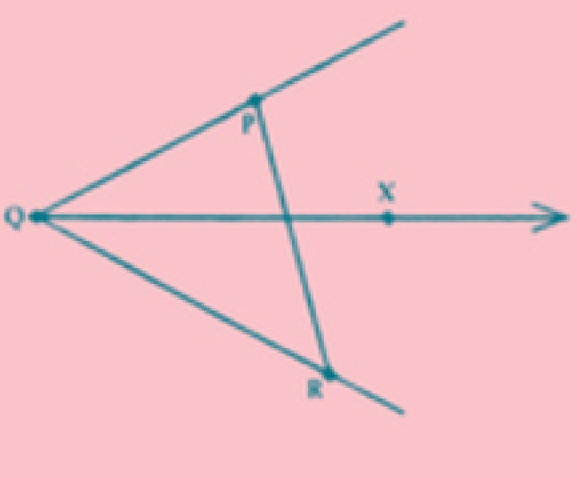
EUCLIDEAN AND NON-EUCLIDEAN GEOMETRY

AN ANALYTIC APPROACH

Patrick J. Ryan



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To my teachers
H. S. M. COXETER
and
K. NOMIZU

Contents

Preface

This book provides a rigorous treatment of the fundamentals of plane geometry: Euclidean, spherical, elliptic, and hyperbolic. It is intended primarily for upper-level undergraduate mathematics students, since they will have acquired the ability to formulate mathematical propositions precisely and to construct and understand mathematical arguments.

The formal prerequisites are minimal, and all the necessary background material is included in the appendixes. However, it is difficult to imagine a student reaching the required level of mathematical maturity without a semester of linear algebra and some familiarity with the elementary transcendental functions. A previous course in group theory is not required. Group concepts used in the text can be developed as needed.

The book serves several purposes. The most obvious one is to acquaint the student with certain geometrical facts. These are basically the classical results of plane Euclidean and non-Euclidean geometry, congruence theorems, concurrence theorems, classification of isometries, angle addition, trigonometrical formulas, and the like. As such, it provides an appropriate background for teachers of high school geometry.

A second purpose is to provide concrete and interesting realizations of concepts students have encountered or will encounter in their other mathematics courses. All vector spaces are at most three-dimensional, so students do not get bogged down in summation signs and indices. The fundamental notions of linear dependence, basis, linear transformation, determinant, inverse, eigenvalue, and eigenvector all occur in simple concrete surroundings, as do many of the principal ideas of group theory. Also, students will be in a better position to integrate geometry with topology and analysis after having worked with the projective plane and the metric space axioms.

A third purpose is to provide students not only with facts and an understanding of the structure of the classical geometries but also with an arsenal of computational techniques and a certain attitude toward geometrical investigation. They should not be concerned merely with questions of existence (e.g., whether two figures are congruent) but with questions of construction (finding the isometry relating two figures in terms of the given data). Many of the proofs and exercises take this approach. This point of view makes it clearer whether or not a student's "proof" is valid. In addition, it is more appropriate for applications in areas such as computer graphics and computer vision. Although this book does not treat such applications explicitly, the concepts and techniques used are playing an important role in these fast-growing areas of computer science. (See, for example, Foley and van Dam [14], Chapters 7 and 8, and [36].)

The fourth purpose is to provide a link between classical geometry and modern geometry, with the aim of preparing students for further study and research in group theory, Lie groups, differential geometry, topology, and mathematical physics. From this viewpoint the book is actually a study of the real two-dimensional space forms, the (flat) Euclidean plane, the sphere (constant positive curvature), and the hyperbolic plane (constant negative curvature). The isometry groups studied are Lie groups, and the notion of homogeneous space implicitly underlies much of the discussion. Although differential calculus is not used in the book, all the constructs lend themselves easily to differential-geometric treatment.

Our approach to the hyperbolic plane allows one to "do" analytic geometry there without burdensome calculations, thus removing some of the mystery of hyperbolic geometry. Familiarity with the concepts and computational techniques of hyperbolic geometry is an asset to any student of modern geometry and topology. Such central areas as Thurston's work on 3-manifolds and Penrose's work on relativity require a working knowledge of hyperbolic geometry. (See Thurston [32] and Penrose [27] for an introduction to these topics.)

In this book each of the geometries is developed separately. Each geometry has the notion of point, line, distance, perpendicularity, ray, angle, triangle, reflection, congruence, and so forth. The amount of detail with which each topic is treated varies with the setting and generally decreases as the book progresses and readers get their bearings. Some topics have been treated extensively in one setting and very briefly in another. Other topics are merely introduced in the exercises. I have tried to avoid repetition of similar arguments in different settings while allowing readers to see a good variety of methods and viewpoints. For theorems presented in a particular setting, readers should ask themselves the questions: Does the statement make sense in other settings? If so, is it true? Does the same proof work? What modifications are required? Certain unifying notions become evident (e.g., the three reflections theorem, true in all settings for all types of pencils), whereas other statements (e.g., that the perpendicular bisectors of the three sides of a triangle are concurrent) are not universally true, but an appropriate reformulation may be.

The entire book may be covered in a two-semester course. In a one-semester course I usually cover Chapter 1, part of Chapter 2, the first half of Chapter 4, Chapter 7, and those sections of Chapters 5 and 6 relevant to Chapter 7. There are numerous opportunities for excursions into areas of interest to the instructor (e.g., projective geometry, Galilean geometry, Lorentzian geometry, geometry over the complex numbers or finite fields). Some source materials for such excursions are listed in the references.

The exercises are closely related to the text material. Some of them require specific numeric computations and provide a means of testing the students' understanding of the formulas presented. Others require students to supply proofs that have been omitted in the text. Students should do a sufficient number of these so that they are confident that that they could do the others on an examination if required. The rest of the exercises extend the results of the text in some way. They can be omitted without loss of continuity. However, these are the most enjoyable exercises, and students are encouraged to work on as many of them as time permits.

I would like to thank several generations of students at the University of Notre Dame and Indiana University at South Bend, whose interest in geometry provided the impetus for developing the material and whose reactions have helped to shape the book. Thanks are also due to a long list of typists at these institutions and at McMaster University who have worked on the various preliminary versions of the manuscript as well as the final one. Finally, I am grateful to those colleagues and students who have made helpful comments on various versions of the manuscript or assisted in its preparation in other ways: in particular, Nancy Bridgeman, Michael Brown, Kristine Broadhead, Thomas Cecil, H. S. M. Coxeter, K. Nomizu, and Debra van Rie.

PATRICK J. RYAN

Notation and special symbols

General Remarks: Points are denoted by uppercase Roman letters. Lines are denoted by lowercase script letters and lowercase Greek letters. Angles are denoted by uppercase script letters. Figures are usually denoted by uppercase script letters. Some symbols are used in more than one way.

Specific Symbols:

$\mathbf{R}$	the set of real numbers
$\mathbf{R}^2$, $\mathbf{R}^3$, $\mathbf{R}^n$	the vector space of all ordered pairs (respectively triples, n-tuples) of real numbers; also called the Cartesian plane (resp. three-space, n-space)
$\mathbf{E}^2$, $\mathbf{E}^3$, $\mathbf{E}^n$	the Euclidean plane (resp. three-space, n-space)
$\mathbf{S}^2$	the two-sphere
$\mathbf{P}^2$	the projective plane
$\mathbf{H}^2$	the hyperbolic plane
$\mathbf{D}^2$	the Klein disk model of $\mathbf{H}^2$
$\mathbf{O(2)}$, $\mathbf{O(3)}$	the orthogonal group of $\mathbf{R}^2$ (resp. $\mathbf{R}^3$)
$\mathbf{SO(2)}$, $\mathbf{SO(3)}$	the special orthogonal group of $\mathbf{R}^2$ (resp. $\mathbf{R}^3$)
$\mathbf{GL(3)}$, $\mathbf{GL}(\boldsymbol{n})$	the group of all invertible 3×3 (resp. $n \times n$) matrices
$\mathbf{SL(3)}$, $\mathbf{SL}(\boldsymbol{n})$	the special linear group
$\mathbf{PGL(2)}$	the group of collineations of $\mathbf{P}^2$
$\mathbf{GAL(2)}$	the Galilean group of $\mathbf{R}^2$
$\mathbf{S}_3$	the symmetric group of permutations of three letters
$\mathbf{C}_m$	the cyclic group of order m
$\mathbf{D}_m$	the dihedral group of order $2m$
$\mathbf{AF(2)}$	the group of all affine transformations of $\mathbf{R}^2$
$\mathbf{Sim}(\mathbf{E}^2)$	the group of all similarities of $\mathbf{E}^2$
$\mathscr{T}(\mathbf{E}^2)$	the group of all translations of $\mathbf{E}^2$
$\mathscr{I}(M)$	the group of all isometries of a geometry M
$\mathscr{S}(A)$	the group of all symmetries of a set A
$\mathscr{A}\mathscr{S}(A)$	the group of all affine symmetries of a set A
$\mathrm{TRANS}(\ell)$	the group of all translations along the line ℓ

$\mathrm{REF}(\mathscr{P})$	the group generated by reflections in all lines of the pencil $\mathscr{P}$
$\mathrm{ROT}(P)$	the group of all rotations leaving P fixed
$\mathrm{DIS}(\mathscr{P})$	the group of all parallel displacements determined by the pencil $\mathscr{P}$
$\mathscr{P}$	the set of all points
$\mathscr{P}$	a pencil of lines
$\mathscr{L}$	the set of all lines
$\mathscr{F}$	a figure
$d(P, Q)$	the distance between the points P and Q
$d(P, \ell)$	the distance between P and the closest point of ℓ
$d(\ell, \mathscr{m})$	the smallest number of the form $d(P, Q)$ where $P \in \ell$ and $Q \in \mathscr{m}$
$\langle x, y \rangle$	the inner product of two vectors in $\mathbf{R}^n$
$\langle A \rangle$	the group generated by a set of elements A
$\|a\|$	the absolute value of a number a
$\|x\|$	the length of the vector x in $\mathbf{R}^n$
$[v]$	the set of all multiples of the vector v
$[a]$	the equivalence class to which a belongs
$[a, b]$	the closed interval of real numbers r satisfying $a \leq r \leq b$
$(a, b]$	the interval of real numbers r satisfying $a < r \leq b$
$[v, w]$	the vector space spanned by vectors v and w
$[G{:}H]$	the index of a subgroup H in a group G
$[P; Q \leftrightarrow R]$	the affine reflection of $\mathbf{E}^2$ that leaves P fixed while interchanging Q and R
$[P; Q \rightarrow R]$	the shear of $\mathbf{E}^2$ that leaves P fixed and sends Q to R
$[C; \ell \rightarrow \ell']$	the perspectivity with center C mapping ℓ to ℓ'
(a, b)	the interval of real numbers r satisfying $a < r < b$
(a, b)	the point of $\mathbf{R}^2$ with coordinates a and b
(PQ)	the permutation that interchanges P and Q
PQ	the segment whose end points are P and Q
(PQR)	the cyclic permutation that sends P to Q, Q to R, and R to P
$T(A)$ or TA	the set of all points of the form TP, where $P \in A$
$\overleftrightarrow{PQ}$	the line containing points P and Q
$\overrightarrow{PQ}$	the ray with origin P that contains Q
π	the natural projection determined by an equivalence relation, $\pi(a) = [a]$
π	the number (approximately equal to 3.14) that is the smallest positive number θ satisfying $\sin\theta = 0$
Π	a plane
$\alpha(t)$	parametric representation of a line
$A + B$	the set of all vectors that can be obtained by adding an

	element of A to an element of B. If $A = \{P\}$ is a singleton, $A + B$ may be written $P + B$
$x \in A$	x is an element of the set A
$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$	standard unit basis vectors for $\mathbf{R}^n$
I	the identity matrix; the identity transformation
J	the complex structure on $\mathbf{R}^2$. In fact, $J = \text{rot}(\pi/2)$
$\hat{\mathscr{F}}$	the rectilinear completion of a figure $\mathscr{F}$
$v^\perp$	in $\mathbf{R}^2$, if $v = (v_1, v_2)$, then $v^\perp = (-v_2, v_1)$
$A^\perp$	the set of all vectors orthogonal to every element of A
$\ell \perp m$	ℓ is perpendicular to m
$\ell \parallel m$	ℓ is parallel to m
$\square$	end of a proof
Ω_ℓ	reflection in the line ℓ
τ_v	translation by the vector v
ref θ	reflection in the line through the origin with direction vector $(\cos\theta, \sin\theta)$
rot θ	the rotation about the origin that takes $(1, 0)$ to $(\cos\theta, \sin\theta)$
H_P	the half-turn about P
$u \times v$	the cross product of vectors u and v in $\mathbf{R}^3$
$\measuredangle PQR$	the angle with vertex Q and sides $\overrightarrow{QP}$ and $\overrightarrow{QR}$
$\triangle PQR$	a triangle with vertices P, Q, and R
det A	the determinant of the matrix A
$\det(u, v, w)$	the determinant of the matrix whose rows are u, v, and w
$Gx = \text{Orbit}(x)$	the orbit of a point x by a group G
$G_x = \text{Stab}(x)$	the stabilizer of a point x in a group G
Σ	summation sign
$\#A$	the number of elements in a set A
$\varnothing$	the empty set
A^t	the transpose of the matrix A
$\cong$	congruence of figures
$\cong$	isomorphism of groups

Theorem Numbers: When theorems are referenced outside of the chapter in which they occur, their numbers are prefixed with the chapter number. For example, Theorem 4.21 refers to Theorem 21 of Chapter 4.

Historical introduction

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In the beginning, geometry was a collection of rules for computing lengths, areas, and volumes. Many were crude approximations arrived at by trial and error. This body of knowledge, developed and used in construction, navigation, and surveying by the Babylonians and Egyptians, was passed on to the Greeks. Blessed with an inclination toward speculative thinking and the leisure to pursue this inclination, the Greeks transformed geometry into a deductive science. About 300 B.C., Euclid of Alexandria organized some of the knowledge of his day in such an effective fashion that all geometers for the next 2000 years used his book, *The Elements*, as their starting point.

First he defined the terms he would use – points, lines, planes, and so on. Then he wrote down five postulates that seemed so clear that one could accept them as true without proof. From this basis he proceeded to derive almost 500 geometrical statements or theorems. The truth of these was in many cases not at all self-evident, but it was guaranteed by the fact that all the theorems had been derived strictly according to the accepted laws of logic from the original (self-evident) assertions.

Although a great breakthrough in their time, the methods of Euclid are imperfect by modern standards. To begin with, he attempted to define everything in terms of a more familiar notion, sometimes creating more confusion than he removed. The following examples provide an illustration:

A *point* is that which has no part. A *line* is breadthless length. A straight line is a line which lies evenly with the points on itself. A *plane angle* is the inclination to one another of two lines which meet. When a straight line set upon a straight line makes adjacent angles equal to one another, each of the equal angles is a *right angle*.

Euclid did not define length, distance, inclination, or "set upon." Once having made his definitions, Euclid never used them. He used instead the "rules of interaction" between the defined objects as set forth in his five postulates and other postulates that he implicitly assumed but did not state. Euclid's five postulates were the following:

I. To draw a straight line from any point to any other point.
II. To produce a finite straight line continuously in a straight line.
III. To describe a circle with any center and distance.
IV. That all right angles are equal to each other.
V. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.

Euclid did not feel it necessary to enunciate the following postulate, even though he used it in his very first theorem.

Two circles, the sum of whose radii is greater than the distance between their centers, and the difference of whose radii is less than that distance, must have a point of intersection.

It is natural to ask why Euclid singled out his five postulates for explicit mention. After Euclid, mathematicians attempted to make explicit the assumptions that Euclid had neglected to mention. The fifth postulate attracted much attention. It was cumbersome but intuitively appealing, and people felt that it might be deduced from the other assumptions of Euclid. Many "proofs" of the fifth postulate were proposed, but they usually contained a hidden assumption equivalent to what was to be proved. Three such equivalent conditions were:

i. Two intersecting straight lines cannot be parallel to the same straight line. (Playfair)
ii. Parallel lines remain at a constant distance from each other. (Proclus)
iii. The interior angles of a triangle add up to two right angles. (Legendre)

In 1763 a man named Klügel wrote a dissertation at Göttingen in which he evaluated all significant attempts to prove the parallel postulate in the 2000 years since Euclid had stated it. Of the 28 proofs he examined, not one was found to be satisfactory. Of particular interest was the work of the Jesuit Saccheri (1667–1733). Saccheri assumed the negation of the fifth postulate and deduced the logical consequences, hoping to arrive at a contradiction. He derived many strange-looking results, some of which he claimed were inconsistent with Euclid's other postulates. Actually, he had discovered some fundamental facts about what we now call hyperbolic geometry.

Gauss (1777–1855) was apparently the first mathematician to whom it occurred that this negation might never lead to a contradiction and that geometries differing from that of Euclid might be possible. The thought struck him as being so revolutionary that he would not make it public. In 1829 he wrote that he feared the "screams of the dullards," so entrenched were the ideas of Euclid. Lobachevsky (1793–1856) and Bolyai (1802–1860) independently worked out geometries that seemed consistent and yet negated Euclid's fifth postulate. These works were published in 1829 and 1832, respectively. Experience proved that Gauss had overestimated the dullards. They paid no attention to the new theories.

Almost 40 years later Beltrami (1835–1900) and Klein (1849–1925) produced models within Euclidean geometry of the geometry of Bolyai and Lobachevsky (now called *hyperbolic geometry*). It was thus established that if Euclid's geometry was free of contradiction, then so was hyperbolic geometry. Because hyperbolic geometry satisfied all the assumptions of Euclid except the parallel postulate, it was finally determined that a proof of the postulate was impossible.

With this branching of geometry into Euclidean and non-Euclidean, it became useful to categorize results according to their dependence on the fifth postulate. Any theorem of Euclid that made no use of the parallel postulate was called a theorem of *absolute geometry*. It was equally valid in Euclidean and hyperbolic geometry. By contrast, certain Euclidean theorems that depended only on postulates I, II, and V became known as *affine geometry*. Theorems common to absolute and affine geometry are called theorems of *ordered geometry*.

The study of central projection was forced upon mathematicians by the problems of perspective faced by artists such as Leonardo da Vinci (1452–1519). The image made by a painter on canvas can be regarded as a projection of the original onto the canvas with the center of projection at the eye of the painter. In this process, lengths are necessarily distorted in a way that depends on the relative positions of the various objects depicted. How is it possible that the geometric structure of the original can still usually be recognized on the canvas? It must be because there are geometric properties invariant under central projection. *Projective geometry* is the body of knowledge that developed from these considerations. Many of the basic facts of projective geometry were discovered by the French engineer Poncelet (1788–1867) in 1813 while a prisoner of war, deprived of books, in Russia. Affine and projective geometry are also closely related, because the study of those properties of figures that remain invariant under parallel projection also leads to affine geometry. This aspect of affine geometry was recognized by Euler (1707–1783).

Because progress in geometry had been frequently hampered by lack of computational facility, the invention of *analytic geometry* by Descartes (1596–1650) made simple approaches to more problems possible. For instance, it allowed an easy treatment of the theory of conics, a subject which had previously been very complicated. Since the time of Descartes, analytic methods have continued to be fruitful because they have allowed geometers to make use of new developments in algebra and calculus.

The scope of geometry was greatly enlarged by Riemann (1826–1866). He realized that the geometry of surfaces provided numerous examples of new geometries. Suppose that a curve lying on the surface is called a line if each small segment of it is the shortest curve joining its end points. Then, for instance, if the surface is a sphere, the lines are the great circles. In this geometry, called *double elliptic geometry*, the following theorems are valid:

i. Every pair of lines has two points of intersection. These points are antipodal; that is, they lie at the opposite ends of the same diameter.
ii. Every pair of nonantipodal points determines exactly one line. An antipodal pair has many lines through them.
iii. The sum of the angles of a triangle is greater than π. It is possible for a triangle to have three right angles.

Riemann and Schläfli (1814–1895) considered higher-dimensional Euclidean and spherical spaces, and in his celebrated inaugural lecture at Göttingen in 1854, Riemann laid the foundations of geometry as a study of general spaces of any dimension, which are now called Riemannian manifolds. These spaces are the principal objects of study in modern *differential geometry*. As the name suggests, the methods used depend on calculus. The geometry of Riemann was used by Einstein (1879–1955) as a basis for his general theory of relativity (1916).

Although Gauss observed the relationship between the angle sum of a triangle and the curvature of the surface on which it occurs, Riemann and those who followed him carried these ideas over to Riemannian manifolds. Thus, curvature is still an important phenomenon in differential geometry, and it indicates how much the geometry of the space being studied differs from being Euclidean.

Although Euclid believed that his geometry contained true facts about the physical world, he realized that he was dealing with an idealization of reality. He did not mean that there was such a thing physically as breadthless length. But he was relying on many of the intuitive properties of real objects. In order to free geometry from reliance on physical concepts for its proofs, Hilbert (1862–1943) rewrote the foundations of geometry in 1899. Hilbert started with undefined objects (e.g., points, lines, planes), undefined relations (e.g., collinearity, congruence, betweenness), and certain axioms expressed in terms of the undefined objects and relations. Anything that could be deduced from this by the usual rules of logic was a geometrical theorem valid in that particular geometry. The choice of axioms was a matter of taste. Of course, some geometries would be interesting and some not, but that is a subjective judgment. The theorems do not depend on the nature of the undefined objects but only on the axioms they satisfy.

Seeing all these geometries around him, Klein, in 1872, proposed to classify them according to the groups of transformations under which their propositions remain true. Since then, group theory has been of increasing importance to geometers. The new geometries of Riemann gave rise to complicated groups of transformations. Soon techniques were developed