Principia Mathematica* forms the basis for the logical methods pursued by the Viennese Positivists. However much it leaves in question the *Principia Mathematica* was, at the time of its publication, undoubtedly the most advanced single body of logical doctrines which had appeared.

There is much in the *Principia Mathematica* which is unquestionably neutral as regards any specifically philosophical issues. However, there are many doctrines presented there which could not possibly be fitted into a rigorously consistent empiricism.

For an example of the possibility of logical analysis of meaning, no better can be found than Russell's theory of description. Russell defined the use of descriptive phrases as follows: " The so-and-so exists (' the ' in the singular) " means " there is one and only one thing possessing the characteristic called ' so-and-so ' and that thing is identical with some specific entity." Symbolically,

$$E!\,(\iota x)\,(\phi x) = df.\ (\exists c)\ \phi x \,.\!\equiv_x\!.\, x = c$$