

GLOBAL
EDITION



Discrete Mathematics

EIGHTH EDITION

Richard Johnsonbaugh



Pearson

List of Symbols

LOGIC

$p \vee q$	p or q ; page 34	$P \equiv Q$	P and Q are logically equivalent; page 45
$p \wedge q$	p and q ; page 34	$\forall$	for all; page 58
$\neg p$	not p ; page 37	$\exists$	there exists; page 60
$p \rightarrow q$	if p , then q ; pages 40–41	$\therefore$	therefore; page 51
$p \leftrightarrow q$	p if and only if q ; pages 44–45		

SET NOTATION

$\{x_1, \dots, x_n\}$	set consisting of the elements $x_1, \dots, x_n$; page 22
$\{x \mid p(x)\}$	set consisting of those elements x satisfying property $p(x)$; page 22
$\mathbb{Z}, \mathbb{Z}^-, \mathbb{Z}^+, \mathbb{Z}^{\text{nonneg}}$	sets of integers, negative integers, positive integers, nonnegative integers; pages 22–23
$\mathbb{Q}, \mathbb{Q}^-, \mathbb{Q}^+, \mathbb{Q}^{\text{nonneg}}$	sets of rational numbers, negative rational numbers, positive rational numbers, nonnegative rational numbers; pages 22–23
$\mathbb{R}, \mathbb{R}^-, \mathbb{R}^+, \mathbb{R}^{\text{nonneg}}$	sets of real numbers, negative real numbers, positive real numbers, nonnegative real numbers; pages 22–23
$x \in X$	x is an element of X ; page 23
$x \notin X$	x is not an element of X ; page 23
$X = Y$	set equality (X and Y have the same elements); page 23
$ X $	cardinality of X (number of elements in X); page 23
$\emptyset$	empty set; page 23
$X \subseteq Y$	X is a subset of Y ; page 24
$X \subset Y$	X is a proper subset of Y ; page 25
$\mathcal{P}(X)$	power set of X (all subsets of X); page 25
$X \cup Y$	X union Y (all elements in X or Y); page 25
$\bigcup_{i=1}^n X_i$	union of $X_1, \dots, X_n$ (all elements that belong to at least one of $X_1, X_2, \dots, X_n$); page 29
$\bigcup_{i=1}^{\infty} X_i$	union of $X_1, X_2, \dots$ (all elements that belong to at least one of $X_1, X_2, \dots$); page 29
$\bigcup \mathcal{S}$	union of $\mathcal{S}$ (all elements that belong to at least one set in $\mathcal{S}$); page 28
$X \cap Y$	X intersect Y (all elements in X and Y); pages 25–26
$\bigcap_{i=1}^n X_i$	intersection of $X_1, \dots, X_n$ (all elements that belong to every one of $X_1, X_2, \dots, X_n$); page 29
$\bigcap_{i=1}^{\infty} X_i$	intersection of $X_1, X_2, \dots$ (all elements that belong to every one of $X_1, X_2, \dots$); page 29
$\bigcap \mathcal{S}$	intersection of $\mathcal{S}$ (all elements that belong to every set in $\mathcal{S}$); page 29
$X - Y$	set difference (all elements in X but not in Y); page 26
$\bar{X}$	complement of X (all elements not in X); page 26
(x, y)	ordered pair; page 29
$(x_1, \dots, x_n)$	n -tuple; page 30

$X \times Y$	Cartesian product of X and Y [pairs (x, y) with x in X and y in Y]; page 29
$X_1 \times X_2 \times \cdots \times X_n$	Cartesian product of $X_1, X_2, \dots, X_n$ (n -tuples with $x_i \in X_i$); page 30
$X \Delta Y$	symmetric difference of X and Y ; page 33

RELATIONS

xRy	(x, y) is in R (x is related to y by the relation R); page 161
$[x]$	equivalence class containing x ; page 173
R^{-1}	inverse relation [all (y, x) with (x, y) in R]; page 167
$R_2 \circ R_1$	composition of relations; page 167
$x \leq y$	xRy ; page 166

FUNCTIONS

$f(x)$	value assigned to x ; page 134
$f: X \rightarrow Y$	function from X to Y ; page 133
$f \circ g$	composition of f and g ; page 142
f^{-1}	inverse function [all (y, x) with (x, y) in f]; pages 141
$f(n) = O(g(n))$	$ f(n) \leq C g(n) $ for n sufficiently large; page 206
$f(n) = \Omega(g(n))$	$c g(n) \leq f(n) $ for n sufficiently large; page 206
$f(n) = \Theta(g(n))$	$c g(n) \leq f(n) \leq C g(n) $ for n sufficiently large; page 206

COUNTING

$C(n, r)$	number of r -combinations of an n -element set $(n! / [(n - r)!r!])$; page 293
$P(n, r)$	number of r -permutations of an n -element set $[n(n - 1) \cdots (n - r + 1)]$; page 291

GRAPHS

$G = (V, E)$	graph G with vertex set V and edge set E ; page 395
(v, w)	edge; page 395
$\delta(v)$	degree of vertex v ; page 408
$(v_1, \dots, v_n)$	path from v_1 to v_n ; page 404–405
$(v_1, \dots, v_n), v_1 = v_n$	cycle; page 407
K_n	complete graph on n vertices; page 400
$K_{m,n}$	complete bipartite graph on m and n vertices; page 401
$w(i, j)$	weight of edge (i, j) ; page 425
F_{ij}	flow in edge (i, j) ; page 527
C_{ij}	capacity of edge (i, j) ; page 527
$(P, \bar{P})$	cut in a network; page 540

PROBABILITY

$P(x)$	probability of outcome x ; page 321
$P(E)$	probability of event E ; page 322
$P(E F)$	conditional probability of E given F $[P(E \cap F) / P(F)]$; page 326

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Discrete Mathematics

Eighth Edition
Global Edition

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Dedication

To Pat, my wife, for her continuous support through my many book projects, for formally and informally copy-editing my books, for maintaining good cheer throughout, and for preventing all *egregious* mistakes that would have otherwise found their way into print. Her contributions are deeply appreciated.

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Preface

This updated edition is intended for a one- or two-term introductory course in discrete mathematics, based on my experience in teaching this course over many years and requests from users of previous editions. Formal mathematics prerequisites are minimal; calculus is not required. There are no computer science prerequisites. The book includes examples, exercises, figures, tables, sections on problem-solving, sections containing problem-solving tips, section reviews, notes, chapter reviews, self-tests, and computer exercises to help the reader master introductory discrete mathematics. In addition, an Instructor's Solutions Manual and website are available.

In the early 1980s there were few textbooks appropriate for an introductory course in discrete mathematics. However, there was a need for a course that extended students' mathematical maturity and ability to deal with abstraction, which also included useful topics such as combinatorics, algorithms, and graphs. The original edition of this book (1984) addressed this need and significantly influenced the development of discrete mathematics courses. Subsequently, discrete mathematics courses were endorsed by many groups for several different audiences, including mathematics and computer science majors. A panel of the Mathematical Association of America (MAA) endorsed a year-long course in discrete mathematics. The Educational Activities Board of the Institute of Electrical and Electronics Engineers (IEEE) recommended a freshman discrete mathematics course. The Association for Computing Machinery (ACM) and IEEE accreditation guidelines mandated a discrete mathematics course. This edition, like its predecessors, includes topics such as algorithms, combinatorics, sets, functions, and mathematical induction endorsed by these groups. It also addresses understanding and constructing proofs and, generally, expanding mathematical maturity.

New to This Edition

The changes in this book, the eighth edition, result from comments and requests from numerous users and reviewers of previous editions of the book. This edition includes the following changes from the seventh edition:

- The web icons in the seventh edition have been replaced by short URLs, making it possible to quickly access the appropriate web page, for example, by using a hand-held device.
- The exercises in the chapter self-tests no longer identify the relevant sections making the self-test more like a real exam. (The hints to these exercises *do* identify the relevant sections.)

- Examples that are worked problems clearly identify where the solution begins and ends.
- The number of exercises in the first three chapters (Sets and Logic; Proofs; and Functions, Sequences, and Relations) has been increased from approximately 1640 worked examples and exercises in the seventh edition to over 1750 in the current edition.
- Many comments have been added to clarify potentially tricky concepts (e.g., “subset” and “element of,” collection of sets, logical equivalence of a sequence of propositions, logarithmic scale on a graph).
- There are more examples illustrating diverse approaches to developing proofs and alternative ways to prove a particular result [see, e.g., Examples 2.2.4 and 2.2.8; Examples 6.1.3(c) and 6.1.12; Examples 6.7.7, 6.7.8, and 6.7.9; Examples 6.8.1 and 6.8.2].
- A number of definitions have been revised to allow them to be more directly applied in proofs [see, e.g., one-to-one function (Definition 3.1.22) and onto function (Definition 3.1.29)].
- Additional real-world examples (see descriptions in the following section) are included.
- The altered definition of sequence (Definition 3.2.1) provides more generality and makes subsequent discussion smoother (e.g., the discussion of subsequences).
- Exercises have been added (Exercises 40–49, Section 5.1) to give an example of an algebraic system in which prime factorization does not hold.
- An application of the binomial theorem is used to prove Fermat’s little theorem (Exercises 40 and 41, Section 6.7).
- There is now a randomized algorithm to search for a Hamiltonian cycle in a graph (Algorithm 8.3.10).
- The Closest-Pair Problem (Section 13.1 in the seventh edition) has been integrated into Chapter 7 (Recurrence Relations) in the current edition. The algorithm to solve the closest-pair problem is based on merge sort, which is discussed and analyzed in Chapter 7. Chapter 13 in the seventh edition, which has now been removed, had only one additional section.
- A number of recent books and articles have been added to the list of references, and several book references have been updated to current editions.
- The number of exercises has been increased to nearly 4500. (There were approximately 4200 in the seventh edition.)

Contents and Structure

Content Overview

Chapter 1 Sets and Logic

Coverage includes quantifiers and features practical examples such as using the Google search engine (Example 1.2.13). We cover translating between English and symbolic expressions as well as logic in programming languages. We also include a logic game (Example 1.6.15), which offers an alternative way to determine whether a quantified propositional function is true or false.

Chapter 2 Proofs

Proof techniques discussed include direct proofs, counterexamples, proof by contradiction, proof by contrapositive, proof by cases, proofs of equivalence, existence proofs (constructive and nonconstructive), and mathematical induction. We present loop invariants as a practical application of mathematical induction. We also include a brief, optional section on resolution proofs (a proof technique that can be automated).

Chapter 3 Functions, Sequences, and Relations

The chapter includes strings, sum and product notations, and motivating examples such as the Luhn algorithm for computing credit card check digits, which opens the chapter. Other examples include an introduction to hash functions (Example 3.1.15), pseudo-random number generators (Example 3.1.16), a real-world example of function composition showing its use in making a price comparison (Example 3.1.45), an application of partial orders to task scheduling (Section 3.3), and relational databases (Section 3.6).

Chapter 4 Algorithms

The chapter features a thorough discussion of algorithms, recursive algorithms, and the analysis of algorithms. We present a number of examples of algorithms before getting into big-oh and related notations (Sections 4.1 and 4.2), thus providing a gentle introduction and motivating the formalism that follows. We then continue with a full discussion of the “big oh,” omega, and theta notations for the growth of functions (Section 4.3). Having all of these notations available makes it possible to make precise statements about the growth of functions and the time and space required by algorithms.

We use the algorithmic approach throughout the remainder of the book. We mention that many modern algorithms do not have all the properties of classical algorithms (e.g., many modern algorithms are not general, deterministic, or even finite). To illustrate the point, we give an example of a randomized algorithm (Example 4.2.4). Algorithms are written in a flexible form of pseudocode, which resembles currently popular languages such as C, C++, and Java. (The book does not assume any computer science prerequisites; the description of the pseudocode used is given in Appendix C.) Among the algorithms presented are:

- Tiling (Section 4.4)
- Euclidean algorithm for finding the greatest common divisor (Section 5.3)
- RSA public-key encryption algorithm (Section 5.4)
- Generating combinations and permutations (Section 6.4)
- Merge sort (Section 7.3)
- Finding a closest pair of points (Section 7.4)
- Dijkstra’s shortest-path algorithm (Section 8.4)
- Backtracking algorithms (Section 9.3)
- Breadth-first and depth-first search (Section 9.3)
- Tree traversals (Section 9.6)
- Evaluating a game tree (Section 9.9)
- Finding a maximal flow in a network (Section 10.2)

Chapter 5 Introduction to Number Theory

The chapter includes classical results (e.g., divisibility, the infinitude of primes, fundamental theorem of arithmetic), as well as algorithmic number theory (e.g., the Euclidean algorithm to find the greatest common divisor, exponentiation using repeated squaring, computing s and t such that $\gcd(a, b) = sa + tb$, computing an inverse modulo an inte-

ger). The major application is the RSA public-key cryptosystem (Section 5.4). The calculations required by the RSA public-key cryptosystem are performed using the algorithms previously developed in the chapter.

Chapter 6 Counting Methods and the Pigeonhole Principle

Coverage includes combinations, permutations, discrete probability (optional Sections 6.5 and 6.6), and the Pigeonhole Principle. Applications include internet addressing (Section 6.1) and real-world pattern recognition problems in telemarketing (Example 6.6.21) and virus detection (Example 6.6.22) using Bayes' Theorem.

Chapter 7 Recurrence Relations

The chapter includes recurrence relations and their use in the analysis of algorithms.

Chapter 8 Graph Theory

Coverage includes graph models of parallel computers, the knight's tour, Hamiltonian cycles, graph isomorphisms, and planar graphs. Theorem 8.4.3 gives a simple, short, elegant proof of the correctness of Dijkstra's algorithm.

Chapter 9 Trees

Coverage includes binary trees, tree traversals, minimal spanning trees, decision trees, the minimum time for sorting, and tree isomorphisms.

Chapter 10 Network Models

Coverage includes the maximal flow algorithm and matching.

Chapter 11 Boolean Algebras and Combinatorial Circuits

Coverage emphasizes the relation of Boolean algebras to combinatorial circuits.

Chapter 12 Automata, Grammars, and Languages

Our approach emphasizes modeling and applications. We discuss the *SR* flip-flop circuit in Example 12.1.11, and we describe fractals, including the von Koch snowflake, which can be described by special kinds of grammars (Example 12.3.19).

Book frontmatter and endmatter

Appendixes include coverage of matrices, basic algebra, and pseudocode. A reference section provides more than 160 references to additional sources of information. Front and back endpapers summarize the mathematical and algorithm notation used in the book.

Features of Content Coverage

- **A strong emphasis on the interplay among the various topics.** Examples of this include:
 - We closely tie mathematical induction to recursive algorithms (Section 4.4).
 - We use the Fibonacci sequence in the analysis of the Euclidean algorithm (Section 5.3).
 - Many exercises throughout the book require mathematical induction.
 - We show how to characterize the components of a graph by defining an equivalence relation on the set of vertices (see the discussion following Example 8.2.13).
 - We count the number of nonisomorphic n -vertex binary trees (Theorem 9.8.12).
- **A strong emphasis on reading and doing proofs.** We illustrate most proofs of theorems with annotated figures and/or motivate them by special Discussion sec-

tions. Separate sections (Problem-Solving Corners) show students how to attack and solve problems and how to do proofs. Special end-of-section Problem-Solving Tips highlight the main problem-solving techniques of the section.

- **A large number of applications, especially applications to computer science.**
- **Figures and tables** illustrate concepts, show how algorithms work, elucidate proofs, and motivate the material. Several figures illustrate proofs of theorems. The captions of these figures provide additional explanation and insight into the proofs.

Textbook Structure

Each chapter is organized as follows:

Chapter X Overview
 Section X.1
 Section X.1 Review Exercises
 Section X.1 Exercises
 Section X.2
 Section X.2 Review Exercises
 Section X.2 Exercises
 ⋮
 Chapter X Notes
 Chapter X Review
 Chapter X Self-Test
 Chapter X Computer Exercises

In addition, most chapters have **Problem-Solving Corners** (see “Hallmark Features” for more information about this feature).

Section review exercises review the key concepts, definitions, theorems, techniques, and so on of the section. All section review exercises have answers in the back of the book. Although intended for reviews of the sections, section review exercises can also be used for placement and pretesting.

Chapter notes contain suggestions for further reading. **Chapter reviews** provide reference lists of the key concepts of the chapters. **Chapter self-tests** contain exercises based on material from throughout the chapter, with answers in the back of the book.

Computer exercises include projects, implementation of some of the algorithms, and other programming related activities. Although there is no programming prerequisite for this book and no programming is introduced in the book, these exercises are provided for those readers who want to explore discrete mathematics concepts with a computer.

Hallmark Features

Exercises

The book contains nearly 4500 exercises, approximately 150 of which are computer exercises. We use a star to label exercises felt to be more challenging than average. Exercise numbers in color (approximately one-third of the exercises) indicate that the exercise has a hint or solution in the back of the book. The solutions to most of the remaining exercises may be found in the Instructor’s Guide. A handful of exercises are clearly identified as requiring calculus. No calculus concepts are used in the main body of the book and, except for these marked exercises, no calculus is needed to solve the exercises.

Examples

The book contains almost 650 worked examples. These examples show students how to tackle problems in discrete mathematics, demonstrate applications of the theory, clarify proofs, and help motivate the material.

Problem-Solving Corners

The Problem-Solving Corner sections help students attack and solve problems and show them how to do proofs. Written in an informal style, each is a self-contained section centered around a problem. The intent of these sections is to go beyond simply presenting a proof or a solution to the problem: we show alternative ways of attacking a problem, discuss what to look for in trying to obtain a solution to a problem, and present problem-solving and proof techniques.

Each Problem-Solving Corner begins with a statement of a problem. We then discuss ways to attack the problem, followed by techniques for finding a solution. After we present a solution, we show how to correctly write it up in a formal manner. Finally, we summarize the problem-solving techniques used in the section. Some sections include a Comments subsection, which discusses connections with other topics in mathematics and computer science, provides motivation for the problem, and lists references for further reading about the problem. Some Problem-Solving Corners conclude with a few exercises.

Supplements and Technology

Instructor's Solution Manual (downloadable)

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The Instructor's Solutions Manual, written by the author, provides worked-out solutions for most exercises in the text. It is available for download to qualified instructors from the Pearson Instructor Resource Center www.pearsonglobaleditions.com.

Web Support

The short URLs in the margin of the text provide students with direct access to relevant content at point-of-use, including:

- Expanded explanations of difficult material and links to other sites for additional information about discrete mathematics topics.
- Computer programs (in C or C++).

The URL `goo.gl/STo1E7` provides access to all of the above resources plus an errata list for the text.

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NOTE:

When you enter URLs that appear in the text, take care to distinguish the following characters:

l = lowercase l

I = uppercase I

1 = one

O = uppercase O

0 = zero

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Richard Johnsonbaugh

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Chapter 1

SETS AND LOGIC

- 1.1 Sets
- 1.2 Propositions
- 1.3 Conditional Propositions and Logical Equivalence
- 1.4 Arguments and Rules of Inference
- 1.5 Quantifiers
- 1.6 Nested Quantifiers

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Chapter 1 begins with sets. A **set** is a collection of objects; order is not taken into account. Discrete mathematics is concerned with objects such as graphs (sets of vertices and edges) and Boolean algebras (sets with certain operations defined on them). In this chapter, we introduce set terminology and notation. In Chapter 2, we treat sets more formally after discussing proof and proof techniques. However, in Section 1.1, we provide a taste of the logic and proofs to come in the remainder of Chapter 1 and in Chapter 2.

Logic is the study of reasoning; it is specifically concerned with whether reasoning is correct. Logic focuses on the relationship among statements as opposed to the content of any particular statement. Consider, for example, the following argument:

All mathematicians wear sandals.

Anyone who wears sandals is an algebraist.

Therefore, all mathematicians are algebraists.

Technically, logic is of no help in determining whether any of these statements is true; however, if the first two statements are true, logic assures us that the statement,

All mathematicians are algebraists,

is also true.

Logic is essential in reading and developing proofs, which we explore in detail in Chapter 2. An understanding of logic can also be useful in clarifying ordinary writing. For example, at one time, the following ordinance was in effect in Naperville, Illinois: “It shall be unlawful for any person to keep more than three dogs and three cats upon his property within the city.” Was one of the citizens, who owned five dogs and no cats, in violation of the ordinance? Think about this question now; then analyze it (see Exercise 75, Section 1.2) after reading Section 1.2.

1.1 Sets

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The concept of set is basic to all of mathematics and mathematical applications. A **set** is simply a collection of objects. The objects are sometimes referred to as elements or members. If a set is finite and not too large, we can describe it by listing the elements in it. For example, the equation

$$A = \{1, 2, 3, 4\} \tag{1.1.1}$$

describes a set A made up of the four elements 1, 2, 3, and 4. A set is determined by its elements and not by any particular order in which the elements might be listed. Thus the set A might just as well be specified as $A = \{1, 3, 4, 2\}$. The elements making up a set are assumed to be distinct, and although for some reason we may have duplicates in our list, only one occurrence of each element is in the set. For this reason we may also describe the set A defined in (1.1.1) as $A = \{1, 2, 2, 3, 4\}$.

If a set is a large finite set or an infinite set, we can describe it by listing a property necessary for membership. For example, the equation

$$B = \{x \mid x \text{ is a positive, even integer}\} \tag{1.1.2}$$

describes the set B made up of all positive, even integers; that is, B consists of the integers 2, 4, 6, and so on. The vertical bar “ $\mid$ ” is read “such that.” Equation (1.1.2) would be read “ B equals the set of all x such that x is a positive, even integer.” Here the property necessary for membership is “is a positive, even integer.” Note that the property appears after the vertical bar. The notation in (1.1.2) is called **set-builder notation**.

A set may contain *any* kind of elements whatsoever, and they need *not* be of the same “type.” For example,

$$\{4.5, \text{Lady Gaga}, \pi, 14\}$$

is a perfectly fine set. It consists of four elements: the number 4.5, the person Lady Gaga, the number $\pi (= 3.1415 \dots)$, and the number 14.

A set may contain elements that are themselves sets. For example, the set

$$\{3, \{5, 1\}, 12, \{\pi, 4.5, 40, 16\}, \text{Henry Cavill}\}$$

consists of five elements: the number 3, the set $\{5, 1\}$, the number 12, the set $\{\pi, 4.5, 40, 16\}$, and the person Henry Cavill.

Some sets of numbers that occur frequently in mathematics generally, and in discrete mathematics in particular, are shown in Figure 1.1.1. The symbol $\mathbf{Z}$ comes from the German word, *Zahlen*, for *integer*. Rational numbers are quotients of integers, thus $\mathbf{Q}$ for *quotient*. The set of real numbers $\mathbf{R}$ can be depicted as consisting of all points on a straight line extending indefinitely in either direction (see Figure 1.1.2).[†]

Symbol	Set	Example of Members
$\mathbf{Z}$	Integers	$-3, 0, 2, 145$
$\mathbf{Q}$	Rational numbers	$-1/3, 0, 24/15$
$\mathbf{R}$	Real numbers	$-3, -1.766, 0, 4/15, \sqrt{2}, 2.666 \dots, \pi$

Figure 1.1.1 Sets of numbers.

[†]The real numbers can be constructed by starting with a more primitive notion such as “set” or “integer,” or they can be obtained by stating properties (axioms) they are assumed to obey. For our purposes, it suffices to think of the real numbers as points on a straight line. The construction of the real numbers and the axioms for the real numbers are beyond the scope of this book.

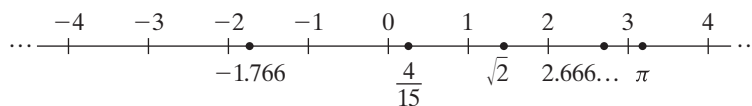


Figure 1.1.2 The real number line.

To denote the negative numbers that belong to one of $\mathbf{Z}$, $\mathbf{Q}$, or $\mathbf{R}$, we use the superscript minus. For example, $\mathbf{Z}^-$ denotes the set of negative integers, namely $-1, -2, -3, \dots$. Similarly, to denote the positive numbers that belong to one of the three sets, we use the superscript plus. For example, $\mathbf{Q}^+$ denotes the set of positive rational numbers. To denote the nonnegative numbers that belong to one of the three sets, we use the superscript *nonneg*. For example, $\mathbf{Z}^{\text{nonneg}}$ denotes the set of nonnegative integers, namely $0, 1, 2, 3, \dots$.

If X is a finite set, we let $|X|$ = number of elements in X . We call $|X|$ the **cardinality** of X . There is also a notion of cardinality of infinite sets, although we will not discuss it in this book. For example, the cardinality of the integers, $\mathbf{Z}$, is denoted $\aleph_0$, read “aleph null.” Aleph is the first letter of the Hebrew alphabet.

Example 1.1.1

For the set A in (1.1.1), we have $|A| = 4$, and the cardinality of A is 4. The cardinality of the set $\{\mathbf{R}, \mathbf{Z}\}$ is 2 since it contains two elements, namely the two sets $\mathbf{R}$ and $\mathbf{Z}$. ◀

Given a description of a set X such as (1.1.1) or (1.1.2) and an element x , we can determine whether or not x belongs to X . If the members of X are listed as in (1.1.1), we simply look to see whether or not x appears in the listing. In a description such as (1.1.2), we check to see whether the element x has the property listed. If x is in the set X , we write $x \in X$, and if x is not in X , we write $x \notin X$. For example, $3 \in \{1, 2, 3, 4\}$, but $3 \notin \{x \mid x \text{ is a positive, even integer}\}$.

The set with no elements is called the **empty** (or **null** or **void**) **set** and is denoted $\emptyset$. Thus $\emptyset = \{ \}$.

Two sets X and Y are **equal** and we write $X = Y$ if X and Y have the same elements. To put it another way, $X = Y$ if the following two conditions hold:

- For every x , if $x \in X$, then $x \in Y$,

and

- For every x , if $x \in Y$, then $x \in X$.

The first condition ensures that every element of X is an element of Y , and the second condition ensures that every element of Y is an element of X .

Example 1.1.2

If $A = \{1, 3, 2\}$ and $B = \{2, 3, 2, 1\}$, by inspection, A and B have the same elements. Therefore $A = B$. ◀

Example 1.1.3

Show that if $A = \{x \mid x^2 + x - 6 = 0\}$ and $B = \{2, -3\}$, then $A = B$.

SOLUTION According to the criteria in the paragraph immediately preceding Example 1.1.2, we must show that for every x ,

$$\text{if } x \in A, \text{ then } x \in B, \quad (1.1.3)$$

and for every x ,

$$\text{if } x \in B, \text{ then } x \in A. \quad (1.1.4)$$

To verify condition (1.1.3), suppose that $x \in A$. Then

$$x^2 + x - 6 = 0.$$

Solving for x , we find that $x = 2$ or $x = -3$. In either case, $x \in B$. Therefore, condition (1.1.3) holds.

To verify condition (1.1.4), suppose that $x \in B$. Then $x = 2$ or $x = -3$. If $x = 2$, then

$$x^2 + x - 6 = 2^2 + 2 - 6 = 0.$$

Therefore, $x \in A$. If $x = -3$, then

$$x^2 + x - 6 = (-3)^2 + (-3) - 6 = 0.$$

Again, $x \in A$. Therefore, condition (1.1.4) holds. We conclude that $A = B$. 

For a set X to *not* be equal to a set Y (written $X \neq Y$), X and Y must *not* have the same elements: There must be at least one element in X that is not in Y or at least one element in Y that is not in X (or both).

Example 1.1.4

Let $A = \{1, 2, 3\}$ and $B = \{2, 4\}$. Then $A \neq B$ since there is at least one element in A (1 for example) that is not in B . [Another way to see that $A \neq B$ is to note that there is at least one element in B (namely 4) that is not in A .] 

Suppose that X and Y are sets. If every element of X is an element of Y , we say that X is a **subset** of Y and write $X \subseteq Y$. In other words, X is a subset of Y if for every x , if $x \in X$, then $x \in Y$.

Example 1.1.5

If $C = \{1, 3\}$ and $A = \{1, 2, 3, 4\}$, by inspection, every element of C is an element of A . Therefore, C is a subset of A and we write $C \subseteq A$. 

Example 1.1.6

Let $X = \{x \mid x^2 + x - 2 = 0\}$. Show that $X \subseteq \mathbf{Z}$.

SOLUTION We must show that for every x , if $x \in X$, then $x \in \mathbf{Z}$. If $x \in X$, then $x^2 + x - 2 = 0$. Solving for x , we obtain $x = 1$ or $x = -2$. In either case, $x \in \mathbf{Z}$. Therefore, for every x , if $x \in X$, then $x \in \mathbf{Z}$. We conclude that X is a subset of $\mathbf{Z}$ and we write $X \subseteq \mathbf{Z}$. 

Example 1.1.7

The set of integers $\mathbf{Z}$ is a subset of the set of rational numbers $\mathbf{Q}$. If $n \in \mathbf{Z}$, n can be expressed as a quotient of integers, for example, $n = n/1$. Therefore $n \in \mathbf{Q}$ and $\mathbf{Z} \subseteq \mathbf{Q}$. 

Example 1.1.8

The set of rational numbers $\mathbf{Q}$ is a subset of the set of real numbers $\mathbf{R}$. If $x \in \mathbf{Q}$, x corresponds to a point on the number line (see Figure 1.1.2) so $x \in \mathbf{R}$. 

For X to *not* be a subset of Y , there must be at least one member of X that is not in Y .

Example 1.1.9

Let $X = \{x \mid 3x^2 - x - 2 = 0\}$. Show that X is not a subset of $\mathbf{Z}$.

SOLUTION If $x \in X$, then $3x^2 - x - 2 = 0$. Solving for x , we obtain $x = 1$ or $x = -2/3$. Taking $x = -2/3$, we have $x \in X$ but $x \notin \mathbf{Z}$. Therefore, X is not a subset of $\mathbf{Z}$. ◀

Any set X is a subset of itself, since any element in X is in X . Also, the empty set is a subset of every set. If $\emptyset$ is *not* a subset of some set Y , according to the discussion preceding Example 1.1.9, there would have to be at least one member of $\emptyset$ that is not in Y . But this cannot happen because the empty set, by definition, has no members.

Notice the difference between the terms “subset” and “element of.” The set X is a *subset* of the set Y ($X \subseteq Y$), if every element of X is an element of Y ; x is an *element of* X ($x \in X$), if x is a member of the set X .

Example 1.1.10

Let $X = \{1, 3, 5, 7\}$ and $Y = \{1, 2, 3, 4, 5, 6, 7\}$. Then $X \subseteq Y$ since every element of X is an element of Y . But $X \not\subseteq Y$, since the *set* X is not a member of Y . Also, $1 \in X$, but 1 is not a subset of X . Notice the difference between the number 1 and the *set* $\{1\}$. The set $\{1\}$ is a subset of X . ◀

If X is a subset of Y and X does not equal Y , we say that X is a **proper subset** of Y and write $X \subset Y$.

Example 1.1.11

Let $C = \{1, 3\}$ and $A = \{1, 2, 3, 4\}$. Then C is a proper subset of A since C is a subset of A but C does not equal A . We write $C \subset A$. ◀

Example 1.1.12

Example 1.1.7 showed that $\mathbf{Z}$ is a subset of $\mathbf{Q}$. In fact, $\mathbf{Z}$ is a proper subset of $\mathbf{Q}$ because, for example, $1/2 \in \mathbf{Q}$, but $1/2 \notin \mathbf{Z}$. ◀

Example 1.1.13

Example 1.1.8 showed that $\mathbf{Q}$ is a subset of $\mathbf{R}$. In fact, $\mathbf{Q}$ is a proper subset of $\mathbf{R}$ because, for example, $\sqrt{2} \in \mathbf{R}$, but $\sqrt{2} \notin \mathbf{Q}$. (In Example 2.2.3, we will show that $\sqrt{2}$ is not the quotient of integers). ◀

The set of all subsets (proper or not) of a set X , denoted $\mathcal{P}(X)$, is called the **power set** of X .

Example 1.1.14

If $A = \{a, b, c\}$, the members of $\mathcal{P}(A)$ are

$$\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}.$$

All but $\{a, b, c\}$ are proper subsets of A . ◀

In Example 1.1.14, $|A| = 3$ and $|\mathcal{P}(A)| = 2^3 = 8$. In Section 2.4 (Theorem 2.4.6), we will give a formal proof that this result holds in general; that is, the power set of a set with n elements has 2^n elements.

Given two sets X and Y , there are various set operations involving X and Y that can produce a new set. The set

$$X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}$$

is called the **union** of X and Y . The union consists of all elements belonging to either X or Y (or both).

The set

$$X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}$$

is called the **intersection** of X and Y . The intersection consists of all elements belonging to both X and Y .

The set

$$X - Y = \{x \mid x \in X \text{ and } x \notin Y\}$$

is called the **difference** (or **relative complement**). The difference $X - Y$ consists of all elements in X that are not in Y .

Example 1.1.15

If $A = \{1, 3, 5\}$ and $B = \{4, 5, 6\}$, then

$$A \cup B = \{1, 3, 4, 5, 6\}$$

$$A \cap B = \{5\}$$

$$A - B = \{1, 3\}$$

$$B - A = \{4, 6\}.$$

Notice that, in general, $A - B \neq B - A$. 

Example 1.1.16

Since $\mathbf{Q} \subseteq \mathbf{R}$,

$$\mathbf{R} \cup \mathbf{Q} = \mathbf{R}$$

$$\mathbf{R} \cap \mathbf{Q} = \mathbf{Q}$$

$$\mathbf{Q} - \mathbf{R} = \emptyset.$$

The set $\mathbf{R} - \mathbf{Q}$, called the set of **irrational numbers**, consists of all real numbers that are not rational. 

We call a set $\mathcal{S}$, whose elements are sets, a **collection of sets** or a **family of sets**. For example, if

$$\mathcal{S} = \{\{1, 2\}, \{1, 3\}, \{1, 7, 10\}\},$$

then $\mathcal{S}$ is a collection or family of sets. The set $\mathcal{S}$ consists of the sets

$$\{1, 2\}, \{1, 3\}, \{1, 7, 10\}.$$

Sets X and Y are **disjoint** if $X \cap Y = \emptyset$. A collection of sets $\mathcal{S}$ is said to be **pairwise disjoint** if, whenever X and Y are distinct sets in $\mathcal{S}$, X and Y are disjoint.

Example 1.1.17

The sets $\{1, 4, 5\}$ and $\{2, 6\}$ are disjoint. The collection of sets $\mathcal{S} = \{\{1, 4, 5\}, \{2, 6\}, \{3\}, \{7, 8\}\}$ is pairwise disjoint. 

Sometimes we are dealing with sets, all of which are subsets of a set U . This set U is called a **universal set** or a **universe**. The set U must be explicitly given or inferred from the context. Given a universal set U and a subset X of U , the set $U - X$ is called the **complement** of X and is written $\overline{X}$.

Example 1.1.18

Let $A = \{1, 3, 5\}$. If U , a universal set, is specified as $U = \{1, 2, 3, 4, 5\}$, then $\overline{A} = \{2, 4\}$. If, on the other hand, a universal set is specified as $U = \{1, 3, 5, 7, 9\}$, then $\overline{A} = \{7, 9\}$. The complement obviously depends on the universe in which we are working. 

Example 1.1.19

Let the universal set be $\mathbf{Z}$. Then $\overline{\mathbf{Z}^-}$, the complement of the set of negative integers, is $\mathbf{Z}^{\text{nonneg}}$, the set of nonnegative integers. 

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Venn diagrams provide pictorial views of sets. In a Venn diagram, a rectangle depicts a universal set (see Figure 1.1.3). Subsets of the universal set are drawn as circles. The inside of a circle represents the members of that set. In Figure 1.1.3 we see two sets A and B within the universal set U . Region 1 represents $\overline{(A \cup B)}$, the elements in neither A nor B . Region 2 represents $A - B$, the elements in A but not in B . Region 3 represents $A \cap B$, the elements in both A and B . Region 4 represents $B - A$, the elements in B but not in A .

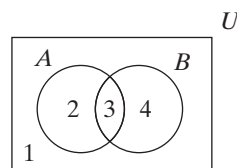


Figure 1.1.3 A Venn diagram.

Example 1.1.20

Particular regions in Venn diagrams are depicted by shading. The set $A \cup B$ is shown in Figure 1.1.4, and Figure 1.1.5 represents the set $A - B$.

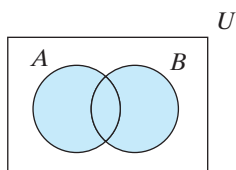


Figure 1.1.4 A Venn diagram of $A \cup B$.

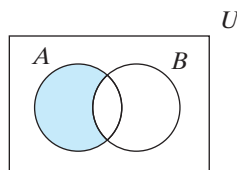


Figure 1.1.5 A Venn diagram of $A - B$.

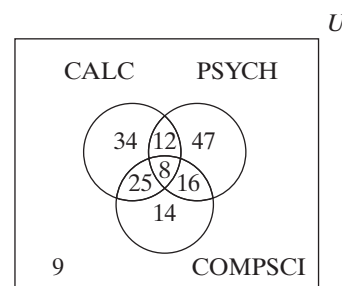


Figure 1.1.6 A Venn diagram of three sets CALC, PSYCH, and COMPSCI. The numbers show how many students belong to the particular region depicted.

To represent three sets, we use three overlapping circles (see Figure 1.1.6).

Example 1.1.21

Among a group of 165 students, 8 are taking calculus, psychology, and computer science; 33 are taking calculus and computer science; 20 are taking calculus and psychology; 24 are taking psychology and computer science; 79 are taking calculus; 83 are taking psychology; and 63 are taking computer science. How many are taking none of the three subjects?

SOLUTION Let CALC, PSYCH, and COMPSCI denote the sets of students taking calculus, psychology, and computer science, respectively. Let U denote the set of all 165 students (see Figure 1.1.6). Since 8 students are taking calculus, psychology, and computer science, we write 8 in the region representing $\text{CALC} \cap \text{PSYCH} \cap \text{COMPSCI}$. Of the 33 students taking calculus and computer science, 8 are also taking psychology; thus 25 are taking calculus and computer science but not psychology. We write 25 in the region representing $\text{CALC} \cap \overline{\text{PSYCH}} \cap \text{COMPSCI}$. Similarly, we write 12 in the region representing $\text{CALC} \cap \text{PSYCH} \cap \overline{\text{COMPSCI}}$ and 16 in the region representing $\overline{\text{CALC}} \cap \text{PSYCH} \cap \text{COMPSCI}$. Of the 79 students taking calculus, 45 have now been accounted for. This leaves 34 students taking only calculus. We write 34 in the region representing $\text{CALC} \cap \overline{\text{PSYCH}} \cap \overline{\text{COMPSCI}}$. Similarly, we write 47 in the region representing $\overline{\text{CALC}} \cap \text{PSYCH} \cap \overline{\text{COMPSCI}}$ and 14 in the region representing

$\overline{\text{CALC}} \cap \overline{\text{PSYCH}} \cap \overline{\text{COMPSCI}}$. At this point, 156 students have been accounted for. This leaves 9 students taking none of the three subjects. ◀

Venn diagrams can also be used to visualize certain properties of sets. For example, by sketching both $\overline{(A \cup B)}$ and $\overline{A} \cap \overline{B}$ (see Figure 1.1.7), we see that these sets are equal. A formal proof would show that for every x , if $x \in \overline{(A \cup B)}$, then $x \in \overline{A} \cap \overline{B}$, and if $x \in \overline{A} \cap \overline{B}$, then $x \in \overline{(A \cup B)}$. We state many useful properties of sets as Theorem 1.1.22.

Theorem 1.1.22

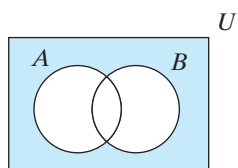


Figure 1.1.7 The shaded region depicts both $\overline{(A \cup B)}$ and $\overline{A} \cap \overline{B}$; thus these sets are equal.

Let U be a universal set and let A , B , and C be subsets of U . The following properties hold.

(a) Associative laws:

$$(A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C)$$

(b) Commutative laws:

$$A \cup B = B \cup A, \quad A \cap B = B \cap A$$

(c) Distributive laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(d) Identity laws:

$$A \cup \emptyset = A, \quad A \cap U = A$$

(e) Complement laws:

$$A \cup \overline{A} = U, \quad A \cap \overline{A} = \emptyset$$

(f) Idempotent laws:

$$A \cup A = A, \quad A \cap A = A$$

(g) Bound laws:

$$A \cup U = U, \quad A \cap \emptyset = \emptyset$$

(h) Absorption laws:

$$A \cup (A \cap B) = A, \quad A \cap (A \cup B) = A$$

(i) Involution law:

$$\overline{\overline{A}} = A^\dagger$$

(j) 0/1 laws:

$$\overline{\emptyset} = U, \quad \overline{U} = \emptyset$$

(k) De Morgan's laws for sets:

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}, \quad \overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

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For a biography of De Morgan, see goo.gl/vfki9p

Proof The proofs are left as exercises (Exercises 46–56, Section 2.1) to be done after more discussion of logic and proof techniques. ◀

We define the union of a collection of sets $\mathcal{S}$ to be those elements x belonging to at least one set X in $\mathcal{S}$. Formally,

$$\cup \mathcal{S} = \{x \mid x \in X \text{ for some } X \in \mathcal{S}\}.$$

[†] $\overline{\overline{A}}$ denotes the complement of the complement of A , that is, $\overline{\overline{A}} = \overline{(\overline{A})}$.

Similarly, we define the intersection of a collection of sets $\mathcal{S}$ to be those elements x belonging to every set X in $\mathcal{S}$. Formally,

$$\cap \mathcal{S} = \{x \mid x \in X \text{ for all } X \in \mathcal{S}\}.$$

Example 1.1.23

Let $\mathcal{S} = \{\{1, 2\}, \{1, 3\}, \{1, 7, 10\}\}$. Then $\cup \mathcal{S} = \{1, 2, 3, 7, 10\}$ since each of the elements 1, 2, 3, 7, 10 belongs to at least one set in $\mathcal{S}$, and no other element belongs to any of the sets in $\mathcal{S}$. Also $\cap \mathcal{S} = \{1\}$ since only the element 1 belong to every set in $\mathcal{S}$. ◀

If

$$\mathcal{S} = \{A_1, A_2, \dots, A_n\},$$

we write

$$\cup \mathcal{S} = \bigcup_{i=1}^n A_i, \quad \cap \mathcal{S} = \bigcap_{i=1}^n A_i,$$

and if

$$\mathcal{S} = \{A_1, A_2, \dots\},$$

we write

$$\cup \mathcal{S} = \bigcup_{i=1}^{\infty} A_i, \quad \cap \mathcal{S} = \bigcap_{i=1}^{\infty} A_i.$$

Example 1.1.24

For $i \geq 1$, define $A_i = \{i, i+1, \dots\}$ and $\mathcal{S} = \{A_1, A_2, \dots\}$. As examples, $A_1 = \{1, 2, 3, \dots\}$ and $A_2 = \{2, 3, 4, \dots\}$. Then

$$\cup \mathcal{S} = \bigcup_{i=1}^{\infty} A_i = \{1, 2, \dots\}, \quad \cap \mathcal{S} = \bigcap_{i=1}^{\infty} A_i = \emptyset. \quad \blacktriangleleft$$

A partition of a set X divides X into nonoverlapping subsets. More formally, a collection $\mathcal{S}$ of nonempty subsets of X is said to be a **partition** of the set X if every element in X belongs to exactly one member of $\mathcal{S}$. Notice that if $\mathcal{S}$ is a partition of X , $\mathcal{S}$ is pairwise disjoint and $\cup \mathcal{S} = X$.

Example 1.1.25

Since each element of $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ is in exactly one member of $\mathcal{S} = \{\{1, 4, 5\}, \{2, 6\}, \{3\}, \{7, 8\}\}$, $\mathcal{S}$ is a partition of X . ◀

At the beginning of this section, we pointed out that a set is an unordered collection of elements; that is, a set is determined by its elements and not by any particular order in which the elements are listed. Sometimes, however, we do want to take order into account. An **ordered pair** of elements, written (a, b) , is considered distinct from the ordered pair (b, a) , unless, of course, $a = b$. To put it another way, $(a, b) = (c, d)$ precisely when $a = c$ and $b = d$. If X and Y are sets, we let $X \times Y$ denote the set of all ordered pairs (x, y) where $x \in X$ and $y \in Y$. We call $X \times Y$ the **Cartesian product** of X and Y .

Example 1.1.26

If $X = \{1, 2, 3\}$ and $Y = \{a, b\}$, then

$$X \times Y = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

$$Y \times X = \{(a, 1), (b, 1), (a, 2), (b, 2), (a, 3), (b, 3)\}$$

$$X \times X = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

$$Y \times Y = \{(a, a), (a, b), (b, a), (b, b)\}. \quad \blacktriangleleft$$

Example 1.1.26 shows that, in general, $X \times Y \neq Y \times X$.

Notice that in Example 1.1.26, $|X \times Y| = |X| \cdot |Y|$ (both are equal to 6). The reason is that there are 3 ways to choose an element of X for the first member of the ordered pair, there are 2 ways to choose an element of Y for the second member of the ordered pair, and $3 \cdot 2 = 6$ (see Figure 1.1.8). The preceding argument holds for arbitrary finite sets X and Y ; it is always true that $|X \times Y| = |X| \cdot |Y|$.

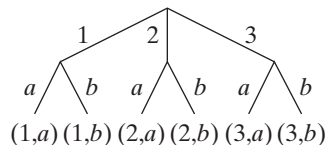


Figure 1.1.8 $|X \times Y| = |X| \cdot |Y|$, where $X = \{1, 2, 3\}$ and $Y = \{a, b\}$. There are 3 ways to choose an element of X for the first member of the ordered pair (shown at the top of the diagram) and, for each of these choices, there are 2 ways to choose an element of Y for the second member of the ordered pair (shown at the bottom of the diagram). Since there are 3 groups of 2, there are $3 \cdot 2 = 6$ elements in $X \times Y$ (labeled at the bottom of the figure).

Example 1.1.27 A restaurant serves four appetizers,

$r = \text{ribs}, \quad n = \text{nachos}, \quad s = \text{shrimp}, \quad f = \text{fried cheese},$

and three entrees,

$c = \text{chicken}, \quad b = \text{beef}, \quad t = \text{trout}.$

If we let $A = \{r, n, s, f\}$ and $E = \{c, b, t\}$, the Cartesian product $A \times E$ lists the 12 possible dinners consisting of one appetizer and one entree. ◀

Ordered lists need not be restricted to two elements. An **n -tuple**, written $(a_1, a_2, \dots, a_n)$, takes order into account; that is,

$$(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n)$$

precisely when

$$a_1 = b_1, a_2 = b_2, \dots, a_n = b_n.$$

The Cartesian product of sets $X_1, X_2, \dots, X_n$ is defined to be the set of all n -tuples $(x_1, x_2, \dots, x_n)$ where $x_i \in X_i$ for $i = 1, \dots, n$; it is denoted $X_1 \times X_2 \times \dots \times X_n$.

Example 1.1.28 If $X = \{1, 2\}$, $Y = \{a, b\}$, and $Z = \{\alpha, \beta\}$, then

$$X \times Y \times Z = \{(1, a, \alpha), (1, a, \beta), (1, b, \alpha), (1, b, \beta), (2, a, \alpha), (2, a, \beta), (2, b, \alpha), (2, b, \beta)\}.$$

Notice that in Example 1.1.28, $|X \times Y \times Z| = |X| \cdot |Y| \cdot |Z|$. In general,

$$|X_1 \times X_2 \times \dots \times X_n| = |X_1| \cdot |X_2| \cdot \dots \cdot |X_n|.$$

We leave the proof of this last statement as an exercise (see Exercise 27, Section 2.4).

Example 1.1.29 If A is a set of appetizers, E is a set of entrees, and D is a set of desserts, the Cartesian product $A \times E \times D$ lists all possible dinners consisting of one appetizer, one entree, and one dessert. ◀

1.1 Problem-Solving Tips

- To verify that two sets A and B are equal, written $A = B$, show that for every x , if $x \in A$, then $x \in B$, and if $x \in B$, then $x \in A$.
- To verify that two sets A and B are *not* equal, written $A \neq B$, find at least one element that is in A but not in B , or find at least one element that is in B but not in A . One or the other conditions suffices; you need not (and may not be able to) show both conditions.
- To verify that A is a subset of B , written $A \subseteq B$, show that for every x , if $x \in A$, then $x \in B$. Notice that if A is a subset of B , it is possible that $A = B$.
- To verify that A is *not* a subset of B , find at least one element that is in A but not in B .
- To verify that A is a proper subset of B , written $A \subset B$, verify that A is a subset of B as described previously, and that $A \neq B$, that is, that there is at least one element that is in B but not in A .
- To visualize relationships among sets, use a Venn diagram. A Venn diagram can suggest whether a statement about sets is true or false.
- A set of elements is determined by its members; order is irrelevant. On the other hand, ordered pairs and n -tuples take order into account.

1.1 Review Exercises

- [†]1. What is a set?
2. What is set notation?
3. Describe the sets $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{Z}^+$, $\mathbf{Q}^+$, $\mathbf{R}^+$, $\mathbf{Z}^-$, $\mathbf{Q}^-$, $\mathbf{R}^-$, $\mathbf{Z}^{\text{nonneg}}$, $\mathbf{Q}^{\text{nonneg}}$, and $\mathbf{R}^{\text{nonneg}}$, and give two examples of members of each set.
4. If X is a finite set, what is $|X|$?
5. How do we denote x is an element of the set X ?
6. How do we denote x is not an element of the set X ?
7. How do we denote the empty set?
8. Define *set X is equal to set Y* . How do we denote X is equal to Y ?
9. Explain a method of verifying that sets X and Y are equal.
10. Explain a method of verifying that sets X and Y are *not* equal.
11. Define *X is a subset of Y* . How do we denote X is a subset of Y ?
12. Explain a method of verifying that X is a subset of Y .
13. Explain a method of verifying that X is *not* a subset of Y .
14. Define *X is a proper subset of Y* . How do we denote X is a proper subset of Y ?
15. Explain a method of verifying that X is a proper subset of Y .
16. What is the power set of X ? How is it denoted?
17. Define *X union Y* . How is the union of X and Y denoted?
18. If $\mathcal{S}$ is a family of sets, how do we define the union of $\mathcal{S}$? How is the union denoted?
19. Define *X intersect Y* . How is the intersection of X and Y denoted?
20. If $\mathcal{S}$ is a family of sets, how do we define the intersection of $\mathcal{S}$? How is the intersection denoted?
21. Define *X and Y are disjoint sets*.
22. What is a pairwise disjoint family of sets?
23. Define the *difference* of sets X and Y . How is the difference denoted?
24. What is a universal set?
25. What is the complement of the set X ? How is it denoted?
26. What is a Venn diagram?
27. Draw a Venn diagram of three sets and identify the set represented by each region.

[†]Exercise numbers in color indicate that a hint or solution appears at the back of the book in the section following the References.

28. State the associative laws for sets.
29. State the commutative laws for sets.
30. State the distributive laws for sets.
31. State the identity laws for sets.
32. State the complement laws for sets.
33. State the idempotent laws for sets.
34. State the bound laws for sets.
35. State the absorption laws for sets.
36. State the involution law for sets.
37. State the 0/1 laws for sets.
38. State De Morgan's laws for sets.
39. What is a partition of a set X ?
40. Define the *Cartesian product* of sets X and Y . How is this Cartesian product denoted?
41. Define the *Cartesian product* of the sets $X_1, X_2, \dots, X_n$. How is this Cartesian product denoted?

1.1 Exercises

In Exercises 1–16, let the universe be the set $U = \{1, 2, 3, \dots, 10\}$. Let $A = \{1, 4, 7, 10\}$, $B = \{1, 2, 3, 4, 5\}$, and $C = \{2, 4, 6, 8\}$. List the elements of each set.

1. $A \cup B$
2. $B \cap C$
3. $A - B$
4. $B - A$
5. $\overline{C}$
6. $U - A$
7. $\overline{U}$
8. $B \cup \emptyset$
9. $B \cap \emptyset$
10. $A \cup U$
11. $B \cap U$
12. $A \cap (B \cup C)$
13. $\overline{B} \cap (C - A)$
14. $A - (B \cap C)$
15. $\overline{A \cap B \cup C}$
16. $(A \cup B) - (C - B)$

In Exercises 17–27, let the universe be the set $\mathbf{Z}^+$. Let $X = \{1, 2, 3, 4, 5\}$ and let Y be the set of positive, even integers. In set-builder notation, $Y = \{2n \mid n \in \mathbf{Z}^+\}$. In Exercises 18–27, give a mathematical notation for the set by listing the elements if the set is finite, by using set-builder notation if the set is infinite, or by using a predefined set such as $\emptyset$.

17. Describe $\overline{Y}$ in words.
18. $\overline{X}$
19. $\overline{Y}$
20. $X \cap Y$
21. $X \cup Y$
22. $\overline{X} \cap Y$
23. $\overline{X} \cup Y$
24. $X \cap \overline{Y}$
25. $X \cup \overline{Y}$
26. $\overline{X} \cap \overline{Y}$
27. $\overline{X} \cup \overline{Y}$

28. What is the cardinality of $\emptyset$?
29. What is the cardinality of $\{\emptyset\}$?
30. What is the cardinality of $\{a, b, a, c\}$?
31. What is the cardinality of $\{\{a\}, \{a, b\}, \{a, c\}, a, b\}$?

In Exercises 32–35, show, as in Examples 1.1.2 and 1.1.3, that $A = B$.

32. $A = \{1, 2, 3, 4\}$, $B = \{4, 3, 2, 1\}$
33. $C = \{1, 2, 3\}$, $D = \{2, 3, 4\}$, $A = \{2, 3\}$, $B = C \cap D$

34. $A = \{1, 2, 3\}$, $B = \{n \mid n \in \mathbf{Z}^+ \text{ and } n^2 < 10\}$
35. $A = \{x \mid x^2 - 4x + 4 = 1\}$, $B = \{1, 3\}$

In Exercises 36–39, show, as in Example 1.1.4, that $A \neq B$.

36. $A = \{1, 2, 3\}$, $B = \emptyset$
37. $A = \{1, 3\}$, $B = \{x \mid x^3 - 2x^2 + 4x - 8 = 0\}$
38. $A = \{1, 3, 5\}$, $B = \{n \mid n \in \mathbf{Z}^+ \text{ and } n^2 - 1 \leq n\}$
39. $B = \{1, 2, 3, 4\}$, $C = \{2, 4, 6, 8\}$, $A = B \cap C$

In Exercises 40–43, determine whether each pair of sets is equal.

40. $\{1, 2, 2, 3\}$, $\{1, 2, 3\}$
41. $\{1, 1, 3\}$, $\{3, 3, 1\}$
42. $\{x \mid x^2 - 2x = 3\}$, $\{-1, 3\}$
43. $\{x \mid x \in \mathbf{R} \text{ and } 0 < x \leq 2\}$, $\{1, 2\}$

In Exercises 44–47, show, as in Examples 1.1.5 and 1.1.6, that $A \subseteq B$.

44. $A = \{1, 2\}$, $B = \{3, 2, 1\}$
45. $A = \{1, 2\}$, $B = \{x \mid x^3 - 6x^2 + 11x = 6\}$
46. $A = \{1\} \times \{1, 2\}$, $B = \{1\} \times \{1, 2, 3\}$
47. $A = \{2n \mid n \in \mathbf{Z}^+\}$, $B = \{n \mid n \in \mathbf{Z}^+\}$

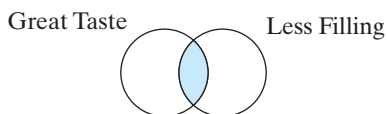
In Exercises 48–51, show, as in Example 1.1.9, that A is not a subset of B .

48. $A = \{1, 2, 3\}$, $B = \{1, 2\}$
49. $A = \{x \mid x^3 - 2x^2 + 4x - 8 = 0\}$, $B = \{1, 3\}$
50. $A = \{1, 2, 3, 4\}$, $C = \{5, 6, 7, 8\}$, $B = \{n \mid n \in A \text{ and } n + m = 8 \text{ for some } m \in C\}$
51. $A = \{1, 2, 3\}$, $B = \emptyset$

In Exercises 52–59, draw a Venn diagram and shade the given set.

52. $A \cap \overline{B}$
53. $\overline{A} - B$
54. $B \cup (B - A)$
55. $(A \cup B) - B$
56. $B \cap \overline{(C \cup A)}$
57. $(\overline{A} \cup B) \cap (\overline{C} - A)$
58. $((C \cap A) - \overline{(B - A)}) \cap C$
59. $(B - \overline{C}) \cup ((B - \overline{A}) \cap (C \cup B))$

60. A television commercial for a popular beverage showed the following Venn diagram



What does the shaded area represent?

Exercises 61–65 refer to a group of 191 students, of which 10 are taking French, business, and music; 36 are taking French and business; 20 are taking French and music; 18 are taking business and music; 65 are taking French; 76 are taking business; and 63 are taking music.

61. How many are taking French and music but not business?
 62. How many are taking music and neither French nor business?
 63. How many are taking French or business (or both)?
 64. How many are taking music or French (or both) but not business?
 65. How many are taking none of the three subjects?
 66. A television poll of 151 persons found that 68 watched “Law and Disorder”; 61 watched “25”; 52 watched “The Tenors”; 16 watched both “Law and Disorder” and “25”; 25 watched both “Law and Disorder” and “The Tenors”; 19 watched both “25” and “The Tenors”; and 26 watched none of these shows. How many persons watched all three shows?
 67. In a group of students, each student is taking a mathematics course or a computer science course or both. One-fifth of those taking a mathematics course are also taking a computer science course, and one-eighth of those taking a computer science course are also taking a mathematics course. Are more than one-third of the students taking a mathematics course?

In Exercises 68–71, let $X = \{1, 2\}$ and $Y = \{a, b, c\}$. List the elements in each set.

68. $X \times Y$ 69. $Y \times X$
 70. $X \times X$ 71. $Y \times Y$

In Exercises 72–75, let $X = \{1, 2\}$, $Y = \{a\}$, and $Z = \{\alpha, \beta\}$. List the elements of each set.

72. $X \times Y \times Z$ 73. $Z \times Y \times X$
 74. $X \times X \times X$ 75. $Y \times X \times Y \times Z$

In Exercises 76–82, give a geometric description of each set in words. Consider the elements of the sets to be coordinates. For example, $\mathbf{R} \times \mathbf{Z}$ is the set $\{(x, n) \mid x \in \mathbf{R} \text{ and } n \in \mathbf{Z}\}$. Interpreting the ordered pairs (x, n) as coordinates in the plane, the graph of all

such ordered pairs is the set of all parallel horizontal lines spaced one unit apart, one of which passes through $(0, 0)$.

76. $\mathbf{R} \times \mathbf{R}$
 77. $\mathbf{Z} \times \mathbf{R}$
 78. $\mathbf{R} \times \mathbf{Z}^{\text{nonneg}}$
 79. $\mathbf{Z} \times \mathbf{Z}$
 80. $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$
 81. $\mathbf{R} \times \mathbf{R} \times \mathbf{Z}$
 82. $\mathbf{R} \times \mathbf{Z} \times \mathbf{Z}$

In Exercises 83–86, list all partitions of the set.

83. $\{1\}$ 84. $\{1, 2\}$
 85. $\{a, b, c\}$ 86. $\{a, b, c, d\}$

In Exercises 87–92, answer true or false.

87. $\{x\} \subseteq \{x\}$ 88. $\{x\} \in \{x\}$
 89. $\{x\} \in \{x, \{x\}\}$ 90. $\{x\} \subseteq \{x, \{x\}\}$
 91. $\{1, 2\} \in \mathcal{P}(\{1, 2\})$ 92. $\{1, 2\} \subseteq \mathcal{P}(\{1, 2\})$
 93. List the members of $\mathcal{P}(\{a, b\})$. Which are proper subsets of $\{a, b\}$?
 94. List the members of $\mathcal{P}(\{a, b, c, d\})$. Which are proper subsets of $\{a, b, c, d\}$?
 95. If X has 12 members, how many members does $\mathcal{P}(X)$ have? How many proper subsets does X have?
 96. If X has n members, how many proper subsets does X have?

In Exercises 97–100, what relation must hold between sets A and B in order for the given condition to be true?

97. $A \cap B = A$ 98. $A \cup B = A$
 99. $\bar{A} \cap U = \emptyset$ 100. $\bar{A} \cap \bar{B} = \bar{B}$

The symmetric difference of two sets A and B is the set

$$A \triangle B = (A \cup B) - (A \cap B).$$

101. If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$, find $A \triangle B$.
 102. Describe the symmetric difference of sets A and B in words.
 103. Given a universe U , describe $A \triangle A$, $A \triangle \bar{A}$, $U \triangle A$, and $\emptyset \triangle A$.
 104. Let C be a circle and let $\mathcal{D}$ be the set of all diameters of C . What is $\cap \mathcal{D}$? (Here, by “diameter” we mean a line segment through the center of the circle with its endpoints on the circumference of the circle.)

- †★105. Let P denote the set of integers greater than 1. For $i \geq 2$, define

$$X_i = \{ik \mid k \in P\}.$$

Describe $P - \bigcup_{i=2}^{\infty} X_i$.

† A starred exercise indicates a problem of above-average difficulty.

1.2 Propositions

Which of sentences (a)–(f) are either true or false (but not both)?

- (a) The only positive integers that divide[†] 7 are 1 and 7 itself.
- (b) Alfred Hitchcock won an Academy Award in 1940 for directing *Rebecca*.
- (c) For every positive integer n , there is a prime number[‡] larger than n .
- (d) Earth is the only planet in the universe that contains life.
- (e) Buy two tickets to the “Unhinged Universe” rock concert for Friday.
- (f) $x + 4 = 6$.

Sentence (a), which is another way to say that 7 is prime, is true.

Sentence (b) is false. Although *Rebecca* won the Academy Award for best picture in 1940, John Ford won the directing award for *The Grapes of Wrath*. It is a surprising fact that Alfred Hitchcock never won an Academy Award for directing.

Sentence (c), which is another way to say that the number of primes is infinite, is true.

Sentence (d) is either true or false (but not both), but no one knows which at this time.

Sentence (e) is neither true nor false [sentence (e) is a command].

The truth of equation (f) depends on the value of the variable x .

A sentence that is either true or false, but not both, is called a **proposition**. Sentences (a)–(d) are propositions, whereas sentences (e) and (f) are not propositions. A proposition is typically expressed as a declarative sentence (as opposed to a question, command, etc.). Propositions are the basic building blocks of any theory of logic.

We will use variables, such as p , q , and r , to represent propositions, much as we use letters in algebra to represent numbers. We will also use the notation

$$p: 1 + 1 = 3$$

to define p to be the proposition $1 + 1 = 3$.

In ordinary speech and writing, we combine propositions using connectives such as *and* and *or*. For example, the propositions “It is raining” and “It is cold” can be combined to form the single proposition “It is raining and it is cold.” The formal definitions of *and* and *or* follow.

Definition 1.2.1 ▶ Let p and q be propositions.

The *conjunction* of p and q , denoted $p \wedge q$, is the proposition

$$p \text{ and } q.$$

The *disjunction* of p and q , denoted $p \vee q$, is the proposition

$$p \text{ or } q.$$

Example 1.2.2

If p : It is raining, and q : It is cold, then the conjunction of p and q is

$$p \wedge q: \text{ It is raining and it is cold.} \quad (1.2.1)$$

The disjunction of p and q is

[†]“Divides” means “divides evenly.” More formally, we say that a nonzero integer d *divides* an integer m if there is an integer q such that $m = dq$. We call q the *quotient*. We will explore the integers in detail in Chapter 5.

[‡]An integer $n > 1$ is *prime* if the only positive integers that divide n are 1 and n itself. For example, 2, 3, and 11 are prime numbers.

$p \vee q$: It is raining or it is cold. 

The truth value of the conjunction $p \wedge q$ is determined by the truth values of p and q , and the definition is based upon the usual interpretation of “and.” Consider the proposition (1.2.1) of Example 1.2.2. If it is raining (i.e., p is true) and it is also cold (i.e., q is also true), then we would consider the proposition (1.2.1) to be true. However, if it is not raining (i.e., p is false) or it is not cold (i.e., q is false) or both, then we would consider the proposition (1.2.1) to be false.

The truth values of propositions such as conjunctions and disjunctions can be described by **truth tables**. The truth table of a proposition P made up of the individual propositions $p_1, \dots, p_n$ lists all possible combinations of truth values for $p_1, \dots, p_n$, T denoting true and F denoting false, and for each such combination lists the truth value of P . We use a truth table to formally define the truth value of $p \wedge q$.

A truth table for a proposition P made up of n propositions has $r = 2^n$ rows. Traditionally, for the first proposition, the first $r/2$ rows list T and the last $r/2$ rows list F. The next proposition has $r/4$ T's alternate with $r/4$ F's. The next proposition has $r/8$ T's alternate with $r/8$ F's, and so on. For example, a proposition P made up of three propositions p_1, p_2 , and p_3 has $8 = 2^3$ rows. Proposition p_1 will list 4 = $8/2$ T's followed by 4 F's. Proposition p_2 will list 2 = $8/4$ T's followed by 2 F's, followed by 2 T's, followed by 2 F's. Proposition p_3 will have one T, followed by one F, followed by one T, and so on. The truth table without the truth values of P would be

p_1	p_2	p_3	P
T	T	T	Here is where the truth values of P go.
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

Notice that all possible combinations of truth values for p_1, p_2, p_3 are listed.

Definition 1.2.3 ► The truth value of the proposition $p \wedge q$ is defined by the truth table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F



Definition 1.2.3 states that the conjunction $p \wedge q$ is true provided that p and q are both true; $p \wedge q$ is false otherwise.

Example 1.2.4

If p : A decade is 10 years, and q : A millennium is 100 years, then p is true, q is false (a millennium is 1000 years), and the conjunction,

$p \wedge q$: A decade is 10 years and a millennium is 100 years,

is false. 

Example 1.2.5 Most programming languages define “and” exactly as in Definition 1.2.3. For example, in the Java programming language, (logical) “and” is denoted `&&`, and the expression

$$x < 10 \ \&\& \ y > 4$$

is true precisely when the value of the variable x is less than 10 (i.e., $x < 10$ is true) and the value of the variable y is greater than 4 (i.e., $y > 4$ is also true). ◀

The truth value of the disjunction $p \vee q$ is also determined by the truth values of p and q , and the definition is based upon the “inclusive” interpretation of “or.” Consider the proposition,

$$p \vee q: \text{ It is raining or it is cold,} \tag{1.2.2}$$

of Example 1.2.2. If it is raining (i.e., p is true) or it is cold (i.e., q is also true) *or both*, then we would consider the proposition (1.2.2) to be true (i.e., $p \vee q$ is true). If it is not raining (i.e., p is false) and it is not cold (i.e., q is also false), then we would consider the proposition (1.2.2) to be false (i.e., $p \vee q$ is false). The **inclusive-or** of propositions p and q is true if p or q , *or both*, is true, and false otherwise. There is also an **exclusive-or** (see Exercise 67) that defines $p \text{ xor } q$ to be true if p or q , *but not both*, is true, and false otherwise.

Definition 1.2.6 ▶ The truth value of the proposition $p \vee q$, called the *inclusive-or* of p and q , is defined by the truth table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 1.2.7 If p : A millennium is 100 years, and q : A millennium is 1000 years, then p is false, q is true, and the disjunction,

$$p \vee q: \text{ A millennium is 100 years or a millennium is 1000 years,}$$

is true. ◀

Example 1.2.8 Most programming languages define (inclusive) “or” exactly as in Definition 1.2.6. For example, in the Java programming language, (logical) “or” is denoted `||`, and the expression

$$x < 10 \ || \ y > 4$$

is true precisely when the value of the variable x is less than 10 (i.e., $x < 10$ is true) or the value of the variable y is greater than 4 (i.e., $y > 4$ is true) or both. ◀

In ordinary language, propositions being combined (e.g., p and q combined to give the proposition $p \vee q$) are normally related; but in logic, these propositions are not required to refer to the same subject matter. For example, in logic, we permit propositions such as

$$3 < 5 \text{ or Paris is the capital of England.}$$

Logic is concerned with the form of propositions and the relation of propositions to each other and not with the subject matter itself. (The given proposition is true because $3 < 5$ is true.)

The final operator on a proposition p that we discuss in this section is the **negation** of p .

Definition 1.2.9 ▶ The *negation* of p , denoted $\neg p$, is the proposition

not p .

The truth value of the proposition $\neg p$ is defined by the truth table

p	$\neg p$
T	F
F	T

In English, we sometimes write $\neg p$ as “It is not the case that p .” For example, if

p : Paris is the capital of England,

the negation of p could be written

$\neg p$: It is not the case that Paris is the capital of England,

or more simply as

$\neg p$: Paris is not the capital of England.

Example 1.2.10

If

p : π was calculated to 1,000,000 decimal digits in 1954,

the negation of p is the proposition

$\neg p$: π was not calculated to 1,000,000 decimal digits in 1954.

It was not until 1973 that 1,000,000 decimal digits of π were computed; so, p is false. (Since then over 12 trillion decimal digits of π have been computed.) Since p is false, $\neg p$ is true.

Example 1.2.11

Most programming languages define “not” exactly as in Definition 1.2.9. For example, in the Java programming language, “not” is denoted `!`, and the expression

`!(x < 10)`

is true precisely when the value of the variable x is not less than 10 (i.e., x is greater than or equal to 10).

In expressions involving some or all of the operators $\neg$, $\wedge$, and $\vee$, in the absence of parentheses, we first evaluate $\neg$, then $\wedge$, and then $\vee$. We call such a convention **operator precedence**. In algebra, operator precedence tells us to evaluate $\cdot$ and $/$ before $+$ and $-$.

Example 1.2.12

Given that proposition p is false, proposition q is true, and proposition r is false, determine whether the proposition $\neg p \vee q \wedge r$ is true or false.

SOLUTION We first evaluate $\neg p$, which is true. We next evaluate $q \wedge r$, which is false. Finally, we evaluate $\neg p \vee q \wedge r$, which is true. ▶

Example 1.2.13

Searching the Web A variety of Web search engines are available (e.g., Google, Yahoo!, Baidu) that allow the user to enter keywords that the search engine then tries to match with Web pages. For example, entering *mathematics* produces a (huge!) list of pages that contain the word “mathematics.” Some search engines allow the user to use *and*, *or*, and *not* operators to combine keywords (see Figure 1.2.1), thus allowing more complex searches. In the Google search engine, *and* is the default operator so that, for example, entering *discrete mathematics* produces a list of pages containing both of the words “discrete” and “mathematics.” The *or* operator is OR, and the *not* operator is the minus sign $-$. Furthermore, enclosing a phrase, typically with embedded spaces, in double quotation marks causes the phrase to be treated as a single word. For example, to search for pages containing the keywords

“Shonda Rhimes” and (Grey’s or Scandal) and (not Murder),

the user could enter

“Shonda Rhimes” Grey’s OR Scandal -Murder ▶



Figure 1.2.1 The Google search engine, which allows the user to use *and* (space), *or* (OR), and *not* ($-$) operators to combine keywords. As shown, Google found about 573,000 Web pages containing “Shonda Rhimes” and (Grey’s or Scandal) and (not Murder).

1.2 Problem-Solving Tips

Although there may be a shorter way to determine the truth values of a proposition P formed by combining propositions $p_1, \dots, p_n$ using operators such as $\neg$ and $\vee$, a truth table will always supply all possible truth values of P for various truth values of the constituent propositions $p_1, \dots, p_n$.

1.2 Review Exercises

1. What is a proposition?
2. What is a truth table?
3. What is the conjunction of p and q ? How is it denoted?
4. Give the truth table for the conjunction of p and q .
5. What is the disjunction of p and q ? How is it denoted?
6. Give the truth table for the disjunction of p and q .
7. What is the negation of p ? How is it denoted?
8. Give the truth table for the negation of p .

1.2 Exercises

Determine whether each sentence in Exercises 1–12 is a proposition. If the sentence is a proposition, write its negation. (You are not being asked for the truth values of the sentences that are propositions.)

1. $3 + 5 = 7$.
2. $6 + 9 = 15$.
3. $x + 9 = 15$.
4. $\pi = 3.14$.
5. Waiter, will you serve the nuts—I mean, would you serve the guests the nuts?
6. For some positive integer n , $19340 = n \cdot 17$.
7. Phil Collins was a member of “Genesis”.
8. Peel me a grape.
9. The line “Play it again, Sam” occurs in the movie *Casablanca*.
10. Every even integer greater than 4 is the sum of two primes.
11. The difference of two primes.
- ★12. This statement is false.

Exercises 13–16 refer to a coin that is flipped 10 times. Write the negation of the proposition.

13. Ten heads were obtained.
14. Some tails were obtained.
15. Some heads and some tails were obtained.
16. At least one head was obtained.

Given that proposition p is false, proposition q is true, and proposition r is false, determine whether each proposition in Exercises 17–22 is true or false.

17. $p \vee q$
18. $\neg p \vee \neg q$
19. $\neg p \vee q$
20. $\neg p \vee \neg(q \wedge r)$
21. $(\neg p \vee q) \wedge \neg(p \vee r)$
22. $(p \vee \neg r) \wedge \neg((q \vee r) \vee \neg(r \vee p))$

Write the truth table of each proposition in Exercises 23–30.

23. $p \wedge \neg q$
24. $(\neg p \vee \neg q) \vee p$
25. $(p \vee q) \wedge \neg p$
26. $(p \wedge q) \wedge \neg p$
27. $(p \vee q) \wedge (\neg p \vee \neg q)$
28. $\neg(p \wedge q) \vee (r \wedge \neg p)$
29. $(p \vee q) \wedge (\neg p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q)$
30. $\neg(p \wedge q) \vee (\neg q \vee r)$

In Exercises 31–33, represent the given proposition symbolically

by letting

$$p: 5 < 9, \quad q: 9 < 7, \quad r: 5 < 7.$$

Determine whether each proposition is true or false.

31. $5 < 9$ and $9 < 7$.
32. It is not the case that $(9 < 7$ or $5 < 7)$.
33. $5 < 9$ or it is not the case that $(9 < 7$ and $5 < 7)$.

In Exercises 34–39, formulate the symbolic expression in words using

p : Lee takes computer science.
 q : Lee takes mathematics.

34. $\neg p$
35. $p \wedge q$
36. $p \vee q$
37. $p \vee \neg q$
38. $p \wedge \neg q$
39. $\neg p \wedge \neg q$

In Exercises 40–44, formulate the symbolic expression in words using

p : You play football.
 q : You miss the midterm exam.
 r : You pass the course.

40. $p \wedge q$
41. $\neg q \wedge r$
42. $p \vee q \vee r$
43. $\neg(p \vee q) \vee r$
44. $(p \wedge q) \vee (\neg q \wedge r)$

In Exercises 45–49, formulate the symbolic expression in words using

p : Today is Monday.
 q : It is raining.
 r : It is hot.

45. $p \vee q$
46. $\neg p \wedge (q \vee r)$
47. $\neg(p \vee q) \wedge r$
48. $(p \wedge q) \wedge \neg(r \vee p)$
49. $(p \wedge (q \vee r)) \wedge (r \vee (q \vee p))$

In Exercises 50–55, represent the proposition symbolically by letting

p : There is a hurricane.
 q : It is raining.

50. There is no hurricane.

51. There is a hurricane and it is raining.
52. There is no hurricane, but it is raining.
53. There is no hurricane and it is not raining.
54. Either there is a hurricane or it is raining (or both).
55. Either there is a hurricane or it is raining, but there is no hurricane.

In Exercises 56–60, represent the proposition symbolically by letting

p : You run 10 laps daily.
 q : You are healthy.
 r : You take multi-vitamins.

56. You do not run 10 laps daily, but you are healthy.
57. You run 10 laps daily, you take multi-vitamins, and you are healthy.
58. You run 10 laps daily or you take multi-vitamins, and you are healthy.
59. You do not run 10 laps daily, you do not take multi-vitamins, and you are not healthy.
60. Either you are healthy or you do not run 10 laps daily, and you do not take multi-vitamins.

In Exercises 61–66, represent the proposition symbolically by letting

p : You heard the “Flying Pigs” rock concert.
 q : You heard the “Y2K” rock concert.
 r : You have sore eardrums.

61. You heard the “Flying Pigs” rock concert, and you have sore eardrums.
62. You heard the “Flying Pigs” rock concert, but you do not have sore eardrums.
63. You heard the “Flying Pigs” rock concert, you heard the “Y2K” rock concert, and you have sore eardrums.

64. You heard either the “Flying Pigs” rock concert or the “Y2K” rock concert, but you do not have sore eardrums.
65. You did not hear the “Flying Pigs” rock concert and you did not hear the “Y2K” rock concert, but you have sore eardrums.
66. It is not the case that: You heard the “Flying Pigs” rock concert or you heard the “Y2K” rock concert or you do not have sore eardrums.
67. Give the truth table for the exclusive-or of p and q in which $p \text{ exor } q$ is true if either p or q , but not both, is true.

In Exercises 68–74, state the meaning of each sentence if “or” is interpreted as the inclusive-or; then, state the meaning of each sentence if “or” is interpreted as the exclusive-or (see Exercise 67). In each case, which meaning do you think is intended?

68. To enter Utopia, you must show a driver’s license or a passport.
69. To enter Utopia, you must possess a driver’s license or a passport.
70. The prerequisite to data structures is a course in Java or C++.
71. The car comes with a cupholder that heats or cools your drink.
72. We offer \$1000 cash or 0 percent interest for two years.
73. Do you want your coffee black or with milk?
74. The meeting will be canceled if fewer than 10 persons sign up or at least 3 inches of snow falls.
75. At one time, the following ordinance was in effect in Naperville, Illinois: “It shall be unlawful for any person to keep more than three [3] dogs and three [3] cats upon his property within the city.” Was Charles Marko, who owned five dogs and no cats, in violation of the ordinance? Explain.
76. Write a command to search the Web for concert venues in North or South London.
77. Write a command to search the Web for information on lung disease other than cancer.
78. Write a command to search the Web for countries in the European Union that are not in the Schengen zone.

1.3 Conditional Propositions and Logical Equivalence

The dean has announced that

If the Mathematics Department gets an additional \$60,000,
 then it will hire one new faculty member. (1.3.1)

Statement (1.3.1) states that on the condition that the Mathematics Department gets an additional \$60,000, then the Mathematics Department will hire one new faculty member. A proposition such as (1.3.1) is called a **conditional proposition**.

Definition 1.3.1 ▶ If p and q are propositions, the proposition

if p then q (1.3.2)

is called a *conditional proposition* and is denoted

$$p \rightarrow q.$$

The proposition p is called the *hypothesis* (or *antecedent*), and the proposition q is called the *conclusion* (or *consequent*). ◀

Example 1.3.2

If we define

p : The Mathematics Department gets an additional \$60,000,

q : The Mathematics Department will hire one new faculty member,

then proposition (1.3.1) assumes the form (1.3.2). The hypothesis is the statement “The Mathematics Department gets an additional \$60,000,” and the conclusion is the statement “The Mathematics Department will hire one new faculty member.” ◀

What is the truth value of the dean’s statement (1.3.1)? First, suppose that the Mathematics Department gets an additional \$60,000. If the Mathematics Department does hire an additional faculty member, surely the dean’s statement is true. (Using the notation of Example 1.3.2, if p and q are both true, then $p \rightarrow q$ is true.) On the other hand, if the Mathematics Department gets an additional \$60,000 and does *not* hire an additional faculty member, the dean is wrong—statement (1.3.1) is false. (If p is true and q is false, then $p \rightarrow q$ is false.) Now, suppose that the Mathematics Department does *not* get an additional \$60,000. In this case, the Mathematics Department might or might not hire an additional faculty member. (Perhaps a member of the department retires and someone is hired to replace the retiree. On the other hand, the department might not hire anyone.) Surely we would not consider the dean’s statement to be false. Thus, if the Mathematics Department does *not* get an additional \$60,000, the dean’s statement must be true, regardless of whether the department hires an additional faculty member or not. (If p is false, then $p \rightarrow q$ is true whether q is true or false.) This discussion motivates the following definition.

Definition 1.3.3 ▶ The truth value of the conditional proposition $p \rightarrow q$ is defined by the following truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

For those who need additional evidence that we should define $p \rightarrow q$ to be true when p is false, we offer further justification. Most people would agree that the proposition,

$$\text{For all real numbers } x, \text{ if } x > 0, \text{ then } x^2 > 0, \quad (1.3.3)$$

is true. (In Section 1.5, we will discuss such “for all” statements formally and in detail.) In the following discussion, we let $P(x)$ denote $x > 0$ and $Q(x)$ denote $x^2 > 0$. That proposition (1.3.3) is true means that no matter which real number we replace x with, the proposition

$$\text{if } P(x) \text{ then } Q(x) \quad (1.3.4)$$

that results is true. For example, if $x = 3$, then $P(3)$ and $Q(3)$ are both true ($3 > 0$ and $3^2 > 0$ are both true), and, by Definition 1.3.3, (1.3.4) is true. Now let us consider the situation when $P(x)$ is false. If $x = -2$, then $P(-2)$ is false ($-2 > 0$ is false) and $Q(-2)$ is true [$(-2)^2 > 0$ is true]. In order for proposition (1.3.4) to be true in this case, we must define $p \rightarrow q$ to be true when p is false and q is true. This is exactly what occurs in the third line of the truth table of Definition 1.3.3. If $x = 0$, then $P(0)$ and $Q(0)$ are both false ($0 > 0$ and $0^2 > 0$ are both false). In order for proposition (1.3.4) to be true in this case, we must define $p \rightarrow q$ to be true when both p and q are false. This is exactly what occurs in the fourth line of the truth table of Definition 1.3.3. Even more motivation for defining $p \rightarrow q$ to be true when p is false is given in Exercises 77 and 78.

Example 1.3.4

Let $p : 1 > 2$ and $q : 4 < 8$. Then p is false and q is true. Therefore, $p \rightarrow q$ is true and $q \rightarrow p$ is false. ◀

In expressions that involve the logical operators $\wedge$, $\vee$, $\neg$, and $\rightarrow$, the conditional operator $\rightarrow$ is evaluated last. For example,

$$p \vee q \rightarrow \neg r$$

is interpreted as

$$(p \vee q) \rightarrow (\neg r).$$

Example 1.3.5

Assuming that p is true, q is false, and r is true, find the truth value of each proposition.

- (a) $p \wedge q \rightarrow r$
- (b) $p \vee q \rightarrow \neg r$
- (c) $p \wedge (q \rightarrow r)$
- (d) $p \rightarrow (q \rightarrow r)$

SOLUTION

- (a) We first evaluate $p \wedge q$ because $\rightarrow$ is evaluated last. Since p is true and q is false, $p \wedge q$ is false. Therefore, $p \wedge q \rightarrow r$ is true (regardless of whether r is true or false).
- (b) We first evaluate $\neg r$. Since r is true, $\neg r$ is false. We next evaluate $p \vee q$. Since p is true and q is false, $p \vee q$ is true. Therefore, $p \vee q \rightarrow \neg r$ is false.
- (c) Since q is false, $q \rightarrow r$ is true (regardless of whether r is true or false). Since p is true, $p \wedge (q \rightarrow r)$ is true.
- (d) Since q is false, $q \rightarrow r$ is true (regardless of whether r is true or false). Thus, $p \rightarrow (q \rightarrow r)$ is true (regardless of whether p is true or false). ◀

A conditional proposition that is true because the hypothesis is false is said to be **true by default** or **vacuously true**. For example, if the proposition,

If the Mathematics Department gets an additional \$60,000, then it will hire one new faculty member,

is true because the Mathematics Department did not get an additional \$60,000, we would say that the proposition is true by default or that it is vacuously true.

Some statements not of the form (1.3.2) may be rephrased as conditional propositions, as the next example illustrates.

Example 1.3.6

Restate each proposition in the form (1.3.2) of a conditional proposition.

- (a) Mary will be a good student if she studies hard.
- (b) John takes calculus only if he has sophomore, junior, or senior standing.
- (c) When you sing, my ears hurt.
- (d) A necessary condition for the Cubs to win the World Series is that they sign a right-handed relief pitcher.
- (e) A sufficient condition for Maria to visit France is that she goes to the Eiffel Tower.

SOLUTION

- (a) The hypothesis is the clause following *if*; thus an equivalent formulation is

If Mary studies hard, then she will be a good student.

- (b) The statement means that in order for John to take calculus, he must have sophomore, junior, or senior standing. In particular, if he is a freshman, he may *not* take calculus. Thus, we can conclude that if he takes calculus, then he has sophomore, junior, or senior standing. Therefore an equivalent formulation is

If John takes calculus, then he has sophomore, junior, or senior standing.

Notice that

If John has sophomore, junior, or senior standing, then he takes calculus,

is *not* an equivalent formulation. If John has sophomore, junior, or senior standing, he may or may *not* take calculus. (Although eligible to take calculus, he may have decided not to.)

The “if p then q ” formulation emphasizes the hypothesis, whereas the “ p only if q ” formulation emphasizes the conclusion; the difference is only stylistic.

- (c) *When* means the same as *if*; thus an equivalent formulation is

If you sing, then my ears hurt.

- (d) A **necessary condition** is just that: a condition that is *necessary* for a particular outcome to be achieved. The condition does *not* guarantee the outcome; but, if the condition does not hold, the outcome will not be achieved. Here, the given statement means that if the Cubs win the World Series, we can be sure that they signed a right-handed relief pitcher since, without such a signing, they would not have won the World Series. Thus, an equivalent formulation of the given statement is

If the Cubs win the World Series, then they signed a right-handed relief pitcher.

The conclusion expresses a necessary condition.

Notice that

If the Cubs sign a right-handed relief pitcher, then they win the World Series,

is *not* an equivalent formulation. Signing a right-handed relief pitcher does not guarantee a World Series win. However, *not* signing a right-handed relief pitcher guarantees that they will not win the World Series.

- (e) Similarly, a **sufficient condition** is a condition that *suffices* to guarantee a particular outcome. If the condition does not hold, the outcome might be achieved in other ways or it might not be achieved at all; but if the condition does hold, the outcome is guaranteed. Here, to be sure that Maria visits France, it suffices for her to go to the Eiffel Tower. (There are surely other ways to ensure that Maria visits France; for example, she could go to Lyon.) Thus, an equivalent formulation of the given statement is

If Maria goes to the Eiffel Tower, then she visits France.

The hypothesis expresses a sufficient condition.

Notice that

If Maria visits France, then she goes to the Eiffel Tower,

is *not* an equivalent formulation. As we have already noted, there are ways other than going to the Eiffel Tower to ensure that Maria visits France. ◀

Example 1.3.4 shows that the proposition $p \rightarrow q$ can be true while the proposition $q \rightarrow p$ is false. We call the proposition $q \rightarrow p$ the **converse** of the proposition $p \rightarrow q$. Thus a conditional proposition can be true while its converse is false.

Example 1.3.7

Write the conditional proposition,

If Jerry receives a scholarship, then he will go to college,

and its converse symbolically and in words. Also, assuming that Jerry does not receive a scholarship, but wins the lottery and goes to college anyway, find the truth value of the original proposition and its converse.

SOLUTION Let p : Jerry receives a scholarship, and q : Jerry goes to college. The given proposition can be written symbolically as $p \rightarrow q$. Since the hypothesis p is false, the conditional proposition is true.

The converse of the proposition is

If Jerry goes to college, then he receives a scholarship.

The converse can be written symbolically as $q \rightarrow p$. Since the hypothesis q is true and the conclusion p is false, the converse is false. ◀

Another useful proposition is

p if and only if q ,

which is considered to be true precisely when p and q have the same truth values (i.e., p and q are both true or p and q are both false).

Definition 1.3.8 ▶ If p and q are propositions, the proposition

p if and only if q

is called a *biconditional proposition* and is denoted

$$p \leftrightarrow q.$$

The truth value of the proposition $p \leftrightarrow q$ is defined by the following truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

It is traditional in mathematical definitions to use “if” to mean “if and only if.” Consider, for example, the definition of set equality: If sets X and Y have the same elements, then X and Y are equal. The meaning of this definition is that sets X and Y have the same elements *if and only if* X and Y are equal.

An alternative way to state “ p if and only if q ” is “ p is a necessary and sufficient condition for q .” The proposition “ p if and only if q ” is sometimes written “ p iff q .”

Example 1.3.9

The proposition

$$1 < 5 \text{ if and only if } 2 < 8 \quad (1.3.5)$$

can be written symbolically as $p \leftrightarrow q$ if we define $p : 1 < 5$ and $q : 2 < 8$. Since both p and q are true, the proposition $p \leftrightarrow q$ is true.

An alternative way to state (1.3.5) is: A necessary and sufficient condition for $1 < 5$ is that $2 < 8$.

In some cases, two different propositions have the same truth values no matter what truth values their constituent propositions have. Such propositions are said to be **logically equivalent**.

Definition 1.3.10 ▶ Suppose that the propositions P and Q are made up of the propositions $p_1, \dots, p_n$. We say that P and Q are *logically equivalent* and write

$$P \equiv Q,$$

provided that, given any truth values of $p_1, \dots, p_n$, either P and Q are both true, or P and Q are both false.

Example 1.3.11

De Morgan’s Laws for Logic We will verify the first of **De Morgan’s laws**

$$\neg(p \vee q) \equiv \neg p \wedge \neg q, \quad \neg(p \wedge q) \equiv \neg p \vee \neg q,$$

and leave the second as an exercise (see Exercise 79).

By writing the truth tables for $P = \neg(p \vee q)$ and $Q = \neg p \wedge \neg q$, we can verify that, given any truth values of p and q , either P and Q are both true or P and Q are both false:

p	q	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

Thus P and Q are logically equivalent.

Example 1.3.12 Show that, in Java, the expressions

$$x < 10 \ || \ x > 20$$

and

$$!(x \geq 10 \ \&\& \ x \leq 20)$$

are equivalent. (In Java, $\geq$ means $\geq$, and $\leq$ means $\leq$.)

SOLUTION If we let p denote the expression $x \geq 10$ and q denote the expression $x \leq 20$, the expression $!(x \geq 10 \ \&\& \ x \leq 20)$ becomes $\neg(p \wedge q)$. By De Morgan's second law, $\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$. Since $\neg p$ translates as $x < 10$ and $\neg q$ translates as $x > 20$, $\neg p \vee \neg q$ translates as $x < 10 \ || \ x > 20$. Therefore, the expressions $x < 10 \ || \ x > 20$ and $!(x \geq 10 \ \&\& \ x \leq 20)$ are equivalent. ◀

Our next example gives a logically equivalent form of the negation of $p \rightarrow q$.

Example 1.3.13 Show that the negation of $p \rightarrow q$ is logically equivalent to $p \wedge \neg q$.

SOLUTION By writing the truth tables for $P = \neg(p \rightarrow q)$ and $Q = p \wedge \neg q$, we can verify that, given any truth values of p and q , either P and Q are both true or P and Q are both false:

p	q	$\neg(p \rightarrow q)$	$p \wedge \neg q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	F	F

Thus P and Q are logically equivalent. ◀

Example 1.3.14 Use the logical equivalence of $\neg(p \rightarrow q)$ and $p \wedge \neg q$ (see Example 1.3.13) to write the negation of

If Jerry receives a scholarship, then he goes to college,
symbolically and in words.

SOLUTION We let p : Jerry receives a scholarship, and q : Jerry goes to college. The given proposition can be written symbolically as $p \rightarrow q$. Its negation is logically equivalent to $p \wedge \neg q$. In words, this last expression is

Jerry receives a scholarship and he does not go to college. ◀

We now show that, according to our definitions, $p \leftrightarrow q$ is logically equivalent to $p \rightarrow q$ and $q \rightarrow p$. In words,

p if and only if q

is logically equivalent to

if p then q and if q then p .

Example 1.3.15

The truth table shows that

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p).$$

p	q	$p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Consider again the definition of set equality: If sets X and Y have the same elements, then X and Y are equal. We noted that the meaning of this definition is that sets X and Y have the same elements if and only if X and Y are equal. Example 1.3.15 shows that an equivalent formulation is: If sets X and Y have the same elements, then X and Y are equal, and if X and Y are equal, then X and Y have the same elements.

We conclude this section by defining the **contrapositive** of a conditional proposition. We will see (in Theorem 1.3.18) that the contrapositive is an alternative, logically equivalent form of the conditional proposition. Exercise 80 gives another logically equivalent form of the conditional proposition.

Definition 1.3.16 ▶ The *contrapositive* (or *transposition*) of the conditional proposition $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$.

Notice the difference between the contrapositive and the converse. The converse of a conditional proposition merely reverses the roles of p and q , whereas the contrapositive reverses the roles of p and q and negates each of them.

Example 1.3.17

Write the conditional proposition,

If the network is down, then Dale cannot access the internet,

symbolically. Write the contrapositive and the converse symbolically and in words. Also, assuming that the network is not down and Dale can access the internet, find the truth value of the original proposition, its contrapositive, and its converse.

SOLUTION Let p : The network is down, and q : Dale cannot access the internet. The given proposition can be written symbolically as $p \rightarrow q$. Since the hypothesis p is false, the conditional proposition is true.

The contrapositive can be written symbolically as $\neg q \rightarrow \neg p$ and, in words,

If Dale can access the internet, then the network is not down.

Since the hypothesis $\neg q$ and conclusion $\neg p$ are both true, the contrapositive is true. (Theorem 1.3.18 will show that the conditional proposition and its contrapositive are logically equivalent, that is, that they always have the same truth value.)

The converse of the given proposition can be written symbolically as $q \rightarrow p$ and, in words,

If Dale cannot access the internet, then the network is down.

Since the hypothesis q is false, the converse is true.

An important fact is that a conditional proposition and its contrapositive are logically equivalent.

Theorem 1.3.18

The conditional proposition $p \rightarrow q$ and its contrapositive $\neg q \rightarrow \neg p$ are logically equivalent.

Proof The truth table

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

shows that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent. ◀

In ordinary language, “if” is often used to mean “if and only if.” Consider the statement

If you fix my computer, then I’ll pay you \$50.

The intended meaning is

If you fix my computer, then I’ll pay you \$50, and
if you do not fix my computer, then I will not pay you \$50,

which is logically equivalent to (see Theorem 1.3.18)

If you fix my computer, then I’ll pay you \$50, and
if I pay you \$50, then you fix my computer,

which, in turn, is logically equivalent to (see Example 1.3.15)

You fix my computer if and only if I pay you \$50.

In ordinary discourse, the intended meaning of statements involving logical operators can often (but, not always!) be inferred. However, in mathematics and science, precision is required. Only by carefully defining what we mean by terms such as “if” and “if and only if” can we obtain unambiguous and precise statements. In particular, logic carefully distinguishes among conditional, biconditional, converse, and contrapositive propositions.

1.3 Problem-Solving Tips

- In formal logic, “if” and “if and only if” are quite different. The conditional proposition $p \rightarrow q$ (if p then q) is true except when p is true and q is false. On the other hand, the biconditional proposition $p \leftrightarrow q$ (p if and only if q) is true precisely when p and q are both true or both false.
- To determine whether propositions P and Q , made up of the propositions $p_1, \dots, p_n$, are logically equivalent, write the truth tables for P and Q . If all of the entries for P and Q are always both true or both false, then P and Q are equivalent. If some entry is true for one of P or Q and false for the other, then P and Q are *not* equivalent.
- De Morgan’s laws for logic

$$\neg(p \vee q) \equiv \neg p \wedge \neg q, \quad \neg(p \wedge q) \equiv \neg p \vee \neg q$$

give formulas for negating “or” ($\vee$) and negating “and” ($\wedge$). Roughly speaking, negating “or” results in “and,” and negating “and” results in “or.”

- Example 1.3.13 states a very important equivalence

$$\neg(p \rightarrow q) \equiv p \wedge \neg q,$$

which we will meet throughout this book. This equivalence shows that the negation of the conditional proposition can be written using the “and” ($\wedge$) operator. Notice that there is no conditional operator on the right-hand side of the equation.

1.3 Review Exercises

- What is a conditional proposition? How is it denoted?
- Give the truth table for the conditional proposition.
- In a conditional proposition, what is the hypothesis?
- In a conditional proposition, what is the conclusion?
- What is a necessary condition?
- What is a sufficient condition?
- What is the converse of $p \rightarrow q$?
- What is a biconditional proposition? How is it denoted?
- Give the truth table for the biconditional proposition.
- What does it mean for P to be logically equivalent to Q ?
- State De Morgan’s laws for logic.
- What is the contrapositive of $p \rightarrow q$?

1.3 Exercises

In Exercises 1–11, restate each proposition in the form (1.3.2) of a conditional proposition.

- Joey will pass the discrete mathematics exam if he studies hard.
- Rosa may graduate if she has 160 quarter-hours of credits.
- A necessary condition for Fernando to buy a computer is that he obtain \$2000.
- A sufficient condition for Katrina to take the algorithms course is that she pass discrete mathematics.
- Getting that job requires knowing someone who knows the boss.
- You can travel to your dream destination unless you can’t afford the travel expenses.
- You may inspect the aircraft only if you have the proper security clearance.
- When better cars are built, Buick will build them.
- The audience will go to sleep if the chairperson gives the lecture.
- The program is readable only if it is well structured.
- A necessary condition for the switch to not be turned properly is that the light is not on.
- Write the converse of each proposition in Exercises 1–11.
- Write the contrapositive of each proposition in Exercises 1–11.

Assuming that p and r are false and that q and s are true, find the truth value of each proposition in Exercises 14–22.

14. $p \rightarrow q$

15. $\neg p \rightarrow \neg q$

16. $\neg p \rightarrow q$

17. $(p \rightarrow q) \wedge (q \rightarrow r)$

18. $(p \rightarrow q) \rightarrow r$

19. $p \rightarrow (q \rightarrow r)$

20. $(s \rightarrow (p \wedge \neg r)) \wedge ((p \rightarrow (r \vee q)) \wedge s)$

21. $((p \wedge \neg q) \rightarrow (q \wedge r)) \rightarrow (s \vee \neg q)$

22. $((p \vee q) \wedge (q \vee s)) \rightarrow ((\neg r \vee p) \wedge (q \vee s))$

Exercises 23–32 refer to the propositions p , q , and r ; p is true, q is false, and r ’s status is unknown at this time. Tell whether each proposition is true, is false, or has unknown status at this time.

23. $p \vee r$

24. $q \vee r$

25. $p \rightarrow r$

26. $q \rightarrow r$

27. $r \rightarrow p$

28. $r \rightarrow q$

29. $(p \wedge r) \leftrightarrow r$

30. $(p \vee r) \leftrightarrow r$

31. $(q \wedge r) \leftrightarrow r$

32. $(q \vee r) \leftrightarrow r$

Determine the truth value of each proposition in Exercises 33–42.

33. If $3 + 5 < 2$, then $1 + 3 = 4$.

34. If $3 + 5 < 2$, then $1 + 3 \neq 4$.

35. If $3 + 5 > 2$, then $1 + 3 = 4$.

36. If $3 + 5 > 2$, then $1 + 3 \neq 4$.

37. $3 + 5 > 2$ if and only if $1 + 3 = 4$.

38. $3 + 5 < 2$ if and only if $1 + 3 = 4$.

39. $3 + 5 < 2$ if and only if $1 + 3 \neq 4$.

40. If Paris is the capital of France, then $2 < 1$.

41. If $2 < 1$, then Paris is the capital of France.

42. If $\pi < 3.14$, then $\pi^2 < 9.85$.

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In Exercises 43–46, represent the given proposition symbolically by letting

$$p: 4 < 2, \quad q: 7 < 10, \quad r: 6 < 6.$$

43. If $4 < 2$, then $7 < 10$.
 44. If $(7 < 10 \text{ or } 6 < 6)$, then $4 < 2$.
 45. If it is not the case that $(6 < 6 \text{ and } 7 \text{ is not less than } 10)$, then $6 < 6$.
 46. $7 < 10$ if and only if $(4 < 2 \text{ and } 6 \text{ is not less than } 6)$.

In Exercises 47–52, represent the given proposition symbolically by letting

$$\begin{aligned} p &: \text{You run 10 laps daily.} \\ q &: \text{You are healthy.} \\ r &: \text{You take multi-vitamins.} \end{aligned}$$

47. If you run 10 laps daily, then you will be healthy.
 48. If you do not run 10 laps daily or do not take multi-vitamins, then you will not be healthy.
 49. Taking multi-vitamins is sufficient for being healthy.
 50. You will be healthy if and only if you run 10 laps daily and take multi-vitamins.
 51. If you are healthy, then you run 10 laps daily or you take multi-vitamins.
 52. If you are run 10 laps daily and you take multi-vitamins, then you are healthy.

In Exercises 53–58, formulate the symbolic expression in words using

$$\begin{aligned} p &: \text{Today is Monday,} \\ q &: \text{It is raining,} \\ r &: \text{It is hot.} \end{aligned}$$

53. $p \rightarrow q$ 54. $p \rightarrow (q \wedge \neg r)$
 55. $\neg p \rightarrow (q \vee r)$ 56. $\neg(p \vee q) \leftrightarrow r$
 57. $(p \wedge (q \vee r)) \rightarrow (r \vee (q \vee p))$
 58. $(p \vee (\neg p \wedge \neg(q \vee r))) \rightarrow (p \vee \neg(r \vee q))$

In Exercises 59–62, write each conditional proposition symbolically. Write the converse and contrapositive of each proposition symbolically and in words. Also, find the truth value of each conditional proposition, its converse, and its contrapositive.

59. If $4 < 6$, then $9 > 12$. 60. If $4 > 6$, then $9 > 12$.
 61. $|1| < 3$ if $-3 < 1 < 3$. 62. $|4| < 3$ if $-3 < 4 < 3$.

For each pair of propositions P and Q in Exercises 63–72, state whether or not $P \equiv Q$.

63. $P = p, Q = p \vee q$ 64. $P = p \wedge q, Q = \neg p \vee \neg q$

65. $P = p \rightarrow q, Q = \neg p \vee q$
 66. $P = \neg p \vee (q \wedge r), Q = (\neg p \vee q) \wedge \neg(q \wedge \neg r)$
 67. $P = p \wedge (q \vee r), Q = (p \vee q) \wedge (p \vee r)$
 68. $P = p \rightarrow q, Q = \neg q \rightarrow \neg p$
 69. $P = p \rightarrow q, Q = p \leftrightarrow q$
 70. $P = (p \rightarrow q) \wedge (q \rightarrow r), Q = p \rightarrow r$
 71. $P = (p \rightarrow q) \rightarrow r, Q = p \rightarrow (q \rightarrow r)$
 72. $P = (s \rightarrow (p \wedge \neg r)) \wedge ((p \rightarrow (r \vee q)) \wedge s), Q = p \vee t$

Using De Morgan's laws for logic, write the negation of each proposition in Exercises 73–76.

73. Pat will use the treadmill or lift weights.
 74. Kate will either eat an apple or drink a glass of water.
 75. Martin will eat a banana and drink a glass of milk.
 76. Red pepper and onions are required to make chili.

Exercises 77 and 78 provide further motivation for defining $p \rightarrow q$ to be true when p is false. We consider changing the truth table for $p \rightarrow q$ when p is false. For the first change, we call the resulting operator *imp1* (Exercise 77), and, for the second change, we call the resulting operator *imp2* (Exercise 78). In both cases, we see that pathologies result.

77. Define the truth table for *imp1* by

p	q	$p \text{ imp1 } q$
T	T	T
T	F	F
F	T	F
F	F	T

Show that $p \text{ imp1 } q \equiv q \text{ imp1 } p$.

78. Define the truth table for *imp2* by

p	q	$p \text{ imp2 } q$
T	T	T
T	F	F
F	T	T
F	F	F

- (a) Show that

$$(p \text{ imp2 } q) \wedge (q \text{ imp2 } p) \not\equiv p \leftrightarrow q. \quad (1.3.6)$$

- (b) Show that (1.3.6) remains true if we change the third row of *imp2*'s truth table to F T F.

79. Verify the second of De Morgan's laws, $\neg(p \wedge q) \equiv \neg p \vee \neg q$.
 80. Show that $(p \rightarrow q) \equiv (\neg p \vee q)$.

1.4 Arguments and Rules of Inference

Consider the following sequence of propositions.

The bug is either in module 17 or in module 81.
 The bug is a numerical error.
 Module 81 has no numerical error. (1.4.1)

Assuming that these statements are true, it is reasonable to conclude

The bug is in module 17. (1.4.2)

This process of drawing a conclusion from a sequence of propositions is called **deductive reasoning**. The given propositions, such as (1.4.1), are called **hypotheses** or **premises**, and the proposition that follows from the hypotheses, such as (1.4.2), is called the **conclusion**. A **(deductive) argument** consists of hypotheses together with a conclusion. Many proofs in mathematics and computer science are deductive arguments.

Any argument has the form

If p_1 and p_2 and $\cdots$ and p_n , then q . (1.4.3)

Argument (1.4.3) is said to be **valid** if the conclusion follows from the hypotheses; that is, if p_1 and p_2 and $\cdots$ and p_n are true, then q must also be true. This discussion motivates the following definition.

Definition 1.4.1 ▶ An *argument* is a sequence of propositions written

$$\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

or

$$p_1, p_2, \dots, p_n / \therefore q.$$

The symbol $\therefore$ is read “therefore.” The propositions $p_1, p_2, \dots, p_n$ are called the *hypotheses* (or *premises*), and the proposition q is called the *conclusion*. The argument is *valid* provided that if p_1 and p_2 and $\cdots$ and p_n are all true, then q must also be true; otherwise, the argument is *invalid* (or a *fallacy*). ◀

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For more on fallacies, see goo.gl/vfki9p

In a valid argument, we sometimes say that the conclusion follows from the hypotheses. Notice that we are not saying that the conclusion is true; we are only saying that if you grant the hypotheses, you must also grant the conclusion. An argument is valid because of its form, not because of its content.

Each step of an extended argument involves drawing intermediate conclusions. For the argument as a whole to be valid, each step of the argument must result in a valid, intermediate conclusion. **Rules of inference**, brief, valid arguments, are used within a larger argument.

Example 1.4.2 Determine whether the argument

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

is valid.

FIRST SOLUTION We construct a truth table for all the propositions involved:

p	q	$p \rightarrow q$	p	q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

We observe that whenever the hypotheses $p \rightarrow q$ and p are true, the conclusion q is also true; therefore, the argument is valid.

SECOND SOLUTION We can avoid writing the truth table by directly verifying that whenever the hypotheses are true, the conclusion is also true.

Suppose that $p \rightarrow q$ and p are true. Then q must be true, for otherwise $p \rightarrow q$ would be false. Therefore, the argument is valid. ◀

The argument in Example 1.4.2 is used extensively and is known as the **modus ponens rule of inference** or **law of detachment**. Several useful rules of inference for propositions, which may be verified using truth tables (see Exercises 33–38), are listed in Table 1.4.1.

TABLE 1.4.1 ■ Rules of Inference for Propositions

<i>Rule of Inference</i>	<i>Name</i>	<i>Rule of Inference</i>	<i>Name</i>
$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$	Modus ponens	$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$	Conjunction
$\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$	Modus tollens	$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$	Hypothetical syllogism
$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$	Addition	$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$	Disjunctive syllogism
$\begin{array}{l} p \wedge q \\ \hline \therefore p \end{array}$	Simplification		

Example 1.4.3 Which rule of inference is used in the following argument?

If the computer has one gigabyte of memory, then it can run “Blast ’em.” If the computer can run “Blast ’em,” then the sonics will be impressive. Therefore, if the computer has one gigabyte of memory, then the sonics will be impressive.

SOLUTION Let p denote the proposition “the computer has one gigabyte of memory,” let q denote the proposition “the computer can run ‘Blast ’em,’” and let r denote the proposition “the sonics will be impressive.” The argument can be written symbolically as

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Therefore, the argument uses the hypothetical syllogism rule of inference. 

Example 1.4.4

Represent the argument

$$\begin{array}{l} \text{If } 2 = 3, \text{ then I ate my hat.} \\ \text{I ate my hat.} \\ \hline \therefore 2 = 3 \end{array}$$

symbolically and determine whether the argument is valid.

SOLUTION If we let $p: 2 = 3$ and $q: \text{I ate my hat}$, the argument may be written

$$\begin{array}{l} p \rightarrow q \\ q \\ \hline \therefore p \end{array}$$

If the argument is valid, then whenever $p \rightarrow q$ and q are both true, p must also be true. Suppose that $p \rightarrow q$ and q are true. This is possible if p is false and q is true. In this case, p is not true; thus the argument is invalid. This fallacy is known as the **fallacy of affirming the conclusion**. 

We can also determine whether the argument in Example 1.4.4 is valid or not by examining the truth table of Example 1.4.2. In the third row of the table, the hypotheses are true and the conclusion is false; thus the argument is invalid.

Example 1.4.5

Represent the argument

$$\begin{array}{l} \text{The bug is either in module 17 or in module 81.} \\ \text{The bug is a numerical error.} \\ \text{Module 81 has no numerical error.} \\ \hline \therefore \text{The bug is in module 17.} \end{array}$$

given at the beginning of this section symbolically and show that it is valid.

SOLUTION If we let

$$\begin{array}{l} p : \text{The bug is in module 17.} \\ q : \text{The bug is in module 81.} \\ r : \text{The bug is a numerical error.} \end{array}$$

the argument may be written

$$\begin{array}{l} p \vee q \\ r \\ r \rightarrow \neg q \\ \hline \therefore p \end{array}$$

From $r \rightarrow \neg q$ and r , we may use modus ponens to conclude $\neg q$. From $p \vee q$ and $\neg q$, we may use the disjunctive syllogism to conclude p . Thus the conclusion p follows from the hypotheses and the argument is valid. ◀

Example 1.4.6

We are given the following hypotheses: If the Chargers get a good linebacker, then the Chargers can beat the Broncos. If the Chargers can beat the Broncos, then the Chargers can beat the Jets. If the Chargers can beat the Broncos, then the Chargers can beat the Dolphins. The Chargers get a good linebacker. Show by using the rules of inference (see Table 1.4.1) that the conclusion, the Chargers can beat the Jets and the Chargers can beat the Dolphins, follows from the hypotheses.

SOLUTION Let p denote the proposition “the Chargers get a good linebacker,” let q denote the proposition “the Chargers can beat the Broncos,” let r denote the proposition “the Chargers can beat the Jets,” and let s denote the proposition “the Chargers can beat the Dolphins.” Then the hypotheses are:

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ q \rightarrow s \\ p. \end{array}$$

From $p \rightarrow q$ and $q \rightarrow r$, we may use the hypothetical syllogism to conclude $p \rightarrow r$. From $p \rightarrow r$ and p , we may use modus ponens to conclude r . From $p \rightarrow q$ and $q \rightarrow s$, we may use the hypothetical syllogism to conclude $p \rightarrow s$. From $p \rightarrow s$ and p , we may use modus ponens to conclude s . From r and s , we may use conjunction to conclude $r \wedge s$. Since $r \wedge s$ represents the proposition “the Chargers can beat the Jets and the Chargers can beat the Dolphins,” we conclude that the conclusion does follow from the hypotheses. ◀

1.4 Problem-Solving Tips

The validity of a very short argument or proof might be verified using a truth table. In practice, arguments and proofs use rules of inference.

1.4 Review Exercises

1. What is deductive reasoning?
2. What is a hypothesis in an argument?
3. What is a premise in an argument?
4. What is a conclusion in an argument?
5. What is a valid argument?
6. What is an invalid argument?
7. State the modus ponens rule of inference.
8. State the modus tollens rule of inference.
9. State the addition rule of inference.
10. State the simplification rule of inference.
11. State the conjunction rule of inference.
12. State the hypothetical syllogism rule of inference.
13. State the disjunctive syllogism rule of inference.

1.4 Exercises

Formulate the arguments of Exercises 1–5 symbolically and determine whether each is valid. Let

p : I study hard. q : I get A's. r : I get rich.

1. If I study hard, then I get A's.

$$\frac{\text{I study hard.}}{\therefore \text{I get A's.}}$$
2. If I study hard, then I get A's.

$$\frac{\text{If I don't get rich, then I don't get A's.}}{\therefore \text{I get rich.}}$$
3. I study hard if and only if I get rich.

$$\frac{\text{I get rich.}}{\therefore \text{I study hard.}}$$
4. If I study hard or I get rich, then I get A's.

$$\frac{\text{I get A's.}}{\therefore \text{If I don't study hard, then I get rich.}}$$
5. If I study hard, then I get A's or I get rich.

$$\frac{\text{I don't get A's and I don't get rich.}}{\therefore \text{I don't study hard.}}$$

Formulate the arguments of Exercises 6–9 symbolically and determine whether each is valid.

p : The Democrats win.
 q : The Republicans win.
 r : Unemployment is up.
 s : The economy is up.

6. If the Democrats win, then the economy is up; and, if the Republicans win, then unemployment is up.

$$\frac{\text{The Democrats win or the Republicans win.}}{\therefore \text{Unemployment is up or the economy is up.}}$$
7. If the Democrats win or the Republicans win, then unemployment is up or the economy is up.

$$\frac{\text{The Democrats win and unemployment is not up.}}{\therefore \text{The economy is up.}}$$
8. If the Democrats win, then unemployment is up.
 If the Republicans win, then the economy is up
 The Republicans and Democrats do not both win.

$$\frac{\text{The Democrats do not win.}}{\therefore \text{The economy is up.}}$$
9. If the Democrats win, then unemployment is up or the economy is up.
 If the Republicans win, then unemployment is up.

The economy is not up.

The Democrats win

$\therefore$ Unemployment is up or the Republicans win.

In Exercises 10–14, write the given argument in words and determine whether each argument is valid. Let

p : 4 gigabytes is better than no memory at all.
 q : We will buy more memory.
 r : We will buy a new computer.

10.
$$\frac{p \rightarrow r}{p \rightarrow q} \quad \frac{p \rightarrow (r \vee q)}{r \rightarrow \neg q} \quad \frac{p \rightarrow r}{r \rightarrow q}$$

$$\therefore p \rightarrow (r \wedge q) \quad \therefore p \rightarrow r \quad \therefore q$$
13.
$$\frac{\neg r \rightarrow \neg p}{r} \quad \therefore p$$
14.
$$\frac{p \rightarrow r}{r \rightarrow q} \quad \frac{p}{\therefore q}$$

In Exercises 15–19, write the given argument in words and determine whether each argument is valid. Let

p : The for loop is faulty.
 q : The while loop is faulty.
 r : The hardware is unreliable.
 s : The output is correct.

15.
$$\frac{(p \vee q) \rightarrow (r \vee s)}{p} \quad \frac{\neg r}{\therefore s}$$
16.
$$\frac{(r \vee s) \rightarrow p}{s \rightarrow q} \quad \frac{p \vee s}{\therefore r}$$
17.
$$\frac{(p \rightarrow r) \rightarrow q}{(q \rightarrow s) \rightarrow p} \quad \frac{r \wedge s}{p \vee q} \quad \therefore p \wedge q$$
18.
$$\frac{p \rightarrow q}{q \rightarrow r} \quad \frac{\neg r}{s \rightarrow r} \quad \therefore \neg s$$
19.
$$\frac{p \rightarrow (q \vee r)}{q \rightarrow (p \vee s)} \quad \frac{p \vee \neg q}{\neg s} \quad \therefore p \vee r$$

Determine whether each argument in Exercises 20–24 is valid.

20.
$$\frac{p \rightarrow q}{\neg p} \quad \therefore \neg q$$
21.
$$\frac{p \rightarrow q}{\neg q} \quad \therefore \neg p$$
22.
$$\frac{p \wedge \neg p}{\therefore q}$$

23.
$$\frac{p \rightarrow (q \rightarrow r) \quad q \rightarrow (p \rightarrow r)}{\therefore (p \vee q) \rightarrow r}$$
24.
$$\frac{(p \rightarrow q) \wedge (r \rightarrow s) \quad p \vee r}{\therefore q \vee s}$$

25. Show that if

$$p_1, p_2 / \therefore p \quad \text{and} \quad p, p_3, \dots, p_n / \therefore c$$

are valid arguments, the argument

$$p_1, p_2, \dots, p_n / \therefore c$$

is also valid.

26. Comment on the following argument:

Hard disk drive storage is better than nothing.
 Nothing is better than a solid state drive.

 $\therefore$ Hard disk drive is better than a solid state drive.

For each argument in Exercises 27–29, tell which rule of inference is used.

27. If James Bond is a successful spy, then Jason Bourne is a successful spy. James Bond is a successful spy. Therefore, Jason Bourne is a successful spy.
28. James Bond is a successful spy or Jason Bourne is a successful spy. Jason Bourne is not a successful spy.

Therefore, James Bond is a successful spy.

29. James Bond is a successful spy. Therefore, James Bond is a successful spy or Jason Bourne is a successful spy.

In Exercises 30–32, give an argument using rules of inference to show that the conclusion follows from the hypotheses.

30. Hypotheses: If there is gas in the car, then I will go to the store. If I go to the store, then I will get a soda. There is gas in the car. Conclusion: I will get a soda.
31. Hypotheses: If there is gas in the car, then I will go to the store. If I go to the store, then I will get a soda. I do not get a soda. Conclusion: There is not gas in the car, or the car transmission is defective.
32. Hypotheses: If Jill can sing or Dweezle can play, then I'll buy the compact disc. Jill can sing. I'll buy the compact disc player. Conclusion: I'll buy the compact disc and the compact disc player.
33. Show that modus tollens (see Table 1.4.1) is valid.
34. Show that addition (see Table 1.4.1) is valid.
35. Show that simplification (see Table 1.4.1) is valid.
36. Show that conjunction (see Table 1.4.1) is valid.
37. Show that hypothetical syllogism (see Table 1.4.1) is valid.
38. Show that disjunctive syllogism (see Table 1.4.1) is valid.

1.5 Quantifiers

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For more on quantifiers, see goo.gl/vfki9p

The logic in Sections 1.2 and 1.3 that deals with propositions is incapable of describing most of the statements in mathematics and computer science. Consider, for example, the statement

p : n is an odd integer.

A proposition is a statement that is either true or false. The statement p is not a proposition, because whether p is true or false depends on the value of n . For example, p is true if $n = 103$ and false if $n = 8$. Since most of the statements in mathematics and computer science use variables, we must extend the system of logic to include such statements.

Definition 1.5.1 ▶ Let $P(x)$ be a statement involving the variable x and let D be a set. We call P a *propositional function* or *predicate* (with respect to D) if for each $x \in D$, $P(x)$ is a proposition. We call D the *domain of discourse* of P . ◀

In Definition 1.5.1, the domain of discourse specifies the allowable values for x .

Example 1.5.2

Let $P(n)$ be the statement

n is an odd integer.

Then P is a propositional function with domain of discourse $\mathbf{Z}^+$ since for each $n \in \mathbf{Z}^+$, $P(n)$ is a proposition [i.e., for each $n \in \mathbf{Z}^+$, $P(n)$ is true or false but not both]. For example, if $n = 1$, we obtain the proposition

$$P(1): 1 \text{ is an odd integer}$$

(which is true). If $n = 2$, we obtain the proposition

$$P(2): 2 \text{ is an odd integer}$$

(which is false). 

A propositional function P , by itself, is neither true nor false. However, for each x in the domain of discourse, $P(x)$ is a proposition and is, therefore, either true or false. We can think of a propositional function as defining a class of propositions, one for each element in the domain of discourse. For example, if P is a propositional function with domain of discourse $\mathbf{Z}^+$, we obtain the class of propositions

$$P(1), P(2), \dots$$

Each of $P(1), P(2), \dots$ is either true or false.

Example 1.5.3

Explain why the following are propositional functions.

- (a) $n^2 + 2n$ is an odd integer (domain of discourse = $\mathbf{Z}^+$).
- (b) $x^2 - x - 6 = 0$ (domain of discourse = $\mathbf{R}$).
- (c) The baseball player hit over .300 in 2015 (domain of discourse = set of baseball players).
- (d) The film is rated over 20% by Rotten Tomatoes [the scale is 0% (awful) to 100% (terrific)]. The domain of discourse is the set of films rated by Rotten Tomatoes.

SOLUTION In statement (a), for each positive integer n , we obtain a proposition; therefore, statement (a) is a propositional function.

Similarly, in statement (b), for each real number x , we obtain a proposition; therefore, statement (b) is a propositional function.

We can regard the variable in statement (c) as “baseball player.” Whenever we substitute a particular baseball player for the variable “baseball player,” the statement is a proposition. For example, if we substitute “Joey Votto” for “baseball player,” statement (c) is

Joey Votto hit over .300 in 2015,

which is true. If we substitute “Andrew McCutchen” for “baseball player,” statement (c) is

Andrew McCutchen hit over .300 in 2015,

which is false. Thus statement (c) is a propositional function.

Statement (d) is similar in form to statement (c). Here the variable is “film.” Whenever we substitute a film rated by Rotten Tomatoes for the variable “film,” the statement is a proposition. For example if we substitute *Spectre* for “film,” statement (d) is

Spectre is rated over 20% by Rotten Tomatoes,

which is true since *Spectre* is rated 64% by Rotten Tomatoes. If we substitute *Blended* for “film,” statement (d) is

Blended is rated over 20% by Rotten Tomatoes,

which is false since *Blended* is rated 14% by Rotten Tomatoes. Thus statement (d) is a propositional function. ◀

Most of the statements in mathematics and computer science use terms such as “for every” and “for some.” For example, in mathematics we have the following theorem:

For every triangle T , the sum of the angles of T is equal to 180° .

In computer science, we have this theorem:

For some program P , the output of P is P itself.

We now extend the logical system of Sections 1.2 and 1.3 so that we can handle statements that include “for every” and “for some.”

Definition 1.5.4 ▶ Let P be a propositional function with domain of discourse D . The statement

for every x , $P(x)$

is said to be a *universally quantified statement*. The symbol $\forall$ means “for every.” Thus the statement

for every x , $P(x)$

may be written

$$\forall x P(x). \quad (1.5.1)$$

The symbol $\forall$ is called a *universal quantifier*.

The statement (1.5.1) is true if $P(x)$ is true for every x in D . The statement (1.5.1) is false if $P(x)$ is false for at least one x in D . ◀

Example 1.5.5

Consider the universally quantified statement

$$\forall x (x^2 \geq 0).$$

The domain of discourse is $\mathbf{R}$. The statement is true because, for every real number x , it is true that the square of x is positive or zero. ◀

According to Definition 1.5.4, the universally quantified statement (1.5.1) is false if for at least one x in the domain of discourse, the proposition $P(x)$ is false. A value x in the domain of discourse that makes $P(x)$ false is called a **counterexample** to the statement (1.5.1).

Example 1.5.6

Determine whether the universally quantified statement $\forall x (x^2 - 1 > 0)$ is true or false. The domain of discourse is $\mathbf{R}$.

SOLUTION The statement is false since, if $x = 1$, the proposition $1^2 - 1 > 0$ is false. The value 1 is a counterexample to the statement $\forall x(x^2 - 1 > 0)$. Although there are values of x that make the propositional function true, the counterexample provided shows that the universally quantified statement is false. ◀

Example 1.5.7

Suppose that P is a propositional function whose domain of discourse is the set $\{d_1, \dots, d_n\}$. The following pseudocode[†] determines whether $\forall x P(x)$ is true or false:

```

for  $i = 1$  to  $n$ 
  if  $(\neg P(d_i))$ 
    return false
return true

```

The for loop examines the members d_i of the domain of discourse one by one. If it finds a value d_i for which $P(d_i)$ is false, the condition $\neg P(d_i)$ in the if statement is true; so the code returns false [to indicate that $\forall x P(x)$ is false] and terminates. In this case, d_i is a counterexample. If $P(d_i)$ is true for every d_i , the condition $\neg P(d_i)$ in the if statement is always false. In this case, the for loop runs to completion, after which the code returns true [to indicate that $\forall x P(x)$ is true] and terminates.

Notice that if $\forall x P(x)$ is true, the for loop necessarily runs to completion so that *every* member of the domain of discourse is checked to ensure that $P(x)$ is true for every x . If $\forall x P(x)$ is false, the for loop terminates as soon as *one* element x of the domain of discourse is found for which $P(x)$ is false. ▶

We call the variable x in the propositional function $P(x)$ a *free variable*. (The idea is that x is “free” to roam over the domain of discourse.) We call the variable x in the universally quantified statement $\forall x P(x)$ a *bound variable*. (The idea is that x is “bound” by the quantifier $\forall$.)

We previously pointed out that a propositional function does not have a truth value. On the other hand, Definition 1.5.4 assigns a truth value to the quantified statement $\forall x P(x)$. In sum, a statement with free (unquantified) variables is not a proposition, and a statement with no free variables (no unquantified variables) is a proposition.

Alternative ways to write $\forall x P(x)$ are

for all x , $P(x)$

and

for any x , $P(x)$.

The symbol $\forall$ may be read “for every,” “for all,” or “for any.”

To prove that $\forall x P(x)$ is *true*, we must, in effect, examine *every* value of x in the domain of discourse and show that for every x , $P(x)$ is true. One technique for proving that $\forall x P(x)$ is true is to let x denote an *arbitrary* element of the domain of discourse D . The argument then proceeds using the symbol x . Whatever is claimed about x must be true *no matter what value* x might have in D . The argument must conclude by proving that $P(x)$ is true.

Sometimes to specify the domain of discourse D , we write a universally quantified statement as

for every x in D , $P(x)$.

[†]The pseudocode used in this book is explained in Appendix C.

Example 1.5.8 Verify that the universally quantified statement

for every real number x , if $x > 1$, then $x + 1 > 1$

is true.

SOLUTION This time we must verify that the statement

$$\text{if } x > 1, \text{ then } x + 1 > 1 \quad (1.5.2)$$

is true *for every* real number x .

Let x be any real number whatsoever. It is true that for any real number x , either $x \leq 1$ or $x > 1$. If $x \leq 1$, the conditional proposition (1.5.2) is vacuously true. (The proposition is true because the hypothesis $x > 1$ is false. Recall that when the hypothesis is false, the conditional proposition is true regardless of whether the conclusion is true or false.) In most arguments, the vacuous case is omitted.

Now suppose that $x > 1$. Regardless of the specific value of x , $x + 1 > x$. Since $x + 1 > x$ and $x > 1$, we conclude that $x + 1 > 1$, so the conclusion is true. If $x > 1$, the hypothesis and conclusion are both true; hence the conditional proposition (1.5.2) is true.

We have shown that for every real number x , the proposition (1.5.2) is true. Therefore, the universally quantified statement

for every real number x , if $x > 1$, then $x + 1 > 1$

is true. ◀

The method of disproving the statement $\forall x P(x)$ is quite different from the method used to prove that the statement is true. To show that the universally quantified statement $\forall x P(x)$ is *false*, it is sufficient to find *one* value x in the domain of discourse for which the proposition $P(x)$ is false. Such a value, we recall, is called a counterexample to the universally quantified statement.

We turn next to existentially quantified statements.

Definition 1.5.9 ▶ Let P be a propositional function with domain of discourse D . The statement

there exists x , $P(x)$

is said to be an *existentially quantified statement*. The symbol $\exists$ means “there exists.” Thus the statement

there exists x , $P(x)$

may be written

$$\exists x P(x). \quad (1.5.3)$$

The symbol $\exists$ is called an *existential quantifier*.

The statement (1.5.3) is true if $P(x)$ is true for at least one x in D . The statement (1.5.3) is false if $P(x)$ is false for every x in D . ◀

Example 1.5.10 Consider the existentially quantified statement

$$\exists x \left(\frac{x}{x^2 + 1} = \frac{2}{5} \right).$$

The domain of discourse is $\mathbf{R}$. The statement is true because it is possible to find *at least one* real number x for which the proposition

$$\frac{x}{x^2 + 1} = \frac{2}{5}$$

is true. For example, if $x = 2$, we obtain the true proposition

$$\frac{2}{2^2 + 1} = \frac{2}{5}.$$

It is not the case that *every* value of x results in a true proposition. For example, if $x = 1$, the proposition

$$\frac{1}{1^2 + 1} = \frac{2}{5}$$

is false. 

According to Definition 1.5.9, the existentially quantified statement (1.5.3) is false if for every x in the domain of discourse, the proposition $P(x)$ is false.

Example 1.5.11

Verify that the existentially quantified statement

$$\exists x \in \mathbf{R} \left(\frac{1}{x^2 + 1} > 1 \right)$$

is false.

SOLUTION We must show that

$$\frac{1}{x^2 + 1} > 1$$

is false for every real number x . Now

$$\frac{1}{x^2 + 1} > 1$$

is false precisely when

$$\frac{1}{x^2 + 1} \leq 1 \tag{1.5.4}$$

is true. Thus, we must show that (1.5.4) is true for every real number x . To this end, let x be any real number whatsoever. Since $0 \leq x^2$, we may add 1 to both sides of this inequality to obtain $1 \leq x^2 + 1$. If we divide both sides of this last inequality by $x^2 + 1$, we obtain (1.5.4). Therefore, the statement (1.5.4) is true for every real number x . Thus the statement

$$\frac{1}{x^2 + 1} > 1$$

is false for every real number x . We have shown that the existentially quantified statement

$$\exists x \left(\frac{1}{x^2 + 1} > 1 \right)$$

is false. 

Example 1.5.12 Suppose that P is a propositional function whose domain of discourse is the set $\{d_1, \dots, d_n\}$. The following pseudocode determines whether $\exists x P(x)$ is true or false:

```

for  $i = 1$  to  $n$ 
    if  $(P(d_i))$ 
        return true
return false

```

The for loop examines the members d_i in the domain of discourse one by one. If it finds a value d_i for which $P(d_i)$ is true, the condition $P(d_i)$ in the if statement is true; so the code returns true [to indicate that $\exists x P(x)$ is true] and terminates. In this case, the code found a value in the domain of discourse, namely d_i , for which $P(d_i)$ is true. If $P(d_i)$ is false for every d_i , the condition $P(d_i)$ in the if statement is always false. In this case, the for loop runs to completion, after which the code returns false [to indicate that $\exists x P(x)$ is true] and terminates.

Notice that if $\exists x P(x)$ is true, the for loop terminates as soon as *one* element x in the domain of discourse is found for which $P(x)$ is true. If $\exists x P(x)$ is false, the for loop necessarily runs to completion so that *every* member in the domain of discourse is checked to ensure that $P(x)$ is false for every x . ◀

Alternative ways to write $\exists x P(x)$ are

there exists x such that, $P(x)$

and

for some x , $P(x)$

and

for at least one x , $P(x)$.

The symbol $\exists$ may be read “there exists,” “for some,” or “for at least one.”

Example 1.5.13 Consider the existentially quantified statement

for some n , if n is prime, then $n + 1$, $n + 2$, $n + 3$, and $n + 4$ are not prime.

The domain of discourse is $\mathbf{Z}^+$. This statement is true because we can find *at least one* positive integer n that makes the conditional proposition

if n is prime, then $n + 1$, $n + 2$, $n + 3$, and $n + 4$ are not prime

true. For example, if $n = 23$, we obtain the true proposition

if 23 is prime, then 24, 25, 26, and 27 are not prime.

(This conditional proposition is true because both the hypothesis “23 is prime” and the conclusion “24, 25, 26, and 27 are not prime” are true.) Some values of n make the conditional proposition true (e.g., $n = 23$, $n = 4$, $n = 47$), while others make it false (e.g., $n = 2$, $n = 101$). The point is that we found *one* value that makes the conditional proposition

if n is prime, then $n + 1$, $n + 2$, $n + 3$, and $n + 4$ are not prime

true. For this reason, the existentially quantified statement

for some n , if n is prime, then $n + 1$, $n + 2$, $n + 3$, and $n + 4$ are not prime
is true. 

In Example 1.5.11, we showed that an existentially quantified statement was false by proving that a related universally quantified statement was true. The following theorem makes this relationship precise. The theorem generalizes De Morgan's laws of logic (Example 1.3.11).

Theorem 1.5.14

Generalized De Morgan's Laws for Logic

If P is a propositional function, each pair of propositions in (a) and (b) has the same truth values (i.e., either both are true or both are false).

$$(a) \neg(\forall x P(x)); \exists x \neg P(x)$$

$$(b) \neg(\exists x P(x)); \forall x \neg P(x)$$

Proof We prove only part (a) and leave the proof of part (b) to the reader (Exercise 73).

Suppose that the proposition $\neg(\forall x P(x))$ is true. Then the proposition $\forall x P(x)$ is false. By Definition 1.5.4, the proposition $\forall x P(x)$ is false precisely when $P(x)$ is false for at least one x in the domain of discourse. But if $P(x)$ is false for at least one x in the domain of discourse, $\neg P(x)$ is true for at least one x in the domain of discourse. By Definition 1.5.9, when $\neg P(x)$ is true for at least one x in the domain of discourse, the proposition $\exists x \neg P(x)$ is true. Thus, if the proposition $\neg(\forall x P(x))$ is true, the proposition $\exists x \neg P(x)$ is true. Similarly, if the proposition $\neg(\forall x P(x))$ is false, the proposition $\exists x \neg P(x)$ is false.

Therefore, the pair of propositions in part (a) always has the same truth values. 

Example 1.5.15

Let $P(x)$ be the statement

$$\frac{1}{x^2 + 1} > 1.$$

In Example 1.5.11 we showed that $\exists x P(x)$ is false by verifying that

$$\forall x \neg P(x) \tag{1.5.5}$$

is true.

The technique can be justified by appealing to Theorem 1.5.14. After we prove that proposition (1.5.5) is true, we may use Theorem 1.5.14, part (b), to conclude that $\neg(\exists x P(x))$ is also true. Thus $\neg\neg(\exists x P(x))$ or, equivalently, $\exists x P(x)$ is false. 

Example 1.5.16

Write the statement

Every rock fan loves U2,

symbolically. Write its negation symbolically and in words.

SOLUTION Let $P(x)$ be the propositional function “ x loves U2.” The given statement can be written symbolically as $\forall x P(x)$. The domain of discourse is the set of rock fans.

By Theorem 1.5.14, part (a), the negation of the preceding proposition $\neg(\forall x P(x))$ is equivalent to $\exists x \neg P(x)$. In words, this last proposition can be stated as: There exists a rock fan who does not love U2. ◀

Example 1.5.17 Write the statement

Some birds cannot fly,

symbolically. Write its negation symbolically and in words.

SOLUTION Let $P(x)$ be the propositional function “ x flies.” The given statement can be written symbolically as $\exists x \neg P(x)$. [The statement could also be written $\exists x Q(x)$, where $Q(x)$ is the propositional function “ x cannot fly.” As in algebra, there are many ways to represent text symbolically.] The domain of discourse is the set of birds.

By Theorem 1.5.14, part (b), the negation $\neg(\exists x \neg P(x))$ of the preceding proposition is equivalent to $\forall x \neg \neg P(x)$ or, equivalently, $\forall x P(x)$. In words, this last proposition can be stated as: Every bird can fly. ◀

A universally quantified proposition generalizes the proposition

$$P_1 \wedge P_2 \wedge \cdots \wedge P_n \quad (1.5.6)$$

in the sense that the individual propositions $P_1, P_2, \dots, P_n$ are replaced by an arbitrary family $P(x)$, where x is in the domain of discourse, and (1.5.6) is replaced by

$$\forall x P(x). \quad (1.5.7)$$

The proposition (1.5.6) is true if and only if P_i is true for every $i = 1, \dots, n$. The truth value of proposition (1.5.7) is defined similarly: (1.5.7) is true if and only if $P(x)$ is true for every x in the domain of discourse.

Example 1.5.18 Suppose that the domain of discourse of the propositional function P is $\{-1, 0, 1\}$. The propositional function $\forall x P(x)$ is equivalent to

$$P(-1) \wedge P(0) \wedge P(1). \quad \blacktriangleleft$$

Similarly, an existentially quantified proposition generalizes the proposition

$$P_1 \vee P_2 \vee \cdots \vee P_n \quad (1.5.8)$$

in the sense that the individual propositions $P_1, P_2, \dots, P_n$ are replaced by an arbitrary family $P(x)$, where x is in the domain of discourse, and (1.5.8) is replaced by $\exists x P(x)$.

Example 1.5.19 Suppose that the domain of discourse of the propositional function P is $\{1, 2, 3, 4\}$. The propositional function $\exists x P(x)$ is equivalent to

$$P(1) \vee P(2) \vee P(3) \vee P(4). \quad \blacktriangleleft$$

The preceding observations explain how Theorem 1.5.14 generalizes De Morgan’s laws for logic (Example 1.3.11). Recall that the first of De Morgan’s law for logic states that the propositions

$$\neg(P_1 \vee P_2 \vee \cdots \vee P_n) \quad \text{and} \quad \neg P_1 \wedge \neg P_2 \wedge \cdots \wedge \neg P_n$$

have the same truth values. In Theorem 1.5.14, part (b),

$$\neg P_1 \wedge \neg P_2 \wedge \cdots \wedge \neg P_n$$

is replaced by $\forall x \neg P(x)$ and

$$\neg(P_1 \vee P_2 \vee \cdots \vee P_n)$$

is replaced by $\neg(\exists x P(x))$.

Example 1.5.20

Statements in words often have more than one possible interpretation. Consider the well-known quotation from Shakespeare's "The Merchant of Venice":

All that glitters is not gold.

One possible interpretation of this quotation is: Every object that glitters is not gold. However, this is surely not what Shakespeare intended. The correct interpretation is: Some object that glitters is not gold.

If we let $P(x)$ be the propositional function " x glitters" and $Q(x)$ be the propositional function " x is gold," the first interpretation becomes

$$\forall x(P(x) \rightarrow \neg Q(x)), \quad (1.5.9)$$

and the second interpretation becomes

$$\exists x(P(x) \wedge \neg Q(x)).$$

Using the result of Example 1.3.13, we see that the truth values of

$$\exists x(P(x) \wedge \neg Q(x))$$

and

$$\exists x \neg(P(x) \rightarrow Q(x))$$

are the same. By Theorem 1.5.14, the truth values of

$$\exists x \neg(P(x) \rightarrow Q(x))$$

and

$$\neg(\forall x P(x) \rightarrow Q(x))$$

are the same. Thus an equivalent way to represent the second interpretation is

$$\neg(\forall x P(x) \rightarrow Q(x)). \quad (1.5.10)$$

Comparing (1.5.9) and (1.5.10), we see that the ambiguity results from whether the negation applies to $Q(x)$ (the first interpretation) or to the entire statement

$$\forall x(P(x) \rightarrow Q(x))$$

(the second interpretation). The correct interpretation of the statement

All that glitters is not gold

results from negating the entire statement.

In positive statements, "any," "all," "each," and "every" have the same meaning. In negative statements, the situation changes:

Not all x satisfy $P(x)$.

Not each x satisfies $P(x)$.

Not every x satisfies $P(x)$.

are considered to have the same meaning as

$$\text{For some } x, \neg P(x);$$

whereas

$$\begin{aligned} &\text{Not any } x \text{ satisfies } P(x). \\ &\text{No } x \text{ satisfies } P(x). \end{aligned}$$

mean

$$\text{For all } x, \neg P(x).$$

See Exercises 61–71 for other examples.

Rules of Inference for Quantified Statements

We conclude this section by introducing some rules of inference for quantified statements and showing how they can be used with rules of inference for propositions (see Section 1.4).

Suppose that $\forall x P(x)$ is true. By Definition 1.5.4, $P(x)$ is true for every x in D , the domain of discourse. In particular, if d is in D , then $P(d)$ is true. We have shown that the argument

$$\frac{\forall x P(x)}{\therefore P(d) \text{ if } d \in D}$$

is valid. This rule of inference is called **universal instantiation**. Similar arguments (see Exercises 79–81) justify the other rules of inference listed in Table 1.5.1.

TABLE 1.5.1 Rules of Inference for Quantified Statements[†]

Rule of Inference	Name
$\frac{\forall x P(x)}{\therefore P(d) \text{ if } d \in D}$	Universal instantiation
$\frac{P(d) \text{ for every } d \in D}{\therefore \forall x P(x)}$	Universal generalization
$\frac{\exists x P(x)}{\therefore P(d) \text{ for some } d \in D}$	Existential instantiation
$\frac{P(d) \text{ for some } d \in D}{\therefore \exists x P(x)}$	Existential generalization

[†] The domain of discourse is D .

Example 1.5.21 Given that

$$\text{for every positive integer } n, n^2 \geq n$$

is true, we may use universal instantiation to conclude that $54^2 \geq 54$ since 54 is a positive integer (i.e., a member of the domain of discourse).

Example 1.5.22

Let $P(x)$ denote the propositional function “ x owns a laptop computer,” where the domain of discourse is the set of students taking MATH 201 (discrete mathematics). Suppose that Taylor, who is taking MATH 201, owns a laptop computer; in symbols, $P(\text{Taylor})$ is true. We may then use existential generalization to conclude that $\exists x P(x)$ is true. ◀

Example 1.5.23

Write the following argument symbolically and then, using rules of inference, show that the argument is valid.

SOLUTION For every real number x , if x is an integer, then x is a rational number. The number $\sqrt{2}$ is not rational. Therefore, $\sqrt{2}$ is not an integer.

If we let $P(x)$ denote the propositional function “ x is an integer” and $Q(x)$ denote the propositional function “ x is rational,” the argument becomes

$$\begin{array}{l} \forall x \in \mathbf{R} (P(x) \rightarrow Q(x)) \\ \neg Q(\sqrt{2}) \\ \hline \therefore \neg P(\sqrt{2}) \end{array}$$

Since $\sqrt{2} \in \mathbf{R}$, we may use universal instantiation to conclude $P(\sqrt{2}) \rightarrow Q(\sqrt{2})$. Combining $P(\sqrt{2}) \rightarrow Q(\sqrt{2})$ and $\neg Q(\sqrt{2})$, we may use modus tollens (see Table 1.4.1) to conclude $\neg P(\sqrt{2})$. Thus the argument is valid. ◀

The argument in Example 1.5.23 is called **universal modus tollens**.

Example 1.5.24

We are given these hypotheses: Everyone loves either Microsoft or Apple. Lynn does not love Microsoft. Show that the conclusion, Lynn loves Apple, follows from the hypotheses.

SOLUTION Let $P(x)$ denote the propositional function “ x loves Microsoft,” and let $Q(x)$ denote the propositional function “ x loves Apple.” The first hypothesis is $\forall x (P(x) \vee Q(x))$. By universal instantiation, we have $P(\text{Lynn}) \vee Q(\text{Lynn})$. The second hypothesis is $\neg P(\text{Lynn})$. The disjunctive syllogism rule of inference (see Table 1.4.1) now gives $Q(\text{Lynn})$, which represents the proposition “Lynn loves Apple.” We conclude that the conclusion does follow from the hypotheses. ◀

1.5 Problem-Solving Tips

- To prove that the universally quantified statement $\forall x P(x)$ is true, show that for *every* x in the domain of discourse, the proposition $P(x)$ is true. Showing that $P(x)$ is true for a *particular* value x does *not* prove that $\forall x P(x)$ is true.
- To prove that the existentially quantified statement $\exists x P(x)$ is true, find *one* value of x in the domain of discourse for which the proposition $P(x)$ is true. *One* value suffices.
- To prove that the universally quantified statement $\forall x P(x)$ is false, find *one* value of x (a counterexample) in the domain of discourse for which the proposition $P(x)$ is false.
- To prove that the existentially quantified statement $\exists x P(x)$ is false, show that for *every* x in the domain of discourse, the proposition $P(x)$ is false. Showing that $P(x)$ is false for a *particular* value x does *not* prove that $\exists x P(x)$ is false.

1.5 Review Exercises

1. What is a propositional function?
2. What is a domain of discourse?
3. What is a universally quantified statement?
4. What is a counterexample?
5. What is an existentially quantified statement?
6. State the generalized De Morgan's laws for logic.
7. Explain how to prove that a universally quantified statement is true.
8. Explain how to prove that an existentially quantified statement is true.
9. Explain how to prove that a universally quantified statement is false.
10. Explain how to prove that an existentially quantified statement is false.
11. State the universal instantiation rule of inference.
12. State the universal generalization rule of inference.
13. State the existential instantiation rule of inference.
14. State the existential generalization rule of inference.

1.5 Exercises

In Exercises 1–6, tell whether the statement is a propositional function. For each statement that is a propositional function, give a domain of discourse.

1. $(2n + 1)^2$ is an odd integer.
2. Choose an integer between 1 and 10.
3. $2 + 3 = 5$.
4. The movie won the Academy Award as the best picture of 1955.
5. Let x be a real number.
6. There exists x such that $x < y$ (x, y real numbers).

Let $P(n)$ be the propositional function “ n divides 77.” Write each proposition in Exercises 7–15 in words and tell whether it is true or false. The domain of discourse is $\mathbf{Z}^+$.

- | | | |
|---------------------------|----------------------------|----------------------------|
| 7. $P(11)$ | 8. $P(1)$ | 9. $P(5)$ |
| 10. $\forall n P(n)$ | 11. $\exists n P(n)$ | 12. $\forall n \neg P(n)$ |
| 13. $\exists n \neg P(n)$ | 14. $\neg(\forall n P(n))$ | 15. $\neg(\exists n P(n))$ |

Let $P(x)$ be the propositional function “ $x \geq x^2$.” Tell whether each proposition in Exercises 16–24 is true or false. The domain of discourse is $\mathbf{R}$.

- | | | |
|----------------------------|---------------------------|----------------------------|
| 16. $P(1)$ | 17. $P(0)$ | 18. $P(-1)$ |
| 19. $\forall x P(x)$ | 20. $\exists x P(x)$ | 21. $\neg(\forall x P(x))$ |
| 22. $\neg(\exists x P(x))$ | 23. $\forall x \neg P(x)$ | 24. $\exists x \neg P(x)$ |

Suppose that the domain of discourse of the propositional function P is $\{1, 2, 3, 4\}$. Rewrite each propositional function in Exercises 25–31 using only negation, disjunction, and conjunction.

- | | | |
|--|---------------------------|----------------------------|
| 25. $\forall x P(x)$ | 26. $\forall x \neg P(x)$ | 27. $\neg(\forall x P(x))$ |
| 28. $\exists x P(x)$ | 29. $\exists x \neg P(x)$ | 30. $\neg(\exists x P(x))$ |
| 31. $\forall x(x \neq 1) \rightarrow P(x)$ | | |

Let $P(x)$ denote the statement “ x is taking a math course.” The domain of discourse is the set of all students. Write each proposition in Exercises 32–37 in words.

- | | |
|----------------------------|----------------------------|
| 32. $\forall x P(x)$ | 33. $\exists x P(x)$ |
| 34. $\forall x \neg P(x)$ | 35. $\exists x \neg P(x)$ |
| 36. $\neg(\forall x P(x))$ | 37. $\neg(\exists x P(x))$ |
38. Write the negation of each proposition in Exercises 32–37 symbolically and in words.

Let $P(x)$ denote the statement “ x is a professional athlete,” and let $Q(x)$ denote the statement “ x plays soccer.” The domain of discourse is the set of all people. Write each proposition in Exercises 39–46 in words. Determine the truth value of each statement.

- | | |
|---|---|
| 39. $\forall x (P(x) \rightarrow Q(x))$ | 40. $\exists x (P(x) \rightarrow Q(x))$ |
| 41. $\forall x (Q(x) \rightarrow P(x))$ | 42. $\exists x (Q(x) \rightarrow P(x))$ |
| 43. $\forall x (P(x) \vee Q(x))$ | 44. $\exists x (P(x) \vee Q(x))$ |
| 45. $\forall x (P(x) \wedge Q(x))$ | 46. $\exists x (P(x) \wedge Q(x))$ |
47. Write the negation of each proposition in Exercises 39–46 symbolically and in words.

Let $P(x)$ denote the statement “ x is an accountant,” and let $Q(x)$ denote the statement “ x owns a Porsche.” Write each statement in Exercises 48–51 symbolically.

48. All accountants own Porsches.
49. Some accountant owns a Porsche.
50. All owners of Porsches are accountants.
51. Someone who owns a Porsche is an accountant.
52. Write the negation of each proposition in Exercises 48–51 symbolically and in words.

Determine the truth value of each statement in Exercises 53–58. The domain of discourse is $\mathbf{R}$. Justify your answers.

- | | |
|--------------------------|--------------------------|
| 53. $\exists x(x^2 < x)$ | 54. $\forall x(x^2 < x)$ |
|--------------------------|--------------------------|

55. $\forall x(x < 1 \rightarrow x^2 < x)$
 56. $\exists x(x < 1 \rightarrow x^2 < x)$
 57. $\forall x(x > 1 \rightarrow x/(x^2 + 1) < 1/3)$
 58. $\exists x(x > 1 \rightarrow x/(x^2 + 1) < 1/3)$
 59. Write the negation of each proposition in Exercises 53–58 symbolically and in words.
 60. Could the pseudocode of Example 1.5.7 be written as follows?

```

for  $i = 1$  to  $n$ 
  if  $(\neg P(d_i))$ 
    return false
else
  return true

```

What is the literal meaning of each statement in Exercises 61–71? What is the intended meaning? Clarify each statement by rephrasing it and writing it symbolically.

61. From *Dear Abby*: All men do not cheat on their wives.
 62. From the *San Antonio Express-News*: All old things don't covet twenty-somethings.
 63. All 63 students did not attend every lecture.
 64. Economist Robert J. Samuelson: Every environmental problem is not a tragedy.
 65. Comment from a Door County alderman: This is still Door County and we all don't have a degree.
 66. Headline over a Martha Stewart column: All lampshades can't be cleaned.
 67. Headline in the *New York Times*: A World Where All Is Not Sweetness and Light.
 68. Headline over a story about subsidized housing: Everyone can't afford home.
 69. George W. Bush: I understand everybody in this country doesn't agree with the decisions I've made.
 70. From *Newsweek*: Formal investigations are a sound practice in the right circumstances, but every circumstance is not right.
 71. Joe Girardi (manager of the New York Yankees): Every move is not going to work out.

72. (a) Use a truth table to prove that if p and q are propositions, at least one of $p \rightarrow q$ or $q \rightarrow p$ is true.
 (b) Let $I(x)$ be the propositional function “ x is an integer” and let $P(x)$ be the propositional function “ x is a positive number.” The domain of discourse is $\mathbf{R}$. Determine whether or not the following proof that all integers are positive or all positive real numbers are integers is correct.
 By part (a),

$$\forall x ((I(x) \rightarrow P(x)) \vee (P(x) \rightarrow I(x)))$$

is true. In words: For all x , if x is an integer, then x is positive; or if x is positive, then x is an integer. Therefore, all integers are positive or all positive real numbers are integers.

73. Prove Theorem 1.5.14, part (b).
 74. Analyze the following comments by film critic Roger Ebert: No good movie is too long. No bad movie is short enough. *Love Actually* is good, but it is too long.
 75. Which rule of inference is used in the following argument? Every rational number is of the form p/q , where p and q are integers. Therefore, 9.345 is of the form p/q .

In Exercises 76–78, give an argument using rules of inference to show that the conclusion follows from the hypotheses.

76. Hypotheses: Everyone in the class has a graphing calculator. Everyone who has a graphing calculator understands the trigonometric functions. Conclusion: Ralphie, who is in the class, understands the trigonometric functions.
 77. Hypotheses: Ken, a member of the Titans, can hit the ball a long way. Everyone who can hit the ball a long way can make a lot of money. Conclusion: Some member of the Titans can make a lot of money.
 78. Hypotheses: Everyone in the discrete mathematics class loves proofs. Someone in the discrete mathematics class has never taken calculus. Conclusion: Someone who loves proofs has never taken calculus.
 79. Show that universal generalization (see Table 1.5.1) is valid.
 80. Show that existential instantiation (see Table 1.5.1) is valid.
 81. Show that existential generalization (see Table 1.5.1) is valid.

1.6 Nested Quantifiers

Consider writing the statement

The sum of any two positive real numbers is positive,

symbolically. We first note that since two numbers are involved, we will need two variables, say x and y . The assertion can be restated as: If $x > 0$ and $y > 0$, then $x + y > 0$. The given statement says that the sum of *any* two positive real numbers is positive, so we need two universal quantifiers. If we let $P(x, y)$ denote the expression $(x > 0) \wedge (y > 0) \rightarrow (x + y > 0)$, the given statement can be written symbolically as

$$\forall x \forall y P(x, y).$$

In words, for every x and for every y , if $x > 0$ and $y > 0$, then $x + y > 0$. The domain of discourse of the two-variable propositional function P is $\mathbf{R} \times \mathbf{R}$, which means that each variable x and y must belong to the set of real numbers. Multiple quantifiers such as $\forall x \forall y$ are said to be **nested quantifiers**. In this section we explore nested quantifiers in detail.

Example 1.6.1

Restate $\forall m \exists n (m < n)$ in words. The domain of discourse is the set $\mathbf{Z} \times \mathbf{Z}$.

SOLUTION We may first rephrase this statement as: For every m , there exists n such that $m < n$. Less formally, this means that if you take any integer m whatsoever, there is an integer n greater than m . Another restatement is then: There is no greatest integer. ◀

Example 1.6.2

Write the assertion

Everybody loves somebody,

symbolically, letting $L(x, y)$ be the statement “ x loves y .”

SOLUTION “Everybody” requires universal quantification and “somebody” requires existential quantification. Thus, the given statement may be written symbolically as

$$\forall x \exists y L(x, y).$$

In words, for every person x , there exists a person y such that x loves y .

Notice that

$$\exists x \forall y L(x, y)$$

is *not* a correct interpretation of the original statement. This latter statement is: There exists a person x such that for all y , x loves y . Less formally, someone loves everyone. The order of quantifiers is important; changing the order can change the meaning. ◀

By definition, the statement $\forall x \forall y P(x, y)$, with domain of discourse $X \times Y$, is true if, for *every* $x \in X$ and for *every* $y \in Y$, $P(x, y)$ is true. The statement $\forall x \forall y P(x, y)$ is false if there is *at least one* $x \in X$ and *at least one* $y \in Y$ such that $P(x, y)$ is false.

Example 1.6.3

Consider the statement

$$\forall x \forall y ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0)).$$

The domain of discourse is $\mathbf{R} \times \mathbf{R}$. This statement is true because, for every real number x and for every real number y , the conditional proposition

$$(x > 0) \wedge (y > 0) \rightarrow (x + y > 0)$$

is true. In words, for every real number x and for every real number y , if x and y are positive, their sum is positive. ◀

Example 1.6.4

Consider the statement

$$\forall x \forall y ((x > 0) \wedge (y < 0) \rightarrow (x + y \neq 0)).$$

The domain of discourse is $\mathbf{R} \times \mathbf{R}$. This statement is false because if $x = 1$ and $y = -1$, the conditional proposition

$$(x > 0) \wedge (y < 0) \rightarrow (x + y \neq 0)$$

is false. We say that the pair $x = 1$ and $y = -1$ is a counterexample. ◀

Example 1.6.5

Suppose that P is a propositional function with domain of discourse $\{d_1, \dots, d_n\} \times \{d_1, \dots, d_n\}$. The following pseudocode determines whether $\forall x \forall y P(x, y)$ is true or false:

```

for  $i = 1$  to  $n$ 
  for  $j = 1$  to  $n$ 
    if  $(\neg P(d_i, d_j))$ 
      return false
return true

```

The for loops examine members of the domain of discourse. If they find a pair d_i, d_j for which $P(d_i, d_j)$ is false, the condition $\neg P(d_i, d_j)$ in the if statement is true; so the code returns false [to indicate that $\forall x \forall y P(x, y)$ is false] and terminates. In this case, the pair d_i, d_j is a counterexample. If $P(d_i, d_j)$ is true for every pair d_i, d_j , the condition $\neg P(d_i, d_j)$ in the if statement is always false. In this case, the for loops run to completion, after which the code returns true [to indicate that $\forall x \forall y P(x, y)$ is true] and terminates. ◀

By definition, the statement $\forall x \exists y P(x, y)$, with domain of discourse $X \times Y$, is true if, for *every* $x \in X$, there is *at least one* $y \in Y$ for which $P(x, y)$ is true. The statement $\forall x \exists y P(x, y)$ is false if there is *at least one* $x \in X$ such that $P(x, y)$ is false for *every* $y \in Y$.

Example 1.6.6

Consider the statement

$$\forall x \exists y (x + y = 0).$$

The domain of discourse is $\mathbf{R} \times \mathbf{R}$. This statement is true because, for every real number x , there is at least one y (namely $y = -x$) for which $x + y = 0$ is true. In words, for every real number x , there is a number that when added to x makes the sum zero. ◀

Example 1.6.7

Consider the statement

$$\forall x \exists y (x > y).$$

The domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$. This statement is false because there is at least one x , namely $x = 1$, such that $x > y$ is false for every positive integer y . ◀

Example 1.6.8

Suppose that P is a propositional function with domain of discourse $\{d_1, \dots, d_n\} \times \{d_1, \dots, d_n\}$. The following pseudocode determines whether $\forall x \exists y P(x, y)$ is true or false:

```

for  $i = 1$  to  $n$ 
  if  $(\neg \text{exists\_dj}(i))$ 
    return false
return true
exists_dj( $i$ ) {
  for  $j = 1$  to  $n$ 
    if  $(P(d_i, d_j))$ 
      return true
  return false
}

```

If for each d_i , there exists d_j such that $P(d_i, d_j)$ is true, then for each i , $P(d_i, d_j)$ is true for some j . Thus, $\text{exists_dj}(i)$ returns true for every i . Since $\neg \text{exists_dj}(i)$ is always false, the first for loop eventually terminates and true is returned to indicate that $\forall x \exists y P(x, y)$ is true.

If for some d_i , $P(d_i, d_j)$ is false for every j , then, for this i , $P(d_i, d_j)$ is false for every j . In this case, the for loop in *exists_dj(i)* runs to termination and false is returned. Since $\neg \text{exists_dj}(i)$ is true, false is returned to indicate that $\forall x \exists y P(x, y)$ is false. ◀

By definition, the statement $\exists x \forall y P(x, y)$, with domain of discourse $X \times Y$, is true if there is *at least one* $x \in X$ such that $P(x, y)$ is true for *every* $y \in Y$. The statement $\exists x \forall y P(x, y)$ is false if, for *every* $x \in X$, there is *at least one* $y \in Y$ such that $P(x, y)$ is false.

Example 1.6.9

Consider the statement $\exists x \forall y (x \leq y)$. The domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$. This statement is true because there is at least one positive integer x (namely $x = 1$) for which $x \leq y$ is true for every positive integer y . In words, there is a smallest positive integer (namely 1). ◀

Example 1.6.10

Consider the statement $\exists x \forall y (x \geq y)$. The domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$. This statement is false because, for every positive integer x , there is at least one positive integer y , namely $y = x + 1$, such that $x \geq y$ is false. In words, there is no greatest positive integer. ◀

By definition, the statement $\exists x \exists y P(x, y)$, with domain of discourse $X \times Y$, is true if there is *at least one* $x \in X$ and *at least one* $y \in Y$ such that $P(x, y)$ is true. The statement $\exists x \exists y P(x, y)$ is false if, for *every* $x \in X$ and for *every* $y \in Y$, $P(x, y)$ is false.

Example 1.6.11

Consider the statement

$$\exists x \exists y ((x > 1) \wedge (y > 1) \wedge (xy = 6)).$$

The domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$. This statement is true because there is at least one integer $x > 1$ (namely $x = 2$) and at least one integer $y > 1$ (namely $y = 3$) such that $xy = 6$. In words, 6 is composite (i.e., not prime). ◀

Example 1.6.12

Consider the statement

$$\exists x \exists y ((x > 1) \wedge (y > 1) \wedge (xy = 7)).$$

The domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$. This statement is false because for every positive integer x and for every positive integer y ,

$$(x > 1) \wedge (y > 1) \wedge (xy = 7)$$

is false. In words, 7 is prime. ◀

The generalized De Morgan's laws for logic (Theorem 1.5.14) can be used to negate a proposition containing nested quantifiers.

Example 1.6.13

Using the generalized De Morgan's laws for logic, we find that the negation of $\forall x \exists y P(x, y)$ is

$$\neg(\forall x \exists y P(x, y)) \equiv \exists x \neg(\exists y P(x, y)) \equiv \exists x \forall y \neg P(x, y).$$

Notice how in the negation, $\forall$ and $\exists$ are interchanged. ◀

Example 1.6.14

Write the negation of $\exists x \forall y (xy < 1)$, where the domain of discourse is $\mathbf{R} \times \mathbf{R}$. Determine the truth value of the given statement and its negation.

SOLUTION Using the generalized De Morgan's laws for logic, we find that the negation is

$$\neg(\exists x \forall y(xy < 1)) \equiv \forall x \neg(\forall y(xy < 1)) \equiv \forall x \exists y \neg(xy < 1) \equiv \forall x \exists y(xy \geq 1).$$

The given statement $\exists x \forall y(xy < 1)$ is true because there is at least one x (namely $x = 0$) such that $xy < 1$ for every y . Since the given statement is true, its negation is false. ◀

We conclude with a logic game, which presents an alternative way to determine whether a quantified propositional function is true or false. André Berthiaume contributed this example.

Example 1.6.15

The Logic Game Given a quantified propositional function such as $\forall x \exists y P(x, y)$, you and your opponent, whom we call Farley, play a logic game. Your goal is to try to make $P(x, y)$ true, and Farley's goal is to try to make $P(x, y)$ false. The game begins with the first (left) quantifier. If the quantifier is $\forall$, Farley chooses a value for that variable; if the quantifier is $\exists$, you choose a value for that variable. The game continues with the second quantifier. After values are chosen for all the variables, if $P(x, y)$ is true, you win; if $P(x, y)$ is false, Farley wins. We will show that if you can always win regardless of how Farley chooses values for the variables, the quantified statement is true, but if Farley can choose values for the variables so that you cannot win, the quantified statement is false.

Consider the statement

$$\forall x \exists y(x + y = 0). \quad (1.6.1)$$

The domain of discourse is $\mathbf{R} \times \mathbf{R}$. Since the first quantifier is $\forall$, Farley goes first and chooses a value for x . Since the second quantifier is $\exists$, you go second. Regardless of what value Farley chose, you can choose $y = -x$, which makes the statement $x + y = 0$ true. You can always win the game, so the statement (1.6.1) is true.

Next, consider the statement

$$\exists x \forall y(x + y = 0). \quad (1.6.2)$$

Again, the domain of discourse is $\mathbf{R} \times \mathbf{R}$. Since the first quantifier is $\exists$, you go first and choose a value for x . Since the second quantifier is $\forall$, Farley goes second. Regardless of what value you chose, Farley can always choose a value for y , which makes the statement $x + y = 0$ false. (If you choose $x = 0$, Farley can choose $y = 1$. If you choose $x \neq 0$, Farley can choose $y = 0$.) Farley can always win the game, so the statement (1.6.2) is false.

We discuss why the game correctly determines the truth value of a quantified propositional function. Consider $\forall x \forall y P(x, y)$. If Farley can always win the game, this means that Farley can find values for x and y that make $P(x, y)$ false. In this case, the propositional function is false; the values Farley found provide a counterexample. If Farley cannot win the game, no counterexample exists; in this case, the propositional function is true.

Consider $\forall x \exists y P(x, y)$. Farley goes first and chooses a value for x . You choose second. If, no matter what value Farley chose, you can choose a value for y that makes $P(x, y)$ true, you can always win the game and the propositional function is true. However, if Farley can choose a value for x so that every value you choose for y makes $P(x, y)$ false, then you will always lose the game and the propositional function is false.

An analysis of the other cases also shows that if you can always win the game, the propositional function is true; but if Farley can always win the game, the propositional function is false.

The logic game extends to propositional functions of more than two variables. The rules are the same and, again, if you can always win the game, the propositional function is true; but if Farley can always win the game, the propositional function is false. ◀

1.6 Problem-Solving Tips

- To prove that $\forall x \forall y P(x, y)$ is true, where the domain of discourse is $X \times Y$, you must show that $P(x, y)$ is true for all values of $x \in X$ and $y \in Y$. One technique is to argue that $P(x, y)$ is true using the symbols x and y to stand for *arbitrary* elements in X and Y .
- To prove that $\forall x \forall y P(x, y)$ is false, where the domain of discourse is $X \times Y$, find one value of $x \in X$ and one value of $y \in Y$ (*two* values suffice—one for x and one for y) that make $P(x, y)$ false.
- To prove that $\forall x \exists y P(x, y)$ is true, where the domain of discourse is $X \times Y$, you must show that for all $x \in X$, there is at least one $y \in Y$ such that $P(x, y)$ is true. One technique is to let x stand for an arbitrary element in X and then find a value for $y \in Y$ (*one* value suffices!) that makes $P(x, y)$ true.
- To prove that $\forall x \exists y P(x, y)$ is false, where the domain of discourse is $X \times Y$, you must show that for at least one $x \in X$, $P(x, y)$ is false for every $y \in Y$. One technique is to find a value of $x \in X$ (again *one* value suffices!) that has the property that $P(x, y)$ is false for every $y \in Y$. Having chosen a value for x , let y stand for an arbitrary element of Y and show that $P(x, y)$ is always false.
- To prove that $\exists x \forall y P(x, y)$ is true, where the domain of discourse is $X \times Y$, you must show that for at least one $x \in X$, $P(x, y)$ is true for every $y \in Y$. One technique is to find a value of $x \in X$ (again *one* value suffices!) that has the property that $P(x, y)$ is true for every $y \in Y$. Having chosen a value for x , let y stand for an arbitrary element of Y and show that $P(x, y)$ is always true.
- To prove that $\exists x \forall y P(x, y)$ is false, where the domain of discourse is $X \times Y$, you must show that for all $x \in X$, there is at least one $y \in Y$ such that $P(x, y)$ is false. One technique is to let x stand for an arbitrary element in X and then find a value for $y \in Y$ (*one* value suffices!) that makes $P(x, y)$ false.
- To prove that $\exists x \exists y P(x, y)$ is true, where the domain of discourse is $X \times Y$, find one value of $x \in X$ and one value of $y \in Y$ (*two* values suffice—one for x and one for y) that make $P(x, y)$ true.
- To prove that $\exists x \exists y P(x, y)$ is false, where the domain of discourse is $X \times Y$, you must show that $P(x, y)$ is false for all values of $x \in X$ and $y \in Y$. One technique is to argue that $P(x, y)$ is false using the symbols x and y to stand for *arbitrary* elements in X and Y .
- To negate an expression with nested quantifiers, use the generalized De Morgan's laws for logic. Loosely speaking, $\forall$ and $\exists$ are interchanged. Don't forget that the negation of $p \rightarrow q$ is equivalent to $p \wedge \neg q$.

1.6 Review Exercises

1. What is the interpretation of $\forall x \forall y P(x, y)$? When is this quantified expression true? When is it false?
2. What is the interpretation of $\forall x \exists y P(x, y)$? When is this quantified expression true? When is it false?
3. What is the interpretation of $\exists x \forall y P(x, y)$? When is this quantified expression true? When is it false?
4. What is the interpretation of $\exists x \exists y P(x, y)$? When is this quantified expression true? When is it false?
5. Give an example to show that, in general, $\forall x \exists y P(x, y)$ and $\exists x \forall y P(x, y)$ have different meanings.
6. Write the negation of $\forall x \forall y P(x, y)$ using the generalized De Morgan's laws for logic.

7. Write the negation of $\forall x \exists y P(x, y)$ using the generalized De Morgan's laws for logic.
8. Write the negation of $\exists x \forall y P(x, y)$ using the generalized De Morgan's laws for logic.
9. Write the negation of $\exists x \exists y P(x, y)$ using the generalized De Morgan's laws for logic.
10. Explain the rules for playing the logic game. How can the logic game be used to determine the truth value of a quantified expression?

1.6 Exercises

In Exercises 1–33, the set D_1 consists of three students: Garth, who is 5 feet 11 inches tall; Erin, who is 5 feet 6 inches tall; and Marty, who is 6 feet tall. The set D_2 consists of four students: Dale, who is 6 feet tall; Garth, who is 5 feet 11 inches tall; Erin, who is 5 feet 6 inches tall; and Marty, who is 6 feet tall. The set D_3 consists of one student: Dale, who is 6 feet tall. The set D_4 consists of three students: Pat, Sandy, and Gale, each of whom is 5 feet 11 inches tall.

In Exercises 1–21, $T_1(x, y)$ is the propositional function “ x is taller than y .” Write each proposition in Exercises 1–4 in words.

1. $\forall x \forall y T_1(x, y)$
2. $\forall x \exists y T_1(x, y)$
3. $\exists x \forall y T_1(x, y)$
4. $\exists x \exists y T_1(x, y)$

5. Write the negation of each proposition in Exercises 1–4 in words and symbolically.

In Exercises 6–21, tell whether each proposition in Exercises 1–4 is true or false if the domain of discourse is $D_i \times D_j$ for the given values of i and j .

6. $i = 1, j = 1$
7. $i = 1, j = 2$
8. $i = 1, j = 3$
9. $i = 1, j = 4$
10. $i = 2, j = 1$
11. $i = 2, j = 2$
12. $i = 2, j = 3$
13. $i = 2, j = 4$
14. $i = 3, j = 1$
15. $i = 3, j = 2$
16. $i = 3, j = 3$
17. $i = 3, j = 4$
18. $i = 4, j = 1$
19. $i = 4, j = 2$
20. $i = 4, j = 3$
21. $i = 4, j = 4$

In Exercises 22–27, $T_2(x, y)$ is the propositional function “ x is taller than or the same height as y .” Write each proposition in Exercises 22–25 in words.

22. $\forall x \forall y T_2(x, y)$
23. $\forall x \exists y T_2(x, y)$
24. $\exists x \forall y T_2(x, y)$
25. $\exists x \exists y T_2(x, y)$

26. Write the negation of each proposition in Exercises 22–25 in words and symbolically.
27. Tell whether each proposition in Exercises 22–25 is true or false if the domain of discourse is $D_i \times D_j$ for each pair of values i, j given in Exercises 6–21. The sets $D_1, \dots, D_4$ are defined before Exercise 1.

In Exercises 28–33, $T_3(x, y)$ is the propositional function “if x and y are distinct persons, then x is taller than y .” Write each proposition in Exercises 28–31 in words.

28. $\forall x \forall y T_3(x, y)$
29. $\forall x \exists y T_3(x, y)$
30. $\exists x \forall y T_3(x, y)$
31. $\exists x \exists y T_3(x, y)$

32. Write the negation of each proposition in Exercises 28–31 in words and symbolically.
33. Tell whether each proposition in Exercises 28–31 is true or false if the domain of discourse is $D_i \times D_j$ for each pair of values i, j given in Exercises 6–21. The sets $D_1, \dots, D_4$ are defined before Exercise 1.

Let $L(x, y)$ be the propositional function “ x loves y .” The domain of discourse is the Cartesian product of the set of all living people with itself (i.e., both x and y take on values in the set of all living people). Write each proposition in Exercises 34–37 symbolically. Which do you think are true?

34. Someone loves everybody.
35. Everybody loves everybody.
36. Somebody loves somebody.
37. Everybody loves somebody.
38. Write the negation of each proposition in Exercises 34–37 in words and symbolically.

Let $A(x, y)$ be the propositional function “ x attended y 's office hours” and let $E(x)$ be the propositional function “ x is enrolled in a discrete math class.” Let S be the set of students and let T denote the set of teachers—all at Hudson University. The domain of discourse of A is $S \times T$ and the domain of discourse of E is S . Write each proposition in Exercises 39–42 symbolically.

39. Brit attended someone's office hours.
40. Someone attended Professor Green's office hours.
41. Some discrete math students attended Professor Green's office hours.
42. All teachers had at least one student attend their office hours.

Let $P(x, y)$ be the propositional function $x \geq y$. The domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$. Tell whether each proposition in Exercises 43–46 is true or false.

43. $\forall x \forall y P(x, y)$
44. $\forall x \exists y P(x, y)$
45. $\exists x \forall y P(x, y)$
46. $\exists x \exists y P(x, y)$

47. Write the negation of each proposition in Exercises 43–46.

Determine the truth value of each statement in Exercises 48–65. The domain of discourse is $\mathbf{R} \times \mathbf{R}$. Justify your answers.

48. $\forall x \forall y (x^2 < y + 1)$
49. $\forall x \exists y (x^2 < y + 1)$
50. $\exists x \forall y (x^2 < y + 1)$
51. $\exists x \exists y (x^2 < y + 1)$
52. $\exists y \forall x (x^2 < y + 1)$
53. $\forall y \exists x (x^2 < y + 1)$
54. $\forall x \forall y (x^2 - y^2 = 1)$
55. $\forall x \forall y (x^2 + y^2 \geq 0)$

56. $\forall x \exists y (x^2 + y^2 \geq 0)$ 57. $\exists x \exists y (x^2 - y^2 = 1)$
 58. $\forall x \exists y (x^2 - y^2 = 1)$ 59. $\exists x \forall y (x^2 - y^2 = 1)$
 60. $\exists x \forall y (x^2 + y^2 \geq 0)$ 61. $\exists x \exists y (x^2 + y^2 \geq 0)$
 62. $\forall x \forall y ((x < y) \rightarrow (x^2 < y^2))$
 63. $\forall x \exists y ((x < y) \rightarrow (x^2 < y^2))$
 64. $\exists x \forall y ((x < y) \rightarrow (x^2 < y^2))$
 65. $\exists x \exists y ((x < y) \rightarrow (x^2 < y^2))$
 66. Write the negation of each proposition in Exercises 48–65.
 67. Suppose that P is a propositional function with domain of discourse $\{d_1, \dots, d_n\} \times \{d_1, \dots, d_n\}$. Write pseudocode that determines whether

$$\exists x \forall y P(x, y)$$

is true or false.

68. Suppose that P is a propositional function with domain of discourse $\{d_1, \dots, d_n\} \times \{d_1, \dots, d_n\}$. Write pseudocode that determines whether

$$\exists x \exists y P(x, y)$$

is true or false.

69. Explain how the logic game (Example 1.6.15) determines whether each proposition in Exercises 48–65 is true or false.
 70. Use the logic game (Example 1.6.15) to determine whether the proposition

$$\forall x \forall y \exists z ((z > x) \wedge (z < y))$$

is true or false. The domain of discourse is $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$.

71. Use the logic game (Example 1.6.15) to determine whether the proposition

$$\forall x \forall y \exists z ((z < x) \wedge (z < y))$$

is true or false. The domain of discourse is $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$.

72. Use the logic game (Example 1.6.15) to determine whether the proposition

$$\forall x \forall y \exists z ((x < y) \rightarrow ((z > x) \wedge (z < y)))$$

is true or false. The domain of discourse is $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$.

73. Use the logic game (Example 1.6.15) to determine whether the proposition

$$\forall x \forall y \exists z ((x < y) \rightarrow ((z > x) \wedge (z < y)))$$

is true or false. The domain of discourse is $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$.

Assume that $\forall x \forall y P(x, y)$ is true and that the domain of discourse is nonempty. Which of Exercises 74–76 must also be true? Prove your answer.

74. $\forall x \exists y P(x, y)$ 75. $\exists x \forall y P(x, y)$ 76. $\exists x \exists y P(x, y)$

Assume that $\forall x \exists y P(x, y)$ is true and that the domain of discourse is nonempty. Which of Exercises 77–79 must also be true? Prove your answer.

77. $\exists x \forall y P(x, y)$ 78. $\exists x \exists y P(x, y)$ 79. $\forall x \forall y P(x, y)$

Assume that $\exists x \exists y P(x, y)$ is true and that the domain of discourse is nonempty. Which of Exercises 80–82 must also be true? Prove your answer.

80. $\forall x \forall y P(x, y)$

81. $\forall x \exists y P(x, y)$

82. $\exists x \forall y P(x, y)$

Assume that $\forall x \forall y P(x, y)$ is false and that the domain of discourse is nonempty. Which of Exercises 83–85 must also be false? Prove your answer.

83. $\forall x \exists y P(x, y)$ 84. $\exists x \forall y P(x, y)$ 85. $\exists x \exists y P(x, y)$

Assume that $\forall x \exists y P(x, y)$ is false and that the domain of discourse is nonempty. Which of Exercises 86–88 must also be false? Prove your answer.

86. $\forall x \forall y P(x, y)$

87. $\exists x \forall y P(x, y)$

88. $\exists x \exists y P(x, y)$

Assume that $\exists x \forall y P(x, y)$ is false and that the domain of discourse is nonempty. Which of Exercises 89–91 must also be false? Prove your answer.

89. $\forall x \forall y P(x, y)$ 90. $\forall x \exists y P(x, y)$ 91. $\exists x \exists y P(x, y)$

Assume that $\exists x \exists y P(x, y)$ is false and that the domain of discourse is nonempty. Which of Exercises 92–94 must also be false? Prove your answer.

92. $\forall x \forall y P(x, y)$

93. $\forall x \exists y P(x, y)$

94. $\exists x \forall y P(x, y)$

Which of Exercises 95–98 is logically equivalent to $\neg(\exists x \forall y P(x, y))$? Explain.

95. $\forall x \forall y \neg P(x, y)$ 96. $\forall x \exists y \neg P(x, y)$

97. $\forall x \neg(\exists y P(x, y))$ 98. $\exists x \neg(\forall y P(x, y))$

99. [Requires calculus] The definition of

$$\lim_{x \rightarrow a} f(x) = L$$

is: For every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. Write this definition symbolically using $\forall$ and $\exists$.

100. [Requires calculus] Write the negation of the definition of limit (see Exercise 99) in words and symbolically using $\forall$ and $\exists$ but not $\neg$.

- ★101. [Requires calculus] Write the definition of “ $\lim_{x \rightarrow a} f(x)$ does not exist” (see Exercise 99) in words and symbolically using $\forall$ and $\exists$ but not $\neg$.

102. Consider the headline: Every school may not be right for every child. What is the literal meaning? What is the intended meaning? Clarify the headline by rephrasing it and writing it symbolically.