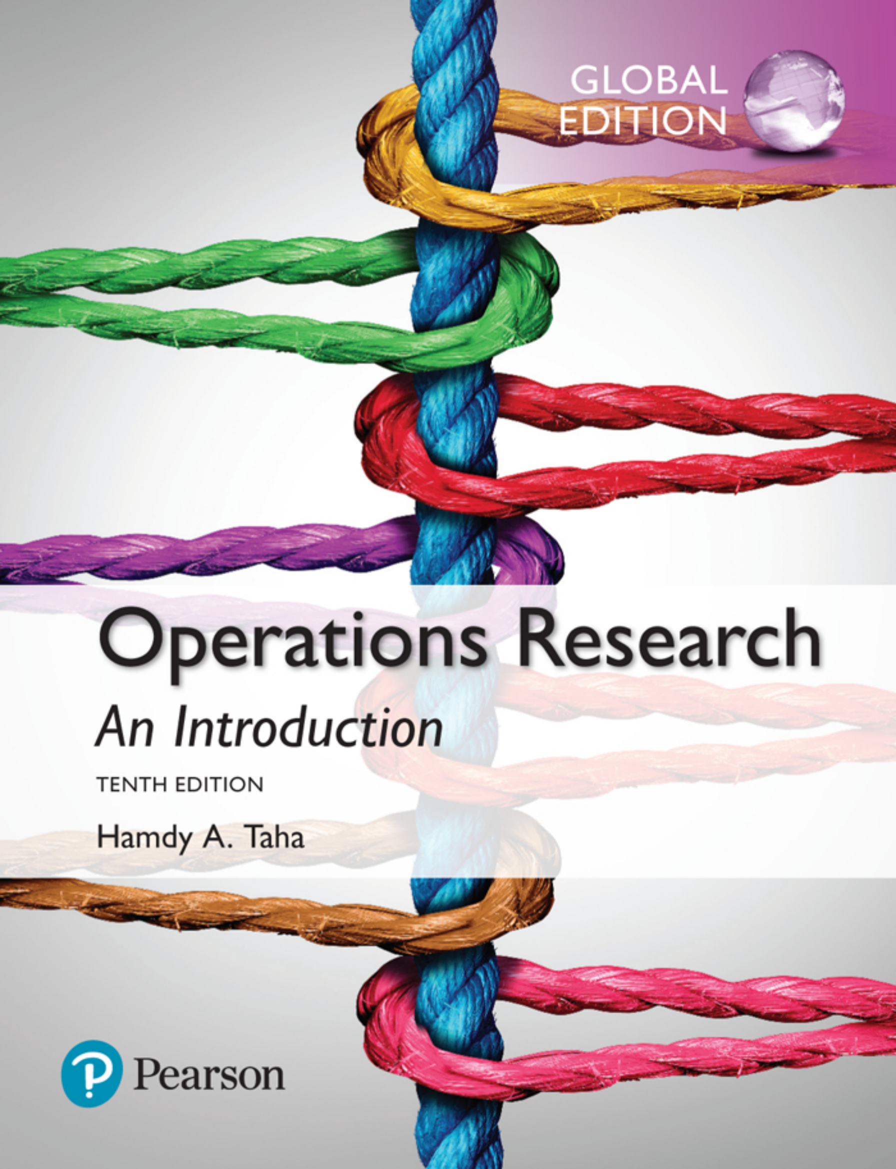

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Operations Research

An Introduction

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Tenth Edition

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Hamdy A. Taha

University of Arkansas, Fayetteville

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To Karen

Los ríos no llevan agua,
el sol las fuentes secó . . .
¡Yo sé donde hay una fuente
que no ha de secar el sol!
La fuente que no se agota
es mi propio corazón . . .

—V. Ruiz Aguilera (1862)

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What's New in the Tenth Edition

Over the past few editions, I agonized over the benefit of continuing to include the hand computational algorithms that, to my thinking, have been made obsolete by present-day great advances in computing. I no longer have this "anxiety" because I sought and received feedback from colleagues regarding this matter. The consensus is that these classical algorithms must be preserved because they are an important part of OR history. Some responses even included possible scenarios (now included in this edition) in which these classical algorithms can be beneficial in practice.

In the spirit of my colleagues collective wisdom, which I now enthusiastically espouse, I added throughout the book some 25 entries titled *Aha! moments.* These entries, written mostly in an informal style, deal with OR anecdotes/stories (some dating back to centuries ago) and OR concepts (theory, applications, computations, and teaching methodology). The goal is to provide a historical perspective of the roots of OR (and, hopefully, render a "less dry" book read).

Additional changes/additions in the tenth edition include:

- Using a brief introduction, inventory modeling is presented within the more encompassing context of supply chains.
- New sections are added about computational issues in the simplex method (Section 7.2.3) and in inventory (Section 13.5).
- This edition adds two new case analyses, resulting in a total of 17 fully developed real-life applications. All the cases appear in Chapter 26 on the website and are cross-referenced throughout the book using abstracts at the start of their most applicable chapters. For convenience, a *select number* of these cases appear in the printed book (I would have liked to move all the cases to their most applicable chapters, but I am committed to limiting the number of hard-copy pages to less than 900).
- By popular demand, all problems now appear at end of their respective chapters and are cross-referenced by text section to facilitate making problem assignments.
- New problems have been added.
- TORA software has been updated.

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I salute my readership—students, professors, and OR practitioners—for their confidence and support over the past 45 years. I am especially grateful to the following colleagues who provided helpful comments in response to my proposal for the tenth edition: Bhaba Sarker (Louisiana State University), Michael Fraboni (Moravian College), Layek Abdel-Malek (New Jersey Institute of Technology), James Smith (University of Massachusetts), Hansuk Sohn (New Mexico State University), Elif Kongar (University of Bridgeport), Sung, Chung-Hsien (University of Illinois), Kash A. Barker (University of Oklahoma).

Professor Michael Trick (Carnegie Mellon University) provided insightful arguments regarding the importance of continuing to include the classical (hand-computational) algorithms of yore in the book and I now enthusiastically share the essence of his statement that "[He] would not be happy to see the day when the Hungarian algorithm is lost to our textbooks."

I wish to thank Professor Donald Erlenkotter (University of California, Los Angles) for his feedback on material in the inventory chapters and Professor Xinhui Zhang (Wright State University) for his input during the preparation of the inventory case study. I also wish to thank Professors Hernan Abeledo (The George Washington University), Ali Diabat (Masdar Institute of Science and Technology, Abu Dhabi, UAE), Robert E. Lewis (University of Alabama, Huntsville), Scott Long (Liberty University), and Daryl Santos (Binghamton University) for pointing out discrepancies in the ninth edition and making suggestions for the tenth.

I offer special thanks and appreciation to the editorial and production teams at Pearson for their valuable help during the preparation of this edition: Marcia Horton (Vice President/Editorial Director, Engineering, Computer Science), Holly Stark (Executive Editor), Scott Disanno (Senior Manager Editor), George Jacob (Project Manager), Erin Ault (Program Manager), Amanda Brands (Editorial Assistant).

It is a great pleasure to recognize Jack Neifert, the first acquisition editor with my former publisher Macmillan, who in 1972, one year after the publication of the first edition, predicted that "this is a book with a long life." The tenth edition is an apt testimonial to the accuracy of Jack's prediction.

I am grateful to Tamara Ellenbecker, Carrie Pennington, Matthew Sparks and Karen Standly, all of the University of Arkansas Industrial Engineering Department, for their able help (and patience) during the preparation of this edition.

My son Sharif, though a neuroscientist, has provided an insightful critique of the *Aha! moments* in this edition.

Hamdy A. Taha
hat@uark.edu

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About the Author

Hamdy A. Taha is a University Professor Emeritus of Industrial Engineering with the University of Arkansas, where he taught and conducted research in operations research and simulation. He is the author of three other books on integer programming and simulation, and his works have been translated to numerous languages. He is also the author of several book chapters, and his technical articles have appeared in operations research and management science journals.

Professor Taha was the recipient of university-wide awards for excellence in research and teaching as well as numerous other research and teaching awards from the College of Engineering, all from the University of Arkansas. He was also named a Senior Fulbright Scholar to Carlos III University, Madrid, Spain. He is fluent in three languages and has held teaching and consulting positions in Europe, Mexico, and the Middle East.

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CHAPTER 1

What Is Operations Research?

1.1 INTRODUCTION

The first formal activities of Operations Research (OR) were initiated in England during World War II, when a team of British scientists set out to assess the best utilization of war materiel based on scientific principles rather than on ad hoc rules. After the war, the ideas advanced in military operations were adapted to improve efficiency and productivity in the civilian sector.

This chapter introduces the basic terminology of OR, including mathematical modeling, feasible solutions, optimization, and iterative algorithmic computations. It stresses that defining the problem correctly is the most important (and most difficult) phase of practicing OR. The chapter also emphasizes that, while mathematical modeling is a cornerstone of OR, unquantifiable factors (such as human behavior) must be accounted for in the final decision. The book presents a variety of applications using solved examples and chapter problems. In particular, the book includes end-of-chapter fully developed case analyses.

1.2 OPERATIONS RESEARCH MODELS

Consider the following **tickets purchasing problem**. A businessperson has a 5-week commitment traveling between Fayetteville (FYV) and Denver (DEN). Weekly departure from Fayetteville occurs on Mondays for return on Wednesdays. A regular roundtrip ticket costs \$400, but a 20% discount is granted if the roundtrip dates span a weekend. A one-way ticket in either direction costs 75% of the regular price. How should the tickets be bought for the 5-week period?

We can look at the situation as a decision-making problem whose solution requires answering three questions:

1. What are the decision **alternatives**?
2. Under what **restrictions** is the decision made?
3. What is an appropriate **objective criterion** for evaluating the alternatives?

Three plausible alternatives come to mind:

1. Buy five regular FYV-DEN-FYV for departure on Monday and return on Wednesday of the same week.
2. Buy one FYV-DEN, four DEN-FYV-DEN that span weekends, and one DEN-FYV.
3. Buy one FYV-DEN-FYV to cover Monday of the first week and Wednesday of the last week and four DEN-FYV-DEN to cover the remaining legs. All tickets in this alternative span at least one weekend.

The restriction on these options is that the businessperson should be able to leave FYV on Monday and return on Wednesday of the same week.

An obvious objective criterion for evaluating the proposed alternatives is the price of the tickets. The alternative that yields the smallest cost is the best. Specifically, we have:

$$\text{Alternative 1 cost} = 5 \times \$400 = \$2000$$

$$\text{Alternative 2 cost} = .75 \times \$400 + 4 \times (.8 \times \$400) + .75 \times \$400 = \$1880$$

$$\text{Alternative 3 cost} = 5 \times (.8 \times \$400) = \$\mathbf{1600}$$

Alternative 3 is the cheapest.

Though the preceding example illustrates the three main components of an OR model—alternatives, objective criterion, and constraints—situations differ in the details of how each component is developed, and how the resulting model is solved. To illustrate this point, consider the following **garden problem**: A home owner is in the process of starting a backyard vegetable garden. The garden must take on a rectangular shape to facilitate row irrigation. To keep critters out, the garden must be fenced. The owner has enough material to build a fence of length $L = 100$ ft. The goal is to fence the largest possible rectangular area.

In contrast with the tickets example, where the number of alternatives is finite, the number of alternatives in the present example is infinite; that is, the *width* and *height* of the rectangle can each assume (theoretically) infinity of values between 0 and L. In this case, the width and the height are **continuous variables**.

Because the variables of the problem are continuous, it is impossible to find the solution by exhaustive enumeration. However, we can *sense* the trend toward the best value of the garden area by fielding increasing values of width (and hence decreasing values of height). For example, for $L = 100$ ft, the combinations (width, height) = (10, 40), (20, 30), (25, 25), (30, 20), and (40, 10) respectively yield (area) = (400, 600, 625, 600, and 400), which demonstrates, but not proves, that the largest area occurs when width = height = $L/4 = 25$ ft. Clearly, this is no way to compute the optimum, particularly for situations with several decision variables. For this reason, it is important to express the problem mathematically in terms of its unknowns, in which case the best solution is found by applying appropriate solution methods.

To demonstrate how the *garden problem* is expressed mathematically in terms of its two unknowns, width and height, define

$$w = \text{width of the rectangle in feet}$$
$$h = \text{height of the rectangle in feet}$$

Based on these definitions, the restrictions of the situation can be expressed verbally as

1. Width of rectangle + Height of rectangle = Half the length of the garden fence
2. Width and height cannot be negative

These restrictions are translated algebraically as

1. $2(w + h) = L$
2. $w \geq 0, h \geq 0$

The only remaining component now is the objective of the problem; namely, maximization of the area of the rectangle. Let z be the area of the rectangle, then the complete model becomes

$$\text{Maximize } z = wh$$

subject to

$$2(w + h) = L$$
$$w, h \geq 0$$

Actually, this model can be simplified further by eliminating one of the variables in the objective function using the constraint equation; that is,

$$w = \tfrac{L}{2} - h$$

The result is

$$z = wh = (\tfrac{L}{2} - h)h = \tfrac{Lh}{2} - h^2$$

The maximization of z is achieved by using differential calculus (Chapter 20), which yields the best solution as $h = \frac{L}{4} = 25$ ft. Back substitution in the constraint equation then yields $w = \frac{L}{4} = 25$ ft. Thus the solution calls for constructing a square-shaped garden.

Based on the preceding two examples, the general OR model can be organized in the following general format:

Maximize or minimize **Objective Function**
subject to
Constraints

A solution is **feasible** if it satisfies all the constraints. It is **optimal** if, in addition to being feasible, it yields the best (maximum or minimum) value of the objective function. In the *ticket purchasing problem*, the problem considers three feasible alternatives, with the third alternative being optimal. In the *garden problem*, a feasible alternative must satisfy the condition $w + h = \frac{L}{2}$, with w and $h \geq 0$, that is, **nonnegative variables**. This definition leads to an infinite number of feasible solutions and, unlike the *ticket purchasing problem*, which uses simple price comparisons, the optimum solution is determined using differential calculus.

Though OR models are designed to *optimize* a specific objective criterion subject to a set of constraints, the quality of the resulting solution depends on the degree of completeness of the model in representing the real system. Take, for example, the *ticket purchasing model*. If *all* the dominant alternatives for purchasing the tickets are not identified, then the resulting solution is optimum only relative to the alternatives represented in the model. To be specific, if for some reason alternative 3 is left out of the model, the resulting "optimum" solution would call for purchasing the tickets for \$1880, which is a **suboptimal** solution. The conclusion is that "the" optimum solution of a model is best only for *that* model. If the model happens to represent the real system reasonably well, then its solution is optimum also for the real situation.

1.3 SOLVING THE OR MODEL

In practice, OR does not offer a single general technique for solving all mathematical models. Instead, the type and complexity of the mathematical model dictate the nature of the solution method. For example, in Section 1.2 the solution of the *tickets purchasing problem* requires simple ranking of alternatives based on the total purchasing price, whereas the solution of the *garden problem* utilizes differential calculus to determine the maximum area.

The most prominent OR technique is **linear programming**. It is designed for models with linear objective and constraint functions. Other techniques include **integer programming** (in which the variables assume integer values), **dynamic programming** (in which the original model can be decomposed into smaller more manageable subproblems), **network programming** (in which the problem can be modeled as a network), and **nonlinear programming** (in which functions of the model are nonlinear). These are only a few among many available OR tools.

A peculiarity of most OR techniques is that solutions are not generally obtained in (formula-like) closed forms. Instead, they are determined by **algorithms**. An algorithm provides fixed computational rules that are applied repetitively to the problem, with each repetition (called **iteration**) attempting to move the solution closer to the optimum. Because the computations in each iteration are typically tedious and voluminous, it is imperative in practice to use the computer to carry out these algorithms.

Some mathematical models may be so complex that it becomes impossible to solve them by any of the available optimization algorithms. In such cases, it may be necessary to abandon the search for the *optimal* solution and simply seek a *good* solution using **heuristics** or **metaheuristics**, a collection of intelligent search *rules of thumb* that move the solution point advantageously toward the optimum.

Aha! Moment: Ada Lovelace, the First-Ever Algorithm Programmer

Though the first conceptual development of an algorithm is attributed to the founder of algebra Muhammad Ibn-Musa Al-Khwarizmi (born c. 780 in Khuwarezm, Uzbekistan, died c. 850 in Baghdad, Iraq),[1] it was British Ada Lovelace (1815–1852) who developed the first computer algorithm. And when we speak of computers, we are referring to the mechanical Difference and Analytical Engines pioneered and designed by the famed British mathematician Charles Babbage (1791–1871).

Lovelace had a keen interest in mathematics. As a teenager, she visited the Babbage home and was fascinated by his invention and its potential uses in doing more than just arithmetic operations. Collaborating with Babbage, she translated into English an article that provided the design details of the Analytical Engine. The article was based on lectures Babbage presented in Italy. In the translated article, Lovelace appended her own notes (which turned out to be longer than the original article and included some corrections of Babbage's design ideas). One of her notes detailed the first-ever *algorithm*, that of computing Bernoulli numbers on the yet-to-be-completed Analytical Engine. She even predicted that the Babbage machine had the potential to manipulate symbols (and not just numbers) and to create complex music scores.[2]

Ada Lovelace died at the young age of 37. In her honor, the computer language *Ada*, developed for the United States Department of Defense, was named after her. The annual mid-October *Ada Lovelace Day* is an international celebration of women in science, technology, engineering, mathematics (STEM). And those of us who have visited St. James Square in London may recall the blue plaque that read "Ada Countess of Lovelace (1815–1852) Pioneer of Computing."

1.4 QUEUING AND SIMULATION MODELS

Queuing and simulation deal with the study of waiting lines. They are not optimization techniques; rather, they determine measures of performance of waiting lines, such as average waiting time in queue, average waiting time for service, and utilization of service facilities, among others.

Queuing models utilize probability and stochastic models to analyze waiting lines, and simulation estimates the measures of performance by "imitating" the behavior of the real system. In a way, simulation may be regarded as the next best thing to observing a real system. The main difference between queuing and simulation is that queuing models are purely mathematical, and hence are subject to specific assumptions that limit their scope of application. Simulation, on the other hand, is flexible and can be used to analyze practically any queuing situation.

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[1]According to Dictionary.com, the word algorithm originates "from Medieval Latin *algorismus*, a mangled transliteration of Arabic al-Khwarizmi."

[2]Lack of funding, among other factors, prevented Babbage from building fully working machines during his lifetime. It was only in 1991 that the London Science Museum built a complete Difference Engine No. 2 using the same materials and technology available to Babbage, thus vindicating his design ideas. There is currently an ongoing long-term effort to construct a fully working Analytical Engine funded entirely by public contributions. It is impressive that modern-day computers are based on the same principal components (memory, CPU, input, and output) advanced by Babbage 100 years earlier.

The use of simulation is not without drawbacks. The process of developing simulation models is costly in both time and resources. Moreover, the execution of simulation models, even on the fastest computer, is usually slow.

1.5 ART OF MODELING

The illustrative models developed in Section 1.2 are exact representations of real situations. This is a rare occurrence in OR, as the majority of applications usually involve (varying degrees of) approximations. Figure 1.1 depicts the levels of abstraction that characterize the development of an OR model. We abstract the assumed real world from the real situation by concentrating on the dominant variables that control the behavior of the real system. The model expresses in an amenable manner the mathematical functions that represent the behavior of the assumed real world.

To illustrate levels of abstraction in modeling, consider the Tyko Manufacturing Company, where a variety of plastic containers are produced. When a production order is issued to the production department, necessary raw materials are acquired from the company's stocks or purchased from outside sources. Once a production batch is completed, the sales department takes charge of distributing the product to retailers.

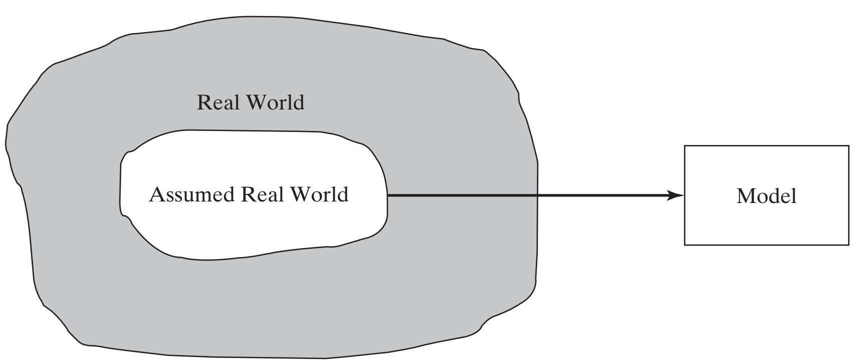
A viable question in the analysis of Tyko's situation is the determination of the size of a production batch. How can this situation be represented by a model?

Looking at the overall system, a number of variables can bear directly on the level of production, including the following (partial) list categorized by department:

1. *Production Department*: Production capacity expressed in terms of available machine and labor hours, in-process inventory, and quality control standards.
2. *Materials Department*: Available stock of raw materials, delivery schedules from outside sources, and storage limitations.
3. *Sales Department*: Sales forecast, capacity of distribution facilities, effectiveness of the advertising campaign, and effect of competition.

FIGURE 1.1

Levels of abstraction in model development



Each of these variables affects the level of production at Tyko. Trying to establish explicit functional relationships between them and the level of production is a difficult task indeed.

A first level of abstraction requires defining the boundaries of the assumed real world. With some reflection, we can approximate the real system by two dominant parameters:

1. Production rate.
2. Consumption rate.

The production rate is determined using data such as production capacity, quality control standards, and availability of raw materials. The consumption rate is determined from the sales data. In essence, simplification from the real world to the assumed real world is achieved by "lumping" several real-world parameters into a single assumed-real-world parameter.

It is easier now to abstract a model from the assumed real world. From the production and consumption rates, measures of excess or shortage inventory can be established. The abstracted model may then be constructed to balance the conflicting costs of excess and shortage inventory—that is, to minimize the total cost of inventory.

1.6 MORE THAN JUST MATHEMATICS

Because of the mathematical nature of OR models, one tends to think that an OR study is *always* rooted in mathematical analysis. Though mathematical modeling is a cornerstone of OR, simpler approaches should be explored first. In some cases, a "commonsense" solution may be reached through simple observations. Indeed, since the human element invariably affects most decision problems, a study of the psychology of people may be key to solving the problem. Six illustrations are presented here to demonstrate the validity of this argument.

1. The stakes were high in 2004 when United Parcel Service (UPS) unrolled its ORION software (based on the sophisticated Traveling Salesman Algorithm—see Chapter 11) to provide its drivers with tailored daily delivery itineraries. The software generally proposed shorter routes than those presently taken by the drivers, with potential savings of millions of dollars a year. For their part, the drivers resented the notion that a machine could "best" them, given their long years of experience on the job. Faced with this human dilemma, ORION developers resolved the issue simply placing a visible banner on the itinerary sheets that read "Beat the Computer." At the same time, they kept ORION-generated routes intact. The drivers took the challenge to heart, with some actually beating the computer suggested route. ORION was no longer putting them down. Instead, they regarded the software as complementing their intuition and experience.[3]

2. Travelers arriving at the Intercontinental Airport in Houston, Texas, complained about the long wait for their baggage. Authorities increased the number of

[3]http://www.fastcompany.com/3004319/brown-down-ups-drivers-vs-ups-algorithm. See also "At UPS, the Algorithm Is the Driver," *Wall Street Journal*, February 16, 2015.

baggage handlers in hope of alleviating the problem, but the complaints persisted. In the end, the decision was made to simply move arrival gates farther away from baggage claim, forcing the passengers to walk longer before reaching the baggage area. The complaints disappeared because the extra walking allowed ample time for the luggage to be delivered to the carousel.[4]

3. In a study of the check-in counters at a large British airport, a U.S.–Canadian consulting team used queuing theory to investigate and analyze the situation. Part of the solution recommended the use of well-placed signs urging passengers within 20 mins of departure time to advance to the head of the queue and request priority service. The solution was not successful because the passengers, being mostly British, were "conditioned to very strict queuing behavior." Hence they were reluctant to move ahead of others waiting in the queue.[5]

4. In a steel mill in India, ingots were first produced from iron ore and then used in the manufacture of steel bars and beams. The manager noticed a long delay between the ingots production and their transfer to the next manufacturing phase (where end products were produced). Ideally, to reduce reheating cost, manufacturing should start soon after the ingots leave the furnaces. Initially, the problem could be perceived as a line-balancing situation, which could be resolved either by reducing the output of ingots or by increasing the capacity of manufacturing. Instead, the OR team used simple charts to summarize the output of the furnaces during the three shifts of the day. They discovered that during the third shift starting at 11:00 P.M., most of the ingots were produced between 2:00 and 7:00 A.M. Investigation revealed that third-shift operators preferred to get long periods of rest at the start of the shift and then make up for lost production during morning hours. Clearly, the third-shift operators have hours to spare to meet their quota. The problem was solved by "leveling out" both the number of operators and the production schedule of ingots throughout the shift.

5. In response to complaints of slow elevator service in a large office building, the OR team initially perceived the situation as a waiting-line problem that might require the use of mathematical queuing analysis or simulation. After studying the behavior of the people voicing the complaint, the psychologist on the team suggested installing full-length mirrors at the entrance to the elevators. The complaints disappeared, as people were kept occupied watching themselves and others while waiting for the elevator.

6. A number of departments in a production facility share the use of three trucks to transport material. Requests initiated by a department are filled on a first-come-first-serve basis. Nevertheless, the departments complained of long wait for service, and demanded adding a fourth truck to the pool. Ensuing simple tallying of the usage of the trucks showed modest daily utilization, obviating a fourth truck. Further investigations revealed that the trucks were parked in an obscure parking lot out of the line of vision for the departments. A requesting supervisor, lacking visual sighting of the trucks, assumed that no trucks were available and hence did not initiate a request.

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[4]Stone, A., "Why Waiting Is Torture," *The New York Times*, August 18, 2012.

[5]Lee, A., *Applied Queuing Theory*, St. Martin's Press, New York, 1966.

The problem was solved simply by installing two-way radio communication between the truck lot and each department.[6]

Four conclusions can be drawn from these illustrations:

1. The OR team should explore the possibility of using "different" ideas to resolve the situation. The (common-sense) solutions proposed for the UPS problem (using *Beat the Computer* banner to engage drivers), the Houston airport (moving arrival gates away from the baggage claim area), and the elevator problem (installing mirrors) are rooted in human psychology rather than in mathematical modeling. This is the reason OR teams may generally seek the expertise of individuals trained in social science and psychology, a point that was recognized and implemented by the first OR team in Britain during World War II.

2. Before jumping to the use of sophisticated mathematical modeling, a bird's eye view of the situation should be adopted to uncover possible nontechnical reasons that led to the problem in the first place. In the steel mill situation, this was achieved by using only simple charting of the ingots production to discover the imbalance in the third-shift operation. A similar simple observation in the case with the transport trucks situation also led to a simple solution of the problem.

3. An OR study should not start with a bias toward using a specific mathematical tool before the use of the tool is justified. For example, because linear programming (Chapter 2 and beyond) is a successful technique, there is a tendency to use it as the modeling tool of choice. Such an approach may lead to a mathematical model that is far removed from the real situation. It is thus imperative to analyze available data, using the simplest possible technique, to understand the essence of the problem. Once the problem is defined, a decision can be made regarding the most appropriate tool for the solution. In the steel mill problem, simple charting of the ingots production was all that was needed to clarify the situation.

4. Solutions are rooted in people and not in technology. Any solution that does not take human behavior into consideration is apt to fail. Even though the solution of the British airport problem may have been mathematically sound, the fact that the consulting team was unaware of the cultural differences between the United States and Britain resulted in an unimplementable recommendation (Americans and Canadians tend to be less formal). The same viewpoint can, in a way, be expressed in the UPS case.

1.7 PHASES OF AN OR STUDY

OR studies are rooted in *teamwork*, where the OR analysts and the client work side by side. The OR analysts' expertise in modeling is complemented by the experience and cooperation of the client for whom the study is being carried out.

[6] G. P. Cosmetatos, "The Value of Queuing Theory—A Case Study," *Interfaces*, Vol. 9, No. 3, pp. 47–51, 1979.

As a decision-making tool, OR is both a science and an art: It is a science by virtue of the mathematical techniques it embodies, and an art because the success of the phases leading to the solution of the mathematical model depends largely on the creativity and experience of the OR team. Willemain (1994) advises that "effective [OR] practice requires more than analytical competence: It also requires, among other attributes, technical judgment (e.g., when and how to use a given technique) and skills in communication and organizational survival."

It is difficult to prescribe specific courses of action (similar to those dictated by the precise theory of most mathematical models) for these intangible factors. We can, however, offer general guidelines for the implementation of OR in practice.

The principal phases for implementing OR in practice include the following:

1. Definition of the problem.
2. Construction of the model.
3. Solution of the model.
4. Validation of the model.
5. Implementation of the solution.

Phase 3, dealing with *model solution*, is the best defined and generally the easiest to implement in an OR study, because it deals mostly with well-defined mathematical models. Implementation of the remaining phases is more an art than a theory.

Problem definition involves delineating the scope of the problem under investigation. This function should be carried out by the entire OR team. The aim is to identify three principal elements of the decision problem: (1) description of the decision alternatives, (2) determination of the objective of the study, and (3) specification of the limitations under which the modeled system operates.

Model construction entails an attempt to translate the problem definition into mathematical relationships. If the resulting model fits one of the standard mathematical models, such as linear programming, we can usually reach a solution by using available algorithms. Alternatively, if the mathematical relationships are too complex to allow the determination of an analytic solution, the OR team may opt to simplify the model and use a heuristic approach, or the team may consider the use of simulation, if appropriate. In some cases, mathematical, simulation, and heuristic models may be combined to solve the decision problem, as some of the end-of-chapter case analyses demonstrate.

Model solution is by far the simplest of all OR phases because it entails the use of well-defined optimization algorithms. An important aspect of the model solution phase is *sensitivity analysis*. It deals with obtaining additional information about the behavior of the optimum solution when the model undergoes some parameter changes. Sensitivity analysis is particularly needed when the parameters of the model cannot be estimated accurately. In these cases, it is important to study the behavior of the optimum solution in the neighborhood of the parameters estimates.

Model validity checks whether or not the proposed model does what it purports to do—that is, does it adequately predict the behavior of the system under study? Initially, the OR team should be convinced that the model's output does not include "surprises." In other words, does the solution make sense? Are the results intuitively acceptable? On the formal side, a common method for validating a model is to compare its output with historical output data. The model is valid if, under similar input conditions, it reasonably duplicates past performance. Generally, however, there is no guarantee that future performance will continue to duplicate past behavior. Also, because the model is usually based on examination of past data, the proposed comparison should usually be favorable. If the proposed model represents a new (non-existing) system, no historical data would be available. In some situations, simulation may be used as an independent tool for validating the output of the mathematical model.

Implementation of the solution of a validated model involves the translation of the results into understandable operating instructions to be issued to the people who will administer the recommended system. The burden of this task lies primarily with the OR team.

1.8 ABOUT THIS BOOK

Morris (1967) states "the teaching of models is not equivalent to the teaching of modeling." I have taken note of this important statement during the preparation of this edition, making every effort to introduce the art of modeling in OR by including realistic models and case studies throughout the book. Because of the importance of computations in OR, the book discusses how the theoretical algorithms fit in commercial computer codes (see Section 3.7). It also presents extensive tools for carrying out the computational task, ranging from tutorial-oriented TORA to the commercial packages Excel, Excel Solver, and AMPL.

OR is both an art and a science—the art of describing and modeling the problem and the science of solving the model using (precise) mathematical algorithms. A first course in the subject should give the student an appreciation of the importance of both areas. This will provide OR users with the kind of confidence that normally would be lacking if training is dedicated solely to the art aspect of OR, under the guise that computers can relieve the user of the need to *understand* why the solution algorithms work.

Modeling and computational capabilities can be enhanced by studying published practical cases. To assist you in this regard, fully developed end-of-chapter case analyses are included. The cases cover most of the OR models presented in this book. There are also some 50 cases that are based on real-life applications in Appendix E on the website that accompanies this book. Additional case studies are available in journals and publications. In particular, *Interfaces* (published by INFORMS) is a rich source of diverse OR applications.

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PROBLEMS[7]

Section	Assigned Problems
1.2	1-1 to 1-11

1-1. In the tickets example,

(a) Provide an infeasible alternative.

(b) Identify a fourth feasible alternative and determine its cost.

1-2. In the garden problem, identify three feasible solutions, and determine which one is better.

1-3. Determine the optimal solution of the garden problem. (*Hint:* Use the constraint to express the objective function in terms of one variable, then use differential calculus.)

***1-4.** Amy, Jim, John, and Kelly are standing on the east bank of a river and wish to cross to the west side using a canoe. The canoe can hold at most two people at a time. Amy, being the most athletic, can row across the river in 1 minute. Jim, John, and Kelly would take 3, 6, and 9 minutes, respectively. If two people are in the canoe, the slower person dictates the crossing time. The objective is for all four people to be on the other side of the river in the shortest time possible.

(a) Define the criterion for evaluating the alternatives (remember, the canoe is the only mode of transportation, and it cannot be shuttled empty).

***(b)** What is the shortest time for moving all four people to the other side of the river?

1-5. In a baseball game, Jim is the pitcher and Joe is the batter. Suppose that Jim can throw either a fast or a curve ball at random. If Joe correctly predicts a curve ball, he can maintain a .400 batting average, else, if Jim throws a curve ball and Joe prepares for a fast ball, his batting average is kept down to .200. On the other hand, if Joe correctly predicts a fast ball, he gets a .250 batting average, else, his batting average is only .125.

(a) Define the alternatives for this situation.

(b) Define the objective function for the problem and discuss how it differs from the familiar optimization (maximization or minimization) of a criterion.

[7]Appendix B gives the solution to asterisk-prefixed problems. The same convention is used in all end-of-chapter problems throughout the book.

1-6. During the construction of a house, six joists of 24 ft each must be trimmed to the correct length of 23 ft. The operations for cutting a joist involve the following sequence:

Operation	Time (seconds)
1. Place joist on saw horses	15
2. Measure correct length (23 ft)	5
3. Mark cutting line for circular saw	5
4. Trim joist to correct length	20
5. Stack trimmed joist in a designated area	20

Three persons are involved: Two loaders must work simultaneously on operations 1, 2, and 5, and one cutter handles operations 3 and 4. There are two pairs of saw horses on which untrimmed joists are placed in preparation for cutting, and each pair can hold up to three side-by-side joists. Suggest a good schedule for trimming the six joists.

1-7. An upright symmetrical triangle is divided into four layers: The bottom layer consists of four (equally-spaced) dots, designated as A, B, C, and D. The next layer includes dots E, F, and G, and the following layer has dots H and I. The top layer has dot J. You want to invert the triangle (bottom layer has one dot and top layer has four) by moving the dots around as necessary.[8]

(a) Identify two feasible solutions.

(b) Determine the smallest number of moves needed to invert the triangle.

1-8. You have five chains, each consisting of four solid links. You need to make a bracelet by connecting all five chains. It costs 2 cents to break a link and 3 cents to re-solder it.

(a) Identify two feasible solutions and evaluate them.

(b) Determine the cheapest cost for making the bracelet.

1-9. The squares of a rectangular board of 11 rows and 9 columns are numbered sequentially 1 through 99 with a *hidden* monetary reward between 0 and 50 dollars assigned to each square. A game using the board requires the player to choose a square by selecting any two digits and then subtracting the sum of its two digits from the selected number. The player then receives the reward assigned the selected square. What monetary values should be assigned to the 99 squares to minimize the player's reward (regardless of how many times the game is repeated)? To make the game interesting, the assignment of $0 to *all* the squares is not an option.

1-10. You have 10 identical cartons each holding 10 water bottles. All bottles weigh 10 oz. each, except for one defective carton in which each of the 10 bottles weighs on 9 oz. only. A scale is available for weighing.

(a) Suggest a method for locating the defective carton.

***(b)** What is the smallest number of times the scale is used that guarantees finding the defective carton? (*Hint:* You will need to be creative in deciding what to weigh.)

***1-11.** You are given two identical balls made of a tough alloy. The hardness test fails if a ball dropped from a floor of a 120-storey building is dented upon impact. A ball can be reused in fresh drops only if it has not been dented in a previous drop. Using only these two identical balls, what is the smallest number of ball drops that will determine the highest floor from which the ball can be dropped without being damaged?

---

[8]Problems 1-7 and 1-8 are adapted from Bruce Goldstein, *Cognitive Psychology: Mind, Research, and Everyday Experience*, Wadsworth Publishing, 2005.

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CHAPTER 2

Modeling with Linear Programming

Real-Life Application—Frontier Airlines Purchases Fuel Economically

The fueling of an aircraft can take place at any of the stopovers along a flight route. Fuel price varies among the stopovers, and potential savings can be realized by tankering (loading) extra fuel at a cheaper location for use on subsequent flight legs. The disadvantage is that the extra weight of tankered fuel will result in higher burn of gasoline. Linear programming (LP) and heuristics are used to determine the optimum amount of tankering that balances the cost of excess burn against the savings in fuel cost. The study, carried out in 1981, resulted in net savings of about $350,000 per year. With the significant rise in the cost of fuel, many airlines are using LP-based tankering software to purchase fuel. Details of the study are given in Case 1, Chapter 26 on the website.

2.1 TWO-VARIABLE LP MODEL

This section deals with the graphical solution of a two-variable LP. Though two-variable problems hardly exist in practice, the treatment provides concrete foundations for the development of the general simplex algorithm presented in Chapter 3.

Example 2.1-1 (The Reddy Mikks Company)

Reddy Mikks produces both interior and exterior paints from two raw materials, $M1$ and $M2$. The following table provides the basic data of the problem:

	Tons of raw material per ton of		Maximum daily availability (tons)
	Exterior paint	*Interior paint*	
Raw material, $M1$	6	4	24
Raw material, $M2$	1	2	6
Profit per ton ($1000)	5	4	

The daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also, the maximum daily demand for interior paint is 2 tons.

Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit.

All OR models, LP included, consist of three basic components:

1. Decision **variables** that we seek to determine.
2. **Objective** (goal) that we need to optimize (maximize or minimize).
3. **Constraints** that the solution must satisfy.

The proper definition of the decision variables is an essential first step in the development of the model. Once done, the task of constructing the objective function and the constraints becomes more straightforward.

For the Reddy Mikks problem, we need to determine the daily amounts of exterior and interior paints to be produced. Thus the variables of the model are defined as:

$$x_1 = \text{Tons produced daily of exterior paint}$$

$$x_2 = \text{Tons produced daily of interior paint}$$

The goal of Reddy Mikks is to *maximize* (i.e., increase as much as possible) the total daily profit of both paints. The two components of the total daily profit are expressed in terms of the variables x_1 and x_2 as:

$$\text{Profit from exterior paint} = 5x_1 \text{ (thousand) dollars}$$

$$\text{Profit from interior paint} = 4x_2 \text{ (thousand) dollars}$$

Letting z represent the total daily profit (in thousands of dollars), the objective (or goal) of Reddy Mikks is expressed as

$$\text{Maximize } z = 5x_1 + 4x_2$$

Next, we construct the constraints that restrict raw material usage and product demand. The raw material restrictions are expressed verbally as

$$\begin{pmatrix}\text{Usage of a raw material}\\ \text{by both paints}\end{pmatrix} \leq \begin{pmatrix}\text{Maximum raw material}\\ \text{availability}\end{pmatrix}$$

The daily usage of raw material $M1$ is 6 tons per ton of exterior paint and 4 tons per ton of interior paint. Thus,

$$\text{Usage of raw material } M1 \text{ by both paints} = 6x_1 + 4x_2 \text{ tons/day}$$

In a similar manner,

$$\text{Usage of raw material } M2 \text{ by both paints} = 1x_1 + 2x_2 \text{ tons/day}$$

The maximum daily availabilities of raw materials $M1$ and $M2$ are 24 and 6 tons, respectively. Thus, the raw material constraints are:

$$\begin{aligned} 6x_1 + 4x_2 &\leq 24 \quad \text{(Raw material } M1\text{)} \\ x_1 + 2x_2 &\leq 6 \quad \text{(Raw material } M2\text{)} \end{aligned}$$

The first restriction on product demand stipulates that the daily production of interior paint cannot exceed that of exterior paint by more than 1 ton, which translates to:

$$x_2 - x_1 \leq 1 \quad \text{(Market limit)}$$

The second restriction limits the daily demand of interior paint to 2 tons—that is,

$$x_2 \le 2 \quad (\text{Demand limit})$$

An implicit (or "understood-to-be") restriction requires (all) the variables, x_1 and x_2, to assume zero or positive values only. The restrictions, expressed as $x_1 \ge 0$ and $x_2 \ge 0$, are referred to as **nonnegativity constraints**.

The complete Reddy Mikks model is

$$\text{Maximize } z = 5x_1 + 4x_2$$

subject to

$$\begin{aligned} 6x_1 + 4x_2 &\le 24 \quad (1) \\ x_1 + 2x_2 &\le \ \ 6 \quad (2) \\ -x_1 + x_2 &\le \ \ 1 \quad (3) \\ x_2 &\le \ \ 2 \quad (4) \\ x_1, x_2 &\ge \ \ 0 \quad (5) \end{aligned}$$

Any values of x_1 and x_2 that satisfy *all* five constraints constitute a **feasible solution**. Otherwise, the solution is **infeasible**. For example, the solution $x_1 = 3$ tons per day and $x_2 = 1$ ton per day is feasible because it does not violate *any* of the five constraints;a result that is confirmed by using substituting $(x_1 = 3, x_2 = 1)$ in the left-hand side of each constraint. In constraint (1), we have $6x_1 + 4x_2 = (6 \times 3) + (4 \times 1) = 22$, which is less than the right-hand side of the constraint $(= 24)$. Constraints 2 to 5 are checked in a similar manner (verify!). On the other hand, the solution $x_1 = 4$ and $x_2 = 1$ is infeasible because it does not satisfy at least one constraint. For example, in constraint (1), $(6 \times 4) + (4 \times 1) = 28$, which is larger than the right-hand side $(= 24)$.

The goal of the problem is to find the **optimum**, the best *feasible* solution that maximizes the total profit z. First, we need to show that the Reddy Mikks problem has an *infinite* number of feasible solutions, a property that is shared by all nontrivial LPs. Hence the problem cannot be solved by enumeration. The graphical method in Section 2.2 and its algebraic generalization in Chapter 3 show how the optimum can be determined in a finite number of steps.

Remarks. The objective and the constraint function in all LPs must be linear. Additionally, all the parameters (coefficients of the objective and constraint functions) of the model are known with certainty.

2.2 GRAPHICAL LP SOLUTION

The graphical solution includes two steps:

1. Determination of the feasible solution space.
2. Determination of the optimum solution from among all the points in the solution space.

The presentation uses two examples to show how maximization and minimization objective functions are handled.

2.2.1 Solution of a Maximization Model

Example 2.2-1

This example solves the Reddy Mikks model of Example 2.1-1.

Step 1. *Determination of the Feasible Solution Space:*
First, consider the nonnegativity constraints $x_1 \geq 0$ and $x_2 \geq 0$. In Figure 2.1, the horizontal axis x_1 and the vertical axis x_2 represent the exterior- and interior-paint variables, respectively. Thus, the nonnegativity constraints restrict the variables to the first quadrant (above the x_1-axis and to the right of the x_2-axis).

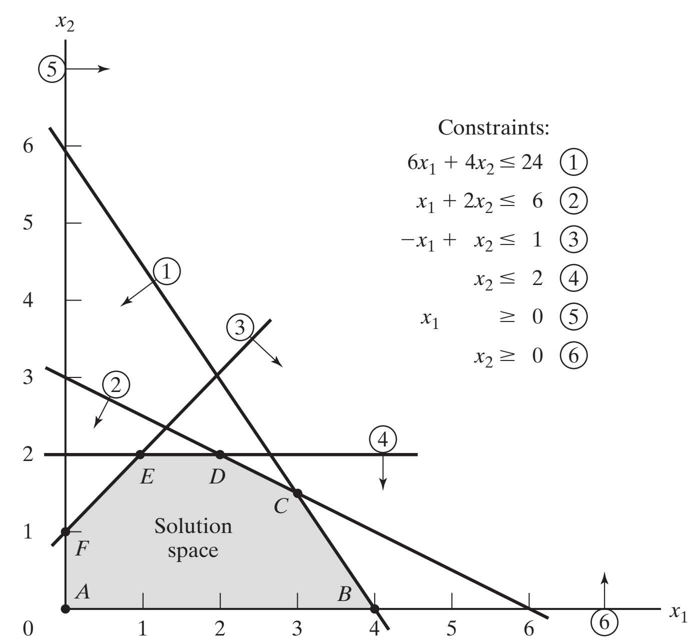
To account for the remaining four constraints, first replace each inequality with an equation, and then graph the resulting straight line by locating two distinct points. For example, after replacing $6x_1 + 4x_2 \leq 24$ with the straight line $6x_1 + 4x_2 = 24$, two distinct points are determined by setting $x_1 = 0$ to obtain $x_2 = \frac{24}{4} = 6$ and then by setting $x_2 = 0$ to obtain $x_1 = \frac{24}{6} = 4$. Thus the line $6x_1 + 4x_2 = 24$ passes through $(0, 6)$ and $(4, 0)$, as shown by line (1) in Figure 2.1.

Next, consider the direction ($>$ or $<$) of the inequality. It divides the (x_1, x_2) plane into two half-spaces, one on each side of the graphed line. Only one of these two halves satisfies the inequality. To determine the correct side, designate any point *not* lying on the straight line as a *reference point*. If the chosen reference point satisfies the inequality, then its side is feasible; otherwise, the opposite side becomes the feasible half-space.

The origin $(0, 0)$ is a convenient reference point and should always be used so long as it does not lie on the line representing the constraint. This happens to be true for all the constraints of this example. Starting with the constraint $6x_1 + 4x_2 \leq 24$,

FIGURE 2.1

Feasible space of the Reddy Mikks model



substitution of $(x_1, x_2) = (0, 0)$ automatically yields zero for the left-hand side. Since it is less than 24, the half-space containing (0, 0) is feasible for inequality (1), as the direction of the arrow in Figure 2.1 shows. A similar application of the reference-point procedure to the remaining constraints produces the **feasible solution space** $ABCDEF$ in which all the constraints are satisfied (verify!). All points outside the boundary of the area $ABCDEF$ are infeasible.

Step 2. *Determination of the Optimum Solution:*
The number of solution points in the feasible space $ABCDEF$ in Figure 2.1 is *infinite*, clearly precluding the use of exhaustive enumeration. A systematic procedure is thus needed to determine the optimum solution.

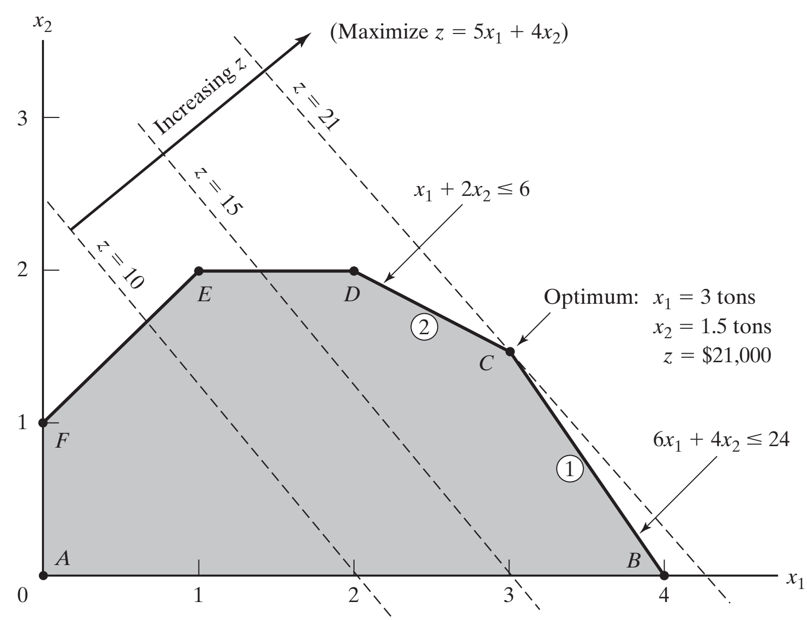
First, the direction in which the profit function $z = 5x_1 + 4x_2$ increases (recall that we are *maximizing* z) is determined by assigning arbitrary *increasing* values to z. In Figure 2.2, the two lines $5x_1 + 4x_2 = 10$ and $5x_1 + 4x_2 = 15$ corresponding to (arbitrary) $z = 10$ and $z = 15$ depict the direction in which z increases. Moving in that direction, the optimum solution occurs at C because it is the feasible point in the solution space beyond which any further increase will render an infeasible solution.

The values of x_1 and x_2 associated with the optimum point C are determined by solving the equations associated with lines (1) and (2):

$$6x_1 + 4x_2 = 24$$
$$x_1 + 2x_2 = 6$$

FIGURE 2.2

Optimum solution of the Reddy Mikks model



The solution is $x_1 = 3$ and $x_2 = 1.5$ with $z = (5 \times 3) + (4 \times 1.5) = 21$. This calls for a daily product mix of 3 tons of exterior paint and 1.5 tons of interior paint. The associated daily profit is \$21,000.

Remarks. In practice, a typical LP may include hundreds or even thousands of variables and constraints. Of what good then is the study of a two-variable LP? The answer is that the graphical solution provides a key result: *The optimum solution of an LP, when it exists, is always associated with a* ***corner point*** *of the solution space, thus limiting the search for the optimum from an infinite number of feasible points to a finite number of corner points.* This powerful result is the basis for the development of the general algebraic *simplex method* presented in Chapter 3.[1]

2.2.2 Solution of a Minimization Model

Example 2.2-2 (Diet Problem)

Ozark Farms uses at least 800 lb of special feed daily. The special feed is a mixture of corn and soybean meal with the following compositions:

	lb per lb of feedstuff		
Feedstuff	*Protein*	*Fiber*	Cost (\$/lb)
Corn	.09	.02	.30
Soybean meal	.60	.06	.90

The dietary requirements of the special feed are at least 30% protein and at most 5% fiber. The goal is to determine the daily minimum-cost feed mix.

The decision variables of the model are:

$$x_1 = \text{lb of corn in the daily mix}$$

$$x_2 = \text{lb of soybean meal in the daily mix}$$

The objective is to minimize the total daily cost (in dollars) of the feed mix—that is,

$$\text{Minimize } z = .3x_1 + .9x_2$$

[1]To reinforce this key result, use TORA to verify that the optimum of the following objective functions of the Reddy Mikks model (Example 2.1-1) will yield the associated corner points as defined in Figure 2.2 (click View/Modify Input Data to modify the objective coefficients and re-solve the problem graphically):

(a) $z = 5x_1 + x_2$ (optimum: point B in Figure 2.2)

(b) $z = 5x_1 + 4x_2$ (optimum: point C)

(c) $z = x_1 + 3x_2$ (optimum: point D)

(d) $z = x_2$ (optimum: point D or E, or any point inbetween—see Section 3.5.2)

(e) $z = -2x_1 + x_2$ (optimum: point F)

(f) $z = -x_1 - x_2$ (optimum: point A)

The constraints represent the daily amount of the mix and the dietary requirements. Ozark Farms needs at least 800 lb of feed a day—that is,

$$x_1 + x_2 \geq 800$$

The amount of protein included in x_1 lb of corn and x_2 lb of soybean meal is $(.09x_1 + .6x_2)$ lb. This quantity should equal at least 30% of the total feed mix $(x_1 + x_2)$ lb—that is,

$$.09x_1 + .6x_2 \geq .3(x_1 + x_2)$$

In a similar manner, the fiber requirement of at most 5% is represented as

$$.02x_1 + .06x_2 \leq .05(x_1 + x_2)$$

The constraints are simplified by moving the terms in x_1 and x_2 to the left-hand side of each inequality, leaving only a constant on the right-hand side. The complete model is

$$\text{Minimize } z = .3x_1 + .9x_2$$

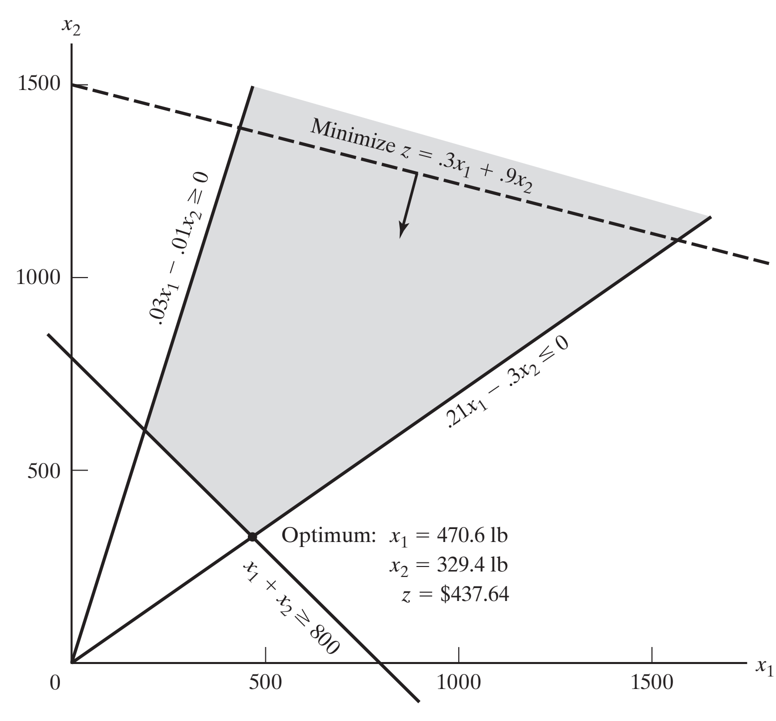
subject to

$$\begin{aligned} x_1 + \quad x_2 &\geq 800 \\ .21x_1 - .30x_2 &\leq 0 \\ .03x_1 - .01x_2 &\geq 0 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Figure 2.3 provides the graphical solution of the model. The second and third constraints pass through the origin. Thus, unlike the Reddy Mikks model of Example 2.2-1, the determination of

FIGURE 2.3

Graphical solution of the diet model



the feasible half-spaces of these two constraints requires using a reference point other than $(0, 0)$ [e.g., $(100, 0)$ or $(0, 100)$].

Solution:

The model minimizes the value of the objective function by reducing z in the direction shown in Figure 2.3. The optimum solution is the intersection of the two lines $x_1 + x_2 = 800$ and $.21x_1 - .3x_2 = 0$, which yields $x_1 = 470.6$ lb and $x_2 = 329.4$ lb. The minimum cost of the feed mix is $z = .3 \times 470.6 + .9 \times 329.4 = \$437.64$ per day.

Remarks. One may wonder why the constraint $x_1 + x_2 \geq 800$ cannot be replaced with $x_1 + x_2 = 800$ because it would not be optimum to produce more than the minimum quantity. Although the solution of the present model did satisfy the equation, a more complex model may impose additional restrictions that would require mixing more than the minimum amount. More importantly, the weak inequality $(\geq)$, by definition, implies the equality case, so that the equation $(=)$ is permitted if optimality requires it. The conclusion is that one should not "preguess" the solution by imposing the additional equality restriction.

2.3 COMPUTER SOLUTION WITH SOLVER AND AMPL

In practice, where typical LP models may involve thousands of variables and constraints, the computer is the only viable venue for solving LP problems. This section presents two commonly used software systems: Excel Solver and AMPL. Solver is particularly appealing to spreadsheet users. AMPL is an algebraic modeling language that, like all higher-order programming languages, requires more expertise. Nevertheless, AMPL, and similar languages,[2] offers great modeling flexibility. Although the presentation in this section concentrates on LPs, both AMPL and Solver can handle integer and nonlinear problems, as will be shown in later chapters.

2.3.1 LP Solution with Excel Solver

In Excel Solver, the spreadsheet is the input and output medium for the LP. Figure 2.4 shows the layout of the data for the Reddy Mikks model (file *solverRM1.xls*). The top of the figure includes four types of information: (1) input data cells (B5:C9 and F6:F9), (2) cells representing the variables and the objective function (B13:D13), (3) algebraic definitions of the objective function and the left-hand side of the constraints (cells D5:D9), and (4) cells that provide (optional) explanatory names or symbols. Solver requires the first three types only. The fourth type enhances readability but serves no other purpose. The relative positioning of the four types of information on the

[2]Other known commercial packages include AIMMS, GAMS, LINGO, MPL, OPL Studio, and Xpress-Mosel.

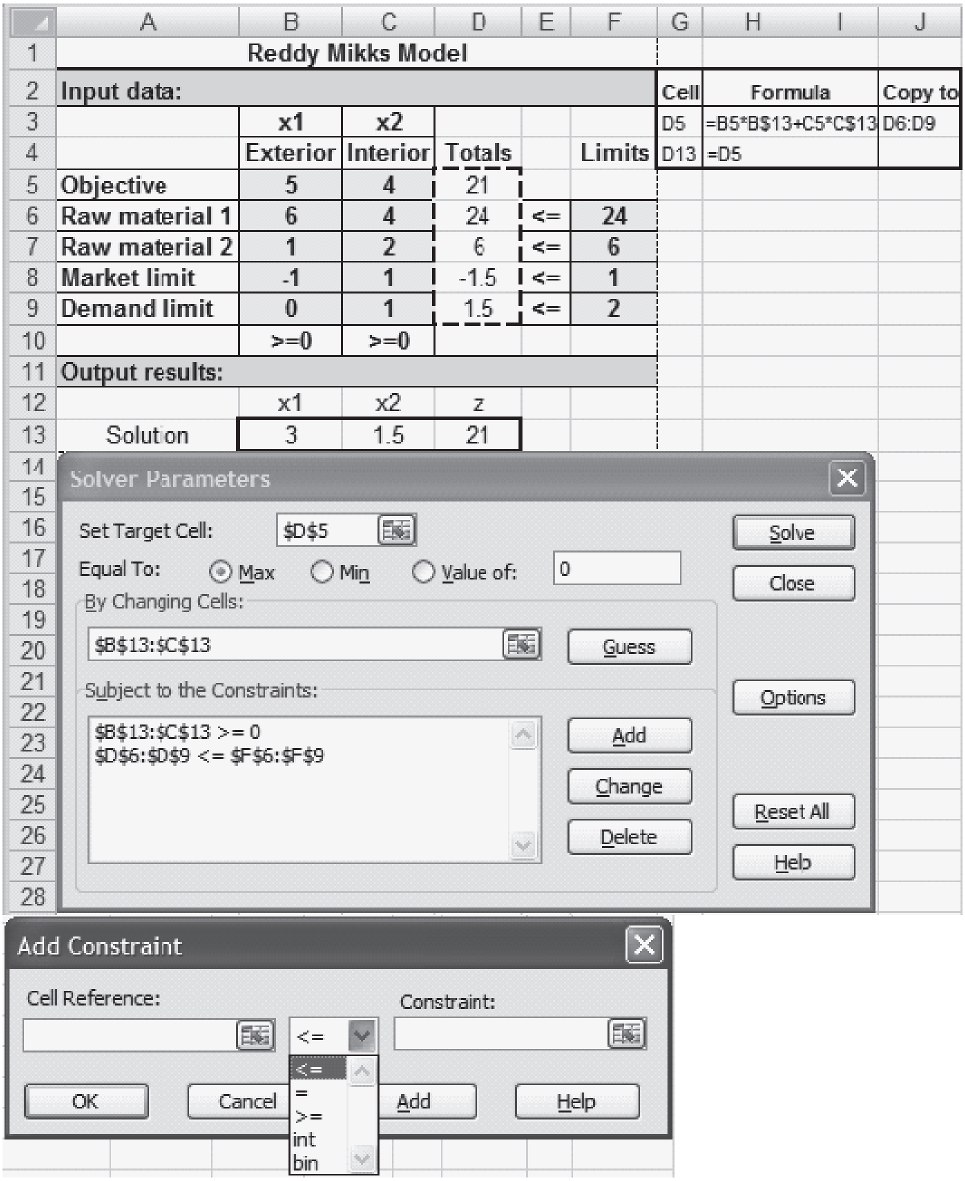


FIGURE 2.4
Defining the Reddy Mikks model with Excel Solver (file *solverRM1.xls*)

spreadsheet (as suggested in Figure 2.4) is convenient for proper cell cross-referencing in Solver, and its use is recommended.

How does Solver link to the spreadsheet data? First, we provide "algebraic" definitions of the objective function and the left-hand side of the constraints using the input data (cells B5:C9 and F6:F9) and the objective function and variables (cells B13:D13). Next, we place the resulting formulas appropriately in cells D5:D9, as the following table shows:

	Algebraic expression	Spreadsheet formula	Entered in cell
Objective, z	$5x_1 + 4x_2$	=B5*$B$13+C5*$C$13	D5
Constraint 1	$6x_1 + 4x_2$	=B6*$B$13+C6*$C$13	D6
Constraint 2	$x_1 + 2x_2$	=B7*$B$13+C7*$C$13	D7
Constraint 3	$-x_1 + x_2$	=B8*$B$13+C8*$C$13	D8
Constraint 4	$0x_1 + x_2$	=B9*$B$13+C9*$C$13	D9

Actually, you only need to enter the formula for cell D5 and then copy it into cells D6:D9. To do so correctly, it is necessary to use *fixed referencing* of the cells representing x_1 and x_2 (i.e., $B$13 and $C$13, respectively).

The explicit formulas just described are impractical for large LPs. Instead, the formula in cell D5 can be written compactly as

= SUMPRODUCT(B5:C5,$B$13:$C$13)

The new formula can then be copied into cells D6:D9.

All the elements of the LP model are now in place. To execute the model, click Solver from the spreadsheet menu bar[3] to access **Solver Parameters** dialogue box (shown in the middle of Figure 2.4). Next, update the dialogue box as follows:

Set Target Cell: $D$5
Equal To: ⊙ Max
By Changing Cells: $B$13:$C$13

This information tells Solver that the LP variables (cells $B$13 and $C$13) are determined by maximizing the objective function in cell $D$5.

To set up the constraints, click Add in the dialogue box to display the **Add Constraint** box (bottom of Figure 2.4) and then enter the left-hand side, inequality type, and right-hand side of the constraints as[4]

$D$6:$D$9 <= $F$6:$F$9

For the nonnegativity restrictions, click Add once again and enter

$B$13:$C$13 >= 0

Another way to enter the nonnegative constraints is to click Options in the **Solver Parameters** box to access **Solver Options** (see Figure 2.5) and then check ☑ Assume Non-Negative . Also, while in the same box, check ☑ Assume Linear Model .

---

[3] If Solver does not appear under Data menu (on Excel menu bar), click Excel Office Button → Excel Options → Add Ins → Solver Add-in → OK; then close and restart Excel.

[4] In the **Add Constraint** box in Figure 2.4, the two additional options, **int** and **bin**, which stand for **int**eger and **bin**ary, are used with integer programs to restrict variables to integer or binary values (see Chapter 9).

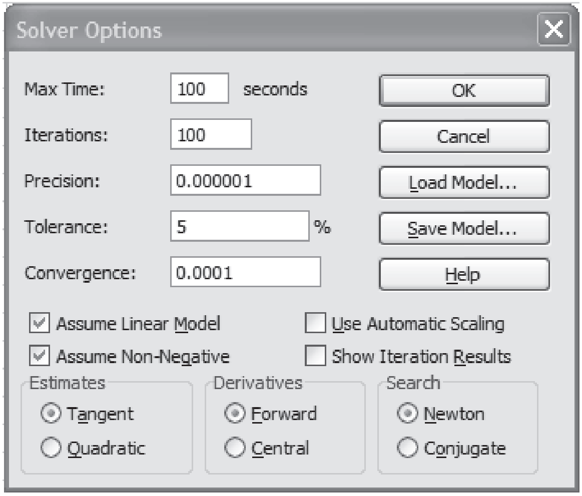


FIGURE 2.5
Solver options dialogue box

In general, the remaining default settings in **Solver Options** need not be changed. However, the default precision of .000001 may be too "high" for some problems, and Solver may incorrectly return the message "Solver could not find a feasible solution". In such cases, less precision (i.e., larger value) needs to be specified. If the message persists, then the problem may be infeasible.

Descriptive Excel range names can be used to enhance readability. A range is created by highlighting the desired cells, typing the range name in the top left box of the sheet, and then pressing Return. Figure 2.6 (file *solverRM2.xls*) provides the details with a summary of the range names used in the model. The model should be contrasted with the file *solverRM1.xls* to see how ranges are used in the formulas.

To solve the problem, click Solve on **Solver Parameters**. A new dialogue box, **Solver Results**, then gives the status of the solution. If the model setup is correct, the optimum value of z will appear in cell D5 and the values of x_1 and x_2 will go to cells B13 and C13, respectively. For convenience, cell D13 exhibits the optimum value of z by entering the formula = D5 in cell D13, thus displaying the entire optimum solution in contiguous cells.

If a problem has no feasible solution, Solver will issue the explicit message "Solver could not find a feasible solution". If the optimal objective value is unbounded (not finite), Solver will issue the somewhat ambiguous message "The Set Cell values do not converge". In either case, the message indicates that there is something wrong with the formulation of the model, as will be discussed in Section 3.5.

The **Solver Results** dialogue box provides the opportunity to request further details about the solution, including the sensitivity analysis report. We will discuss these additional results in Section 3.6.4.

The solution of the Reddy Mikks by Solver is straightforward. Other models may require a "bit of ingenuity" before they can be set up. A class of LP models that falls in this category deals with network optimization, as will be demonstrated in Chapter 6.

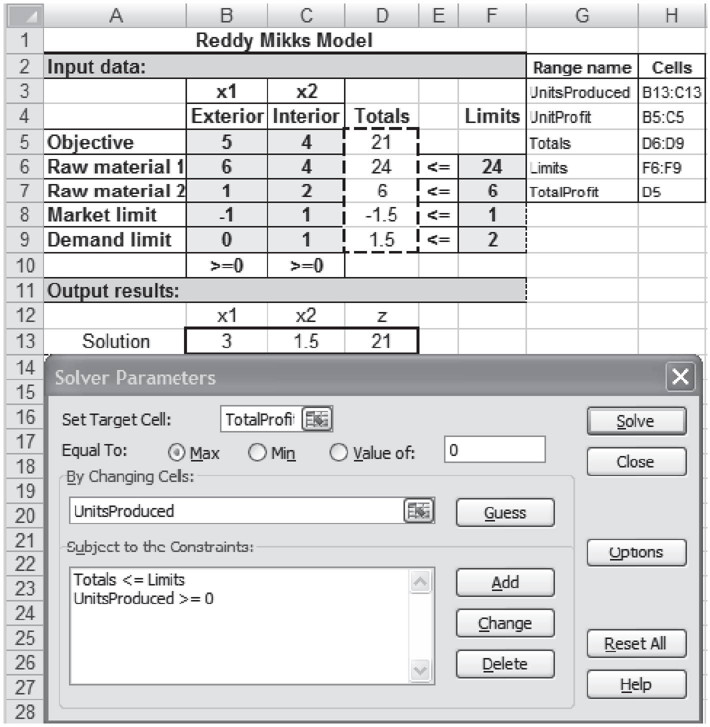


FIGURE 2.6

Use of range names in Excel Solver (file *solverRM2.xls)*

2.3.2 LP Solution with AMPL[5]

This section provides a brief introduction to AMPL. The material in Appendix C on the website details AMPL syntax. It will be cross-referenced with the presentation in this section and with other AMPL presentations in the book. The two examples presented here deal with the basics of AMPL.

Reddy Mikks Problem—A Rudimentary Model. AMPL provides a facility for modeling an LP in a rudimentary longhand format. Figure 2.7 gives the self-explanatory code for the Reddy Mikks model (file *amplRM1.txt*). All reserved keywords are in bold. All other names are user generated. The objective function and each of the constraints must have distinct (user-generated) names followed by a colon. Each statement closes with a semicolon.

The longhand format is problem-specific, in the sense that a new code is needed whenever the input data are changed. For practical problems (with complex structure and a large number of variables and constraints), the longhand format is at best cumbersome. AMPL alleviates this difficulty by devising a code that divides the problem into two components: (1) a general algebraic model for a specific class of problems

[5]For convenience, the AMPL student version is on the website. Future updates may be downloaded from www.ampl.com. AMPL uses line commands and does not operate in Windows environment.

```
maximize z: 5*x1+4*x2;
subject to
  c1: 6*x1+4*x2<=24;
  c2: x1+2*x2<=6;
  c3: -x1+x2<=1;
  c4: x2<=2;
solve;
display z,x1,x2;
```

FIGURE 2.7

Rudimentary AMPL model for the Reddy Mikks problem (file *amplRM1.txt*)

applicable to any number of variables and constraints, and (2) data for driving the algebraic model. The implementation of these two points is addressed in the following section using the Reddy Mikks problem.

Reddy Mikks Problem—An Algebraic Model. Figure 2.8 lists the statements of the model (file *amplRM2.txt*). The file must be strictly text (ASCII). The symbol # designates the start of explanatory comments. Comments may appear either on a separate line or following the semicolon at the end of a statement. The language is case sensitive, and all of its keywords, with few exceptions, are in lower case. (Section C.2 provides more details.)

The algebraic model in AMPL views the general LP problem with n variables and m constraints in the following generic format (*restr* is a user-generated name):

$$\textbf{Maximize } z\text{: } \sum_{j=1}^{n} c_j x_j$$

$$\textbf{subject to } restr_i\text{: } \sum_{j=1}^{n} a_{ij}x_j \leq b_i, i = 1, 2, \ldots, m$$

$$x_j \geq 0, j = 1, 2, \ldots, n$$

It gives the objective function and constraint i the (user-specified) names z and $restr_i$.

The model starts with the **`param`** statements that declare m, n, c, b, and a_{ij} as parameters (or constants) whose specific values are given in the input data section of the model. It translates $c_j (j = 1, 2, \ldots, n)$ as `c{1..n}`, $b_i (i = 1, 2, \ldots, m)$ as `b{1..m}`, and $a_{ij} (i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)$ as `a{1..m,1..n}`. Next, the variables x_j $(j = 1, 2, \ldots, n)$ together with the nonnegativity restriction are defined by the **`var`** statement

```
var x{1..n}>=0;
```

A variable is considered unrestricted if `>=0` is removed from its definition. The notation in `{}` represents the *set* of subscripts over which a `param` or a `var` is defined.

The model is developed in terms of the parameters and the variables in the following manner. The objective function and constraints carry distinct names followed by a colon (:). The objective statement is a direct translation of maximize $z = \sum_{j=1}^{n} c_j x_j$:

```
maximize z: sum{j in 1..n}c[j]*x[j];
```

Constraint i is given the (arbitrary) root name `restr` indexed over the set `{1..m}`:

```
restr{i in 1..m}:sum{j in 1..n}a[i,j]*x[j]<=b[i];
```

```
#-------------------------------------------algebraic model
param m;
param n;
param c{1..n};
param b{1..m};
param a{1..m,1..n};

var x{1..n}==0;

maximize z: sum{j in 1..n}c[j]*x[j];
subject to restr{i in 1..m}:
           sum{j in 1..n}a[i,j]*x[j]<=b[i];
#-------------------------------------------specify model data
data;
param n:=2;
param m:=4;
param c:=1 5 2 4;
param b:=1 24  2 6  3 1  4 2;
param a:    1    2 :=
       1    6    4
       2    1    2
       3   -1    1
       4    0    1;
#-------------------------------------------solve the problem
solve;
display z, x;
```

FIGURE 2.8

AMPL model of the Reddy Mikks problem using hard-coded input data (file *amplRM2.txt*)

The statement is a direct translation of $restr_i \sum_{j=1}^{n} a_{ij}x_j \leq b_i$.

The algebraic model may now be used with any set of applicable data that can be entered following the statement `data;`. For the Reddy Mikks model, the data tells AMPL that the problem has two variables (`param n:=2;`) and four constraints (`param m:=4;`). The compound operator `:=` must be used, and the statement must start with the keyword `param`. For the single-subscripted parameters, `c` and `b`, each element is represented by its index followed by its value and separated by at least one blank space. Thus, $c_1 = 5$ and $c_2 = 4$ are entered as

```
param c:= 1 5 2 4;
```

The data for `param b` is entered in a similar manner.

For the double-subscripted parameter a_{ij}, that data set reads as a two-dimensional matrix with its rows designating i and its columns designating j. The top line defines the subscript j, and the subscript i is entered at the start of each row as

```
param a:   1    2 :=
      1    6    4
      2    1    2
      3   -1    1
      4    0    1;
```

The data set must terminate with a semicolon. Note the *mandatory* location of the separator **:** and the compound operator **:=** after `param a`.

The model and its data are now ready. The command `solve;` invokes the solution algorithm and the command `display z, x;` provides the solution.

To execute the model, first invoke AMPL (by clicking ampl.exe in the AMPL directory). At the `ampl:` prompt, enter the following **model** command, and then press Return:

```
model amplRM2.txt;
```

The output of the system will then appear on the screen as follows:

```
MINOS 5.5: Optimal solution found.
2 iterations, objective = 21

z = 21
x[*]:=

1 = 3
2 = 1.5
```

The bottom four lines are the result of executing `display z,x;`. Actually, AMPL has formatting capabilities that enhance the readability of the output results (see Section C.5.2).

AMPL allows separating the algebraic model and the data into two independent files. This arrangement is more convenient because only the data file needs to be changed once the model has been developed. See the end of Section C.2 for details.

AMPL offers a wide range of programming capabilities. For example, the input/output data can be secured from/sent to external files, spreadsheets, and databases, and the model can be executed interactively for a wide variety of options. The details are given in Appendix C on the website.

2.4 LINEAR PROGRAMMING APPLICATIONS

This section presents realistic LP models in which the definition of the variables and the construction of the objective function and the constraints are not as straightforward as in the case of the two-variable model. The areas covered by these applications include the following:

1. Investment.
2. Production planning and inventory control.

3. Workforce planning.
4. Urban development planning.
5. Oil refining and blending.

Each model is detailed, and its optimum solution is interpreted.

2.4.1 Investment

Multitudes of investment opportunities are available to today's investor. Examples of investment problems are capital budgeting for projects, bond investment strategy, stock portfolio selection, and establishment of bank loan policy. In many of these situations, LP can be used to select the optimal mix of opportunities that will maximize return while meeting requirements set by the investor and the market.

Example 2.4-1 (Bank Loan Model)

Bank One is in the process of devising a loan policy that involves a maximum of \$12 million. The following table provides the pertinent data about available loans.

Type of loan	Interest rate	Bad-debt ratio
Personal	.140	.10
Car	.130	.07
Home	.120	.03
Farm	.125	.05
Commercial	.100	.02

Bad debts are unrecoverable and produce no interest revenue.

Competition with other financial institutions dictates the allocation of at least 40% of the funds to farm and commercial loans. To assist the housing industry in the region, home loans must equal at least 50% of the personal, car, and home loans. The bank limits the overall ratio of bad debts on all loans to at most 4%.

Mathematical Model: The situation deals with determining the amount of loan in each category, thus leading to the following definitions of the variables:

$$x_1 = \text{personal loans (in millions of dollars)}$$

$$x_2 = \text{car loans}$$

$$x_3 = \text{home loans}$$

$$x_4 = \text{farm loans}$$

$$x_5 = \text{commercial loans}$$

The objective of the Bank One is to maximize net return, the difference between interest revenue and lost bad debts. Interest revenue is accrued on loans in good standing. For example, when 10% of personal loans are lost to bad debt, the bank will receive interest on 90% of the loan—that is, it will receive 14% interest on $.9x_1$ of the original loan x_1. The same reasoning applies to the remaining four types of loans. Thus,

$$\text{Total interest} = .14(.9x_1) + .13(.93x_2) + .12(.97x_3) + .125(.95x_4) + .1(.98x_5)$$
$$= .126x_1 + .1209x_2 + .1164x_3 + .11875x_4 + .098x_5$$

We also have

$$\text{Bad debt} = .1x_1 + .07x_2 + .03x_3 + .05x_4 + .02x_5$$

The objective function combines interest revenue and bad debt as:

$$\begin{aligned}\text{Maximize } z &= \text{Total interest} - \text{Bad debt}\\ &= (.126x_1 + .1209x_2 + .1164x_3 + .11875x_4 + .098x_5)\\ &\quad - (.1x_1 + .07x_2 + .03x_3 + .05x_4 + .02x_5)\\ &= .026x_1 + .0509x_2 + .0864x_3 + .06875x_4 + .078x_5\end{aligned}$$

The problem has five constraints:

1. *Total funds should not exceed \$12 (million):*

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 12$$

2. *Farm and commercial loans equal at least 40% of all loans:*

$$x_4 + x_5 \geq .4(x_1 + x_2 + x_3 + x_4 + x_5)$$

or

$$.4x_1 + .4x_2 + .4x_3 - .6x_4 - .6x_5 \leq 0$$

3. *Home loans should equal at least 50% of personal, car, and home loans:*

$$x_3 \geq .5(x_1 + x_2 + x_3)$$

or

$$.5x_1 + .5x_2 - .5x_3 \leq 0$$

4. *Bad debts should not exceed 4% of all loans:*

$$.1x_1 + .07x_2 + .03x_3 + .05x_4 + .02x_5 \leq .04(x_1 + x_2 + x_3 + x_4 + x_5)$$

or

$$.06x_1 + .03x_2 - .01x_3 + .01x_4 - .02x_5 \leq 0$$

5. *Nonnegativity:*

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$$

A subtle assumption in the preceding formulation is that all loans are issued at approximately the same time. This allows us to ignore differences in the time value of the funds allocated to the different loans.

Solution:

The optimal solution is computed using AMPL (file *amplEx2.4-1.txt*):

$$z = .99648, x_1 = 0, x_2 = 0, x_3 = 7.2, x_4 = 0, x_5 = 4.8$$

Remarks.

1. You may be wondering why we did not define the right-hand side of the second constraint as $.4 \times 12$ instead of $.4(x_1 + x_2 + x_3 + x_4 + x_5)$. After all, it appears plausible that the bank would want to loan out all \$12 million. The answer is that the usage given in the formulation does not disallow this possibility. But there are two more reasons why you should not use $.4 \times 12$: (1) If other constraints in the model are such that all \$12 million *cannot* be used (e.g., the bank may set caps on the different loans), then the choice $.4 \times 12$ could lead to an infeasible or incorrect solution. (2) If you want to experiment with the effect of changing available funds (say from \$12 to \$13 million) on the optimum solution, there is a real chance that you may forget to change $.4 \times 12$ to $.4 \times 13$, in which case the solution will not be correct. A similar reasoning applies to the left-hand side of the fourth constraint.
2. The optimal solution calls for allocating all \$12 million: \$7.2 million to home loans and \$4.8 million to commercial loans. The remaining categories receive none. The return on the investment is

$$\text{Rate of return} = \frac{z}{12} = \frac{.99648}{12} = .08034$$

This shows that the combined annual rate of return is 8.034%, which is less than the best *net* interest rate (= 8.64% for home loans), and one wonders why the model does not take full advantage of this opportunity. The answer is that the stipulation that farm and commercial loans must account for at least 40% of all loans (constraint 2) forces the solution to allocate \$4.8 million to commercial loans at the lower *net* rate of 7.8%, hence lowering the overall interest rate to $100\left(\frac{.0864 \times 7.2 + .078 \times 4.8}{12}\right) = 8.034\%$. In fact, if we remove constraint 2, the optimum will allocate all the funds to home loans at the higher 8.64% rate (try it using the AMPL model!).

2.4.2 Production Planning and Inventory Control

There is a wealth of LP applications in the area of production planning and inventory control. This section presents three examples. The first deals with production scheduling to meet a single-period demand. The second deals with the use of inventory in a multiperiod production system to meet future demand, and the third deals with the use of inventory and worker hiring/firing to "smooth" production over a multiperiod planning horizon.

Example 2.4-2 (Single-Period Production Model)

In preparation for the winter season, a clothing company is manufacturing parka and goose overcoats, insulated pants, and gloves. All products are manufactured in four different departments: cutting, insulating, sewing, and packaging. The company has received firm orders for its products. The contract stipulates a penalty for undelivered items. Devise an optimal production plan for the company based on the following data:

	Time per unit (hr)				
Department	*Parka*	*Goose*	*Pants*	*Gloves*	Capacity (hr)
Cutting	.30	.30	.25	.15	1000
Insulating	.25	.35	.30	.10	1000
Sewing	.45	.50	.40	.22	1000
Packaging	.15	.15	.1	.05	1000
Demand	800	750	600	500	
Unit profit	\$30	\$40	\$20	\$10	
Unit penalty	\$15	\$20	\$10	\$8	

Mathematical Model: The variables of the problem are as follows:

$$x_1 = \text{number of parka jackets}$$
$$x_2 = \text{number of goose jackets}$$
$$x_3 = \text{number of pairs of pants}$$
$$x_4 = \text{number of pairs of gloves}$$

The company is penalized for not meeting demand. The objective then is to maximize net profit, defined as

$$\text{Net profit} = \text{Total profit} - \text{Total penalty}$$

The total profit is $30x_1 + 40x_2 + 20x_3 + 10x_4$. To compute the total penalty, the demand constraints can be written as

$$x_1 + s_1 = 800, x_2 + s_2 = 750, x_3 + s_3 = 600, x_4 + s_4 = 500,$$
$$x_j \geq 0, s_j \geq 0, j = 1, 2, 3, 4$$

The new variable s_j represents the shortage in demand for product j, and the total penalty can be computed as $15s_1 + 20s_2 + 10s_3 + 8s_4$. The complete model thus becomes

$$\text{Maximize } z = 30x_1 + 40x_2 + 20x_3 + 10x_4 - (15s_1 + 20s_2 + 10s_3 + 8s_4)$$

subject to

$$.30x_1 + .30x_2 + .25x_3 + .15x_4 \leq 1000$$
$$.25x_1 + .35x_2 + .30x_3 + .10x_4 \leq 1000$$
$$.45x_1 + .50x_2 + .40x_3 + .22x_4 \leq 1000$$
$$.15x_1 + .15x_2 + .10x_3 + .05x_4 \leq 1000$$
$$x_1 + s_1 = 800, x_2 + s_2 = 750, x_3 + s_3 = 600, x_4 + s_4 = 500$$
$$x_j \geq 0, s_j \geq 0, j = 1, 2, 3, 4$$

Solution:

The optimum solution (obtained using file *amplEx2.4-2.txt*) is $z = \$64{,}625, x_1 = 800, x_2 = 750, x_3 = 387.5, x_4 = 500, s_1 = s_2 = s_4 = 0, s_3 = 212.5$. The solution satisfies all the demand for both types of jackets and the gloves. A shortage of 213 (rounded up from 212.5) pairs of pants will result in a penalty cost of $213 \times \$10 = \$2130$.

Example 2.4-3 (Multiple Period Production-Inventory Model)

Acme Manufacturing Company has a contract to deliver 100, 250, 190, 140, 220, and 110 home windows over the next 6 months. Production cost (labor, material, and utilities) per window varies by period and is estimated to be \$50, \$45, \$55, \$48, \$52, and \$50 over the next 6 months. To take advantage of the fluctuations in manufacturing cost, Acme can produce more windows than needed in a given month and hold the extra units for delivery in later months. This will incur a storage cost at the rate of \$8 per window per month, assessed on end-of-month inventory. Develop a linear program to determine the optimum production schedule.

Mathematical Model: The variables of the problem include the monthly production amount and the end-of-month inventory. For $i = 1, 2, \ldots, 6$, let

$$x_i = \text{Number of units produced in month } i$$

$$I_i = \text{Inventory units left at the end of month } i$$

The relationship between these variables and the monthly demand over the 6-month horizon is represented schematically in Figure 2.9. The system starts empty $(I_0 = 0)$.

The objective is to minimize the total cost of production and end-of-month inventory.

$$\text{Total production cost} = 50x_1 + 45x_2 + 55x_3 + 48x_4 + 52x_5 + 50x_6$$

$$\text{Total inventory (storage) cost} = 8(I_1 + I_2 + I_3 + I_4 + I_5 + I_6)$$

Thus the objective function is

$$\text{Minimize } z = 50x_1 + 45x_2 + 55x_3 + 48x_4 + 52x_5 + 50x_6 + 8(I_1 + I_2 + I_3 + I_4 + I_5 + I_6)$$

The constraints of the problem can be determined directly from the representation in Figure 2.9. For each period we have the following balance equation:

$$\text{Beginning inventory} + \text{Production amount} - \text{Ending inventory} = \text{Demand}$$

This is translated mathematically for the individual months as

$$x_1 - I_1 = 100 \quad (\text{Month } 1)$$

$$I_1 + x_2 - I_2 = 250 \quad (\text{Month } 2)$$

$$I_2 + x_3 - I_3 = 190 \quad (\text{Month } 3)$$

$$I_3 + x_4 - I_4 = 140 \quad (\text{Month } 4)$$

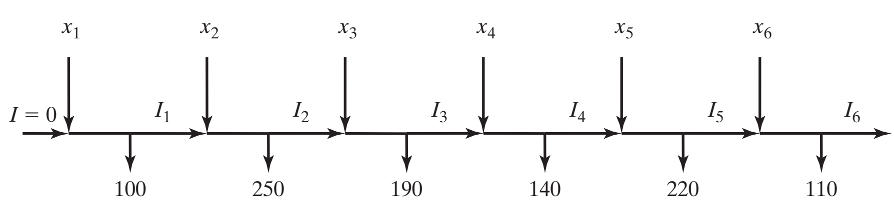
$$I_4 + x_5 - I_5 = 220 \quad (\text{Month } 5)$$

$$I_5 + x_6 = 110 \quad (\text{Month } 6)$$

$$x_i, i = 1, 2, \ldots, 6, I_i \geq 0, i = 1, 2, \ldots, 5$$

FIGURE 2.9

Schematic representation of the production-inventory system



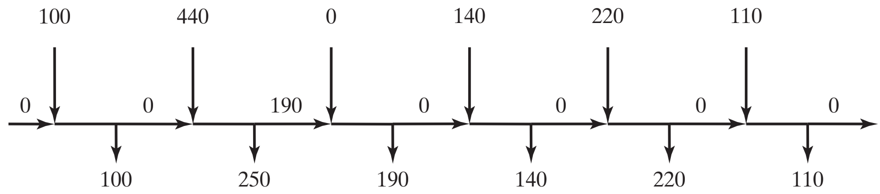


FIGURE 2.10

Optimum solution of the production-inventory problem

Note that the initial inventory, I_0, is zero. Also, in any optimal solution, the ending inventory I_6 will be zero because it is not economical to incur unnecessary additional storage cost.

Solution:

The optimum solution (obtained using file *amplEx2.4-3.txt*) is summarized in Figure 2.10. It shows that each month's demand is satisfied from the same month's production, except for month 2, where the production quantity $(= 440 \text{ units})$ covers the demand for both months 2 and 3. The total associated cost is $z = \$49{,}980$.

Example 2.4-4 (Multiperiod Production Smoothing Model)

A company is planning the manufacture of a product for March, April, May, and June of next year. The demand quantities are 520, 720, 520, and 620 units, respectively. The company has a steady workforce of 10 employees but can meet fluctuating production needs by hiring and firing temporary workers. The extra costs of hiring and firing a temp in any month are \$200 and \$400, respectively. A permanent worker produces 12 units per month, and a temporary worker, lacking equal experience, produces 10 units per month. The company can produce more than needed in any month and carry the surplus over to a succeeding month at a holding cost of \$50 per unit per month. Develop an optimal hiring/firing policy over the 4-month planning horizon.

Mathematical Model: This model is similar to that of Example 2.4-3 in the sense that each month has its production, demand, and ending inventory. The only exception deals with handling a permanent versus temporary workforce.

The permanent workers (10 in all) can be accounted for by subtracting the units they produce from the respective monthly demand. The remaining demand is then satisfied through the hiring and firing of temps. Thus,

$$\text{Remaining demand for March} = 520 - 12 \times 10 = 400 \text{ units}$$

$$\text{Remaining demand for April} = 720 - 12 \times 10 = 600 \text{ units}$$

$$\text{Remaining demand for May} = 520 - 12 \times 10 = 400 \text{ units}$$

$$\text{Remaining demand for June} = 620 - 12 \times 10 = 500 \text{ units}$$

The variables of the model for month i can be defined as

x_i = Net number of temps at the start of month i *after* any hiring or firing

S_i = Number of temps hired or fired at the start of month i

I_i = Units of ending inventory for month i

By definition, x_i and I_i are nonnegative, whereas S_i is *unrestricted in sign* because it equals the number of hired or fired workers in month i. This is the first instance in this chapter of using an unrestricted variable. As we will see shortly, special substitution is needed to allow the implementation of hiring and firing in the model.

In this model, the development of the objective function requires constructing the constraints first. The number of units produced in month i by x_i temps is $10x_i$. Thus, we have the following inventory constraints:

$$\begin{aligned}
10x_1 &= 400 + I_1 \quad (\text{March})\\
I_1 + 10x_2 &= 600 + I_2 \quad (\text{April})\\
I_2 + 10x_3 &= 400 + I_3 \quad (\text{May})\\
I_3 + 10x_4 &= 500 \qquad\quad (\text{June})\\
x_1, x_2, x_3, x_4 &\geq 0, I_1, I_2, I_3 \geq 0
\end{aligned}$$

For hiring and firing, the temp workforce starts with x_1 workers at the beginning of March. At the start of April, x_1 will be adjusted (up or down) by S_2 temps to generate x_2. The same idea applies to x_3 and x_4, thus leading to the following constraint equations:

$$\begin{aligned}
&x_1 = S_1\\
&x_2 = x_1 + S_2\\
&x_3 = x_2 + S_3\\
&x_4 = x_3 + S_4\\
&S_1, S_2, S_3, S_4 \text{ unrestricted in sign}\\
&x_1, x_2, x_3, x_4 \geq 0
\end{aligned}$$

Next, we develop the objective function. The goal is to minimize the inventory cost plus the cost of hiring and firing. As in Example 2.4-3,

$$\text{Inventory holding cost} = 50(I_1 + I_2 + I_3)$$

Modeling the cost of hiring and firing is a bit involved. Given the costs of hiring and firing a temp are \$200 and \$400, respectively, we have

$$\begin{pmatrix}\text{Cost of hiring}\\ \text{and firing}\end{pmatrix} = 200\begin{pmatrix}\text{Number of } \textit{hired} \text{ temps}\\ \text{at the start of each month}\end{pmatrix} + 400\begin{pmatrix}\text{Number of } \textit{fired} \text{ temps}\\ \text{at the start of each month}\end{pmatrix}$$

If the variable S_i is positive, hiring takes place in month i. If it is negative, then firing occurs. This "qualitative" assessment can be translated mathematically by using the substitution

$$S_i = S_i^- - S_i^+, \text{ where } S_i^-, S_i^+ \geq 0$$

The unrestricted variable S_i is now the difference between the two nonnegative variables S_i^- and S_i^+. We can think of S_i^- as the number of temps hired and S_i^+ as the number fired. For example, if $S_i^- = 5$ and $S_i^+ = 0$, then $S_i = 5 - 0 = +5$, which represents hiring. If $S_i^- = 0$ and $S_i^+ = 7$, then $S_i = 0 - 7 = -7$, which represents firing. In the first case, the corresponding cost of hiring is $200S_i^- = 200 \times 5 = \$1000$, and in the second case, the corresponding cost of firing is $400S_i^+ = 400 \times 7 = \$2800$.

The substitution $S_i = S_i^- - S_i^+$ is the basis for the development of cost of hiring and firing. First we need to address a possible question: What if both S_i^- and S_i^+ are positive? The answer is that this cannot happen because it implies both hiring and firing in the same month. Interestingly, the theory of LP (see Chapter 7) tells us that S_i^- and S_i^+ cannot be positive simultaneously, a mathematical result that confirms intuition.

We can now write the total cost of hiring and firing as

$$\text{Cost of hiring} = 200(S_1^- + S_2^- + S_3^- + S_4^-)$$
$$\text{Cost of firing} = 400(S_1^+ + S_2^+ + S_3^+ + S_4^+)$$

It may appear necessary to add to z the amount $400x_4$ representing the cost of end-of-horizon-firing of x_4 temps. From the standpoint of optimization, this factor is accounted for by the presence of S_4^+ in the objective function. Hence the optimum will not change, except for inflating optimum z by $400x_4$ (try it!).

The complete model is as follows:

$$\text{Minimize}\, z = 50(I_1 + I_2 + I_3) + 200(S_1^- + S_2^- + S_3^- + S_4^-) + 400(S_1^+ + S_2^+ + S_3^+ + S_4^+)$$

subject to

$$\begin{aligned}
10x_1 &= 400 + I_1\\
I_1 + 10x_2 &= 600 + I_2\\
I_2 + 10x_3 &= 400 + I_3\\
I_3 + 10x_4 &= 500\\
x_1 &= S_1^- - S_1^+\\
x_2 &= x_1 + S_2^- - S_2^+\\
x_3 &= x_2 + S_3^- - S_3^+\\
x_4 &= x_3 + S_4^- - S_4^+\\
S_1^-, S_1^+ S_2^-, S_2^+, S_3^-, S_3^+, S_4^-, S_4^+ &\geq 0\\
x_1, x_2, x_3, x_4 &\geq 0\\
I_1, I_2, I_3 &\geq 0
\end{aligned}$$

Solution:

The optimum solution (obtained using file *amplEx2.4-4.txt*) is $z = \$19,500, x_1 = 50, x_2 = 50, x_2 = 50, x_3 = 45, x_4 = 45,\ S_1^- = 50, S_3^+ = 5,\ \ I_1 = 100, I_3 = 50$. All the remaining variables are zero. The solution calls for hiring 50 temps in March $(S_1^- = 50)$ and holding the workforce steady till May when five temps are fired $(S_3^+ = 5)$. No further hiring or firing is recommended until the end of June, when, presumably, all temps are terminated. This solution requires 100 units of inventory to be carried into May and 50 units to be carried into June.

2.4.3 Workforce Planning

Real-Life Application—Telephone Sales Workforce Planning at Qantas Airways

Australian airline Qantas operates its main reservation offices from 7:00 till 22:00 using six shifts that start at different times of the day. Qantas used LP (with imbedded queuing analysis) to staff its main telephone sales reservation office efficiently while providing convenient service to its customers. The study, carried out in the late 1970s, resulted in annual savings of over 200,000 Australian dollars per year. The study is detailed in Case 15, Chapter 26, on the website.

Fluctuations in a labor force to meet variable demand over time can be achieved through the process of hiring and firing, as demonstrated in Example 2.4-4. There are situations in which the effect of fluctuations in demand can be "absorbed" by adjusting the start and end times of a work shift. For example, instead of following the traditional three 8-hr-shift start times at 8:00 A.M., 3:00 P.M., and 11:00 P.M., we can use overlapping 8-hr shifts in which the start time of each is made in response to increase or decrease in demand.

The idea of redefining the start of a shift to accommodate fluctuation in demand can be extended to other operating environments as well. Example 2.4-5 deals with the determination of the minimum number of buses needed to meet rush-hour and off-hour transportation needs.

Example 2.4-5 (Bus Scheduling Model)

Progress City is studying the feasibility of introducing a mass-transit bus system to reduce in-city driving. The study seeks the minimum number of buses that can handle the transportation needs. After gathering necessary information, the city engineer noticed that the minimum number of buses needed fluctuated with time of the day, and that the required number of buses could be approximated by constant values over successive 4-hr intervals. Figure 2.11 summarizes the engineer's findings. To carry out the required daily maintenance, each bus can operate only 8 successive hours a day.

Mathematical Model: The variables of the model are the number of buses needed in each shift, and the constraints deal with satisfying demand. The objective is to minimize the number of buses in operation.

The stated definition of the variables is somewhat "vague." We know that each bus will run for 8 consecutive hours, but we do not know when a shift should start. If we follow a normal three-shift schedule (8:01 A.M. to 4:00 P.M., 4:01 P.M. to 12:00 midnight, and 12:01 A.M. to 8:00 A.M.) and assume that x_1, x_2, and x_3 are the number of buses starting in the first, second, and third shifts, we can see in Figure 2.11 that $x_1 \geq 10$, $x_2 \geq 12$, and $x_3 \geq 8$. The corresponding minimum number of daily buses is $x_1 + x_2 + x_3 = 10 + 12 + 8 = 30$.

The given solution is acceptable only if the shifts *must* coincide with the normal three-shift schedule. However, it may be advantageous to allow the optimization process to choose the "best" starting time for a shift. A reasonable way to accomplish this goal is to allow a shift to start every 4 hr. The bottom of Figure 2.11 illustrates this idea with overlapping 8-hr shifts starting at 12:01 A.M., 4:01 A.M., 8:01 A.M., 12:01 P.M., 4:01 P.M., and 8:01 P.M. Thus, the variables are defined as

$$x_1 = \text{number of buses starting at 12:01 A.M.}$$

$$x_2 = \text{number of buses starting at 4:01 A.M.}$$

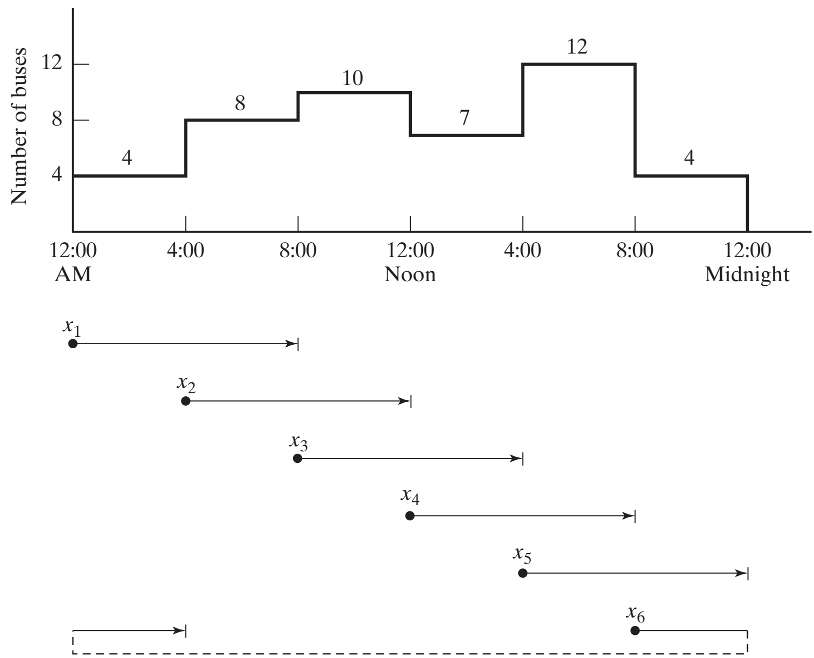


FIGURE 2.11
Number of buses as a function of the time of the day

x_3 = number of buses starting at 8:01 A.M.

x_4 = number of buses starting at 12:01 P.M.

x_5 = number of buses starting at 4:01 P.M.

x_6 = number of buses starting at 8:01 P.M.

We can see from Figure 2.11 that because of the overlapping of the shifts, the number of buses for the successive 4-hr periods can be computed as follows:

Time period	Number of buses in operation
12:01 A.M. to 4:00 A.M.	$x_1 + x_6$
4:01 A.M. to 8:00 A.M.	$x_1 + x_2$
8:01 A.M. to 12:00 noon	$x_2 + x_3$
12:01 P.M. to 4:00 P.M.	$x_3 + x_4$
4:01 P.M. to 8:00 P.M.	$x_4 + x_5$
8:01 A.M. to 12:00 A.M.	$x_5 + x_6$

The complete model thus becomes

$$\text{Minimize } z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

subject to

$$\begin{aligned}
x_1 \qquad\qquad\qquad\qquad\qquad + x_6 &\geq 4 \text{ (12:01 A.M.–4:00 A.M.)}\\
x_1 + x_2 \qquad\qquad\qquad\qquad &\geq 8 \text{ (4:01 A.M.–8:00 A.M.)}\\
x_2 + x_3 \qquad\qquad\qquad &\geq 10 \text{ (8:01 A.M.–12:00 noon)}\\
x_3 + x_4 \qquad\qquad &\geq 7 \text{ (12:01 P.M.–4:00 P.M.)}\\
x_4 + x_5 \qquad &\geq 12 \text{ (4:01 P.M.–8:00 P.M.)}\\
x_5 + x_6 &\geq 4 \text{ (8:01 P.M.–12:00 P.M.)}\\
x_j \geq 0, j = 1, 2, \ldots, 6 &
\end{aligned}$$

Solution:

The optimal solution (obtained using file *amplEx2.4-5.txt*, *solverEx2.4-5.xls*, or *toraEx2.4-5.txt*) calls for scheduling 26 buses (compared with 30 buses when the three traditional shifts are used). The schedule calls for $x_1 = 4$ buses to start at 12:01 A.M., $x_2 = 10$ at 4:01 A.M., $x_4 = 8$ at 12:01 P.M., and $x_5 = 4$ at 4:01 P.M. (N*ote*: File *solverEx2.4-5.xls* yields the alternative optimum $x_1 = 2, x_2 = 6, x_3 = 4, x_4 = 6, x_5 = 6$, and $x_6 = 2$, with $z = 26$.)

2.4.4 Urban Development Planning[6]

Urban planning deals with three general areas: (1) building new housing developments, (2) upgrading inner-city deteriorating housing and recreational areas, and (3) planning public facilities (such as schools and airports). The constraints associated with these projects are both economic (land, construction, and financing) and social (schools, parks, and income level). The objectives in urban planning vary. In new housing developments, profit is usually the motive for undertaking the project. In the remaining two categories, the goals involve social, political, economic, and cultural considerations. Indeed, in a publicized case in 2004, the mayor of a city in Ohio wanted to condemn an old area of the city to make way for a luxury housing development. The motive was to increase tax collection to help alleviate budget shortages. The example presented in this section is fashioned after the Ohio case.

Example 2.4-6 (Urban Renewal Model)

The city of Erstville is faced with a severe budget shortage. Seeking a long-term solution, the city council votes to improve the tax base by condemning an inner-city housing area and replacing it with a modern development.

The project involves two phases: (1) demolishing substandard houses to provide land for the new development and (2) building the new development. The following is a summary of the situation.

1. As many as 300 substandard houses can be demolished. Each house occupies a .25-acre lot. The cost of demolishing a condemned house is \$2000.
2. Lot sizes for new single-, double-, triple-, and quadruple-family homes (units) are .18, .28, .4, and .5 acre, respectively. Streets, open space, and utility easements account for 15% of available acreage.

[6]This section is based on Laidlaw (1972).

3. In the new development, the triple and quadruple units account for at least 25% of the total. Single units must be at least 20% of all units, and double units at least 10%.
4. The tax levied per unit for single, double, triple, and quadruple units is \$1000, \$1900, \$2700, and \$3400, respectively.
5. The construction cost per unit for single-, double-, triple-, and quadruple-family homes is \$50,000, \$70,000, \$130,000, and \$160,000, respectively.
6. Financing through a local bank is limited to \$15 million.

How many units of each type should be constructed to maximize tax collection?

Mathematical Model: Besides determining the number of units of each type of housing to be constructed, we also need to decide how many houses must be demolished to make room for the new development. Thus, the variables of the problem can be defined as follows:

$$x_1 = \text{Number of units of single-family homes}$$

$$x_2 = \text{Number of units of double-family homes}$$

$$x_3 = \text{Number of units of triple-family homes}$$

$$x_4 = \text{Number of units of quadruple-family homes}$$

$$x_5 = \text{Number of condemned homes to be demolished}$$

The objective is to maximize total tax collection from all four types of homes—that is,

$$\text{Maximize } z = 1000x_1 + 1900x_2 + 2700x_3 + 3400x_4$$

The first constraint of the problem deals with land availability.

$$\begin{pmatrix}\text{Acreage used for new} \\ \text{homes construction}\end{pmatrix} \leq \begin{pmatrix}\text{Net available} \\ \text{acreage}\end{pmatrix}$$

From the data of the problem, we have

$$\text{Acreage needed for new homes} = .18x_1 + .28x_2 + .4x_3 + .5x_4$$

To determine the available acreage, each demolished home occupies a .25-acre lot, thus netting $.25x_5$ acres. Allowing for 15% open space, streets, and easements, the net acreage available is $.85(.25x_5) = .2125x_5$. The resulting constraint is

$$.18x_1 + .28x_2 + .4x_3 + .5x_4 \leq .2125x_5$$

or

$$.18x_1 + .28x_2 + .4x_3 + .5x_4 - .2125x_5 \leq 0$$

The number of demolished homes cannot exceed 300, which translates to

$$x_5 \leq 300$$

Next, we add the constraints limiting the number of units of each home type.

$$(\text{Number of single units}) \geq (20\% \text{ of all units})$$

$$(\text{Number of double units}) \geq (10\% \text{ of all units})$$

$$(\text{Number of triple and quadruple units}) \geq (25\% \text{ of all units})$$

These constraints translate mathematically to

$$x_1 \geq .2(x_1 + x_2 + x_3 + x_4)$$

$$x_2 \geq .1(x_1 + x_2 + x_3 + x_4)$$

$$x_3 + x_4 \geq .25(x_1 + x_2 + x_3 + x_4)$$

The only remaining constraint deals with keeping the demolition/construction cost within the allowable budget—that is,

$$(\text{Construction and demolition cost}) \leq (\text{Available budget})$$

Expressing all the costs in thousands of dollars, we get

$$(50x_1 + 70x_2 + 130x_3 + 160x_4) + 2x_5 \leq 15000$$

The complete model thus becomes

$$\text{Maximize } z = 1000x_1 + 1900x_2 + 2700x_3 + 3400x_4$$

subject to

$$\begin{aligned}
.18x_1 + .28x_2 + .4x_3 + .5x_4 - .2125x_5 &\leq 0 \\
x_5 &\leq 300 \\
-.8x_1 + .2x_2 + .2x_3 + .2x_4 &\leq 0 \\
.1x_1 - .9x_2 + .1x_3 + .1x_4 &\leq 0 \\
.25x_1 + .25x_2 - .75x_3 - .75x_4 &\leq 0 \\
50x_1 + 70x_2 + 130x_3 + 160x_4 + 2x_5 &\leq 15000 \\
x_1, x_2, x_3, x_4, x_5 &\geq 0
\end{aligned}$$

Solution:

The optimum solution (obtained using file *amplEX2.4-6.txt* or *solverEx2.4-6.xls*) is

Total tax collection $= z = \$343,965$
Number of single homes $= x_1 = 35.83 \simeq 36$ units
Number of double homes $= x_2 = 98.53 \simeq 99$ units
Number of triple homes $= x_3 = 44.79 \simeq 45$ units
Number of quadruple homes $= x_4 = 0$ units
Number of homes demolished $= x_5 = 244.49 \simeq 245$ units

Remarks. Linear programming does not automatically guarantee an integer solution, and this is the reason for rounding the continuous values to the closest integer. The rounded solution calls for constructing $180 (= 36 + 99 + 45)$ units and demolishing 245 old homes, which yields \$345,600 in taxes. Keep in mind, however, that, in general, the rounded solution may not be feasible. In fact, the current rounded solution violates the budget constraint by \$70,000 (verify!). Interestingly, the true optimum integer solution (using the algorithms in Chapter 9) is $x_1 = 36, x_2 = 98, x_3 = 45, x_4 = 0$, and $x_5 = 245$ with $z = \$343{,}700$. Carefully note that the rounded solution yields a better objective value, which appears contradictory. The reason is that the rounded solution calls for producing an extra double home, which is feasible only if the budget is increased by \$70,000.

2.4.5 Blending and Refining

A number of LP applications deal with blending different input materials to manufacture products that meet certain specifications while minimizing cost or maximizing profit. The input materials could be ores, metal scraps, chemicals, or crude oils, and the output products could be metal ingots, paints, or gasoline of various grades. This section presents a (simplified) model for oil refining. The process starts with distilling crude oil to produce intermediate gasoline stocks, and then blending these stocks to produce final gasoline products. The final products must satisfy certain quality specifications (such as octane rating). In addition, distillation capacities and demand limits can directly affect the level of production of the different grades of gasoline. One goal of the model is to determine the optimal mix of final products that will maximize an appropriate profit function. In some cases, the goal may be to minimize a cost function.

Example 2.4-7 (Crude Oil Refining and Gasoline Blending)

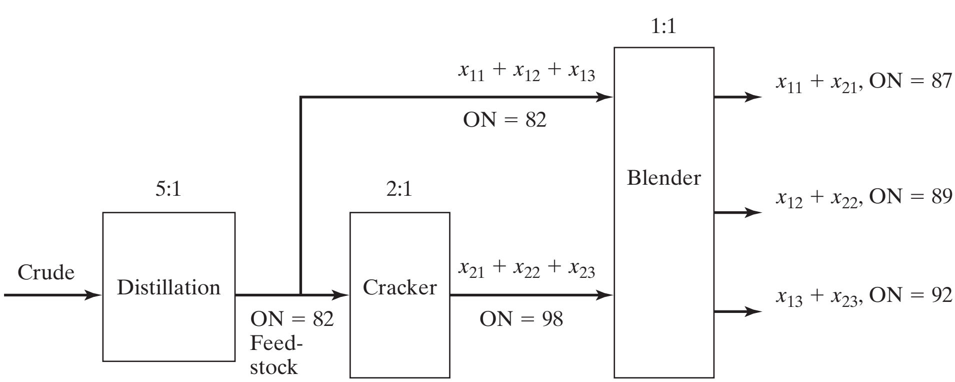
Shale Oil, located on the island of Aruba, has a capacity of 1,500,000 bbl of crude oil per day. The final products from the refinery include three types of unleaded gasoline with different octane numbers (ON): regular with ON = 87, premium with ON = 89, and super with ON = 92. The refining process encompasses three stages: (1) a distillation tower that produces feedstock (ON = 82) at the rate of .2 bbl per bbl of crude oil, (2) a cracker unit that produces gasoline stock (ON = 98) by using a portion of the feedstock produced from the distillation tower at the rate of .5 bbl per bbl of feedstock, and (3) a blender unit that blends the gasoline stock from the cracker unit and the feedstock from the distillation tower. The company estimates the net profit per barrel of the three types of gasoline to be \$6.70, \$7.20, and \$8.10, respectively. The input capacity of the cracker unit is 200,000 bbl of feedstock a day. The demand limits for regular, premium, and super gasoline are 50,000, 30,000, and 40,000 bbl, respectively, per day. Develop a model for determining the optimum production schedule for the refinery.

Mathematical Model: Figure 2.12 summarizes the elements of the model. The variables can be defined in terms of two input streams to the blender (feedstock and cracker gasoline) and the three final products. Let

$$x_{ij} = \text{bbl/day of input stream } i \text{ used to blend final product } j,\ i = 1, 2;\ j = 1, 2, 3$$

FIGURE 2.12

Product flow in the refinery problem



Using this definition, we have

Daily production of regular gasoline $= x_{11} + x_{21}$ bbl/day

Daily production of premium gasoline $= x_{12} + x_{22}$ bbl/day

Daily production of super gasoline $= x_{13} + x_{23}$ bbl/day

$$\begin{pmatrix}\text{Daily output}\\ \text{of blender unit}\end{pmatrix} = \begin{pmatrix}\text{Daily } \textit{regular}\\ \text{production}\end{pmatrix} + \begin{pmatrix}\text{Daily } \textit{premium}\\ \text{production}\end{pmatrix} + \begin{pmatrix}\text{Daily } \textit{super}\\ \text{production}\end{pmatrix}$$

$$= (x_{11} + x_{21}) + (x_{12} + x_{22}) + (x_{13} + x_{23}) \text{ bbl/day}$$

$$\begin{pmatrix}\text{Daily feedstock}\\ \text{to blender}\end{pmatrix} = x_{11} + x_{12} + x_{13} \text{ bbl/day}$$

$$\begin{pmatrix}\text{Daily cracker unit}\\ \text{feed to blender}\end{pmatrix} = x_{21} + x_{22} + x_{23} \text{ bbl/day}$$

$$\begin{pmatrix}\text{Daily feedstock}\\ \text{to cracker}\end{pmatrix} = 2(x_{21} + x_{22} + x_{23}) \text{ bbl/day}$$

$$\begin{pmatrix}\text{Daily crude oil used}\\ \text{in the refinery}\end{pmatrix} = 5(x_{11} + x_{12} + x_{13}) + 10(x_{21} + x_{22} + x_{23}) \text{ bbl/day}$$

The objective of the model is to maximize the total profit resulting from the sale of all three grades of gasoline. From the definitions given earlier, we get

$$\text{Maximize } z = 6.70(x_{11} + x_{21}) + 7.20(x_{12} + x_{22}) + 8.10(x_{13} + x_{23})$$

The constraints of the problem are developed as follows:

1. *Daily crude oil supply does not exceed 1,500,000 bbl/day:*

$$5(x_{11} + x_{12} + x_{13}) + 10(x_{21} + x_{22} + x_{23}) \leq 1{,}500{,}000$$

2. *Cracker unit input capacity does not exceed 200,000 bbl/day:*

$$2(x_{21} + x_{22} + x_{23}) \leq 200{,}000$$

3. *Daily demand for regular does not exceed 50,000 bbl:*

$$x_{11} + x_{21} \leq 50{,}000$$

4. *Daily demand for premium does not exceed 30,000 bbl:*

$$x_{12} + x_{22} \leq 30{,}000$$

5. *Daily demand for super does not exceed 40,000 bbl:*

$$x_{13} + x_{23} \leq 40{,}000$$

6. *Octane number (ON) for regular is at least 87:*

The octane number of a gasoline product is the weighted average of the octane numbers of the input streams used in the blending process and can be computed as

$$\begin{pmatrix}\text{Average ON of}\\ \text{regular gasoline}\end{pmatrix} = \frac{\text{Feedstock ON} \times \text{feedstock bbl/day} + \text{Cracker unit ON} \times \text{Cracker unit bbl/day}}{\text{Total bbl/day of regular gasoline}}$$

$$= \frac{82x_{11} + 98x_{21}}{x_{11} + x_{21}}$$

Thus, octane number constraint for regular gasoline becomes

$$\frac{82x_{11} + 98x_{21}}{x_{11} + x_{21}} \geq 87$$

The constraint is linearized as

$$82x_{11} + 98x_{21} \geq 87(x_{11} + x_{21})$$

7. *Octane number for premium is at least 89:*

$$\frac{82x_{12} + 98x_{22}}{x_{12} + x_{22}} \geq 89$$

which is linearized as

$$82x_{12} + 98x_{22} \geq 89(x_{12} + x_{22})$$

8. *Octane number for super is at least 92:*

$$\frac{82x_{13} + 98x_{23}}{x_{13} + x_{23}} \geq 92$$

or

$$82x_{13} + 98x_{23} \geq 92(x_{13} + x_{23})$$

The complete model is thus summarized as

$$\text{Maximize } z = 6.70(x_{11} + x_{21}) + 7.20(x_{12} + x_{22}) + 8.10(x_{13} + x_{23})$$

subject to

$$\begin{aligned}
5(x_{11} + x_{12} + x_{13}) + 10(x_{21} + x_{22} + x_{23}) &\leq 1{,}500{,}000\\
2(x_{21} + x_{22} + x_{23}) &\leq 200{,}000\\
x_{11} + x_{21} &\leq 50{,}000\\
x_{12} + x_{22} &\leq 30{,}000\\
x_{13} + x_{23} &\leq 40{,}000\\
82x_{11} + 98x_{21} &\geq 87(x_{11} + x_{21})\\
82x_{12} + 98x_{22} &\geq 89(x_{12} + x_{22})\\
82x_{13} + 98x_{23} &\geq 92(x_{13} + x_{23})\\
x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} &\geq 0
\end{aligned}$$

The last three constraints can be simplified to produce a constant right-hand side.

Solution:

The optimum solution (obtained using file *toraEx2.4-7.txt* or *amplEx2.4-7.txt*) is $z = 875{,}000$, $x_{11} = 34{,}375$, $x_{21} = 15{,}625$, $x_{12} = 16{,}875$, $x_{22} = 13{,}125$, $x_{13} = 15{,}000$, $x_{23} = 25{,}000$. This translates to

Daily profit = \$875,000
Daily amount of regular gasoline = $x_{11} + x_{21} = 34{,}375 + 13{,}125 = 30{,}000$ bbl/day
Daily amount of premium gasoline = $x_{12} + x_{22} = 16{,}875 + 13{,}125 = 30{,}000$ bbl/day
Daily amount of super gasoline = $x_{13} + x_{23} = 15{,}000 + 25{,}000 = 40{,}000$ bbl/day

The solution shows that regular gasoline production is 20,000 bbl/day short of satisfying the maximum demand. The demand for the remaining two grades is satisfied.

2.4.6 Additional LP Applications

The preceding sections have demonstrated representative LP applications in five areas. Problems 2-77 to 2-87 provide additional areas of application, ranging from agriculture to military.

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PROBLEMS

Section	Assigned Problems	Section	Assigned Problems
2.1	2-1 to 2-4	2.4.2	2-47 to 2-54
2.2.1	2-5 to 2-27	2.4.3	2-55 to 2-60
2.2.2	2-28 to 2-35	2.4.4	2-61 to 2-66
2.3.1	2-36 to 2-37	2.4.5	2-67 to 2-76
2.3.2	2-38 to 2-39	2.4.6	2-77 to 2-87
2.4.1	2-40 to 2-46		

2-1. For the Reddy Mikks model, construct each of the following constraints, and express it with a linear left-hand side and a constant right-hand side:

*__(a)__ The daily demand for interior paint exceeds that of exterior paint by *at least* 1 ton.

(b) The daily usage of raw material $M1$ in tons is *at most* 8 and *at least* 5.

*__(c)__ The demand for exterior paint cannot be less than the demand for interior paint.

(d) The maximum quantity that should be produced of both the interior and the exterior paint is 15 tons.

*__(e)__ The proportion of exterior paint to the total production of both interior and exterior paints must not exceed .5.

2-2. Determine the best *feasible* solution among the following (feasible and infeasible) solutions of the Reddy Mikks model:

(a) $x_1 = 1, x_2 = 2.$

(b) $x_1 = 3, x_2 = 1.$

(c) $x_1 = 3, x_2 = 1.5.$

(d) $x_1 = 2, x_2 = 1.$

(e) $x_1 = 2, x_2 = -1.$

*__2-3.__ For the feasible solution $x_1 = 1, x_2 = 2$ of the Reddy Mikks model, determine the unused amounts of raw materials $M1$ and $M2$.

2-4. Suppose that Reddy Mikks sells its exterior paint to a single wholesaler at a quantity discount. The profit per ton is $5000 if the contractor buys no more than 5 tons daily and $4300 otherwise. Express the objective function mathematically. Is the resulting function linear?

2-5. Determine the feasible space for each of the following independent constraints, given that $x_1, x_2 \geq 0$.

*__(a)__ $-3x_1 + x_2 \leq 6.$

(b) $x_1 - 2x_2 \geq 5.$

(c) $2x_1 - 3x_2 \leq 12.$

(d) $x_1 - x_2 \leq 0.$

*__(e)__ $-x_1 + x_2 \geq 0.$

2-6. Identify the direction of increase in z in each of the following cases:

*__(a)__ Maximize $z = x_1 - x_2.$

(b) Maximize $z = -8x_1 - 3x_2.$

(c) Maximize $z = -x_1 + 3x_2.$

*__(d)__ Maximize $z = -3x_1 + x_2.$

2-7. Determine the solution space and the optimum solution of the Reddy Mikks model for each of the following independent changes:

(a) The maximum daily demand for interior paint is 1.9 tons and that for exterior paint is at most 2.5 tons.

(b) The daily demand for interior paint is at least 2.5 tons.

(c) The daily demand for interior paint is exactly 1 ton higher than that for exterior paint.

(d) The daily availability of raw material $M1$ is at least 24 tons.

(e) The daily availability of raw material $M1$ is at least 24 tons, and the daily demand for interior paint exceeds that for exterior paint by at least 1 ton.

2-8. A company that operates 10 hrs a day manufactures two products on three sequential processes. The following table summarizes the data of the problem:

	Minutes per unit			
Product	*Process 1*	*Process 2*	*Process 3*	Unit profit
1	10	6	8	\$20
2	5	20	10	\$30

Determine the optimal mix of the two products.

***2-9.** A company produces two products, A and B. The sales volume for A is at least 80% of the total sales of both A and B. However, the company cannot sell more than 110 units of A per day. Both products use one raw material, of which the maximum daily availability is 300 lb. The usage rates of the raw material are 2 lb per unit of A, and 4 lb per unit of B. The profit units for A and B are \$40 and \$90, respectively. Determine the optimal product mix for the company.

2-10. Alumco manufactures aluminum sheets and aluminum bars. The maximum production capacity is estimated at either 800 sheets or 600 bars per day. The maximum daily demand is 550 sheets and 560 bars. The profit per ton is \$40 per sheet and \$35 per bar. Determine the optimal daily production mix.

***2-11.** An individual wishes to invest \$5000 over the next year in two types of investment: Investment A yields 5%, and investment B yields 8%. Market research recommends an allocation of at least 25% in A and at most 50% in B. Moreover, investment in A should be at least half the investment in B. How should the fund be allocated to the two investments?

2-12. The Continuing Education Division at the Ozark Community College offers a total of 30 courses each semester. The courses offered are usually of two types: practical and humanistic. To satisfy the demands of the community, at least 10 courses of each type must be offered each semester. The division estimates that the revenues of offering practical and humanistic courses are approximately \$1500 and \$1000 per course, respectively.

(a) Devise an optimal course offering for the college.

(b) Show that the worth per additional course is \$1500, which is the same as the revenue per practical course. What does this result mean in terms of offering additional courses?

2-13. ChemLabs uses raw materials I and II to produce two domestic cleaning solutions, A and B. The daily availabilities of raw materials I and II are 150 and 145 units, respectively. One unit of solution A consumes .5 unit of raw material I and .6 unit of raw material II. One unit of solution B uses .5 unit of raw material I and .4 unit of raw material II. The profits per unit of solutions A and B are \$8 and \$10, respectively. The daily demand for solution A lies between 30 and 150 units, and that for solution B between 40 and 200 units. Find the optimal production amounts of A and B.

2-14. In the Ma-and-Pa grocery store, shelf space is limited and must be used effectively to increase profit. Two cereal items, Grano and Wheatie, compete for a total shelf space of 60 ft^2. A box of Grano occupies .2 ft^2 and a box of Wheatie needs .4 ft^2. The maximum daily demands of Grano and Wheatie are 200 and 120 boxes, respectively. A box of Grano nets \$1.00 in profit and a box of Wheatie \$1.35. Ma-and-Pa thinks that because the unit profit of Wheatie is 35% higher than that of Grano, Wheatie should be allocated 35% more space than Grano, which amounts to allocating about 57% to Wheatie and 43% to Grano. What do you think?

2-15. Jack is an aspiring freshman at Ulern University. He realizes that "all work and no play make Jack a dull boy." Jack wants to apportion his available time of about 10 hrs a day between work and play. He estimates that play is twice as much fun as work. He also wants to study at least as much as he plays. However, Jack realizes that if he is going to get all his homework assignments done, he cannot play more than 4 hrs a day. How should Jack allocate his time to maximize his pleasure from both work and play?

2-16. Wild West produces two types of cowboy hats. A Type 1 hat requires twice as much labor time as a Type 2. If all the available labor time is dedicated to Type 2 alone, the company can produce a total of 400 Type 2 hats a day. The respective market limits for Type 1 and Type 2 are 150 and 200 hats per day, respectively. The profit is $8 per Type 1 hat and $5 per Type 2 hat. Determine the number of hats of each type that maximizes profit.

2-17. Show & Sell can advertise its products on local radio and television (TV). The advertising budget is limited to $10,000 a month. Each minute of radio advertising costs $15, and each minute of TV commercials $300. Show & Sell likes to advertise on radio at least twice as much as on TV. In the meantime, it is not practical to use more than 400 minutes of radio advertising a month. From past experience, advertising on TV is estimated to be 25 times as effective as on radio. Determine the optimum allocation of the budget to radio and TV advertising.

***2-18.** Wyoming Electric Coop owns a steam-turbine power-generating plant. Because Wyoming is rich in coal deposits, the plant generates its steam from coal. This, however, may result in emission that does not meet the Environmental Protection Agency (EPA) standards. EPA regulations limit sulfur dioxide discharge to 2000 parts per million per ton of coal burned and smoke discharge from the plant stacks to 20 lb per hour. The Coop receives two grades of pulverized coal, $C1$ and $C2$, for use in the steam plant. The two grades are usually mixed together before burning. For simplicity, it can be assumed that the amount of sulfur pollutant discharged (in parts per million) is a weighted average of the proportion of each grade used in the mixture. The following data is based on the consumption of 1 ton per hr of each of the two coal grades.

Coal grade	Sulfur discharge in parts per million	Smoke discharge in lb per hour	Steam generated in lb per hour
$C1$	1800	2.1	12,000
$C2$	2100	.9	9,000

(a) Determine the optimal ratio for mixing the two coal grades.

(b) Determine the effect of relaxing the smoke discharge limit by 1 lb on the amount of generated steam per hour.

2-19. Top Toys is planning a new radio and TV advertising campaign. A radio commercial costs $300 and a TV ad costs $2000. A total budget of $20,000 is allocated to the campaign. However, to ensure that each medium will have at least one radio commercial and one TV ad, the most that can be allocated to either medium cannot exceed 80% of the total budget. It is estimated that the first radio commercial will reach 5000 people, with each additional commercial reaching only 2000 new ones. For TV, the first ad will reach 4500 people, and each additional ad an additional 3000. How should the budgeted amount be allocated between radio and TV?

2-20. The Burroughs Garment Company manufactures men's shirts and women's blouses for Walmark Discount Stores. Walmark will accept all the production supplied by Burroughs. The production process includes cutting, sewing, and packaging. Burroughs employs 25 workers in the cutting department, 35 in the sewing department, and 5 in the packaging department. The factory works one 8-hr shift, 5 days a week. The following table gives the time requirements and profits per unit for the two garments.

	Minutes per unit			
Garment	*Cutting*	*Sewing*	*Packaging*	Unit profit ($)
Shirts	20	70	12	8
Blouses	60	60	4	12

Determine the optimal weekly production schedule for Burroughs.

2-21. A furniture company manufactures desks and chairs. The sawing department cuts the lumber for both products, which is then sent to separate assembly departments. Assembled items are sent to the painting department for finishing. The daily capacity of the sawing department is 200 chairs or 80 desks. The chair assembly department can produce 120 chairs daily, and the desk assembly department 60 desks daily. The paint department has a daily capacity of either 150 chairs or 110 desks. Given that the profit per chair is $50 and that of a desk is $100, determine the optimal production mix for the company.

***2-22.** An assembly line consisting of three consecutive stations produces two radio models: HiFi-1 and HiFi-2. The following table provides the assembly times for the three workstations.

	Minutes per unit	
Workstation	*HiFi-1*	*HiFi-2*
1	6	4
2	5	5
3	4	6

The daily maintenance for stations 1, 2, and 3 consumes 10%, 14%, and 12%, respectively, of the maximum 480 minutes available for each station each day. Determine the optimal product mix that will minimize the idle (or unused) times in the three workstations.

2-23. *Determination of the Optimum LP Solution by Enumerating All Feasible Corner Points.* The remarkable observation gleaned from the graphical LP solution is that the optimum, when finite, is always associated with a corner point of the feasible solution space. Show how this idea is applied to the Reddy Mikks model by evaluating all of its feasible corner points A, B, C, D, E, and F.

2-24. *TORA Experiment.* Enter the following LP into TORA, and select the graphic solution mode to reveal the LP graphic screen.

$$\text{Minimize } z = 3x_1 + 8x_2$$

subject to

$$\begin{aligned}
x_1 + x_2 &\geq 8 \\
2x_1 - 3x_2 &\leq 0 \\
x_1 + 2x_2 &\leq 30 \\
3x_1 - x_2 &\geq 0 \\
x_1 &\leq 10 \\
x_2 &\geq 9 \\
x_1, x_2 &\geq 0
\end{aligned}$$

Next, on a sheet of paper, graph and scale the x_1- and x_2-axes for the problem (you may also click Print Graph on the top of the right window to obtain a ready-to-use scaled sheet). Now, graph a constraint manually on the prepared sheet, and then click on the left window of the screen to check your answer. Repeat the same for each constraint, and then terminate the procedure with a graph of the objective function. The suggested process is designed to test and reinforce your understanding of the graphical LP solution through immediate feedback from TORA.

2-25. *TORA Experiment*. Consider the following LP model:

$$\text{Maximize } z = 5x_1 + 4x_2$$

subject to

$$\begin{aligned} 6x_1 + 4x_2 &\leq 24 \\ 6x_1 + 3x_2 &\leq 22.5 \\ x_1 + x_2 &\leq 5 \\ x_1 + 2x_2 &\leq 6 \\ -x_1 + x_2 &\leq 1 \\ x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

In LP, a constraint is said to be *redundant* if its removal from the model leaves the feasible solution space unchanged. Use the graphical facility of TORA to identify the redundant constraints, and then show that their removal (simply by not graphing them) does not affect the solution space or the optimal solution.

2-26. *TORA Experiment.* In the Reddy Mikks model, use TORA to show that the removal of the raw material constraints (constraints 1 and 2) would result in an *unbounded solution space*. What can be said in this case about the optimal solution of the model?

2-27. *TORA Experiment.* In the Reddy Mikks model, suppose that the following constraint is added to the problem:

$$x_2 \geq 3$$

Use TORA to show that the resulting model has conflicting constraints that cannot be satisfied simultaneously, and hence it has *no feasible solution*.

2-28. Identify the direction of decrease in z in each of the following cases:

*(a) Minimize $z = 4x_1 - 2x_2$.

(b) Minimize $z = -6x_1 + 2x_2$.

(c) Minimize $z = -3x_1 - 6x_2$.

2-29. For the diet model, suppose that the daily availability of corn is limited to 400 lb. Identify the new solution space, and determine the new optimum solution.

2-30. For the diet model, determine the optimum solution given the feed mix does not exceed 500 lb a day? Does the solution make sense?

2-31. John must work at least 20 hours a week to supplement his income while attending school. He has the opportunity to work in two retail stores. In store 1, he can work between 4.5 and 12 hours a week, and in store 2, he is allowed between 5.5 and 10 hours. Both stores pay the same hourly wage. In deciding how many hours to work in each store, John wants to base his decision on work stress. Based on interviews with present employees, John estimates that, on an ascending scale of 1 to 10, the stress factors are 8 and 6 at stores 1 and 2, respectively. Because stress mounts by the hour, he assumes that the total stress for each store at the end of the week is proportional to the number of hours he works in the store. How many hours should John work in each store?

***2-32.** OilCo is building a refinery to produce four products: diesel, gasoline, lubricants, and jet fuel. The minimum demand (in bbl/day) for each of these products is 14,000, 30,000, 10,000, and 8000, respectively. Iraq and Dubai are under contract to ship crude to OilCo. Because of the production quotas specified by OPEC (Organization of Petroleum Exporting Countries), the new refinery can receive at least 40% of its crude from Iraq and the remaining amount from Dubai. OilCo predicts that the demand and crude oil quotas will remain steady over the next 10 years.

The specifications of the two crude oils lead to different product mixes. One barrel of Iraq crude yields .2 bbl of diesel, .25 bbl of gasoline, .1 bbl of lubricant, and .15 bbl of jet fuel. The corresponding yields from Dubai crude are .1, .6, .15, and .1, respectively. OilCo needs to determine the minimum capacity of the refinery (in bbl/day).

2-33. Day Trader wants to invest a sum of money that would generate an annual yield of at least \$10,000. Two stock groups are available: blue chips and high tech, with average annual yields of 10% and 25%, respectively. Though high-tech stocks provide higher yield, they are more risky, and Trader wants to limit the amount invested in these stocks to no more than 60% of the total investment. What is the minimum amount Trader should invest in each stock group to accomplish the investment goal?

***2-34.** An industrial recycling center uses two scrap aluminum metals, A and B, to produce a special alloy. Scrap A contains 6% aluminum, 3% silicon, and 4% carbon. Scrap B has 3% aluminum, 6% silicon, and 3% carbon. The costs per ton for scraps A and B are \$100 and \$80, respectively. The specifications of the special alloy require that (1) the aluminum content must be at least 3% and at most 6%, (2) the silicon content must be between 3% and 5%, and (3) the carbon content must be between 3% and 7%. Determine the optimum mix of the scraps that should be used in producing 1000 tons of the alloy.

2-35. *TORA Experiment*. Consider the Diet Model, and let the objective function be given as

$$\text{Minimize } z = .8x_1 + .8x_2$$

Use TORA to show that the optimum solution is associated with *two* distinct corner points, and that both points yield the same objective value. In this case, the problem is said to have *alternative optima*. Explain the conditions leading to this situation, and show that, in effect, the problem has an infinite number of alternative optima. Then provide a formula for determining all such solutions.

2-36. Modify the Reddy Mikks Solver model of Figure 2.4 to account for a third type of paint named "marine." Requirements per ton of raw materials 1 and 2 are .6 and .85 ton, respectively. The daily demand for the new paint lies between .6 ton and 1.9 tons. The profit per ton is \$3700.

2-37. Develop the Excel Solver model for the following problems:

(a) The diet model of Example 2.2-2.

(b) Problem 2-21.

(c) Problem 2-34.

2-38. In the Reddy Mikks model, suppose that a third type of paint, named "marine," is produced. The requirements per ton of raw materials $M1$ and $M2$ are .7 and .95 ton, respectively. The daily demand for the new paint lies between .4 ton and 2.1 tons, and the profit per ton is \$4500. Modify the Excel Solver model *solverRM2.xls* and the AMPL model *amplRM2.txt* to account for the new situation and determine the optimum solution. Compare the additional effort associated with each modification.

2-39. Develop AMPL models for the following problems:

(a) The diet problem of Example 2.2-2 and find the optimum solution.

(b) Problem 2-22.

(c) Problem 2-34.

2-40. Fox Enterprises is considering six projects for possible construction over the next four years. Fox can undertake any of the projects partially or completely. A partial undertaking of a project will prorate both the return and cash outlays proportionately. The expected (present value) returns and cash outlays for the projects are given in the following table.

	Cash outlay ($1000)				
Project	*Year 1*	*Year 2*	*Year 3*	*Year 4*	Return ($1000)
1	10.5	14.4	2.2	2.4	324.00
2	8.3	12.6	9.5	3.1	358.00
3	10.2	14.2	5.6	4.2	177.50
4	7.2	10.5	7.5	5.0	148.00
5	12.3	10.1	8.3	6.3	182.00
6	9.2	7.8	6.9	5.1	123.50
Available funds ($1000)	60.0	70.0	35.0	20.0	

(a) Formulate the problem as a linear program, and determine the optimal project mix that maximizes the total return using AMPL, Solver, or TORA. Ignore the time value of money.

(b) Suppose that if a portion of project 2 is undertaken, then at least an equal portion of project 6 must be undertaken. Modify the formulation of the model, and find the new optimal solution.

(c) In the original model, suppose that any funds left at the end of a year are used in the next year. Find the new optimal solution, and determine how much each year "borrows" from the preceding year. For simplicity, ignore the time value of money.

(d) Suppose in the original model the yearly funds available for any year can be exceeded, if necessary, by borrowing from other financial activities within the company. Ignoring the time value of money, reformulate the LP model, and find the optimum solution. Would the new solution require borrowing in any year? If so, what is the rate of return on borrowed money?

***2-41.** Investor Doe has $10,000 to invest in four projects. The following table gives the cash flow for the four investments.

	Cash flow ($1000) at the start of				
Project	*Year 1*	*Year 2*	*Year 3*	*Year 4*	*Year 5*
1	−1.00	0.50	0.30	1.80	1.20
2	−1.00	0.60	0.20	1.50	1.30
3	0.00	−1.00	0.80	1.90	0.80
4	−1.00	0.40	0.60	1.80	0.95

The information in the table can be interpreted as follows: For project 1, \$1.00 invested at the start of year 1 will yield \$.50 at the start of year 2, \$.30 at the start of year 3, \$1.80 at the start of year 4, and \$1.20 at the start of year 5. The remaining entries can be interpreted similarly. The entry 0.00 indicates that no transaction is taking place. Doe has the additional option of investing in a bank account that earns 6.5% annually. All funds accumulated at the end of 1 year can be reinvested in the following year. Formulate the problem as a linear program to determine the optimal allocation of funds to investment opportunities. Solve the model using Solver or AMPL.

2-42. HiRise Construction can bid on two 1-year projects. The following table provides the quarterly cash flow (in millions of dollars) for the two projects.

	Cash flow (in millions of \$) at				
Project	*January 1*	*April 1*	*July 1*	*October 1*	*December 31*
I	−1.0	−3.1	−1.5	1.8	5.0
II	−3.0	−2.5	1.5	1.8	2.8

HiRise has cash funds of \$1 million at the beginning of each quarter and may borrow at most \$1 million at a 10% nominal annual interest rate. Any borrowed money must be returned at the end of the quarter. Surplus cash can earn quarterly interest at an 8% nominal annual rate. Net accumulation at the end of one quarter is invested in the next quarter.

(a) Assume that HiRise is allowed partial or full participation in the two projects. Determine the level of participation that will maximize the net cash accumulated on December 31. Solve the model using Solver or AMPL.

(b) Is it possible in any quarter to borrow money and simultaneously end up with surplus funds? Explain.

2-43. In anticipation of the immense college expenses, Joe and Jill started an annual investment program on their child's eighth birthday that will last until the eighteenth birthday. They plan to invest the following amounts at the beginning of each year:

Year	1	2	3	4	5	6	7	8	9	10
Amount (\$)	2000	2000	2500	2500	3000	3500	3500	4000	4000	5000

To avoid unpleasant surprises, they want to invest the money safely in the following options: insured savings with 7.5% annual yield, 6-year government bonds that yield 7.9% and have a current market price equal to 98% of face value, and 9-year municipal bonds yielding 8.5% and having a current market price of 1.02 of face value. How should the money be invested?

***2-44.** A business executive has the option to invest money in two plans: Plan A guarantees that each dollar invested will earn \$.70 a year later, and plan B guarantees that each dollar invested will earn \$2 after 2 years. In plan A, investments can be made annually, and in plan B, investments are allowed for periods that are multiples of 2 years only. How should the executive invest \$100,000 to maximize the earnings at the end of 3 years? Solve the model using Solver or AMPL.

2-45. A gambler plays a game that requires dividing bet money among four choices. The game has three outcomes. The following table gives the corresponding gain or loss per dollar for the different options of the game.