Introduction to Management Science

THIRTEENTH EDITION

Bernard W. Taylor III

Introduction to
Management Science

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Management Science

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To Diane, Kathleen, and Lindsey

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Brief Contents

The following items can be found on the Companion Web
site that accompanies this text:

Web Site Modules

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Preface

New to This Edition

Management science is the application of mathematical models and computing technology to help decision makers solve problems. Therefore, new text revisions like this one tend to focus on the latest technological advances used by businesses and organizations for solving problems, as well as new features that students and instructors have indicated would be helpful to them in learning about management science. Following is a list of the substantial new changes made for this 13th edition of the text:

- This revision incorporates the latest version of Excel® 2016 and includes more than 175 new spreadsheet screenshots.
- More than 60 new exhibit screenshots have been added to show the latest versions of Microsoft® Project 2016, QM for Windows, Excel QM, TreePlan, and Crystal Ball.
- This edition includes 20 new end-of-chapter homework problems, so it now contains more than 800 homework problems and 69 cases.
- All 800-plus Excel homework files on the Instructor's Web site have been replaced with new Excel 2016 files.
- Updated "Chapter Web links" are included for every chapter. More than 550 Web links are provided to access tutorials, summaries, and notes available on the Internet for the various topics in the chapters. Also included are links to YouTube videos that provide additional learning resources.
- Twelve of the 48 "Management Science Application" boxes are new for this edition. All of these new boxes provide applications of management science techniques by companies and organizations.

Solving Teaching and Learning Challenges

The objective of management science is to solve the decision-making problems that confront and confound managers in both the public and private sectors by developing mathematical models of those problems. These models have traditionally been solved with various mathematical techniques, all of which lend themselves to specific types of problems. Thus, management science as a field of study has always been inherently mathematical in nature, and as a result sometimes complex and rigorous. My main goal through 13 editions of this book has always been to make these mathematical topics seem less complex and thus more palatable to undergraduate business students. To achieve this goal I try to provide simple, straightforward explanations of often difficult mathematical topics. I try to use lots of examples that demonstrate in detail the fundamental mathematical steps of the modeling and solution techniques. Although in the past three decades the emphasis in management science has shifted away from strictly mathematical to mostly computer solutions, my objective has not changed. I have provided clear, concise explanations of the techniques used in management science to model problems and provided many examples of how to solve these models on the computer while still including some of the fundamental mathematics of the techniques.

The stuff of management science can seem abstract, and students sometimes have trouble perceiving the usefulness of quantitative courses in general. I remember that when I was a student, I could not foresee how I would use such mathematical topics (in addition to a lot of the other things I learned in college) in any job after graduation. Part of the problem is that the examples used in books often do not seem realistic. Unfortunately, examples must be made simple to facilitate the learning process. Larger, more complex examples reflecting actual applications would be too complex to help the student learn the modeling technique. The modeling techniques presented in this text are, in fact, used extensively in the business world, and their use is increasing rapidly because of computer and information technology, and the emerging field of business analytics. Therefore, the chances that students will use the modeling techniques that they learn from this text in a future job are very great indeed.

Even if these techniques are not used on the job, the logical approach to problem solving embodied in management science is valuable for all types of jobs in all types of organizations. Management science consists of more than just a collection of mathematical modeling techniques; it embodies a philosophy of approaching a problem in a logical manner, as does any science. Thus, this text not only teaches specific techniques but also provides a very useful method for approaching problems.

My primary objective throughout all revisions of this text is readability. The modeling techniques presented in each chapter are explained with straightforward examples that avoid lengthy written explanations. These examples are organized in a logical step-by-step fashion that the student can subsequently apply to the problems at the end of each chapter. I have tried to avoid complex mathematical notation and formulas wherever possible. These various factors will, I hope, help make the material more interesting and less intimidating to students.

Management Science Application

Management Science and Analytics

As we discussed in the section "Management Science and Business Analytics," when applied to business problems, analytics often combines the management science approach to problem solving and decision making, including model building, with the use of data. Following are a few examples of the many recent applications of analytics for problem solving in agriculture, media, urban planning, and sports.

Although the total world population is expected to grow by one-third to 9.6 billion in 2050, there will be less natural resources and land to support the necessary food production to feed an additional 2.4 billion people. Plant seed developer Syngenta is using analytics and management science models in its research and development efforts to develop and implement a plant-breeding strategy for soybeans that will improve the quality and quantity of the soybeans that farmers produce per acre. Their application of analytics enables better decisions that result in reducing the time and cost required to develop higher-productivity crops, saving Syngenta an estimated $287 million in a five-year period, while making a contribution to meeting the world's growing food needs.

iHeartMedia, Inc. (IHM) owns over 850 radio stations in more than 150 cities and provides programming (i.e., news, sports, traffic reports and weather) to over 2,250 stations. The company uses a set of management science models and sales data to maximize revenue from their inventory of radio advertising spots. Advertisers expect IHM to distribute their spots fairly and equitably across available inventory according to their order specifications, including dates, times, spot length, programs, stations, and demographic targets. IHM uses two linear programming models to assign advertising spots. The use of analytics has resulted in a more efficient use of available inventory, improved customer service, and enhanced sales from more accurate inventory visibility, resulting in a financial benefit of over a half million dollars annually.

ParkPGH is a decision analytics application that provides real-time and predictive information for garage parking space availability within the downtown Pittsburgh Cultural District. The model collects real time parking information for garage gate counts and uses historical data and event schedules to predict parking availability and provide downtown visitors with information on available parking via mobile devices and the Internet. The system has reduced parking space search times and changed the perception of downtown patrons about the downtown parking situation (including security and availability), and also helped garage operators better manage parking demand. In one year the parking application received over 300,000 inquiries.

One of the most visible applications of analytics in the sports industry has been the development and use of "sabermetrics"

San Gabriel Valley Tribune/ZUMA Press Inc./Alamy Stock Photo

in major league baseball, popularized by the book and movie *Moneyball*. It was originally defined in 1980 by Bill James (currently an analyst with the Boston Red Sox) as the "search for objective knowledge about baseball," and it is derived from the acronym SABR (e.g., Society for American Baseball Research). It has generally evolved into the application of statistical analysis of baseball records to develop predictive models and measures to evaluate and compare the in-game performance of individual players, usually in terms of runs or team wins. Sabermetrics attempts to answer questions such as, which players on a team will contribute most to the team's offense? For example, the sabermetric measure, VORP (value over replacement player), attempts to predict how much a hitter contributes offensively to his team in comparison to a fictitious average replacement player. A player might be worth 50 more runs in a season than a replacement level player at the same position (acquired at minimal cost). Currently every major league team has some employees in administrative positions dedicated to quantitative analytics for the evaluation of player performance to determine player acquisitions, trades, and contracts.

Sources: J. Byrum, C. Davis, G. Doonan, T. Doubler, D. Foster, B. Luzzi, R. Mowers, C. Zinselmeir, J. Klober, D. Culhane, and S. Mack, "Advanced Analytics for Agricultural Product Development," *Interfaces* 46, no. 1 (January–February 2016): 5–17; S. Venkatachalam, F. Wong, E. Uyar, S. Ward, and A. Aggarwal, "Media Company Uses Analytics to Schedule Radio Advertisement Spots," *Interfaces* 45, no. 6 (November–December 2015): 485–500; T. Fabusuyi, R. Hampshire, V. Hill, and K. Sasanuma, "Decision Analytics for Parking Availability in Downtown Pittsburgh," *Interfaces* 44, no. 3 (May–June 2014): 286–299.

Developing Employability Skills

For students to succeed in a rapidly changing job market, they need to know how to develop a variety of analytical and quantitative skills that they should be aware of for their career options. In this 13th edition of *Introduction to Management Science* we focus on developing these skills in the following ways.

Management Science Applications

Management Science Application boxes are located in every chapter in the text. They describe how a company, an organization, or an agency uses the particular management science technique being presented and demonstrated in the chapter to compete in a global environment. There are 48 of these boxes, 12 of which are new, throughout the text. They encompass a broad range of business and public-sector applications, both foreign and domestic.

Excel Spreadsheets

This new edition continues to emphasize Excel spreadsheet solutions of problems. Spreadsheet solutions are demonstrated in all the chapters in the text (except for Chapter 2, on linear programming modeling and graphical solution) for virtually every management science modeling technique presented. These spreadsheet solutions are presented in optional subsections, allowing the instructor to decide whether to cover them. The text includes more than 140 new Excel spreadsheet screenshots for Excel 2016. Most of these screenshots include reference callout boxes that describe the solution steps within the spreadsheet. Files that include all the Excel spreadsheet

Exhibit3.4 - Excel — The Beaver Creek Pottery Company

Formula bar: B12 | =C4*B10+D4*B11

	A	B	C	D	E	F	G	H	I	J	K	L
1	The Beaver Creek Pottery Company											
2												
3	Products:		Bowl	Mug						maximize $Z = 40x_1 + 50x_2$		
4	Profit per unit:		40	50		Resources						
5	Resources:				Usage	Constraint	Available	Left over		subject to:		
6	labor (hr/unit)		1	2	40	<=	40	0				
7	clay (lb/unit)		4	3	120	<=	120	0		$1x_1 + 2x_2 <= 40$		
8										$4x_1 + 3x_2 <= 120$		
9	Production:									$x_1, x_2 >= 0$		
10	Bowls =	24										
11	Mugs =	8										
12	Profit =	1360										
13												

model solutions for the examples in the text (data files) are included on the Companion Web site and can be easily downloaded by the student to determine how the spreadsheet was set up and the solution derived, and to use as templates to work homework problems. In addition, Appendix B at the end of the text provides a tutorial on how to set up and edit spreadsheets for problem solution. At top left is an example of one of the Excel spreadsheet files (from Chapter 3) that is available on the Companion Web site accompanying the text.
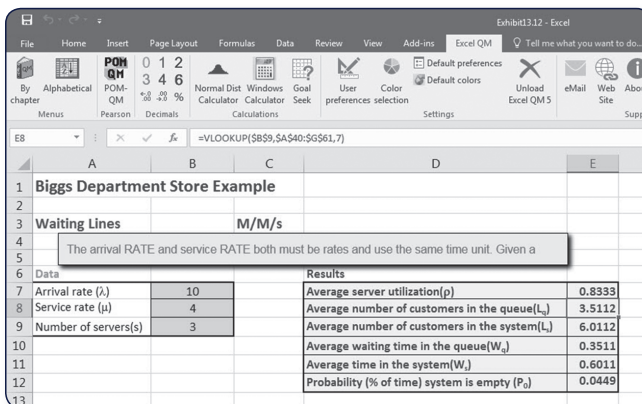
Spreadsheet Add-Ins

Several spreadsheet add-in packages are available with this book, often in trial and premium versions. For complete information on options for downloading each package, please visit *http://www.pearsonglobaleditions.com*

Excel QM

For some management science topics, the Excel formulas that are required for solution are lengthy and complex and thus are very tedious and time consuming to type into a spreadsheet. In several of these instances in the book, including Chapter 6 on transportation and assignment problems, Chapter 12 on decision analysis, Chapter 13 on queuing, Chapter 15 on forecasting, and Chapter 16 on inventory control, spreadsheet "add-ins" called Excel QM are demonstrated. These add-ins provide a generic spreadsheet setup with easy-to-use dialog boxes and all of the formulas already typed in for specific problem types. Unlike other "black box" software, these add-ins allow users to see the formulas used in each cell. The input, results, and the graphics are easily seen and can be easily changed, making this software ideal for classroom demonstrations and student explorations. At left is an example of an Excel QM file (from Chapter 13 on queuing analysis) that is on the Companion Web site that accompanies the text.

Exhibit13.12 - Excel — Biggs Department Store Example

Formula bar: E8 | =VLOOKUP($B$9,$A$40:$G$61,7)

	A	B	C	D	E
1	Biggs Department Store Example				
2					
3	Waiting Lines		M/M/s		
4			The arrival RATE and service RATE both must be rates and use the same time unit. Given a		
5					
6	Data			Results	
7	Arrival rate (λ)	10		Average server utilization(ρ)	0.8333
8	Service rate (μ)	4		Average number of customers in the queue(L_q)	3.5112
9	Number of servers(s)	3		Average number of customers in the system(L_s)	6.0112
10				Average waiting time in the queue(W_q)	0.3511
11				Average time in the system(W_s)	0.6011
12				Probability (% of time) system is empty (P_0)	0.0449
13					

Risk Solver Platform for Education

This program is a tool for risk analysis, simulation, and optimization in Excel. The Companion Web site will direct you to a trial version of the software.

TreePlan

Another spreadsheet add-in program that is demonstrated in the text is TreePlan, a program that will set up a generic spreadsheet for the solution of decision-tree problems in Chapter 12 on decision analysis. This is also available on the Companion Web site. At left is an example of one of the TreePlan files (from Chapter 12) that is on the text Companion Web site.

Exhibit12.14 - Excel — TreePlan for Real Estate Example

Formula bar: C20 | =MAX(G10,G20,G30)

TreePlan for Real Estate Example

Apartment Building
- 42000 / 0
- 0.6 Good Conditions — 50000 / 50000 — 50000
- 0.4 Poor Conditions — 30000 / 30000 — 30000

Office Building (44000)
- 44000 / 0
- 0.6 Good Conditions — 100000 / 100000 — 100000
- 0.4 Poor Condition — -40000 / -40000 — -40000

Warehouse
- 22000 / 0
- 0.6 Good Conditions — 30000 / 30000 — 30000
- 0.4 Poor Conidtions — 10000 / 10000 — 10000

Crystal Ball

Another spreadsheet add-in program is Crystal Ball by Oracle. Crystal Ball is demonstrated in Chapter 14 on simulation and shows how to perform simulation analysis for certain types of risk analysis and forecasting problems. Here is an example of one of the Crystal Ball files (from Chapter 14) that is on the Companion Web site. The Companion Web site will direct you to a trial version of the software.

QM for Windows Software Package

QM for Windows is a computer package that is included on the text Companion Web site, and many students and instructors will prefer to use it with this text. This software is very user-friendly, requiring virtually no preliminary instruction except for the "help" screens that can be accessed directly from the program. It is demonstrated throughout the text in conjunction with virtually every management science modeling technique, except simulation. The text includes

50 QM for Windows screens used to demonstrate example problems. Thus, for most topics problem solution is demonstrated via both Excel spreadsheets and QM for Windows. Files that include all the QM for Windows solutions, for example, in the text are included on the accompanying Companion Web site. Here is an example of one of the QM for Windows files (from Chapter 4 on linear programming) that is on the Companion Web site.

Microsoft Project

Chapter 8 on project management includes the popular software package Microsoft Project. Here is an example of one of the Microsoft Project files (from Chapter 8) that is available on the text Companion Web site. The Companion Web site will direct you to a trial version of the software.

Problems and Cases

Previous editions of the text always provided a substantial number of homework questions, problems, and cases for students to practice on. This edition includes more than 800 homework problems, 20 of which are new, and 69 end-of-chapter case problems.

Example Problem Solutions

As a prelude to the problems, this section presents example solutions to two linear programming problems.

Problem Statement

Moore's Meatpacking Company produces a hot dog mixture in 1,000-pound batches. The mixture contains two ingredients—chicken and beef. The cost per pound of each of these ingredients is as follows:

Ingredient	Cost/lb.
Chicken	$3
Beef	$5

Each batch has the following recipe requirements:

a. At least 500 pounds of chicken
b. At least 200 pounds of beef

The ratio of chicken to beef must be at least 2 to 1. The company wants to know the optimal mixture of ingredients that will minimize cost. Formulate a linear programming model for this problem.

Solution

Step 1: Identify Decision Variables

Recall that the problem should not be "swallowed whole." Identify each part of the model separately, starting with the decision variables:

$$x_1 = \text{lb. of chicken}$$
$$x_2 = \text{lb. of beef}$$

Step 2: Formulate the Objective Function

$$\text{minimize } Z = \$3x_1 + \$5x_2$$
where
$$Z = \text{cost per 1,000-lb batch}$$
$$\$3x_1 = \text{cost of chicken}$$
$$\$5x_2 = \text{cost of beef}$$

Step 3: Establish Model Constraints

The constraints of this problem are embodied in the recipe restrictions and (not to be overlooked) the fact that each batch must consist of 1,000 pounds of mixture:

$$x_1 + x_2 = 1,000 \text{ lb.}$$
$$x_1 \geq 500 \text{ lb. of chicken}$$
$$x_2 \geq 200 \text{ lb. of beef}$$
$$x_1/x_2 \geq 2/1 \text{ or } x_1 - 2x_2 \geq 0$$

and

$$x_1, x_2 \geq 0$$

The Model

$$\text{minimize } Z = \$3x_1 + \$5x_2$$
$$\text{subject to}$$
$$x_1 + x_2 = 1,000$$
$$x_1 \geq 500$$
$$x_2 \geq 200$$
$$x_1 - 2x_2 \geq 0$$
$$x_1, x_2 \geq 0$$

Marginal Notes

Notes in the margins of this text serve the same basic function as notes that students themselves might write in the margin. They highlight certain topics to make it easier for students to locate them, summarize topics and important points, and provide brief definitions of key terms and concepts.

Examples

The primary means of teaching the various quantitative modeling techniques presented in this text is through examples. Thus, examples are liberally inserted throughout the text, primarily to demonstrate how problems are solved with the different quantitative techniques and to make them easier to understand. These examples are organized in a logical step-by-step solution approach that the student can subsequently apply to the homework problems.

Example Problem Solutions

At the end of each chapter, just prior to the homework questions and problems, is a section that provides solved examples to serve as a guide for doing the homework problems. These examples are solved in a detailed, step-by-step fashion. Here is an example from Chapter 2.

Chapter Web Links

The files on the Companion Web site contains Chapter Web links for every chapter in the text. These Web links access tutorials, summaries, and notes available on the Internet for the various techniques and topics in every chapter in the text. Also included are YouTube videos that provide additional learning resources and tutorials about many of the topics and techniques, links to the development and developers of the techniques in the text, and links to the Web sites for the companies and organizations that are featured in the "Management Science Application" boxes in every chapter. The "Chapter Web links" file includes more than 550 Web links.

Chapter Modules

Several of the strictly mathematical topics—such as the simplex and transportation solution methods—are included as chapter modules on the Companion Web site, at *http://www.pearson globaleditions.com*

Table of Contents Overview

An important objective is to have a well-organized text that flows smoothly and follows a logical progression of topics, placing the different management science modeling techniques in their proper perspective. The following Figure 1.6 from Chapter 1 outlines the organization of topics in the book.

The first 10 chapters are related to mathematical programming that can be solved using Excel spreadsheets, including linear, integer, nonlinear, and goal programming, as well as network techniques.

Within these mathematical programming chapters, the traditional simplex procedure for solving linear programming problems mathematically is located in Module A on the Companion Web site, at *http://www.pearson globaleditions.com*, that accompanies this text. It can still be covered by the student on the computer as part of linear programming, or it can be excluded, without leaving

FIGURE 1.6
Classification of management science techniques
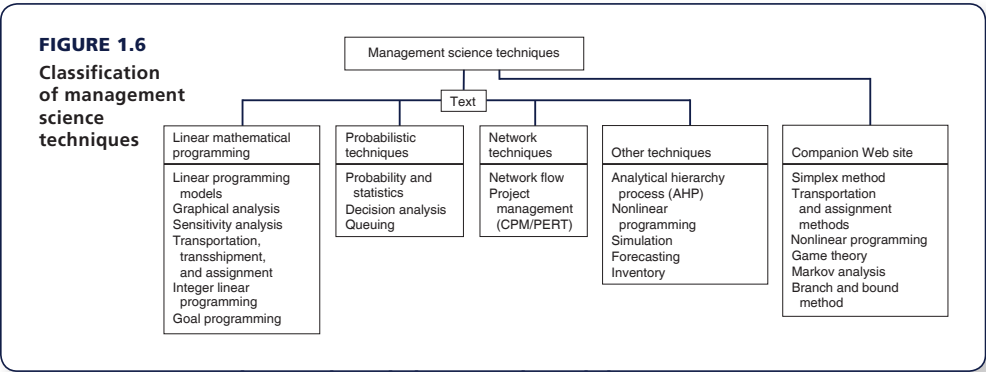
a "hole" in the presentation of this topic. The integer programming mathematical branch and bound solution method (Chapter 5) is located in Module C on the Companion Web site. In Chapter 6, on the transportation and assignment problems, the strictly mathematical solution approaches, including the northwest corner, VAM, and stepping-stone methods, are located in Module B on the Companion Web site. Because transportation and assignment problems are specific types of network problems, the two chapters that cover network flow models and project networks that can be solved with linear programming, as well as traditional model-specific solution techniques and software, follow Chapter 6 on transportation and assignment problems. In addition, in Chapter 10, on nonlinear programming, the traditional mathematical solution techniques, including the substitution method and the method of Lagrange multipliers, are located in Module D on the Companion Web site.

Chapters 11 through 14 include topics generally thought of as being probabilistic, including probability and statistics, decision analysis, queuing, and simulation. Module F on Markov analysis and Module E on game theory are on the Companion Web site. Forecasting in Chapter 15 and inventory management in Chapter 16 are both unique topics related to operations management.

Instructor Teaching Resources

This text comes with the following teaching resources.

Supplements available to instructors at *http://www.pearsonglobaleditions.com*	Features of the Supplement
Instructor's Solutions Manual developed by the author	• Detailed solutions for all end-of-chapter exercises and cases • One file per chapter provided in MS Word format
Excel Homework Solutions developed by the author	• A corresponding Excel solution file for almost all 840 end-of-chapter homework and case problems in the text • Organized by chapter and problem number • Also includes Excel homework solution files for TreePlan, Crystal Ball, and Microsoft Project
Test Bank authored by Geoff Willis of the University of Central Oklahoma	• 2,000 questions, including true/false, multiple-choice, and problem-solving questions for each chapter • Each question followed by the correct answer, page references, main headings, difficulty rating, and key words
TestGen® Computerized Test Bank	• Pearson Education's test-generating software, PC and Mac compatible, and preloaded with all of the Test Bank questions • Can manually or randomly view test questions and drag and drop to create a test • Can add or modify test bank questions as needed

PowerPoint Presentations authored by Geoff Willis of the University of Central Oklahoma	• Available for every chapter
	• Features figures, tables, Excel spreadsheets, and main points
	• They meet accessibility standards for students with disabilities. Features include, but not limited to:
	  ○ Keyboard and Screen Reader access
	  ○ Alternative text for images
	  ○ High color contrast between background and foreground colors

Chapter Web Links developed by the author	• Internet links to tutorials, summaries, notes and videos

Acknowledgments

As with any other large project, the revision of a textbook is not accomplished without the help of many people. The 13th edition of this book is no exception, and I would like to take this opportunity to thank those who have contributed to its preparation.

I thank the reviewers of this and previous editions: Dr. B. S. Bal, Nagraj Balakrishnan, Edward M. Barrow, Ali Behnezhad, Weldon J. Bowling, Rod Carlson, Petros Christofi, Yar M. Ebadi, Richard Ehrhardt, Warren W. Fisher, James Flynn, Wade Furgeson, Soumen Ghosh,

Global Edition Acknowledgments

Management Science

Management science is the application of a scientific approach to solving management problems to help managers make better decisions. As implied by this definition, management science encompasses a number of mathematically oriented techniques that have either been developed within the field of management science or been adapted from other disciplines, such as the natural sciences, mathematics, statistics, and engineering. This text provides an introduction to the techniques that make up management science and demonstrates their applications to management problems.

Management science is a scientific approach to solving management problems.

Management science is a recognized and established discipline in business. The applications of management science techniques are widespread, and they have been frequently credited with increasing the efficiency and productivity of business firms. In various surveys of businesses, many indicate that they use management science techniques, and most rate the results to be very good. Management science (also referred to as *operations research*, *quantitative methods*, *quantitative analysis*, *decision sciences*, and *business analytics*) is part of the fundamental curriculum of most programs in business.

Management science can be used in a variety of organizations to solve many different types of problems.

As you proceed through the various management science models and techniques contained in this text, you should remember several things. First, most of the examples presented in this text are for business organizations because businesses represent the main users of management science. However, management science techniques can be applied to solve problems in different types of organizations, including services, government, military, business and industry, and health care.

Second, in this text all the modeling techniques and solution methods are mathematically based. In some instances the manual, mathematical solution approach is shown because it helps one understand how the modeling techniques are applied to different problems. However, a computer solution is possible for each of the modeling techniques in this text, and in many cases the computer solution is emphasized. The more detailed mathematical solution procedures for many of the modeling techniques are included as supplemental modules on the companion Web site for this text.

Management science encompasses a logical approach to problem solving.

Finally, as the various management science techniques are presented, keep in mind that management science is more than just a collection of techniques. Management science also involves the philosophy of approaching a problem in a logical manner (i.e., a scientific approach). The logical, consistent, and systematic approach to problem solving can be as useful (and valuable) as the knowledge of the mechanics of the mathematical techniques themselves. This understanding is especially important for those readers who do not always see the immediate benefit of studying mathematically oriented disciplines such as management science.

The Management Science Approach to Problem Solving

The steps of the scientific method are (1) observation, (2) problem definition, (3) model construction, (4) model solution, and (5) implementation.

As indicated in the previous section, management science encompasses a logical, systematic approach to problem solving, which closely parallels what is known as the **scientific method** for attacking problems. This approach, as shown in Figure 1.1, follows a generally recognized and ordered series of steps: (1) observation, (2) definition of the problem, (3) model construction, (4) model solution, and (5) implementation of solution results. We will analyze each of these steps individually in this text.

Observation

The first step in the management science process is the identification of a problem that exists in the system (organization). The system must be continuously and closely observed so that problems can be identified as soon as they occur or are anticipated. Problems are not always the result of a crisis that must be reacted to but, instead, frequently involve an anticipatory or planning situation. The person who normally identifies a problem is the manager because managers work in places where problems might occur. However, problems can often be identified by a

FIGURE 1.1

The management science process
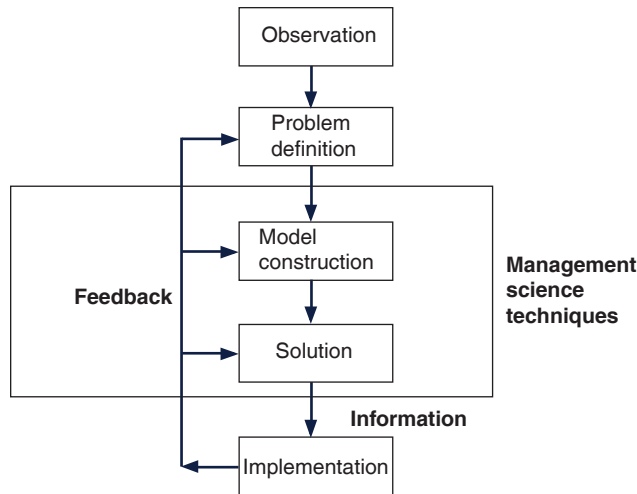
A ***management scientist*** *is a person skilled in the application of management science techniques.*

management scientist, a person skilled in the techniques of management science and trained to identify problems, who has been hired specifically to solve problems using management science techniques.

Definition of the Problem

Once it has been determined that a problem exists, the problem must be clearly and concisely *defined*. Improperly defining a problem can easily result in no solution or an inappropriate solution. Therefore, the limits of the problem and the degree to which it pervades other units of the organization must be included in the problem definition. Because the existence of a problem implies that the objectives of the firm are not being met in some way, the goals (or objectives) of the organization must also be clearly defined. A stated objective helps to focus attention on what the problem actually is.

Model Construction

A ***model*** *is an abstract mathematical representation of a problem situation.*

A management science **model** is an abstract representation of an existing problem situation. It can be in the form of a graph or chart, but most frequently a management science model consists of a set of mathematical relationships. These mathematical relationships are made up of numbers and symbols.

As an example, consider a business firm that sells a product. The product costs \$5 to produce and sells for \$20. A model that computes the total profit that will accrue from the items sold is

$$Z = \$20x - 5x$$

A ***variable*** *is a symbol used to represent an item that can take on any value.*

In this equation, x represents the number of units of the product that are sold, and Z represents the total profit that results from the sale of the product. The *symbols x and Z* are *variables*. The term **variable** is used because no set numeric value has been specified for these items. The number of units sold, x, and the profit, Z, can be any amount (within limits); they can vary. These two variables can be further distinguished. Z is a *dependent variable* because its value is dependent on the number of units sold; x is an *independent variable* because the number of units sold is *not* dependent on anything else (in this equation).

Parameters *are known, constant values that are often coefficients of variables in equations.*

The numbers \$20 and \$5 in the equation are referred to as **parameters**. Parameters are constant values that are generally coefficients of the variables (symbols) in an equation.

Data are pieces of information from the problem environment.

Parameters usually remain constant during the process of solving a specific problem. The parameter values are derived from **data** (i.e., pieces of information) from the problem environment. Sometimes the data are readily available and quite accurate. For example, presumably the selling price of $20 and product cost of $5 could be obtained from the firm's accounting department and would be very accurate. However, sometimes data are not as readily available to the manager or firm, and the parameters must be either estimated or based on a combination of the available data and estimates. In such cases, the model is only as accurate as the data used in constructing the model.

A model is a functional relationship that includes variables, parameters, and equations.

The equation as a whole is known as a **functional relationship** (also called *function* and *relationship*). The term is derived from the fact that profit, Z, is a *function* of the number of units sold, x, and the equation *relates* profit to units sold.

Because only one functional relationship exists in this example, it is also the *model*. In this case, the relationship is a model of the determination of profit for the firm. However, this model does not really replicate a problem. Therefore, we will expand our example to create a problem situation.

Let us assume that the product is made from steel and that the business firm has 100 pounds of steel available. If it takes 4 pounds of steel to make each unit of the product, we can develop an additional mathematical relationship to represent steel usage:

$$4x = 100 \text{ lb. of steel}$$

This equation indicates that for every unit produced, 4 of the available 100 pounds of steel will be used. Now our model consists of two relationships:

$$Z = \$20x - 5x$$
$$4x = 100$$

We say that the profit equation in this new model is an *objective function*, and the resource equation is a *constraint*. In other words, the objective of the firm is to achieve as much profit, Z, as possible, but the firm is constrained from achieving an infinite profit by the limited amount of steel available. To signify this distinction between the two relationships in this model, we will add the following notations:

$$\text{maximize } Z = \$20x - 5x$$
$$\text{subject to}$$
$$4x = 100$$

This model now represents the manager's problem of determining the number of units to produce. You will recall that we defined the number of units to be produced as x. Thus, when we determine the value of x, it represents a potential (or recommended) *decision* for the manager. Therefore, x is also known as a *decision variable*. The next step in the management science process is to solve the model to determine the value of the decision variable.

Model Solution

A management science technique usually applies to a specific model type.

Once models have been constructed in management science, they are solved using the management science techniques presented in this text. A management science solution technique usually applies to a specific type of model. Thus, the model type and solution method are both part of the management science technique. We are able to say that *a model is solved* because the model represents a problem. When we refer to model solution, we also mean problem solution.

Time Out for Pioneers in Management Science

Throughout this text, TIME OUT boxes introduce you to the individuals who developed the various techniques that are described in the chapters. This provides a historical perspective on the development of the field of management science. In this first instance, we will briefly outline the development of management science.

Although a number of the mathematical techniques that make up management science date to the turn of the twentieth century or before, the field of management science itself can trace its beginnings to military operations research (OR) groups formed during World War II in Great Britain circa 1939. These OR groups typically consisted of a team of about a dozen individuals from different fields of science, mathematics, and the military, brought together to find solutions to military-related problems. One of the most famous of these groups—called "Blackett's circus" after its leader, Nobel Laureate P. M. S. Blackett of the University of Manchester and a former naval officer—included three physiologists, two mathematical physicists, one astrophysicist, one general physicist, two mathematicians, an Army officer, and a surveyor. Blackett's group and the other OR teams made significant contributions in improving Britain's early-warning radar system (which was instrumental in their victory in the Battle of Britain), aircraft gunnery, antisubmarine warfare, civilian defense, convoy size determination, and bombing raids over Germany.

The successes achieved by the British OR groups were observed by two Americans working for the U.S. military, Dr. James B. Conant and Dr. Vannevar Bush, who recommended that OR teams be established in the U.S. branches of the military. Subsequently, both the Air Force and Navy created OR groups.

After World War II, the contributions of the OR groups were considered so valuable that the Army, Air Force, and Navy set up various agencies to continue research of military problems. Two of the more famous agencies were the Navy's Operations Evaluation Group at MIT and Project RAND, established by the Air Force to study aerial warfare. Many of the individuals who developed OR and management science techniques did so while working at one of these agencies after World War II or as a result of their work there.

As the war ended and the mathematical models and techniques that were kept secret during the war began to be released, there was a natural inclination to test their applicability to business problems. At the same time, various consulting firms were established to apply these techniques to industrial and business problems, and courses in the use of quantitative techniques for business management began to surface in American universities. In the early 1950s, the use of these quantitative techniques to solve management problems became known as management science, and it was popularized by a book of that name by Stafford Beer of Great Britain.

For the example model developed in the previous section,

$$\text{maximize } Z = \$20x - 5x$$
$$\text{subject to}$$
$$4x = 100$$

the solution technique is simple algebra. Solving the constraint equation for x, we have

$$4x = 100$$
$$x = 100/4$$
$$x = 25 \text{ units}$$

Substituting the value of 25 for x into the profit function results in the total profit:

$$Z = \$20x - 5x$$
$$= 20(25) - 5(25)$$
$$= \$375$$

A management science solution can be either a recommended decision or information that helps a manager make a decision.

Thus, if the manager decides to produce 25 units of the product and all 25 units sell, the business firm will receive $375 in profit. Note, however, that the value of the decision variable does not constitute an actual decision; rather, it is *information* that serves as a recommendation or guideline, helping the manager make a decision.

Some management science techniques do not generate an answer or a recommended decision. Instead, they provide *descriptive results:* results that describe the system being modeled. For

example, suppose the business firm in our example desires to know the average number of units sold each month during a year. The monthly *data* (i.e., sales) for the past year are as follows:

Month	Sales	Month	Sales
January	30	July	35
February	40	August	50
March	25	September	60
April	60	October	40
May	30	November	35
June	25	December	50
		Total	480 units

Monthly sales average 40 units (480 ÷ 12). This result is not a decision; it is information that describes what is happening in the system. The results of the management science techniques

Management Science Application

Room Pricing with Management Science and Analytics at Marriott

Marriott International, Inc., headquartered in Bethesda, Maryland, has more than 140,000 employees working at more than 3,300 hotels in 70 countries. Its hotel franchises include Marriott, JW Marriott, The Ritz-Carlton, Renaissance, Residence Inn, Courtyard, TownePlace Suites, Fairfield Inn, and Springhill Suites. *Fortune* magazine ranks Marriott as the lodging industry's most admired company and one of the best companies to work for.

Marriott uses a revenue management system for individual hotel bookings. This system provides forecasts of customer demand and pricing controls, makes optimal inventory allocations, and interfaces with a reservation system that handles more than 75 million transactions each year. The system makes a demand forecast for each rate category and length of stay for each arrival day up to 90 days in advance, and it provides inventory allocations to the reservation system. This inventory of hotel rooms is then sold to individual customers through channels such as Marriott.com, the company's toll-free reservation number, the hotels directly, and global distribution systems.

One of the most significant revenue streams for Marriott is for group sales, which can contribute more than half of a full-service hotel's revenue. However, group business has challenging characteristics that introduce uncertainty and make modeling it difficult, including longer booking windows (as compared to those for individuals), price negotiation as part of the booking process, demand for blocks of rooms, and lack of demand data. For a group request, a hotel must know if it has sufficient rooms and determine a recommended rate. A key challenge is estimating the value of the business the hotel is turning away if the room

David Zanzinger/Alamy Stock Photo

inventory is given to a group rather than being held for individual bookings.

To address the group booking process, Marriott developed a decision support system, Group Pricing Optimizer (GPO), that provides guidance to Marriott personnel on pricing hotel rooms for group customers. GPO uses various management science modeling techniques and tools, including simulation, forecasting, and optimization techniques, to recommend an optimal price rate. Marriott estimates that GPO provided an improvement in profit of over $120 million derived from $1.3 billion in group business in its first 2 years of use.

Source: Based on S. Hormby, J. Morrison, P. Dave, M. Myers, and T. Tenca, "Marriott International Increases Revenue by Implementing a Group Pricing Optimizer," *Interfaces* 40, no. 1 (January–February 2010): 47–57.

in this text are examples of the two types shown in this section: (1) solutions/decisions and (2) descriptive results.

Implementation

Implementation is the actual use of a model once it has been developed.

The final step in the management science process for problem solving described in Figure 1.1 is implementation. **Implementation** is the actual use of the model once it has been developed or the solution to the problem the model was developed to solve. This is a critical but often overlooked step in the process. It is not always a given that once a model is developed or a solution found, it is automatically used. Frequently the person responsible for putting the model or solution to use is not the same person who developed the model, and thus the user may not fully understand how the model works or exactly what it is supposed to do. Individuals are also sometimes hesitant to change the normal way they do things or to try new things. In this situation, the model and solution may get pushed to the side or ignored altogether if they are not carefully explained and their benefit fully demonstrated. If the management science model and solution are not implemented, then the effort and resources used in their development have been wasted.

Management Science and Business Analytics

Analytics is the latest hot topic and new buzzword in business. Companies are establishing analytics departments, and the demand for employees with analytics skills and expertise is growing faster than almost any other business skill set. Universities and business schools are developing new degree programs and courses in analytics. So exactly what is this new and very popular area called business analytics, and how does it relate to management science?

Business analytics uses large amounts of data with management science techniques and modeling to help managers makes decisions.

Business analytics is a somewhat general term that seems to have a number of different definitions, but in broad terms it is considered to be a process for using large amounts of data combined with information technology, statistics, management science techniques, and mathematical modeling to help managers solve problems and make decisions that will improve their business performance. It makes use of these technological tools to help businesses understand their past performance and to help them plan and make decisions for the future; thus analytics is said to be descriptive, predictive, and prescriptive.

Much as the word *science* groups together a number of disciplines such as chemistry, biology, physics, geology, and so on, the word *analytics* seems to group together disciplines such as management science, operations research, statistics, computer science, engineering, data science, and so on. All of these disciplines (and thus analytics, in general) have in common the scientific method for addressing problems that was discussed in the previous section.

A key component of business analytics is the recent availability of large amounts of data—called "big data"—that is now accessible to businesses, and that is perceived to be an integral part and starting point of the analytical process. Data are considered to be the engine that drives the process of analysis and decision making in business analytics. For example, a bank might apply analytics by using data to determine different customer characteristics to match them with the bank services they provide; or a retail store might apply analytics by using data to determine which styles of denim jeans match their customer preferences, determine how many jeans to order from their foreign suppliers, how much inventory to keep on hand, and when the best time is to sell the jeans and what is the best price.

If you have not already noticed, analytics is very much like the "management science approach to problem solving" that we have already described in the previous section. In fact, many in business perceive business analytics to just be a repackaged version of management science. In some business schools, management science courses are simply being renamed as "analytics." Business students are being advised that in the future companies will expect them to have an analytics skill set, and these skills need to include knowledge of statistics, mathematical modeling, and quantitative tools—the topics traditionally considered to be management science and that are covered in this text.

For our purposes in studying management science, it is clear that the quantitative tools and techniques that are included in this book are an important major part of business analytics, no

matter what the definition of the business analytics process is. As such, becoming skilled in the use of these management science techniques is a necessary and important step for someone who wants to become a business analytics professional.

Developing Analytical Career Skills

As you learn the management science techniques in this text, you may think that they are not relevant to what you imagine your future job might be. However, you can be assured that this is not the case. Whether or not you plan on a career in which management science or business analytics is a key part, the logical, analytical approach to problem solving and decision making employed in management science will help you in any career path you choose. It is through the aggregate of your educational experiences that you will develop the skills that employers have identified as critical to success in the workplace.

Critical thnking purpsoeful and goal directed thinking used to define and solve problems, make decsions and form judgements related to a particular situation.

Management science provides students with many of these skill sets besides just the quantitative techniques it comprises that employers will seek and value in business graduates who market themselves as having expertise in business analytics. The ability to bring **critical thinking** to problem-solving scenarios is an important aspect of business analytics, which management science provides. Critical thinking involves purposeful and goal directed thinking used to define and solve problems, make decisions and form judgements related to a particular situation or set of circumstances. This is what management science does by providing a structured format for looking at problems and defining them, formulating them as mathematical models, and providing approaches to solving them that result in decisions that achieve the organization's goals.

Collaboration a necessary skill for decision scenarios that take place in a project team-based environment.

Many decision-making scenarios take place in a project team-based environment where **collaboration** is a necessary skill. Management science provides an approach to problem solving in which individuals may actively work together on a problem using their combined efforts to construct meaning and knowledge as a group through dialogue and negotiation that ultimately results in a modeling approach that reflects their joint actions. Chapter 8 on "Project Management" directly addresses this collaborative project-based approach to decision making.

Information technology and computing skills important attributes to employers because of the reliance on computer software to solve decison problems.

Because of its reliance on computer software to solve decision problems that students will learn about in this text, management science teaches **information technology and computing skills** that are very important to employers. Implicit in the management science approach is the ability to select and use the appropriate technology to solve a particular type of modeling problem. The student of management science learns how to apply computing skills to solve problems and show proficiency with the various computer software programs introduced in the text, including Excel, QM for Windows, MS Project, Crystal Ball, and Treeplan, among others.

Data Literacy the ability to acccess, interpret, manipulate, summarize and commnicte data in a decison-making situation.

We have already noted the importance of data in business analytics, and management science provides a platform for a student to develop **data literacy**, the ability to access, interpret, manipulate, summarize, and communicate data in a decision-making situation.

Model Building: Break-Even Analysis

In the previous section, we gave a brief, general description of how management science models are formulated and solved, using a simple algebraic example. In this section, we will continue to explore the process of building and solving management science models, using **break-even analysis**, also called *profit analysis*. Break-even analysis is a good topic to expand our discussion of model building and solution because it is straightforward, relatively familiar to most people, and not overly complex. In addition, it provides a convenient means to demonstrate the different ways management science models can be solved—mathematically (by hand), graphically, and with a computer.

Break-even analysis is a modeling technique to determine the number of units to sell or produce that will result in zero profit.

The purpose of break-even analysis is to determine the number of units of a product (i.e., the volume) to sell or produce that will equate total revenue with total cost. The point where total revenue equals total cost is called the *break-even point*, and at this point profit is zero. The break-even point gives a manager a point of reference in determining how many units will be needed to ensure a profit.

Components of Break-Even Analysis

The three components of break-even analysis are volume, cost, and profit. *Volume* is the level of sales or production by a company. It can be expressed as the number of units (i.e., quantity) produced and sold, as the dollar volume of sales, or as a percentage of total capacity available.

Two types of cost are typically incurred in the production of a product: fixed costs and variable costs. **Fixed costs** are generally independent of the volume of units produced and sold. That is, fixed costs remain constant, regardless of how many units of product are produced within a given range. Fixed costs can include such items as rent on plant and equipment, taxes, staff and management salaries, insurance, advertising, depreciation, heat and light, and plant maintenance. Taken together, these items result in total fixed costs.

Fixed costs are independent of volume and remain constant.

Variable costs are determined on a per-unit basis. Thus, total variable costs depend on the number of units produced. Variable costs include such items as raw materials and resources, direct labor, packaging, material handling, and freight.

Variable costs depend on the number of items produced.

Total variable costs are a function of the *volume* and the *variable cost per unit*. This relationship can be expressed mathematically as

$$\text{total variable cost} = vc_v$$

where c_v = variable cost per unit and v = volume (number of units) sold.

The **total cost** of an operation is computed by summing total fixed cost and total variable cost, as follows:

Total cost (TC) equals the fixed cost (c_f) plus the variable cost per unit (c_v) multiplied by volume (v).

$$\text{total cost} = \text{total fixed cost} + \text{total variable cost}$$

or

$$TC = c_f + vc_v$$

where c_f = fixed cost.

As an example, consider Western Clothing Company, which produces denim jeans. The company incurs the following monthly costs to produce denim jeans:

$$\text{fixed cost} = c_f = \$10{,}000$$
$$\text{variable cost} = c_v = \$8 \text{ per pair}$$

If we arbitrarily let the monthly sales volume, v, equal 400 pairs of denim jeans, the total cost is

$$TC = c_f + vc_v = \$10{,}000 + (400)(8) = \$13{,}200$$

Profit is the difference between total revenue (volume multiplied by price) and total cost.

The third component in our break-even model is **profit**. Profit is the difference between *total revenue* and total cost. Total revenue is the volume multiplied by the price per unit,

$$\text{total revenue} = vp$$

where p = price per unit.

For our clothing company example, if denim jeans sell for $23 per pair and we sell 400 pairs per month, then the total monthly revenue is

$$\text{total revenue} = vp = (400)(23) = \$9,200$$

Now that we have developed relationships for total revenue and total cost, profit (Z) can be computed as follows:

$$\text{total profit} = \text{total revenue} - \text{total cost}$$
$$Z = vp - (c_f + vc_v)$$
$$= vp - c_f - vc_v$$

Computing the Break-Even Point

For our clothing company example, we have determined total revenue and total cost to be $9,200 and $13,200, respectively. With these values, there is no profit but, instead, a loss of $4,000:

$$\text{total profit} = \text{total revenue} - \text{total cost} = \$9,200 - 13,200 = -\$4,000$$

We can verify this result by using our total profit formula,

$$Z = vp - c_f - vc_v$$

and the values $v = 400$, $p = \$23$, $c_f = \$10,000$, and $c_v = \$8$:

$$Z = vp - c_f - vc_v$$
$$= \$(400)(23) - 10,000 - (400)(8)$$
$$= \$9,200 - 10,000 - 3,200$$
$$= -\$4,000$$

Obviously, the clothing company does not want to operate with a monthly loss of $4,000 because doing so might eventually result in bankruptcy. If we assume that price is static because of market conditions and that fixed costs and the variable cost per unit are not subject to change, then the only part of our model that can be varied is *volume*. Using the modeling terms we developed earlier in this chapter, price, fixed costs, and variable costs are parameters, whereas the volume, v, is a decision variable. In break-even analysis, we want to compute the value of v that will result in zero profit.

*The **break-even point** is the volume (v) that equates total revenue with total cost where profit is zero.*

At the **break-even point**, where total revenue equals total cost, the profit, Z, equals zero. Thus, if we let profit, Z, equal zero in our total profit equation and solve for v, we can determine the break-even volume:

$$Z = vp - c_f - vc_v$$
$$0 = v(23) - 10,000 - v(8)$$
$$0 = 23v - 10,000 - 8v$$
$$15v = 10,000$$
$$v = 666.7 \text{ pairs of jeans}$$

In other words, if the company produces and sells 666.7 pairs of jeans, the profit (and loss) will be zero, and the company will *break even*. This gives the company a point of reference from which to determine how many pairs of jeans it needs to produce and sell to gain a profit

(subject to any capacity limitations). For example, a sales volume of 800 pairs of denim jeans will result in the following monthly profit:

$$Z = vp - c_f - vc_v$$
$$= \$(800)(23) - 10,000 - (800)(8) = \$2,000$$

In general, the break-even volume can be determined using the following formula:

$$Z = vp - c_f - vc_v$$
$$0 = v(p - c_v) - c_f$$
$$v(p - c_v) = c_f$$
$$v = \frac{c_f}{p - c_v}$$

For our example,

$$v = \frac{c_f}{p - c_v}$$
$$= \frac{10,000}{23 - 8}$$
$$= 666.7 \text{ pairs of jeans}$$

Graphical Solution

It is possible to represent many of the management science models in this text graphically and use these graphical models to solve problems. Graphical models also have the advantage of providing a "picture" of the model that can sometimes help us understand the modeling process better than mathematics alone can. We can easily graph the break-even model for our Western Clothing Company example because the functions for total cost and total revenue are *linear*. That means we can graph each relationship as a straight line on a set of coordinates, as shown in Figure 1.2.

FIGURE 1.2

Break-even model
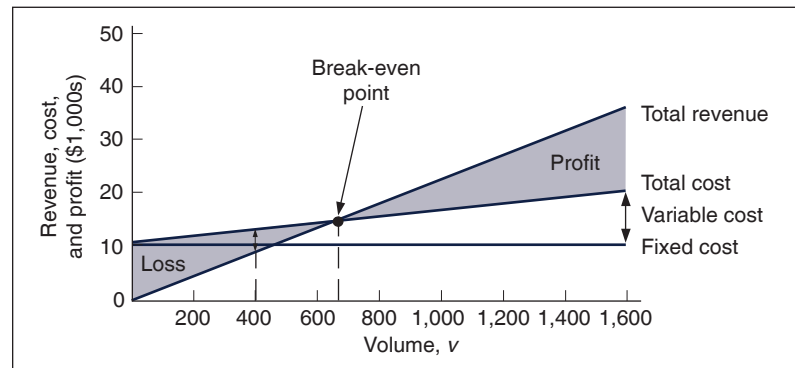
In Figure 1.2, the fixed cost, c_f, has a constant value of \$10,000, regardless of the volume. The total cost line, *TC*, represents the sum of variable cost and fixed cost. The total cost line increases because variable cost increases as the volume increases. The total revenue line also increases as volume increases, but at a faster rate than total cost. The point where these two lines intersect indicates that total revenue equals total cost. The volume, *v*, that corresponds to this point is the *break-even volume*. The break-even volume in Figure 1.2 is 666.7 pairs of denim jeans.

Sensitivity Analysis

We have now developed a general relationship for determining the break-even volume, which was the objective of our modeling process. This relationship enables us to see how the level of profit (and loss) is directly affected by changes in volume. However, when we developed this

model, we assumed that our parameters, fixed and variable costs and price, were constant. In reality such parameters are frequently uncertain and can rarely be assumed to be constant, and changes in any of the parameters can affect the model solution. The study of changes on a management science model is called **sensitivity analysis**—that is, seeing how sensitive the model is to changes.

Sensitivity analysis sees how sensitive a management model is to changes.

Sensitivity analysis can be performed on all management science models in one form or another. In fact, sometimes companies develop models for the primary purpose of experimentation to see how the model will react to different changes the company is contemplating or that management might expect to occur in the future. As a demonstration of how sensitivity analysis works, we will look at the effects of some changes on our break-even model.

In general, an increase in price lowers the break-even point, all other things held constant.

The first thing we will analyze is price. As an example, we will increase the price for denim jeans from $23 to $30. As expected, this increases the total revenue, and it therefore reduces the break-even point from 666.7 pairs of jeans to 454.5 pairs of jeans:

$$v = \frac{c_f}{p - c_v}$$

$$= \frac{10,000}{30 - 8} = 454.5 \text{ pairs of denim jeans}$$

The effect of the price change on break-even volume is illustrated in Figure 1.3.
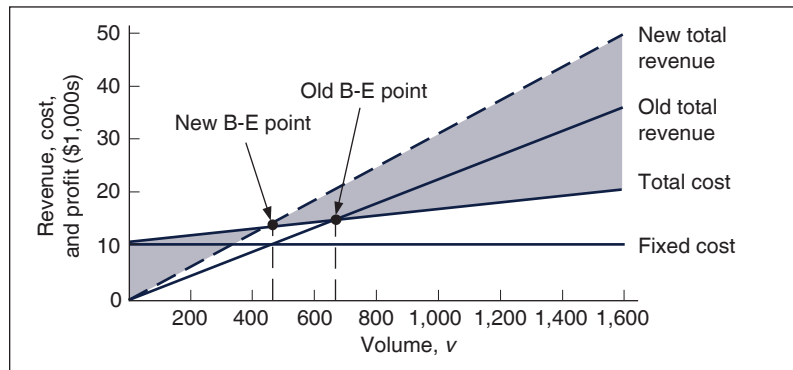
Although a decision to increase price looks inviting from a strictly analytical point of view, it must be remembered that the lower break-even volume and higher profit are *possible* but not guaranteed. A higher price can make it more difficult to sell the product. Thus, a change in price often must be accompanied by corresponding increases in costs, such as those for advertising, packaging, and possibly production (to enhance quality). However, even such direct changes as these may have little effect on product demand because price is often sensitive to numerous factors, such as the type of market, monopolistic elements, and product differentiation.

In general, an increase in variable costs will increase the break-even point, all other things held constant.

When we increased price, we mentioned the possibility of raising the quality of the product to offset a potential loss of sales due to the price increase. For example, suppose the stitching on the denim jeans is changed to make the jeans more attractive and stronger. This change results in an increase in variable costs of $4 per pair of jeans, thus raising the variable cost per unit, c_v, to $12 per pair. This change (in conjunction with our previous price change to $30) results in a new break-even volume:

$$v = \frac{c_f}{p - c_v}$$

$$= \frac{10,000}{30 - 12} = 555.5 \text{ pairs of denim jeans}$$

This new break-even volume and the change in the total cost line that occurs as a result of the variable cost change are shown in Figure 1.4.
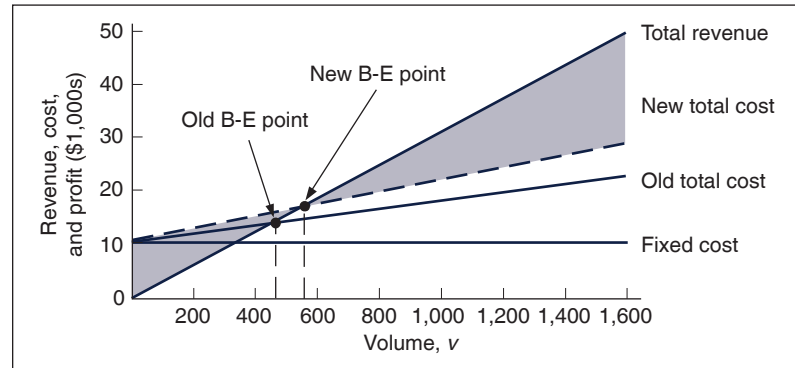
Next let's consider an increase in advertising expenditures to offset the potential loss in sales resulting from a price increase. An increase in advertising expenditures is an addition to fixed costs. For example, if the clothing company increases its monthly advertising budget by $3,000, then the total fixed cost, c_f, becomes $13,000. Using this fixed cost, as well as the increased variable cost per unit of $12 and the increased price of $30, we compute the break-even volume as follows:

$$v = \frac{c_f}{p - c_v}$$

$$= \frac{13,000}{30 - 12}$$

$$= 722.2 \text{ pairs of denim jeans}$$

In general, an increase in fixed costs will increase the break-even point, all other things held constant.

This new break-even volume, representing changes in price, fixed costs, and variable costs, is illustrated in Figure 1.5. Notice that the break-even volume is now higher than the original volume of 666.7 pairs of jeans, as a result of the increased costs necessary to offset the potential loss in sales. This indicates the necessity to analyze the effect of a change in one of the break-even components on the whole break-even model. In other words, generally it is not sufficient to consider a change in one model component without considering the overall effect.
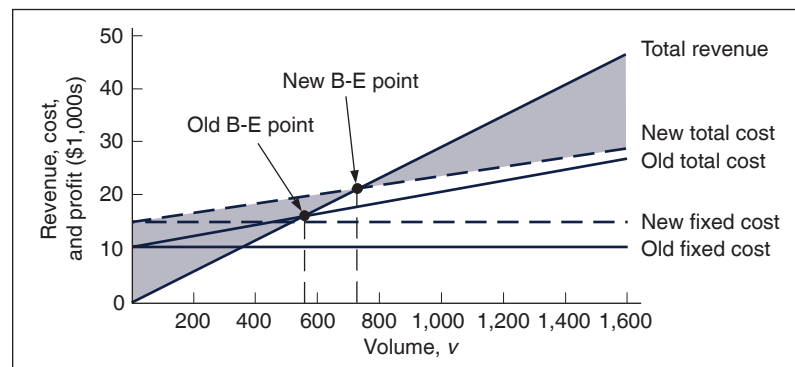
Computer Solution

Throughout the text, we will demonstrate how to solve management science models on the computer by using Excel spreadsheets and QM for Windows, a general-purpose quantitative methods software package by Howard Weiss. QM for Windows has program modules to solve almost every

type of management science problem you will encounter in this text. There are a number of similar quantitative methods software packages available on the market, with characteristics and capabilities similar to those of QM for Windows. In most cases, you simply input problem data (i.e., model parameters) into a model template, click on a solve button, and the solution appears in a Windows format. QM for Windows is included on the companion Web site for this text.

Spreadsheets are not always easy to use, and you cannot conveniently solve every type of management science model by using a spreadsheet. Most of the time, you must not only input the model parameters but also set up the model mathematics, including formulas, as well as your own model template with headings to display your solution output. However, spreadsheets provide a powerful reporting tool in which you can present your model and results in any format you choose. Spreadsheets such as Excel have become almost universally available to anyone who owns a computer. In addition, spreadsheets have become very popular as a teaching tool because they tend to guide the student through a modeling procedure, and they can be interesting and fun to use. However, because spreadsheets are somewhat more difficult to set up and apply than is QM for Windows, we will spend more time explaining their use to solve various types of problems in this text.

One of the difficult aspects of using spreadsheets to solve management science problems is setting up a spreadsheet with some of the more complex models and formulas. For the most complex models in the text, we will show how to use Excel QM, a supplemental spreadsheet macro that is included on the companion Web site for this text. A *macro* is a template or an overlay that already has the model format with the necessary formulas set up on the spreadsheet so that the user only has to input the model parameters. We will demonstrate Excel QM in six chapters, including this chapter, Chapter 6 ("Transportation, Transshipment, and Assignment Problems"), Chapter 12 ("Decision Analysis"), Chapter 13 ("Queuing Analysis"), Chapter 15 ("Forecasting"), and Chapter 16 ("Inventory Management").

Later in this text, we will also demonstrate two spreadsheet add-ins, TreePlan and Crystal Ball. TreePlan is a program for setting up and solving decision trees that we use in Chapter 12 ("Decision Analysis"), whereas Crystal Ball is a simulation package that we use in Chapter 14 ("Simulation"). Also, in Chapter 8 ("Project Management"), we will demonstrate Microsoft Project.

In this section, we will demonstrate how to use Excel, Excel QM, and QM for Windows, using our break-even model example for Western Clothing Company.

Excel Spreadsheets

To solve the break-even model using Excel, you must set up a spreadsheet with headings to identify your model parameters and variables and then input the appropriate mathematical formulas into the cells where you want to display your solution. Exhibit 1.1 (which can be downloaded from the text website) shows the spreadsheet for the Western Clothing Company example. Setting up the different headings to describe the parameters and the solution is not difficult, but it does require that you know your way around Excel a little. Appendix B provides a brief tutorial titled "Setting Up and Editing a Spreadsheet" for solving management science problems.

EXHIBIT 1.1

Formula for *v*, break-even point, =D4/(D8–D6)
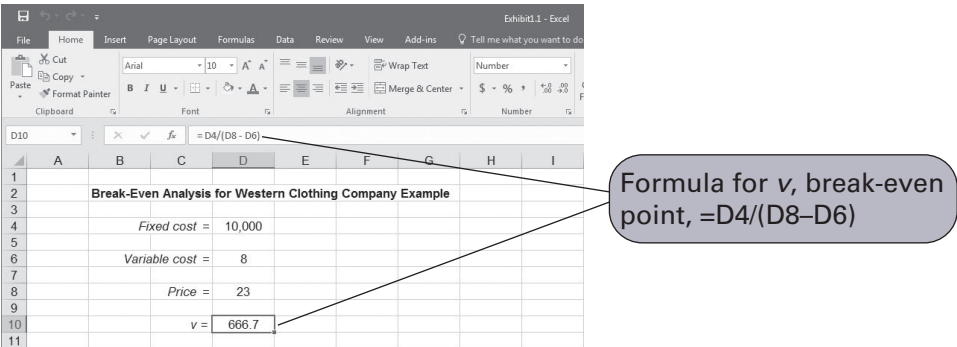
Notice that cell D10 contains the break-even formula, which is displayed on the toolbar near the top of the screen. The fixed cost of $10,000 is typed in cell D4, the variable cost of $8 is in cell D6, and the price of $23 is in cell D8.

As we present more complex models and problems in the chapters to come, the spreadsheets we develop to solve these problems will become more involved and will enable us to demonstrate different features of Excel and spreadsheet modeling.

The Excel QM Macro for Spreadsheets

Excel QM is included on the companion Web site for this text. You can install Excel QM onto your computer by following a brief series of steps displayed when the program is first accessed.

After Excel is started, Excel QM is normally accessed from the computer's program files, where it is usually loaded. When Excel QM is activated, "Add-Ins" will appear at the top of the spreadsheet (as indicated in Exhibit 1.2). Clicking on "Excel QM" or "Taylor" will pull down a menu of the topics in Excel QM, one of which is break-even analysis. Clicking on "Break-Even Analysis" will result in the window for spreadsheet initialization. Every Excel QM macro listed on the menu will start with a Spreadsheet Initialization window.

EXHIBIT 1.2
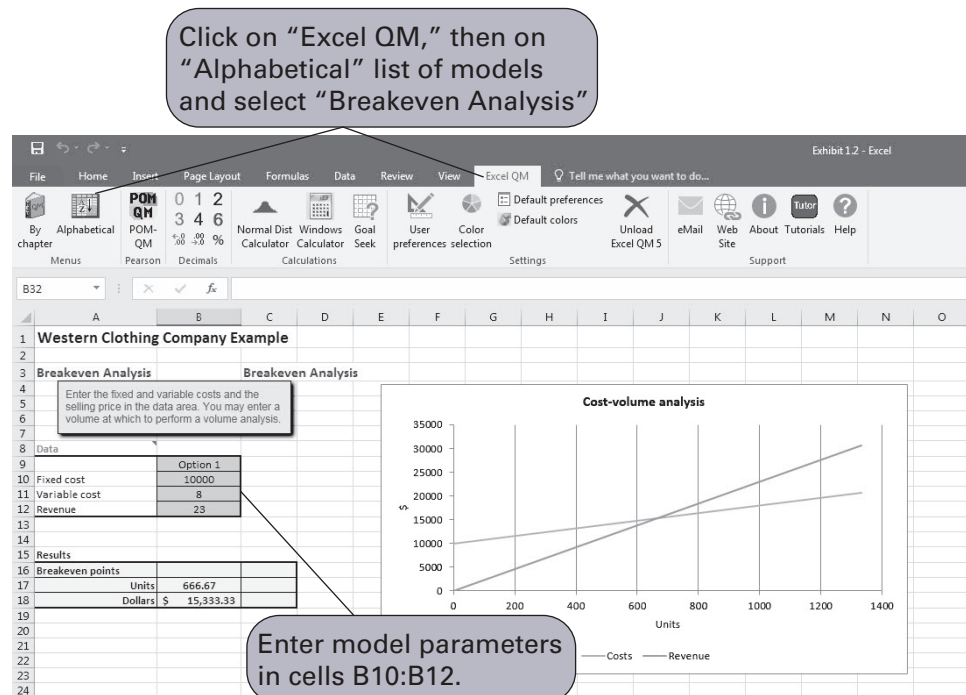
In this window, you can enter a spreadsheet title and choose under "Options" whether you also want volume analysis and a graph. Clicking on "OK" will result in the spreadsheet shown in Exhibit 1.2. The first step is to input the values for the Western Clothing Company example in cells B10 to B12, as shown in Exhibit 1.2. The spreadsheet shows the break-even volume in cell B17.

QM for Windows

You begin using QM for Windows by clicking on the "Module" button on the toolbar at the top of the main window that appears when you start the program. This will pull down a window with
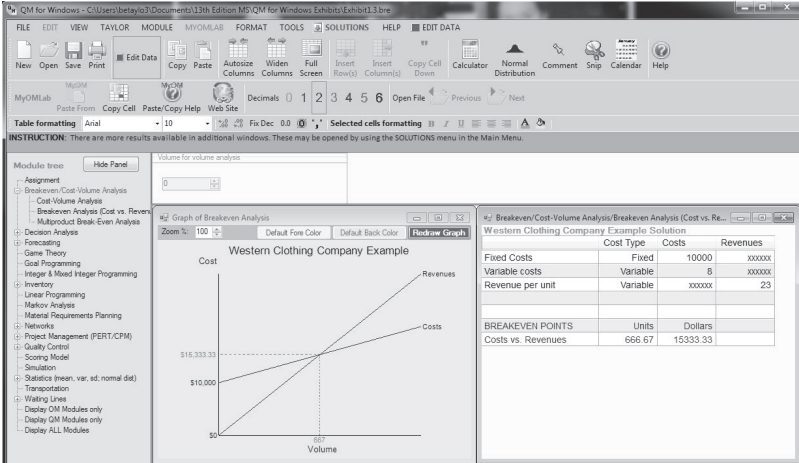
a list of all the model solution modules available in QM for Windows. Clicking on the "Break-even Analysis" module will access a new screen for typing in the problem title. Clicking again will access a screen with input cells for the model parameters—that is, fixed cost, variable cost, and price (or revenue). Next, clicking on the "Solve" button at the top of the screen will provide the solution and the break-even graph for the Western Clothing Company example, as shown in Exhibit 1.3.

Management Science Modeling Techniques

This text focuses primarily on two of the five steps of the management science process described in Figure 1.1—model construction and solution. These are the two steps that use the management science techniques. In a textbook, it is difficult to show how an unstructured real-world problem is identified and defined because the problem must be written out. However, once a problem statement has been given, we can show how a model is constructed and a solution is derived. The techniques presented in this text can be loosely classified into four categories, as shown in Figure 1.6.

FIGURE 1.6

Classification of management science techniques
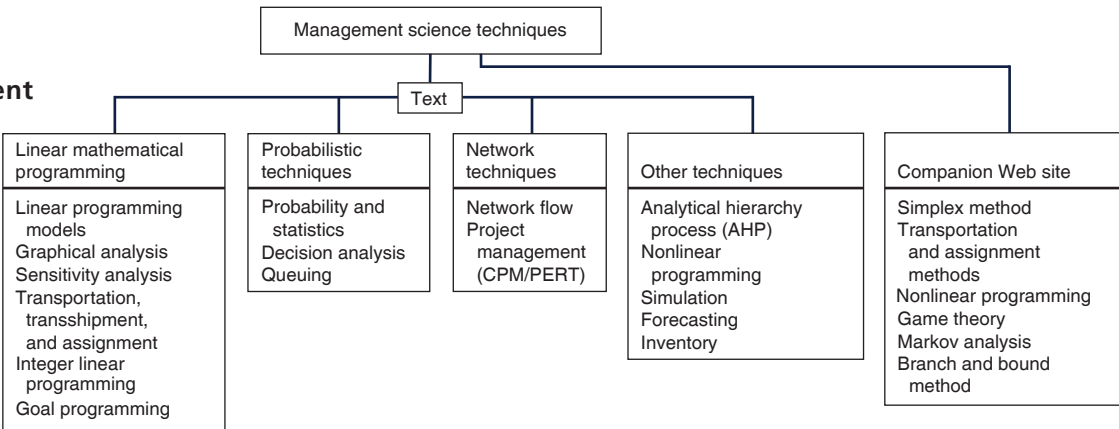
Linear Mathematical Programming Techniques

Chapters 2 through 6 and 9 present techniques that together make up *linear mathematical programming*. (The first example used to demonstrate model construction earlier in this chapter is a very rudimentary linear programming model.) The term *programming* used to identify this technique

Management Science Application

Management Science and Analytics

As we discussed in the section "Management Science and Business Analytics," when applied to business problems, analytics often combines the management science approach to problem solving and decision making, including model building, with the use of data. Following are a few examples of the many recent applications of analytics for problem solving in agriculture, media, urban planning, and sports.

Although the total world population is expected to grow by one-third to 9.6 billion in 2050, there will be less natural resources and land to support the necessary food production to feed an additional 2.4 billion people. Plant seed developer Syngenta is using analytics and management science models in its research and development efforts to develop and implement a plant-breeding strategy for soybeans that will improve the quality and quantity of the soybeans that farmers produce per acre. Their application of analytics enables better decisions that result in reducing the time and cost required to develop higher-productivity crops, saving Syngenta an estimated $287 million in a five-year period, while making a contribution to meeting the world's growing food needs.

iHeartMedia, Inc. (IHM) owns over 850 radio stations in more than 150 cities and provides programming (i.e., news, sports, traffic reports and weather) to over 2,250 stations. The company uses a set of management science models and sales data to maximize revenue from their inventory of radio advertising spots. Advertisers expect IHM to distribute their spots fairly and equitably across available inventory according to their order specifications, including dates, times, spot length, programs, stations, and demographic targets. IHM uses two linear programming models to assign advertising spots. The use of analytics has resulted in a more efficient use of available inventory, improved customer service, and enhanced sales from more accurate inventory visibility, resulting in a financial benefit of over a half million dollars annually.

ParkPGH is a decision analytics application that provides real-time and predictive information for garage parking space availability within the downtown Pittsburgh Cultural District. The model collects real time parking information for garage gate counts and uses historical data and event schedules to predict parking availability and provide downtown visitors with information on available parking via mobile devices and the Internet. The system has reduced parking space search times and changed the perception of downtown patrons about the downtown parking situation (including security and availability), and also helped garage operators better manage parking demand. In one year the parking application received over 300,000 inquiries.

One of the most visible applications of analytics in the sports industry has been the development and use of "sabermetrics"

San Gabriel Valley Tribune/ZUMA Press Inc./Alamy Stock Photo

in major league baseball, popularized by the book and movie *Moneyball*. It was originally defined in 1980 by Bill James (currently an analyst with the Boston Red Sox) as the "search for objective knowledge about baseball," and it is derived from the acronym SABR (e.g., Society for American Baseball Research). It has generally evolved into the application of statistical analysis of baseball records to develop predictive models and measures to evaluate and compare the in-game performance of individual players, usually in terms of runs or team wins. Sabermetrics attempts to answer questions such as, which players on a team will contribute most to the team's offense? For example, the sabermetric measure, VORP (value over replacement player), attempts to predict how much a hitter contributes offensively to his team in comparison to a fictitious average replacement player. A player might be worth 50 more runs in a season than a replacement level player at the same position (acquired at minimal cost). Currently every major league team has some employees in administrative positions dedicated to quantitative analytics for the evaluation of player performance to determine player acquisitions, trades, and contracts.

Sources: J. Byrum, C. Davis, G. Doonan, T. Doubler, D. Foster, B. Luzzi, R. Mowers, C. Zinselmeir, J. Klober, D. Culhane, and S. Mack, "Advanced Analytics for Agricultural Product Development," *Interfaces* 46, no. 1 (January–February 2016): 5–17; S. Venkatachalam, F. Wong, E. Uyar, S. Ward, and A. Aggarwal, "Media Company Uses Analytics to Schedule Radio Advertisement Spots," *Interfaces* 45, no. 6 (November–December 2015): 485–500; T. Fabusuyi, R. Hampshire, V. Hill, and K. Sasanuma, "Decision Analytics for Parking Availability in Downtown Pittsburgh," *Interfaces* 44, no. 3 (May–June 2014): 286–299.

does not refer to computer programming but rather to a predetermined set of mathematical steps used to solve a problem. This particular class of techniques holds a predominant position in this text because it includes some of the more frequently used and popular techniques in management science.

In general, linear programming models help managers determine solutions (i.e., make decisions) for problems that will achieve some objective in which there are restrictions, such as limited resources or a recipe or perhaps production guidelines. For example, you could actually develop a linear programming model to help determine a breakfast menu for yourself that would meet dietary guidelines you may have set, such as number of calories, fat content, and vitamin level, while minimizing the cost of the breakfast. Manufacturing companies develop linear programming models to help decide how many units of different products they should produce to maximize their profit (or minimize their cost), given scarce resources such as capital, labor, and facilities.

Six chapters in this text are devoted to this topic because there are several variations of linear programming models that can be applied to specific types of problems. Chapter 4 is devoted entirely to describing example linear programming models for several different types of problem scenarios. Chapter 6, for example, focuses on one particular type of linear programming application for transportation, transshipment, and assignment problems. An example of a transportation problem is a manager trying to determine the lowest-cost routes to use to ship goods from several sources (such as plants or warehouses) to several destinations (such as retail stores), given that each source may have limited goods available and each destination may have limited demand for the goods. Also, Chapter 9 includes the topic of goal programming, which is a form of linear programming that addresses problems with more than one objective or goal.

As mentioned previously in this chapter, some of the more mathematical topics in the text are included as supplementary modules on the companion Web site for the text. Among the linear programming topics included on the companion Web site are modules on the simplex method, the transportation and assignment solution methods, and the branch and bound solution method for integer programming models. Also included on the companion Web site are modules on nonlinear programming, game theory, and Markov analysis.

Probabilistic Techniques

*A **deterministic** technique assumes certainty in the solution.*

Probabilistic techniques are presented in Chapters 11 through 13. These techniques are distinguished from mathematical programming techniques in that the results are probabilistic. Mathematical programming techniques assume that all parameters in the models are known with *certainty*. Therefore, the solution results are assumed to be known with certainty, with no probability that other solutions might exist. A technique that assumes certainty in its solution is referred to as **deterministic**. In contrast, the results from a probabilistic technique *do* contain uncertainty, with some possibility that alternative solutions might exist. In the model solution presented earlier in this chapter, the result of the first example ($x = 25$ units to produce) is deterministic, whereas the result of the second example (estimating an average of 40 units sold each month) is probabilistic.

An example of a probabilistic technique is decision analysis, the subject of Chapter 12. In decision analysis, it is shown how to select among several different decision alternatives, given uncertain (i.e., probabilistic) future conditions. For example, a developer may want to decide whether to build a shopping mall, build an office complex, build condominiums, or not build anything at all, given future economic conditions that might be good, fair, or poor, each with a probability of occurrence. Chapter 13, on queuing analysis, presents probabilistic techniques for analyzing waiting lines that might occur, for example, at the grocery store, at a bank, or at a movie. The results of waiting line analysis are statistical averages showing, among other things, the average number of customers in line waiting to be served or the average time a customer might have to wait for service.

Network Techniques

Networks, the topic of Chapters 7 and 8, consist of models that are represented as diagrams rather than as strictly mathematical relationships. As such, these models offer a pictorial representation of the system under analysis. These models represent either probabilistic or deterministic systems.

For example, in shortest-route problems, one of the topics in Chapter 7 ("Network Flow Models"), a network diagram can be drawn to help a manager determine the shortest route among a number of different routes from a source to a destination. For example, you could use this technique to determine the shortest or quickest car route from St. Louis to Daytona Beach for a spring break vacation. In Chapter 8 ("Project Management"), a network is drawn that shows the relationships of all the tasks and activities for a project, such as building a house or developing a new computer system. This type of network can help a manager plan the best way to accomplish each of the tasks in the project so that it will take the shortest amount of time possible. You could use this type of technique to plan for a concert or an intramural volleyball tournament on your campus.

Other Techniques

Some topics in the text are not easily categorized; they may overlap several categories, or they may be unique. The analytical hierarchy process (AHP) in Chapter 9 is such a topic that is not easily classified. It is a mathematical technique for helping the decision maker choose between several alternative decisions, given more than one objective; however, it is not a form of linear programming, as is goal programming, the shared topic in Chapter 9 ("Multicriteria Decision Making"). The structure of the mathematical models for nonlinear programming problems in Chapter 10 is similar to the linear programming problems in Chapters 2 through 6; however, the mathematical equations and functions in nonlinear programming can be nonlinear instead of linear, thus requiring the use of calculus to solve them. Simulation, the subject of Chapter 14, is probably the single most unique topic in the text. It has the capability to solve probabilistic and deterministic problems and is often the technique of last resort when no other management science technique will work. In simulation, a mathematical model is constructed (typically using a computer) that replicates a real-world system under analysis, and then that simulation model is used to solve problems in the "simulated" real-world system. For example, with simulation you could build a model to simulate the traffic patterns of vehicles at a busy intersection to determine how to set the traffic light signals.

Forecasting, the subject of Chapter 15, and inventory management, in Chapter 16, are topics traditionally considered to be part of the field of operations management. However, because they are both important business functions that also rely heavily on quantitative models for their analysis, they are typically considered important topics in the study of management science as well. Both topics also include probabilistic as well as deterministic aspects. In Chapter 15, we will look at several different quantitative models that help managers predict what the future demand for products and services will look like. In general, historical sales and demand data are used to build a mathematical function or formula that can be used to estimate product demand in the future. In Chapter 16, we will look at several different quantitative models that help organizations determine how much inventory to keep on hand to minimize inventory costs, which can be significant.

Business Usage of Management Science Techniques

Not all management science techniques are equally useful or equally used by business firms and other organizations. Some techniques are used quite frequently by business practitioners and managers; others are used less often. The most frequently used techniques are linear and integer programming, simulation, network analysis (including critical path method/project evaluation and review technique [CPM/PERT]), inventory control, decision analysis, and queuing theory, as well as probability and statistics. An attempt has been made in this text to provide a comprehensive treatment of all the topics generally considered within the field of management science, regardless of how frequently they are used. Although some topics may have limited direct applicability, their study can reveal informative and unique means of approaching a problem and can often enhance one's understanding of the decision-making process.

The variety and breadth of management science applications and of the potential for applying management science, not only in business and industry but also in government, health care, and service organizations, are extensive. Areas of application include project planning, capital

Management Science Application

Management Science in Health Care

Over 17 percent of the U.S. GDP is spent on health care each year (over $3 trillion), making it the single largest industry in the United States. However, it is estimated that as much as 30 percent of health care costs result from waste through inefficient processes. Management science is really good at making inefficient processes more efficient. Thus, it is not surprising that one of the most frequent areas of application of management science techniques is in health care. Following are several brief examples of its many successful applications.

Improving patient flow through the hospital is a critical factor in improving hospital operating efficiency and reducing costs, and optimizing bed assignments is critical for patient flow. It is estimated that an average 300-bed hospital could add $10 million to its contribution margin with a 27 percent increase in bed utilization. At Mount Sinai Medical Center in New York a bed-assignment solution approach using a combination of integer programming (Chapter 5) and goal programming (Chapter 9) reduced the average time from bed requests to bed assignments by 23 percent (from almost 4 hours to 3 hours). At the Duke Cancer Center a simulation model (Chapter 14) was used to predict patient waiting times and resource utilization in various departments throughout the hospital, including the outpatient clinic, radiology, and the oncology treatment center. This model identified nurse unavailability during oncology treatment as creating a serious bottleneck in patient flow. An integer programming model (Chapter 5) was used to develop optimal weekly and monthly nurse schedules that relieved the bottleneck. At Grady Memorial Hospital in Atlanta, the emergency department (ED) receives more than 125,000 patient visits per year. Using an analytics approach, including management science modeling (i.e., integer programming and simulation), the hospital was able to reduce patient length of stay by 33 percent to approximately 7 hours, reduce readmissions by 28 percent, reduce patient waiting times, and improved efficiencies in the ED reduced patient throughput by over 16 percent, resulting in annual revenues plus savings of $190 million, without additional funds or resources.

The East Carolina University (ECU) Student Health Service is a clinic that serves the 23,000 student body at this public university located in Greenville, South Carolina. In a one-year period, slightly over 35,000 appointments were scheduled with approximately 3,800 no-shows. The problem of no-shows at health care clinics is a significant problem, with estimated costs at the ECU clinic of over $400,000 per year (resulting from reduced patient access). Researchers at East Carolina used a combination of several management science techniques, including forecasting (Chapter 15), decision analysis (Chapter 12), and simulation (Chapter 14) to develop a solution approach employing an overbooking policy (similar to what airlines do for flights). In the first semester the clinic implemented the policy,

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appointment times were overbooked by only 7.3 percent with few patients overscheduled and an estimated cost savings of about $95,000.

At the University of Tennessee Medical Center in Knoxville, management science (including integer programming) was used to schedule a nine-physician group that provides 24-7 coverage for the 63-bed Neonatal Intensive Care Unit. Assigning physicians to shifts is problematic because of myriad constraints, including work rules and patterns based on quality-of-care and safety issues, workload and lifestyle choices, plus workload fairness that all members of the group consider equal in terms of apportioned work and times. The modeling approach created schedules that were individually preferred over acceptable equality schedules.

Sources: B. G. Thomas, S. Bollapragada, K. Akbay, D. Toledano, P. Katlic, O. Dulgeroglu, and D. Yang, "Automated Bed Assignments in a Complex Dynamic Hospital Environment," *Interfaces* 43, no. 5 (September–October 2013): 435–448; J. C. Woodall, T. Gosselin, A. Boswell, M. Murr and B. T. Denton, "Improving Patient Access to Chemotherapy Treatment at Duke Cancer Center," *Interfaces* 43, no. 5 (September–October 2013): 449–461; E. Lee, H. Atallah, M. Wright, E. Post, C. Thomas, D. Wu, and L. Haley, "Transforming Hospital Emergency Department Workflow and Patient Care," *Interfaces* 45, no. 1 (January–February 2015): 58–82; J. Kros, S. Dellana, and D. West, "Overbooking Increases Patient Access at East Carolina University's Student Health Services Clinic," *Interfaces* 39, no. 3 (May–June 2009): 271–287; and M. Bowers, C. Noon, W. Wu, and J. Bass, "Neonatal Physician Scheduling at the University of Tennessee Medical Center," *Interfaces* 46, no. 2 (March–April 2016): 168–182.

budgeting, production planning, inventory analysis, scheduling, marketing planning, quality control, plant location, maintenance policy, personnel management, and product demand forecasting, among others. In this text, the applicability of management science to a variety of problem areas is demonstrated via individual chapter examples and the problems that accompany each chapter.

A small portion of the thousands of applications of management science that occur each year are recorded in various academic and professional journals. Frequently, these journal articles are as complex as the applications themselves and are very difficult to read. However, one particular journal, *Interfaces*, is devoted specifically to the application of management science and is written not just for college professors but for businesspeople, practitioners, and students as well. *Interfaces* is published by INFORMS (Institute for Operations Research and Management Sciences), an international professional organization whose members include college professors, businesspeople, scientists, students, and a variety of professional people interested in the practice and application of management science and operations research.

Interfaces regularly publishes articles that report on the application of management science to a wide variety of problems. The chapters that follow present examples of applications of management science from *Interfaces* and other professional journals. These applications are from a variety of U.S. and overseas companies and organizations in business, industry, services, and government. These examples, as presented here, do not detail the actual models and the model components. Instead, they briefly indicate the type of problem the company or organization faced, the objective of the solution approach developed to solve the problem, and the benefits derived from the model or technique (i.e., what was accomplished). The interested reader who desires more detailed information about these and other management science applications is encouraged to go to the library and peruse *Interfaces* and the many other journals that contain articles on the application of management science.

Management Science Models in Decision Support Systems

Historically, management science models have been applied to the solution of specific types of problems; for example, a waiting line model is used to analyze a specific waiting line system at a store or bank. However, the evolution of computer and information technology has enabled the development of expansive computer systems that combine several management science models and solution techniques to address more complex, interrelated organizational problems. A **decision support system (DSS)** is a computer-based system that helps decision makers address complex problems that cut across different parts of an organization and operations.

Decision support system (DSS) is a computer-based information system that a manager can use to assist in and support decision making.

A DSS is normally *interactive*, combining various databases and different management science models and solution techniques with a user interface that enables the decision maker to ask questions and receive answers. In its simplest form any computer-based software program that helps a decision maker make a decision can be referred to as a DSS. For example, an Excel spreadsheet like the one shown for break-even analysis in Exhibit 1.1 or the QM for Windows model shown in Exhibit 1.3 can realistically be called a DSS. Alternatively, enterprisewide DSSs can encompass many different types of models and large data warehouses, and they can serve many decision makers in an organization. They can provide decision makers with interrelated information and analyses about almost anything in a company.

Figure 1.7 illustrates the basic structure of a DSS with a database component, a modeling component, and a user interface with the decision maker. As noted earlier, a DSS can be small and singular, with one analytical model linked to a database, or it can be very large and complex, linking many models and large databases. A DSS can be primarily a data-oriented system, or it can be a model-oriented system. A new type of DSS, called an *online analytical processing* system, or *OLAP*, focuses on the use of analytical techniques such as management science models and statistics for decision making. A desktop DSS for a single user can be a spreadsheet program such as Excel to develop specific solutions to individual problems. Exhibit 1.1 includes all the components of a DSS—cost, volume, and price data, a break-even model, and the opportunity for the user to manipulate the data and see the results (i.e., a user interface).

FIGURE 1.7

**A decision
support system**
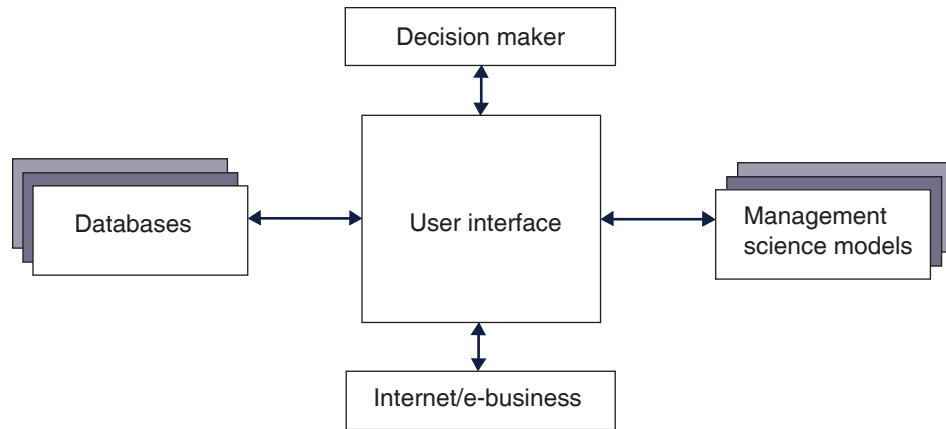
On the other end of the DSS spectrum, an *enterprise resource planning (ERP)* system is software that can connect the components and functions of an entire company. It can transform data, such as individual daily sales, directly into information that supports immediate decisions in other parts of the company, such as ordering, manufacturing, inventory, and distribution. A large-scale DSS such as an ERP system in a company might include a forecasting model (Chapter 15) to analyze sales data and help determine future product demand; an inventory model (Chapter 16) to determine how much inventory to keep on hand; a linear programming model (Chapters 2–5) to determine how much material to order and product to produce, and when to produce it; a transportation model (Chapter 6) to determine the most cost-effective method of distributing a product to customers; and a network flow model (Chapter 7) to determine the best delivery routes. All these different management science models and the data necessary to support them can be linked in a single enterprisewide DSS that can provide many decisions to many different decision makers.

In addition to helping managers answer specific questions and make decisions, a DSS may be most useful in answering "What-if?" questions and performing sensitivity analysis. In other words, a DSS provides a computer-based laboratory to perform experiments. By linking various management science models with different databases, a user can change a parameter in one model related to one company function and see what the effect will be in a model related to a different operation in the company. For example, by changing the data in a forecasting model, a manager could see the impact of a hypothetical change in product demand on the production schedule, as determined by a linear programming model.

Advances in information and computer technology have provided the opportunity to apply management science models to a broad array of complex organizational problems by linking different models to databases in a DSS. These advances have also made the application of management science models more readily available to individual users in the form of desktop DSSs that can help managers make better decisions relative to their day-to-day operations. In the future, it will undoubtedly become even easier to apply management science to the solution of problems with the development of newer software, and management science will become even more important and pervasive as an aid to decision makers as managers are linked within companies with sophisticated computer systems and to other companies via the Internet.

Many companies now interface with new types of DSS over the Internet. In e-business applications, companies can link to other business units around the world through computer systems called *intranets*, with other companies through systems called *extranets*, and over the Internet. For example, electronic data interchange (EDI) and point-of-sale data (through bar codes) can provide companies with instantaneous records of business transactions and sales at retail stores that are immediately entered into a company's DSS to update inventory and production scheduling, using management science models. Internet transportation exchanges enable companies to arrange cost-effective transportation of their products at Web sites that match shipping loads with available trucks at the lowest cost and fastest delivery speed, using sophisticated management science models.

Summary

Management science is an art.

In the chapters that follow, the model construction and solutions that constitute each management science technique are presented in detail and illustrated with examples. In fact, the primary method of presenting the techniques is through examples. Thus, the text offers you a broad spectrum of knowledge of the mechanics of management science techniques and the types of problems to which these techniques are applied. However, the ultimate test of a management scientist or a manager who uses management science techniques is the ability to transfer textbook knowledge to the business world. In such instances, there is an *art* to the application of management science, but it is an art predicated on practical experience and sound textbook knowledge. Providing the first of these necessities is beyond the scope of textbooks; providing the second is the objective of this text.

Problems

1. The Willow Furniture Company produces tables. The fixed monthly cost of production is $8,000, and the variable cost per table is $65. The tables sell for $180 apiece.
   a. For a monthly volume of 300 tables, determine the total cost, total revenue, and profit.
   b. Determine the monthly break-even volume for the Willow Furniture Company.

2. Gobblecakes is a bakery that specializes in cupcakes. The annual fixed cost to make cupcakes is $18,000. The variable cost including ingredients and labor to make a cupcake is $0.90. The bakery sells cupcakes for $3.20 apiece.
   a. If the bakery sells 12,000 cupcakes annually, determine the total cost, total revenue, and profit.
   b. How many cupcakes will the bakery need to sell to break even?

3. The Rolling Creek Textile Mill produces denim. The fixed monthly cost is $21,000, and the variable cost per yard of denim is $0.45. The mill sells a yard of denim for $1.30.
   a. For a monthly volume of 18,000 yards of denim, determine the total cost, total revenue, and profit.
   b. Determine the annual break-even volume for the Rolling Creek Textile Mill.

4. Evergreen Fertilizer Company produces fertilizer. The company's fixed monthly cost is $25,000, and its variable cost per pound of fertilizer is $0.15. Evergreen sells the fertilizer for $0.40 per pound. Determine the monthly break-even volume for the company.

5. Graphically illustrate the break-even volume for the Gobblecakes bakery determined in Problem 2.

6. Graphically illustrate the break-even volume for the Evergreen Fertilizer Company determined in Problem 4.

7. Andy Mendoza makes handcrafted dolls, which he sells at craft fairs. He is considering mass-producing the dolls to sell in stores. He estimates that the initial investment for plant and equipment will be $25,000, whereas labor, material, packaging, and shipping will be about $10 per doll. If the dolls are sold for $30 each, what sales volume is necessary for Andy to break even?

8. If the maximum operating capacity of the Gobblecakes bakery described in Problem 2 is 12,000 cupcakes annually, determine the break-even volume as a percentage of that capacity.

9. If the maximum operating capacity of the Rolling Creek Textile Mill described in Problem 3 is 25,000 yards of denim per month, determine the break-even volume as a percentage of capacity.

10. If the maximum operating capacity of Evergreen Fertilizer Company described in Problem 4 is 120,000 pounds of fertilizer per month, determine the break-even volume as a percentage of capacity.

11. If the Gobblecakes bakery in Problem 2 changes the selling price for a cupcake from $3.20 to $2.75, what effect will the change have on the break-even volume?

12. If Evergreen Fertilizer Company in Problem 4 changes the price of its fertilizer from $0.40 per pound to $0.60 per pound, what effect will the change have on the break-even volume?

13. If Evergreen Fertilizer Company changes its production process to add a weed killer to the fertilizer to increase sales, the variable cost per pound will increase from $0.15 to $0.22. What effect will this change have on the break-even volume computed in Problem 12?

14. If Evergreen Fertilizer Company increases its advertising expenditures by $14,000 per year, what effect will the increase have on the break-even volume computed in Problem 13?

15. Pastureland Dairy makes cheese, which it sells at local supermarkets. The fixed monthly cost of production is $4,000, and the variable cost per pound of cheese is $0.21. The cheese sells for $0.75 per pound; however, the dairy is considering raising the price to $0.95 per pound. The dairy currently produces and sells 9,000 pounds of cheese per month, but if it raises its price per pound, sales will decrease to 5,700 pounds per month. Should the dairy raise the price?

16. David wants to be an entrepreneur after his graduation. He is interested in toy manufacturing. After conducting market research, he estimates that his initial investment to buy brand-new toy production equipment will be $30,000; whereas for each toy, he must spend $15 toward toy materials, labor, packaging, and shipping. He plans to produce as many toys as the market demands. He expects that he can sell each toy for $30.
    a. How many toys does David need to sell to reach a breakeven?
    b. David determines that by spending $9,000 toward advertising through local media, he can increase the sales volume by 500 toys. Should he then spend on advertising?

17. David from Problem 16 is now concerned as the market demand does not look very promising, and he may not be able to sell more than the breakeven volume. Therefore, instead of buying brand-new equipment, he plans on buying used equipment for $20,000 to reduce his fixed cost. Since the equipment is secondhand, it forces his employees to work at a slower pace, thus increasing the variable cost per toy to $20. Will these changes reduce his breakeven point?

18. The General Store at State University is an auxiliary bookstore located near the dormitories that sells academic supplies, toiletries, sweatshirts and T-shirts, magazines, packaged food items, and canned soft drinks and fruit drinks. The manager of the store has noticed that several pizza delivery services near campus make frequent deliveries. The manager is therefore considering selling pizza at the store. She could buy premade frozen pizzas and heat them in an oven. The cost of the oven and freezer would be $27,000. The frozen pizzas cost $3.75 each to buy from a distributor and to prepare (including labor and a box). To be competitive with the local delivery services, the manager believes she should sell the pizzas for $8.95 apiece. The manager needs to write up a proposal for the university's director of auxiliary services.
    a. Determine how many pizzas would have to be sold to break even.
    b. If the General Store sells 20 pizzas per day, how many days would it take to break even?
    c. The manager of the store anticipates that once the local pizza delivery services start losing business, they will react by cutting prices. If after a month (30 days) the manager has to lower the price of a pizza to $7.95 to keep demand at 20 pizzas per day, as she expects, what will the new break-even point be, and how long will it take the store to break even?

19. Kim Davis has decided to purchase a cellular phone, but she is unsure about which rate plan to select. The "regular" plan charges a fixed fee of $55 per month for 1,000 minutes of airtime plus $0.33 per minute for any time over 1,000 minutes. The "executive" plan charges a fixed fee of $100 per month for 1,200 minutes of airtime plus $0.25 per minute over 1,200 minutes.
    a. If Kim expects to use the phone for 21 hours per month, which plan should she select?
    b. At what level of use would Kim be indifferent between the two plans?

20. A university student, Benny Orsini, is planning to open a walk-in pizza restaurant near campus that he will call Benny's Big Slice Pizza. He will have limited seating and will sell two kinds of oversized slices of pizza to go, plain cheese and pepperoni. The fixed annual cost including rent and equipment is $26,000, and the variable cost including ingredients and labor to make a pizza is $5.36. If Benny plans to sells pizza slices for $3.75 apiece, and one pizza has 8 slices, how many slices will he need to sell to break even? If Benny forecasts demand to be 540 slices per week, how much annual profit will he make?

21. In problem 20, the production process for making pizzas that Benny's Big Slice Pizza uses is very labor intensive. Benny is considering a more automated process design that requires the purchase of additional equipment that will increase annual fixed costs by $4,000 annually, and reduce the variable cost to $3.84 per pizza. What is the approximate volume of pizza slices that will make Benny indifferent between the old and new processes? Should Benny use the old labor-intensive process or the process with new equipment?

22. Annie Russell, a student at Tech, plans to open a hot dog stand inside Tech's football stadium during home games. There are seven home games scheduled for the upcoming season. She must pay the Tech athletic department a vendor's fee of $3,000 for the season. Her stand and other equipment will cost her $4,500 for the season. She estimates that each hot dog she sells will cost her $0.35. She has talked to friends at other universities who sell hot dogs at games. Based on their information and the athletic department's forecast that each game will sell out, she anticipates that she will sell approximately 2,000 hot dogs during each game.
    a. What price should she charge for a hot dog to break even?
    b. What factors might occur during the season that would alter the volume sold and thus the break-even price Annie might charge?
    c. What price would you suggest that Annie charge for a hot dog to provide her with a reasonable profit while remaining competitive with other food vendors?

23. Hannah Byers and Kathleen Taylor are considering the possibility of teaching swimming to kids during the summer. A local swim club opens its pool at noon each day, so it is available to rent during the morning. The cost of renting the pool during the 10-week period for which Hannah and Kathleen would need it is $1,700. The pool would also charge Hannah and Kathleen an admission, towel service, and life guarding fee of $7 per pupil, and Hannah and Kathleen estimate an additional $5 cost per student to hire several assistants. Hannah and Kathleen plan to charge $75 per student for the 10-week swimming class.
    a. How many pupils do Hannah and Kathleen need to enroll in their class to break even?
    b. If Hannah and Kathleen want to make a profit of $5,000 for the summer, how many pupils do they need to enroll?
    c. Hannah and Kathleen estimate that they might not be able to enroll more than 60 pupils. If they enroll this many pupils, how much would they need to charge per pupil to realize their profit goal of $5,000?

24. The College of Business at Tech is planning to begin an online MBA program. The initial start-up cost for computing equipment, facilities, course development, and staff recruitment and development is $350,000. The college plans to charge tuition of $18,000 per student per year. However,

the university administration will charge the college $12,000 per student for the first 100 students enrolled each year for administrative costs and its share of the tuition payments.

   a. How many students does the college need to enroll in the first year to break even?

   b. If the college can enroll 75 students the first year, how much profit will it make?

   c. The college believes it can increase tuition to $24,000, but doing so would reduce enrollment to 35. Should the college consider doing this?

25. The Youth Cricket Club of a small city has a team of 20 members. The club provides professional coaching to its members. The club needs a budget of $75,000 every year to support its team and to pay the coaches' salaries. To generate this amount, the club sets up a tournament each fall. The cost of setting up a tournament is $10,000, which includes costs for advertising, communicating with other teams in the region through various channels, etc. The participation fee per team is $300. For each team participating in the tournament, the club spends $70 to pay their logistical expenses and to pay umpires. How many teams does the club need to invite to generate $75,000?

26. A group of developers is opening a health club near a new housing development. The health club—which will have exercise and workout equipment, basketball courts, swimming pools, an indoor walking/running track, and tennis courts—is one of the amenities the developers are building to attract new homebuyers. However, they want the health club to at least break even the first year or two. The annual fixed cost for the building, equipment, utilities, staff, and so on is $875,000, and annual variable costs are $200 per member for things like water, towels, laundry, soap, shampoo, and other member services. The membership fee is $225 per month. How many members will the club need to break even? If the club doubles its break-even membership after a year, what will its profit be?

27. The Tech Student Government Association (SGA) has several campus projects it undertakes each year, and its primary source of funding to support these projects is a T-shirt sale in the fall for what is known as the "orange effect" football game (with orange being one of Tech's colors). The club's publicized (media) objective is for everyone in the stadium to wear orange. The club's financial goal is to make a profit of $150,000, but to have a significant number of fans buy the shirts and wear them to the game, it doesn't want to price the T-shirts much more than $6. The stadium seats 62,000 fans, and the SGA would like to sell approximately 45,000 orange T-shirts to achieve the desired orange effect, which it's relatively confident it can do. It will cost $100,000 to purchase, silk-screen print, and ship this many T-shirts. The SGA sells the shirts through three sources: online, the two Tech bookstores, and a local independent bookstore. Although the bookstores don't expect to share in the profits from the sale of the shirts, they do expect for their direct costs to be covered, including labor, space, and other costs. The two Tech bookstores charge the SGA $0.35 per shirt, and the local independent store charges $0.50 per shirt. The cost per sale online (including handling, packaging, and shipping) is $2.30 per shirt. The SGA estimates that it will sell 50% of the shirts at the two Tech bookstores, 35% at the local bookstore, and 15% online. If the SGA sells the T-shirts for $6 and if it sells all the shirts it orders, will it make enough profit to achieve its financial goal? If not, at what price would the SGA need to sell the T-shirts, or how many would the SGA have to sell to achieve its financial goal?

28. The owners of Backstreets Italian Restaurant are considering starting a delivery service for pizza and their other Italian dishes in the small college town where they are located. They can purchase a used delivery van and have it painted with their name and logo for $21,500. They can hire part-time drivers who will work in the evenings from 5 P.M. to 10 P.M. for $8 per hour. The drivers are mostly college students who study at the restaurant when they are not making deliveries. During the day, there are so few deliveries that the regular employees can handle them. The owners estimate that the van will last 5 years (365 days per year) before it has to be replaced and that each delivery will cost about $1.35 in gas and other maintenance costs (including tires, oil, scheduled service, etc.). They also estimate that on average each delivery order will cost $15 for direct labor and ingredients to prepare and package and will generate $34 in revenue.

a. How many delivery orders must Backstreets make each month for the service to break even?

b. The owners believe that if they have approximately the break-even number of deliveries during the week, they will at least double that number on Fridays, Saturdays, and Sundays. If that's the case, how much profit will they make, at a minimum, from their delivery service each month (4 weeks per month)?

29. Kathleen Taylor is a high school student who has been investigating the possibility of mowing lawns for a summer job. She has a couple of friends she thinks she could hire on an hourly basis per job. The equipment, including two new lawnmowers and weed-eaters, would cost her $500, and she estimates her cost per lawn, based on the time required to pay her friends to mow an average residential lawn (and not including her own labor) and gas for driving to the jobs and mowing, would be about $14.

    a. If she charges customers $30 per lawn, how many lawns would she need to mow to break even?

    b. Kathleen has 8 weeks available to mow lawns before school starts again, and she estimates that she can get enough customers to mow at least three lawns per day, 6 days per week. How much money can she expect to make over the summer?

    c. Kathleen believes she can get more business if she lowers her price per lawn. If she lowers her price to $25 per lawn and increases her number of jobs to four per day (which is about all she can handle anyway), should she make this decision?

30. Ahmed wants to earn some extra pocket money while on his summer break, which is 5 weeks long. He plans to work for three days a week and enjoy his vacation the rest of the week. He has identified that car cleaning is a good business in his locality. He buys a trolley for $500 to carry water, detergent, brushes, towels, and other items, and he spends about $2 per car on these items.

    a. If Ahmed charges $10 to clean one car, how many cars would he need to clean to breakeven?

    b. If Ahmed cleans about 10 cars per day, how much money does he expect to make?

    c. Another possibility for Ahmed is to clean the open parking lots (areas, not cars). To do this, he need to hire a truck for $1,500 and hire a driver on an hourly basis, which, including gas, would cost him about $25 per parking lot. Ahmed estimates that the truck helps him complete four parking lots per day. If he charges $125 per parking lot, would this be a better alternative than cleaning cars?

31. Whitney Eggleston has quit her job in retail sales and has moved back into her parent's house with the idea of starting an online home decor business. She plans to purchase items like decorative pillows, lamps, bowls, mirrors, etc., at craft fairs, home shows and wholesale marts, and resell them on her site. Part of her marketing plan is to have customers describe their room or area and allow Whitney to suggest a group of decorative items. She has hired a local website developer to construct her site for her and get it up and running for $7,500. In her first year of business she has budgeted $2,300 per month for her product costs. She has researched other online retail sites and estimates that her return for each hit on her site is approximately $0.24. How many hits will Whitney need per month to break even in the first year? If she budgets $3,500 per month for product costs in the second year, how many hits will she need per month to make a profit of $45,000 so she can move out of her parent's house?

32. A bakery specializes in cupcakes and chocolate cookies. The bakery estimates that cupcakes will account for 75% of its total sales, whereas chocolate cookies will account for 25% of its total sales. The annual fixed cost to make these products is $20,000. The variable cost including ingredients and labor to make a cupcake is $1.00 and to make a chocolate cookie is $ 0.50. The bakery sells cupcakes for $3.50 per piece and chocolate cookies for $2.50 per piece.

    a. How many units of each item, cupcakes and chocolate cookies, will the bakery need to sell to breakeven?

    b. What is the breakeven point in sales in dollars?

33. Although Annie Russell (in problem 22) has been told by the athletic department at Tech that all the games will likely be sellouts, she is aware that the weather will affect how many fans will actually attend the games and buy hot dogs. Fans are less likely to attend the game if the weather is too hot or too cold and will not buy hot dogs if the weather is cold. Considering these new factors Annie has decided to analyze her decision-making situation to consider the possible outcomes given different weather conditions. If the weather is good all season she estimates that if she sells hotdogs for $3.25, she will sell 7,000, but if the weather is bad she will sell only 5,500. On the other hand, if she charges $4.00 for a hotdog and the weather is good, she will sell 6,000, but if the weather is bad she will sell only 3,500. Which price do you think Annie should sell the hotdogs for? Explain your reason.

34. In Problem 33, Annie has used historical weather data to determine that next season there is a 60% chance that the weather will be good and a 40% chance the weather will be bad. Given this additional information, what do you think Annie's decision should be, and why?

35. In the example used to demonstrate model construction in this chapter, a firm sells a product, x, for $20 that costs $5 to make, it has 100 pounds of steel to make the product, and it takes 4 pounds of steel to make each unit. The model that was constructed is

$$\text{maximize } Z = 20x - 5x$$
$$\text{subject to}$$
$$4x = 100$$

Now suppose that there is a second product, y, that has a profit of $10 and requires 2 pounds of steel to make, such that the model becomes

$$\text{maximize } Z = 15x + 10y$$
$$\text{subject to}$$
$$4x + 2y = 100$$

Can you determine a solution to this new model that will achieve the objective? Explain your answer.

36. Consider a model in which two products, x and y, are produced. There are 100 pounds of material and 80 hours of labor available. It requires 2 pounds of material and 1 hour of labor to produce a unit of x, and 4 pounds of material and 5 hours of labor to produce a unit of y. The profit for x is $30 per unit, and the profit for y is $50 per unit. If we want to know how many units of x and y to produce to maximize profit, the model is

$$\text{maximize } Z = 30x + 50y$$
$$\text{subject to}$$
$$2x + 4y = 100$$
$$x + 5y = 80$$

Determine the solution to this problem, and explain your answer.

37. Maria Eagle is a Native American artisan. She works part-time making bowls and mugs by hand from special pottery clay and then sells her items to the Beaver Creek Pottery Company, a Native American crafts guild. She has 60 hours available each month to make bowls and mugs, and it takes her 12 hours to make a bowl and 15 hours to make a mug. She uses 9 pounds of special clay to make a bowl, and she needs 5 pounds to make a mug; Maria has 30 pounds of clay available

each month. She makes a profit of $300 for each bowl she delivers, and she makes $250 for each mug. Determine all the possible combinations of bowls and mugs Maria can make each month, given her limited resources, and select the most profitable combination of bowls and mugs Maria should make each month. Develop an Excel spreadsheet and a graph to help solve this problem.

38. A car manufacturing company has its manufacturing plants in Chennai and Mumbai in India. The company transports its cars from these manufacturing plants to its distribution centers located in Bangalore and Hyderabad to meet the demands. Currently, the Mumbai plant has 500 cars and the Chennai plant has 500 cars. The Hyderabad center demands 600 cars and the Bangalore center demands 400 cars. The cost of transportation between these locations depends on several factors including but not limited to toll, interstate tax, fuel cost, etc. The company estimates that the transportation cost to ship a car from Mumbai to Hyderabad is $40 and Mumbai to Bangalore is $80. The transportation cost to ship a car from Chennai to Hyderabad is $50 and Mumbai to Bangalore is $70. The company wants to meet the demand while minimizing the transportation costs. Develop a mathematical model for this problem and use logic to determine the solution.

39. Ed Norris has developed a Web site for his used textbook business at State University. To sell advertising, he needs to forecast the number of site visits he expects in the future. For the past 6 months he has had the following number of site visits:

Month	1	2	3	4	5	6
Site visits	6,300	10,200	14,700	18,500	25,100	30,500

Determine a forecast for Ed to use for month 7, and explain the logic used to develop your forecast.

40. When Tracy McCoy wakes up on Saturday morning, she remembers that she promised the PTA she would make some cakes and/or homemade bread for its bake sale that afternoon. However, she does not have time to go to the store and get ingredients, and she has only a short time to bake things in her oven. Because cakes and breads require different baking temperatures, she cannot bake them simultaneously, and she has only 3 hours available to bake. A cake requires 3 cups of flour, and a loaf of bread requires 8 cups; Tracy has 20 cups of flour. A cake requires 45 minutes to bake, and a loaf of bread requires 30 minutes. The PTA will sell a cake for $10 and a loaf of bread for $6. Tracy wants to decide how many cakes and loaves of bread she should make. Identify all the possible solutions to this problem (i.e., combinations of cakes and loaves of bread Tracy has the time and flour to bake), and select the best one.

41. The local Food King grocery store has eight possible checkout stations with registers. On Saturday mornings customer traffic is relatively steady from 8 A.M. to noon. The store manager would like to determine how many checkout stations to staff during this time period. The manager knows from information provided by the store's national office that each minute past 3 minutes a customer must wait in line costs the store on average $50 in ill will and lost sales. Alternatively, each additional checkout station the store operates on Saturday morning costs the store $60 in salary and benefits. The following table shows the waiting time for the different staff levels.

Registers staffed	1	2	3	4	5	6	7	8
Waiting time (min.)	20.0	14.0	9.0	4.0	1.7	1.0	0.5	0.1

How many registers should the store staff, and why?

42. A furniture manufacturer in Roanoke, Virginia, must deliver a tractor trailer load of furniture to a retail store in Washington, DC. There are a number of different routes the truck can take from Roanoke to DC, as shown in the following road network, with the distance for each segment shown in miles.
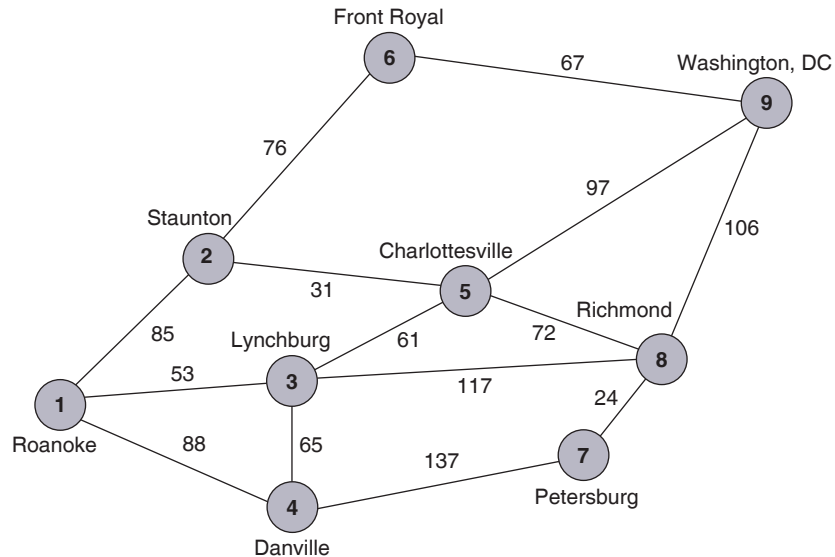
Determine the shortest route the truck can take from Roanoke to Washington, DC.

Case Problem

THE CLEAN CLOTHES CORNER LAUNDRY

When Molly Lai purchased the Clean Clothes Corner Laundry, she thought that because it was in a good location near several high-income neighborhoods, she would automatically generate good business if she improved the laundry's physical appearance. Thus, she initially invested a lot of her cash reserves in remodeling the exterior and interior of the laundry. However, she just about broke even in the year following her acquisition of the laundry, which she didn't feel was a sufficient return, given how hard she had worked. Molly didn't realize that the dry-cleaning business is very competitive and that success is based more on price and quality service, including quickness of service, than on the laundry's appearance.

In order to improve her service, Molly is considering purchasing new dry-cleaning equipment, including a pressing machine that could substantially increase the speed at which she can dry-clean clothes and improve their appearance. The new machinery costs $16,200 installed and can clean 40 clothes items per hour (or 320 items per day). Molly estimates her variable costs to be $0.25 per item

dry-cleaned, which will not change if she purchases the new equipment. Her current fixed costs are $1,700 per month. She charges customers $1.10 per clothing item.

A. What is Molly's current monthly volume?

B. If Molly purchases the new equipment, how many additional items will she have to dry-clean each month to break even?

C. Molly estimates that with the new equipment she can increase her volume to 4,300 items per month. What monthly profit would she realize with that level of business during the next 3 years? After 3 years?

D. Molly believes that if she doesn't buy the new equipment but lowers her price to $0.99 per item, she will increase her business volume. If she lowers her price, what will her new break-even volume be? If her price reduction results in a monthly volume of 3,800 items, what will her monthly profit be?

E. Molly estimates that if she purchases the new equipment and lowers her price to $0.99 per item, her volume will increase to about 4,700 units per month. Based on the local market, that is the largest volume she can realistically expect. What should Molly do?

Case Problem

THE OCOBEE RIVER RAFTING COMPANY

Vicki Smith, Penny Miller, and Darryl Davis are students at State University. In the summer, they often go rafting with other students down the Ocobee River in the nearby Blue Ridge Mountain foothills. The river has a number of minor rapids but is not generally dangerous. The students' rafts basically consist of large rubber tubes, sometimes joined together with ski rope. They have noticed that a number of students who come to the river don't have rubber rafts and often ask to borrow theirs, which can be very annoying. In discussing this nuisance, it occurred to Vicki, Penny, and Darryl that the problem might provide an opportunity to make some extra money. They considered starting a new enterprise, the Ocobee River Rafting Company, to sell rubber rafts at the river. They determined that their initial investment would be about $3,000 to rent a small parcel of land next to the river on which to make and sell the rafts; to purchase a tent to operate out of; and to buy some small equipment such as air pumps and a rope cutter. They estimated that the labor and material cost per raft will be about $12, including the purchase and shipping costs for the rubber tubes and rope. They plan to sell the rafts for $20 apiece, which they think is about the maximum price students will pay for a preassembled raft.

Soon after they determined these cost estimates, the newly formed company learned about another rafting company in North Carolina that was doing essentially what they planned to do. Vicki got in touch with one of the operators of that company, and he told her the company would be willing to supply rafts to the Ocobee River Rafting Company for an initial fixed fee of $9,000 plus $8 per raft, including shipping. (The Ocobee River Rafting Company would still have to rent the parcel of riverside land and tent for $1,000.) The rafts would already be inflated and assembled. This alternative appealed to Vicki, Penny, and Darryl because it would reduce the amount of time they would have to work pumping up the tubes and putting the rafts together, and it would increase time for their schoolwork.

Although the students prefer the alternative of purchasing the rafts from the North Carolina company, they are concerned about the large initial cost and worried about whether they will lose money. Of course, Vicki, Penny, and Darryl realize that their profit, if any, will be determined by how many rafts they sell. As such, they believe that they first need to determine how many rafts they must sell with each alternative to make a profit and which alternative would be best given different levels of demand. Furthermore, Penny has conducted a brief sample survey of people at the river and estimates that demand for rafts for the summer will be around 1,000 rafts.

Perform an analysis for the Ocobee River Rafting Company to determine which alternative would be best for different levels of demand. Indicate which alternative should be selected if demand is approximately 1,000 rafts and how much profit the company would make.

Case Problem

CONSTRUCTING A DOWNTOWN PARKING LOT IN DRAPER

The town of Draper, with a population of 20,000, sits adjacent to State University, which has an enrollment of 27,000 students. Downtown Draper merchants have long complained about the lack of parking available to their customers. This is one primary reason for the steady migration of downtown businesses to a mall several miles outside town. The local chamber of commerce has finally convinced the town council to consider the construction of a new multilevel indoor parking facility downtown. Kelly Mattingly, the town's public works director, has developed plans for a facility that would cost $4.5 million to construct. To pay for the project, the town would sell municipal bonds with a duration of 30 years at 8% interest. Kelly also estimates that five employees would be required to operate the lot on a daily basis, at a total annual cost of $140,000. It is estimated that each car that enters the lot would park for an average of 2.5 hours and pay an average fee of $3.20. Further, it is estimated that each car that parks in the lot would (on average) cost the town $0.60 in annual maintenance for cleaning and repairs to the facility. Most of the downtown businesses (which include a number of restaurants) are open 7 days per week.

A. Using break-even analysis, determine the number of cars that would have to park in the lot on an annual basis to pay off the project in the 30-year time frame.
B. From the results in (A), determine the approximate number of cars that would have to park in the lot on a daily basis. Does this seem to be a reasonable number to achieve, given the size of the town and college population?

Case Problem

A Bus Service For Draper

Draper, where Tech is located, does not have a commercial airport; instead residents who are flying on an airline must travel to the airport in Roanoke 40 miles away. The town council is considering a joint economic development plan with the local Chamber of Commerce and the university to start a bus service from the town to the airport in Roanoke with other stops on the way and back, that will operate 365 days per year. In order to have continuous daily service for the 1.5-hour round trip, 3 passenger buses, each with a seating capacity of 55 passengers, will need to be purchased at a cost of $400,000 apiece. Each bus will make 4 trips spread out during the day starting at 5 A.M. in order to deliver passengers to the airport in time for the first flight of the day, and they will pick up passengers up at the airport after the last scheduled flight of the day lands. Passengers who are going to work or school will also embark and disembark at various stops along the route. The operating cost per hour (including fuel and driver salary) for each bus is estimated to be $90 per hour per trip. The planned fare per rider is

$4, regardless of where they get on or off the bus. Based on customer surveys, it is estimated that an average of 37 passengers will ride the buses each trip in the first year.

A. The town anticipates that the service will lose money the first year; how much will it lose?
B. How many years will it take the service to break even, if at all, during the 6-year life span of the buses?
C. What would be the effect if the bus service ridership averages 45 passengers per trip? 50 passengers per trip?
D. If the trips per day per bus were reduced to 3 with a corresponding increase in passengers to 50 (due to increased demand for the fewer trips), what effect would this have?
E. Because the bus service project has a "green" footprint by taking cars off of the road, and will also spur economic development by providing low-cost regular transportation from Draper to Roanoke, and from Roanoke to the university, the town believes it can get a federal grant to pay for the purchase of the buses. How would this effect the financial loss in the first year of operation, and the years it might take to break even?

Chapter 2

Linear Programming: Model Formulation and Graphical Solution

Objectives of a business frequently are to maximize profit or minimize cost.

Many major decisions faced by a manager of a business focus on the best way to achieve the objectives of the firm, subject to the restrictions placed on the manager by the operating environment. These restrictions can take the form of limited resources, such as time, labor, energy, material, or money; or they can be in the form of restrictive guidelines, such as a recipe for making cereal or engineering specifications. One of the most frequent objectives of business firms is to gain the most profit possible or, in other words, to *maximize* profit. The objective of individual organizational units within a firm (such as a production or packaging department) is often to *minimize* cost. When a manager attempts to solve a general type of problem by seeking an objective that is subject to restrictions, the management science technique called **linear programming** is frequently used. In the *Interfaces* journal that reports on real-world management science applications as mentioned in Chapter 1, one of the most frequently cited management science modeling techniques used in analytics applications is linear programming. There are three steps in applying the linear programming technique. First, the problem must be identified as being solvable by linear programming. Second, the unstructured problem must be formulated as a mathematical model. Third, the model must be solved by using established mathematical techniques. The linear programming technique derives its name from the fact that the functional relationships in the mathematical model are *linear*, and the solution technique consists of predetermined mathematical steps—that is, a *program*. In this chapter, we will concern ourselves with the formulation of the mathematical model that represents the problem and then with solving this model by using a graph.

Linear programming is a model that consists of linear relationships representing a firm's decision(s), given an objective and resource constraints.

Model Formulation

A linear programming model consists of certain common components and characteristics. The model components include decision variables, an objective function, and model constraints, which consist of decision variables and parameters. **Decision variables** are mathematical symbols that represent levels of activity by the firm. For example, an electrical manufacturing firm desires to produce x_1 radios, x_2 toasters, and x_3 clocks, where x_1, x_2, and x_3 are symbols representing unknown variable quantities of each item. The final values of x_1, x_2, and x_3, as determined by the firm, constitute a *decision* (e.g., the equation $x_1 = 100$ radios is a decision by the firm to produce 100 radios).

Decision variables are mathematical symbols that represent levels of activity.

The **objective function** is a linear mathematical relationship that describes the objective of the firm in terms of the decision variables. The objective function always consists of either *maximizing* or *minimizing* some value (e.g., maximize the profit or minimize the cost of producing radios).

*The **objective function** is a linear relationship that reflects the objective of an operation.*

The **model constraints** are also linear relationships of the decision variables; they represent the restrictions placed on the firm by the operating environment. The restrictions can be in the form of limited resources or restrictive guidelines. For example, only 40 hours of labor may be available to produce radios during production. The actual numeric values in the objective function and the constraints, such as the 40 hours of available labor, are **parameters**.

*A **model constraint** is a linear relationship that represents a restriction on decision making.*

The next section presents an example of how a linear programming model is formulated. Although this example is simplified, it is realistic and represents the type of problem to which linear programming can be applied. In the example, the model components are distinctly identified and described. By carefully studying this example, you can become familiar with the process of formulating linear programming models.

Parameters are numerical values that are included in the objective functions and constraints.

A Maximization Model Example

Beaver Creek Pottery Company is a small crafts operation run by a Native American tribal council. The company employs skilled artisans to produce clay bowls and mugs with authentic Native American designs and colors. The two primary resources used by the company are special pottery clay and skilled labor. Given these limited resources, the company desires to know how many bowls and mugs to produce each day to maximize profit. This is generally referred to as a *product mix* problem type. This scenario is illustrated in Figure 2.1.

FIGURE 2.1

Beaver Creek Pottery Company
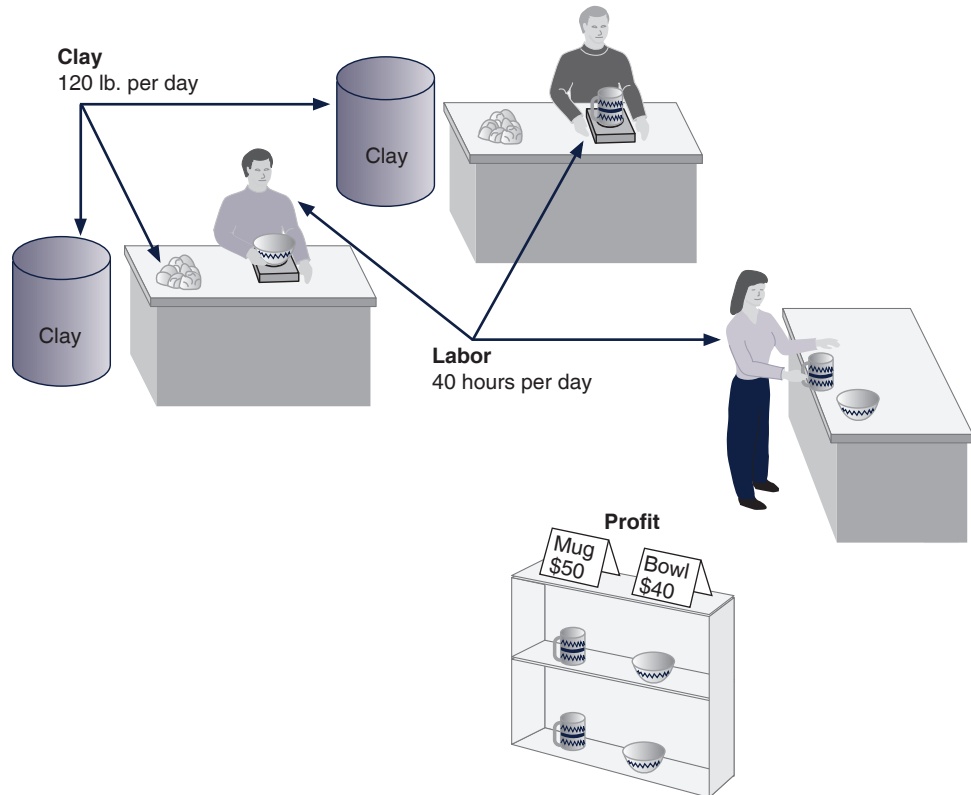
The two products have the following resource requirements for production and profit per item produced (i.e., the model parameters):

	Resource Requirements		
Product	**Labor (hr./unit)**	**Clay (lb./unit)**	**Profit ($/unit)**
Bowl	1	4	40
Mug	2	3	50

There are 40 hours of labor and 120 pounds of clay available each day for production. We will formulate this problem as a linear programming model by defining each component of the

A linear programming model consists of decision variables, an objective function, and constraints.

model separately and then combining the components into a single model. The steps in this formulation process are summarized as follows:

Summary of LP Model Formulation Steps

Step 1: Define the decision variables

How many bowls and mugs to produce

Step 2: Define the objective function

Maximize profit

Step 3: Define the constraints

The resources (clay and labor) available

Decision Variables

The decision confronting management in this problem is how many bowls and mugs to produce. The two decision variables represent the number of bowls and mugs to be produced on a daily basis. The quantities to be produced can be represented symbolically as

$$x_1 = \text{number of bowls to produce}$$
$$x_2 = \text{number of mugs to produce}$$

The Objective Function

The objective of the company is to maximize total profit. The company's profit is the sum of the individual profits gained from each bowl and mug. Profit derived from bowls is determined by multiplying the unit profit of each bowl, $40, by the number of bowls produced, x_1. Likewise, profit derived from mugs is derived from the unit profit of a mug, $50, multiplied by the number of mugs produced, x_2. Thus, total profit, which we will define symbolically as Z, can be expressed mathematically as $\$40x_1 + \$50x_2$. By placing the term *maximize* in front of the profit function, we express the objective of the firm—to maximize total profit:

$$\text{maximize } Z = \$40x_1 + 50x_2$$
$$\text{where}$$
$$Z = \text{total profit per day}$$
$$\$40x_1 = \text{profit from bowls}$$
$$\$50x_2 = \text{profit from mugs}$$

Model Constraints

In this problem, two resources are used for production—labor and clay—both of which are limited. Production of bowls and mugs requires both labor and clay. For each bowl produced, 1 hour of labor is required. Therefore, the labor used for the production of bowls is $1x_1$ hours. Similarly, each mug requires 2 hours of labor; thus, the labor used to produce mugs every day is $2x_2$ hours. The total labor used by the company is the sum of the individual amounts of labor used for each product:

$$1x_1 + 2x_2$$

However, the amount of labor represented by $1x_1 + 2x_2$ is limited to 40 hours per day; thus, the complete labor constraint is

$$1x_1 + 2x_2 \leq 40 \text{ hr.}$$

The "less than or equal to" ($\leq$) inequality is employed instead of an equality ($=$) because the 40 hours of labor is a maximum limitation that *can be used*, not an amount that *must be used*. This constraint allows the company some flexibility; the company is not restricted to using exactly 40 hours but can use whatever amount is necessary to maximize profit, up to and including 40 hours. This means that it is possible to have idle, or excess, capacity (i.e., some of the 40 hours may not be used).

The constraint for clay is formulated in the same way as the labor constraint. Because each bowl requires 4 pounds of clay, the amount of clay used daily for the production of bowls is $4x_1$ pounds, and because each mug requires 3 pounds of clay, the amount of clay used daily for mugs is $3x_2$. Given that the amount of clay available for production each day is 120 pounds, the material constraint can be formulated as

$$4x_1 + 3x_2 \leq 120 \text{ lb.}$$

A final restriction is that the number of bowls and mugs produced must be either zero or a positive value because it is impossible to produce negative items. These restrictions are referred to as **nonnegativity constraints** and are expressed mathematically as

$$x_1 \geq 0, x_2 \geq 0$$

Nonnegativity constraints restrict the decision variables to zero or positive values.

The complete linear programming model for this problem can now be summarized as follows:

$$\text{maximize } Z = \$40x_1 + 50x_2$$
$$\text{subject to}$$
$$1x_1 + 2x_2 \leq 40$$
$$4x_1 + 3x_2 \leq 120$$
$$x_1, x_2 \geq 0$$

The solution of this model will result in numeric values for x_1 and x_2 that will maximize total profit, Z. As *one possible* solution, consider $x_1 = 5$ bowls and $x_2 = 10$ mugs. First, we will substitute this hypothetical solution into each of the constraints to make sure that the solution does not require more resources than the constraints show are available:

$$1(5) + 2(10) \leq 40$$
$$25 \leq 40$$

and

$$4(5) + 3(10) \leq 120$$
$$50 \leq 120$$

Because neither of the constraints is violated by this hypothetical solution, we say the solution is **feasible** (i.e., possible). Substituting these solution values in the objective function gives $Z = 40(5) + 50(10) = \$700$. However, for the time being, we do not have any way of knowing whether $700 is the *maximum* profit.

*A **feasible solution** does not violate any of the constraints.*

Now consider a solution of $x_1 = 10$ bowls and $x_2 = 20$ mugs. This solution results in a profit of

$$Z = \$40(10) + 50(20)$$
$$= 400 + 1,000$$
$$= \$1,400$$

*An **infeasible problem** violates at least one of the constraints.*

Although this is certainly a better solution in terms of profit, it is **infeasible** (i.e., not possible) because it violates the resource constraint for labor:

$$1(10) + 2(20) \leq 40$$
$$50 \nleq 40$$

Management Science Application

Allocating Seat Capacity on Indian Railways Using Linear Programming

Indian Railways, with more than 1,600 trains, serves more than 7 million passengers each day. Its reservation system books passengers in three coach classes: reserved — air-conditioned, reserved non-air-conditioned, and unreserved non-air-conditioned. A train can make multiple stops from its origin to its destination, and thus passengers can book many combinations of tickets from a train's origin to its destination or to and/or from intermediate stations. Because passengers can depart the train or board en route, multiple passengers can occupy a single seat during a train's journey from origin to destination. This also means that a seat might be vacant for some segments of the trip and thus will not earn any revenue. If there are an abnormally high number of reservations for intermediate trips, then there might be a high number of partially vacant seats, which might deny subsequent passengers the ability to book complete origin-to-destination trips. This results in suboptimal utilization of a train's capacity. However, in many cases, passenger demand is not high at the train's origin station, and the highest passenger demand occurs en route, at intermediate stations. As a result, the railway has traditionally allocated various seat quotas to intermediate stations and limited the seats allocated for end-to-end trips, in order to maximize capacity utilization and the number of confirmed seat reservations and reduce the number of passengers wait-listed at intermediate stations.

In this application, a linear programming model was formulated, with an objective of minimizing the total seats required to

Neil McAllister/Alamy Stock Photo

fill all possible seat demand (in a specific coach class) between any two major stations, subject to constraints for station-to-station quotas based on historical seat demand. In a test case of 17 trains in the Western Railway zone (based in Mumbai) of Indian Railways, revenue was increased between 2.6% and 29.3%, and the number of passengers carried increased from 8.4% to 29%.

Source: Based on R. Gopalakrishnan and N. Rangaraj, "Capacity Management on Long-Distance Passenger Trains of Indian Railways," *Interfaces* 40, no. 4 (July–August 2010): 291–302.

The solution to this problem must maximize profit without violating the constraints. The solution that achieves this objective is $x_1 = 24$ bowls and $x_2 = 8$ mugs, with a corresponding profit of $1,360. The determination of this solution is shown using the graphical solution approach in the following section.

Graphical Solutions of Linear Programming Models

Graphical solutions are limited to linear programming problems with only two decision variables.

Following the formulation of a mathematical model, the next stage in the application of linear programming to a decision-making problem is to find the solution of the model. A common solution approach is to solve algebraically the set of mathematical relationships that form the model either manually or using a computer program, thus determining the values for the decision variables. However, because the relationships are *linear*, some models and solutions can be illustrated *graphically*.

The graphical method is realistically limited to models with only two decision variables, which can be represented on a graph of two dimensions. Models with three decision variables can be graphed in three dimensions, but the process is quite cumbersome, and models of four or more decision variables cannot be graphed at all.

The graphical method provides a picture of how a solution is obtained for a linear programming problem.

Although the graphical method is limited as a solution approach, it is very useful at this point in our presentation of linear programming in that it gives a picture of how a solution is derived. Graphs can provide a clearer understanding of how the computer and mathematical solution approaches presented in subsequent chapters work and, thus, a better understanding of the solutions.

Graphical Solution of a Maximization Model

The product mix model will be used to demonstrate the graphical interpretation of a linear programming problem. Recall that the problem describes Beaver Creek Pottery Company's attempt to decide how many bowls and mugs to produce daily, given limited amounts of labor and clay. The complete linear programming model was formulated as

$$\text{maximize } Z = \$40x_1 + 50x_2$$

subject to

$$x_1 + 2x_2 \leq 40 \text{ hr. of labor}$$
$$4x_1 + 3x_2 \leq 120 \text{ lb. of clay}$$
$$x_1, x_2 \geq 0$$

where

$$x_1 = \text{number of bowls produced}$$
$$x_2 = \text{number of mugs produced}$$

Figure 2.2 is a set of coordinates for the decision variables x_1 and x_2, on which the graph of our model will be drawn. Note that only the positive quadrant is drawn (i.e., the quadrant where x_1 and x_2 will always be positive) because of the nonnegativity constraints, $x_1 \geq 0$ and $x_2 \geq 0$.

FIGURE 2.2

Coordinates for graphical analysis
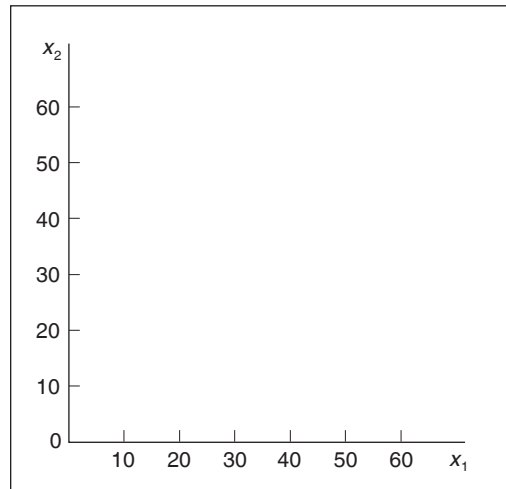
Constraint lines are plotted as equations.

The first step in drawing the graph of the model is to plot the constraints on the graph. This is done by treating both constraints as equations (or straight lines) and plotting each line on the graph. Let's consider the labor constraint line first:

$$x_1 + 2x_2 = 40$$

A simple procedure for plotting this line is to determine two points that are on the line and then draw a straight line through the points. One point can be found by letting $x_1 = 0$ and solving for x_2:

$$(0) + 2x_2 = 40$$
$$x_2 = 20$$

Thus, one point is at the coordinates $x_1 = 0$ and $x_2 = 20$. A second point can be found by letting $x_2 = 0$ and solving for x_1:

$$x_1 + 2(0) = 40$$
$$x_1 = 40$$

Now we have a second point, $x_1 = 40$, $x_2 = 0$. The line on the graph representing this equation is drawn by connecting these two points, as shown in Figure 2.3. However, this is only the graph of the constraint *line* and does not reflect the entire constraint, which also includes the values that are less than or equal to ($\leq$) this line. The *area* representing the entire constraint is shown in Figure 2.4.

FIGURE 2.3

Graph of the labor constraint line
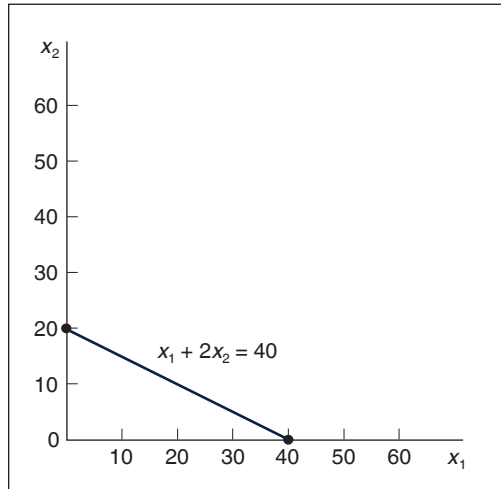
FIGURE 2.4

The labor constraint area
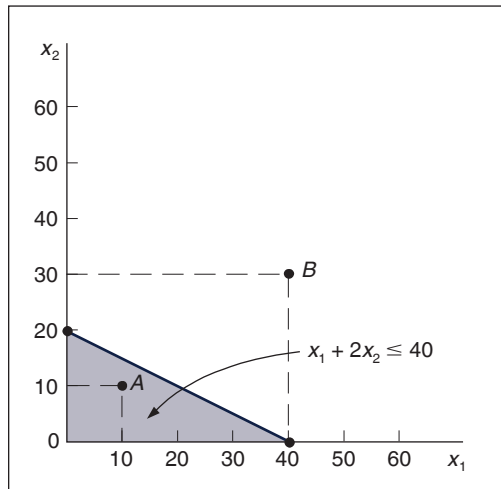
To test the correctness of the constraint area, we check any two points—one inside the constraint area and one outside. For example, check point A in Figure 2.4, which is at the intersection of $x_1 = 10$ and $x_2 = 10$. Substituting these values into the following labor constraint:

$$10 + 2(10) \leq 40$$
$$30 \leq 40 \text{ hr.}$$

shows that point A is indeed within the constraint area, as these values for x_1 and x_2 yield a quantity that does not exceed the limit of 40 hours. Next, we check point B at $x_1 = 40$ and $x_2 = 30$:

$$40 + 2(30) \leq 40$$
$$100 \nleq 40 \text{ hr.}$$

Point *B* is obviously outside the constraint area because the values for x_1 and x_2 yield a quantity (100) that exceeds the limit of 40 hours.

We draw the line for the clay constraint the same way as the one for the labor constraint—by finding two points on the constraint line and connecting them with a straight line. First, let $x_1 = 0$ and solve for x_2:

$$4(0) + 3x_2 = 120$$
$$x_2 = 40$$

Performing this operation results in a point, $x_1 = 0$, $x_2 = 40$. Next, we let $x_2 = 0$ and then solve for x_1:

$$4x_1 + 3(0) = 120$$
$$x_1 = 30$$

This operation yields a second point, $x_1 = 30$, $x_2 = 0$. Plotting these points on the graph and connecting them with a line gives the constraint line and area for clay, as shown in Figure 2.5.

FIGURE 2.5

The constraint area for clay
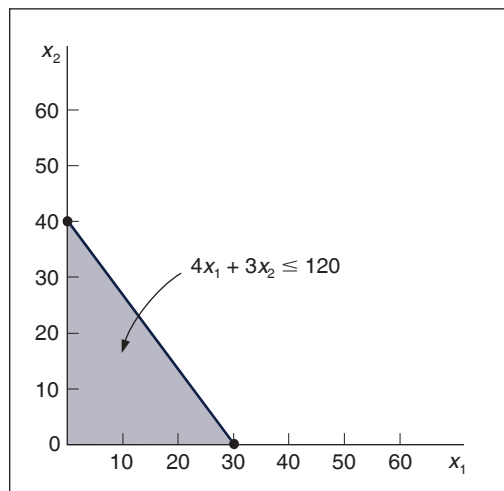
Combining the two individual graphs for both labor and clay (Figures 2.4 and 2.5) produces a graph of the model constraints, as shown in Figure 2.6. The shaded area in Figure 2.6 is the

FIGURE 2.6

Graph of both model constraints
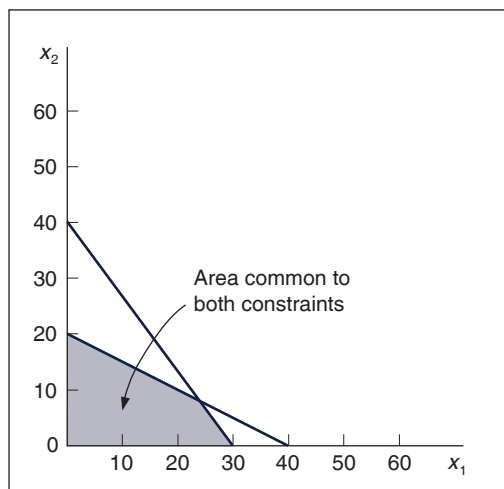
area that is common to both model constraints. Therefore, this is the only area on the graph that contains points (i.e., values for x_1 and x_2) that will satisfy both constraints simultaneously. For example, consider the points R, S, and T in Figure 2.7. Point R satisfies both constraints; thus, we say it is a *feasible* solution point. Point S satisfies the clay constraint ($4x_1 + 3x_2 \leq 120$) but exceeds the labor constraint; thus, it is infeasible. Point T satisfies neither constraint; thus, it is also infeasible.

The feasible solution area is an area on the graph that is bounded by the constraint equations.

The shaded area in Figure 2.7 is referred to as the *feasible solution area* because all the points in this area satisfy both constraints. Some point within this feasible solution area will result in *maximum profit* for Beaver Creek Pottery Company. The next step in the graphical solution approach is to locate this point.

FIGURE 2.7

The feasible solution area constraints
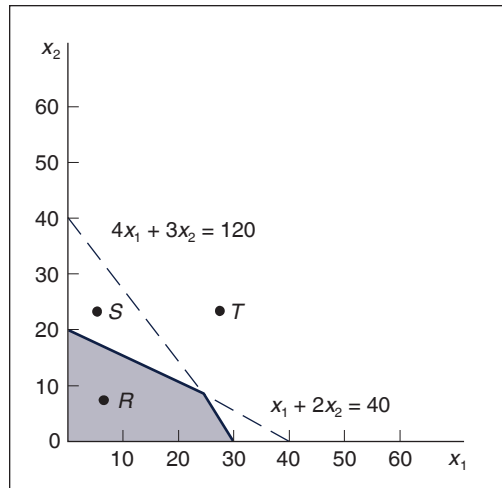
The Optimal Solution Point

The second step in the graphical solution method is to locate the point in the feasible solution area that will result in the greatest total profit. To begin the solution analysis, we first plot the objective function line for an *arbitrarily* selected level of profit. For example, if we say profit, Z, is $800, the objective function is

$$\$800 = 40x_1 + 50x_2$$

Plotting this line just as we plotted the constraint lines results in the graph shown in Figure 2.8. Every point on this line is in the feasible solution area and will result in a profit of $800 (i.e., every combination of x_1 and x_2 on this line will give a Z value of $800). However, let us see whether an even greater profit will still provide a feasible solution. For example, consider profits of $1,200 and $1,600, as shown in Figure 2.9.

A portion of the objective function line for a profit of $1,200 is outside the feasible solution area, but part of the line remains within the feasible area. Therefore, this profit line indicates that there are feasible solution points that give a profit greater than $800. Now let us increase profit

again, to \$1,600. This profit line, also shown in Figure 2.9, is completely outside the feasible solution area. The fact that no points on this line are feasible indicates that a profit of \$1,600 is not possible.

FIGURE 2.8

Objective function line for *Z* = \$800

$$800 = 40x_1 + 50x_2$$
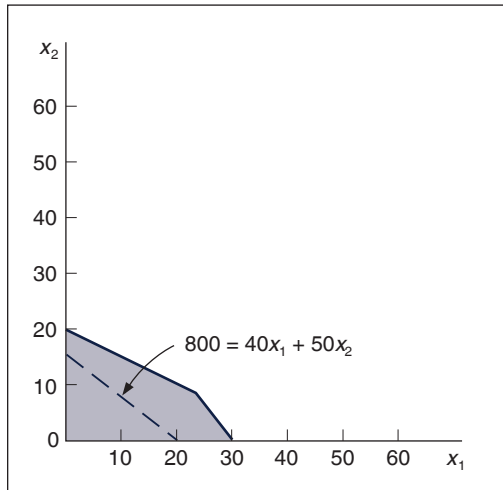
FIGURE 2.9

Alternative objective function lines for profits, *Z*, of \$800, \$1,200, and \$1,600

$$800 = 40x_1 + 50x_2$$
$$1{,}200 = 40x_1 + 50x_2$$
$$1{,}600 = 40x_1 + 50x_2$$
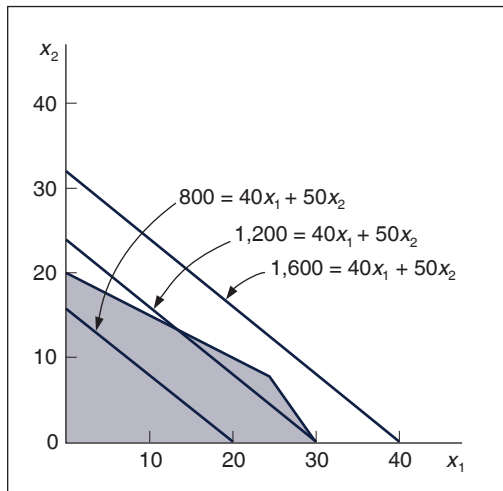
Because a profit of \$1,600 is too great for the constraint limitations, as shown in Figure 2.9, the question of the maximum profit value remains. We can see from Figure 2.9 that profit increases as the objective function line moves away from the origin (i.e., the point $x_1 = 0$, $x_2 = 0$). Given this characteristic, the maximum profit will be attained at the point where the objective function line is farthest from the origin *and* is still touching a point in the feasible solution area. This point is shown as point *B* in Figure 2.10.

To find point *B*, we place a straightedge parallel to the objective function line *The **optimal solution*** $800 = 40x_1 + 50x_2$ in Figure 2.10 and move it outward from the origin as far as we can without *is the best feasible* losing contact with the feasible solution area. Point *B* is referred to as the **optimal** (i.e., best) *solution.* **solution**.

The Solution Values

The third step in the graphical solution approach is to solve for the values of x_1 and x_2 once the optimal solution point has been found. It is possible to determine the x_1 and x_2 coordinates of point B in Figure 2.10 directly from the graph, as shown in Figure 2.11. The graphical coordinates corresponding to point B in Figure 2.11 are $x_1 = 24$ and $x_2 = 8$. This is the optimal solution for the decision variables in the problem. However, unless an absolutely accurate graph is drawn, it is frequently difficult to determine the correct solution directly from the graph. A more exact approach is to determine the solution values mathematically once the optimal point on the graph has been determined. The mathematical approach for determining the solution is described in the following pages. First, however, we will consider a few characteristics of the solution.

FIGURE 2.10

Identification of optimal solution point
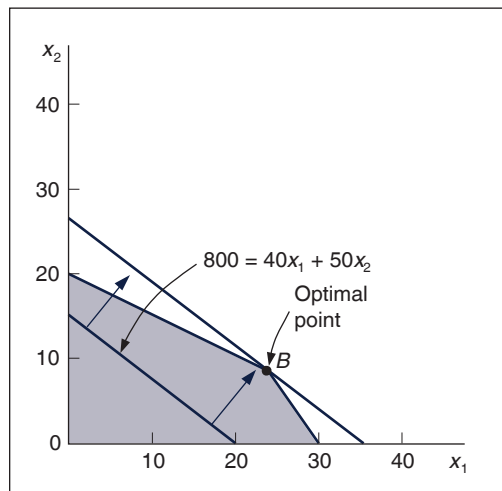
FIGURE 2.11

Optimal solution coordinates
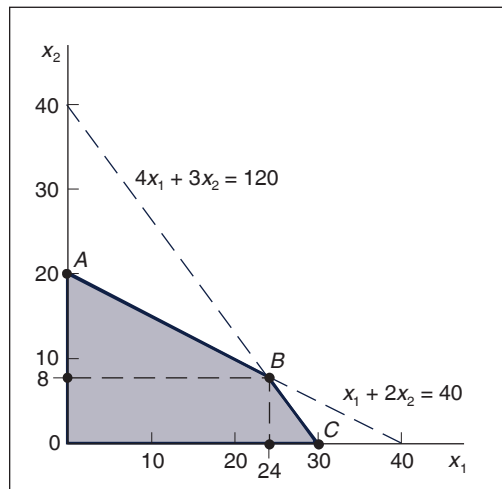
The optimal solution point is the last point the objective function touches as it leaves the feasible solution area.

In Figure 2.10, as the objective function was increased, the last point it touched in the feasible solution area was on the boundary of the feasible solution area. The solution point is always on this boundary because the boundary contains the points farthest from the origin (i.e., the points corresponding to the greatest profit). This characteristic of linear programming problems reduces the number of possible solution points considerably, from all points in the solution area to just those points on the boundary. However, the number of possible solution points is reduced even more by another characteristic of linear programming problems.

The solution point will be on the boundary of the feasible solution area and at one of the *corners* of the boundary where two constraint lines intersect. (The graphical axes, you will recall, are also constraints because $x_1 \geq 0$ and $x_2 \geq 0$.) These corners (points A, B, and C in Figure 2.11) are protrusions, or *extremes*, in the feasible solution area; they are called **extreme points**. It has been proven mathematically that the optimal solution in a linear programming model will always occur at an extreme point. Therefore, in our sample problem the possible solution points are limited to the three extreme points, A, B, and C. The optimal extreme point is the extreme point the objective function touches last as it leaves the feasible solution area, as shown in Figure 2.10.

Extreme points are corner points on the boundary of the feasible solution area.

From the graph shown in Figure 2.10, we know that the optimal solution point is B. Because point B is formed by the intersection of two constraint lines, as shown in Figure 2.11, these two lines are *equal* at point B. Thus, the values of x_1 and x_2 at that intersection can be found by solving the two equations *simultaneously*.

Constraint equations are solved simultaneously at the optimal extreme point to determine the variable solution values.

First, we convert both equations to functions of x_1:

$$x_1 + 2x_2 = 40$$
$$x_1 = 40 - 2x_2$$

and

$$4x_1 + 3x_2 = 120$$
$$4x_1 = 120 - 3x_2$$
$$x_1 = 30 - (3x_2/4)$$

Now, we let x_1 in the first equation equal x_1 in the second equation,

$$40 - 2x_2 = 30 - (3x_2/4)$$

and solve for x_2:

$$5x_2/4 = 10$$
$$x_2 = 8$$

Substituting $x_2 = 8$ into either one of the original equations gives a value for x_1:

$$x_1 = 40 - 2x_2$$
$$x_1 = 40 - 2(8)$$
$$= 24$$

Thus, the optimal solution at point B in Figure 2.11 is $x_1 = 24$ and $x_2 = 8$. Substituting these values into the objective function gives the maximum profit,

$$Z = \$40x_1 + 50x_2$$
$$Z = \$40(24) + 50(8)$$
$$= \$1,360$$

In terms of the original problem, the solution indicates that if the pottery company produces 24 bowls and 8 mugs, it will receive $1,360, the maximum daily profit possible (given the resource constraints).

Given that the optimal solution will be at one of the extreme corner points, A, B, or C, we can also find the solution by testing each of the three points to see which results in the greatest profit, rather than by graphing the objective function and seeing which point it last touches as it

moves out of the feasible solution area. Figure 2.12 shows the solution values for all three points, A, B, and C, and the amount of profit, Z, at each point.

As indicated in the discussion of Figure 2.10, point B is the optimal solution point because it is the last point the objective function touches before it leaves the solution area. In other words, the objective function determines which extreme point is optimal. This is because the objective function designates the profit that will accrue from each combination of x_1 and x_2 values at the extreme points. If the objective function had had different coefficients (i.e., different x_1 and x_2 profit values), one of the extreme points other than B might have been optimal.

FIGURE 2.12

Solutions at all corner points
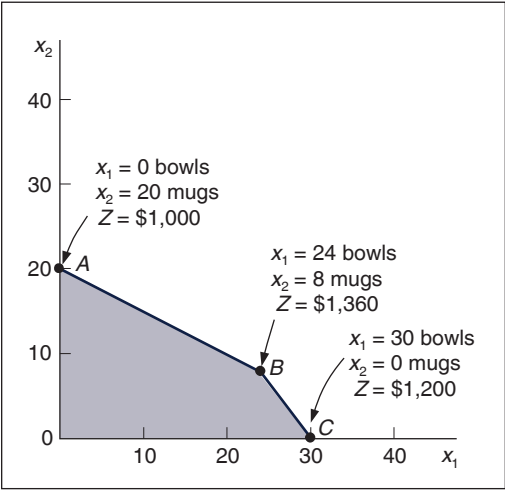
Let's assume for a moment that the profit for a bowl is $70 instead of $40, and the profit for a mug is $20 instead of $50. These values result in a new objective function, $Z = \$70x_1 + 20x_2$. If the model constraints for labor or clay are not changed, the feasible solution area remains the same, as shown in Figure 2.13. However, the location of the objective function in Figure 2.13 is different from that of the original objective function in Figure 2.10. The reason for this change is that the new profit coefficients give the linear objective function a new *slope*.

FIGURE 2.13

The optimal solution with $Z = 70x_1 = 20x_2$
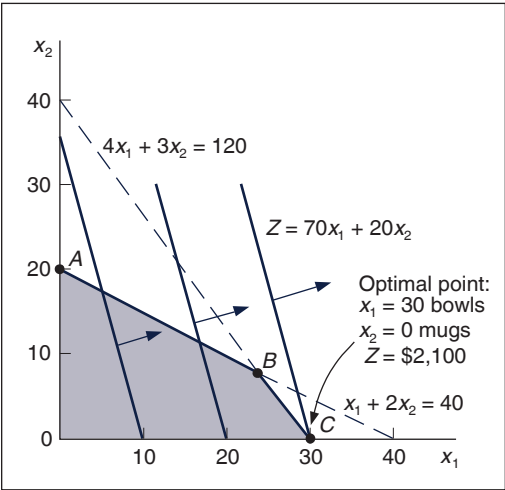
*The **slope** is computed as the "rise" over the "run."*

The **slope** can be determined by transforming the objective function into the general equation for a straight line, $y = a + bx$, where y is the dependent variable, a is the y intercept, b is the slope, and x is the independent variable. For our sample objective function, x_2 is the dependent

variable corresponding to y (i.e., it is on the vertical axis), and x_1 is the independent variable. Thus, the objective function can be transformed into the general equation of a line as follows:

$$Z = 70x_1 + 20x_2$$
$$20x_2 = Z - 70x_1$$
$$x_2 = \frac{Z}{20} - \frac{7}{2}x_1$$
$$\uparrow \quad \uparrow \quad \uparrow$$
$$y \quad a \quad b$$

This transformation identifies the slope of the new objective function as $-7/2$ (the minus sign indicates that the line slopes downward). In contrast, the slope of the original objective function was $-4/5$.

If we move this new objective function out through the feasible solution area, the last extreme point it touches is point C. Simultaneously solving the constraint lines at point C results in the following solution:

$$x_1 = 30$$
$$4x_1 + 3x_2 = 120$$

and

$$x_2 = 40 - (4x_1/3)$$
$$x_2 = 40 - 4(30)/3$$
$$x_2 = 0$$

Thus, the optimal solution at point C in Figure 2.13 is $x_1 = 30$ bowls, $x_2 = 0$ mugs, and $Z = \$2,100$ profit. Altering the objective function coefficients results in a new solution.

This brief example of the effects of altering the objective function highlights two useful points. First, the optimal extreme point is determined by the objective function, and an extreme point on one axis of the graph is as likely to be the optimal solution as is an extreme point on a different axis. Second, the solution is sensitive to the values of the coefficients in the objective function. If the objective function coefficients are changed, as in our example, the solution may change. Likewise, if the constraint coefficients are changed, the solution space and solution points may change also. This information can be of consequence to the decision maker trying to deter-

Sensitivity analysis mine how much of a product to produce. **Sensitivity analysis**—the use of linear programming to
is used to analyze evaluate the effects of changes in model parameters—is discussed in Chapter 3.
changes in model It should be noted that some problems do not have a single extreme point solution. For example,
parameters. when the objective function line parallels one of the constraint lines, an entire line segment is bounded by two adjacent corner points that are optimal; there is no single extreme point on the objective
Multiple optimal function line. In this situation, there are **multiple optimal solutions**. This and other irregular types
solutions *can occur* of solution outcomes in linear programming are discussed at the end of this chapter.
when the objective
function is parallel to
a constraint line.

Slack Variables

Once the optimal solution was found at point B in Figure 2.12, simultaneous equations were solved to determine the values of x_1 and x_2. Recall that the solution occurs at an extreme point
A slack variable where constraint equation lines intersect with each other or with the axis. Thus, the model con-
is added to a $\leq$ straints are considered as *equations* ($=$) rather than $\leq$ or $\geq$ inequalities.
constraint to convert There is a standard procedure for transforming $\leq$ inequality constraints into equations. This
it to an equation ($=$). transformation is achieved by adding a new variable, called a **slack variable**, to each constraint.
For the pottery company example, the model constraints are
*A **slack variable***
represents unused
resources.

$$x_1 + 2x_2 \leq 40 \text{ hr. of labor}$$
$$4x_1 + 3x_2 \leq 120 \text{ lb. of clay}$$

The addition of a unique slack variable, s_1, to the labor constraint and s_2 to the constraint for clay results in the following equations:

$$x_1 + 2x_2 + s_1 = 40 \text{ hr. of labor}$$
$$4x_1 + 3x_2 + s_2 = 120 \text{ lb. of clay}$$

The slack variables in these equations, s_1 and s_2, will take on any value necessary to make the left-hand side of the equation equal to the right-hand side. For example, consider a hypothetical solution of $x_1 = 5$ and $x_2 = 10$. Substituting these values into the foregoing equations yields

$$x_1 + 2x_2 + s_1 = 40 \text{ hr. of labor}$$
$$5 + 2(10) + s_1 = 40 \text{ hr. of labor}$$
$$s_1 = 15 \text{ hr. of labor}$$

and

$$4x_1 + 3x_2 + s_2 = 120 \text{ lb. of clay}$$
$$4(5) + 3(10) + s_2 = 120 \text{ lb. of clay}$$
$$s_2 = 70 \text{ lb. of clay}$$

In this example, $x_1 = 5$ bowls and $x_2 = 10$ mugs represent a solution that does not make use of the total available amount of labor and clay. In the labor constraint, 5 bowls and 10 mugs require only 25 hours of labor. This leaves 15 hours that are not used. Thus, s_1 represents the amount of *unused* labor, or slack.

In the clay constraint, 5 bowls and 10 mugs require only 50 pounds of clay. This leaves 70 pounds of clay unused. Thus, s_2 represents the amount of *unused* clay. In general, slack variables represent the amount of *unused resources*.

The ultimate instance of unused resources occurs at the origin, where $x_1 = 0$ and $x_2 = 0$. Substituting these values into the equations yields

$$x_1 + 2x_2 + s_1 = 40$$
$$0 + 2(0) + s_1 = 40$$
$$s_1 = 40 \text{ hr. of labor}$$

and

$$4x_1 + 3x_2 + s_2 = 120$$
$$4(0) + 3(0) + s_2 = 120$$
$$s_2 = 120 \text{ lb. of clay}$$

Because no production takes place at the origin, all the resources are unused; thus, the slack variables equal the total available amounts of each resource: $s_1 = 40$ hours of labor and $s_2 = 120$ pounds of clay.

What is the effect of these new slack variables on the objective function? The objective function for our example represents the profit gained from the production of bowls and mugs,

$$Z = \$40x_1 + \$50x_2$$

A slack variable contributes nothing to the objective function value.

The coefficient $40 is the contribution to profit of each bowl; $50 is the contribution to profit of each mug. What, then, do the slack variables s_1 and s_2 contribute? They contribute *nothing* to profit because they represent unused resources. Profit is made only after the resources are put to use in making bowls and mugs. Using slack variables, we can write the objective function as

$$\text{maximize } Z = \$40x_1 + \$50x_2 + 0s_1 + 0s_2$$

As in the case of decision variables (x_1 and x_2), slack variables can have only nonnegative values because negative resources are not possible. Therefore, for this model formulation, x_1, x_2, s_1, and $s_2 \geq 0$.

The complete linear programming model can be written in what is referred to as *standard form* with slack variables as follows:

$$\text{maximize } Z = \$40x_1 + \$50x_2 + 0s_1 + 0s_2$$

subject to

$$x_1 + 2x_2 + s_1 = 40$$
$$4x_1 + 3x_2 + s_2 = 120$$
$$x_1, x_2, s_1, s_2 \geq 0$$

The solution values, including the slack at each solution point, are summarized as follows:

Solution Summary with Slack			
Point	Solution Values	Z	Slack
A	$x_1 = 0$ bowls, $x_2 = 20$ mugs	$1,000	$s_1 = 0$ hr.; $s_2 = 60$ lb.
B	$x_1 = 24$ bowls, $x_2 = 8$ mugs	$1,360	$s_1 = 0$ hr.; $s_2 = 0$ lb.
C	$x_1 = 30$ bowls, $x_2 = 0$ mugs	$1,200	$s_1 = 10$ hr.; $s_2 = 0$ lb.

Figure 2.14 shows the graphical solution of this example, with slack variables included at each solution point.
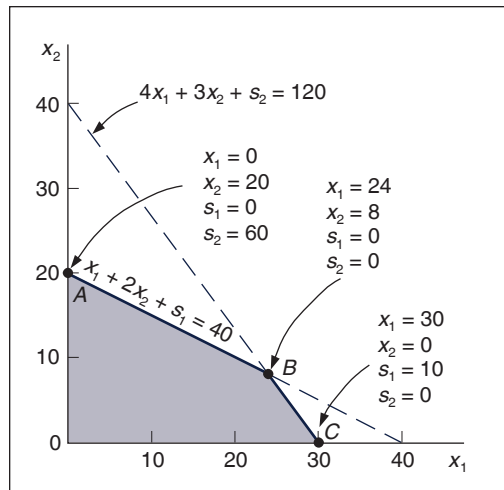
Summary of the Graphical Solution Steps

The steps for solving a graphical linear programming model are summarized here:

1. Plot the model constraints as equations on the graph; then, considering the inequalities of the constraints, indicate the feasible solution area.
2. Plot the objective function; then, move this line out from the origin to locate the optimal solution point.
3. Solve simultaneous equations at the solution point to find the optimal solution values.

Management Science Application

Renewable Energy Investment Decisions at GE Energy

General Electric Energy is the world's second-largest manufacturer of wind turbines and has a growing solar business, and GE Capitals' Energy Financial Services invests billions of dollars each year in renewable energy power plants. To help it make investment decisions GE Energy developed a mathematical programming model to project the financial performance of its renewable energy plants and forecast new renewable energy capacity and generation over a 10-year horizon. Building renewable energy power plants is more expensive than building conventional plants, so their construction is driven by government incentives and mandates called renewable portfolio standards (RPS), which require utilities to meet a percentage of demand with renewable energy sources. Utilities that cannot meet these mandates either purchase renewable energy credits from utilities that have excess renewable energy or pay a compliance payment. The model's objective function minimizes the cost of meeting the RPS mandates over a 10-year horizon (which is equal to the cost of producing energy at new plants plus energy transportation costs plus penalties paid for noncompliance), subject to constraints for meeting RPS demand by program and year, meeting transmission capacity constraints, new building capacity constraints, and meeting energy capacity constraints. Decision variables include (for the time horizon) renewable energy plants built by technology, resource, and year; renewable energy transported across states by year; and penalties paid to states for not

Michele Tantussi/Getty Images

meeting RPS mandates by year. Since it was implemented, the model has helped GE estimate renewable energy credit prices used in renewable project evaluations, forecast renewable energy demand, evaluate the effectiveness of government policies, and advocate for effective policies.

Source: Based on S. Bollapragada, B. Owens, and S. Taub, "Practice Summaries: An Optimization Model to Support Renewable Energy Investment Decisions," *Interfaces* 41, no. 4 (July–August 2011): 394–95.

Or

1. Solve simultaneous equations at each corner point to find the solution values at each point.
2. Substitute these values into the objective function to find the set of values that results in the maximum *Z* value.

A Minimization Model Example

As mentioned at the beginning of this chapter, there are two types of linear programming problems: maximization problems (like the Beaver Creek Pottery Company example) and minimization problems. A minimization problem is formulated the same basic way as a maximization problem, except for a few minor differences. The following sample problem will demonstrate the formulation of a minimization model.

A farmer is preparing to plant a crop in the spring and needs to fertilize a field. There are two brands of fertilizer to choose from, Super-gro and Crop-quick. Each brand yields a specific amount of nitrogen and phosphate per bag, as follows:

	Chemical Contribution	
Brand	**Nitrogen (lb./bag)**	**Phosphate (lb./bag)**
Super-gro	2	4
Crop-quick	4	3

The farmer's field requires at least 16 pounds of nitrogen and at least 24 pounds of phosphate. Super-gro costs $6 per bag, and Crop-quick costs $3. The farmer wants to know how many bags of each brand to purchase to minimize the total cost of fertilizing. This scenario is illustrated in Figure 2.15.

FIGURE 2.15
Fertilizing farmer's field
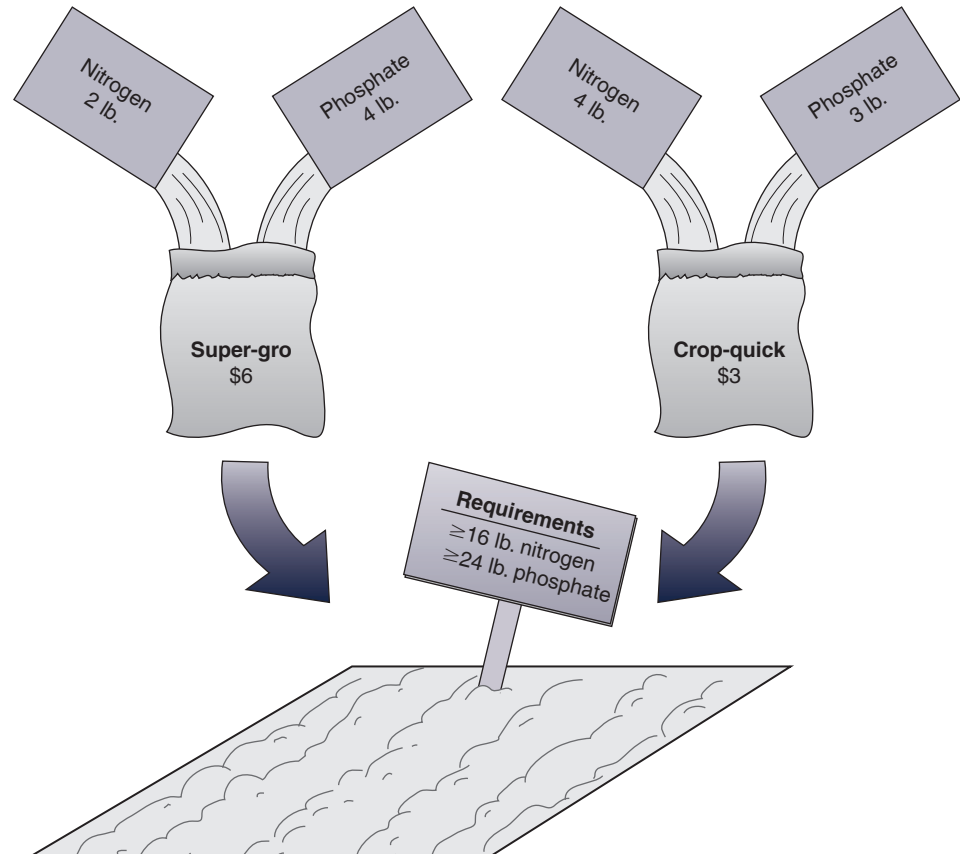
The steps in the linear programming model formulation process are summarized as follows:

Summary of LP Model Formulation Steps

Step 1: Define the decision variables
How many bags of Super-gro and Crop-quick to buy

Step 2: Define the objective function
Minimize cost

Step 3: Define the constraints
The field requirements for nitrogen and phosphate

Decision Variables

This problem contains two decision variables, representing the number of bags of each brand of fertilizer to purchase:

$$x_1 = \text{bags of Super-gro}$$
$$x_2 = \text{bags of Crop-quick}$$

The Objective Function

The farmer's objective is to minimize the total cost of fertilizing. The total cost is the sum of the individual costs of each type of fertilizer purchased. The objective function that represents total cost is expressed as

$$\text{minimize } Z = \$6x_1 + 3x_2$$

where

$\$6x_1 = $ cost of bags of Super-gro

$\$3x_2 = $ cost of bags of Crop-quick

Model Constraints

The requirements for nitrogen and phosphate represent the constraints of the model. Each bag of fertilizer contributes a number of pounds of nitrogen and phosphate to the field. The constraint for nitrogen is

$$2x_1 + 4x_2 \geq 16 \text{ lb.}$$

where

$2x_1 = $ the nitrogen contribution (lb.) per bag of Super-gro
$4x_2 = $ the nitrogen contribution (lb.) per bag of Crop-quick

Rather than a $\leq$ (less than or equal to) inequality, as used in the Beaver Creek Pottery Company model, this constraint requires a $\geq$ (greater than or equal to) inequality. This is because the nitrogen content for the field is a minimum requirement specifying that at least 16 pounds of nitrogen be deposited on the farmer's field. If a minimum cost solution results in more than 16 pounds of nitrogen on the field, that is acceptable; however, the amount cannot be less than 16 pounds.

The constraint for phosphate is constructed like the constraint for nitrogen:

$$4x_1 + 3x_2 \geq 24 \text{ lb.}$$

The three types of linear programming constraints are $\leq$, $=$, and $\geq$.   With this example, we have shown two of the three types of linear programming model constraints, $\leq$ and $\geq$. The third type is an exact equality, $=$. This type specifies that a constraint requirement must be exact. For example, if the farmer had said that the phosphate requirement for the field was exactly 24 pounds, the constraint would have been

$$4x_1 + 3x_2 = 24 \text{ lb.}$$

As in our maximization model, there are also nonnegativity constraints in this problem to indicate that negative bags of fertilizer cannot be purchased:

$$x_1, x_2 \geq 0$$

The complete model formulation for this minimization problem is

$$\text{minimize } Z = \$6x_1 + 3x_2$$
$$\text{subject to}$$
$$2x_1 + 4x_2 \geq 16 \text{ lb of nitrogen}$$
$$4x_1 + 3x_2 \geq 24 \text{ lb of phosphate}$$
$$x_1, x_2 \geq 0$$

Graphical Solution of a Minimization Model

We follow the same basic steps in the graphical solution of a minimization model as in a maximization model. The fertilizer example will be used to demonstrate the graphical solution of a minimization model.

The first step is to graph the equations of the two model constraints, as shown in Figure 2.16. Next, the feasible solution area is chosen, to reflect the $\geq$ inequalities in the constraints, as shown in Figure 2.17.
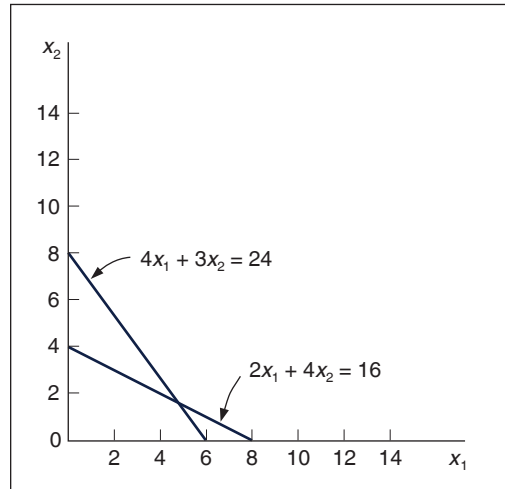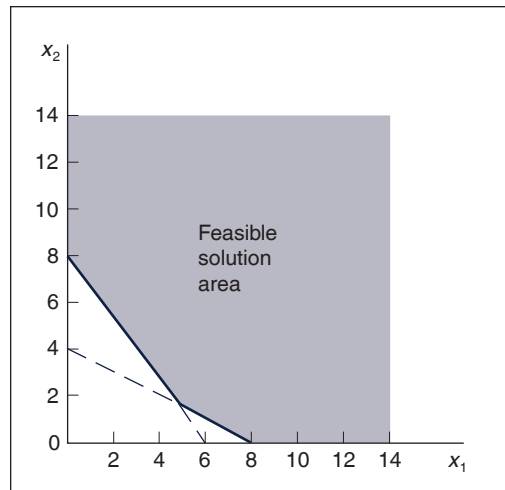
In a minimization problem the boundary of the feasible solution area closest to the origin contains the optimal extreme point.

After the feasible solution area has been determined, the second step in the graphical solution approach is to locate the optimal point. Recall that in a maximization problem, the optimal solution is on the boundary of the feasible solution area that contains the point(s) farthest from the origin. The optimal solution point in a minimization problem is also on the boundary of the feasible solution area; however, the boundary contains the point(s) *closest* to the origin (zero being the lowest cost possible).

As in a maximization problem, the optimal solution is located at one of the extreme points of the boundary. In this case, the corner points represent extremities in the boundary of the feasible
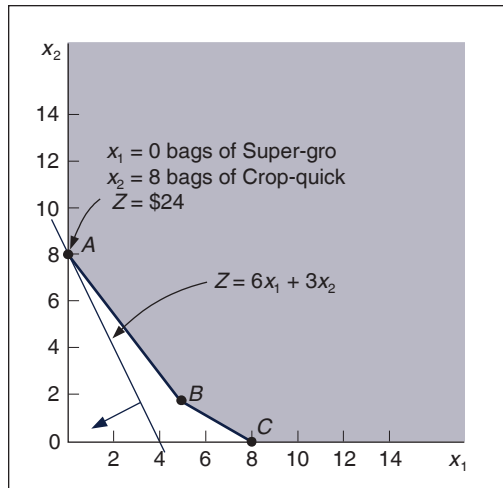
solution area that are *closest* to the origin. Figure 2.18 shows the three corner points—*A*, *B*, and *C*—and the objective function line.

As the objective function edges *toward* the origin, the last point it touches in the feasible solution area is *A*. In other words, point *A* is the closest the objective function can get to the origin without encompassing infeasible points. Thus, it corresponds to the lowest cost that can be attained.

Management Science Application

Determining Optimal Fertilizer Mixes at Soquimich (South America)

Soquimich, a Chilean fertilizer manufacturer, is the leading producer and distributor of specialty fertilizers in the world, with revenues of almost US$0.5 billion in more than 80 countries. Soquimich produces four main specialty fertilizers and more than 200 fertilizer blends, depending on the needs of its customers. Farmers want the company to quickly recommend optimal fertilizer blends that will provide the appropriate quantity of ingredients for their particular crop at the lowest possible cost. A farmer will provide a company sales representative with information about previous crop yields and his or her target yields, and then company representatives will visit the farm to obtain soil samples, which are analyzed in the company labs. A report is generated, which indicates the soil requirements for nutrients, including nitrogen, phosphorus, potassium, boron, magnesium, sulfur, and zinc. Given these soil requirements, company experts determine an optimal fertilizer blend, using a linear programming model that includes constraints for the nutrient quantities required by the soil (for a particular crop) and an objective function that minimizes production costs. Previously the company determined fertilizer blend recommendations by using a time-consuming manual procedure conducted by experts. The linear programming model enables the company to provide accurate, quick, low-cost (discounted)

Alex Havret/Dk Images

estimates to its customers, which has helped the company gain new customers and increase its market share.

Source: Based on A. M. Angel, L. A. Taladriz, and R. Weber, "Soquimich Uses a System Based on Mixed-Integer Linear Programming and Expert Systems to Improve Customer Service," *Interfaces* 33, no. 4 (July–August 2003): 41–52.

The final step in the graphical solution approach is to solve for the values of x_1 and x_2 at point A. Because point A is on the x_2 axis, $x_1 = 0$; thus,

$$4(0) + 3x_2 = 24$$
$$x_2 = 8$$

Given that the optimal solution is $x_1 = 0$, $x_2 = 8$, the minimum cost, Z, is

$$Z = \$6x_1 + \$3x_2$$
$$Z = 6(0) + 3(8)$$
$$= \$24$$

This means the farmer should not purchase any Super-gro but, instead, should purchase eight bags of Crop-quick, at a total cost of $24.

Surplus Variables

Greater-than or equal-to constraints cannot be converted to equations by adding slack variables, as with $\leq$ constraints. Recall our fertilizer model, formulated as

$$\text{minimize } Z = \$6x_1 + \$3x_2$$

subject to

$$2x_1 + 4x_2 \geq 16 \text{ lb. of nitrogen}$$
$$4x_1 + 3x_2 \geq 24 \text{ lb. of phosphate}$$
$$x_1, x_2 \geq 0$$

where

$x_1 = $ bags of Super-gro fertilizer
$x_2 = $ bags of Crop-quick fertilizer
$Z = $ farmers total cost ($) of purchasing fertilizer

A surplus variable is subtracted from a $\geq$ constraint to convert it to an equation $(=)$.

*A **surplus variable** represents an excess above a constraint requirement level.*

Because this problem has $\geq$ constraints as opposed to the $\leq$ constraints of the Beaver Creek Pottery Company maximization example, the constraints are converted to equations a little differently.

Instead of adding a slack variable as we did with a $\geq$ constraint, we subtract a **surplus variable**. Whereas a slack variable is added and reflects unused resources, a surplus variable is subtracted and reflects the excess above a minimum resource requirement level. Like a slack variable, a surplus variable is represented symbolically by s_1 and must be nonnegative.

For the nitrogen constraint, the subtraction of a surplus variable gives

$$2x_1 + 4x_2 - s_1 = 16$$

The surplus variable s_1 transforms the nitrogen constraint into an equation.

As an example, consider the hypothetical solution

$$x_1 = 0$$
$$x_2 = 10$$

Substituting these values into the previous equation yields

$$2(0) + 4(10) - s_1 = 16$$
$$-s_1 = 16 - 40$$
$$s_1 = 24 \text{ lb. of nitrogen}$$

In this equation, s_1 can be interpreted as the *extra* amount of nitrogen above the minimum requirement of 16 pounds that would be obtained by purchasing 10 bags of Crop-quick fertilizer.

In a similar manner, the constraint for phosphate is converted to an equation by subtracting a surplus variable, s_2:

$$4x_1 + 3x_2 - s_2 = 24$$

As is the case with slack variables, surplus variables contribute nothing to the overall cost of a model. For example, putting additional nitrogen or phosphate on the field will not affect the farmer's cost; the only thing affecting cost is the number of bags of fertilizer purchased. As such, the standard form of this linear programming model is summarized as

$$\text{minimize } Z = \$6x_1 + 3x_2 + 0s_1 + 0s_2$$
$$\text{subject to}$$
$$2x_1 + 4x_2 - s_1 = 16$$
$$4x_1 + 3x_2 - s_2 = 24$$
$$x_1, x_2, s_1, s_2 \geq 0$$

Figure 2.19 shows the graphical solutions for our example, with surplus variables included at each solution point.

FIGURE 2.19

Graph of the fertilizer example
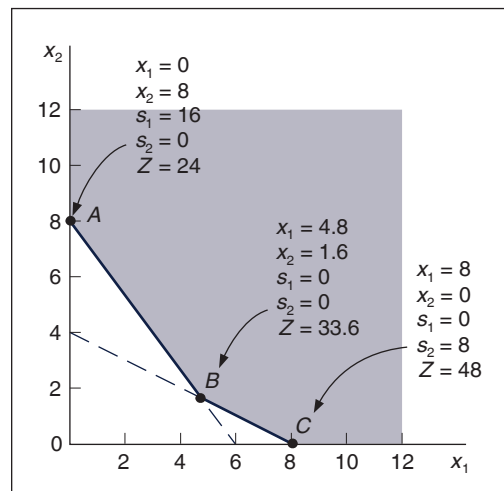
Irregular Types of Linear Programming Problems

For some linear programming models, the general rules do not always apply.

The basic forms of typical maximization and minimization problems have been shown in this chapter. However, there are several special types of atypical linear programming problems. Although these special cases do not occur frequently, they will be described so that you can recognize them when they arise. These special types include problems with more than one optimal solution, infeasible problems, and problems with unbounded solutions.

Multiple Optimal Solutions

Consider the Beaver Creek Pottery Company example, with the objective function changed from $Z = 40x_1 + 50x_2$ to $Z = 40x_1 + 30x_2$:

$$\text{maximize } Z = 40x_1 + 30x_2$$

subject to

$$x_1 + 2x_2 \leq 40 \text{ hr. of labor}$$
$$4x_1 + 3x_2 \leq 120 \text{ lb. of clay}$$
$$x_1, x_2 \geq 0$$

where
$$x_1 = \text{bowls produced}$$
$$x_2 = \text{mugs produced}$$

Alternate optimal solutions are at the endpoints of the constraint line segment that the objective function parallels.

The graph of this model is shown in Figure 2.20. The slight change in the objective function makes it now *parallel* to the constraint line, $4x_1 + 3x_2 = 120$. Both lines now have the same slope of $-4/3$. Therefore, as the objective function edge moves outward from the origin, it touches the whole line segment BC rather than a single extreme corner point before it leaves the feasible solution area. This means that every point along this line segment is optimal (i.e., each point results in the same profit of $Z = \$1,200$). The endpoints of this line segment, B and C, are typically referred to as the **alternate optimal solutions**. It is understood that these points represent the endpoints of a range of optimal solutions.

FIGURE 2.20

Graph of the Beaver Creek Pottery Company example with multiple optimal solutions
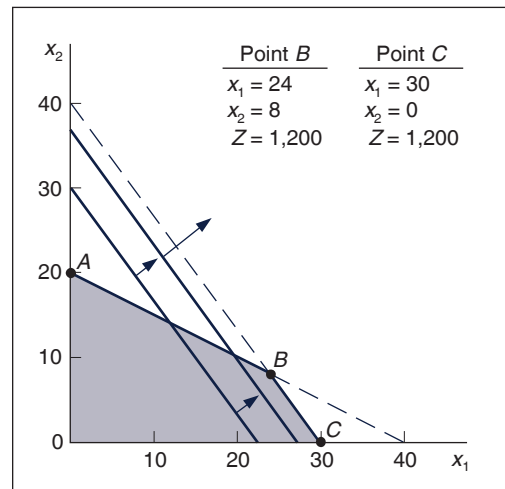
Multiple optimal solutions provide greater flexibility to the decision maker.

The pottery company, therefore, has several options in deciding on the number of bowls and mugs to produce. Multiple optimal solutions can benefit the decision maker because the number of decision options is enlarged. The multiple optimal solutions (along the line segment BC in Figure 2.20) allow the decision maker greater flexibility. For example, in the case of Beaver Creek Pottery Company, it may be easier to sell bowls than mugs; thus, the solution at point C, where only bowls are produced, would be more desirable than the solution at point B, where a mix of bowls and mugs is produced.

*An **infeasible problem** has no feasible solution area; every possible solution point violates one or more constraints.*

An Infeasible Problem

In some cases, a linear programming problem has no feasible solution area; thus, there is no solution to the problem. An example of an **infeasible problem** is formulated next and depicted graphically in Figure 2.21:

$$\text{maximize } Z = 5x_1 + 3x_2$$

subject to

$$4x_1 + 2x_2 \leq 8$$
$$x_1 \geq 4$$
$$x_2 \geq 6$$
$$x_1, x_2 \geq 0$$

Point A in Figure 2.21 satisfies only the constraint $4x_1 + 2x_2 \leq 8$, whereas point C satisfies only the constraints $x_1 \geq 4$ and $x_2 \geq 6$. Point B satisfies none of the constraints. The three constraints do not overlap to form a feasible solution area. Because no point satisfies all three constraints simultaneously, there is no solution to the problem. Infeasible problems do not typically occur, but when they do, they are usually a result of errors in defining the problem or in formulating the linear programming model.

FIGURE 2.21

Graph of an infeasible problem
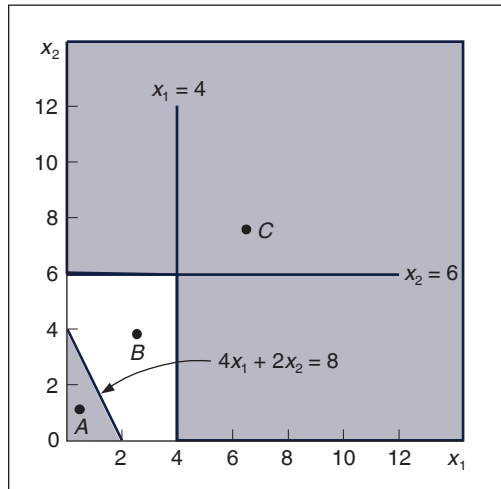
In an **unbounded problem** the objective function can increase indefinitely without reaching a maximum value.

An Unbounded Problem

In some problems, the feasible solution area formed by the model constraints is not closed. In these cases, it is possible for the objective function to increase indefinitely without ever reaching a maximum value because it never reaches the boundary of the feasible solution area.

An example of this type of **unbounded problem** is formulated next and shown graphically in Figure 2.22:

$$\text{maximize } Z = 4x_1 + 2x_2$$
$$\text{subject to}$$
$$x_1 \geq 4$$
$$x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

FIGURE 2.22

An unbounded problem
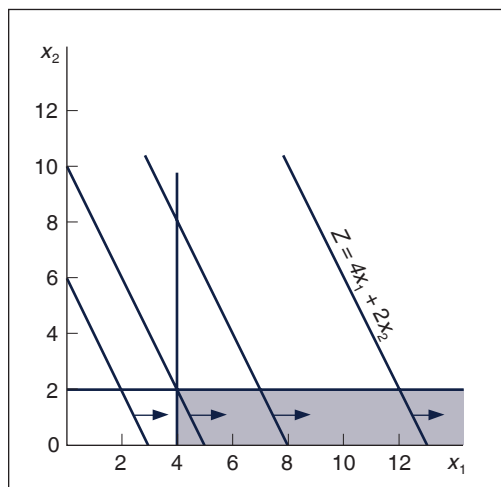
The solution space is not completely closed in.

In Figure 2.22, the objective function is shown to increase without bound; thus, a solution is never reached.

Unlimited profits are not possible in the real world; an unbounded solution, like an infeasible solution, typically reflects an error in defining the problem or in formulating the model.

Characteristics of Linear Programming Problems

Now that we have had the opportunity to construct several linear programming models, let's review the characteristics that identify a linear programming problem.

The components of a linear programming model are an objective function, decision variables, and constraints.

A linear programming problem requires a choice between alternative courses of action (i.e., a decision). The decision is represented in the model by decision variables. A typical choice task for a business firm is deciding how much of several different products to produce, as in the Beaver Creek Pottery Company example presented earlier in this chapter. Identifying the choice task and defining the decision variables is usually the first step in the formulation process because it is quite difficult to construct the objective function and constraints without first identifying the decision variables.

The problem encompasses an objective that the decision maker wants to achieve. The two most frequently encountered objectives for a business are maximizing profit and minimizing cost.

A third characteristic of a linear programming problem is that restrictions exist, making unlimited achievement of the objective function impossible. In a business firm, these restrictions often take the form of limited resources, such as labor or material; however, the sample models in this chapter exhibit a variety of problem restrictions. These restrictions, as well as the objective, must be definable by mathematical functional relationships that are linear. Defining these relationships is typically the most difficult part of the formulation process.

Properties of Linear Programming Models

In addition to encompassing only linear relationships, a linear programming model also has several other implicit properties, which have been exhibited consistently throughout the examples in this chapter. The term *linear* not only means that the functions in the models are graphed as a straight line; it also means that the relationships exhibit **proportionality**. In other words, the rate of change, or slope, of the function is constant; therefore, changes of a given size in the value of a decision variable will result in exactly the same relative changes in the functional value.

Proportionality means the slope of a constraint or objective function line is constant.

*The terms in the objective function or constraints are **additive**.*

Linear programming also requires that the objective function terms and the constraint terms be **additive**. For example, in the Beaver Creek Pottery Company model, the total profit (Z) must equal the sum of profits earned from making bowls ($\$40x_1$) and mugs ($\$50x_2$). Also, the total resources used must equal the sum of the resources used for each activity in a constraint (e.g., labor).

*The values of decision variables are continuous or **divisible**.*

Another property of linear programming models is that the solution values (of the decision variables) cannot be restricted to integer values; the decision variables can take on any fractional value. Thus, the variables are said to be *continuous* or **divisible**, as opposed to *integer* or *discrete*. For example, although decision variables representing bowls or mugs or airplanes or automobiles should realistically have integer (whole number) solutions, the solution methods for linear programming will not necessarily provide such solutions. This is a property that will be discussed further as solution methods are presented in subsequent chapters.

*All model parameters are assumed to be known with **certainty**.*

The final property of linear programming models is that the values of all the model parameters are assumed to be constant and known with **certainty**. In real situations, however, model parameters are frequently uncertain because they reflect the future as well as the present, and future conditions are rarely known with certainty.

To summarize, a linear programming model has the following general properties: linearity, proportionality, additivity, divisibility, and certainty. As various linear programming solution methods are presented throughout this book, these properties will become more obvious, and their impact on problem solution will be discussed in greater detail.

Summary

The two example problems in this chapter were formulated as linear programming models to demonstrate the modeling process. These problems were similar in that they concerned achieving some objective subject to a set of restrictions or requirements. Linear programming models exhibit certain common characteristics:

- An objective function to be maximized or minimized
- A set of constraints
- Decision variables for measuring the level of activity
- Linearity among all constraint relationships and the objective function

The graphical approach to the solution of linear programming problems is not a very efficient means of solving problems. For one thing, drawing accurate graphs is tedious. Moreover, the graphical approach is limited to models with only two decision variables. However, the analysis of the graphical approach provides valuable insight into linear programming problems and their solutions.

In the graphical approach, once the feasible solution area and the optimal solution point have been determined from the graph, simultaneous equations are solved to determine the values of x_1 and x_2 at the solution point. In Chapter 3 we will show how linear programming solutions can be obtained using computer programs.

Example Problem Solutions

As a prelude to the problems, this section presents example solutions to two linear programming problems.

Problem Statement

Moore's Meatpacking Company produces a hot dog mixture in 1,000-pound batches. The mixture contains two ingredients—chicken and beef. The cost per pound of each of these ingredients is as follows:

Ingredient	Cost/lb.
Chicken	$3
Beef	$5

Each batch has the following recipe requirements:

a. At least 500 pounds of chicken
b. At least 200 pounds of beef

The ratio of chicken to beef must be at least 2 to 1. The company wants to know the optimal mixture of ingredients that will minimize cost. Formulate a linear programming model for this problem.

Solution

Step 1: Identify Decision Variables

Recall that the problem should not be "swallowed whole." Identify each part of the model separately, starting with the decision variables:

$$x_1 = \text{lb. of chicken}$$
$$x_2 = \text{lb. of beef}$$

Step 2: Formulate the Objective Function

$$\text{minimize } Z = \$3x_1 + \$5x_2$$

where

$$Z = \text{cost per 1,000-lb batch}$$
$$\$3x_1 = \text{cost of chicken}$$
$$\$5x_2 = \text{cost of beef}$$

Step 3: Establish Model Constraints

The constraints of this problem are embodied in the recipe restrictions and (not to be overlooked) the fact that each batch must consist of 1,000 pounds of mixture:

$$x_1 + x_2 = 1,000 \text{ lb.}$$
$$x_1 \geq 500 \text{ lb. of chicken}$$
$$x_2 \geq 200 \text{ lb. of beef}$$
$$x_1/x_2 \geq 2/1 \text{ or } x_1 - 2x_2 \geq 0$$

and

$$x_1, x_2 \geq 0$$

The Model

$$\text{minimize } Z = \$3x_1 + \$5x_2$$

subject to

$$x_1 + x_2 = 1,000$$
$$x_1 \geq 500$$
$$x_2 \geq 200$$
$$x_1 - 2x_2 \geq 0$$
$$x_1, x_2 \geq 0$$

Problem Statement

Solve the following linear programming model graphically:

$$\text{maximize } Z = 4x_1 + 5x_2$$

subject to

$$x_1 + 2x_2 \leq 10$$
$$6x_1 + 6x_2 \leq 36$$
$$x_1 \leq 4$$
$$x_1, x_2 \geq 0$$

Solution

Step 1: Plot the Constraint Lines as Equations

A simple method for plotting constraint lines is to set one of the constraint variables equal to zero and solve for the other variable to establish a point on one of the axes. The three constraint lines are graphed in the following figure:

The constraint equations
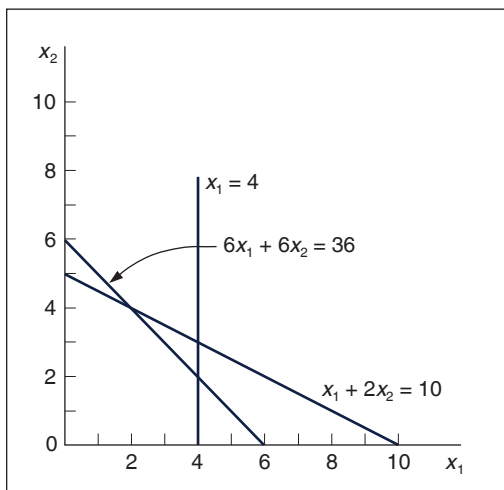
Step 2: Determine the Feasible Solution Area

The feasible solution area is determined by identifying the space that jointly satisfies the $\leq$ conditions of all three constraints, as shown in the following figure:

The feasible solution space and extreme points
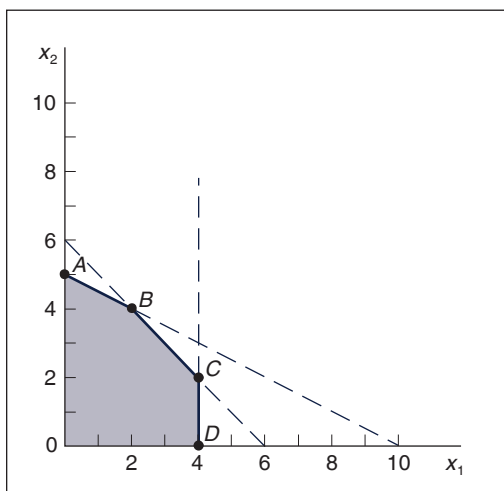
Step 3: Determine the Solution Points

The solution at point A can be determined by noting that the constraint line intersects the x_2 axis at 5; thus, $x_2 = 5$, $x_1 = 0$, and $Z = 25$. The solution at point D on the other axis can be determined similarly; the constraint intersects the axis at $x_1 = 4$, $x_2 = 0$, and $Z = 16$.

The values at points B and C must be found by solving simultaneous equations. Note that point B is formed by the intersection of the lines $x_1 + 2x_2 = 10$ and $6x_1 + 6x_2 = 36$. First, convert both of these equations to functions of x_1:

$$x_1 + 2x_2 = 10$$
$$x_1 = 10 - 2x_2$$

and

$$6x_1 + 6x_2 = 36$$
$$6x_1 = 36 - 6x_2$$
$$x_1 = 6 - x_2$$

Now, set the equations equal and solve for x_2:

$$10 - 2x_2 = 6 - x_2$$
$$-x_2 = -4$$
$$x_2 = 4$$

Substituting $x_2 = 4$ into either of the two equations gives a value for x_1:

$$x_1 = 6 - x_2$$
$$x_1 = 6 - (4)$$
$$x_1 = 2$$

Thus, at point B, $x_1 = 2$, $x_2 = 4$, and $Z = 28$.

At point C, $x_1 = 4$. Substituting $x_1 = 4$ into the equation $x_1 = 6 - x_2$ gives a value for x_2:

$$4 = 6 - x_2$$
$$x_2 = 2$$

Thus, $x_1 = 4$, $x_2 = 2$, and $Z = 26$.

Step 4: Determine the Optimal Solution

The optimal solution is at point B, where $x_1 = 2$, $x_2 = 4$, and $Z = 28$. The optimal solution and solutions at the other extreme points are summarized in the following figure:

Optimal solution point
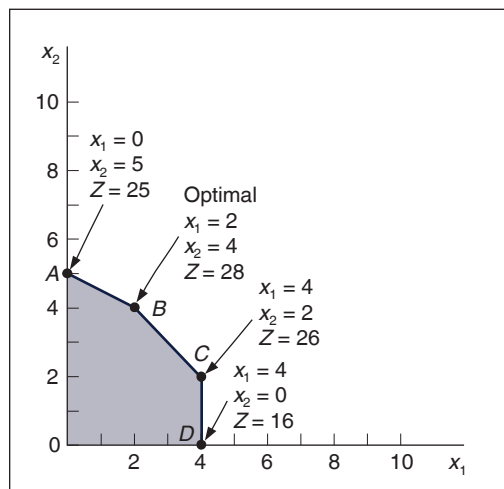
Problems

1. In Problem 40 in Chapter 1, when Tracy McCoy wakes up Saturday morning, she remembers that she promised the PTA she would make some cakes and/or homemade bread for its bake sale that afternoon. However, she does not have time to go to the store to get ingredients, and she has only a short time to bake things in her oven. Because cakes and breads require different baking temperatures, she cannot bake them simultaneously, and she has only 3 hours available to bake. A cake requires 3 cups of flour, and a loaf of bread requires 8 cups; Tracy has 20 cups of flour. A cake requires 45 minutes to bake, and a loaf of bread requires 30 minutes. The PTA will sell a cake for $10 and a loaf of bread for $6. Tracy wants to decide how many cakes and loaves of bread she should make.
   a. Formulate a linear programming model for this problem.
   b. Solve this model by using graphical analysis.

2. The Munchies Cereal Company makes a cereal from several ingredients. Two of the ingredients, oats and rice, provide vitamins A and B. The company wants to know how many ounces of oats and rice it should include in each box of cereal to meet the minimum requirements of 48 milligrams of vitamin A and 12 milligrams of vitamin B while minimizing cost. An ounce of oats contributes 8 milligrams of vitamin A and 1 milligram of vitamin B, whereas an ounce of rice contributes 6 milligrams of A and 2 milligrams of B. An ounce of oats costs $0.05, and an ounce of rice costs $0.03.
   a. Formulate a linear programming model for this problem.
   b. Solve this model by using graphical analysis.

3. What would be the effect on the optimal solution in Problem 2 if the cost of rice increased from $0.03 per ounce to $0.06 per ounce?

4. The Kalo Fertilizer Company makes a fertilizer using two chemicals that provide nitrogen, phosphate, and potassium. A pound of ingredient 1 contributes 10 ounces of nitrogen and 6 ounces of phosphate, whereas a pound of ingredient 2 contributes 2 ounces of nitrogen, 6 ounces of phosphate, and 1 ounce of potassium. Ingredient 1 costs $3 per pound, and ingredient 2 costs $5 per pound. The company wants to know how many pounds of each chemical ingredient to put into a bag of fertilizer to meet the minimum requirements of 20 ounces of nitrogen, 36 ounces of phosphate, and 2 ounces of potassium while minimizing cost.
   a. Formulate a linear programming model for this problem.
   b. Solve this model by using graphical analysis.

5. A metal shop fabricates windows and doors to construction companies. To fabricate these, the shop uses iron, glass, and labor. The company has 80 units of standard size iron, 60 square feet of glass, and unlimited hours of labor but the shop has to use 50 hours of labor. The following table shows the requirement of these resources to produce one window and one door.

Product	Window (x1)	Doors (x2)
Iron	3	4
Glass	7	2
Labor	3	4
Cost	€ 500	€ 400

   The metal shop wants to determine the number of windows and doors it needs to fabricate to minimize its cost. Assume that fractional values of windows and doors are acceptable.
   a. Formulate a linear programming model for this problem.
   b. Solve this model by using graphical analysis.

6. Refer to Problem 5.
   a. How much iron and labor will be unused if the optimal number of windows and doors are produced?
   b. Explain the effect on the optimal solution of changing the profit on a window from € 500 to € 300.

7. The Crumb and Custard Bakery makes coffee cakes and Danish pastries in large pans. The main ingredients are flour and sugar. There are 25 pounds of flour and 16 pounds of sugar available, and the demand for coffee cakes is 5. Five pounds of flour and 2 pounds of sugar are required to make a pan of coffee cakes, and 5 pounds of flour and 4 pounds of sugar are required to make a pan of Danish. A pan of coffee cakes has a profit of $1, and a pan of Danish has a profit of $5. Determine the number of pans of cakes and Danish to produce each day so that profit will be maximized.
   a. Formulate a linear programming model for this problem.
   b. Solve this model by using graphical analysis.

8. In Problem 7, how much flour and sugar will be left unused if the optimal numbers of cakes and Danish are baked?

9. Solve the following linear programming model graphically:

$$\text{maximize } Z = 3x_1 + 6x_2$$
$$\text{subject to}$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 + x_2 \geq 5$$
$$x_1 \leq 4$$
$$x_1, x_2 \geq 0$$

10. The Elixer Drug Company produces a drug from two ingredients. Each ingredient contains the same three antibiotics, in different proportions. One gram of ingredient 1 contributes 3 units, and 1 gram of ingredient 2 contributes 1 unit of antibiotic 1; the drug requires 6 units. At least 4 units of antibiotic 2 are required, and the ingredients contribute 1 unit each per gram. At least 12 units of antibiotic 3 are required; a gram of ingredient 1 contributes 2 units, and a gram of ingredient 2 contributes 6 units. The cost for a gram of ingredient 1 is $80, and the cost for a gram of ingredient 2 is $50. The company wants to formulate a linear programming model to determine the number of grams of each ingredient that must go into the drug to meet the antibiotic requirements at the minimum cost.
    a. Formulate a linear programming model for this problem.
    b. Solve this model by using graphical analysis.

11. A jewelry store makes necklaces and bracelets from gold and platinum. The store has 18 ounces of gold and 20 ounces of platinum. Each necklace requires 3 ounces of gold and 2 ounces of platinum, whereas each bracelet requires 2 ounces of gold and 4 ounces of platinum. The demand for bracelets is no more than four. A necklace earns $300 in profit and a bracelet, $400. The store wants to determine the number of necklaces and bracelets to make to maximize profit.
    a. Formulate a linear programming model for this problem.
    b. Solve this model by using graphical analysis.

12. In Problem 11, explain the effect on the optimal solution of increasing the profit on a bracelet from $400 to $600. What will be the effect of changing the platinum requirement for a necklace from 2 ounces to 3 ounces?

13. In Problem 11:
    a. The maximum demand for bracelets is 4. If the store produces the optimal number of bracelets and necklaces, will the maximum demand for bracelets be met? If not, by how much will it be missed?
    b. What profit for a necklace would result in no bracelets being produced, and what would be the optimal solution for this profit?

14. A clothier makes coats and slacks. The two resources required are wool cloth and labor. The clothier has 150 square yards of wool and 200 hours of labor available. Each coat requires 3 square yards of wool and 10 hours of labor, whereas each pair of slacks requires 5 square yards of wool

and 4 hours of labor. The profit for a coat is $50, and the profit for slacks is $40. The clothier wants to determine the number of coats and pairs of slacks to make so that profit will be maximized.
   a. Formulate a linear programming model for this problem.
   b. Solve this model by using graphical analysis.

15. In Problem 14, what would be the effect on the optimal solution if the available labor were increased from 200 to 240 hours?

16. Amelia wants to do a part time job during her spring break. She has two jobs now—one in an admissions office and the other in a nearby restaurant. She does not want to work for more than 40 hours per week. She believes that working in the restaurant is easier than working in the admissions office. Since the job at the admissions office demands a high degree of attention, she wants to work more in the restaurant. For every 1 hour of working at the admissions office, she plans to work 2 hours in the restaurant. She wants to work up to 22 hours in the restaurant. She will get a revenue of £9 per hour of work at the admissions office and £7 per hour in the restaurant. She wants to determine the number of hours she should work in the admissions office (x_1) and the restaurant (x_2) to maximize her earning per week during the spring break.
   a. Formulate a linear programming model for this problem.
   b. Solve this model by using graphical analysis.

17. In Problem 16:
   a. If Amelia works 35 hours weekly, will she make more money?
   b. If the admissions office reduces Amelia's wage to £8 per hour, and restaurant raises the wage to £9 per hour, and Amelia works up to 25 hours at the restaurant, will these changes affect her revenue?
   c. Alternatively, if she plans to work for 50 hours per week, will this affect her revenue?
   d. If Amelia combines alternatives proposed in b and c, will this affect her profit?

18. Solve the following linear programming model graphically:

$$\text{maximize } Z = 5x_1 + 8x_2$$
$$\text{subject to}$$
$$3x_1 + 5x_2 \leq 50$$
$$2x_1 + 4x_2 \leq 40$$
$$x_1 \leq 8$$
$$x_2 \leq 10$$
$$x_1, x_2 \geq 0$$

19. Transform the model in Problem 18 into standard form, and indicate the value of the slack variables at each corner point solution.

20. Solve the following linear programming model graphically:

$$\text{maximize } Z = 6.5x_1 + 10x_2$$
$$\text{subject to}$$
$$2x_1 + 4x_2 \leq 40$$
$$x_1 + x_2 \leq 15$$
$$x_1 \geq 8$$
$$x_1, x_2 \geq 0$$

21. In Problem 20, if the constraint $x_1 \geq 8$ is changed to $x_1 \leq 8$, what effect does this have on the feasible solution space and the optimal solution?

22. Universal Claims Processors processes insurance claims for large national insurance companies. Most claim processing is done by a large pool of computer operators, some of whom are permanent