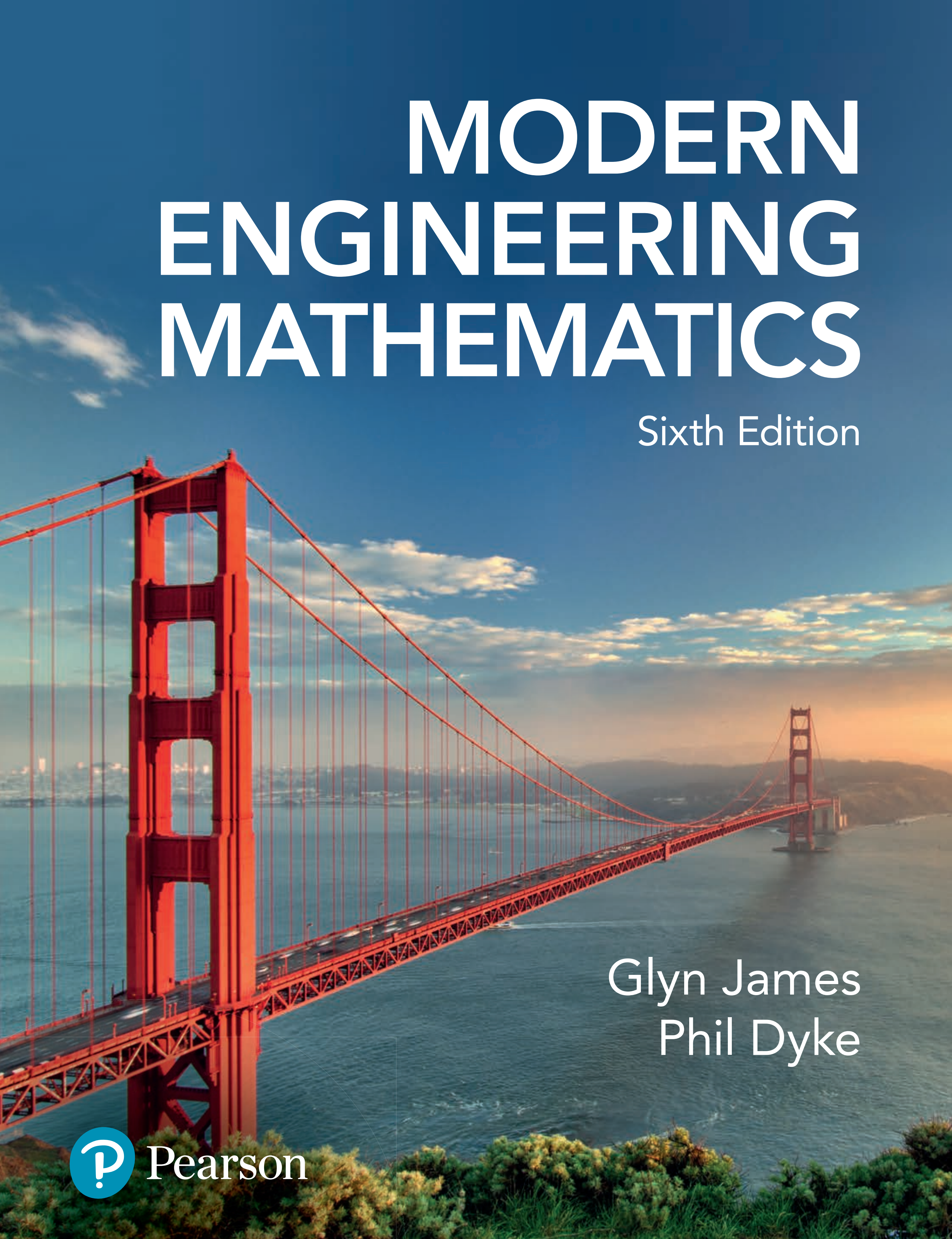

MODERN ENGINEERING MATHEMATICS

Sixth Edition

Glyn James
Phil Dyke

Pearson

Modern Engineering Mathematics

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Sixth Edition

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and
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ON THE WEBSITE

For open-access **student resources** specifically written to complement this textbook and support your learning, please visit **go.pearson.com/uk/he/resources**

Lecturer Resources

For password-protected online resources tailored to support the use of this textbook in teaching, please visit **go.pearson.com/uk/he/resources**

Preface

The first edition of this book appeared in 1992; this is the sixth edition and there have been a few changes, mostly a few corrections and additions, but also more substantive changes to Chapter 13 Data Handling and Probability Theory. Echoing the words of my predecessor Professor Glyn James, the range of material covered in this sixth edition is regarded as appropriate for a first-level core studies course in mathematics for undergraduate courses in all engineering disciplines. Whilst designed primarily for use by engineering students it is believed that the book is also highly suitable for students of the physical sciences and applied mathematics. Additional material appropriate for second-level undergraduate core studies, or possibly elective studies for some engineering disciplines, is contained in the companion text *Advanced Modern Engineering Mathematics*.

The objective of the authoring team remains that of achieving a balance between the development of understanding and the mastering of solution techniques, with the emphasis being on the development of the student's ability to use mathematics with understanding to solve engineering problems. Consequently, the book is not a collection of recipes and techniques designed to teach students to solve routine exercises, nor is mathematical rigour introduced for its own sake. To achieve the desired objective the text contains:

- Worked examples
  Approximately 500 worked examples, many of which incorporate mathematical models and are designed both to provide relevance and to reinforce the role of mathematics in various branches of engineering. In response to feedback from users, additional worked examples have been incorporated within this revised edition.
- Applications
  To provide further exposure to the use of mathematical models in engineering practice, each chapter contains sections on engineering applications. These sections form an ideal framework for individual, or group, case study assignments leading to a written report and/or oral presentation, thereby helping to develop the skills of mathematical modelling necessary to prepare for the more open-ended modelling exercises at a later stage of the course.
- Exercises
  There are numerous exercise sections throughout the text, and at the end of each chapter there is a comprehensive set of review exercises. While many of the exercise problems are designed to develop skills in mathematical techniques,

others are designed to develop understanding and to encourage learning by doing, and some are of an open-ended nature. This book contains over 1200 exercises and answers to all the questions are given. It is hoped that this provision, together with the large number of worked examples and style of presentation, also make the book suitable for private or directed study. Again in response to feedback from users, the frequency of exercise sections has been increased and additional questions have been added to many of the sections.

- Numerical methods
  Recognizing the increasing use of numerical methods in engineering practice, which often complement the use of analytical methods in analysis and design and are of ultimate relevance when solving complex engineering problems, there is wide agreement that they should be integrated within the mathematics curriculum. Consequently the treatment of numerical methods is integrated within the analytical work throughout the book.

The position of software use is an important aspect of engineering education. The decision has been taken to use mainly MATLAB but also, in later chapters, MAPLE. Students are encouraged to make intelligent use of software, and where appropriate codes are included, but there is a health warning. The pace of technology shows little signs of lessening, and so in the space of six years, the likely time lapse before a new edition of this text, it is probable that software will continue to be updated, probably annually. There is therefore a real risk that much coding, though correct and working at the time of publication, could be broken by these updates. Therefore, in this edition the decision has been made not to overemphasize specific code but to direct students to the Companion Website or to general principles instead. The software packages, particularly MAPLE, have become easier to use without the need for programming skills. Much is menu driven these days. Here is more from Glyn on the subject that is still true:

> Students are strongly encouraged to use one of these packages to check the answers to the examples and exercises. It is stressed that the MATLAB (and a few MAPLE) inserts are not intended to be a first introduction of the package to students; it is anticipated that they will receive an introductory course elsewhere and will be made aware of the excellent ‘help’ facility available. The purpose of incorporating the inserts is not only to improve efficiency in the use of the package but also to provide a facility to help develop a better understanding of the related mathematics. Whilst use of such packages takes the tedium out of arithmetic and algebraic manipulations it is important that they are used to enhance understanding and not to avoid it. It is recognized that not all users of the text will have access to either MATLAB or MAPLE, and consequently all the inserts are highlighted and can be ‘omitted’ without loss of continuity in developing the subject content.

Throughout the text two icons are used:

- An open screen indicates that use of a software package would be useful (for example, for checking solutions) but not essential.
- A closed screen indicates that the use of a software package is essential or highly desirable.

Specific changes in this sixth edition are an improvement in many of the diagrams, taking advantage of present-day software, and modernization of the examples and language. Also, Chapter 13 Data Handling and Probability Theory has been significantly modernized by interfacing the presentation with the very powerful software package R. It is free; simply search for 'R Software' and download it. I have been much aided in getting this edition ready for publication by my hardworking colleagues Matthew, John and Yinghui who now comprise the team.

Feedback from users of the previous edition on the subject content has been favourable, and consequently no new chapters have been introduced. However, in response to the feedback, chapters have been reviewed and amended/updated accordingly. Whilst subject content at this level has not changed much over the years the mode of delivery is being driven by developments in computer technology. Consequently there has been a shift towards online teaching and learning, coupled with student self-study programmes. In support of such programmes, worked examples and exercise sections are seen by many as the backbone of the text. Consequently in this new edition emphasis is given to strengthening the 'Worked Examples' throughout the text and increasing the frequency and number of questions in the 'Exercise Sections'. This has involved the restructuring, sometimes significantly, of material within individual chapters.

A comprehensive Solutions Manual is obtainable free of charge to lecturers using this textbook. It will be available for download online at go.pearson.com/uk/he/resources.

Also available online is a set of 'Refresher Units' covering topics students should have encountered at school but may not have used for some time.

This text is also paired with a MyLab™ - a teaching and learning platform that empowers you to reach every student. By combining trusted author content with digital tools and a flexible platform, MyLab personalizes the learning experience and improves results for each student. MyLab Math for this textbook has over 1150 questions to assign to your students, including exercises requiring different types of mathematics applications for a variety of industry types. Note that students require a course ID and an access card in order to use MyLab Math (see inside front cover for more information or contact your Pearson account manager at the link go.pearson.com/findarep).

Acknowledgements

The authoring team is extremely grateful to all the reviewers and users of the text who have provided valuable comments on previous editions of this book. Most of this has been highly constructive and very much appreciated. The team has continued to enjoy the full support of a very enthusiastic production team at Pearson Education and wishes to thank all those concerned.

Phil Dyke
Plymouth
and
Glyn James
Coventry
July 2019

About the authors

New authors Matthew Craven and Yinghui Wei join one of the original authors John Searl under the new editor, also one of the original authors, Phil Dyke, to produce this the sixth edition of Modern Engineering Mathematics.

Phil Dyke is Professor of Applied Mathematics at the University of Plymouth. He was a Head of School for twenty-two years, eighteen of these as Head of Mathematics and Statistics. He has over forty-five years' teaching and research experience in Higher Education, much of this teaching engineering students not only mathematics but also marine and coastal engineering. Apart from his contributions to both *Modern Engineering Mathematics* and *Advanced Modern Engineering Mathematics* he is the author of eleven other textbooks ranging in topic from advanced calculus, Laplace transforms and Fourier series to mechanics and marine physics. He is now semi-retired, but still teaches, is involved in research, and writes. He is a Fellow of the Institute of Mathematics and its Applications.

Matthew Craven is a Lecturer in Applied Mathematics at the University of Plymouth. For fifteen years, he has taught foundation year, postgraduate and everything in between. He is also part of the author team for the 5th edition of the companion text, *Advanced Modern Engineering Mathematics*. He has research interests in computational simulation, real-world operational research, high performance computing and optimization.

Yinghui Wei is an Associate Professor of Statistics at the University of Plymouth. She has taught probability and statistics modules for mathematics programmes as well as for programmes in other subject areas, including engineering, business and medicine. She has broad research interests in statistical modelling, data analysis and evidence synthesis.

John Searl was Director of the Edinburgh Centre for Mathematical Education at the University of Edinburgh before his retirement. As well as lecturing on mathematical education, he taught service courses for engineers and scientists. His most recent research concerned the development of learning environments that make for the effective learning of mathematics for 16–20 year olds. As an applied mathematician he worked collaboratively with (amongst others) engineers, physicists, biologists and pharmacologists, he is keen to develop problem-solving skills of students and to provide them with opportunities to display their mathematical knowledge within a variety of practical contexts. The contexts develop the extended reasoning needed in all fields of engineering.

The original editor was **Glyn James** who retired as Dean of the School of Mathematical and Information Sciences at Coventry University in 2001 and then became Emeritus Professor in Mathematics at the University. He graduated from the University College of Wales, Cardiff in the late 1950s, obtaining first-class honours degrees in both Mathematics and Chemistry. He obtained a PhD in Engineering Science in 1971 as an external student of the University of Warwick. He was employed at Coventry in 1964 and held the position of the Head of Mathematics Department prior to his appointment as Dean in 1992. His research interests were in control theory and its applications to industrial problems. He also had a keen interest in mathematical education, particularly in relation to the teaching of engineering mathematics and mathematical modelling. He was co-chairman of the European Mathematics Working Group established by the European Society for Engineering Education (SEFI) in 1982, a past chairman of the Education Committee of the Institute of Mathematics and its Applications (IMA), and a member of the Royal Society Mathematics Education Subcommittee. In 1995 he was chairman of the Working Group that produced the report *Mathematics Matters in Engineering* on behalf of the professional bodies in engineering and mathematics within the UK. He was also a member of the editorial/advisory board of three international journals. He published numerous papers and was co-editor of five books on various aspects of mathematical modelling. He was a past Vice-President of the IMA and also served a period as Honorary Secretary of the Institute. He was a Chartered Mathematician and a Fellow of the IMA. Sadly, Glyn James passed away in October 2019 during the production of this edition; his enthusiastic input was sorely missed, but this and its companion text remain a fitting legacy.

The original authors are David Burley, Dick Clements, Jerry Wright together with Phil Dyke and John Searl. The short biographies that are not here can be found in the previous editions.

1 Number, Algebra and Geometry

Chapter 1 Contents

1.1 Introduction

Mathematics plays an important role in our lives. It is used in everyday activities from buying food to organizing maintenance schedules for aircraft. Through applications developed in various cultural and historical contexts, mathematics has been one of the decisive factors in shaping the modern world. It continues to grow and to find new uses, particularly in engineering and technology, from electronic circuit design to machine learning.

Mathematics provides a powerful, concise and unambiguous way of organizing and communicating information. It is a means by which aspects of the physical universe can be explained and predicted. It is a problem-solving activity supported by a body of knowledge. Mathematics consists of facts, concepts, skills and thinking processes – aspects that are closely interrelated. It is a hierarchical subject in that new ideas and skills are developed from existing ones. This sometimes makes it a difficult subject for learners who, at every stage of their mathematical development, need to have ready recall of material learned earlier.

In the first two chapters we shall summarize the concepts and techniques that most students will already understand and we shall extend them into further developments in mathematics. There are four key areas of which students will already have considerable knowledge.

- numbers
- algebra
- geometry
- functions

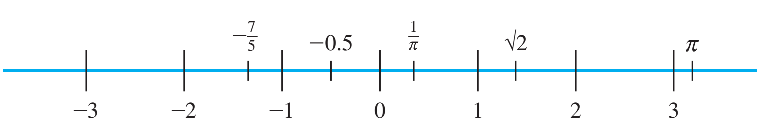
These areas are vital to making progress in engineering mathematics (indeed, they will solve many important problems in engineering). Here we will aim to consolidate that knowledge, to make it more precise and to develop it. In this first chapter we will deal with the first three topics; functions are considered next (see Chapter 2).

1.2 Number and arithmetic

1.2.1 Number line

Mathematics has grown from primitive arithmetic and geometry into a vast body of knowledge. The most ancient mathematical skill is counting, using, in the first instance, the natural numbers and later the integers. The term **natural numbers** commonly refers to the set $\mathbb{N} = \{1, 2, 3, \ldots\}$, and the term **integers** to the set $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$. The integers can be represented as equally spaced points on a line called the **number line** as shown in Figure 1.1. In a computer the integers can be stored exactly. The set of all points (not just those representing integers) on the number line represents the **real numbers** (so named to distinguish them from the complex numbers, which are

Figure 1.1
The number line.



discussed in Chapter 3). The set of real numbers is denoted by $\mathbb{R}$. The general real number is usually denoted by the letter x and we write 'x in $\mathbb{R}$', meaning x is a real number. A real number that can be written as the ratio of two integers, like $\frac{3}{2}$ or $-\frac{7}{5}$, is called a **rational number**. Other numbers, like $\sqrt{2}$ and π, that cannot be expressed in that way are called **irrational numbers**. In a computer the real numbers can be stored only to a limited number of figures. This is a basic difference between the ways in which computers treat integers and real numbers, and is the reason why the computer languages commonly used by engineers distinguish between integer values and variables on the one hand and real number values and variables on the other.

1.2.2 Representation of numbers

For everyday purposes we use a system of representation based on ten **numerals**: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. These ten symbols are sufficient to represent all numbers if a **position notation** is adopted. For whole numbers this means that, starting from the right-hand end of the number, the least significant end, the figures represent the number of units, tens, hundreds, thousands, and so on. Thus one thousand, three hundred and sixty-five is represented by 1365, and two hundred and nine is represented by 209. Notice the role of the 0 in the latter example, acting as a position keeper. The use of a decimal point makes it possible to represent fractions as well as whole numbers. This system uses ten symbols. The number system is said to be 'to base ten' and is called the **decimal** system. Other bases are possible: for example, the Babylonians used a number system to base sixty, a fact that still influences our measurement of time. In some societies a number system evolved with more than one base, a survival of which can be seen in imperial measures (inches, feet, yards, ...). For some applications it is more convenient to use a base other than ten. Early electronic computers used **binary** numbers (to base two); modern computers use **hexadecimal** numbers (to base sixteen). For elementary (pen-and-paper) arithmetic a representation to base twelve would be more convenient than the usual decimal notation because twelve has more integer divisors (2, 3, 4, 6) than ten (2, 5).

In a decimal number the positions to the left of the decimal point represent units (10^0), tens (10^1), hundreds (10^2) and so on, while those to the right of the decimal point represent tenths (10^{-1}), hundredths (10^{-2}) and so on. Thus, for example,

$$\begin{array}{cccccc} 2 & 1 & 4 & \cdot & 3 & 6 \\ \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow \\ 10^2 & 10^1 & 10^0 & & 10^{-1} & 10^{-2} \end{array}$$

so

$$\begin{aligned} 214.36 &= 2(10^2) + 1(10^1) + 4(10^0) + 3(\tfrac{1}{10}) + 6(\tfrac{1}{100}) \\ &= 200 + 10 + 4 + \tfrac{3}{10} + \tfrac{6}{100} \\ &= \tfrac{21436}{100} = \tfrac{5359}{25} \end{aligned}$$

In other number bases the pattern is the same: in base b the position values are b^0, b^1, b^2, ... and b^{-1}, b^{-2}, ... Thus in binary (base two) the position values are units, twos, fours, eights, sixteens and so on, and halves, quarters, eighths and so on. In hexadecimal (base sixteen) the position values are units, sixteens, two hundred and fifty-sixes and so on, and sixteenths, two hundred and fifty-sixths and so on.

Example 1.1

Write (a) the binary number 1011101_2 as a decimal number and (b) the decimal number 115_{10} as a binary number.

Solution

(a) $1011101_2 = 1(2^6) + 0(2^5) + 1(2^4) + 1(2^3) + 1(2^2) + 0(2^1) + 1(2^0)$

$$= 64_{10} + 0 + 16_{10} + 8_{10} + 4_{10} + 0 + 1_{10}$$

$$= 93_{10}$$

(b) We achieve the conversion to binary by repeated division by 2. Thus

$115 \div 2 = 57$	remainder 1	(2^0)
$57 \div 2 = 28$	remainder 1	(2^1)
$28 \div 2 = 14$	remainder 0	(2^2)
$14 \div 2 = 7$	remainder 0	(2^3)
$7 \div 2 = 3$	remainder 1	(2^4)
$3 \div 2 = 1$	remainder 1	(2^5)
$1 \div 2 = 0$	remainder 1	(2^6)

so that

$$115_{10} = 1110011_2$$

Example 1.2

Represent the numbers (a) two hundred and one, (b) two hundred and seventy-five, (c) five and three-quarters and (d) one-third in

(i) decimal form using the figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9;

(ii) binary form using the figures 0, 1;

(iii) duodecimal (base twelve) form using the figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, Δ, ε.

Solution

(a) two hundred and one

(i) = 2 (hundreds) + 0 (tens) and 1 (units) = 201_{10}

(ii) = 1 (one hundred and twenty-eight) + 1 (sixty-four) + 1 (eight) + 1 (unit) = 11001001_2

(iii) = 1 (gross) + 4 (dozens) + 9 (units) = 149_{12}

Here the subscripts 10, 2, 12 indicate the number base.

(b) two hundred and seventy-five

(i) = 2 (hundreds) + 7 (tens) + 5 (units) = 275_{10}

(ii) = 1 (two hundred and fifty-six) + 1 (sixteen) + 1 (two) + 1 (unit) = 100010011_2

(iii) $= 1$ (gross) $+ 10$ (dozens) $+$ eleven (units) $= 1\Delta\varepsilon_{12}$
(Δ represents ten and ε represents eleven)

(c) five and three-quarters

(i) $= 5$ (units) $+ 7$ (tenths) $+ 5$ (hundredths) $= 5.75_{10}$

(ii) $= 1$ (four) $+ 1$ (unit) $+ 1$ (half) $+ 1$ (quarter) $= 101.11_2$

(iii) $= 5$ (units) $+ 9$ (twelfths) $= 5.9_{12}$

(d) one-third

(i) $= 3$ (tenths) $+ 3$ (hundredths) $+ 3$ (thousandths) $+ \ldots = 0.333\ldots_{10}$

(ii) $= 1$ (quarter) $+ 1$ (sixteenth) $+ 1$ (sixty-fourth) $+ \ldots = 0.010101\ldots_2$

(iii) $= 4$ (twelfths) $= 0.4_{12}$

1.2.3 Rules of arithmetic

The basic arithmetical operations of addition, subtraction, multiplication and division are performed subject to the **Fundamental Rules of Arithmetic**. For any three numbers a, b and c:

(a1) the commutative law of addition

$$a + b = b + a$$

(a2) the commutative law of multiplication

$$a \times b = b \times a$$

(b1) the associative law of addition

$$(a + b) + c = a + (b + c)$$

(b2) the associative law of multiplication

$$(a \times b) \times c = a \times (b \times c)$$

(c1) the distributive law of multiplication over addition and subtraction

$$(a + b) \times c = (a \times c) + (b \times c)$$

$$(a - b) \times c = (a \times c) - (b \times c)$$

(c2) the distributive law of division over addition and subtraction

$$(a + b) \div c = (a \div c) + (b \div c)$$

$$(a - b) \div c = (a \div c) - (b \div c)$$

Here the brackets indicate which operation is performed first. These operations are called **binary** operations because they associate with every two members of the set of real numbers a unique third member; for example,

$$2 + 5 = 7 \quad \text{and} \quad 3 \times 6 = 18$$

Example 1.3 Find the value of $(100 + 20 + 3) \times 456$.

Solution Using the distributive law we have

$$\begin{aligned}(100 + 20 + 3) \times 456 &= 100 \times 456 + 20 \times 456 + 3 \times 456\\ &= 45\,600 + 9120 + 1368 = 56\,088\end{aligned}$$

Here 100×456 has been evaluated as

$$100 \times 400 + 100 \times 50 + 100 \times 6$$

and similarly 20×456 and 3×456.

This, of course, is normally set out in the traditional school arithmetic way:

$$\begin{array}{r} 456 \\ \underline{123\ \times} \\ 1\,368 \\ 9\,120 \\ \underline{45\,600} \\ \underline{\underline{56\,088}} \end{array}$$

Example 1.4 Rewrite $(a + b) \times (c + d)$ as the sum of products.

Solution Using the distributive law we have

$$\begin{aligned}(a + b) \times (c + d) &= a \times (c + d) + b \times (c + d)\\ &= (c + d) \times a + (c + d) \times b\\ &= c \times a + d \times a + c \times b + d \times b\\ &= a \times c + a \times d + b \times c + b \times d\end{aligned}$$

applying the commutative laws several times.

A further operation used with real numbers is that of **powering**. For example, $a \times a$ is written as a^2, and $a \times a \times a$ is written as a^3. In general the product of n a's where n is a positive integer is written as a^n. (Here the n is called the **index** or **exponent**.) Operations with powering also obey simple rules:

$$a^n \times a^m = a^{n+m} \tag{1.1a}$$

$$a^n \div a^m = a^{n-m} \tag{1.1b}$$

$$(a^n)^m = a^{nm} \tag{1.1c}$$

From rule (1.1b) it follows, by setting $n = m$ and $a \neq 0$, that $a^0 = 1$. It is also convention to take $0^0 = 1$. The process of powering can be extended to include the fractional powers like $a^{1/2}$. Using rule (1.1c),

$$(a^{1/n})^n = a^{n/n} = a^1$$

and we see that

$$a^{1/n} = {}^n\sqrt{a}$$

the nth root of a. Also, we can define a^{-m} using rule (1.1b) with $n = 0$, giving

$$1 \div a^m = a^{-m}, \qquad a \neq 0$$

Thus a^{-m} is the reciprocal of a^m. In contrast with the binary operations $+$, $\times$, $-$ and $\div$, which operate on two numbers, the powering operation $(\ )^r$ operates on just one element and is consequently called a **unary** operation. Notice that the fractional power

$$a^{m/n} = ({}^n\sqrt{a})^m = {}^n\sqrt{(a^m)}$$

is the nth root of a^m. If n is an even integer, then $a^{m/n}$ is not defined when a is negative. When ${}^n\sqrt{a}$ is an irrational number then such a root is called a **surd**.

Numbers like $\sqrt{2}$ were described by the Greeks as **a-logos**, without a ratio number. An Arabic translator took the alternative meaning 'without a word' and used the Arabic word for 'deaf', which subsequently became **surdus**, Latin for deaf, when translated from Arabic to Latin in the mid-twelfth century.

Example 1.5

Find the values of

(a) $27^{1/3}$ (b) $(-8)^{2/3}$ (c) $16^{-3/2}$

(d) $(-2)^{-2}$ (e) $(-1/8)^{-2/3}$ (f) $(9)^{-1/2}$

Solution

(a) $27^{1/3} = {}^3\sqrt{27} = 3$

(b) $(-8)^{2/3} = ({}^3\sqrt{(-8)})^2 = (-2)^2 = 4$

(c) $16^{-3/2} = (16^{1/2})^{-3} = (4)^{-3} = \frac{1}{4^3} = \frac{1}{64}$

(d) $(-2)^{-2} = \dfrac{1}{(-2)^2} = \frac{1}{4}$

(e) $(-1/8)^{-2/3} = [{}^3\sqrt{(-1/8)}]^{-2} = [{}^3\sqrt{(-1)}/{}^3\sqrt{(8)}]^{-2} = [-1/2]^{-2} = 4$

(f) $(9)^{-1/2} = (3)^{-1} = \frac{1}{3}$

Example 1.6

Express (a) in terms of $\sqrt{2}$ and simplify (b) to (f).

(a) $\sqrt{18} + \sqrt{32} - \sqrt{50}$ (b) $6/\sqrt{2}$ (c) $(1 - \sqrt{3})(1 + \sqrt{3})$

(d) $\dfrac{2}{1 - \sqrt{3}}$ (e) $(1 + \sqrt{6})(1 - \sqrt{6})$ (f) $\dfrac{1 - \sqrt{2}}{1 + \sqrt{6}}$

Solution (a) $\sqrt{18} = \sqrt{(2 \times 9)} = \sqrt{2} \times \sqrt{9} = 3\sqrt{2}$

$\sqrt{32} = \sqrt{(2 \times 16)} = \sqrt{2} \times \sqrt{16} = 4\sqrt{2}$

$\sqrt{50} = \sqrt{(2 \times 25)} = \sqrt{2} \times \sqrt{25} = 5\sqrt{2}$

Thus $\sqrt{18} + \sqrt{32} - \sqrt{50} = 2\sqrt{2}$.

(b) $6/\sqrt{2} = 3 \times 2/\sqrt{2}$

Since $2 = \sqrt{2} \times \sqrt{2}$, we have $6/\sqrt{2} = 3\sqrt{2}$.

(c) $(1 - \sqrt{3})(1 + \sqrt{3}) = 1 + \sqrt{3} - \sqrt{3} - 3 = -2$

(d) Using the result of part (c), $\dfrac{2}{1-\sqrt{3}}$ can be simplified by multiplying 'top and bottom' by $1 + \sqrt{3}$ (notice the sign change in front of the $\sqrt{\ }$). Thus

$$\frac{2}{1-\sqrt{3}} = \frac{2(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})}$$

$$= \frac{2(1+\sqrt{3})}{1-3}$$

$$= -1 - \sqrt{3}$$

(e) $(1 + \sqrt{6})(1 - \sqrt{6}) = 1 - \sqrt{6} + \sqrt{6} - 6 = -5$

(f) Using the same technique as in part (d) we have

$$\frac{1-\sqrt{2}}{1+\sqrt{6}} = \frac{(1-\sqrt{2})(1-\sqrt{6})}{(1+\sqrt{6})(1-\sqrt{6})}$$

$$= \frac{1-\sqrt{2}-\sqrt{6}+\sqrt{12}}{1-6}$$

$$= -(1 - \sqrt{2} - \sqrt{6} + 2\sqrt{3})/5$$

This process of expressing the irrational number so that all of the surds are in the numerator is called **rationalization**.

When evaluating arithmetical expressions the following rules of precedence are observed:

- the powering operation $(\ )^r$ is performed first
- then multiplication $\times$ and/or division $\div$
- then addition $+$ and/or subtraction $-$

When two operators of equal precedence are adjacent in an expression the left-hand operation is performed first. For example,

$$12 - 4 + 13 = 8 + 13 = 21$$

and

$$15 \div 3 \times 2 = 5 \times 2 = 10$$

The precedence rules are overridden by brackets; thus

$$12 - (4 + 13) = 12 - 17 = -5$$

and

$$15 \div (3 \times 2) = 15 \div 6 = 2.5$$

This order of precedence is commonly referred to as BODMAS/BIDMAS (meaning: brackets, order/index, multiplication, addition, subtraction).

Example 1.7 Evaluate $7 - 5 \times 3 \div 2^2$.

Solution Following the rules of precedence, we have

$$7 - 5 \times 3 \div 2^2 = 7 - 5 \times 3 \div 4 = 7 - 15 \div 4 = 7 - 3.75 = 3.25$$

1.2.4 Exercises

1. Find the decimal equivalent of 110110.101_2.

2. Find the binary and octal (base eight) equivalents of the decimal number 16 321. Obtain a simple rule that relates these two representations of the number, and hence write down the octal equivalent of 1011100101101_2.

3. Find the binary and octal equivalents of the decimal number 30.6. Does the rule obtained in Question 2 still apply?

4. Use binary arithmetic to evaluate

   (a) $100011.011_2 + 1011.001_2$

   (b) $111.10011_2 \times 10.111_2$

5. Simplify the following expressions, giving the answers with positive indices and without brackets:

   (a) $2^3 \times 2^{-4}$ (b) $2^3 \div 2^{-4}$ (c) $(2^3)^{-4}$

   (d) $3^{1/3} \times 3^{5/3}$ (e) $(36)^{-1/2}$ (f) $16^{3/4}$

6. The expression $7 - 2 \times 3^2 + 8$ may be evaluated using the usual implicit rules of precedence. It could be rewritten as $((7 - (2 \times (3^2))) + 8)$ using brackets to make the precedence explicit. Similarly rewrite the following expressions in fully bracketed form:

   (a) $21 + 4 \times 3 \div 2$

   (b) $17 - 6^{2+3}$

   (c) $4 \times 2^3 - 7 \div 6 \times 2$

   (d) $2 \times 3 - 6 \div 4 + 3^{2-5}$

7. Express the following in the form $x + y\sqrt{2}$ with x and y rational numbers:

   (a) $(7 + 5\sqrt{2})^3$ (b) $(2 + \sqrt{2})^4$

   (c) $\sqrt[3]{(7 + 5\sqrt{2})}$ (d) $\sqrt{(\tfrac{11}{2} - 3\sqrt{2})}$

8. Show that

   $$\frac{1}{a + b\sqrt{c}} = \frac{a - b\sqrt{c}}{a^2 - b^2c}$$

   Hence express the following numbers in the form $x + y\sqrt{n}$ where x and y are rational numbers and n is an integer:

   (a) $\dfrac{1}{7 + 5\sqrt{2}}$ (b) $\dfrac{2 + 3\sqrt{2}}{9 - 7\sqrt{2}}$

   (c) $\dfrac{4 - 2\sqrt{3}}{7 - 3\sqrt{3}}$ (d) $\dfrac{2 + 4\sqrt{5}}{4 - \sqrt{5}}$

9. Find the difference between 2 and the squares of

   $$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}$$

   (a) Verify that successive terms of the sequence stand in relation to each other as m/n does to $(m + 2n)/(m + n)$.

   (b) Verify that if m/n is a good approximation to $\sqrt{2}$ then $(m + 2n)/(m + n)$ is a better one, and that the errors in the two cases are in opposite directions.

   (c) Find the next three terms of the above sequence.

1.2.5 Inequalities

The number line (Figure 1.1) makes explicit a further property of the real numbers – that of **ordering**. This enables us to make statements like 'seven is greater than two' and 'five is less than six'. We represent this using the comparison symbols

$>$, 'greater than'
$<$, 'less than'

It also makes obvious two other comparators:

$=$, 'equals'
$\neq$, 'does not equal'

These comparators obey simple rules when used in conjunction with the arithmetical operations. For any four numbers a, b, c and d:

$$(a < b \text{ and } c < d) \quad \text{implies} \quad a + c < b + d \tag{1.2a}$$

$$(a < b \text{ and } c > d) \quad \text{implies} \quad a - c < b - d \tag{1.2b}$$

$$(a < b \text{ and } b < c) \quad \text{implies} \quad a < c \tag{1.2c}$$

$$a < b \quad \text{implies} \quad a + c < b + c \tag{1.2d}$$

$$(a < b \text{ and } c > 0) \quad \text{implies} \quad ac < bc \tag{1.2e}$$

$$(a < b \text{ and } c < 0) \quad \text{implies} \quad ac > bc \tag{1.2f}$$

$$(a < b \text{ and } ab > 0) \quad \text{implies} \quad \frac{1}{a} > \frac{1}{b} \tag{1.2g}$$

Example 1.8 Show, without using a calculator, that $\sqrt{2} + \sqrt{3} > 2(\sqrt[4]{6})$.

Solution By squaring we have that

$$(\sqrt{2} + \sqrt{3})^2 = 2 + 2\sqrt{2}\sqrt{3} + 3 = 5 + 2\sqrt{6}$$

Also

$$(2\sqrt{6})^2 = 24 < 25 = 5^2$$

implying that $5 > 2\sqrt{6}$. Thus

$$(\sqrt{2} + \sqrt{3})^2 > 2\sqrt{6} + 2\sqrt{6} = 4\sqrt{6}$$

and, since $\sqrt{2} + \sqrt{3}$ is a positive number, it follows that

$$\sqrt{2} + \sqrt{3} > \sqrt{(4\sqrt{6})} = 2(\sqrt[4]{6})$$

1.2.6 Modulus and intervals

The size of a real number x is called its modulus (or absolute value) and is denoted by $|x|$ (or sometimes by mod(x)). Thus

$$|x| = \begin{cases} x & (x \geqslant 0) \\ -x & (x < 0) \end{cases} \tag{1.3}$$

where the comparator $\geqslant$ indicates 'greater than or equal to'. (Likewise $\leqslant$ indicates 'less than or equal to'.)

Geometrically $|x|$ is the distance of the point representing x on the number line from the point representing zero. Similarly $|x - a|$ is the distance of the point representing x on the number line from that representing a.

The set of numbers between two distinct numbers, a and b say, defines an **open interval** on the real line. This is the set $\{x{:}a < x < b, x \text{ in } \mathbb{R}\}$ and is usually denoted by (a, b). (Set notation will be fully described later (see Chapter 6); here $\{x{:}P\}$ denotes the set of all x that have property P.) Here the double-sided inequality means that x is greater than a and less than b; that is, the inequalities $a < x$ and $x < b$ apply simultaneously. An interval that includes the end points is called a **closed interval**, denoted by $[a, b]$, with

$$[a, b] = \{x{:}a \leqslant x \leqslant b, x \text{ in } \mathbb{R}\}$$

Note that the distance between two numbers a and b might be either $a - b$ or $b - a$ depending on which was the larger. An immediate consequence of this is that

$$|a - b| = |b - a|$$

since a is the same distance from b as b is from a.

Example 1.9

Find the values of x so that

$$|x - 4.3| = 5.8$$

Solution

$|x - 4.3| = 5.8$ means that the distance between the real numbers x and 4.3 is 5.8 units, but does not tell us whether $x > 4.3$ or whether $x < 4.3$. The situation is illustrated in Figure 1.2, from which it is clear that the two possible values of x are -1.5 and 10.1.

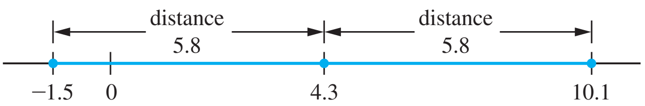


Figure 1.2 Illustration of $|x - 4.3| = 5.8$.

Example 1.10

Express the sets (a) $\{x{:}|x - 3| < 5, x \text{ in } \mathbb{R}\}$ and (b) $\{x{:}|x + 2| \leqslant 3, x \text{ in } \mathbb{R}\}$ as intervals.

Solution

(a) $|x - 3| < 5$ means that the distance of the point representing x on the number line from the point representing 3 is less than 5 units, as shown in Figure 1.3(a). This implies that

$$-5 < x - 3 < 5$$

Adding 3 to each member of this inequality, using rule (1.2d), gives

$$-2 < x < 8$$

and the set of numbers satisfying this inequality is the open interval $(-2, 8)$.

(b) Similarly $|x + 2| \leqslant 3$, which may be rewritten as $|x - (-2)| \leqslant 3$, means that the distance of the point x on the number line from the point representing -2 is less than or equal to 3 units, as shown in Figure 1.3(b). This implies

$$-3 \leqslant x + 2 \leqslant 3$$

Subtracting 2 from each member of this inequality, using rule (1.2d), gives

$$-5 \leqslant x \leqslant 1$$

and the set of numbers satisfying this inequality is the closed interval $[-5, 1]$.

It is easy (and sensible) to check these answers using spot values. For example, putting $x = -4$ in (b) gives $|-4 + 2| < 3$ correctly. Sometimes the sets $|x + 2| \leqslant 3$ and $|x + 2| < 3$ are described verbally as 'lies in the interval x equals -2 ± 3'.

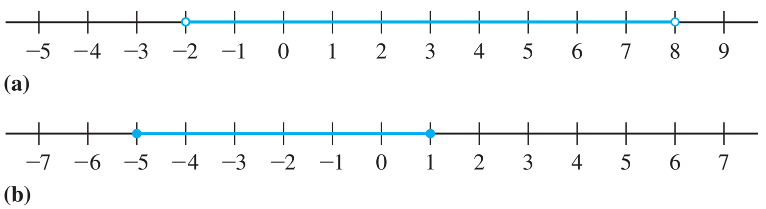


Figure 1.3
(a) The open interval $(-2, 8)$. (b) The closed interval $[-5, 1]$.

We note in passing the following results. For any two real numbers x and y:

$$|xy| = |x|\,|y| \tag{1.4a}$$

$$|x| < a \text{ for } a > 0, \quad \text{implies} \quad -a < x < a \tag{1.4b}$$

$$|x + y| \leqslant |x| + |y|, \quad \text{known as the 'triangle inequality'} \tag{1.4c}$$

$$\tfrac{1}{2}(x + y) \geqslant \sqrt{(xy)}, \quad \text{when } x \geqslant 0 \text{ and } y \geqslant 0 \tag{1.4d}$$

Result (1.4d) is proved in Example 1.11 below and may be stated in words as

the arithmetic mean $\frac{1}{2}(x + y)$ of two positive numbers x and y is greater than or equal to the geometric mean $\sqrt{(xy)}$. Equality holds only when $y = x$.

Results (1.4a) to (1.4c) should be verified by the reader, who may find it helpful to try some particular values first, for example setting $x = -2$ and $y = 3$ in (1.4c).

Example 1.11 Prove that for any two positive numbers x and y, the arithmetic–geometric inequality

$$\tfrac{1}{2}(x + y) \geqslant \sqrt{(xy)}$$

holds.

Deduce that $x + \frac{1}{x} \geqslant 2$ for any positive number x.

We have to prove that $\frac{1}{2}(x + y) - \sqrt{(xy)}$ is greater than or equal to zero. Let E denote the expression $(x + y) - 2\sqrt{(xy)}$. Then

$$E \times [(x + y) + 2\sqrt{(xy)}] = (x + y)^2 - 4(xy)$$

(see Example 1.13)

$$\begin{aligned} E &= x^2 + 2xy + y^2 - 4xy \\ &= x^2 - 2xy + y^2 \\ &= (x-y)^2 \end{aligned}$$

which is greater than zero unless $x = y$. Since $(x + y) + 2\sqrt{(xy)}$ is positive, this implies

$E \geqslant 0$ or $\frac{1}{2}(x + y) \geqslant \sqrt{(xy)}$. Setting $y = \frac{1}{x}$, we obtain

$$\frac{1}{2}\left(x + \frac{1}{x}\right) \geqslant \sqrt{\left(x.\frac{1}{x}\right)} = 1$$

or

$$\left(x + \frac{1}{x}\right) \geqslant 2$$

1.2.7 Exercises

10 Show that $(\sqrt{5} + \sqrt{13})^2 > 34$ and determine without using a calculator the larger of $\sqrt{5} + \sqrt{13}$ and $\sqrt{3} + \sqrt{19}$.

11 Show the following sets on number lines and express them as intervals:

(a) $\{x:|x - 4| \leqslant 6\}$ (b) $\{x:|x + 3| < 2\}$

(c) $\{x:|2x - 1| \leqslant 7\}$ (d) $\{x:|\frac{1}{4}x + 3| < 3\}$

12 Show the following intervals on number lines and express them as sets in the form $\{x:|ax + b| < c\}$ or $\{x:|ax + b| \leqslant c\}$:

(a) $(1, 7)$ (b) $[-4, -2]$

(c) $(17, 26)$ (d) $[-\frac{1}{2}, \frac{3}{4}]$

13 Given that $a < b$ and $c < d$, which of the following statements are always true?

(a) $a - c < b - d$ (b) $a - d < b - c$

(c) $ac < bd$ (d) $\dfrac{1}{b} < \dfrac{1}{a}$

In each case either prove that the statement is true or give a numerical example to show it can be false.

If, additionally, a, b, c and d are all greater than zero, how does that modify your answer?

14 The average speed for a journey is the distance covered divided by the time taken.

(a) A journey is completed by travelling for the first half of the *time* at speed v_1 and the second half at speed v_2. Find the average speed v_a for the journey in terms of v_1 and v_2.

(b) A journey is completed by travelling at speed v_1 for half the *distance* and at speed v_2 for the second half. Find the average speed v_b for the journey in terms of v_1 and v_2.

Deduce that a journey completed by travelling at two different speeds for equal distances will take longer than the same journey completed at the same two speeds for equal times.

1.3 Algebra

The origins of algebra are to be found in Arabic mathematics as the name suggests, coming from the word *aljabara* meaning ‘combination’ or ‘re-uniting’. Algorithms are rules for solving problems in mathematics by standard step-by-step methods. Such methods were first described by the ninth-century mathematician Abu Ja’far Mohammed ben Musa from Khwarizm, modern Khiva on the southern border of Uzbekistan. The Arabic al-Khwarizm (‘from Khwarizm’) was Latinized to algorithm in the late Middle Ages. Often the letter x is used to denote an unassigned (or free) variable. It is thought that this is a corruption of the script letter ꝛ abbreviating the Latin word *res*, thing. The use of unassigned variables enables us to form mathematical models of practical situations as illustrated in the following example. First we deal with a specific case and then with the general case using unassigned variables.

The idea, first introduced in the seventeenth century, of using letters to represent unspecified quantities led to the development of algebraic manipulation based on the elementary laws of arithmetic. This development greatly enhanced the problem-solving power of mathematics – so much so that it is difficult now to imagine doing mathematics without this resource.

Example 1.12

A pipe has the form of a hollow cylinder as shown in Figure 1.4. Find its mass when

(a) its length is 1.5 m, its external diameter is 205 mm, its internal diameter is 160 mm and its density is 5500 kg m^{-3};

(b) its length is l m, its external diameter is D mm, its internal diameter is d mm and its density is ρ kg m^{-3}. Notice here that the unassigned variables l, D, d, ρ are pure numbers and do not include units of measurement.

Solution (a) Standardizing the units of length, the internal and external diameters are 0.16 m and 0.205 m respectively. The area of cross-section of the pipe is

$$0.25\pi(0.205^2 - 0.160^2)\ \text{m}^2$$

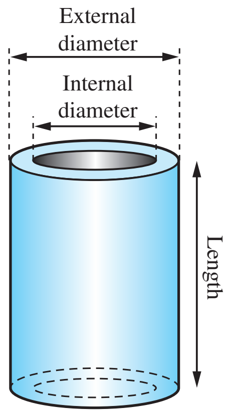


Figure 1.4 Cylindrical pipe of Example 1.12.

(*Reminder*: The area of a circle of diameter D is $\pi D^2/4$.)

Hence the volume of the material of the pipe is

$$0.25\pi(0.205^2 - 0.160^2) \times 1.5\ \text{m}^3$$

and the mass (volume $\times$ density) of the pipe is

$$0.25 \times 5500 \times \pi(0.205^2 - 0.160^2) \times 1.5\ \text{kg}$$

Evaluating this last expression by calculator gives the mass of the pipe as 106 kg to the nearest kilogram.

(b) The internal and external diameters of the pipe are $d/1000$ and $D/1000$ metres, respectively, so that the area of cross-section is

$$0.25\pi(D^2 - d^2)/1\,000\,000\ \text{m}^2$$

The volume of the pipe is

$$0.25\pi l(D^2 - d^2)/10^6\ \text{m}^3$$

Hence the mass M kg of the pipe of density ρ is given by the formulae

$$M = 0.25\pi\rho l(D^2 - d^2)/10^6 = 2.5\pi\rho l(D + d)(D - d) \times 10^{-5}$$

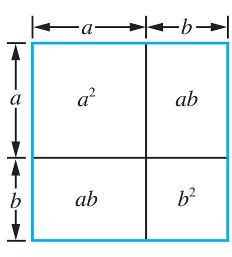
1.3.1 Algebraic manipulation

Algebraic manipulation made possible concise statements of well-known results, such as

$$(a + b)^2 = a^2 + 2ab + b^2 \qquad \textbf{(1.5)}$$

Previously these results had been obtained by a combination of verbal reasoning and elementary geometry as illustrated in Figure 1.5.

Figure 1.5 Illustration of $(a + b)^2 = a^2 + 2ab + b^2$.



Example 1.13 Prove that

$$ab = \tfrac{1}{4}[(a + b)^2 - (a - b)^2]$$

Given $70^2 = 4900$ and $36^2 = 1296$, calculate 53×17.

Solution Since

$$(a + b)^2 = a^2 + 2ab + b^2$$

we deduce

$$(a - b)^2 = a^2 - 2ab + b^2$$

and

$$(a + b)^2 - (a - b)^2 = 4ab$$

and

$$ab = \tfrac{1}{4}[(a + b)^2 - (a - b)^2]$$

The result is illustrated geometrically in Figure 1.6. Setting $a = 53$ and $b = 17$, we have

$$53 \times 17 = \tfrac{1}{4}[70^2 - 36^2] = 901$$

This method of calculating products was used by the Babylonians and is sometimes called 'quarter-square' multiplication. It has been used in some analogue devices and simulators.

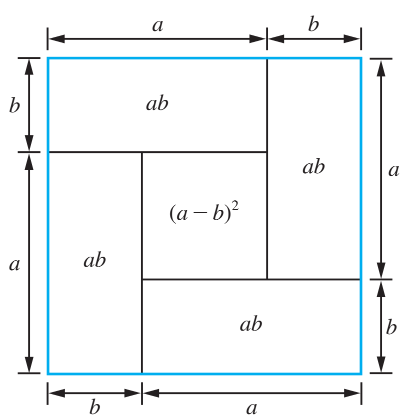


Figure 1.6 Illustration of $ab = \tfrac{1}{4}[(a + b)^2 - (a - b)^2]$.

Example 1.14 Show that

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

Solution Rewriting $a + b + c$ as $(a + b) + c$ we have

$$\begin{aligned}((a + b) + c)^2 &= (a + b)^2 + 2(a + b)c + c^2 \quad \text{using (1.5a)}\\ &= a^2 + 2ab + b^2 + 2ac + 2bc + c^2\\ &= a^2 + b^2 + c^2 + 2ab + 2bc + 2ac\end{aligned}$$

Example 1.15 Verify that

$$(x + p)^2 + q - p^2 = x^2 + 2px + q$$

and deduce that

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

Solution $(x + p)^2 = x^2 + 2px + p^2$

so that

$$(x + p)^2 + q - p^2 = x^2 + 2px + q$$

Working in the reverse direction is more difficult

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

Comparing $x^2 + \frac{b}{a}x + \frac{c}{a}$ with $x^2 + 2px + q$, we can identify

$$\frac{b}{a} = 2p \quad \text{and} \quad \frac{c}{a} = q$$

Thus we can write

$$ax^2 + bx + c = a[(x + p)^2 + q - p^2]$$

where $p = \frac{b}{2a}$ and $q = \frac{c}{a}$

giving

$$\begin{aligned} ax^2 + bx + c &= a\left(x + \frac{b}{2a}\right)^2 + a\left(\frac{c}{a} - \frac{b^2}{4a^2}\right) \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \end{aligned}$$

This algebraic process is called 'completing the square'.

We may summarize the results so far

$$(a + b)^2 = a^2 + 2ab + b^2 \quad \textbf{(1.5a)}$$

$$(a - b)^2 = a^2 - 2ab + b^2 \quad \textbf{(1.5b)}$$

$$a^2 - b^2 = (a + b)(a - b) \quad \textbf{(1.5c)}$$

$$a^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \quad \textbf{(1.5d)}$$

As shown in the previous examples, the ordinary rules of arithmetic carry over to the generalized arithmetic of algebra. This is illustrated again in the following example.

Example 1.16

Express as a single fraction

(a) $\dfrac{1}{12} - \dfrac{2}{3} + \dfrac{3}{4}$

(b) $\dfrac{1}{(x+1)(x+2)} - \dfrac{2}{x+1} + \dfrac{3}{x+2}$

Solution

(a) The lowest common denominator of these fractions is 12, so we may write

$$\frac{1}{12} - \frac{2}{3} + \frac{3}{4} = \frac{1-8+9}{12}$$
$$= \frac{2}{12} = \frac{1}{6}$$

(b) The lowest common multiple of the denominators of these fractions is $(x+1)(x+2)$, so we may write

$$\frac{1}{(x+1)(x+2)} - \frac{2}{x+1} + \frac{3}{x+2}$$
$$= \frac{1}{(x+1)(x+2)} - \frac{2(x+2)}{(x+1)(x+2)} + \frac{3(x+1)}{(x+1)(x+2)}$$
$$= \frac{1-2(x+2)+3(x+1)}{(x+1)(x+2)}$$
$$= \frac{1-2x-4+3x+3}{(x+1)(x+2)}$$
$$= \frac{x}{(x+1)(x+2)}$$

Example 1.17

Use the method of completing the square to manipulate the following quadratic expressions into the form of a number + (or −) the square of a term involving x.

(a) $x^2 + 3x - 7$ (b) $5 - 4x - x^2$

(c) $3x^2 - 5x + 4$ (d) $1 + 2x - 2x^2$

Solution

Remember $(a+b)^2 = a^2 + 2ab + b^2$.

(a) To convert $x^2 + 3x$ into a perfect square we need to add $(\frac{3}{2})^2$. Thus we have

$$x^2 + 3x - 7 = [(x + \tfrac{3}{2})^2 - (\tfrac{3}{2})^2] - 7$$
$$= (x + \tfrac{3}{2})^2 - \tfrac{37}{4}$$

(b) $5 - 4x - x^2 = 5 - (4x + x^2)$

To convert $x^2 + 4x$ into a perfect square we need to add 2^2. Thus we have

$$x^2 + 4x = (x+2)^2 - 2^2$$

and

$$5 - 4x - x^2 = 5 - [(x + 2)^2 - 2^2] = 9 - (x + 2)^2$$

(c) First we 'take outside' the coefficient of x^2:

$$3x^2 - 5x + 4 = 3(x^2 - \tfrac{5}{3}x + \tfrac{4}{3})$$

Then we rearrange

$$x^2 - \tfrac{5}{3}x = (x - \tfrac{5}{6})^2 - \tfrac{25}{36}$$

so that $3x^2 - 5x + 4 = 3[(x - \frac{5}{6})^2 - \frac{25}{36} + \frac{4}{3}] = 3[(x - \frac{5}{6})^2 + \frac{23}{36}]$.

(d) Similarly

$$1 + 2x - 2x^2 = 1 - 2(x^2 - x)$$

and

$$x^2 - x = (x - \tfrac{1}{2})^2 - \tfrac{1}{4}$$

so that

$$1 + 2x - 2x^2 = 1 - 2[(x - \tfrac{1}{2})^2 - \tfrac{1}{4}] = \tfrac{3}{2} - 2(x - \tfrac{1}{2})^2$$

The reader should confirm that these results agree with identity (1.5d).

The number 45 can be factorized as $3 \times 3 \times 5$. Any product of numbers from 3, 3 and 5 is also a factor of 45. Algebraic expressions can be factorized in a similar fashion. An algebraic expression with more than one term can be factorized if each term contains common factors (either numerical or algebraic). These factors are removed by division from each term and the non-common factors remaining are grouped into brackets.

Example 1.18 Factorize $xz + 2yz - 2y - x$.

Solution There is no common factor to all four terms so we take them in pairs:

$$\begin{aligned} xz + 2yz - 2y - x &= (x + 2y)z - (2y + x) \\ &= (x + 2y)z - (x + 2y) \\ &= (x + 2y)(z - 1) \end{aligned}$$

Alternatively, we could have written

$$\begin{aligned} xz + 2yz - 2y - x &= (xz - x) + (2yz - 2y) \\ &= x(z - 1) + 2y(z - 1) \\ &= (x + 2y)(z - 1) \end{aligned}$$

to obtain the same result.

In many problems we are able to facilitate the solution by factorizing a quadratic expression $ax^2 + bx + c$ 'by hand', using knowledge of the factors of the numerical coefficients a, b and c.

Example 1.19 Factorize the expressions

(a) $x^2 + 12x + 35$ (b) $2x^2 + 9x - 5$

Solution (a) Since

$$(x + \alpha)(x + \beta) = x^2 + (\alpha + \beta)x + \alpha\beta$$

we examine the factors of the constant term of the expression

$$35 = 5 \times 7 = 35 \times 1$$

and notice that $5 + 7 = 12$ while $35 + 1 = 36$. So we can choose $\alpha = 5$ and $\beta = 7$ and write

$$x^2 + 12x + 35 = (x + 5)(x + 7)$$

(b) Since

$$(mx + \alpha)(nx + \beta) = mnx^2 + (n\alpha + m\beta)x + \alpha\beta$$

we examine the factors of the coefficient of x^2 and of the constant to give the coefficient of x. Here

$$2 = 2 \times 1 \text{ and } -5 = (-5) \times 1 = 5 \times (-1)$$

and we see that

$$2 \times 5 + 1 \times (-1) = 9$$

Thus we can write

$$(2x - 1)(x + 5) = 2x^2 + 9x - 5$$

It is sensible to do a 'spot-check' on the factorization by inserting a sample value of x, for example $x = 1$

$$(1)(6) = 2 + 9 - 5$$

Comment Some quadratic expressions, for example $x^2 + y^2$, do not have real factors.

The expansion of $(a + b)^2$ in (1.5a) is a special case of a general result for $(a + b)^n$ known as the binomial expansion. This is discussed again later (see Sections 1.3.6 and 7.7.2). Here we shall look at the cases for $n = 0, 1, \dots, 6$.

Writing these out, we have

$$(a + b)^0 = 1$$

$$(a + b)^1 = a + b$$

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

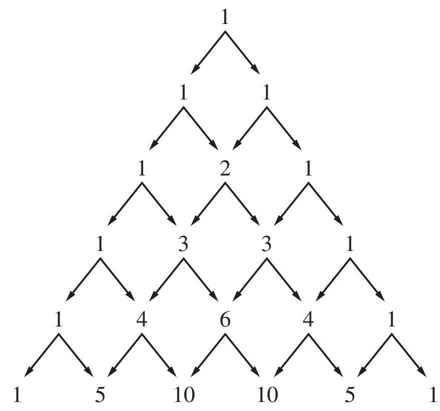
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

$$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

Figure 1.7
Pascal's triangle.

$$(a+b)^0 = 1$$
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2+2ab+b^2$$
$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$
$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$
$$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$$



This table can be extended indefinitely. Each line can easily be obtained from the previous one. Thus, for example,

$$\begin{aligned}(a+b)^4 &= (a+b)(a+b)^3 \\ &= a(a^3+3a^2b+3ab^2+b^3)+b(a^3+3a^2b+3ab^2+b^3) \\ &= a^4+3a^3b+3a^2b^2+ab^3+a^3b+3a^2b^2+3ab^3+b^4 \\ &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\end{aligned}$$

The coefficients involved form a pattern of numbers called Pascal's triangle, shown in Figure 1.7. Each number in the interior of the triangle is obtained by summing the numbers to its right and left in the row above, as indicated by the arrows in Figure 1.7. This number pattern had been discovered prior to Pascal by the Chinese mathematician Jia Xian (in the mid-eleventh century).

Example 1.20

Expand

(a) $(2x+3y)^2$ (b) $(2x-3)^3$ (c) $\left(2x-\dfrac{1}{x}\right)^4$

Solution

(a) Here we use the expansion

$$(a+b)^2 = a^2+2ab+b^2$$

with $a = 2x$ and $b = 3y$ to obtain

$$\begin{aligned}(2x+3y)^2 &= (2x)^2+2(2x)(3y)+(3y)^2 \\ &= 4x^2+12xy+9y^2\end{aligned}$$

(b) Here we use the expansion

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

with $a = 2x$ and $b = -3$ to obtain

$$(2x-3)^3 = 8x^3-36x^2+54x-27$$

(c) Here we use the expansion

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

with $a = 2x$ and $b = -1/x$ to obtain

$$\left(2x - \frac{1}{x}\right)^4 = (2x)^4 + 4(2x)^3(-1/x) + 6(2x)^2(-1/x)^2 + 4(2x)(-1/x)^3 + (-1/x)^4$$
$$= 16x^4 - 32x^2 + 24 - 8/x^2 + 1/x^4$$

1.3.2 Exercises

15 Simplify the following expressions:

(a) $x^3 \times x^{-4}$ (b) $x^3 \div x^{-4}$ (c) $(x^3)^{-4}$

(d) $x^{1/3} \times x^{5/3}$ (e) $(4x^8)^{-1/2}$ (f) $\left(\dfrac{3}{2\sqrt{x}}\right)^{-2}$

(g) $\sqrt{x}\left(x^2 - \dfrac{2}{x}\right)$ (h) $\left(5x^{1/3} - \dfrac{1}{2x^{1/3}}\right)^2$

(i) $\dfrac{2x^{1/2} - x^{-1/2}}{x^{1/2}}$ (j) $\dfrac{(a^2b)^{1/2}}{(ab^{-2})^2}$

(k) $(4ab^2)^{-3/2}$

16 Factorize

(a) $x^2y - xy^2$

(b) $x^2yz - xy^2z + 2xyz^2$

(c) $ax - 2by - 2ay + bx$

(d) $x^2 + 3x - 10$

(e) $x^2 - \frac{1}{4}y^2$ (f) $81x^4 - y^4$

17 Simplify

(a) $\dfrac{x^2 - x - 12}{x^2 - 16}$ (b) $\dfrac{x - 1}{x^2 - 2x - 3} - \dfrac{2}{x + 1}$

(c) $\dfrac{1}{x^2 + 3x - 10} + \dfrac{1}{x^2 + 17x + 60}$

(d) $(3x + 2y)(x - 2y) + 4xy$

18 An isosceles trapezium has non-parallel sides of length 20 cm and the shorter parallel side is 30 cm, as illustrated in Figure 1.8. The perpendicular distance between the parallel sides is h cm. Show that the area of the trapezium is $h(30 + \sqrt{(400 - h^2)})\,\text{cm}^2$.

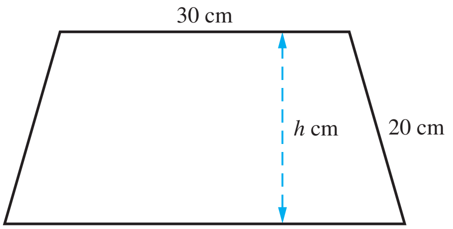


Figure 1.8

19 An open container is made from a sheet of cardboard of size 200 mm × 300 mm using a simple fold, as shown in Figure 1.9. Show that the capacity C ml of the box is given by

$$C = x(150 - x)(100 - x)/250$$

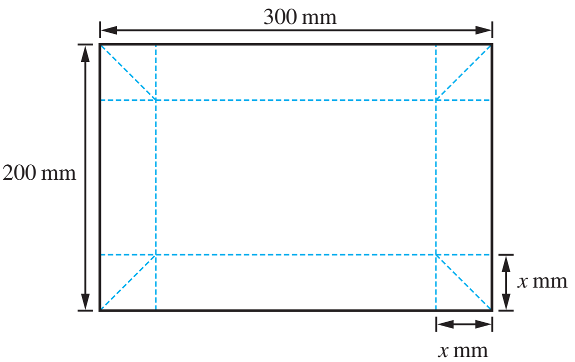


Figure 1.9 Sheet of cardboard of Question 19.

20 Rearrange the following quadratic expressions by completing the square.

(a) $x^2 + x - 12$ (b) $3 - 2x + x^2$

(c) $(x - 1)^2 - (2x - 3)^2$ (d) $1 + 4x - x^2$

1.3.3 Equations, inequalities and identities

It commonly occurs in the application of mathematics to practical problem solving that the numerical value of an expression involving unassigned variables is specified and we have to find the values of the unassigned variables which yield that value. We illustrate the idea with the elementary examples that follow.

Example 1.21

A hollow cone of base diameter 100 mm and height 150 mm is held upside down and completely filled with a liquid. The liquid is then transferred to a hollow circular cylinder of base diameter 80 mm. To what height is the cylinder filled?

Solution

The situation is illustrated in Figure 1.10. The capacity of the cone is

$$\tfrac{1}{3}(\text{base area}) \times (\text{perpendicular height})$$

Thus the volume of liquid contained in the cone is

$$\tfrac{1}{3}\pi(50^2)(150) = 125\,000\pi \text{ mm}^3$$

The volume of the liquid in the circular cylinder is

$$(\text{base area}) \times (\text{height}) = \pi(40^2)h \text{ mm}^3$$

where h mm is the height of the liquid in the cylinder. Equating these quantities (assuming no liquid is lost in the transfer) we have

$$1600\,\pi h = 125\,000\pi$$

This **equation** enables us to find the value of the unassigned variable h:

$$h = 1250/16 = 78.125$$

Thus the height of the liquid in the cylinder is 78 mm to the nearest millimetre.

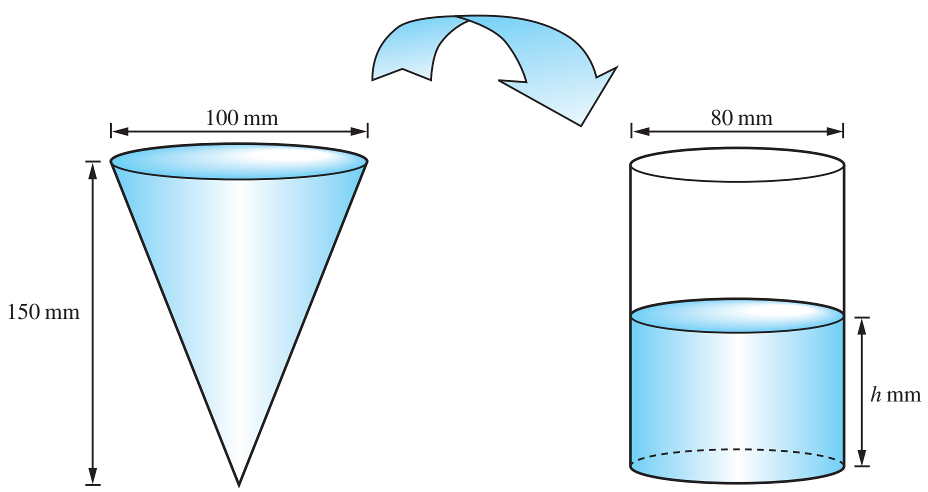


Figure 1.10
The cone and cylinder of Example 1.21.

In the previous example we made use of the formula for the volume V of a cone of base diameter D and height H. We normally write this as

$$V = \tfrac{1}{12}\pi D^2 H$$

understanding that the units of measurement are compatible. This formula also tells us the height of such a cone in terms of its volume and base diameter

$$H = \frac{12V}{\pi D^2}$$

This type of rearrangement is common and is generally described as 'changing the subject of the formula'.

Example 1.22

A dealer bought a number of equally priced articles for a total cost of £120. He sold all but one of them, making a profit of £1.50 on each article with a total revenue of £135. How many articles did he buy?

Solution

Let n be the number of articles bought. Then the cost of each article was £$(120/n)$. Since $(n - 1)$ articles were sold the selling price of each article was £$(135/(n - 1))$. Thus the profit per item was

$$\pounds\left\{\frac{135}{n-1} - \frac{120}{n}\right\}$$

which we are told is equal to £1.50. Thus

$$\frac{135}{n-1} - \frac{120}{n} = 1.50$$

This implies

$$135n - 120(n - 1) = 1.50(n - 1)n$$

Dividing both sides by 1.5 gives

$$90n - 80(n - 1) = n^2 - n$$

Simplifying and collecting terms we obtain

$$n^2 - 11n - 80 = 0$$

This **equation** for n can be simplified further by factorizing the quadratic expression on the left-hand side

$$(n - 16)(n + 5) = 0$$

This implies either $n = 16$ or $n = -5$, so the dealer initially bought 16 articles (the solution $n = -5$ is not feasible).

Example 1.23

Using the method of completing the square (1.5a), obtain the formula for finding the roots of the general quadratic equation

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

Solution Dividing throughout by a gives

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Completing the square leads to

$$\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} = \left(\frac{b}{2a}\right)^2$$

giving

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

which on taking the square root gives

$$x + \frac{b}{2a} = +\frac{\sqrt{(b^2 - 4ac)}}{2a} \quad \text{or} \quad -\frac{\sqrt{(b^2 - 4ac)}}{2a}$$

or

$$\boxed{x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}} \tag{1.6}$$

Here the $\pm$ symbol provides a neat shorthand for the two solutions.

Comments (a) The formula given in (1.6) makes clear the three cases: where for $b^2 > 4ac$ we have two real roots to the equation, for $b^2 < 4ac$ we have no real roots, and for $b^2 = 4ac$ we have one repeated real root.

(b) The condition for equality of the roots of a quadratic equation occurs in practical applications, and we shall illustrate this in Example 2.48 after considering the trigonometric functions.

(c) The quadratic equation has many important applications. One, which is of historical significance, concerned the electrical engineer Oliver Heaviside. In 1871 the telephone cable between England and Denmark developed a fault caused by a short circuit under the sea. His task was to locate that fault. The cable had a uniform resistance per unit length. His method of solution was brilliantly simple. The situation can be represented schematically as shown in Figure 1.11.

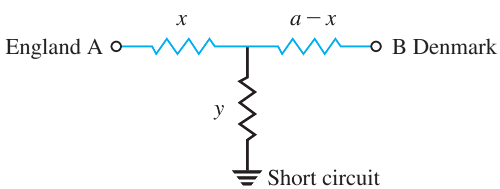


Figure 1.11
The circuit for the telephone line fault.

In the figure the total resistance of the line between A and B is a ohms and is known; x and y are unknown. If we can find x, we can locate the distance along the cable where the fault has occurred. Heaviside solved the problem by applying two tests. First he applied a battery, having voltage E, at A with the circuit open at B, and measured the resulting current I_1. Then he applied the same battery at A but with the cable earthed at B, and again measured the resulting current I_2. Using Ohm's law and the rules for combining resistances in parallel and in series, this yields the pair of equations

$$E = I_1(x + y)$$

$$E = I_2\left[x + \left(\frac{1}{y} + \frac{1}{a - x}\right)^{-1}\right]$$

Writing $b = E/I_1$ and $c = E/I_2$, we can eliminate y from these equations to obtain an equation for x:

$$x^2 - 2cx + c(a + b) - ab = 0$$

which, using (1.6), has solutions

$$x = c \pm \sqrt{[(a - c)(b - c)]}$$

From his experimental data Heaviside was able to predict accurately the location of the fault.

In some problems we have to find the values of unassigned variables such that the value of an expression involving those variables satisfies an inequality condition (that is, it is either greater than, or alternatively less than, a specified value). Solving such inequalities requires careful observance of the rules for inequalities (1.2a–1.2g) set out previously (see Section 1.2.5).

Example 1.24 Find the values of x for which

$$\frac{1}{3 - x} < 2 \tag{1.7}$$

Solution (a) When $3 - x > 0$, that is $x < 3$, we may, using (1.2e), multiply (1.7) throughout by $3 - x$ to give

$$1 < 2(3 - x)$$

which, using (1.2d, e), reduces to

$$x < \tfrac{5}{2}$$

so that (1.7) is satisfied when both $x < 3$ and $x < \frac{5}{2}$ are satisfied; that is, $x < \frac{5}{2}$.

(b) When $3 - x < 0$, that is $x > 3$, we may, using (1.2f), multiply (1.7) throughout by $3 - x$ to give

$$1 > 2(3 - x)$$

which reduces to $x > \frac{5}{2}$ so that (1.7) is also satisfied when both $x > 3$ and $x > \frac{5}{2}$; that is, $x > 3$.

Thus inequality (1.7) is satisfied by values of x in the ranges $x > 3$ and $x < \frac{5}{2}$.

Comment A common mistake made is simply to multiply (1.7) throughout by $3 - x$ to give the answer $x < \frac{5}{2}$, forgetting to consider both cases of $3 - x > 0$ and $3 - x < 0$. We shall return to consider this example from the graphical point of view in Example 2.36.

Example 1.25

Find the values of x such that

$$x^2 + 2x + 2 > 50$$

Solution

Completing the square on the left-hand side of the inequality we obtain

$$(x + 1)^2 + 1 > 50$$

which gives

$$(x + 1)^2 > 49$$

Taking the square root of both sides of this inequality we deduce that

$$\text{either } (x + 1) < -7 \text{ or } (x + 1) > 7$$

Note particularly the first of these inequalities. From these we deduce that

$$x^2 + 2x + 2 > 50 \text{ for } x < -8 \text{ or } x > 6$$

The reader should check these results using spot values of x, say $x = -10$ and $x = 10$.

Example 1.26

A food manufacturer found that the sales figure for a certain item depended on its selling price. The company's market research department advised that the maximum number of items that could be sold weekly was 20 000 and that the number sold decreased by 100 for every 1p increase in its price. The total production cost consisted of a set-up cost of £200 **plus** 50p for every item manufactured. What price should the manufacturer adopt?

Solution

The data supplied by the market research department suggests that if the price of the item is p pence, then the number sold would be $20\,000 - 100p$. (So the company would sell none with $p = 200$, when the price is £2.) The production cost in pounds would be $200 + 0.5 \times$ (number sold), so that in terms of p we have the production cost £C given by

$$C = 200 + 0.5(20\,000 - 100p)$$

The revenue £R accrued by the manufacturer for the sales is (number sold) $\times$ (price), which gives

$$R = (20\,000 - 100p)p/100$$

(remember to express the amount in pounds). Thus, the profit £P is given by

$$\begin{aligned} P &= R - C \\ &= (20\,000 - 100p)p/100 - 200 - 0.5(20\,000 - 100p) \\ &= -p^2 + 250p - 10\,200 \end{aligned}$$

Completing the square we have

$$\begin{aligned} P &= 125^2 - (p - 125)^2 - 10\,200 \\ &= 5425 - (p - 125)^2 \end{aligned}$$

Since $(p - 125)^2 \geqslant 0$, we deduce that $P \leqslant 5425$ and that the maximum value of P is 5425. To achieve this weekly profit, the manufacturer should adopt the price £1.25.

It is important to distinguish between those equalities that are valid for a restricted set of values of the unassigned variable x and those that are true for all values of x. For example,

$$(x - 5)(x + 7) = 0$$

is true only if $x = 5$ or $x = -7$. In contrast

$$(x - 5)(x + 7) = x^2 + 2x - 35 \tag{1.8}$$

is true for all values of x. The word ‘equals’ here is being used in subtly different ways. In the first case ‘=’ means ‘is numerically equal to’; in the second case ‘=’ means ‘is algebraically equal to’. Sometimes we emphasize the different meaning by means of the special symbol $\equiv$, meaning ‘algebraically equal to’. (However, it is fairly common practice in engineering to use ‘=’ in both cases.) Such equations are often called **identities**. Identities that involve an unassigned variable x as in (1.8) are valid for all values of x, and we can sometimes make use of this fact to simplify algebraic manipulations.

Example 1.27

Find the numbers A, B and C such that

$$x^2 + 2x - 35 \equiv A(x - 1)^2 + B(x - 1) + C$$

Solution

Method (a): Since $x^2 + 2x - 35 \equiv A(x - 1)^2 + B(x - 1) + C$ it will be true for any value we give to x. So we choose values that make finding A, B and C easy.

Choosing $x = 0$ gives $-35 = A - B + C$
Choosing $x = 1$ gives $-32 = C$
Choosing $x = 2$ gives $-27 = A + B + C$

So we obtain $C = -32$, with $A - B = -3$ and $A + B = 5$. Hence $A = 1$ and $B = 4$ to give the identity

$$x^2 + 2x - 35 \equiv (x - 1)^2 + 4(x - 1) - 32$$

Method (b): Expanding the terms on the right-hand side, we have

$$x^2 + 2x - 35 \equiv Ax^2 + (B - 2A)x + A - B + C$$

The expressions on either side of the equals sign are algebraically equal, which means that the coefficient of x^2 on the left-hand side must equal the coefficient of x^2 on the right-hand side and so on. Thus

$$\begin{aligned} 1 &= A \\ 2 &= B - 2A \\ -35 &= A - B + C \end{aligned}$$

Hence we find $A = 1$, $B = 4$ and $C = -32$, as before.

Note: Method (a) assumes that a valid A, B and C exist. Sometimes a combination of methods (a) and (b) is helpful.

Example 1.28

Find numbers A, B and C such that

$$\frac{x^2}{x-1} \equiv Ax + B + \frac{C}{x-1}, \qquad x \neq 1$$

Solution

Expressing the right-hand side as a single term, we have

$$\frac{x^2}{x-1} \equiv \frac{(Ax + B)(x-1) + C}{x-1}$$

which, with $x \neq 1$, is equivalent to

$$x^2 \equiv (Ax + B)(x - 1) + C$$

Choosing $x = 0$ gives $0 = -B + C$
Choosing $x = 1$ gives $1 = C$
Choosing $x = 2$ gives $4 = 2A + B + C$

Thus we obtain

$C = 1$, $B = 1$ and $A = 1$, yielding

$$\frac{x^2}{x-1} \equiv x + 1 + \frac{1}{x-1}$$

1.3.4 Exercises

21 Rearrange the following formula to make s the subject

$$m = p\sqrt{\frac{s+t}{s-t}}$$

22 Given $u = \dfrac{x^2+t}{x^2-t}$, find t in terms of u and x.

23 Solve for t

$$\frac{1}{1-t} - \frac{1}{1+t} = 1$$

24 If

$$\frac{3c^2 + 3xc + x^2}{3c^2 + 3yc + y^2} = \frac{yV_1}{xV_2}$$

find the positive value of c when

$$x = 4,\ y = 6,\ V_1 = 120,\ V_2 = 315$$

25 Solve for p the equation

$$\frac{2p+1}{p+5} + \frac{p-1}{p+1} = 2$$

26 A rectangle has a perimeter of 30 m. If its length is twice its breadth, find the length.

27 (a) A4 paper is such that a half sheet has the same shape as the whole sheet. Find the ratio of the lengths of the sides of the paper.

(b) Foolscap paper is such that cutting off a square whose sides equal the shorter side of the paper leaves a rectangle which has the same shape as the original sheet. Find the ratio of the sides of the original page.

28 Find the values of x for which

(a) $\dfrac{5}{x} < 2$ (b) $\dfrac{1}{2-x} < 1$

(c) $\dfrac{3x-2}{x-1} > 2$ (d) $\dfrac{3}{3x-2} > \dfrac{1}{x+4}$

29 Find the values of x for which

$$x^2 < 2 + |x|$$

30 Prove that

(a) $x^2 + 3x - 10 \geqslant -(\frac{7}{2})^2$

(b) $18 + 4x - x^2 \leqslant 22$

(c) $x + \dfrac{4}{x} \geqslant 4 \quad \text{where} \quad x > 0$

(*Hint*: First complete the square of the left-hand members.)

31 Find the values of A and B such that

(a) $\dfrac{1}{(x+1)(x-2)} \equiv \dfrac{A}{x+1} + \dfrac{B}{x-2}$

(b) $3x + 2 \equiv A(x-1) + B(x-2)$

(c) $\dfrac{5x+1}{\sqrt{(x^2+x+1)}} \equiv \dfrac{A(2x+1)+B}{\sqrt{(x^2+x+1)}}$

32 Find the values of A, B and C such that

$$2x^2 - 5x + 12 \equiv A(x-1)^2 + B(x-1) + C$$

1.3.5 Suffix and sigma notation

We have seen in previous sections how letters are used to denote general or unspecified values or numbers. This process has been extended in a variety of ways. In particular, the introduction of suffixes enables us to deal with problems that involve a high degree of generality or whose solutions have the flexibility to apply in a large number of situations. Consider for the moment an experiment involving measuring the temperature of an object (for example, a piece of machinery or a cooling fin in a heat exchanger) at intervals over a period of time. In giving a theoretical description of the experiment we would talk about the total period of time in general terms, say T minutes, and the time interval between measurements as h minutes, so that the total number n of time intervals would be given by T/h. Assuming that the initial and final temperatures are recorded there are $(n + 1)$ measurements. In practice we would obtain a set of experimental results, as illustrated partially in Figure 1.12.

Figure 1.12 Experimental results: temperature against lapsed time.

Lapsed time (minutes)	0	5	10	15	…	170	175	180
Temperature (°C)	97.51	96.57	93.18	91.53	…	26.43	24.91	23.57

Here we could talk about the twenty-first reading and look it up in the table. In the theoretical description we would need to talk about any one of the $(n + 1)$ temperature measurements. To facilitate this we introduce a suffix notation. We label the times at which the temperatures are recorded $t_0, t_1, t_2, \dots, t_n$, where t_0 corresponds to the time when the initial measurement is taken, t_n to the time when the final measurement is taken, and

$$t_1 = t_0 + h, t_2 = t_0 + 2h, \dots, t_n = t_0 + nh$$

so that $t_n = t_0 + T$. We label the corresponding temperatures by $\theta_0, \theta_1, \theta_2, \dots, \theta_n$. We can then talk about the general result θ_k as measuring the temperature at time t_k.

In the analysis of the experimental results we may also wish to manipulate the data we have obtained. For example, we might wish to work out the average value of the temperature over the time period. With the thirty-seven specific experimental results given in Figure 1.12 it is possible to compute the average directly as

$$(97.51 + 96.57 + 93.18 + 91.53 + \dots + 23.57)/37$$

In general, however, we have

$$(\theta_0 + \theta_1 + \theta_2 + \dots + \theta_n)/(n + 1)$$

A compact way of writing this is to use the **sigma notation** for the extended summation $\theta_0 + \theta_1 + \dots + \theta_n$. We write

$$\sum_{k=0}^{n} \theta_k \qquad (\Sigma \text{ is the upper-case Greek letter sigma})$$

to denote

$$\theta_0 + \theta_1 + \theta_2 + \dots + \theta_n$$

Thus

$$\sum_{k=0}^{3} \theta_k = \theta_0 + \theta_1 + \theta_2 + \theta_3$$

and

$$\sum_{k=5}^{10} \theta_k = \theta_5 + \theta_6 + \theta_7 + \theta_8 + \theta_9 + \theta_{10}$$

The suffix k appearing in the quantity to be summed and underneath the sigma symbol is the ‘counting variable’ or ‘counter’. We may use any letter we please as a counter, provided that it is not being used at the same time for some other purpose. Thus

$$\sum_{i=0}^{3}\theta_i = \theta_0 + \theta_1 + \theta_2 + \theta_3 = \sum_{n=0}^{3}\theta_n = \sum_{j=0}^{3}\theta_j$$

Thus, in general, if $a_0, a_1, a_2, \ldots, a_n$ is a sequence of numbers or expressions, we write

$$\sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \ldots + a_n$$

Example 1.29

Given $a_0 = 1$, $a_1 = 5$, $a_2 = 2$, $a_3 = 7$, $a_4 = -1$ and $b_0 = 0$, $b_1 = 2$, $b_2 = -2$, $b_3 = 11$, $b_4 = 3$, calculate

(a) $\sum_{k=0}^{4} a_k$ (b) $\sum_{i=2}^{3} a_i$ (c) $\sum_{k=1}^{3} a_k b_k$ (d) $\sum_{k=0}^{4} b_k^2$

Solution

(a) $\sum_{k=0}^{4} a_k = a_0 + a_1 + a_2 + a_3 + a_4$

Substituting the given values for a_k $(k = 0, \ldots, 4)$ gives

$$\sum_{k=0}^{4} a_k = 1 + 5 + 2 + 7 + (-1) = 14$$

(b) $\sum_{i=2}^{3} a_i = a_2 + a_3 = 2 + 7 = 9$

(c) $\sum_{k=1}^{3} a_k b_k = a_1 b_1 + a_2 b_2 + a_3 b_3 = (5 \times 2) + (2 \times (-2)) + (7 \times 11) = 83$

(d) $\sum_{k=0}^{4} b_k^2 = b_0^2 + b_1^2 + b_2^2 + b_3^2 + b_4^2 = 0 + 4 + 4 + 121 + 9 = 138$

1.3.6 Factorial notation and the binomial expansion

The product of integers

$$1 \times 2 \times 3 \times \ldots \times n = n \times (n-1) \times (n-2) \times \ldots \times 1$$

has a special notation and name. It is called ***n* factorial** and is denoted by $n!$. Thus with

$$n! = n(n-1)(n-2)\ldots(1)$$

two examples are

$$5! = 5 \times 4 \times 3 \times 2 \times 1 \quad \text{and} \quad 8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

Notice that $5! = 5(4!)$ so that we can write in general

$$n! = (n-1)! \times n$$

This relationship enables us to define 0!, since $1! = 1 \times 0!$ and 1! also equals 1. Thus 0! is defined by

$$\boxed{0! = 1}$$

Example 1.30

Evaluate

(a) 4! (b) $3! \times 2!$ (c) 6! (d) $7!/(2! \times 5!)$

Solution

(a) $4! = 4 \times 3 \times 2 \times 1 = 24$

(b) $3! \times 2! = (3 \times 2 \times 1) \times (2 \times 1) = 12$

(c) $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

Notice that $2! \times 3! \neq (2 \times 3)!$.

(d) $$\frac{7!}{2! \times 5!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{7 \times 6}{2} = 21$$

Notice that we could have simplified the last item by writing

$$7! = 7 \times 6 \times (5!)$$

then

$$\frac{7!}{2! \times 5!} = \frac{7 \times 6 \times (5!)}{2! \times 5!} = \frac{7 \times 6}{2 \times 1} = 21$$

An interpretation of $n!$ is the total number of different ways it is possible to arrange n different objects in a single line. For example, the word SEAT comprises four different letters, and we can arrange the letters in $4! = 24$ different ways.

SEAT	EATS	ATSE	TSEA
SETA	EAST	ATES	TSAE
SAET	ESAT	AETS	TESA
SATE	ESTA	AEST	TEAS
STAE	ETSA	ASET	TAES
STEA	ETAS	ASTE	TASE

This is because we can choose the first letter in four different ways (S, E, A or T). Once that choice is made, we can choose the second letter in three different ways, then we can choose the third letter in two different ways. Having chosen the first three letters, the last letter is automatically fixed. For each of the four possible first choices, we have three possible choices for the second letter, giving us twelve (4×3) possible choices of the first two letters. To each of these twelve possible choices we have two possible choices of the third letter, giving us twenty-four $(4 \times 3 \times 2)$ possible choices of the first three letters. Having chosen the first three letters, there is only one possible choice of last letter. So in all we have 4! possible choices.

Example 1.31 In how many ways can the letters of the word REGAL be arranged in a line, and in how many of those do the two letters A and E appear in adjacent positions?

Solution The word REGAL has five distinct letters, so they can be arranged in a line in $5! = 120$ different ways. To find out in how many of those arrangements the A and E appear together, we consider how many arrangements can be made of RGL(AE) and RGL(EA), regarding the bracketed terms as a single symbol. There are 4! possible arrangements of both of these, so of the 120 different ways in which the letters of the word REGAL can be arranged, 48 contain the letters A and E in adjacent positions.

The introduction of the factorial notation facilitates the writing down of many complicated expressions. In particular it enables us to write down the general form of the binomial expansion discussed earlier (see Section 1.3.1). There we wrote out long-hand the expansion of $(a + b)^n$ for $n = 0, 1, 2, \dots, 6$ and noted the relationship between the coefficients of $(a + b)^n$ and those of $(a + b)^{n-1}$, shown clearly in Pascal's triangle of Figure 1.7.

If

$$(a + b)^{n-1} = c_0a^{n-1} + c_1a^{n-2}b + c_2a^{n-3}b^2 + c_3a^{n-4}b^3 + \dots + c_{n-1}b^{n-1}$$

and

$$(a + b)^n = d_0a^n + d_1a^{n-1}b + d_2a^{n-2}b^2 + \dots + d_{n-1}ab^{n-1} + d_nb^n$$

then, as described previously when developing Pascal's triangle,

$$c_0 = d_0 = 1, \quad d_1 = c_1 + c_0, \quad d_2 = c_2 + c_1, \quad d_3 = c_3 + c_2, \quad \dots$$

and in general

$$d_r = c_r + c_{r-1}$$

It is easy to verify that this relationship is satisfied by

$$d_r = \frac{n!}{r!(n-r)!}, \quad c_r = \frac{(n-1)!}{r!(n-1-r)!}, \quad c_{r-1} = \frac{(n-1)!}{(r-1)!(n-1-r+1)!}$$

and it can be shown that the coefficient of $a^{n-r}b^r$ in the expansion of $(a + b)^n$ is

$$\frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r(r-1)(r-2)\dots(1)} \tag{1.9}$$

This is a very important result, with many applications. Using it we can write down the general binomial expansion

$$(a + b)^n = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!}a^{n-r}b^r \tag{1.10}$$

The coefficient $\dfrac{n!}{r!(n-r)!}$ is called the **binomial coefficient** and has the special notation

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

This may be written as ${}^nC_r$. Thus we may write

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r \tag{1.11}$$

which is referred to as the general **binomial expansion**.

Example 1.32 Expand the expression $(2+x)^5$.

Solution Setting $a = 2$ and $b = x$ in the general binomial expansion we have

$$\begin{aligned}(2+x)^5 &= \sum_{r=0}^{5} \binom{5}{r} 2^{5-r} x^r \\ &= \binom{5}{0}2^5 + \binom{5}{1}2^4 x + \binom{5}{2}2^3x^2 + \binom{5}{3}2^2x^3 + \binom{5}{4}2x^4 + \binom{5}{5}x^5 \\ &= (1)(2^5) + (5)(2^4)x + (10)(2^3)x^2 + (10)(2^2)x^3 + (5)(2)x^4 + 1x^5\end{aligned}$$

since $\binom{5}{0} = \dfrac{5!}{0!5!} = 1$, $\binom{5}{1} = \dfrac{5!}{1!4!} = 5$, $\binom{5}{2} = \dfrac{5!}{2!3!} = 10$ and so on. Thus

$$(2+x)^5 = 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$$

1.3.7 Exercises

33 Given $a_0 = 2$, $a_1 = -1$, $a_2 = -4$, $a_3 = 5$, $a_4 = 3$ and $b_0 = 1$, $b_1 = 1$, $b_2 = 2$, $b_3 = -1$, $b_4 = 2$, calculate

(a) $\sum_{k=0}^{4} a_k$ (b) $\sum_{i=1}^{3} a_i$

(c) $\sum_{k=1}^{2} a_k b_k$ (d) $\sum_{j=0}^{4} b_j^2$

34 Evaluate

(a) $5!$ (b) $3!/4!$ (c) $7!/(3! \times 4!)$

(d) $\binom{5}{2}$ (e) $\binom{9}{3}$ (f) $\binom{8}{4}$

35 Using the general binomial expansion expand the following expressions:

(a) $(x-3)^4$ (b) $(x+\frac{1}{2})^3$

(c) $(2x+3)^5$ (d) $(3x+2y)^4$

1.4 Geometry

1.4.1 Coordinates

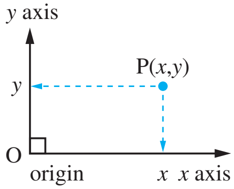


Figure 1.13
Cartesian coordinates.

In addition to the introduction of algebraic manipulation another innovation made in the seventeenth century was the use of coordinates to represent the position of a point P on a plane as shown in Figure 1.13. Conventionally the point P is represented by an ordered pair of numbers contained in brackets thus: (x, y). This innovation was largely due to the philosopher and scientist René Descartes (1596–1650) and consequently we often refer to (x, y) as the **cartesian coordinates** of P. This notation is the same as that for an open interval on the number line introduced previously (see Section 1.2.1), but has an entirely separate meaning and the two should not be confused. Whether (x, y) denotes an open interval or a coordinate pair is usually clear from the context.

1.4.2 Straight lines

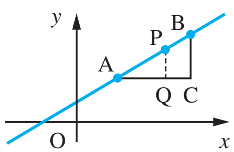


Figure 1.14
Straight line.

The introduction of coordinates made possible the algebraic description of the plane curves of classical geometry and the proof of standard results by algebraic methods.

Consider, for example, the point P lying on the line AB as shown in Figure 1.14. Let P divide AB in the ratio $\lambda:1-\lambda$. Then AP/AB $= \lambda$ and, by similar triangles,

$$\frac{\text{AP}}{\text{AB}} = \frac{\text{PQ}}{\text{BC}} = \frac{\text{AQ}}{\text{AC}}$$

Let A, B and P have coordinates (x_0, y_0), (x_1, y_1) and (x, y) respectively; then from the diagram

$$\text{AQ} = x - x_0,\ \text{AC} = x_1 - x_0,\ \text{PQ} = y - y_0,\ \text{BC} = y_1 - y_0$$

Thus

$$\frac{\text{PQ}}{\text{BC}} = \frac{\text{AQ}}{\text{AC}} \quad \text{implies} \quad \frac{y - y_0}{y_1 - y_0} = \frac{x - x_0}{x_1 - x_0}$$

from which we deduce, after some rearrangement,

$$y = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) + y_0 \tag{1.12}$$

which represents the equation of a straight line passing through two points (x_0, y_0) and (x_1, y_1).

More simply, the equation of a straight line passing through the two points having coordinates (x_0, y_0) and (x_1, y_1) may be written as

$$y = mx + c \tag{1.13}$$

where $m = \dfrac{y_1 - y_0}{x_1 - x_0}$ is the gradient (slope) of the line and $c = \dfrac{y_0x_1 - y_1x_0}{x_1 - x_0}$ is the intercept on the y axis.

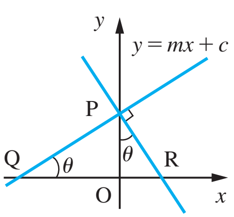


Figure 1.15
Perpendicular lines.

A line perpendicular to $y = mx + c$ has gradient $-1/m$ as shown in Figure 1.15. The gradient of the line PQ is OP/QO $= m$. The gradient of the line PR is $-$OP/OR. By similar triangles POQ, POR we have OP/OR $=$ OQ/OP $= 1/m$.

Equations of the form

$$y = mx + c$$

represent straight lines on the plane and, consequently, are called **linear equations**.

Example 1.33 Find the equation of the straight line that passes through the points (1, 2) and (3, 3).

Solution Taking $(x_0, y_0) = (1, 2)$ and $(x_1, y_1) = (3, 3)$

$$\text{slope of line} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{3-2}{3-1} = \frac{1}{2}$$

so from formula (1.12) the equation of the straight line is

$$y = \tfrac{1}{2}(x - 1) + 2$$

which simplifies to

$$y = \tfrac{1}{2}x + \tfrac{3}{2}$$

Example 1.34 Find the equation of the straight line passing through the point (3, 2) and parallel to the line $2y = 3x + 4$. Determine its x and y intercepts.

Solution Writing $2y = 3x + 4$ as

$$y = \tfrac{3}{2}x + 2$$

we have from (1.13) that the slope of this line is $\frac{3}{2}$. Since the required line is parallel to this line, it will also have a slope of $\frac{3}{2}$. (The slope of the line perpendicular to it is $-\frac{2}{3}$.) Thus from (1.13) it has equation

$$y = \tfrac{3}{2}x + c$$

To determine the constant c, we use the fact that the line passes through the point (3, 2), so that

$$2 = \tfrac{9}{2} + c \qquad \text{giving} \qquad c = -\tfrac{5}{2}$$

Thus the equation of the required line is

$$y = \tfrac{3}{2}x - \tfrac{5}{2} \qquad \text{or} \qquad 2y = 3x - 5$$

The y intercept is $c = -\frac{5}{2}$.

To obtain the x intercept we substitute $y = 0$, giving $x = \frac{5}{3}$, so that the x intercept is $\frac{5}{3}$.

The graph of the line is shown in Figure 1.16.

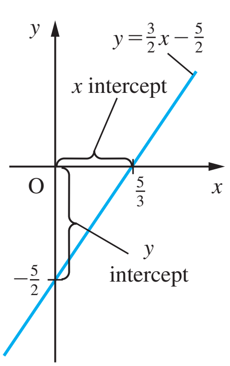


Figure 1.16
The straight line $2y = 3x - 5$.

1.4.3 Circles

A circle is the planar curve whose points are all equidistant from a fixed point called the centre of the circle. The simplest case is a circle centred at the origin with radius r, as shown in Figure 1.17(a). Applying Pythagoras' theorem to triangle OPQ we obtain

$$x^2 + y^2 = r^2$$

(Note that r is a constant.) When the centre of the circle is at the point (a, b), rather than the origin, the equation of the circle is

$$(x - a)^2 + (y - b)^2 = r^2 \tag{1.14a}$$

obtained by applying Pythagoras' theorem in triangle O′PN of Figure 1.17(b). This expands to

$$x^2 + y^2 - 2ax - 2by + (a^2 + b^2 - r^2) = 0$$

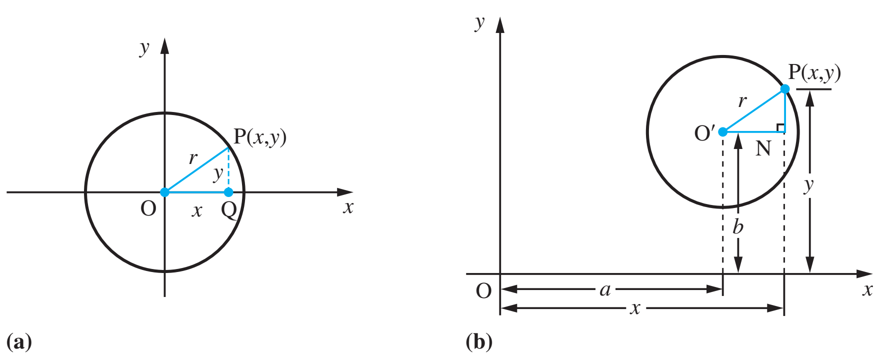


Figure 1.17 (a) A circle of centre origin, radius r. (b) A circle of centre (a, b), radius r.

Thus the general equation

$$x^2 + y^2 + 2fx + 2gy + c = 0 \tag{1.14b}$$

represents a circle having centre $(-f, -g)$ and radius $\sqrt{(f^2 + g^2 - c)}$. Notice that the general circle has three constants f, g and c in its equation. This implies that we need three points to specify a circle completely.

Example 1.35 Find the equation of the circle with centre (1, 2) and radius 3.

Solution Using Pythagoras' theorem, if the point P(x, y) lies on the circle then from (1.14a)

$$(x - 1)^2 + (y - 2)^2 = 3^2$$

Thus

$$x^2 - 2x + 1 + y^2 - 4y + 4 = 9$$

giving the equation as

$$x^2 + y^2 - 2x - 4y - 4 = 0$$

Example 1.36 Find the radius and the coordinates of the centre of the circle whose equation is

$$2x^2 + 2y^2 - 3x + 5y + 2 = 0$$

Solution Dividing through by the coefficient of x^2 we obtain

$$x^2 + y^2 - \tfrac{3}{2}x + \tfrac{5}{2}y + 1 = 0$$

Now completing the square on the x terms and the y terms separately gives

$$(x - \tfrac{3}{4})^2 + (y + \tfrac{5}{4})^2 = \tfrac{9}{16} + \tfrac{25}{16} - 1 = \tfrac{18}{16}$$

Hence, from (1.14a), the circle has radius $(3\sqrt{2})/4$ and centre $(3/4, -5/4)$.

Example 1.37 Find the equation of the circle which passes through the points (0, 0), (0, 2), (4, 0).

Solution Method (a): From (1.14b) the general equation of a circle is

$$x^2 + y^2 + 2fx + 2gy + c = 0$$

Substituting the three points into this equation gives three equations for the unknowns f, g and c.

Thus substituting (0, 0) gives $c = 0$, substituting (0, 2) gives $4 + 4g + c = 0$ and substituting (4, 0) gives $16 + 8f + c = 0$. Solving these equations gives $g = -1, f = -2$ and $c = 0$, so the required equation is

$$x^2 + y^2 - 4x - 2y = 0$$

Method (b): From Figure 1.18 using the geometrical properties of the circle, we see that its centre lies at (2, 1) and since it passes through the origin its radius is $\sqrt{5}$. Hence, from (1.14a), its equation is

$$(x - 2)^2 + (y - 1)^2 = (\sqrt{5})^2$$

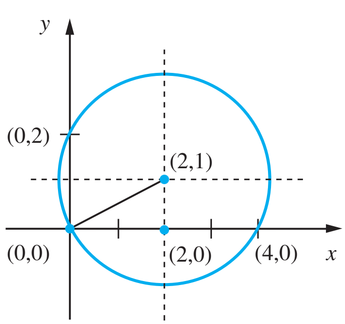


Figure 1.18 The circle of Example 1.37.

which simplifies to

$$x^2 + y^2 - 4x - 2y = 0$$

as before.

Example 1.38 Find the point of intersection of the line $y = x - 1$ with the circle $x^2 + y^2 - 4y - 1 = 0$.

Solution Substituting $y = x - 1$ into the formula for the circle gives

$$x^2 + (x - 1)^2 - 4(x - 1) - 1 = 0$$

which simplifies to

$$x^2 - 3x + 2 = 0$$

This equation may be factored to give

$$(x - 2)(x - 1) = 0$$

so that $x = 1$ and $x = 2$ are the roots. Thus the points of intersection are (1, 0) and (2, 1).

Example 1.39 Find the equation of the tangent at the point (2, 1) of the circle $x^2 + y^2 - 4y - 1 = 0$.

Solution A tangent is a line, which is the critical case between a line intersecting the circle in two distinct points and its not intersecting at all. We can describe this as the case when the line cuts the circle in two coincident points. Thus the line, which passes through (2, 1) with slope m

$$y = m(x - 2) + 1$$

is a tangent to the circle when the equation

$$x^2 + [m(x - 2) + 1]^2 - 4[m(x - 2) + 1] - 1 = 0$$

has two equal roots. Multiplying these terms out we obtain the equation

$$(m^2 + 1)x^2 - 2m(2m + 1)x + 4(m^2 + m - 1) = 0$$

The condition for this equation to have equal roots is (using comment (a) of Example 1.23)

$$4m^2(2m + 1)^2 = 4[4(m^2 + m - 1)(m^2 + 1)]$$

This simplifies to

$$m^2 - 4m + 4 = 0 \quad \text{or} \quad (m - 2)^2 = 0$$

giving the result $m = 2$ and the equation of the tangent $y = 2x - 3$.

1.4.4 Exercises

36 Find the equation of the straight line

(a) with gradient $\frac{3}{2}$ passing through the point (2, 1);

(b) with gradient -2 passing through the point $(-2, 3)$;

(c) passing through the points (1, 2) and (3, 7);

(d) passing through the points (5, 0) and (0, 3);

(e) parallel to the line $3y - x = 5$, passing through (1, 1);

(f) perpendicular to the line $3y - x = 5$, passing through (1, 1).

37 Write down the equation of the circle with centre (1, 2) and radius 5.

38 Find the radius and the coordinates of the centre of the circle with equation

$$x^2 + y^2 + 4x - 6y = 3$$

39 Find the equation of the circle with centre $(-2, 3)$ that passes through $(1, -1)$.

40 Find the equation of the circle that passes through the points (1, 0), (3, 4) and (5, 0).

41 Find the equation of the tangent to the circle

$$x^2 + y^2 - 4x - 1 = 0$$

at the point (1, 2).

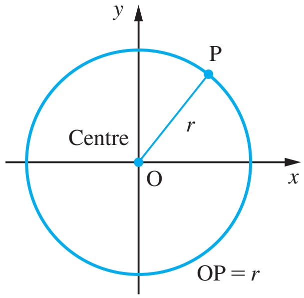
42 A rod, 50 cm long, moves in a plane with its ends on two perpendicular wires. Find the equation of the curve followed by its midpoint.

43 The feet of the altitudes of triangle A(0, 0), B(b, 0) and C(c, d) are D, E and F respectively. Show that the altitudes meet at the point O(c, $c(b-c)/d$). Further, show that the circle through D, E and F also passes through the midpoint of each side as well as the midpoints of the lines AO, BO and CO.

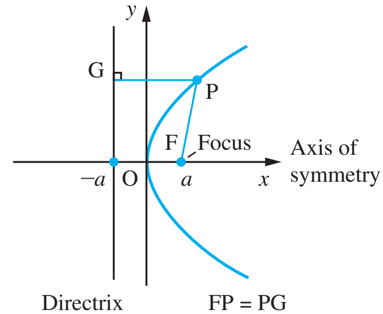
1.4.5 Conics

The circle is one of the conic sections (Figure 1.19) introduced around 200 BC by Apollonius, who published an extensive study of their properties in a textbook that he called *Conics*. He used this title because he visualized them as cuts made by a 'flat' or plane surface when it intersects the surface of a cone in different directions, as illustrated in Figures 1.20(a–d). Note that the conic sections degenerate into a point and straight lines at the extremities, as illustrated in Figures 1.20(e–g). Although at the time of Apollonius his work on conics appeared to be of little value in terms of applications, it has since turned out to have considerable importance. This is primarily due to the fact that the conic sections are the paths followed by projectiles, artificial satellites, moons and the Earth under the influence of gravity around planets or stars. The early Greek astronomers thought that the planets moved in circular orbits, and it was not until 1609 that the German astronomer Johannes Kepler described their paths correctly as being elliptic, with the Sun at one focus. It is quite possible for an orbit to be a curve other than an ellipse. Imagine a meteor or comet approaching the Sun from some distant region in space. The path that the body will follow depends very much on the speed at which it is moving. If the body is small compared to the Sun, say of planetary dimensions, and its speed relative to the Sun is not very high, it will never escape and will describe an *elliptic* path about it. An example is the comet observed by Edmond Halley in 1682 and now known as Halley's comet. He computed its elliptic orbit, found that it was the same comet that had been seen in 1066, 1456, 1531 and 1607, and correctly forecast its reappearance in 1758. It was most recently seen in 1986. If the speed of the body is very high, its path will be deviated by the Sun but it will not orbit for ever around the Sun. Rather, it will bend around the Sun in a path in the form of a **hyperbola** and continue on its journey back to outer space. Somewhere between these two extremes there is a certain critical speed that is just too great to allow the body to

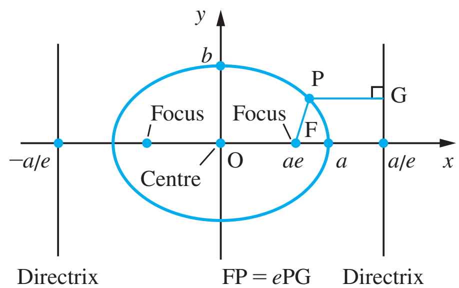
Figure 1.19
Standard equations of the four conics.



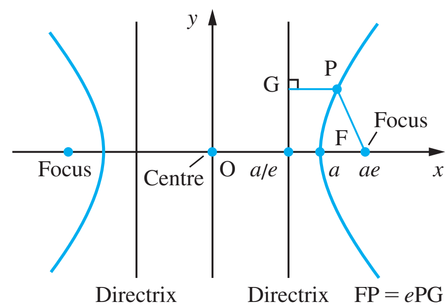
(a) Circle: $x^2 + y^2 = r^2$



(b) Parabola: $y^2 = 4ax$
$e = 1$

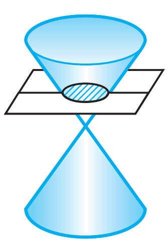


(c) Ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
$b^2 = a^2(1 - e^2)$
and eccentricity $e < 1$

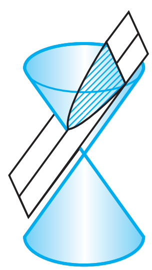


(d) Hyperbola: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$
$b^2 = a^2(e^2 - 1)$
and eccentricity $e > 1$

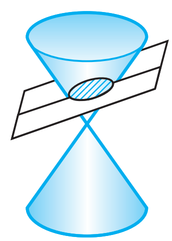
Figure 1.20



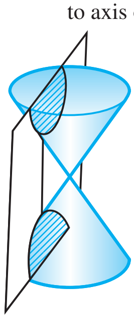
(a) Circle: section perpendicular to axis of cone



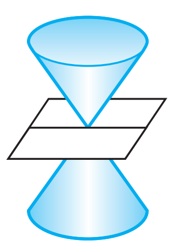
(b) Parabola: section parallel to generator of cone



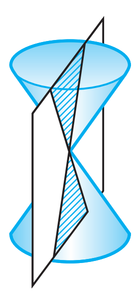
(c) Ellipse: section slanted to axis of cone



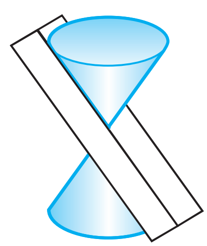
(d) Hyperbola: section slanted to axis of cone



(e) Point



(f) Two intersecting lines



(g) Line

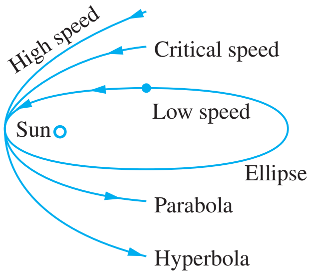
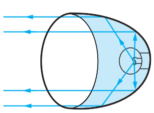


Figure 1.21
Orbital path.

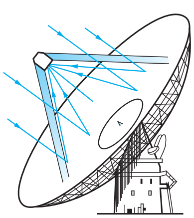
orbit the Sun, but not great enough for the path to be a hyperbola. In this case the path is a **parabola**, and once again the body will bend around the Sun and continue on its journey into outer space. These possibilities are illustrated in Figure 1.21.

Examples of where conic sections appear in engineering practice include the following.

(a) A parabolic surface, obtained by rotating a parabola about its axis of symmetry, has the important property that an energy source placed at the focus will cause rays to be reflected at the surface such that after reflection they will be parallel. Reversing the process, a beam parallel to the axis impinging on the surface will be reflected onto the focus (Example 8.6). This property is involved in many engineering design projects: for example, the design of a car headlamp or a radio telescope, as illustrated in Figures 1.22(a) and (b) respectively. Other examples involving a parabola are the path of a projectile (Example 2.39) and the shape of the cable on certain types of suspension bridge (Example 8.69).



(a)



(b)

Figure 1.22
(a) Car headlamp.
(b) Radio telescope.

(b) A ray of light emitted from one focus of an elliptic mirror and reflected by the mirror will pass through the other focus, as illustrated in Figure 1.23. This property is sometimes used in designing mirror combinations for a reflecting telescope. Ellipses have been used in other engineering designs, such as aircraft wings and stereo styli. Elliptical pipes are used for foul and surface water drainage because the elliptical profile is hydraulically efficient. As described earlier, every planet orbits around the Sun in an elliptic path with the Sun at one of its foci. The planet's speed depends on its distance from the Sun; it speeds up as it nears the Sun and slows down as it moves further away. The reason for this is that for an ellipse the line drawn from the focus S (Sun) to a point P (planet) on the ellipse sweeps out areas at a constant rate as P moves around the ellipse. Thus in Figure 1.24 the planet will take the same time to travel the two different distances shown, assuming that the two shaded regions are of equal area.

(c) Consider a supersonic aircraft flying over land. As it breaks the sound barrier (that is, it travels faster than the speed of sound, which is about 750 mph ($331.4\,\mathrm{m\,s^{-1}}$)), it will create a shock wave, which we hear on the ground as a *sonic boom* – this being one of the major disadvantages of supersonic aircraft. This shock wave will trail behind

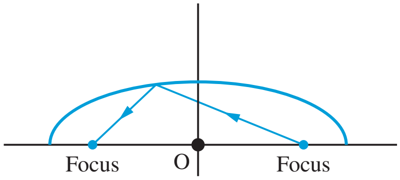


Figure 1.23 Reflection of a ray by an elliptic mirror.

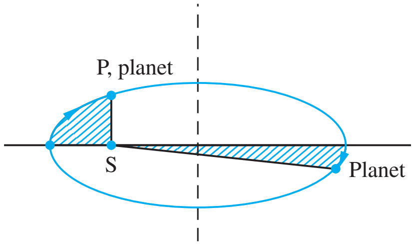


Figure 1.24 Regions of equal area.

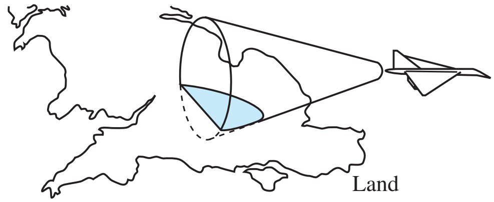


Figure 1.25
Sonic boom.

the aircraft in the form of a cone with the aircraft as vertex. This cone will intersect the ground in a *hyperbolic curve*, as illustrated in Figure 1.25. The sonic boom will hit every point on this curve at the same instant of time, so that people living on the curve will hear it simultaneously. No boom will be heard by people living outside this curve, but eventually it will be heard at every point inside it.

Figure 1.19 illustrates the conics in their standard positions, and the corresponding equations may be interpreted as the standard equations for the four curves. More generally the conic sections may be represented by the general second-order equation

$$ax^2 + by^2 + 2fx + 2gy + 2hxy + c = 0 \tag{1.15}$$

Provided its graph does not degenerate into a point or straight lines, (1.15) is representative of

- a circle if $a = b \neq 0$ and $h = 0$
- a parabola if $h^2 = ab$
- an ellipse if $h^2 < ab$
- a hyperbola if $h^2 > ab$

The conics can be defined mathematically in a number of (equivalent) ways, as we shall illustrate in the next examples.

Example 1.40

A point P moves in such a way that its total distance from two fixed points A and B is constant. Show that it describes an ellipse.

Solution

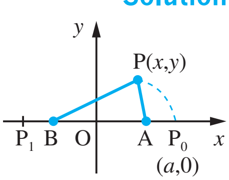


Figure 1.26
Path of Example 1.40.

The definition of the curve implies that AP + BP = constant with the origin O being the midpoint of AB. From symmetry considerations we choose x and y axes as shown in Figure 1.26. Suppose the curve crosses the x axis at P_0, then

$$AP_0 + BP_0 = AB + 2AP_0 = 2OP_0$$

so the constant in the definition is $2OP_0$ and for any point P on the curve

$$AP + BP = 2OP_0$$

Let $P = (x, y)$, $P_0 = (a, 0)$, $P_1 = (-a, 0)$, $A = (c, 0)$ and $B = (-c, 0)$. Then using Pythagoras' theorem we have

$$AP = \sqrt{[(x - c)^2 + y^2]}$$

$$BP = \sqrt{[(x + c)^2 + y^2]}$$

so that the defining equation of the curve becomes

$$\sqrt{[(x - c)^2 + y^2]} + \sqrt{[(x + c)^2 + y^2]} = 2a$$

To obtain the required equation we need to 'remove' the square root terms. This can only be done by squaring both sides of the equation. First we rewrite the equation as

$$\sqrt{[(x - c)^2 + y^2]} = 2a - \sqrt{[(x + c)^2 + y^2]}$$

and then square to give

$$(x - c)^2 + y^2 = 4a^2 - 4a\sqrt{[(x + c)^2 + y^2]} + (x + c)^2 + y^2$$

Expanding the squared terms we have

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{[(x + c)^2 + y^2]} + x^2 + 2cx + c^2 + y^2$$

Collecting together terms, we obtain

$$a\sqrt{[(x + c)^2 + y^2]} = a^2 + cx$$

Squaring both sides again gives

$$a^2[x^2 + 2cx + c^2 + y^2] = a^4 + 2a^2cx + c^2x^2$$

which simplifies to

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

Noting that $a > c$ we write $a^2 - c^2 = b^2$, to obtain

$$b^2x^2 + a^2y^2 = a^2b^2$$

which yields the standard equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The points A and B are the foci of the ellipse, and the property that the sum of the focal distances is a constant is known as the **string property** of the ellipse since it enables us to draw an ellipse using a piece of string.

For a hyperbola, the *difference* of the focal distances is constant.

Example 1.41

A point moves in such a way that its distance from a fixed point F is equal to its perpendicular distance from a fixed line. Show that it describes a parabola.

Solution

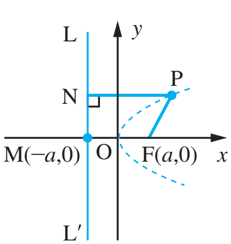


Figure 1.27
Path of point in Example 1.41.

Suppose the fixed line is LL′ shown in Figure 1.27, choosing the coordinate axes shown. Since PF = PN for points on the curve we deduce that the curve bisects FM, so that if F is $(a, 0)$, then M is $(-a, 0)$. Let the general point P on the curve have coordinates (x, y). Then by Pythagoras' theorem

$$PF = \sqrt{[(x - a)^2 + y^2]}$$

Also $PN = x + a$, so that PN = PF implies that

$$x + a = \sqrt{[(x - a)^2 + y^2]}$$

Squaring both sides gives

$$(x + a)^2 = (x - a)^2 + y^2$$

which simplifies to

$$y^2 = 4ax$$

the standard equation of a parabola. The line LL′ is called the **directrix** of the parabola.

Example 1.42

(a) Find the equation of the tangent at the point (1, 1) to the parabola $y = x^2$. Show that it is parallel to the line through the points $(\frac{1}{2}, \frac{1}{4})$, $(\frac{3}{2}, \frac{9}{4})$, which also lie on the parabola.

(b) Find the equation of the tangent at the point (a, a^2) to the parabola $y = x^2$. Show that it is parallel to the line through the points $(a - h, (a - h)^2)$, $(a + h, (a + h)^2)$.

Solution

(a) Consider the general line through (1, 1). It has equation $y = m(x - 1) + 1$. This cuts the parabola when

$$m(x - 1) + 1 = x^2$$

that is, when

$$x^2 - mx + m - 1 = 0$$

Factorizing this quadratic, we have

$$(x - 1)(x - m + 1) = 0$$

giving the roots $x = 1$ and $x = m - 1$

These two roots are equal when $m - 1 = 1$; that is, when $m = 2$. Hence the equation of the tangent is $y = 2x - 1$.

The line through the points $(\frac{1}{2}, \frac{1}{4})$, $(\frac{3}{2}, \frac{9}{4})$ has gradient

$$\frac{\frac{9}{4} - \frac{1}{4}}{\frac{3}{2} - \frac{1}{2}} = 2$$

so that it is parallel to the tangent at (1, 1).

(b) Consider the general line through (a, a^2). It has equation $y = m(x - a) + a^2$. This cuts the parabola $y = x^2$ when

$$m(x - a) + a^2 = x^2$$

that is, where

$$x^2 - mx + ma - a^2 = 0$$

This factorizes into

$$(x - a)(x - m + a) = 0$$

giving the roots $x = a$ and $x = m - a$. These two roots are equal when $a = m - a$; that is, when $m = 2a$. Thus the equation of the tangent at (a, a^2) is $y = 2ax - a^2$.

The line through the points $(a - h, (a - h)^2)$, $(a + h, (a + h)^2)$ has gradient

$$\frac{(a + h)^2 - (a - h)^2}{(a + h) - (a - h)} = \frac{a^2 + 2ah + h^2 - (a^2 - 2ah + h^2)}{2h}$$

$$= \frac{4ah}{2h} = 2a$$

So the symmetrically disposed chord through $(a - h, (a - h)^2)$, $(a + h, (a + h)^2)$ is parallel to the tangent at $x = a$. This result is true for all parabolas.

1.4.6 Exercises

44 Find the coordinates of the focus and the equation of the directrix of the parabola whose equation is

$$3y^2 = 8x$$

The chord which passes through the focus parallel to the directrix is called the **latus rectum** of the parabola. Show that the latus rectum of the above parabola has length 8/3.

45 For the ellipse $25x^2 + 16y^2 = 400$ find the coordinates of the foci, the eccentricity, the equations of the directrices and the lengths of the semi-major and semi-minor axes.

46 For the hyperbola $9x^2 - 16y^2 = 144$ find the coordinates of the foci and the vertices and the equations of its asymptotes.

1.5 Number and accuracy

Arithmetic that only involves integers can be performed to obtain an exact answer (that is, one without rounding errors). In general, this is not possible with real numbers, and when solving practical problems such numbers are rounded to an appropriate number of digits. In this section we shall review the methods of recording numbers, obtain estimates for the effect of rounding errors in elementary calculations and discuss the implementation of arithmetic on computers.

1.5.1 Rounding, decimal places and significant figures

The Fundamental Laws of Arithmetic are, of course, independent of the choice of representation of the numbers. Similarly, the representation of irrational numbers will always be incomplete. Because of these numbers and because some rational numbers have recurring representations (whether the representation of a particular rational number is recurring or not will of course depend on the number base used – see Example 1.2d), any arithmetical calculation will contain errors caused by truncation. In practical problems it is usually known how many figures are meaningful, and the numbers are 'rounded' accordingly. In the decimal representation, for example, the numbers are approximated by the closest decimal number with some prescribed number of figures after the decimal point. Thus, to two decimal places (dp),

$$\pi = 3.14 \quad \text{and} \quad \tfrac{5}{12} = 0.42$$

and to five decimal places

$$\pi = 3.141\,59 \quad \text{and} \quad \tfrac{5}{12} = 0.416\,67$$

Normally this is abbreviated to

$$\pi = 3.141\,59 \text{ (5dp)} \quad \text{and} \quad \tfrac{5}{12} = 0.416\,67 \text{ (5dp)}$$

Similarly

$$\sqrt{2} = 1.4142 \text{ (4dp)} \quad \text{and} \quad \tfrac{2}{3} = 0.667 \text{ (3dp)}$$

In hand computation, by convention, when shortening a number ending with a five we 'round to the even'. For example,

$$1.2345 \quad \text{and} \quad 1.2335$$

are both represented by 1.234 to three decimal places. In contrast, most calculators and computers would 'round up' in the ambiguous case, giving 1.2345 and 1.2335 as 1.235 and 1.234 respectively.

Any number occurring in practical computation will either be given an error bound or be correct to within half a unit in the least significant figure (sf). For example,

$$\pi = 3.14 \pm 0.005 \quad \text{or} \quad \pi = 3.14$$

Any number given in scientific or mathematical tables observes this convention. Thus

$$g_0 = 9.806\,65$$

implies

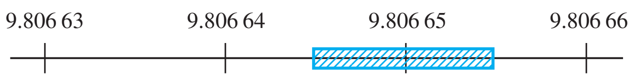
$$g_0 = 9.806\,65 \pm 0.000\,005$$

that is,

$$9.806\,645 < g_0 < 9.806\,655$$

as illustrated in Figure 1.28,

Figure 1.28



Sometimes the decimal notation may create a false impression of accuracy. When we write that the distance of the Earth from the Sun is ninety-three million miles, we mean that the distance is nearer to 93 000 000 than to 94 000 000 or to 92 000 000, not that it is nearer to 93 000 000 than to 93 000 001 or to 92 999 999. This possible misinterpretation of numerical data is avoided by stating the number of significant figures, giving an error estimate or using scientific notation. In this example the distance d miles is given in the forms

$$d = 93\,000\,000 \text{ (2sf)}$$

or

$$d = 93\,000\,000 \pm 500\,000$$

or

$$d = 9.3 \times 10^7$$

Notice how information about accuracy is discarded by the rounding-off process. The value ninety-three million miles is actually correct to within fifty thousand miles, while the convention about rounded numbers would imply an error bound of five hundred thousand.

The number of significant figures tells us about the relative accuracy of a number when it is related to a measurement. Thus a number given to 3sf is relatively ten times more accurate than one given to 2sf. The number of decimal places, dp, merely tells us the number of digits including leading zeros after the decimal point. Thus

$$2.321 \quad \text{and} \quad 0.000\,059\,71$$

both have 4sf, while the former has 3dp and the latter 8dp.

It is not clear how many significant figures a number like 3200 has. It might be 2, 3 or 4. To avoid this ambiguity it must be written in the form 3.2×10^3 (when it is correct to 2sf) or 3.20×10^3 (3sf) or 3.200×10^3 (4sf). This is usually called **scientific notation**. It is widely used to represent numbers that are very large or very small. Essentially, a number x is written in the form

$$x = a \times 10^n$$

where $1 \leqslant |a| < 10$ and n is an integer. Thus the mass of an electron at rest is 9.11×10^{-28} g, while the velocity of light in a vacuum is $2.9978 \times 10^{10}\,\text{cm}\,\text{s}^{-1}$.

Example 1.43

Express the number 150.4152

(a) correct to 1, 2 and 3 decimal places; (b) correct to 1, 2 and 3 significant figures.

Solution

(a)
$$\begin{aligned} 150.4152 &= 150.4 && \text{(1dp)} \\ &= 150.42 && \text{(2dp)} \\ &= 150.415 && \text{(3dp)} \end{aligned}$$

(b)
$$\begin{aligned} 150.4152 &= 1.504\,152 \times 10^2 \\ &= 2 \times 10^2 && \text{(1sf)} \\ &= 1.5 \times 10^2 && \text{(2sf)} \\ &= 1.50 \times 10^3 && \text{(3sf)} \end{aligned}$$

1.5.2 Estimating the effect of rounding errors

Numerical data obtained experimentally will often contain rounding errors due to the limited accuracy of measuring instruments. Also, because irrational numbers and some rational numbers do not have a terminating decimal representation, arithmetical operations inevitably contain errors arising from rounding off. The effect of such errors can accumulate in an arithmetical procedure and good engineering computations will include an estimate for it. This process has become more important with the widespread use of computers. When users are isolated from the computational chore, they often fail to develop a sense of the limits of accuracy of an answer. Indeed, with certain calculations the error can balloon as the calculation proceeds. In this section we shall develop the basic ideas for such sensitivity in analyses of calculations.

Example 1.44

Compute

(a) $3.142 + 4.126$ (b) $5.164 - 2.341$ (c) 235.12×0.531

Calculate estimates for the effects of rounding errors in each answer and give the answer as a correctly rounded number.

Solution (a) $3.142 + 4.126 = 7.268$

Because of the convention about rounded numbers, 3.142 represents all the numbers a between 3.1415 and 3.1425, and 4.126 represents all the numbers b between 4.1255 and 4.1265. Thus if a and b are correctly rounded numbers, their sum $a + b$ lies between $c_1 = 7.2670$ and $c_2 = 7.2690$. Rounding c_1 and c_2 to 3dp gives $c_1 = 7.267$ and $c_2 = 7.269$. Since these disagree, we cannot give an answer to 3dp. Rounding c_1 and c_2 to 2dp gives $c_1 = 7.27$ and $c_2 = 7.27$. Since these agree, we can give the answer to 2dp; thus $a + b = 7.27$, as shown in Figure 1.29.

Figure 1.29

(b) $5.164 - 2.341 = 2.823$

Applying the same 'worst case' analysis to this implies that the difference lies between $5.1635 - 2.3415$ and $5.1645 - 2.3405$; that is, between 2.8220 and 2.8240. Thus the answer should be written 2.823 ± 0.001 or, as a correctly rounded number, 2.82.

(c) $235.12 \times 0.531 = 124.848\,72$

Clearly, writing an answer with so many decimal places is unjustified if we are using rounded numbers, but how many decimal places are sensible? Using the 'worst case' analysis again, we deduce that the product lies between 235.115×0.5305 and 235.125×0.5315; that is, between $c_1 = 124.728\,507\,5$ and $c_2 = 124.968\,937\,5$. Thus the answer should be written 124.85 ± 0.13. In this example, because of the place where the number occurs on the number line, c_1 and c_2 only agree when we round them to 3sf (0dp). Thus the product as a correctly rounded number is 125.

A competent computation will contain within it estimates of the effect of rounding errors. Analysing the effect of such errors for complicated expressions has to be approached systematically.

Definitions

(a) The **error** in a value is defined by

$$\text{error} = \text{approximate value} - \text{true value}$$

This is sometimes termed the dead error. Notice that the true value equals the approximate value minus the error.

(b) Similarly the **correction** is defined by

$$\text{true value} = \text{approximate value} + \text{correction}$$

so that

$$\text{correction} = -\text{error}$$

(c) The **error modulus** is the size of the error, $|\text{error}|$, and the **error bound** (or **absolute error bound**) is the maximum possible error modulus.

(d) The **relative error** is the ratio of the size of the error to the size of the true value:

$$\text{relative error} = \left|\frac{\text{error}}{\text{value}}\right|$$

The **relative error bound** is the maximum possible relative error.

(e) The **percent error** (or percentage error) is 100 × relative error and the **percent error bound** is the maximum possible per cent error.

In some contexts we think of the true value as an approximation and a remainder. In such cases the remainder is given by

$$\begin{aligned}\text{remainder} &= -\text{error}\\ &= \text{correction}\end{aligned}$$

Example 1.45

Give the absolute and relative error bounds of the following correctly rounded numbers

(a) 29.92 (b) −0.015 23 (c) 3.9×10^{10}

Solution

(a) The number 29.92 is given to 2dp, which implies that it represents a number within the domain 29.92 ± 0.005. Thus its absolute error bound is 0.005, half a unit of the least significant figure, and its relative error bound is 0.005/29.92 or 0.000 17.

(b) The absolute error bound of −0.015 23 is half a unit of the least significant figure, that is 0.000 005. Notice that it is a positive quantity. Its relative error bound is 0.000 005/0.015 23 or 0.000 33.

(c) The absolute error bound of 3.9×10^{10} is $0.05 \times 10^{10} = 5 \times 10^8$ and its relative error bound is 0.05/3.9 or 0.013.

Usually, because we do not know the true values, we estimate the effects of error in a calculation in terms of the error bounds, the 'worst case' analysis illustrated in Example 1.44. The error bound of a value v is denoted by ε_v.

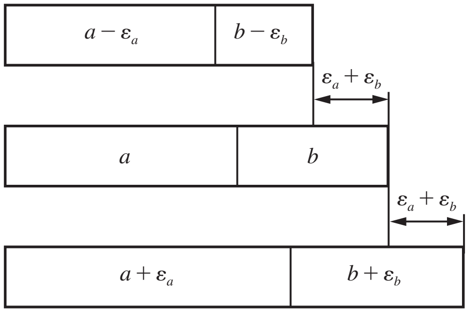
Consider, first, the sum $c = a + b$. When we add together the two rounded numbers a and b their sum will inherit a rounding error from both a and b. The true value of a lies between $a - \varepsilon_a$ and $a + \varepsilon_a$ and the true value of b lies between $b - \varepsilon_b$ and $b + \varepsilon_b$. Thus the smallest value that the true value of c can have is $a - \varepsilon_a + b - \varepsilon_b$, and its largest possible value is $a + \varepsilon_a + b + \varepsilon_b$. (Remember that ε_a and ε_b are positive.) Thus $c = a + b$ has an error bound

$$\varepsilon_c = \varepsilon_a + \varepsilon_b$$

as illustrated in Figure 1.30. A similar 'worst case' analysis shows that the difference $d = a - b$ has an error bound that is the sum of the error bounds of a and b:

$$d = a - b, \qquad \varepsilon_d = \varepsilon_a + \varepsilon_b$$

Figure 1.30



Thus for both addition and subtraction the error bound of the result is the sum of the individual error bounds.

Next consider the product $p = a \times b$, where a and b are positive numbers. The smallest possible value of p will be equal to the product of the least possible values of a and b; that is,

$$p > (a - \varepsilon_a) \times (b - \varepsilon_b)$$

Similarly

$$p < (a + \varepsilon_a) \times (b + \varepsilon_b)$$

Thus, on multiplying out the brackets, we obtain

$$ab - a\varepsilon_b - b\varepsilon_a + \varepsilon_a\varepsilon_b < p < ab + a\varepsilon_b + b\varepsilon_a + \varepsilon_a\varepsilon_b$$

Ignoring the very small term $\varepsilon_a\varepsilon_b$, we obtain an estimate for the error bound of the product:

$$\varepsilon_p = a\varepsilon_b + b\varepsilon_a, \qquad p = a \times b$$

Dividing both sides of the equation by p, we obtain

$$\frac{\varepsilon_p}{p} = \frac{\varepsilon_a}{a} + \frac{\varepsilon_b}{b}$$

Now the relative error of a is defined as the ratio of the error in a to the size of a. The above equation connects the relative error bounds for a, b and p:

$$r_p = r_a + r_b$$

Here $r_a = \varepsilon_a/|a|$ allowing for a to be negative, and so on.

A similar worst case analysis for the quotient $q = a/b$ leads to the estimate

$$r_q = r_a + r_b$$

Thus for both multiplication and division, the relative error bound of the result is the sum of the individual relative error bounds.

These elementary rules for estimating error bounds can be combined to obtain more general results. For example, consider $z = x^2$; then $r_z = 2r_x$. In general, if $z = x^y$, where x is a rounded number and y is exact, then

$$r_z = yr_x$$

Example 1.46 Evaluate 13.92×5.31 and $13.92 \div 5.31$.

Assuming that these values are correctly rounded numbers, calculate error bounds for each answer and write them as correctly rounded numbers which have the greatest possible number of significant digits.

Solution $13.92 \times 5.31 = 73.9152$; $13.92 \div 5.31 = 2.621\,468\,927$

Let $a = 13.92$ and $b = 5.31$; then $r_a = 0.000\,36$ and $r_b = 0.000\,94$, so that $a \times b$ and $a \div b$ have relative error bounds $0.000\,36 + 0.000\,94 = 0.0013$. We obtain the absolute error bound of $a \times b$ by multiplying the relative error bound by $a \times b$. Thus the absolute error bound of $a \times b$ is $0.0013 \times 73.9152 = 0.0961$. Similarly, the absolute error bound of $a \div b$ is $0.0013 \times 2.6215 = 0.0034$. Hence the values of $a \times b$ and $a \div b$ lie in the error intervals

$$73.9152 - 0.0961 < a \times b < 73.9152 + 0.0961$$

and

$$2.6215 - 0.0034 < a \div b < 2.6215 + 0.0034$$

Thus $73.8191 < a \times b < 74.0113$ and $2.6181 < a \div b < 2.6249$.

From these inequalities we can deduce the correctly rounded values of $a \times b$ and $a \div b$

$$a \times b = 74 \quad \text{and} \quad a \div b = 2.62$$

and we see how the rounding convention discards information. In a practical context, it would probably be more helpful to write

$$73.81 < a \times b < 74.02$$

and

$$2.618 < a \div b < 2.625$$

Example 1.47 Evaluate

$$6.721 - \frac{4.931 \times 71.28}{89.45}$$

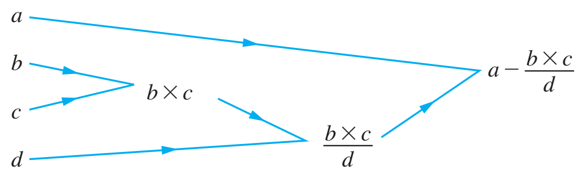
Assuming that all the values given are correctly rounded numbers, calculate an error bound for your answer and write it as a correctly rounded number.

Solution Using a calculator, the answer obtained is

$$6.721 - \frac{4.931 \times 71.28}{89.45} = 2.791\,635\,216$$

To estimate the effect of the rounding error of the data, we first draw up a tree diagram representing the order in which the calculation is performed. Remember that $+$, $-$, $\times$

Figure 1.31



(a)

Label	*Value*	*Absolute error bound*	*Relative error bound*
b	4.931	0.0005	0.0005/4.931 = 0.0001
c	71.28	0.005	0.005/71.28 = 0.000 07
p	351.481 68		0.000 17
d	89.45	0.005	0.005/89.45 = 0.000 06
q	3.929 364 784	0.0009 = 0.000 23 × 3.9	0.000 23
a	6.721	0.0005	
e	2.791 635 216	0.0014	

(b)

and ÷ are binary operations, so only one operation can be performed at each step. Here we are evaluating

$$a - \frac{b \times c}{d} = e$$

We calculate this as $b \times c = p$, then $p \div d = q$ and then $a - q = e$, as shown in Figure 1.31(a). We set this calculation out in a table as shown in Figure 1.31(b), where the arrows show the flow of the error analysis calculation. Thus the value of e lies between 2.790 235… and 2.793 035…, and the answer may be written as 2.7916 ± 0.0015 or as the correctly rounded number 2.79.

The calculations shown in Figure 1.31 indicate the way in which errors may accumulate in simple arithmetical calculations. The error bounds given are rarely extreme and their behaviour is 'random'. This is discussed later in Example 13.31 in the work on statistics.

1.5.3 Exercises

47 State the numbers of decimal places and significant figures of the following correctly rounded numbers:

(a) 980.665 (b) 9.11×10^{-28}
(c) 2.9978×10^{10} (d) 2.00×10^{33}
(e) 1.759×10^{7} (f) 6.67×10^{-8}

48 In a right-angled triangle the height is measured as 1 m and the base as 2 m, both measurements being accurate to the nearest centimetre. Using Pythagoras' theorem, the hypotenuse is calculated as 2.236 07 m. Is this a sensible deduction? What other source of error will occur?

49 Determine the error bound and relative error bound for x, where

(a) $x = 35\,\text{min} \pm 5\,\text{s}$

(b) $x = 35\,\text{min} \pm 4\%$

(c) $x = 0.58$ and x is correctly rounded to 2dp.

50 A value is calculated to be 12.9576, with a relative error bound of 0.0003. Calculate its absolute error bound and give the value as a correctly rounded number with as many significant digits as possible.

51 Using exact arithmetic, compute the values of the expressions below. Assuming that all the numbers given are correctly rounded, find absolute and relative error bounds for each term in the expressions and for your answers. Give the answers as correctly rounded numbers.

(a) $1.316 - 5.713 + 8.010$

(b) 2.51×1.01

(c) $19.61 + 21.53 - 18.67$

52 Evaluate $12.42 \times 5.675/15.63$, giving your answer as a correctly rounded number with the greatest number of significant figures.

53 Evaluate

$$a + b, \quad a - b, \quad a \times b, \quad a/b$$

for $a = 4.99$ and $b = 5.01$. Give absolute and relative error bounds for each answer.

54 Complete the table below for the computation

$$9.21 + (3.251 - 3.115)/0.112$$

and give the result as the correctly rounded answer with the greatest number of significant figures.

Label	*Value*	*Absolute error bound*	*Relative error bound*
a	3.251		
b	3.115		
$a - b$			
c	0.112		
$(a - b)/c$			
d	9.21		
$d + (a - b)/c$			

55 Evaluate $uv/(u + v)$ for $u = 1.135$ and $v = 2.332$, expressing your answer as a correctly rounded number.

56 Working to 4dp, evaluate

$$E = 1 - 1.65 + \tfrac{1}{2}(1.65)^2 - \tfrac{1}{6}(1.65)^3 + \tfrac{1}{24}(1.65)^4$$

(a) by evaluating each term and then summing,

(b) by 'nested multiplication'

$$E = 1 + 1.65(-1 + 1.65(\tfrac{1}{2} + 1.65(-\tfrac{1}{6} + \tfrac{1}{24}(1.65))))$$

Assuming that the number 1.65 is correctly rounded and that all other numbers are exact, obtain error bounds for both answers.

1.5.4 Computer arithmetic

The error estimate outlined in Example 1.44 is a 'worst case' analysis. The actual error will usually be considerably less than the error bound. For example, the maximum error in the sum of 100 numbers, each rounded to three decimal places, is 0.05. This would only occur in the unlikely event that each value has the greatest possible rounding error. In contrast, the chance of the error being as large as one-tenth of this is only about 1 in 20.

When calculations are performed on a computer the situation is modified a little by the limited space available for number storage. Arithmetic is usually performed using floating-point notation. Each number x is stored in the **normal form**

$$x = (\text{sign})b^n(a)$$

where b is the number base, usually 2 or 16, n is an integer, and the **mantissa** a is a proper fraction with a fixed number of digits such that $1/b \leqslant a < 1$. As there are a limited number of digits available to represent the mantissa, calculations will involve intermediate rounding. As a consequence, the order in which a calculation is performed may affect the outcome – in other words the Fundamental Laws of Arithmetic may no longer hold! We shall illustrate this by means of an exaggerated example for a small computer using a decimal representation whose capacity for recording numbers is limited to four figures only. In large-scale calculations in engineering such considerations are sometimes important.

Consider a computer with storage capacity for real numbers limited to four figures; each number is recorded in the form $(\pm)10^n(a)$ where the exponent n is an integer, $0.1 \leqslant a < 1$ and a has four digits. For example,

$$\begin{aligned} \pi &= +10^1(0.3142) \\ -\tfrac{1}{3} &= -10^0(0.3333) \\ 5764 &= +10^4(0.5764) \\ -0.000\,971\,3 &= -10^{-3}(0.9713) \\ 5\,764\,213 &= +10^7(0.5764) \end{aligned}$$

Addition is performed by first adjusting the exponent of the smaller number to that of the larger, then adding the numbers, which now have the same multiplying power of 10, and lastly truncating the number to four digits. Thus $7.182 + 0.053\,81$ becomes

$$\begin{aligned} +10^1(0.7182) + 10^{-1}(0.5381) &= 10^1(0.7182) + 10^1(0.005\,381) \\ &= 10^1(0.723\,581) \\ &= 10^1(0.7236) \end{aligned}$$

With $a = 31.68$, $b = -31.54$ and $c = 83.21$, the two calculations $(a + b) + c$ and $(a + c) + b$ yield different results on this computer:

$$(a + b) + c = 83.35, \qquad (a + c) + b = 83.34$$

Notice how the symbol ‘=’ is being used in the examples above. Sometimes it means ‘equals to 4sf’. This computerized arithmetic is usually called **floating-point arithmetic**, and the number of digits used is normally specified.

1.5.5 Exercises

57 Two possible methods of adding five numbers are

$$(((a + b) + c) + d) + e$$

and

$$(((e + d) + c) + b) + a$$

Using 4dp floating-point arithmetic, evaluate the sum

$$10^1(0.1000) + 10^1(0.1000) - 10^0(0.5000) + 10^0(0.1667) + 10^{-1}(0.4167)$$

by both methods. Explain any discrepancy in the results.

58 Find $(10^{-2}(0.3251) \times 10^{-5}(0.2011))$ and $(10^{-1}(0.2168) \div 10^2(0.3211))$ using four-digit floating-point arithmetic.

59 Find the relative error resulting when four-digit floating-point arithmetic is used to evaluate

$$10^4(0.1000) + 10^2(0.1234) - 10^4(0.1013)$$

1.6 Engineering applications

In this section we illustrate through two examples how some of the results developed in this chapter may be used in an engineering application.

Example 1.48

A continuous belt of length L m passes over two wheels of radii r and R m with their centres a distance l m apart, as illustrated in Figure 1.32. The belt is sufficiently tight for any sag to be negligible. Show that L is given approximately by

$$L \approx 2[l^2 - (R - r)^2]^{1/2} + \pi(R + r)$$

Find the error inherent in this approximation and obtain error bounds for L given the rounded data $R = 1.5$, $r = 0.5$ and $l = 3.5$.

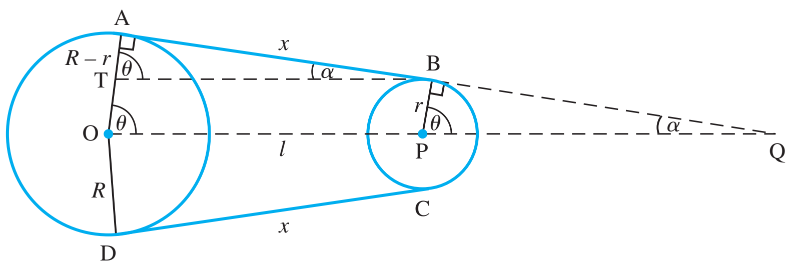


Figure 1.32 Continuous belt of Example 1.48.

Solution The length of the belt consists of the straight sections AB and CD and the wraps round the wheels $\widehat{BC}$ and $\widehat{DA}$. From Figure 1.32 it is clear that BT = OP = l and $\angle$OAB is a right angle. Also, AT = AO − OT and OT = PB so that AT = $R - r$. Applying Pythagoras' theorem to the triangle TAB gives

$$AB^2 = l^2 - (R - r)^2$$

Since the length of an arc of a circle is the product of its radius and the angle (measured in radians) subtended at the centre (see (2.17)), the length of wrap $\widehat{DA}$ is given by

$$(2\pi - 2\theta)R$$

where the angle is measured in radians. By geometry, $\theta = \dfrac{\pi}{2} - \alpha$, so that

$$\widehat{DA} = \pi R + 2R\alpha$$

Similarly, the arc $\widehat{BC} = \pi r - 2r\alpha$. Thus the total length of the belt is

$$L = 2[l^2 - (R - r)^2]^{1/2} + \pi(R + r) + 2(R - r)\alpha$$

Taking the length to be given approximately by

$$L \approx 2[l^2 - (R - r)^2]^{1/2} + \pi(R + r)$$

the error of the approximation is given by $-2(R - r)\alpha$, where the angle α is expressed in radians (remember that error = approximation − true value). The angle α is found by elementary trigonometry, since $\sin\alpha = (R - r)/l$. (Trigonometric functions will be reviewed later in Section 2.6.)

For the (rounded) data given, we deduce, following earlier procedures (see of Section 1.5.2), that for $R = 1.5$, $r = 0.5$ and $l = 3.5$ we have an error interval for α of

$$\left[\sin^{-1}\left(\frac{1.45 - 0.55}{3.55}\right), \sin^{-1}\left(\frac{1.55 - 0.45}{3.45}\right)\right] = [0.256, 0.325]$$

Thus $\alpha = 0.29 \pm 0.035$, and similarly $2(R - r)\alpha = 0.572 \pm 0.111$.

Evaluating the approximation for L gives

$$2[l^2 - (R - r)^2]^{1/2} + \pi(R + r) = 12.991 \pm 0.478$$

and the corresponding value for L is

$$L = 13.563 \pm 0.589$$

Thus, allowing both for the truncation error of the approximation and for the rounding errors in the data, the value 12.991 given by the approximation has an error interval [12.974, 14.152]. Its error bound is the larger of $|12.991 - 14.152|$ and $|12.991 - 12.974|$, that is 1.16. Its relative error is 0.089 and its percent error is 8.9%, where the terminology follows the definitions given earlier (see Section 1.5.2).

Example 1.49

A cable company is to run an optical cable from a relay station, A, on the shore to an installation, B, on an island, as shown in Figure 1.33. The island is 6 km from the shore at its nearest point, P, and A is 9 km from P measured along the shore. It is proposed to run the cable from A along the shoreline and then underwater to the island. It costs 25% more to run the cable underwater than along the shoreline. At what point should the cable leave the shore in order to minimize the total cost?

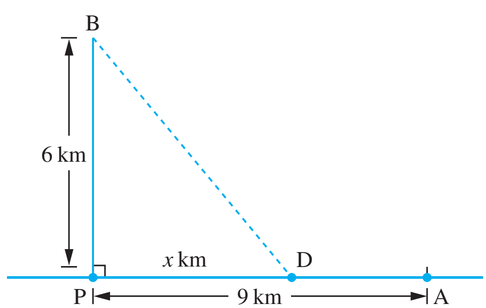


Figure 1.33 Optical cable of Example 1.49.

Solution

Optimization problems frequently occur in engineering and technology and often their solution is found algebraically.

If the cable leaves the shore at D, a distance x km from P, then the underwater distance is $\sqrt{(x^2 + 36)}$ km and the overland distance is $(9 - x)$ km, assuming $0 < x < 9$. If the overland cost of laying the cable is £c per kilometre, then the total cost £C is given by

$$C(x) = [(9 - x) + 1.25\sqrt{(x^2 + 36)}]c$$

We wish to find the value of x, $0 \leqslant x \leqslant 9$, which minimizes C. To do this we first change the variable x by substituting

$$x = 3\left(t - \frac{1}{t}\right)$$

such that $x^2 + 36$ becomes a perfect square:

$$x^2 + 36 = 36 + 9(t^2 - 2 + 1/t^2)$$
$$= 9(t + 1/t)^2$$

Hence C(x) becomes

$$C(t) = [9 - 3(t - 1/t) + 3.75(t + 1/t)]c$$
$$= [9 + 0.75(t + 9/t)]c$$

Using the arithmetic–geometric inequality $x + y \geqslant 2\sqrt{(xy)}$, see (1.4d), we know that

$$t + \frac{9}{t} \geqslant 6$$

and that the equality occurs where $t = 9/t$; that is, where $t = 3$.

Thus the minimum cost is achieved where $t = 3$ and $x = 3(3 - 1/3) = 8$. Hence the cable should leave the shore after laying the cable 1 km from its starting point at A.

1.7 Review exercises (1–25)

1 (a) A formula in the theory of ventilation is

$$Q = \frac{\sqrt{H}}{K}\sqrt{\frac{A^2D^2}{A^2 + D^2}}$$

Express A in terms of the other symbols.

(b) Solve the equation

$$\frac{1}{x+2} - \frac{2}{x} = \frac{3}{x-1}$$

2 Factorize the following:

(a) $ax - 2x - a + 2$ (b) $a^2 - b^2 + 2bc - c^2$

(c) $4k^2 + 4kl + l^2 - 9m^2$ (d) $p^2 - 3pq + 2q^2$

(e) $l^2 + lm + ln + mn$

3 (a) Two small pegs are 8 cm apart on the same horizontal line. An inextensible string of length 16 cm has equal masses fastened at either end and is placed symmetrically over the pegs. The middle point of the string is pulled down vertically until it is in line with the masses. How far does each mass rise?

(b) Find an ‘acceptable’ value of x to three decimal places if the shaded area in Figure 1.34 is 10 square units.

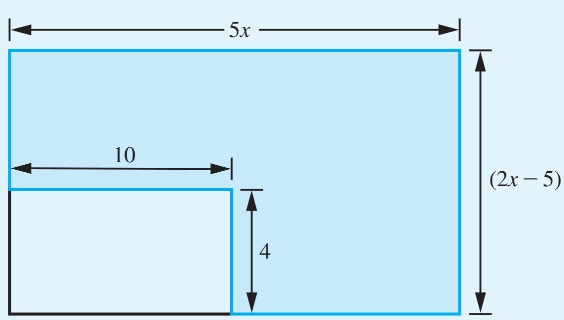


Figure 1.34 Shaded area of Question 3(b).

4 The impedance Z ohms of a circuit containing a resistance R ohms, inductance L henries and capacity C farads, when the frequency of the oscillation is n per second, is given by

$$Z = \sqrt{\left(R^2 + \left(2\pi nL - \frac{1}{2\pi nC}\right)^2\right)}$$

(a) Make L the subject of this formula.

(b) If $n = 50$, $R = 15$ and $C = 10^{-4}$ show that there are two values of L which make $Z = 20$ but only one value of L which will make $Z = 100$. Find the values of Z in each case to two decimal places.

5 Expand out (a) and (b) and rationalize (c) to (e).

(a) $(3\sqrt{2} - 2\sqrt{3})^2$

(b) $(\sqrt{5} + 7\sqrt{3})(2\sqrt{5} - 3\sqrt{3})$

(c) $\dfrac{4 + 3\sqrt{2}}{5 + \sqrt{2}}$

(d) $\dfrac{\sqrt{3} + \sqrt{2}}{2 - \sqrt{3}}$

(e) $\dfrac{1}{1 + \sqrt{2} - \sqrt{3}}$

6 Find integers m and n such that

$$\sqrt{(11 + 2\sqrt{30})} = \sqrt{m} + \sqrt{n}$$

7 Show that

$$\sqrt{(n+1)} - \sqrt{n} = \frac{1}{\sqrt{(n+1)} + \sqrt{n}}$$

and deduce that

$$\sqrt{(n+1)} - \sqrt{n} < \frac{1}{2\sqrt{n}} < \sqrt{n} - \sqrt{(n-1)}$$

for any integer $n \geqslant 1$. Deduce that the sum

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{(9999)}} + \frac{1}{\sqrt{(10\,000)}}$$

lies between 198 and 200.

8 Express each of the following subsets of $\mathbb{R}$ in terms of intervals:

(a) $\{x: 4x^2 - 3 < 4x,\ x \text{ in } \mathbb{R}\}$

(b) $\{x: 1/(x + 2) > 2/(x - 1),\ x \text{ in } \mathbb{R}\}$

(c) $\{x: |x + 1| < 2,\ x \text{ in } \mathbb{R}\}$

(d) $\{x: |x + 1| < 1 + \frac{1}{2}x,\ x \text{ in } \mathbb{R}\}$

9 It is known that of all plane curves that enclose a given area, the circle has the least perimeter. Show that if a plane curve of perimeter L encloses an area A then $4\pi A \leqslant L^2$. Verify this inequality for a square and a semicircle.

10 The arithmetic–geometric inequality

$$\frac{x + y}{2} \geqslant \sqrt{xy}$$

implies

$$\left(\frac{x + y}{2}\right)^2 \geqslant xy$$

Use the substitution $x = \frac{1}{2}(a + b)$, $y = \frac{1}{2}(c + d)$, where a, b, c and $d > 0$, to show that

$$\left(\frac{a + b}{2}\right)\left(\frac{c + d}{2}\right) \leqslant \left(\frac{a + b + c + d}{4}\right)^2$$

and hence that

$$\left(\frac{a + b}{2}\right)^2\left(\frac{c + d}{2}\right)^2 \leqslant \left(\frac{a + b + c + d}{4}\right)^4$$

By applying the arithmetic–geometric inequality to the first two terms of this inequality, deduce that

$$abcd \leqslant \left(\frac{a + b + c + d}{4}\right)^4$$

and hence

$$\frac{a + b + c + d}{4} \geqslant \sqrt[4]{abcd}$$

11 Show that if $a < b$, $b > 0$ and $c > 0$ then

$$\frac{a}{b} < \frac{a + c}{b + c} < 1$$

Obtain a similar inequality for the case $a > b$.

12 (a) If $n = n_1 + n_2 + n_3$ show that

$$\binom{n}{n_1}\binom{n_2 + n_3}{n_2} = \frac{n!}{n_1!\,n_2!\,n_3!}$$

(This represents the number of ways in which n objects may be divided into three groups containing respectively n_1, n_2 and n_3 objects.)

(b) Expand the following expressions

(i) $\left(1-\frac{x}{2}\right)^5$ (ii) $(3-2x)^6$

13 (a) Evaluate $\sum_{n=-2}^{3}[n^{n+1}+3(-1)^n]$

(b) A square grid of dots may be divided up into a set of L-shaped groups as illustrated in Figure 1.35.

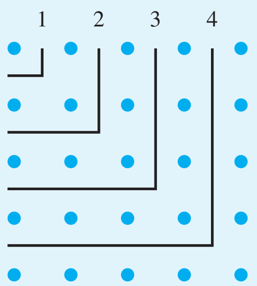


Figure 1.35

How many dots are inside the third L shape? How many extra dots are needed to extend the 3 by 3 square to one of side 4 by 4? How many dots are needed to extend an $(r-1)$ by $(r-1)$ square to one of size r by r? Denoting this number by P_r, use a geometric argument to obtain an expression for $\sum_{r=1}^{n} P_r$ and verify your conclusion by direct calculation in the case $n = 10$.

14 Find the equations of the straight line

(a) which passes through the points $(-6, -11)$ and $(2, 5)$;

(b) which passes through the point $(4, -1)$ and has gradient $\frac{1}{3}$;

(c) which has the same intercept on the y axis as the line in (b) and is parallel to the line in (a).

15 Find the equation of the circle which touches the y axis at the point $(0, 3)$ and passes through the point $(1, 0)$.

16 Find the centres and radii of the following circles:

(a) $x^2 + y^2 + 2x - 4y + 1 = 0$

(b) $4x^2 - 4x + 4y^2 + 12y + 9 = 0$

(c) $9x^2 + 6x + 9y^2 - 6y = 25$

17 For each of the two parabolas

(i) $y^2 = 8x + 4y - 12$, and

(ii) $x^2 + 12y + 4x = 8$

determine

(a) the coordinates of the vertex,

(b) the coordinates of the focus,

(c) the equation of the directrix,

(d) the equation of the axis of symmetry.

Sketch each parabola.

18 Find the coordinates of the centre and foci of the ellipse with equation

$$25x^2 + 16y^2 - 100x - 256y + 724 = 0$$

What are the coordinates of its vertices and the equations of its directrices? Sketch the ellipse.

19 Find the duodecimal equivalent of the decimal number 10.386 23.

20 Show that if $y = x^{1/2}$ then the relative error bound of y is one-half that of x. Hence complete the table in Figure 1.36.

	Value	*Absolute error bound*		*Relative error bound*
a	7.01	0.005	$\longrightarrow$	0.0007
$\sqrt{a}$	2.6476	0.0009	$\longleftarrow$	0.000 35
b	52.13			
$\sqrt{b}$				
c	0.010 11			
$\sqrt{c}$				
d	5.631×10^{11}			
$\sqrt{d}$				

Correctly rounded values	$\sqrt{a}$	$\sqrt{b}$	$\sqrt{c}$	$\sqrt{d}$
	2.65			

Figure 1.36

21 Assuming that all the numbers given are correctly rounded, calculate the positive root together with its error bound of the quadratic equation

$$1.4x^2 + 5.7x - 2.3 = 0$$

Give your answer also as a correctly rounded number.

22 The quantities f, u and v are connected by

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

Find f when $u = 3.00$ and $v = 4.00$ are correctly rounded numbers. Compare the error bounds obtained for f when

(a) it is evaluated by taking the reciprocal of the sum of the reciprocals of u and v,

(b) it is evaluated using the formula

$$f = \frac{uv}{u + v}$$

23 If the number whose decimal representation is 14 732 has the representation $152\,112_b$ to base b, what is b?

24 A milk carton has capacity 2 pints (1136 ml). It is made from a rectangular waxed card using the net shown in Figure 1.37. Show that the total area A (mm^2) of card used is given by

$$A(h, w) = (2w + 145)(h + 80)$$

with $hw = 113\,600/7$. Show that

$$A(h, w) = C(h, w) + \frac{308\,400}{7}$$

where $C(h, w) = 145h + 160w$.

Use the arithmetic–geometric inequality to show that

$$C(h, w) \geqslant 2\sqrt{(160w \times 145h)}$$

with equality when $160w = 145h$. Hence show that the minimum values of $C(h, w)$ and $A(h, w)$ are achieved when $h = 133.8$ and $w = 121.3$. Give these answers to more sensible accuracy.

25 A family of straight lines in the $x-y$ plane is such that each line joins the point $(-p, p)$ on the line $y = -x$ to the point $(10 - p, 10 - p)$ on the line $y = x$, as shown in Figure 1.38, for different values of p. On a piece of graph paper, draw the lines corresponding to $p = 1, 2, 3, \dots, 9$. The resulting family is seen to envelop a curve. Show that the line which joins $(-p, p)$ to $(10 - p, 10 - p)$ has equation

$$5y = 5x - px + 10p - p^2$$

Show that two lines of the family pass through the point (x_0, y_0) if $x_0^2 > 20(y_0 - 5)$, but no lines pass through (x_0, y_0) if $x_0^2 < 20(y_0 - 5)$. Deduce that the enveloping curve of the family of straight lines is

$$y = \tfrac{1}{20}x^2 + 5$$

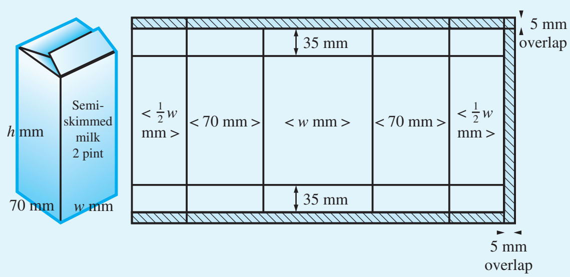


Figure 1.37 Milk carton of Question 24.

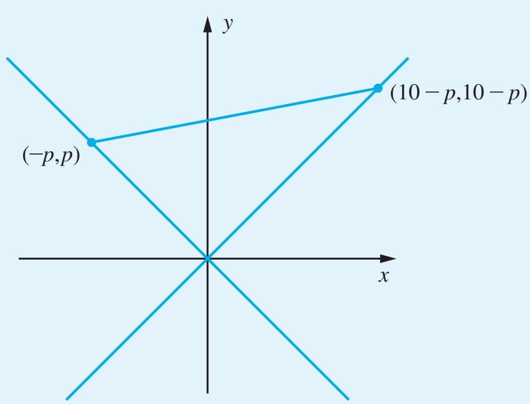


Figure 1.38

2 Functions

Chapter 2 Contents

2.1 Introduction

As we have remarked in the introductory section of Chapter 1, mathematics provides a means of solving the practical problems that occur in engineering. To do this, it uses concepts and techniques that operate on and within the concepts. In this chapter we shall describe the concept of a function – a concept that is both fundamental to mathematics and intuitive. We shall make the intuitive idea mathematically precise by formal definitions and also describe why such formalism is needed for practical problem solving.

The function concept has taken many centuries to evolve. The intuitive basis for the concept is found in the analysis of cause and effect, which underpins developments in science, technology and commerce. As with many mathematical ideas, many people use the concept in their everyday activities without being aware that they are using mathematics, and many would be surprised if they were told that they were. The abstract manner in which the developed form of the concept is expressed by mathematicians often intimidates learners, but the essential idea is very simple. A consequence of the long period of development is that the way in which the concept is described often makes an idiomatic use of words. Ordinary words which in common parlance have many different shades of meaning are used in mathematics with very specific meanings.

The key idea is that of the values of two variable quantities being related. For example, the amount of tax paid depends on the selling price of an item; the deflection of a beam depends on the applied load; the cost of an article varies with the number produced; and so on. Historically, this idea has been expressed in a number of ways. The oldest gave a verbal recipe for calculating the required value. Thus, in the early Middle Ages, a very elaborate verbal recipe was given for calculating the monthly interest payments on a loan which would now be expressed very compactly by a single formula. John Napier, when he developed the logarithm function at the beginning of the seventeenth century, expressed the functional relationship in terms of two particles moving along a straight line. One particle moved with constant velocity and the other with a velocity that depended on its distance from a fixed point on the line. The relationship between the distances travelled by the particles was used to define the logarithms of numbers. This would now be described by the solution of a differential equation. The introduction of algebraic notation led to the representation of functions by algebraic rather than verbal formulae. That produced many theoretical problems. For example, a considerable controversy was caused by Fourier when he used functions that did not have the same algebraic formula for all values of the independent variable. Similarly, the existence of functions that do not have a simple algebraic representation caused considerable difficulties for mathematicians in the early nineteenth century.

2.2 Basic definitions

2.2.1 Concept of a function

The essential idea that flows through all of the developments is that of two quantities whose values are related. One of these variables, the **independent** or **free variable**,

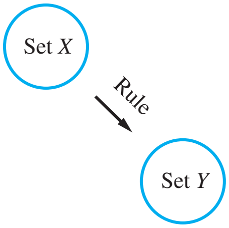


Figure 2.1 Schematic representation of a function.

may take any value in a set of values. The value it actually takes fixes uniquely the value of the second quantity, the **dependent** or **slave variable**. Thus for each value of the independent variable there is one and only one value of the dependent variable. The way in which that value is calculated will vary between functions. Sometimes it will be by means of a formula, sometimes by means of a graph and sometimes by means of a table of values. Here the words 'value' and 'quantity' cover many very different contexts, but in each case what we have is two sets of values X and Y and a rule that assigns to each value x in the set X precisely one value y from the set Y. The elements of X and Y need not be numbers, but the essential idea is that to every x in the set X there corresponds exactly one y in the set Y. Whenever this situation arises we say that there is a **function** f that maps the set X to the set Y. Such a function may be illustrated schematically as in Figure 2.1.

We represent a functional relationship symbolically in two ways: either

$$f{:}x \rightarrow y \quad (x \text{ in } X)$$

or

$$y = f(x) \quad (x \text{ in } X)$$

The first emphasizes the fact that a function f associates each element (value) x of X with exactly one element (value) y of Y: it 'maps x to y'. The second method of notation emphasizes the dependence of the elements of Y on the elements of X under the function f. In this case the value or variable appearing within the brackets is known as the **argument** of the function; we might say 'the argument x of a function $f(x)$'. In engineering it is more common to use the second notation $y = f(x)$ and to refer to this as the function $f(x)$, while modern mathematics textbooks prefer the mapping notation, on the grounds that it is less ambiguous. The set X is called the **domain** of the function and the set Y is called its **codomain**. Knowing the domain and codomain is important in computing. We need to know the type of variables, whether they are integers or reals, and their size. When $y = f(x)$, y is said to be the **image** of x under f. The set of all images $y = f(x)$, x in X, is called the **image set** or **range** of f. It is not necessary for all elements y of the codomain set Y to be images under f. In the terminology to be presented later (see Chapter 6), the range is a subset of the codomain. We may regard x as being a variable that can be replaced by any element of the set X. The rule giving f is then completely determined if we know $f(x)$, and consequently in engineering it is common to refer to the function as being $f(x)$ rather than f. Likewise we can regard $y = f(x)$ as being a variable. However, while x can freely take any value from the set X, the variable $y = f(x)$ depends on the particular element chosen for x. We therefore refer to x as the **free** or **independent** variable and to y as the **slave** or **dependent** variable. The function $f(x)$ is therefore specified completely by the set of ordered pairs (x, y) for all x in X. For real variables a graphical representation of the function may then be obtained by plotting a graph determined by this set of ordered pairs (x, y), with the independent variable x measured along the horizontal axis and the dependent variable y measured along the vertical axis. Obtaining a good graph by hand is not always easy but there are now available excellent graphics facilities on computers and calculators which assist in the task. Even so, some practice is required to ensure that a good choice of 'drawing window' is selected to obtain a meaningful graph.

Example 2.1 For the functions with formulae below, identify their domains, codomains and ranges and calculate values of $f(2)$, $f(-3)$ and $f(-x)$.

(a) $f(x) = 3x^2 + 1$ (b) $f{:}x \to \sqrt{[(x + 4)(3 - x)]}$

Solution (a) The formula for $f(x)$ can be evaluated for all real values of x and so we can take a domain which includes all the real numbers, $\mathbb{R}$. The values obtained are also real numbers, so we may take $\mathbb{R}$ as the codomain. The range of $f(x)$ is actually less than $\mathbb{R}$ in this example because the minimum value of $y = 3x^2 + 1$ occurs at $y = 1$ where $x = 0$. Thus the range of f is the set

$$\{x{:}1 \leqslant x, x \text{ in } \mathbb{R}\} = [1, \infty)$$

Notice the convention here that the set is specified using the *dummy* variable x. We could also write $\{y{:}1 \leqslant y, y \text{ in } \mathbb{R}\}$ – any letter could be used but conventionally x is used. Using the formula we find that $f(2) = 13$, $f(-3) = 28$ and $f(-x) = 3(-x)^2 + 1 = 3x^2 + 1$. The function is *even* (see Section 2.2.6).

(b) The formula $f{:}x \to \sqrt{[(x + 4)(3 - x)]}$ only gives real values for $-4 \leqslant x \leqslant 3$, since we cannot take square roots of negative numbers. Thus the domain of f is $[-4,3]$. Within its domain the function has real values so that its codomain is $\mathbb{R}$ but its range is less than $\mathbb{R}$. The least value of f occurs at $x = -4$ and $x = 3$ when $f(-4) = f(3) = 0$. The largest value of f occurs at $x = -\frac{1}{2}$ when $f(-\frac{1}{2}) = \sqrt{(35)}/2$.

So the range of f in this example is $[0, \sqrt{(35)}/2]$. Using the formula we have $f(2) = \sqrt{6}$, $f(-3) = \sqrt{6}$, $f(-x) = \sqrt{[(4 - x)(x + 3)]}$.

Example 2.2 The function $y = f(x)$ is given by the minimum diameter y of a circular pipe that can contain x circular pipes of unit diameter, where $x = 1, 2, 3, 4, 5, 6, 7$. Find the domain, codomain and range of $f(x)$.

Solution This function is illustrated in Figure 2.2.

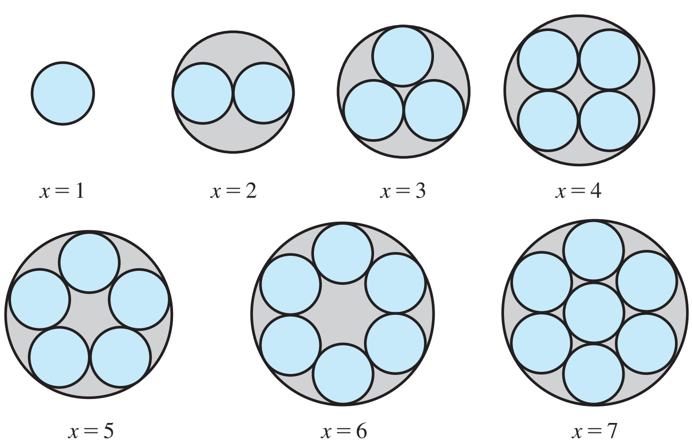


Figure 2.2 Enclosing x circular pipes in a circular pipe.

Here the domain is the set {1, 2, 3, 4, 5, 6, 7} and the codomain is $\mathbb{R}$. Calculating the range is more difficult as there is not a simple algebraic formula relating x and y. From geometry we have

$$f(1) = 1, f(2) = 2, f(3) = 1 + 2/\sqrt{3}, f(4) = 1 + \sqrt{2}, f(5) = \tfrac{1}{4}\sqrt{[2(5 - \sqrt{5})]},$$
$$f(6) = 3, f(7) = 3$$

The range of $f(x)$ is the set of these values.

Example 2.3

The relationship between the temperature T_1 measured in degrees Celsius (°C) and the corresponding temperature T_2 measured in degrees Fahrenheit (°F) is

$$T_2 = \tfrac{9}{5}T_1 + 32$$

Interpreting this as a function with T_1 as the independent variable and T_2 as the dependent variable:

(a) What are the domain and codomain of the function?

(b) What is the function rule?

(c) Plot a graph of the function.

(d) What is the image set or range of the function?

(e) Use the function to convert the following into °F:

(i) 60°C, (ii) 0°C, (iii) −50°C

Solution

(a) Since temperature can vary continuously, the domain is the set $T_1 \geqslant T_0 = -273.16$ (absolute zero). The codomain can be chosen as the set of real numbers $\mathbb{R}$.

(b) The function rule in words is

multiply by $\tfrac{9}{5}$ and then add 32

or algebraically

$$f(T_1) = \tfrac{9}{5}T_1 + 32$$

(c) Since the domain is the set $T_1 \geqslant T_0$, there must be an image for every value of T_1 on the horizontal axis which is greater than −273.16. The graph of the function is that part of the line $T_2 = \tfrac{9}{5}T_1 + 32$ for which $T_1 > -273.16$, as illustrated in Figure 2.3.

(d) Since each value of T_2 is an image of some value T_1 in its domain, it follows that the range of $f(T_1)$ is the set of real numbers greater than −459.69.

(e) The conversion may be done graphically by reading values of the graph, as illustrated by the dashed lines in Figure 2.3, or algebraically using the rule

$$T_2 = \tfrac{9}{5}T_1 + 32$$

giving the values

(i) 140°F, (ii) 32°F, (iii) −58°F

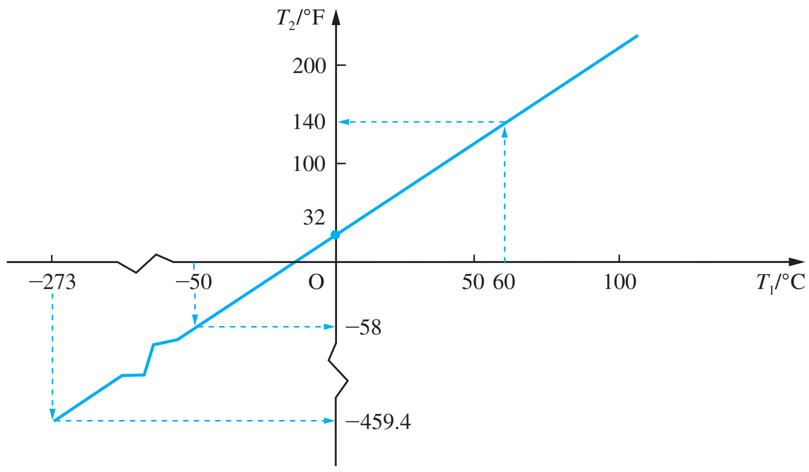


Figure 2.3
Graph of
$T_2 = f(T_1) = \frac{9}{5}T_1 + 32$.

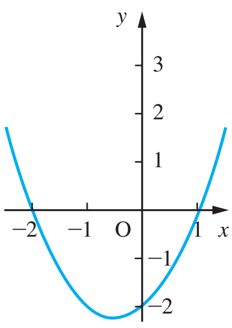


Figure 2.4
Graph of
$y = (x - 1)(x + 2)$.

A value of the independent variable for which the value of a function is zero is called a **zero** of that function. Thus the function $f(x) = (x - 1)(x + 2)$ has two zeros, $x = 1$ and $x = -2$. These correspond to where the graph of the function crosses the x axis, as shown in Figure 2.4. We can see from the diagram that, for this function, its values decrease as the values of x increase from (say) -5 up to $-\frac{1}{2}$, and then its values increase with x. We can demonstrate this algebraically by rearranging the formula for $f(x)$:

$$\begin{aligned} f(x) &= (x - 1)(x + 2) \\ &= x^2 + x - 2 \\ &= (x + \tfrac{1}{2})^2 - \tfrac{9}{4} \end{aligned}$$

From this we can see that $f(x)$ achieves its smallest value $(-\frac{9}{4})$ where $x = -\frac{1}{2}$ and that the value of the function is greater than $-\frac{9}{4}$ both sides of $x = -\frac{1}{2}$ because $(x + \frac{1}{2})^2 \geqslant 0$. The function is said to be a **decreasing function** for $x < -\frac{1}{2}$ and an **increasing function** for $x > -\frac{1}{2}$. More formally, a function is said to be increasing on an interval (a, b) if $f(x_2) > f(x_1)$ when $x_2 > x_1$ for all x_1 and x_2 lying in (a, b). Similarly for decreasing functions, we have $f(x_2) < f(x_1)$ when $x_2 > x_1$.

The value of a function at the point where its behaviour changes from decreasing to increasing is a **minimum** (*plural* **minima**) of the function. Often this is denoted by an asterisk superscript f^* and the corresponding value of the independent variable by x^* so that $f(x^*) = f^*$. Similarly a **maximum** (*plural* **maxima**) occurs when a function changes from being increasing to being decreasing. In many cases the terms maximum and minimum refer to the local values of the function, as illustrated in Example 2.4(a). Sometimes, in practical problems, it is necessary to distinguish between the largest value the function achieves on its domain and the *local maxima* it achieves elsewhere. Similarly for *local minima*. Maxima and minima are jointly referred to as **optimal values** and as **extremal values** of the function.

The point (x^*, f^*) of the graph of $f(x)$ is often called a turning point of the graph, whether it is a maximum or a minimum. These properties will be discussed in more detail later (see Sections 8.2.7 and 8.5). For smooth functions as in Figure 2.5, the tangent to the graph of the function is horizontal at a turning point. This property can be used to locate maxima and minima.

Example 2.4

Draw graphs of the functions below, locating their zeros, intervals in which they are increasing, intervals in which they are decreasing and their optimal values.

(a) $y = 2x^3 + 3x^2 - 12x + 32$ (b) $y = (x - 1)^{2/3} - 1$

Solution

(a) The graph of the function is shown in Figure 2.5. From the graph we can see that the function has one zero at $x = -4$. It is an increasing function on the intervals $-\infty < x < -2$ and $1 < x < \infty$ and a decreasing function on the interval $-2 < x < 1$. It achieves a maximum value of 52 at $x = -2$ and a minimum value of 25 at $x = 1$. In this example the extremal values at $x = -2$ and $x = 1$ are *local maximum* and *local minimum* values. The function is defined on the set of real numbers $\mathbb{R}$. Thus it does not have finite upper and lower values. If the domain were restricted to $[-4, 4]$, say, then the *global minimum* would be $f(-4) = 0$ and the *global maximum* would be $f(4) = 160$.

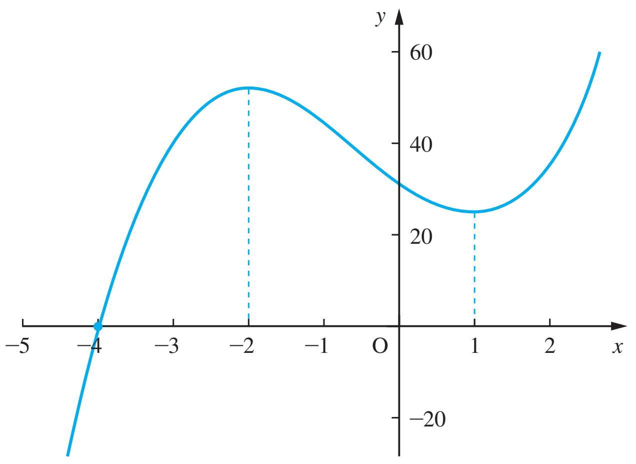


Figure 2.5
Graph of $y = 2x^3 + 3x^2 - 12x + 32$.

(b) The graph of the function is shown in Figure 2.6. (Note that to evaluate $(x - 1)^{2/3}$ on some calculators/computer packages it has to be expressed as $((x - 1)^2)^{1/3}$ for $x < 1$.)

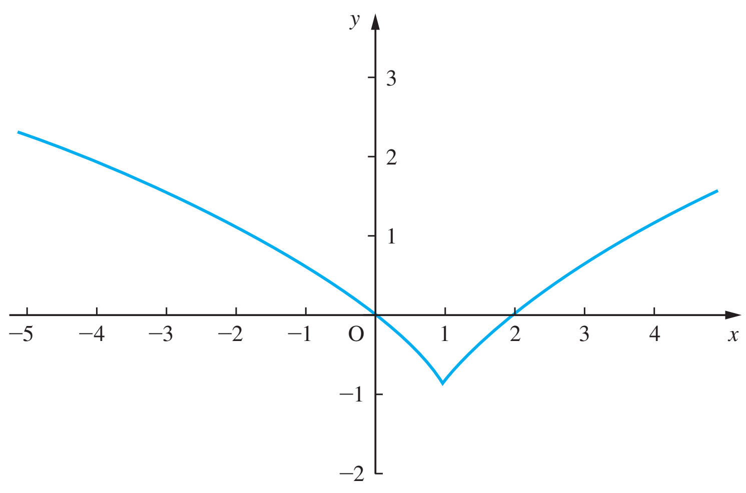


Figure 2.6
Graph of $y = (x - 1)^{2/3} - 1$.

From the graph, we see that the function has two zeros, one at $x = 0$ and the other at $x = 2$. It is a decreasing function for $x < 1$ and an increasing function for $x > 1$. This is obvious algebraically since $(x - 1)^{2/3}$ is greater than or equal to zero. This example also provides an illustration of the behaviour of some algebraic functions at a maximum or minimum value. In contrast to (a) where the function changes from decreasing to increasing at $x = 1$ quite smoothly, in this case the function changes from decreasing to increasing abruptly at $x = 1$. Such a minimum value is called a **cusp**. In this example, the value at $x = 1$ is both a local minimum and a global minimum.

It is important to appreciate the difference between a function and a formula. A function is a mapping that associates one and only one member of the codomain with every member of its domain. It may be possible to express this association, as in Example 2.3, by a formula. Some functions may be represented by different formulae on different parts of their domain.

Example 2.5

A gas company charges its industrial users according to their gas usage. Their tariff is as follows:

Quarterly usage/10^3 units	*Standing charge/£*	*Charge per 10^3 units/£*
0–19.999	200	60
20–49.999	400	50
50–99.999	600	46
$\geqslant 100$	800	44

What is the quarterly charge paid by a user?

Solution

The charge £c paid by a user for a quarter's gas is a function, since for any number of units used there is a unique charge. The charging tariff is expressed in terms of the number u of thousands of units of gas consumed. In this situation the independent variable is the gas consumption u since that determines the charge £c which accrues to the customer. The function f: usage $\rightarrow$ cost must, however, be expressed in the form $c = f(u)$, where

$$f(u) = \begin{cases} 200 + 60u & (0 \leqslant u < 20) \\ 400 + 50u & (20 \leqslant u < 50) \\ 600 + 46u & (50 \leqslant u < 100) \\ 800 + 44u & (100 \leqslant u) \end{cases}$$

Functions that are represented by different formulae on different parts of their domains arise frequently in engineering and management applications.

The basic MATLAB package is primarily a number-crunching package. Symbolic manipulation and algebra can be undertaken by the Symbolic Math Toolbox, which incorporates many MAPLE commands to implement the algebraic work. Consequently, most of the commands in Symbolic Math Toolbox are identical to the MAPLE commands. In order to use any symbolic variables, such as x and y, in MATLAB these must be declared by entering a command, such as `syms x y;`. Inserting a semicolon at the end of a statement suppresses display on screen of the output to the command.

The MATLAB operators for the basic arithmetic operations are + for addition, – for subtraction, * for multiplication, / for division and ^ for power. The colon command `x = a:dx:b` generates an array of numbers which are the values of x between a and b in steps of dx. For example, the command

```
x = 0:0.1:1
```

generates the array

```
x = 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
```

When using the operations of multiplication, division and power on such arrays *, / and ^ are replaced respectively by .*, ./ and .^ in which the 'dot' implies element by element operations. For example, if $x = [1\ 2\ 3]$ and $y = [4\ -3\ 5]$ are two arrays then `x.*y` denotes the array $[4\ -6\ 15]$ and `x.^2` denotes the array $[1\ 4\ 9]$. Note that to enter an array it must be enclosed within square brackets `[ ]`.

To plot the graph of $y = f(x)$, $a \leqslant x \leqslant b$, an array of x values is first produced and then a corresponding array of y values is produced. Then the command `plot(x,y)` plots a graph of y against x. Check that the sequence of commands

```
x = -5:0.1:3;
y = 2*x.^3 + 3*x.^2 - 12*x + 32;
plot(x,y)
```

plots the graph of Figure 2.5. Entering a further command

```
grid
```

draws gridlines on the existing plot. The following commands may be used for labelling the graph:

`title('text')`	prints 'text' at the top of the plot
`xlabel('text')`	labels the x axis with 'text'
`ylabel('text')`	labels the y axis with 'text'

Plotting the graphs of $y_1 = f(x)$ and $y_2 = g(x)$, $a \leqslant x \leqslant b$, can be achieved using the commands

```
x = [a:dx:b]'; y1 = f(x); y2 = g(x);
plot(x,y1, '-',x,y2, '- -')
```

with '-' and '- -' indicating that the graph of $y_1 = f(x)$ will appear as a 'solid line' and that of $y_2 = g(x)$ as a 'dashed line'. These commands can be extended to include more than two graphs as well as colour. To find out more, use the *help* facility in MATLAB.

Using the Symbolic Math Toolbox the `sym` command enables us to construct symbolic variables and expressions. For example,

```
x = sym('x')
```

creates the variable x that prints as x; whilst the command

```
f = sym(2*x + 3)
```

assigns the symbolic expression $2x + 3$ to the variable f. If f includes parameters then these must be declared as symbolic terms at the outset. For example, the sequence of commands

```
syms x a b
f = sym(a*x + b)
```

prints

```
f = ax + b
```

(Note the use of spacing when specifying variables under `syms`.)

The command `ezplot(y)` produces the plot of $y = f(x)$, making a reasonable choice for the range of the x axis and resulting scale of the y axis, the default domain of the x axis being $-2\pi \leqslant x \leqslant 2\pi$. The domain can be changed to $a \leqslant x \leqslant b$ using the command `ezplot(y,[a,b])`. Check that the commands

```
syms x
y = sym(2*x^3 + 3*x^2 - 12*x + 32);
ezplot(y,[-5,3])
```

reproduce the graph of Figure 2.5 and that the commands

```
syms x
y = sym(((x - 1)^2)^(1/3) - 1)
ezplot(y,[-5,3])
```

reproduce the graph of Figure 2.6. (Note that in the second case the function is expressed in the form indicated in the solution to Example 2.4(b).)

2.2.2 Exercises

Check your answers using MATLAB whenever possible.

1 Determine the largest valid domains for the functions whose formulae are given below. Identify the corresponding codomains and ranges and evaluate $f(5)$, $f(-4)$, $f(-x)$.

(a) $f(x) = \sqrt{(25 - x^2)}$ (b) $f: x \rightarrow \sqrt[3]{(x + 3)}$

2 A straight horizontal road is to be constructed through rough terrain. The width of the road is to be 10 m, with the sides of the embankment sloping at 1 (vertical) in 2 (horizontal), as shown in Figure 2.7. Obtain a formula for the cross-sectional area of the road and its embankment, taken at right angles to the road, where the rough ground lies at a depth x below the level of the proposed road. Use your formula to complete the table below, and draw a graph to represent this function.

x/m	0	1	2	3	4	5
Area/m^2	0		28			100

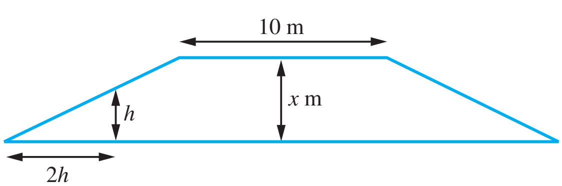


Figure 2.7

What is the value given by the formula when $x = -2$, and what is the meaning of that value?

3 A hot-water tank has the form of a circular cylinder of internal radius r, topped by a hemisphere as shown in Figure 2.8. Show that the internal surface area A is given by

$$A = 2\pi rh + 3\pi r^2$$

and the volume V enclosed is

$$V = \pi r^2 h + \tfrac{2}{3}\pi r^3$$

Find the formula relating the value of A to the value of r for tanks with capacity 0.15 m^3. Complete the table below for A in terms of r and draw a graph to represent the function.

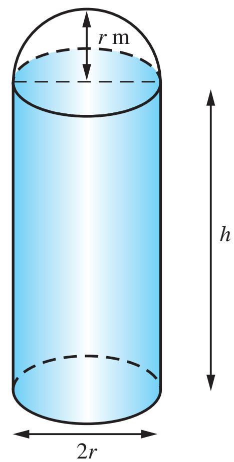


Figure 2.8

r/m	0.10	0.15	0.20	0.25	0.30	0.35	0.40
A/m^2	3.05		1.71			1.50	

The cost of the tank is proportional to the amount of metal used in its manufacture. Estimate the value of r that will minimize that cost, carefully listing the assumptions you make in your analysis. [Recall: the volume of a sphere of radius a is $4\pi a^3/3$ and its surface area is $4\pi a^2$.]

4 An oil storage tank has the form of a circular cylinder with its axis horizontal, as shown in Figure 2.9. The volume of oil in the tank when the depth is h is given in the table below.

h/m	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
V/1000 l	7.3	19.7	34.4	50.3	66.1	80.9	93.9	100.5

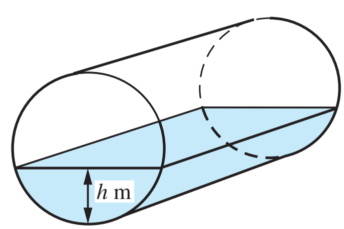


Figure 2.9

Draw a careful graph of V against h, and use it to design the graduation marks on a dipstick to be used to assess the volume of oil in the tank.

5 The initial cost of buying a car is £6000. Over the years, its value depreciates and its running costs increase, as shown in the table below.

t	1	2	3	4	5	6
Value after t years	4090	2880	2030	1430	1010	710
Running cost in year t	600	900	1200	1500	1800	2100

Draw up a table showing (a) the cumulative running cost after t years, (b) the total cost (that is, running cost plus depreciation) after t years and (c) the average cost per year over t years. Estimate the optimal time to replace the car.

6 Plot graphs of the functions below, locating their zeros, intervals in which they are increasing, intervals in which they are decreasing and their optimal values.

(a) $y = x(x - 2)$ (b) $y = 2x^3 - 3x^2 - 12x + 20$

(c) $y = x^2(x^2 - 2)$ (d) $y = 1/[x(x - 2)]$

2.2.3 Inverse functions

In some situations we may need to use the functional dependence in the reverse sense. For example, we may wish to use the function

$$T_2 = f(T_1) = \tfrac{9}{5}T_1 + 32 \tag{2.1}$$

of Example 2.3, relating T_2 in °F to the corresponding T_1 in °C to convert degrees Fahrenheit to degrees Celsius. In this simple case we can rearrange the relationship (2.1) algebraically

$$T_1 = \tfrac{5}{9}(T_2 - 32)$$

giving us the function

$$T_1 = g(T_2) = \tfrac{5}{9}(T_2 - 32) \tag{2.2}$$

having T_2 as the independent variable and T_1 as the dependent variable. We may then use this to convert degrees Fahrenheit into degrees Celsius.

Looking more closely at the two functions $f(T_1)$ and $g(T_2)$ associated with (2.1) and (2.2), we have the function rule for $f(T_1)$ as

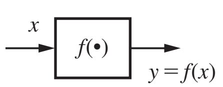
multiply by $\tfrac{9}{5}$ and then add 32

If we reverse the process, we have the rule

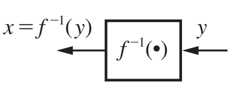
take away 32 and then multiply by $\tfrac{5}{9}$

which is precisely the function rule for $g(T_2)$. Thus the function $T_1 = g(T_2)$ reverses the operations carried out by the function $T_2 = f(T_1)$, and for this reason is called the **inverse function** of $T_2 = f(T_1)$.

In general, the inverse function of a function f is a function that reverses the operations carried out by f. It is denoted by f^{-1}. Writing $y = f(x)$, the function f may be represented by the block diagram of Figure 2.10(a), which indicates that the function operates on the input variable x to produce the output variable $y = f(x)$. The inverse function f^{-1} will reverse the process, and will take the value of y back to the original corresponding values of x. It can be represented by the block diagram of Figure 2.10(b).



(a)



(b)

Figure 2.10
Block diagram of (a) function and (b) inverse function.

We therefore have

$$x = f^{-1}(y), \quad \text{where } y = f(x) \tag{2.3}$$

that is, the independent variable x for f acts as the dependent variable for f^{-1}, and correspondingly the dependent variable y for f becomes the independent variable for f^{-1}. At the same time the range of f becomes the domain of f^{-1} and the domain of f becomes the range of f^{-1}.

Since it is usual to denote the independent variable of a function by x and the dependent variable by y, we interchange the variables x and y in (2.3) and define the inverse function by

$$\text{if } y = f^{-1}(x) \quad \text{then } x = f(y) \tag{2.4}$$

Again in engineering it is common to denote an inverse function by $f^{-1}(x)$ rather than f^{-1}. Writing x as the independent variable for both $f(x)$ and $f^{-1}(x)$ sometimes leads to confusion, so you need to be quite clear as to what is meant by an inverse function. It is also important not to confuse $f^{-1}(x)$ with $[f(x)]^{-1}$, which means $1/f(x)$. You also then need to watch out for values x at which $f(x) = 0$ and act accordingly.

Finding an explicit formula for $f^{-1}(x)$ is often impossible and its values are calculated by special numerical methods. Sometimes it is possible to find the formula for $f^{-1}(x)$ by algebraic methods. We illustrate the technique in the next two examples.

Example 2.6 Obtain the inverse function of the real function $y = f(x) = \frac{1}{5}(4x - 3)$.

Solution Here the formula for the inverse function can be found algebraically. First rearranging

$$y = f(x) = \tfrac{1}{5}(4x - 3)$$

to express x in terms of y gives

$$x = f^{-1}(y) = \tfrac{1}{4}(5y + 3)$$

Interchanging the variables x and y then gives

$$y = f^{-1}(x) = \tfrac{1}{4}(5x + 3)$$

as the inverse function of

$$y = f(x) = \tfrac{1}{5}(4x - 3)$$

As a check, we have

$$f(2) = \tfrac{1}{5}(4 \times 2 - 3) = 1$$

while

$$f^{-1}(1) = \tfrac{1}{4}(5 \times 1 + 3) = 2$$

Example 2.7 Obtain the inverse function of $y = f(x) = \dfrac{x+2}{x+1}$, $x \neq -1$.

Solution We rearrange $y = \dfrac{x+2}{x+1}$ to obtain x in terms of y. (Notice that y is not defined where $x = -1$.) Thus

$$y(x + 1) = x + 2 \quad \text{so that} \quad x(y - 1) = 2 - y$$

giving $x = \dfrac{2-y}{y-1}$, $y \neq 1$. (Notice that x is not defined where $y = 1$. Putting $y = 1$ into the formula for y results in the equation $x + 1 = x + 2$ which is not possible.) Thus

$$f^{-1}(x) = \frac{2-x}{x-1}, \quad x \neq 1$$

If we are given the graph of $y = f(x)$ and wish to obtain the graph of the inverse function $y = f^{-1}(x)$ then what we really need to do is interchange the roles of x and y. Thus we need to manipulate the graph of $y = f(x)$ so that the x and y axes are interchanged. This can be achieved by taking the mirror image in the line $y = x$ and relabelling the axes as illustrated in Figures 2.11(a) and (b). It is important to recognize that the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetrical about the line $y = x$, since this property is frequently used in mathematical arguments. Notice that the x and y axes have the same scale.

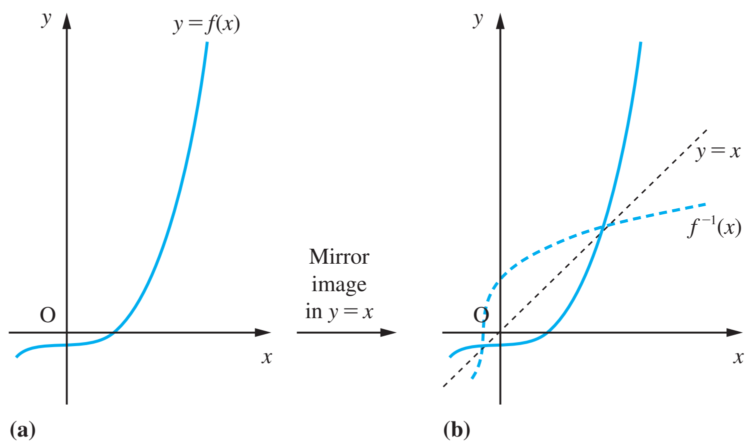


Figure 2.11 The graph of $y = f^{-1}(x)$.

Example 2.8 Obtain the graph of $f^{-1}(x)$ when (a) $f(x) = \frac{9}{5}x + 32$, (b) $f(x) = \dfrac{x+2}{x+1}$, $x \neq -1$, (c) $f(x) = x^2$.

Solution (a) This is the formula for converting the temperature measured in °C to the temperature in °F and its graph is shown by the blue line in Figure 2.12(a). Reflecting the graph in the line $y = x$ yields the graph of the inverse function $y = g(x) = \frac{5}{9}(x - 32)$ as illustrated by the black line in Figure 2.12(a).

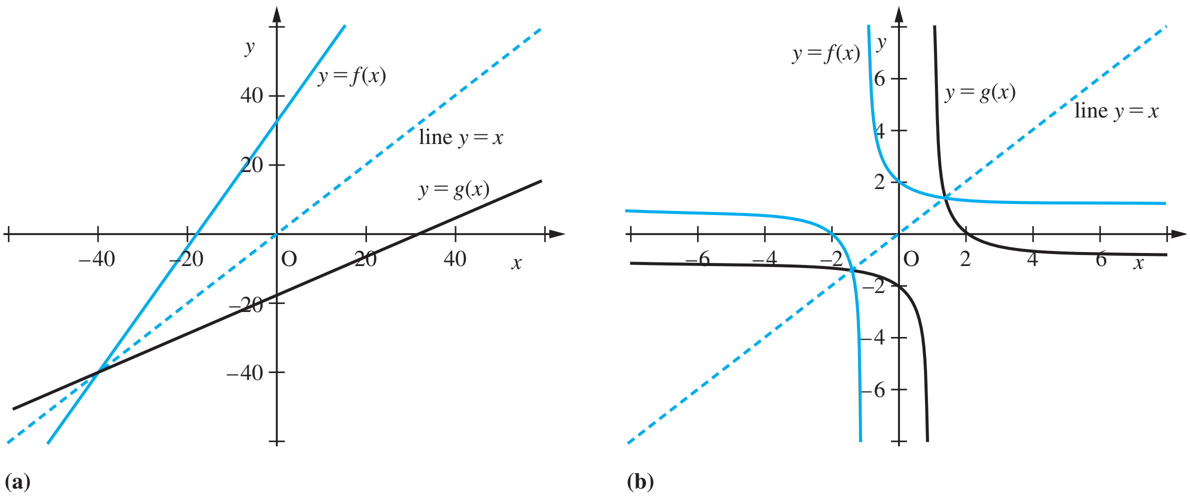
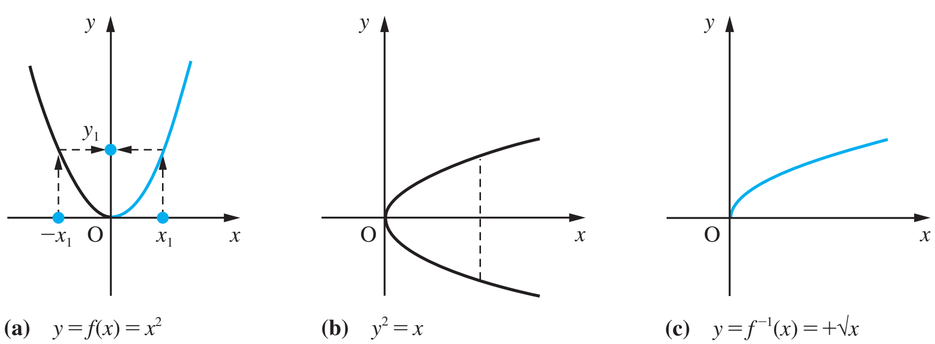


Figure 2.12 (a) Graph of $f(x) = \frac{9}{5}x + 32$ and its inverse $g(x)$. (b) Graph of $f(x) = \dfrac{x+2}{x+1}$ and its inverse $g(x)$.

(b) The graph of $y = f(x) = \dfrac{x+2}{x+1}$, $x \neq -1$, is shown in blue in Figure 2.12(b). The graph of its inverse function $y = g(x) = \dfrac{2-x}{x-1}$, $x \neq 1$, can be seen as the mirror image illustrated in black in Figure 2.12(b).

(c) The graph of $y = x^2$ is shown in Figure 2.13(a). Its mirror image in the line $y = x$ gives the graph of Figure 2.13(b). We note that this graph is not representative of a function according to our definition, since for all values of $x > 0$ there are two images – one positive and one negative – as indicated by the dashed line. This follows because $y = x^2$ corresponds to $x = +\sqrt{y}$ or $x = -\sqrt{y}$. In order to avoid this ambiguity, we define the inverse function of $f(x) = x^2$ to be $f^{-1}(x) = +\sqrt{x}$, which corresponds to the upper half of the graph as illustrated in Figure 2.13(c). $\sqrt{x}$ therefore denotes a positive number (cf. calculators), so the range of $\sqrt{x}$ is $x \geqslant 0$. Thus the inverse function of $y = f(x) = x^2$ $(x \geqslant 0)$ is $y = f^{-1}(x) = \sqrt{x}$. Note that the domain of $f(x)$ had to be restricted to $x \geqslant 0$ in order that an inverse could be defined. In modern usage, the symbol $\sqrt{x}$ denotes a positive number.

Figure 2.13
Graphs of $f(x) = x^2$ and its inverse.



We see from Example 2.8(c) that there is no immediate inverse function corresponding to $f(x) = x^2$. This arises because for the function $f(x) = x^2$ there is a codomain element that is the image of two domain elements x_1 and $-x_1$, as indicated by the dashed arrowed lines in Figure 2.13(a). That is, $f(x_1) = f(-x_1) = y_1$. If a function $y = f(x)$ is to have an immediate inverse $f^{-1}(x)$, without any imposed conditions, then *every* element of its range must occur *precisely once* as an image under $f(x)$. Such a function is known as a one-to-one (1:1) injective function.

2.2.4 Composite functions

In many practical problems the mathematical model will involve several different functions. For example, the kinetic energy T of a moving particle is a function of its velocity v, so that

$$T = f(v)$$

Also, the velocity v itself is a function of time t, so that

$$v = g(t)$$

Clearly, by eliminating v, it is possible to express the kinetic energy as a function of time according to

$$T = f(g(t))$$

A function of the form $y = f(g(x))$ is called a **function of a function** or a **composite** of the functions $f(x)$ and $g(x)$. In modern mathematical texts it is common to denote the composite function by $f \circ g$ so that

$$y = f \circ g(x) = f(g(x)) \tag{2.5}$$

We can represent the composite function (2.5) schematically by the block diagram of Figure 2.14, where $u = g(x)$ is called the intermediate variable.

It is important to recognize that the composition of functions is not in general commutative. That is, for two general functions $f(x)$ and $g(x)$

$$f(g(x)) \neq g(f(x))$$

Algebraically, given two functions $y = f(x)$ and $y = g(x)$, the composite function $y = f(g(x))$ may be obtained by replacing x in the expression for $f(x)$ by $g(x)$. Likewise, the composite function $y = g(f(x))$ may be obtained by replacing x in the expression for $g(x)$ by $f(x)$.

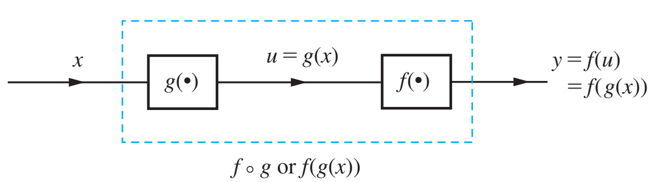


Figure 2.14 The composite function $f(g(x))$.

Example 2.9 If $y = f(x) = x^2 + 2x$ and $y = g(x) = x - 1$, obtain the composite functions $f(g(x))$ and $g(f(x))$.

Solution To obtain $f(g(x))$, replace x in the expression for $f(x)$ by $g(x)$, giving

$$y = f(g(x)) = (g(x))^2 + 2(g(x))$$

But $g(x) = x - 1$, so that

$$\begin{aligned} y = f(g(x)) &= (x - 1)^2 + 2(x - 1) \\ &= x^2 - 2x + 1 + 2x - 2 \end{aligned}$$

That is,

$$f(g(x)) = x^2 - 1$$

Similarly,

$$\begin{aligned} y = g(f(x)) &= (f(x)) - 1 \\ &= (x^2 + 2x) - 1 \end{aligned}$$

That is,

$$g(f(x)) = x^2 + 2x - 1$$

Note that this example confirms the result that, in general, $f(g(x)) \neq g(f(x))$.

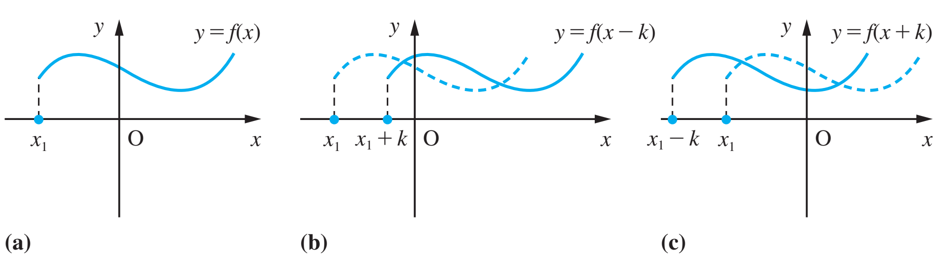
Given a function $y = f(x)$, two composite functions that occur frequently in engineering are

$$y = f(x + k) \quad \text{and} \quad y = f(x - k)$$

where k is a positive constant. As illustrated in Figures 2.15(b) and (c), the graphs of these two composite functions are readily obtained given the graph of $y = f(x)$ as in Figure 2.15(a). The graph of $y = f(x - k)$ is obtained by displacing the graph of $y = f(x)$ by k units to the right, while the graph of $y = f(x + k)$ is obtained by displacing the graph of $y = f(x)$ by k units to the left.

Viewing complicated functions as composites of simpler functions often enables us to 'get to the heart' of a practical problem, and to obtain and understand the solution. For example, recognizing that $y = x^2 + 2x - 3$ is the composite function $y = (x + 1)^2 - 4$ tells us that the function is essentially the squaring function. Its graph is a parabola with minimum point at $x = -1$, $y = -4$ (rather than at $x = 0$, $y = 0$). A similar process

Figure 2.15 Graphs of $f(x)$, $f(x - k)$ and $f(x + k)$, with $k > 0$.



of reducing a complicated problem to a simpler one occurred in the solution of the practical problem discussed in Example 1.49.

Example 2.10

An open conical container is made from a sector of a circle of radius 10 cm as illustrated in Figure 2.16, with sectional angle θ (radians). The capacity $C\,\text{cm}^3$ of the cone depends on θ. Find the formula for C in terms of θ and the simplest associated function that could be studied if we wish to maximize C with respect to θ.

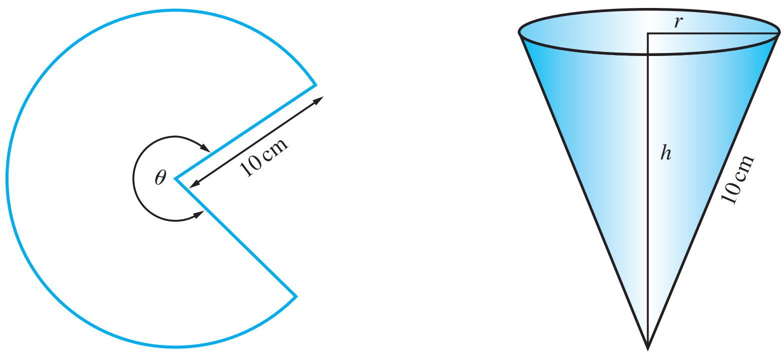


Figure 2.16 Conical container of Example 2.10.

Solution

Let the cone have base radius r cm and height h cm. Then its capacity is given by $C = \frac{1}{3}\pi r^2 h$ with r and h dependent upon the sectorial angle θ (since the perimeter of the sector has to equal the circumference of the base of the cone). Thus, by Pythagoras' theorem,

$$10\theta = 2\pi r \quad \text{and} \quad h^2 = 10^2 - r^2$$

so that

$$C(\theta) = \frac{1}{3}\pi\left(\frac{10\theta}{2\pi}\right)^2\left[10^2 - \left(\frac{10\theta}{2\pi}\right)^2\right]^{1/2}$$

$$= \frac{1000}{3}\pi\left(\frac{\theta}{2\pi}\right)^2\left[1 - \left(\frac{\theta}{2\pi}\right)^2\right]^{1/2}, \quad 0 \leqslant \theta \leqslant 2\pi$$

Maximizing $C(\theta)$ with respect to θ is essentially the same problem as maximizing

$$D(x) = x(1 - x)^{1/2}, \quad 0 \leqslant x \leqslant 1$$

(where $x = (\theta/2\pi)^2$).

Maximizing $D(x)$ with respect to x is essentially the same problem as maximizing

$$E(x) = x^2(1 - x), \quad 0 \leqslant x \leqslant 1$$

which is considerably easier than the original problem.

Plotting the graph of $E(x)$ suggests that it has a minimum at $x = \frac{2}{3}$ where its value is $\frac{4}{27}$. We can prove that this is true by showing that the horizontal line $y = \frac{4}{27}$ is a tangent to the graph at $x = \frac{2}{3}$; that is, the line cuts the graph at two coincident points at $x = \frac{2}{3}$.

Setting $x^2(1 - x) = \frac{4}{27}$ gives $27x^3 - 27x^2 + 4 = 0$ which factorizes into

$$(3x - 2)^2(3x + 1) = 0$$

Thus the equation has a double root at $x = \frac{2}{3}$ and a single root at $x = -\frac{1}{3}$. Thus $E(x)$ has a maximum at $x = \frac{2}{3}$ and the corresponding optimal value of θ is $2\pi\sqrt{(\frac{2}{3})}$. (Later, in Section 8.5 (see also Question 5 in Review exercises 8.13), we shall consider theoretical methods of confirming such results.)

When we compose a function with its inverse function, we usually obtain the identity function $y = x$. Thus from Example 2.6, we have

$$f(x) = \tfrac{1}{5}(4x - 3) \quad \text{and} \quad f^{-1}(x) = \tfrac{1}{4}(5x + 3)$$

and

$$f(f^{-1}(x)) = \tfrac{1}{5}\{4[\tfrac{1}{4}(5x + 3)] - 3\} = x$$

and

$$f^{-1}(f(x)) = \tfrac{1}{4}\{5[\tfrac{1}{5}(4x - 3)] + 3\} = x$$

We need to take care with the exceptional cases that occur, like the square root function, where the inverse function is defined only after restricting the domain of the original function. Thus for $f(x) = x^2$ $(x \geqslant 0)$ and $f^{-1}(x) = \sqrt{x}$ $(x \geqslant 0)$, we obtain

$$f(f^{-1}(x)) = x, \quad \text{for } x \geqslant 0 \text{ only}$$

and

$$f^{-1}(f(x)) = \begin{cases} x, & \text{for } x \geqslant 0 \\ -x, & \text{for } x \leqslant 0 \end{cases}$$

2.2.5 Exercises

7 A function $f(x)$ is defined by $f(x) = \frac{1}{2}(10^x + 10^{-x})$, for x in $\mathbb{R}$. Show that

(a) $2(f(x))^2 = f(2x) + 1$

(b) $2f(x)f(y) = f(x + y) + f(x - y)$

8 Draw separate graphs of the functions f and g where

$$f(x) = (x + 1)^2 \text{ and } g(x) = x - 2$$

The functions F and G are defined by

$$F(x) = f(g(x)) \text{ and } G(x) = g(f(x))$$

Find formulae for $F(x)$ and $G(x)$ and sketch their graphs. What relationships do the graphs of F and G bear to those of f and g?

9 A function f is defined by

$$f(x) = \begin{cases} 0 & (x < -1) \\ x + 1 & (-1 \leqslant x < 0) \\ 1 - x & (0 \leqslant x \leqslant 1) \\ 0 & (x > 1) \end{cases}$$

Sketch on separate diagrams the graphs of $f(x)$, $f(x + \frac{1}{2})$, $f(x + 1)$, $f(x + 2)$, $f(x - \frac{1}{2})$, $f(x - 1)$ and $f(x - 2)$.

10 Find the inverse function (if it is defined) of the following functions:

(a) $f(x) = 2x - 3 \quad (x \text{ in } \mathbb{R})$

(b) $f(x) = \dfrac{2x-3}{x+4} \quad (x \text{ in } \mathbb{R}, x \neq -4)$

(c) $f(x) = x^2 + 1 \quad (x \text{ in } \mathbb{R})$

If $f(x)$ does not have an inverse function, suggest a suitable restriction of the domain of $f(x)$ that will allow the definition of an inverse function.

11 Show that

$$f(x) = \frac{2x-3}{x+4}$$

may be expressed in the form

$$f(x) = g(h(l(x)))$$

where

$$l(x) = x + 4$$

$$h(x) = 1/x$$

$$g(x) = 2 - 11x$$

Interpret this result graphically.

12 The stiffness of a rectangular beam varies directly with the cube of its height and directly with its breadth. A beam of rectangular section is to be cut from a circular log of diameter d. Show that the optimal choice of height and breadth of the beam in terms of its stiffness is related to the value of x which maximizes the function

$$E(x) = x^3(d^2 - x), \quad 0 \leqslant x \leqslant d^2$$

13 A beam is used to support a building as shown in Figure 2.17. The beam has to pass over a 3 m brick wall which is 2 m from the building. Show that the minimum length of the beam is associated with the value of x which minimizes

$$E(x) = (x+2)^2\left(1 + \frac{9}{x^2}\right)$$

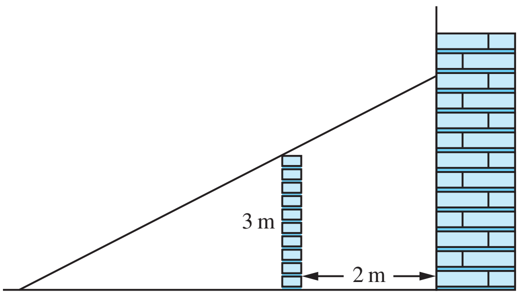


Figure 2.17 Beam of Question 13.

2.2.6 Odd, even and periodic functions

Some commonly occurring functions in engineering contexts have the special properties of oddness or evenness or periodicity. These properties are best understood from the graphs of the functions.

An **even function** is one that satisfies the functional equation

$$f(-x) = f(x)$$

Thus the value of $f(-2)$ is the same as $f(2)$, and so on. The graph of such a function is symmetrical about the y axis, as shown in Figure 2.18.

In contrast, an **odd function** has a graph which is antisymmetrical about the origin, as shown in Figure 2.19, and satisfies the equation

$$f(-x) = -f(x)$$

We notice that $f(0) = 0$ or is undefined.

Polynomial functions like $y = x^4 - x^2 - 1$, involving only even powers of x, are examples of even functions, while those like $y = x - x^5$, involving only odd powers of x, provide examples of odd functions. Of course, not all functions have the property of oddness or evenness.

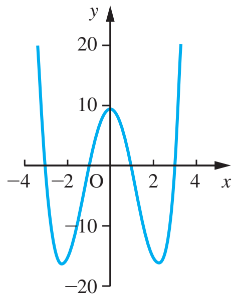


Figure 2.18 Graph of an even function.

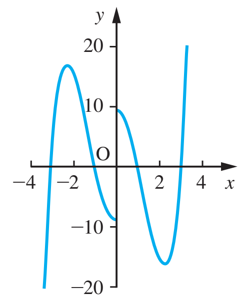


Figure 2.19 Graph of an odd function.

Example 2.11

Which of the functions $y = f(x)$ whose graphs are shown in Figure 2.20 are odd, even or neither odd nor even?

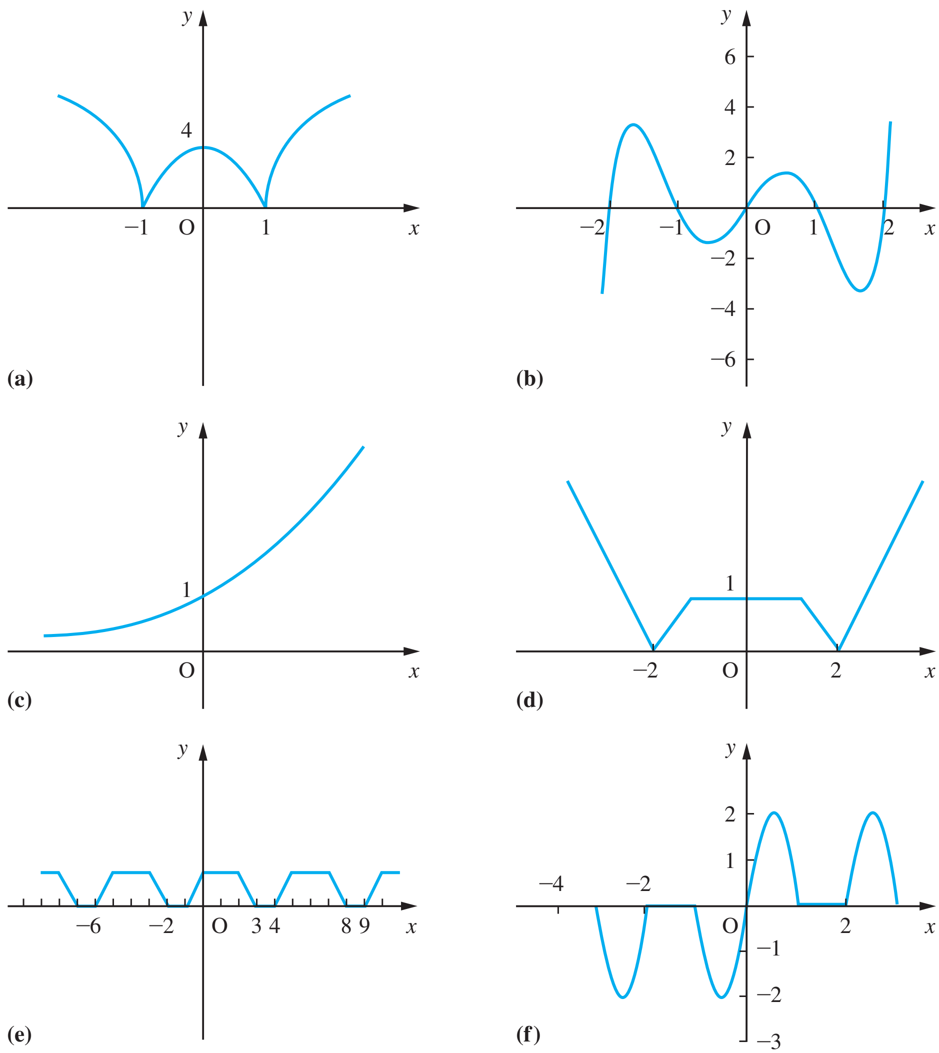
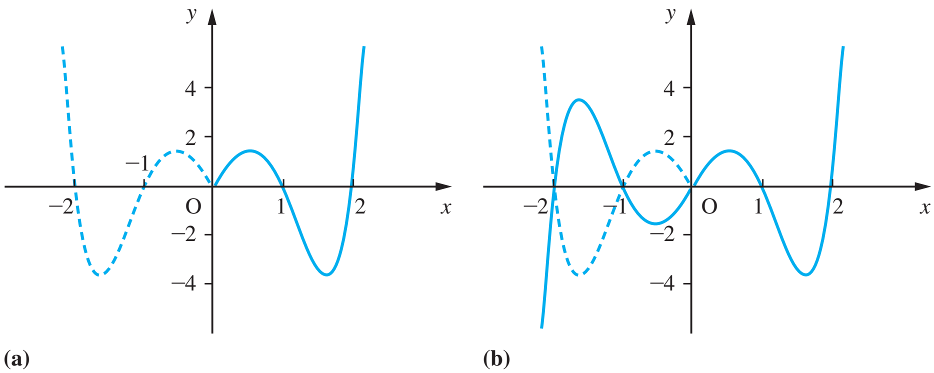


Figure 2.20 Graphs of Example 2.11.

Solution (a) The graph for $x < 0$ is the mirror image of the graph for $x > 0$ when the mirror is placed on the y axis. Thus the graph represents an even function.

(b) The mirror image of the graph for $x > 0$ in the y axis is shown in Figure 2.21(a). Now reflecting that image in the x axis gives the graph shown in Figure 2.21(b). Thus Figure 2.20(b) represents an odd function since its graph is antisymmetrical about the origin.

Figure 2.21



(c) The graph is neither symmetrical nor antisymmetrical about the origin, so the function it represents is neither odd nor even.

(d) The graph is symmetrical about the y axis so it is an even function.

(e) The graph is neither symmetrical nor antisymmetrical about the origin, so it is neither an even nor an odd function.

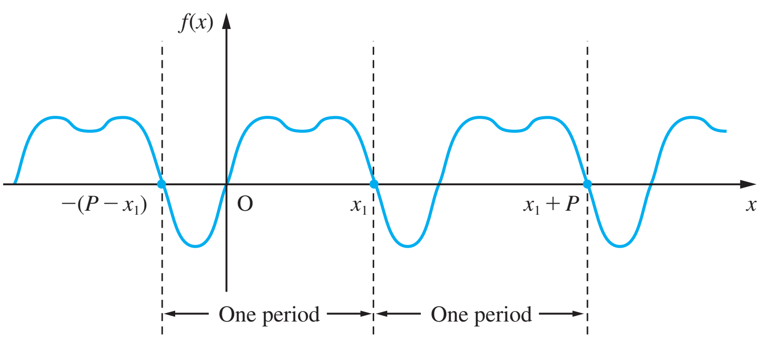
(f) The graph is antisymmetrical about the origin, so it represents an odd function.

A **periodic function** is such that its image values are repeated at regular intervals in its domain. Thus the graph of a periodic function can be divided into 'vertical strips' that are replicas of each other, as shown in Figure 2.22. The width of each strip is called the **period** of the function. We therefore say that a function $f(x)$ is periodic with period P if for all its domain values x

$$f(x + nP) = f(x)$$

for any integer n.

Figure 2.22 A periodic function of period P.



To provide a measure of the number of repetitions per unit of x, we define the **frequency** of a periodic function to be the reciprocal of its period, so that

$$\text{frequency} = \frac{1}{\text{period}}$$

The Greek letter ν ('nu') is usually used to denote the frequency, so that $\nu = 1/P$. The term **circular frequency** is also used in some engineering contexts. This is denoted by the Greek letter ω ('omega') and is defined by

$$\omega = 2\pi\nu = \frac{2\pi}{P}$$

It is measured in radians per unit of x, the free variable. When the meaning is clear from the context, the adjective 'circular' is commonly omitted.

Example 2.12

A function $f(x)$ has the graph on [0, 1] shown in Figure 2.23. Sketch its graph on [−3, 3] given that

(a) $f(x)$ is periodic with period 1;

(b) $f(x)$ is periodic with period 2 and is even;

(c) $f(x)$ is periodic with period 2 and is odd.

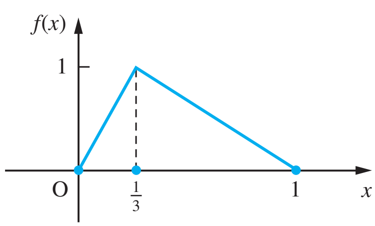


Figure 2.23 $f(x)$ of Example 2.12 defined on [0, 1].

Solution

(a) Since $f(x)$ has period 1, strips of width 1 unit are simply replicas of the graph between 0 and 1. Hence we obtain the graph shown in Figure 2.24.

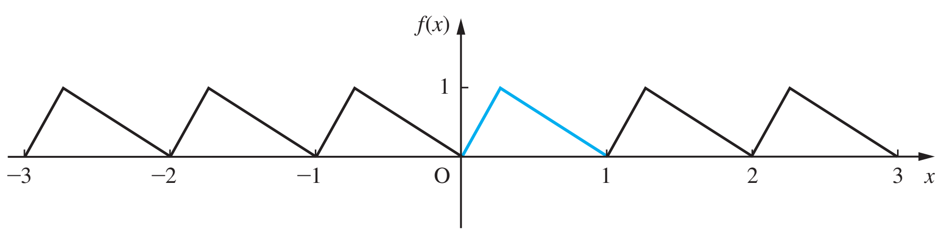
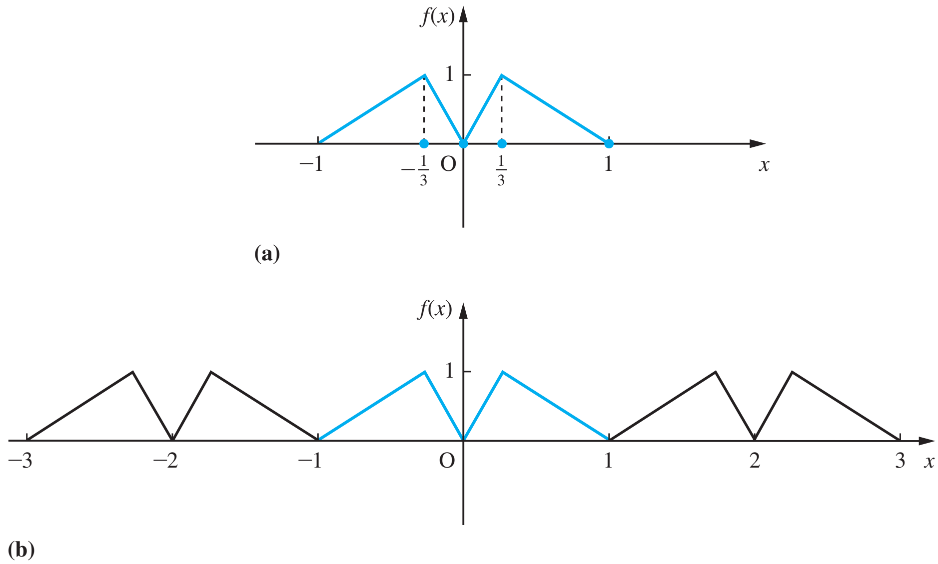


Figure 2.24 $f(x)$ having period 1.

(b) Since $f(x)$ has period 2 we need to establish the graph over a complete period before we can replicate it along the domain of $f(x)$. Since it is an even function and

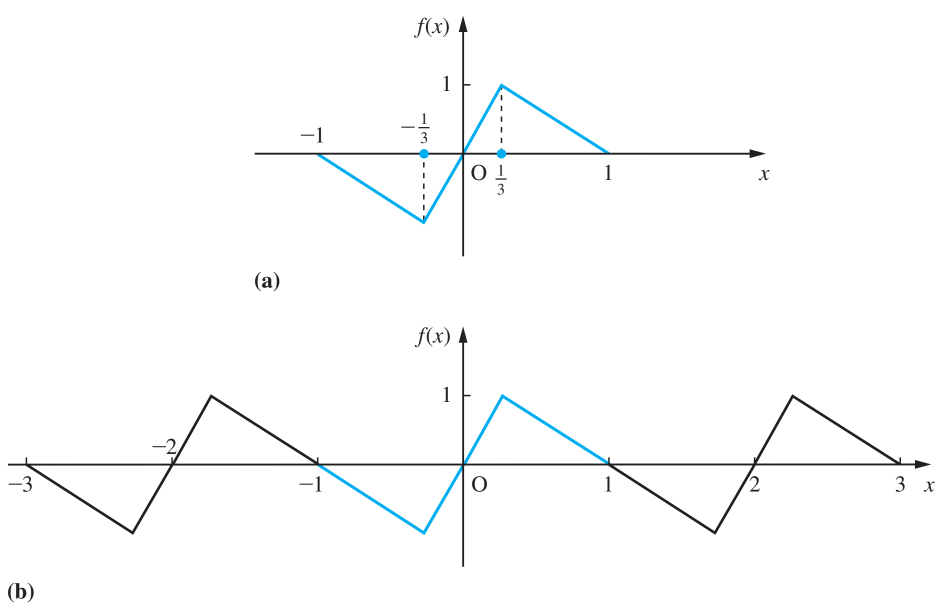
Figure 2.25
$f(x)$ periodic with period 2 and is even.



we know its values between 0 and 1, we also know its values between -1 and 0. We can obtain the graph of $f(x)$ between -1 and 0 by reflecting in the y axis, as shown in Figure 2.25(a). Thus we have the graph over a complete period, from -1 to $+1$, and so we can replicate along the x axis, as shown in Figure 2.25(b).

(c) Similarly, if $f(x)$ is an odd function we can obtain the graph for the interval $[-1, 0]$ using antisymmetry and the graph for the interval $[0, 1]$. This gives us Figure 2.26(a) and we then obtain the whole graph, Figure 2.26(b), by periodic extension.

Figure 2.26
$f(x)$ periodic with period 2 and is odd.



2.2.7 Exercises

14 Which of the functions $y = f(x)$ whose graphs are shown in Figure 2.27 are odd, even or neither odd nor even?

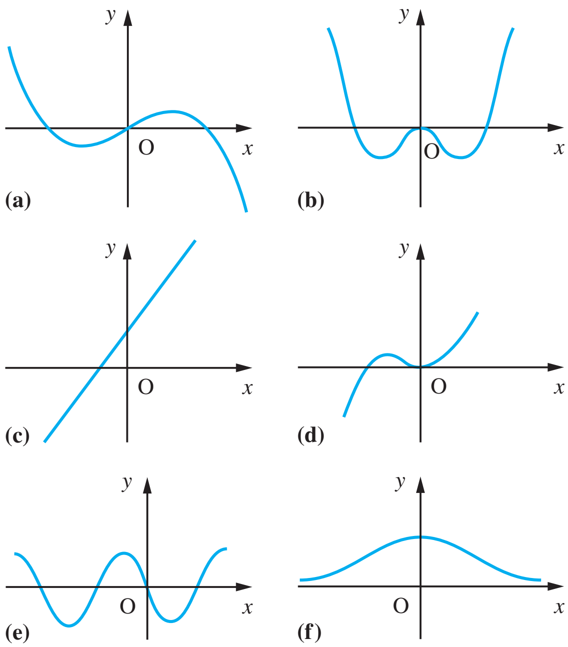


Figure 2.27 Graphs of Question 14.

15 Three different functions, $f(x)$, $g(x)$ and $h(x)$, have the same graph on $[0, 2]$ as shown in Figure 2.28. On separate diagrams, sketch their graphs for $[-4, 4]$ given that

(a) $f(x)$ is periodic with period 2;

(b) $g(x)$ is periodic with period 4 and is even;

(c) $h(x)$ is periodic with period 4 and is odd.

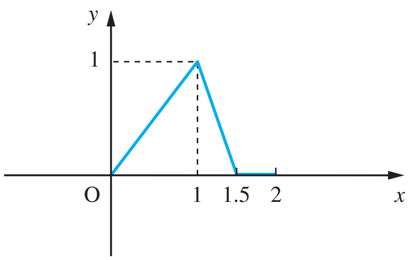


Figure 2.28 Graph of Question 15.

16 Show that

$$h(x) = \tfrac{1}{2}[f(x) - f(-x)]$$

is an odd function and that any function $f(x)$ may be written as the sum of an odd and an even function.

Illustrate this result with $f(x) = (x - 1)^3$.

2.3 Linear and quadratic functions

Among the more commonly used functions in engineering contexts are the linear and quadratic functions. This is because the mathematical models of practical problems often involve linear functions and also because more complicated functions are often well approximated locally by linear or quadratic functions. We shall review the properties of these functions and in the process describe some of the contexts in which they occur.

2.3.1 Linear functions

The **linear function** is the simplest function that occurs in practical problems. It has the formula $f(x) = mx + c$ where m and c are constant numbers and x is the unassigned or independent variable as usual. The graph of $f(x)$ is the set of points (x, y) where $y = mx + c$, which is the equation of a straight line on a cartesian coordinate plot (see Section 1.4.2). Hence, the function is called the linear function. An example of a linear function is the conversion of a temperature T_1°C to the temperature T_2°F. Here

$$T_2 = \tfrac{9}{5}T_1 + 32$$

and $m = \frac{9}{5}$ with $c = 32$.

To determine the formula for a particular linear function the two constants m and c have to be found. This implies that we need two pieces of information to determine $f(x)$.

Example 2.13

A manufacturer produces 5000 items at a total cost of £10 000 and sells them at £2.75 each. What is the manufacturer's profit as a function of the number x of items sold?

Solution

Let the manufacturer's profit be £P. If x items are sold then the total revenue is £$2.75x$, so that the amount of profit $P(x)$ is given by

$$P(x) = \text{revenue} - \text{cost} = 2.75x - 10\,000$$

Here the domain of the function is [0, 5000] and the range is [−10 000, 3750]. This function has a zero at $x = 3636\frac{4}{11}$. Thus to make a profit, the manufacturer has to sell more than 3636 items. (Note the modelling approximation in that, strictly, x is an integer variable, not a general real variable.)

If we know the values that the function $f(x)$ takes at two values, x_0 and x_1, of the independent variable x we can find the formula for $f(x)$. Let $f(x_0) = f_0$ and $f(x_1) = f_1$; then

$$f(x) = \frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_0}{x_1 - x_0} f_1 \qquad \textbf{(2.6)}$$

This formula is known as **Lagrange's formula**. It is obvious that the function is linear since we can arrange it as

$$f(x) = x\left[\frac{f_1 - f_0}{x_1 - x_0}\right] + \left[\frac{x_1 f_0 - x_0 f_1}{x_1 - x_0}\right]$$

The reader should verify from (2.6) that $f(x_0) = f_0$ and $f(x_1) = f_1$.

Example 2.14

Use Lagrange's formula to find the linear function $f(x)$ where $f(10) = 1241$ and $f(15) = 1556$.

Solution

Taking $x_0 = 10$ and $x_1 = 15$ so that $f_0 = 1241$ and $f_1 = 1556$ we obtain

$$\begin{aligned} f(x) &= \frac{x - 15}{10 - 15}(1241) + \frac{x - 10}{15 - 10}(1556) \\ &= \frac{x}{5}(1556 - 1241) + 3(1241) - 2(1556) \\ &= \frac{x}{5}(315) + (3723 - 3112) = 63x + 611 \end{aligned}$$

The **rate of change** of a function, between two values $x = x_0$ and $x = x_1$ in its domain, is defined by the ratio of the change in the values of the function to the change in the values of x. Thus

$$\text{rate of change} = \frac{\text{change in values of } f(x)}{\text{change in values of } x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

For a linear function with formula $f(x) = mx + c$ we have

$$\begin{aligned}\text{rate of change} &= \frac{(mx_1 + c) - (mx_0 + c)}{x_1 - x_0} \\ &= \frac{m(x_1 - x_0)}{x_1 - x_0} = m\end{aligned}$$

which is a constant. If we know the rate of change m of a linear function $f(x)$ and the value f_0 at a point $x = x_0$, then we can write the formula for $f(x)$ as

$$f(x) = mx + f_0 - mx_0$$

For a linear function, the slope (gradient) of the graph is the rate of change of the function.

Example 2.15 The labour cost of producing a certain item is £21 per 10 000 items and the raw materials cost is £4 for 1000 items. Each time a new production run is begun, there is a set-up cost of £8. What is the cost, £$C(x)$, of a production run of x items?

Solution Here the cost function has a rate of change comprising the labour cost per item (21/10 000) and the materials cost per item (4/1000). Thus the rate of change is 0.0061. We also know that if there is a production run with zero items, there is still a set-up cost of £8 so $f(0) = 8$. Thus the required function is

$$C(x) = 0.0061x + 8$$

2.3.2 Least squares fit of a linear function to experimental data

Because the linear function occurs in many mathematical models of practical problems, we often have to ‘fit’ linear functions to experimental data. That is, we have to find the values of m and c which yield the best overall description of the data. There are two distinct mathematical models that occur. These are given by the functions with formulae

(a) $y = ax$ and (b) $y = mx + c$

For example, the extension of an ideal spring under load may be represented by a function of type (a), while the velocity of a projectile launched vertically may be represented by a function of type (b).

From experiments we obtain a set of data points (x_k, y_k), $k = 1, 2, \dots, n$. We wish to find the value of the constant(s) of the linear function that best describes the phenomenon the data represents.