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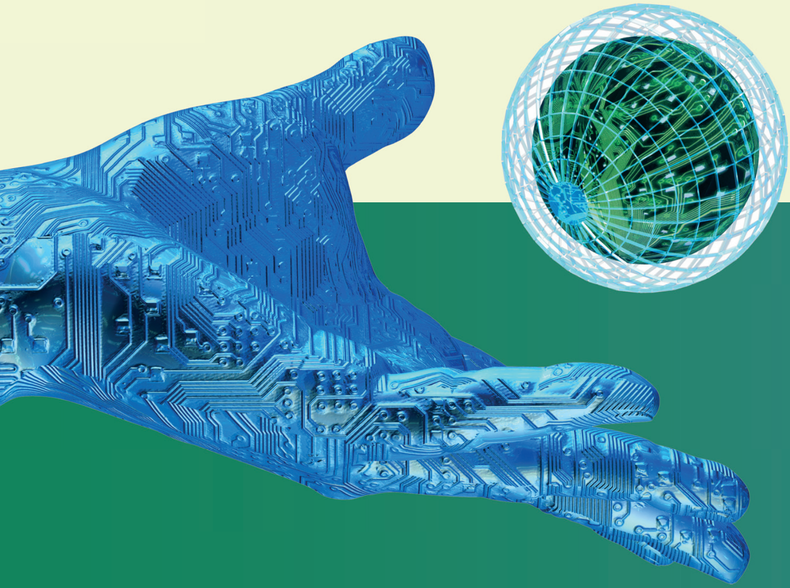

Dmitry A. Novikov
Alexander G. Chkhartishvili

Reflexion and Control

Mathematical Models

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Reflexion and Control

Mathematical Models

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Alexander G. Chkhartishvili

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– “See how the minnows come out and dart around where they please! That’s what fish really enjoy!”
– “You’re not a fish – how do you know what fish enjoy?”
– “You’re not me, so how do you know I don’t know what fish enjoy?”

From a Taoist parable

– “Well, of course, Archbishop, the point is that you believe what you believe because of the way you were brought up.”
– “That is as it may be. But the fact remains that you believe I believe what I believe because of the way I was brought up, because of the way you were brought up.”

From D. Myers’ book “Social Psychology”

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Editorial board

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About the authors

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Introduction

This book is dedicated to modern approaches to mathematical modeling of *reflexion in control* (including an important class of game-theoretic models – *reflexive games* describing the interaction of subjects making decisions based on an hierarchy of beliefs about essential parameters, beliefs about beliefs, etc.).

Reflexion. A fundamental property of human entity lies in the following. In addition to *natural* ("objective") *reality*, there exists its image in human minds. Furthermore, an inevitable gap (mismatch) takes place between the latter and the former. In the sequel, the described image will be called a part of *reflexive reality*.

Traditionally, purposeful study of this phenomenon relates to the term "reflexion." The term reflexion (from Latin *reflex* "bent back") means:

- a principle of human thinking, guiding humans towards comprehension and perception of one's own forms and premises;
- subjective consideration of a knowledge, critical analysis of its content and cognition methods;
- the activity of self-actualization, revealing the internal structure and specifics of the spiritual world of a human.

The term "*reflexion*" was first suggested by John Locke. However, in different philosophical systems (those of Locke, Leibniz, Hume, Hegel, etc.), reflexion has various interpretations. In psychology, systematical treatment of reflexion dates back to the 1960s (Lefebvre's scientific school). Note one may view reflexion in another interpretation connected with a *reflex* (a reaction of a living organism to excited receptors). In the present book, we employ the first (philosophical) definition of reflexion.

To elucidate the whole essence of reflexion, let us consider the case of a single subject. He/she possesses certain beliefs about natural reality; however, a subject may perform reflexion (construct images) with respect to these beliefs (thus, generating new beliefs). Generally, this process is infinite and results in formation of reflexive reality. The reflexion of a subject with respect to his/her own beliefs of reality, principles of his/her activity, etc., is said to be *self-reflexion* or *reflexion of the first kind*. We emphasize that most social research works concentrate on self-reflexion. In philosophy, self-reflexion represents the process of the individual's thinking about beliefs in his/her own mind [127].

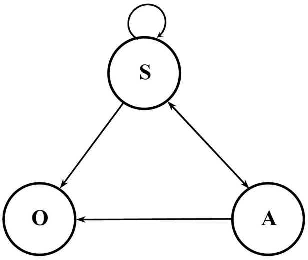

Figure 1 Ways of estimating.

Reflexion of the second kind takes place with respect to other subjects; it covers the beliefs of a subject about possible beliefs, decision principles and self-reflexion of other subjects.

Reflexion ranks. To provide a common description of reflexive imaging, psychology involves the following approach [127]. Consider interrelations among three elements (see Fig. 1), *viz.*, the subject of activity (S), the object of activity (O) and another subject (A). The arrows designate separate acts of "thinking" ("constructing images").

We use various sequences of characters ("S," "O," and "A") to characterize relations among the elements. The order of characters corresponds to (a) who assesses (constructs images of) what or (b) who performs reflexion with respect to what. The object of activity is assumed "passive" (thus, performing no reflexion).

First order relations (*zero-rank reflexion*) include the following *estimates*:

SO – the estimate of the results of the subject's activity by himself/herself (*self-appraisal of the results*);
SS – the estimate of the subject by himself/herself (individual *self-appraisal*);
SA – the estimate of another subject by the subject of activity (as an individual);
AO – the estimate of the results of the subject's activity by another subject;
AS – the estimate of the subject by another subject (as an individual).

Being passive, the object appears unable to estimate; moreover, we do not consider self-appraisal of another subject (AA). Therefore, the above five relations exhaust feasible combinations of the relations of the first order.

The subject of activity and another subject may think about the relations shown in Fig. 1. This yields first-rank reflexion.

Second order relations (first-rank reflexion). Here one should distinguish between:

– self-reflexion (reflexion of the first kind), which corresponds to SS-type sequences, i.e., subject's thoughts about his/her self-appraisal and self-appraisal of his/her results:

  SSO – the subject's thoughts about his/her appraisal of his/her results;
  SSS – the subject's thoughts about his/her self-appraisal;

  and

- reflexion of the second kind (the remaining sequences):
  SAO – the subject's thoughts about the estimate of his/her activity results by another subject ("what others think about the results of my activity");
  SAS – the subject's thoughts about the estimate given to him/her by another subject ("what others think about me");
  ASS – another subject's thoughts about the subject's self-appraisal;
  ASO – another subject's thoughts about the subject's self-appraisal of his/her activity results;
  ASA – another subject's thoughts about the estimate given to him/her by the subject of activity.

Third order relations (*second-rank reflexion*). Naturally, in this case we find numerous combinations. Some are provided below: SASO – the subject's thoughts about another subject's thoughts about his/her self-appraisal of the subject's results ("what others think about my estimates of my results"); ASAO – another subject's thoughts about the subject's thoughts about the estimate given to his/her activity results by another subject, and so on.

Similarly, this framework serves to describe relations of higher orders (higher reflexion ranks).

Examples. Below we choose several examples of second-order reflexion that illustrate the following. In many cases, making correct conclusions is possible only by taking the position of other subjects and analyzing their feasible reasoning.

The first example concerns the classical *dirty face game*) [65], also known as the three wise men puzzle or the problem of husbands and unfaithful wives [121].

Imagine the United Kingdom of the Victorian period; two passengers, Bob and his niece Alice, sit in a compartment of a railway carriage. They both have dirty faces. However, neither of them blushes with shame (Victorians would definitely blush if somebody observed them with a dirty face). And so, we conclude that neither passenger knows anything about his/her dirty face (even though observing the dirty face of the companion).

Suddenly, the Conductor enters the compartment and takes notice of a sitting passenger with a dirty face. Subsequently, Alice blushed. Actually, she understood that her face was dirty. However, how could she realize that? Hadn't the Conductor reported what she knew before?

Let us follow Alice's line of reasoning. Alice: "Suppose that my face is dirty. Being aware of that one of us is dirty, Bob should then have concluded that he is dirty and should have blushed. Meanwhile, Bob is not ashamed; this implies the premise of my clean face is false – my face is actually dirty, and I should have blushed."

As a matter of fact, the Conductor added information on Bob's knowledge to Alice's awareness. Previously, she knew nothing about Bob's awareness that one of them has dirty face. In other words, the Conductor's message made the knowledge of a sitting passenger with a dirty face common knowledge.

Another example involves the *coordinated attack problem*) [70]. There also exist close problems, *viz*., the electronic mail game [151] and others (see the reviews in [56, 76, 166]).

Consider the following situation. Two army divisions are located on two hills, whereas their enemy is in the valley. Gaining a victory over the enemy is possible only through a coordinated attack by both divisions. The commander of division 1 (*General A*) sends to the commander of division 2 (*General B*) a herald with the message "We attack at dawn." The herald can be intercepted by the enemy; and so, *General A* has to wait for message 2 from *General B* (which confirms that message 1 has been received). But message 2 can be intercepted by the enemy, as well. Hence, *General B* has to wait for the confirmation that *General A* has received his confirmation. And so on – *ad infinitum*. The problem lies in defining the maximal number of messages (confirmations) required for attacking. Still, the conclusion is as follows. Within the stated conditions, a coordinated attack is impossible, and the way out consists in adopting probabilistic models [116, 117].

The third example deals with the classical *problem of two brokers* [138]. Assume that two brokers gambling on a stock exchange apply different decision support systems. It happens that a network administrator illegally duplicates these systems and sells the opponent's system to each broker. Afterwards, the administrator tries to sell the following information to each broker: "Your opponent has your decision support system." The next initiative of the administrator is attempting to sell the following information: "Your opponent knows that you have his decision support system." And so forth. The problem is how brokers should use information acquired from the administrator. What information is important at different iterations?

Thus, we have briefly studied the examples of second-rank reflexion. Now, let us discuss when reflexion is essential. Suppose that the only subject performing reflexion is an economic agent striving for maximization of his/her goal function by choosing one of several ethically admissible actions. In this case, natural reality is incorporated in the goal function as a certain parameter, whereas the results of reflexion (beliefs about beliefs, etc.) do not represent arguments of the goal function. Consequently, one may claim that self-reflexion is useless as not modifying the agent's choice.

Note that the subject's actions can depend on reflexion when actions appear nonequivalent ethically. That is, the *utilitarian* aspect runs parallel to the deontological (*ethical*) aspect, see [95, 96]. However, generally economic decisions are ethically neutral. Thus, we analyze the interaction among several subjects.

Consider a situation of decision-making with several participating subjects (such decision situations are called *interactive*). The goal function of each subject includes the actions of other subjects. In other words, these actions represent a part of natural reality (they are conditioned by reflexive reality though). Reflexion and analysis of reflexive reality become necessary. Prior to exploring the basic approaches to mathematical modeling of reflexion effects, we describe the correlation between two major categories of this book – "reflexion" and "control."

Reflexion and control. Let us start with defining the essence of control. *Control* is an element, a function of organized systems of different nature (biological, social, technical, etc.), preserving their definite structure, sustaining their mode of activity and implementing the program or goal of their activity; *control* is a purposeful impact exerted on a controlled system to ensure its required behavior [135].

In what follows, we discuss the general statement of a control problem. Assume there is a *control subject* (a principal) and a *controlled system* (*control object* – in terminology of technical systems – or a *controlled subject*). The state of a controlled system depends on external disturbances, control actions applied by a principal and possibly on actions performed by the controlled system (if the latter represents an

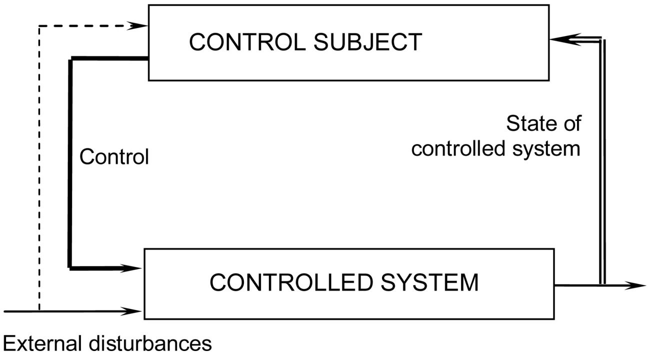

Figure 2 The structure of a control system.

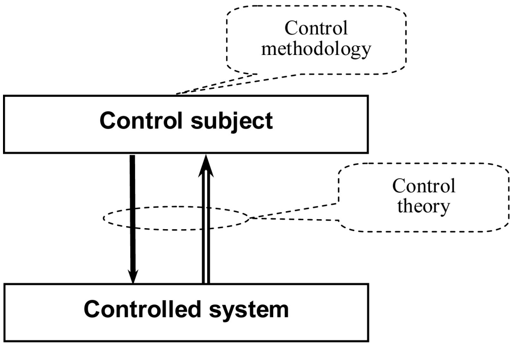

Figure 3 Control methodology and control theory.

active subject), see Fig. 2. The principal's problem lies in choosing control actions (see the thick line in Fig. 2) to ensure the required behavior of a controlled system taking into account information on external disturbances (see the dashed line in Fig. 2).

The so-called input-output structure of a control system (Fig. 2) is typical for control theory dealing with control problems in systems of different nature. The presence of *feedback* (see the double line in Fig. 2) which provides a principal with information on the state of a controlled system is the key (but not compulsory!) property of a control system. Some researchers **interpret feedback as reflexion** (as an image of the controlled system's state in the "mind" of a control subject). This forms the first aspect of interrelation between control and reflexion.

A series of scientific directions investigate the interaction and activity of a control subject and controlled system. Control science (or *control theory* in the terminology of corresponding experts) mostly focuses on the interaction between a control subject and controlled system – see Fig. 3. *Control methodology* [130] is the theory of organizing of control activity, i.e., the activity performed by a control subject. We emphasize that *activity* can be mentioned only with respect to active subjects (e.g., a human being, a group, a collective). In the case of passive (e.g., technical) systems, the term "functioning" is used instead. In the sequel, we believe that a control subject and controlled system appear active (otherwise, there is a clear provision for the opposite). Hence, **each of them may perform (at least) self-reflexion**, constructing "images" of the process, organization principles and results of his/her own activity. This is the second aspect of interrelation between control and reflexion.

Searching for *optimal control* (i.e., the most efficient admissible control) requires the control subject's ability to predict the controlled system's response to certain control actions. One of the prerequisites is a model of the controlled system. Generally speaking, a *model* is an image of a certain system; an analog (a scheme, a structure or a sign system) of a certain fragment of the natural or social reality, a "substitute" for the original in cognition process and practice. A model can be considered as an image of a controlled system in the mind of a control subject. **Modeling** (as a process of "reflecting," i.e., constructing this image) **can be viewed as reflexion**. Furthermore, a controlled system may predict and assess the activity performed by a control subject. And so, we obtain the third aspect of interrelation between control and reflexion.

The fourth aspect lies in the following. **A control subject or controlled system performs reflexion with respect to external subjects and objects**, phenomena or processes, their properties and laws of activity/functioning. For instance, the matter concerns an external environment (for a control subject), an external environment and/or other elements of a controlled system (for a fixed element of a controlled system). Indeed, suppose that a controlled system includes several active agents; each of them may perform reflexion with respect to the others. It is exactly this aspect – mutual reflexion of controlled subjects – which is discussed in detail below (see Chapters 2–3).

The listed quartet of aspects corresponds to zero reflexion rank ("estimating," see above). By analogy to Fig. 1, one can give a uniform description to reflexion of higher ranks. First-rank reflexion covers the control subject's beliefs about the estimate of other controlled subjects (agents) by a given agent. Second-rank reflexion touches the estimate of these beliefs by a controlled system. And so on.

Of crucial importance here is that **the process and/or result of reflexion can be controlled**, i.e., can represent a component of the controlled system's activity, being modified by a control subject for a definite goal. Precisely this relationship between control and reflexion enables informational control and reflexive control studied in this book! Actually, we present theoretical results and applications in the field of controlling reflexion.

In this context, let us make a digression as follows. The results of modeling and informational/reflexive control derived for social, economic, organizational and other systems (including human beings) have recently been extended to artificial technical systems. For instance, consider the so-called *multi-agent systems* (MAS) [157]. Such systems consist of numerous interacting autonomous agents having technical or informational nature (a classical example is a group of mobile robots). Multi-agent systems are remarkable for interaction decentralization and agents' multiplicity; these features lead to fundamentally new and important emergence properties (autonomy, lower vulnerability to adverse effects, etc.).

MAS have a complex (hierarchical) internal structure. The typical functional structure of an agent includes several hierarchical levels – see Fig. 4. *Operational level* (execution level) serves for implementing certain actions (e.g., stabilization of motion along a given trajectory). *Tactical level* is intended for choosing actions (e.g., planning

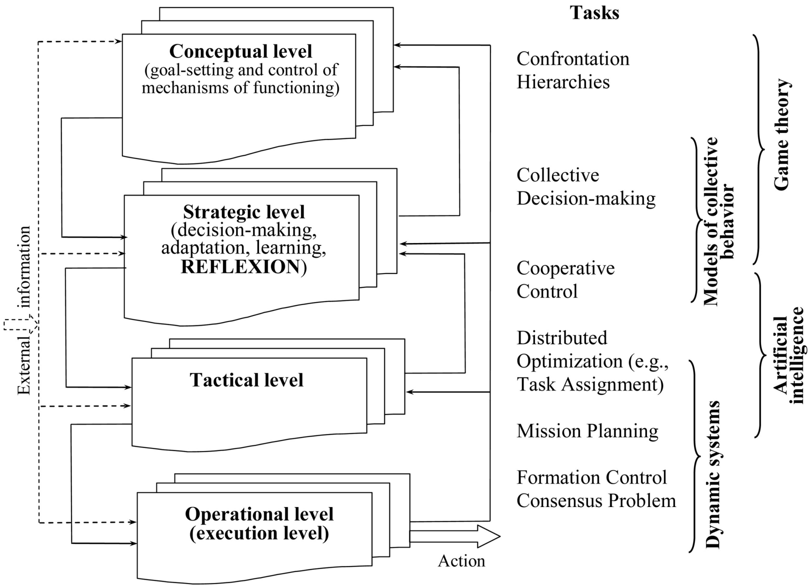

Figure 4 MAS: The hierarchical architecture of an agent.

of actions – trajectories selection or solution of distributed optimization problems). Actions can be chosen taking into account interaction with other agents. *Strategic level* answers for decision-making, learning and adaptivity of agents, as well as for control cooperativity (coordinated solution of a common task by a set of agents). An agent should have the capacity for strategic decision-making, adaptation, learning and **reflexion**. Finally, *conceptual level* corresponds to goal-setting principles. Each level employs a certain framework (as a rule, methods being applicable at a certain level can be used at higher hierarchical levels – see Fig. 4).

One modern tendency of the theory of multi-agent systems, game theory (see below) and artificial intelligence theory lies in that researchers strive to integrate these scientific directions. Yet, game theory and artificial intelligence theory aim at higher levels of agents' architecture. Within algorithmic, computational and evolutionary game theories [3, 106, 175]), one would observe "transition downwards," i.e., from the uniform description of a game to its decentralization and analysis of the feasibility of implementing autonomously the mechanisms of equilibrium behavior and realization. In fact, similar "decentralization" trends can be found in operations research [168]. On the other hand, the theory of MAS moves "upwards" in a parallel way due to the local character of scientific communities. The theory of MAS aspires after better consideration of strategic behavior and "intellectuality of agents" (including their capacity for reflexion). The behavior and interaction of active subjects is described by game theory. Today, game theory is a major tool for studying systems with incorporated human beings.

Game theory. Formal (mathematical) models of human behavior have been constructed and studied for over 150 years. Gradually, these models find wider application in control theory, economics, psychology, sociology, etc., as well as in practical problems. Most intensive development dates from the 1940s, the appearance of *game theory* often connected with J. von Neumann and O. Morgenstern's famous book *Theory of Games and Economic Behavior* [125] published in 1944.

In the sequel, we will understand a *game* as the interaction of subjects with noncoinciding interests. Still, an alternative interpretation treats a game as a type of unproductive activity whose motive consists not in the corresponding results, but in the process of activity itself. In addition, we refer to [88, 127], where the notion of a game is assigned a broader sense.

Game theory represents a branch of applied mathematics, which analyzes models of decision-making in the conditions of noncoinciding interests of opponents (*players*); each player strives to influence the situation in his/her favor [67, 75, 121]. In what follows, a decision-maker (a player) is called an *agent*. In the present book, we focus *par excellence* on noncooperative static normal-form games, where agents choose their *actions* one-time, simultaneously and independently. The only exception lies in dynamic models of collective decision-making discussed in Section 3.4.

Therefore, the major task of game theory is describing the interaction among several agents with noncoinciding interests, where the results of agent's activity (payoff, utility, etc.) generally depend on actions of all agents [75, 121]. Such description yields a forecast of a rational and "stable" outcome of the game – the so-called *game solution* (*equilibrium*).

Describing a *game* means specifying the following parameters:

- *a set of agents*;
- *preferences of agents* (relationships between payoffs and actions). Each agent is supposed to strive to maximize his/her payoff (and so, the behavior of each agent appears purposeful);
- *a set of feasible actions of agents*;
- *awareness of agents* (information on essential parameters, being available to agents at the moment of their choice);
- *sequence of moves* (the sequence of choosing actions).

Roughly speaking, a set of agents determines *who* participates in a game. Next, preferences reflect what agents *want*, sets of feasible actions describe what agents *can*. Finally, awareness and sequence of moves correspond to what agents *know* and *when* they choose actions, respectively.

The above parameters define a game; unfortunately, they are insufficient for forecasting its outcome, i.e., a solution (or an equilibrium) of the game – the set of rational and stable actions of agents. Nowadays, game theory suggests no universal concept of equilibria. By adopting different assumptions regarding principles of the agent's decision-making, one can construct different solutions. Thus, designing an equilibrium concept forms a basic problem for any game-theoretic research; this book does not represent an exception, either. Reflexive games are defined as a direct interaction among agents, where they make decisions based on hierarchies of their beliefs. In other words, awareness of agents is extremely important. And so, let us discuss this component in greater detail.

The role of awareness. Common knowledge. In game theory, psychology, distributed systems and other fields of science (see the overviews in [66, 117]), one should consider not only agents' *beliefs* about essential parameters, but also their beliefs about the beliefs of other agents, etc. The set of such beliefs is called the *hierarchy of beliefs*. We will model it using the tree of awareness structure of a reflexive game (see below). In other words, situations of interactive decision-making (modeled in game theory) require that each agent "forecasts" opponents' behavior prior to his/her choice. And so, each agent should possess definite beliefs about the view of the game by his/her opponents. On the other hand, opponents should do the same. Consequently, the uncertainty regarding the game to-be-played generates an infinite hierarchy of beliefs of game participants.

Consider an example of such an hierarchy. Suppose there exist two agents, namely, A and B. Each agent can have individual nonreflexive beliefs about an uncertain parameter θ (the *state of nature*). Denote these beliefs by θ_A and θ_B, respectively. Yet, performing the *first-rank reflexion*, each agent may think about their opponent's beliefs. The described beliefs (known as *beliefs of the second order*) will be designated by θ_{AB} and θ_{BA}, where θ_{AB} are the beliefs of agent A about the beliefs of agent B, θ_{BA} are the beliefs of agent B about the beliefs of agent A. Moreover, the process continues – within the framework of further reflexion (the *second-rank reflexion*) each agent may think about the opponent's beliefs about his/her beliefs. This yields beliefs of the *third order*, θ_{ABA} and θ_{BAB}. The process of generating beliefs of higher orders can be infinite (indeed, there are no logic restrictions to further increase of reflexion rank). The whole set of all beliefs – θ_A, θ_B, θ_{AB}, θ_{BA}, θ_{ABA}, θ_{BAB}, etc. – forms the hierarchy of beliefs.

A special case of awareness concerns *common knowledge*, when beliefs of all orders coincide. A rigorous definition of common knowledge was introduced in [100]. Notably, common knowledge is a fact with the following properties:

1) all agents know it;
2) all agents know 1;
3) all agents know 2 and so on – *ad infinitum*.

The formal model of common knowledge was originally proposed in [8]. Later on, many investigators refined and redeveloped it [9, 11, 57, 58, 59, 76, 83, 84, 102, 116, 159].

The present book is almost completely dedicated to models of agents' awareness in *game theory* (*viz*., hierarchies of beliefs and common knowledge). Thus, we give several examples demonstrating the role of common knowledge in different fields of science – philosophy, psychology, etc. (see also the overview in [56]).

In *philosophy*, common knowledge has been studied in *convention* analysis [100, 172, 173]. For instance, consider Road Regulations in a certain country; they state that each participant of road traffic must follow these regulations and has the right to expect the same behavior from other participants. But other participants of road traffic must be also sure that the rest observe the Road Regulations, and so on. Hence, the convention "Follow the Road Regulations" must be a common knowledge.

In *psychology*, one would face the notion of *discourse* (from Latin *discursus* 'argument'). It means human thinking in words, being mediated by past experience; discourse acts as the process of connected logical reasoning, where a next idea stems from the previous one. The importance of common knowledge in discourse comprehension has the following illustration in [48, 56].

Two persons leave a movie theater. One asks another, "What did you think of the movie?" The second person understands the question only in the following case.

He/she understands the matter concerns the movie they have just seen. In addition, the second person must understand that this fact is understood by the first person. On the other hand, the asking person must be sure that the responding person understands that the matter concerns the movie they have just seen, and so on. Notably, adequate interaction (communication) between these persons requires that the movie forms their common knowledge (people must agree about language usage [100]).

Mutual awareness of agents turns out to be significant in *distributed computer systems* [57, 59, 76], *artificial intelligence* [70, 110] and other fields.

Game theory often assumes that all[1] parameters of a game are a *common knowledge*. In other words, each agent knows (a) all parameters of the game, (b) the fact that the rest of the agents know (a), and so on – *ad infinitum*. Such assumption corresponds to the *objective description of a game* and enables addressing the *Nash equilibrium*[2]concept [124] as a forecasted outcome of a noncooperative game (a game, where agents do not agree about coalitions, data exchange, joint actions, redistribution of payoffs, etc.). Thus, the assumption regarding common knowledge allows claiming that all agents know which game they play and that their beliefs about the game coincide.

Instead of agents' actions, one may consider something more complicated – agents' *strategies*. A strategy represents a mapping of all information available to an agent into a set of his/her feasible actions. For instance, we mention strategies in a multi-step game, mixed strategies, strategies in Howard's metagames [86, 87] (see also informational extensions of games in [67]). However, in these cases the rules of play are a common knowledge. Finally, it seems possible to believe that a game is chosen randomly according to a certain probability distribution making up a common knowledge – the so-called *Bayesian games* [63, 78, 121].

Generally, each agent may possess individual beliefs about parameters of a game. And so, each belief corresponds to a *subjective description of the game* [67] (see also modern models of awareness in [50, 61, 81, 149]). Consequently, agents participate in the game, having no objective views of it or interpreting this game in different ways (rules, goals, the roles and awareness of opponents, etc.). Unfortunately, still no universal approaches have been proposed for equilibria design under insufficient common knowledge.

On the other part, within the "reflexive tradition" of the humanities, the surrounding world of each agent includes the rest of the agents; moreover, beliefs about other agents get reflected during the process of reflexion (in particular, variations of beliefs may result from nonidentical awareness). However, researchers have not succeeded in deriving constructive formal outcomes in this field to date.

Hence, an urgent problem lies in designing and analyzing mathematical models of games, where agents' awareness is not a common knowledge and agents make decisions based on hierarchies of their beliefs. Such a class of games is called **reflexive games** [42, 137, 138]. We will provide a formal definition later.

[1]If the initial model incorporates uncertain factors, specific procedures of uncertainty elimination are involved to obtain a deterministic model.

[2]An agents' action vector is a Nash equilibrium if none of them benefits by unilateral deviation from it (provided that the rest of the agents choose the corresponding components of the Nash equilibrium). A more rigorous definition can be found below.

The term "reflexive games" was introduced by V. Lefebvre in 1965, see [99]. However, the cited work and his other publications [96-99] represented qualitative discussions of reflexion effects in interaction among subjects (actually, no general concept of solution was suggested for this class of games). Similar remarks apply to [55, 69, 161, 169], where a series of special cases of players' awareness was studied. The research work [138] concentrated on systematical treatment of reflexive games and an endeavor of constructing *a uniform equilibrium concept* for these games.

Prior to outlining the major content of this book, let us describe the basic approaches used below.

The basic approaches and structure of this book. In fact, the monograph [138] is our first work dedicated to models of reflexion in the game-theoretic context. Many years have elapsed since that time, and this line of investigations gained further development (e.g., see [39, 74, 132]). The present book reflects recent advances in the corresponding field. It includes the primary results derived by us and our colleagues, as well as reviewing the approaches adopted by other researchers.

Chapter 1 (Reflexion in Decision-making) possesses an introductory character; notably, models of individual and interactive decision-making are considered, the awareness required for implementing well-known equilibrium concepts is analyzed, and famous models of common knowledge and hierarchy of beliefs are described.

Recall that a reflexive game is a game, where agents' awareness does not form a common knowledge[3] and agents make decisions based on hierarchies of their beliefs. According to game theory and reflexive models of decision-making, it seems reasonable to distinguish between strategic reflexion and informational reflexion.

Informational reflexion is the process and result of agent's thinking about (a) the values of uncertain parameters and (b) what his/her opponents (other agents) know about these values. Here the "game" component actually disappears – an agent makes no decisions.

Strategic reflexion is the process and result of agent's thinking about which decision-making principles his/her opponents (other agents) employ under the awareness assigned by him/her via informational reflexion.

Therefore, informational reflexion often relates to insufficient mutual awareness, and its result serves for decision-making (including informational reflexion). Strategic reflexion takes place even in the case of complete awareness, preceding an agent's choice of action. In other words, informational and strategic reflexion can be studied independently, but both occur in the case of incomplete or insufficient awareness.

Chapter 2 (Informational Reflexion and Control) deals with formal models of informational reflexion and informational control. A key factor in reflexive games consists in agents' awareness (hierarchy of beliefs). Hence, its formal description involves the notion of an *awareness structure* – a (generally, infinite) tree whose nodes correspond to information (beliefs) of agents about essential parameters, beliefs of other agents, etc. An example of such an hierarchy can be found below.

The concept of an awareness structure enables giving a formal definition to certain intuitively apprehensible notions such as adequate awareness of agent 1 about agent 2, mutual awareness, identical awareness, etc.

[3]Naturally enough, research results in the field of reflexive games turn into corresponding results in the field of classic games if awareness is a common knowledge (see below).

The notion of a *phantom agent* is a key to reflexive games analysis in this book. Let us discuss it at the qualitative level (omitting the technicalities, see Chapter 2).

Suppose that two agents, namely, A and B, interact in a certain situation. Of course, each agent possesses an image of the other; agent A has an image of agent B (denoted by AB), while agent B has an image of agent A (denoted by BA). These images coincide with or differ from the reality. For instance, agent A may possess an adequate belief about agent B (the identity $AB = B$ holds true) or may not.

The following question rises immediately. Is the identity $AB = B$ possible in principle? You know, B represents a real agent, whereas AB is merely his/her image! This philosophical question requires going into subtleties; let us merely emphasize a couple of important facts. First, the matter concerns modeling of individual behavior in a specific situation rather than complete understanding of an individual. During everyday communication with different people, we often face situations of adequate and inadequate perception of an individual by other individuals.

Second, within the framework of formal (game-theoretic) modeling of human behavior, an agent as a participant of a certain situation is described by a (relatively) small set of characteristics. The latter can be completely known to another agent (exactly as to a researcher).

Consider the case when B and AB differ (formally, due to incomplete information on B available to A or due to trusting false information). Choosing certain actions, A takes into account not B, but the latter's image, i.e., AB. Reformulating this statement, one may say that A subjectively interacts with AB. And so, AB can be called a *phantom agent*. Really, this agent does not exist, but appears in the mind of *real agent A*. Consequently, the phantom agent affects actions of agent A, i.e., it affects the reality.

We give an elementary example. A believes that B is his/her friend. At the same time, B knows this fact and is a foe of A (the so-called "betrayal"). Obviously, such a situation includes phantom agent AB described by "B is a friend of A"; actually, this subject is missed. On the other hand, B is adequately informed of A, notably, $BA = A$.

Thus, the idea consists in studying phantom agents (existing in the minds of real and other agents) in addition to real agents (actually participating in a game). Real and phantom agents perform reflexion, enduing phantom agents with some awareness reflected in an awareness structure.

There may be infinitely many (real and phantom) agents participating in a game. This means a potentially infinite number of acts of reflexive imaging (infinite depth of the tree of an awareness structure). In any common situations, one can construct infinitely large assertions such as "I know…," "I know that you know…," "I know that you know that I know …," "I know that you know that I know that you know…," and so on. Yet, in practice such a "stupid infinity" seems pointless – starting from a definite moment, beliefs get stabilized and further increase in reflexion ranks yields nothing new. Therefore, in real situations an awareness structure possesses finite *complexity*, i.e., the corresponding tree has a finite number of pairwise-different subtrees. In other words, a game comprises a finite number of real and phantom agents[4].

[4]In the limiting case of common knowledge, a phantom agent of level 1 coincides with its prototype (real agent) and the corresponding tree has the depth of 1. More specifically, the rest of the subtrees duplicate higher-level trees.

The notion of a phantom agent enables the following. First, determining a reflexive game as a game of real and phantom agents. Second, defining an *informational equilibrium* as the generalization of a Nash equilibrium to the case of a reflexive game; here each (real and phantom) agent evaluates his/her *subjective equilibrium* (an equilibrium in the game he/she believes they are playing) using the available hierarchy of beliefs about objective and reflexive reality [40].

A convenient tool of informational equilibrium analysis lies in the *graph of a reflexive game*. In this graph, nodes answer to real and phantom agents and each node-agent has incoming arcs from nodes-agents whose actions affect the payoff of the given agent (in his/her subjective equilibrium). The number of incoming arcs equals the number of real agents minus unity. The graph of a reflexive game can be constructed without specifying the goal functions of agents. In this case, the graph reflects the qualitative interrelation of awareness of reflexing agents (instead of the quantitative ratio of their interests). Moreover, this graph provides a comfortable and expressive means of describing reflexion effects (see Section 2.4).

Let us get back to the example above. The graph of the reflexive game between two agents acquires the form $B \leftarrow A \leftrightarrow AB$. Real agent B (the betrayer) is adequately aware of agent A, which interacts with phantom agent AB (B representing a friend of A).

Strategic reflexion is considered in *Chapter 3* of the present book. The following observation can be made. Suppose that an agent models opponents' behavior by assigning definite reflexion ranks to them and him/her. Then the initial game turns into a new game, where the agent's strategy consists in choosing reflexion ranks.

Studying the process of reflexion in the new game leads to another new game, and so on. Furthermore, even if the former game incorporates a finite set of feasible actions, the latter game would have an infinite set of feasible actions (the number of different reflexion ranks). Hence, the primary problem of strategic reflexion analysis is evaluating the maximal reasonable rank of reflexion. In Chapter 3, this problem is solved for bimatrix games (Section 3.2) and models accounting for the bounded abilities of a human being in the field of data processing (Section 3.3).

We provide an example of strategic reflexion – *Penalty kick* in soccer (we also refer to *Hide-and-seek* and *Misère in Preferans*, see Section 3.2). Agents represent a kicker and a goal-keeper. For simplicity, suppose that the kicker chooses between two actions, *viz.*, "shooting in the left corner of goal" and "shooting in the right corner of goal". Accordingly, the goal-keeper has two actions, "catching the ball in the left corner" and "catching the ball in the right corner". If the goal-keeper guesses right, he/she catches the ball.

Let us model the reasoning of agents. Assume that the goal-keeper knows the kicker often chooses the right corner of the goal. Hence, he/she should catch the ball in the right corner. Yet, if the kicker knows that the goal-keeper knows his/her common way of shooting, the goal-keeper should model the reasoning of the kicker. He/she may think as follows, "The kicker knows that I know his/her common way of shooting. And so, the kicker expects me to catch the ball in the right corner and may shoot in the left corner. In this case, I should catch the ball in the left corner." If the kicker possesses sufficient reflexion depth, he/she may guess the reasoning of the goal-keeper and try outwitting the opponent by shooting in the right corner. The same line of reasoning can be followed by the goal-keeper; as the result, he/she would catch the ball in the right corner.

Both the kicker and goal-keeper may infinitely increase reflexion depths by thinking in the place of each other. Furthermore, none of them have rational grounds to stop at a certain step. Hence, in modeling of mutual reasoning, one would not a priori define the outcome of this game. The game, where agents choose between two actions, can be substituted by another game, where agents choose reflexion ranks assigned to an opponent. But this game also admits no rational solution, since each agent may model an opponent's behavior by considering a "twice reflexive game" and so on – *ad infinitum*.

In such a situation, the only aspect assisting agents lies in bounding the depth of their reflexion. The initial set of feasible actions is finite; consequently, the situation repeats itself starting from reflexion rank 2. Indeed, the kicker shoots in the right corner, being at zero and second (any even) level of reflexion. Thus, the goal-keeper has to guess whether the kicker's reflexion rank is even or not.

The maximal reflexion rank to-be-possessed by an agent for embracing the whole variety of game outcomes is called the *maximal rational rank of reflexion*. By failing to bear certain opponent's strategies in mind, an agent runs the risk of decreasing his/her payoff. As it turns out, the maximal rational rank of reflexion is finite in many cases; the corresponding formal results are presented in Sections 2.6 and 3.2. In the example *Penalty kick*, the maximal rational rank of reflexion performed by agents constitutes 2.

Suppose that the goal-keeper has no information on the common way of shooting by the kicker. Hence, the latter's actions are symmetric (i.e., the left and right corners appear "equivalent"). Still, it seems possible to introduce asymmetry for pursuing one's own goals. For instance, the goal-keeper may twitch to a certain corner of goal, "inviting" the kicker to shoot in another corner (subsequently, the goal-keeper jumps exactly in the "distant" corner). A more sophisticated strategy consists in the following. A goal-keeper's team-mate shows him/her the corner the kicker would shoot in (such that the kicker notices the hint). Subsequently, the goal-keeper jumps in the opposite corner. Finally, we emphasize that both techniques have been successfully adopted in football many times.

The concepts of an awareness structure, informational equilibrium and the graph of a reflexive game form the model of a reflexive game, which enables the following. First, it provides the uniform methodology and mathematical framework to describe and analyze various situations of collective decision-making by agents possessing different awareness, to study the impact of reflexion ranks on agents' payoffs, to obtain conditions of existence and implementability of informational equilibria, etc. Many examples of possible applications are discussed below.

Second, the suggested model of a reflexive game allows investigating the influence of reflexion ranks (the depth of an awareness structure) on agents' payoffs. The results derived in Sections 2.5, 2.6 and 3.2 indicate that (under slight assumptions) the maximal rational rank of reflexion is bounded. In other words, in many cases infinite increase in reflexion rank seems unreasonable in the sense of agents' payoffs.

Third, the suggested model of a reflexive game makes it possible to establish existence conditions and properties of an informational equilibrium, as well as to pose constructively and correctly the ***problem of informational control***. In this problem, a principal has to find an awareness structure such that the informational equilibrium implemented in it appears most beneficial to him/her. The problem of informational control is stated and solved in Section 2.11 for several special cases. The corresponding theoretical results are adopted in applied models discussed in *Chapter 4*.

In *Chapter 3* we consider models of strategic reflexion. This is done according to the logic of describing informational reflexion used in Chapter 2. Similarly to informational control for informational reflexion, in Section 3.4 we formulate the ***problem of reflexive control*** (for strategic reflexion). In addition, *Chapter 4* presents some models of reflexive control.

Finally – in the fourth place – the language of reflexive games (awareness structures, graphs of a reflexive game, etc.) is convenient to describe reflexion effects in psychology (see *Playing chess*, *Transactional analysis*, *Ethical choice*, etc.), to analyze art works, as well as to model organizational, economic, social and other systems. Details can be found in Chapter 4.

Alternatively, the structure of this book can be viewed from decision theory positions (see Fig. 5 and Fig. 1.1 below). The elementary (basic) model of *decision-making* lies in the choice problem solved by an individual (a decision-maker, DM) under complete awareness. Possible extensions of this model are the cases of natural or/and game uncertainty. The latter comprises uncertainty (incomplete awareness of a DM) regarding opponents' awareness (informational reflexion) or their decision-making principles

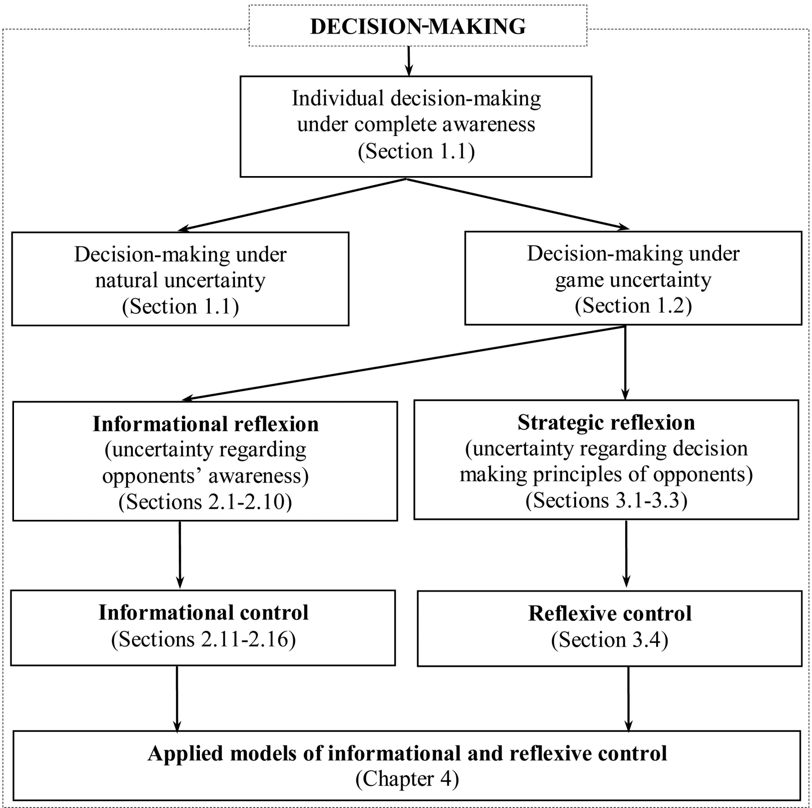

Figure 5 The logic and structure of this book.

(strategic reflexion). Purposeful impacts on DM's beliefs about opponents' awareness or decision-making principles are the essence of informational and reflexive control.

Thus, we have presented the structure and content of this book. Actually, several approaches to reading can be proposed. The first one is linear (successive reading of all chapters). The second approach is intended for a reader mostly interested in formal models (observational reading of Chapters 2–3 and glancing over the examples in Chapter 4). The third approach aims at a reader concerned with practical interpretations rather than mathematical subtleties (observational reading of the Introduction, the examples in Chapter 4 and the Conclusion).

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Chapter 1

Reflexion in decision-making

Chapter 1 presents the model of individual decision-making (Section 1.1), overviews some major solution concepts of noncooperative games, discusses necessary assumptions imposed on awareness and mutual awareness of agents according to these solution concepts (Section 1.2), as well as analyzes conventional models of awareness and common knowledge (Section 1.3).

1.1 INDIVIDUAL DECISION-MAKING

Following [135, 136], let us state the model of agent's decision-making. Suppose that an agent can choose an *action* x from a set of *feasible actions* X. By choosing the action $x \in X$, the agent obtains the payoff $f(x)$, where $f: X \to \Re^1$ represents a real-valued *goal function* reflecting agent's preferences.

Accept *the hypothesis of rational behavior* which states the following. Under all available information, an agent chooses actions leading to the most beneficial values of his/her goal function. This hypothesis is not the only possible one – for instance, see the concept of bounded rationality [158]. According to the hypothesis of rational behavior, an agent chooses an alternative from the set of "best" alternatives. In the present case, this is a set of alternatives, where the goal function attains its maximum.

Hence, the agent's choice is determined by the *rule of individual rational choice* $P(f, X) \subseteq X$, which separates a set of the most beneficial actions[1] (from the agent's view):

$$P(f, X) = \operatorname{Arg} \max_{x \in X} f(x).$$

Now, complicate the model by assuming the following. In addition to his/her actions, the agent's payoff depends on the value of an uncertain parameter $\theta \in \Theta$ – *the state of nature*. Notably, choosing the action $x \in X$ and realizing the state of nature $\theta \in \Theta$ lead to the agent's payoff $f(\theta, x)$, where $f: \Theta \times X \to \Re^1$.

In this general case (under an uncertain parameter – the state of nature), there exists no unambiguously "best" action. Choosing an action, an agent should "predict" or guess the state of nature.

[1]Appropriate maxima or minima are supposed to exist.