PEARSON NEW INTERNATIONAL EDITION

Mathematical Proofs: A Transition to Advanced Mathematics

Chartrand Polimeni Zhang

Third Edition

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Communicating Mathematics

Quite likely, the mathematics you have already encountered consists of doing problems using a specific approach or procedure. These may include solving equations in algebra, simplifying algebraic expressions, verifying trigonometric identities, using certain rules to find and simplify the derivatives of functions and setting up and evaluating a definite integral that give the area of a region or the volume of a solid. Accomplishing all of these is often a matter of practice.

Many of the methods that one uses to solve problems in mathematics are based on results in mathematics that were discovered by people and shown to be true. This kind of mathematics may very well be new to you and, as with anything that's new, there are things to be learned. But learning something new can be (in fact should be) fun. There are several steps involved here. The first step is discovering something in mathematics that we believe to be true. How does one discover new mathematics? This usually comes about by considering examples and observing that a pattern seems to be occurring with the examples. This may lead to a guess on our part as to what appears to be happening. We then have to convince ourselves that our guess is correct. In mathematics this involves constructing a proof of what we believe to be true is, in fact, true. But this is not enough. We need to convince others that we are right. So we need to write a proof that is written so clearly and so logically that people who know the methods of mathematics will be convinced. Where mathematics differs from all other scholarly fields is that once a proof has been given of a certain mathematical statement, there is no longer any doubt. This statement is true. Period. There is no other alternative.

Our main emphasis here will be in learning how to construct mathematical proofs and learning to write the proof in such a manner that it will be clear to and understood by others. Even though learning to guess new mathematics is important and can be fun, we will spend only a little time on this as it often requires an understanding of more mathematics than can be discussed at this time. But why would we want to discover new mathematics? While one possible answer is that it comes from the curiosity that most mathematicians possess, a more common explanation is that we have a problem to solve that requires knowing that some mathematical statement is true.

Learning Mathematics

One of the major goals of this text is to assist you as you progress from an individual who uses mathematics to an individual who understands mathematics. Perhaps this will mark the beginning of you becoming someone who actually develops mathematics of your own. This is an attainable goal if you have the desire.

The fact that you've gone this far in your study of mathematics suggests that you have ability in mathematics. This is a real opportunity for you. Much of the mathematics that you will encounter in the future is based on what you are about to learn here. The better you learn the material and the mathematical thought process now, the more you will understand later. To be sure, any area of study is considerably more enjoyable when you understand it. But getting to that point will require effort on your part.

There are probably as many excuses for doing poorly in mathematics as there are strategies for doing well in mathematics. We have all heard students say (sometimes, remarkably, even with pride) that they are not good at mathematics. That's only an alibi. Mathematics can be learned like any other subject. Even some students who have done well in mathematics say that they are not good with proofs. This, too, is unacceptable. What is required is determination and effort. To have done well on an exam with little or no studying is nothing to be proud of. Confidence based on being well-prepared is good, however.

Here is some advice that has worked for several students. First, it is important to understand what goes on in class each day. This means being present and being prepared for every class. After each class, recopy any lecture notes. When recopying the notes, express sentences in your own words and add details so that everything is as clear as possible. If you run into snags (and you will), talk them over with a classmate or your instructor. In fact, it's a good idea (at least in our opinion) to have someone with whom to discuss the material on a regular basis. Not only does it often clarify ideas, it gets you into the habit of using correct terminology and notation.

In addition to learning mathematics from your instructor, solidifying your understanding by careful note-taking and by talking with classmates, your text is (or at least should be) an excellent source as well. Read your text carefully with pen (or pencil) and paper in hand. Make a serious effort to do every homework problem assigned and, eventually, be certain that you know how to solve them. If there are exercises in the text that have not been assigned, you might even try to solve these as well. Another good idea is to try to create your own problems. In fact, when studying for an exam, try creating your own exam. If you start doing this for all of your classes, you might be surprised at how good you become. Creativity is a major part of mathematics. Discovering mathematics not only contributes to your understanding of the subject but has the potential to contribute to mathematics itself. Creativity can come in all forms. The following quote is due to the well-known writer J. K. Rowling (author of the *Harry Potter* novels).

> *Sometimes ideas just come to me. Other times I have to sweat and almost bleed to make ideas come. It's a mysterious process, but I hope I never find out exactly how it works.*

In the book *Defying Gravity: The Creative Career of Stephen Schwartz from Godspell to Wicked*, the author Carol de Giere writes a biography of Stephen Schwartz, one of the most successful composer-lyricists, in which she discusses not only creativity in music but how an idea can lead to something special and interesting and how creative people may have to deal with disappointment. Indeed, de Giere dedicates her book to the *creative spirit within each of us*. While he wrote the music for such famous shows as *Godspell* and *Wicked*, Schwartz discusses creativity head-on in the song "The Spark of Creation" he wrote for the musical *Children of Eden*. In her book, de Giere writes:

> *In many ways, this song expresses the theme of Stephen Schwartz's life—the naturalness and importance of the creative urge within us. At the same time he created an anthem for artists.*

In mathematics our goal is to seek the truth. Finding answers to mathematical questions is important, but we cannot be satisfied with this alone. We must be certain that we are right and that our explanation for why we believe we are correct is convincing to others. The reasoning we use as we proceed from what we know to what we wish to show must be logical. It must make sense to others, not just to ourselves.

There is joint responsibility here. As writers, it is our responsibility to give an accurate clear argument with enough details provided to allow the reader to understand what we have written and to be convinced. It is the reader's responsibility to know the basics of logic and to study the concepts involved so that a well-presented argument will be understood. Consequently, in mathematics writing is important, *very* important. Is it *really* important to write mathematics well? After all, isn't mathematics mainly equations and symbols? Not at all. It is not only important to write mathematics well, it is important to write well. You will be writing the rest of your life, at least reports, letters and e-mail. Many people who never meet you will know you only by what you write and how you write.

Mathematics is a sufficiently complicated subject that we don't need vague, hazy and boring writing to add to it. A teacher has a very positive impression of a student who hands in well-written and well-organized assignments and examinations. You want people to enjoy reading what you've written. It is important to have a good reputation as a writer. It's part of being an educated person. Especially with the large number of e-mail letters that so many of us write, it has become commonplace for writing to be more casual. Although all people would probably subscribe to this (since it is more efficient), we should know how to write well formally and professionally when the situation requires it.

You might think that considering how long you've been writing and that you're set in your ways, it will be very difficult to improve your writing. Not really. If you want to improve, you can and will. Even if you are a good writer, your writing can always be improved. Ordinarily, people don't think much about their writing. Often just thinking about your writing is the first step to writing better.

What Others Have Said about Writing

Many people who are well known in their areas of expertise have expressed their thoughts about writing. Here are quotes by some of these individuals.

Anything that helps communication is good. Anything that hurts it is bad.

I like words more than numbers, and I always did—conceptual more than computational.

Paul Halmos, mathematician

Writing is easy. All you have to do is cross out all the wrong words.

Mark Twain, author (*The Adventures of Huckleberry Finn*)

You don't write because you want to say something; you write because you've got something to say.

F. Scott Fitzgerald, author (*The Great Gatsby*)

Writing comes more easily if you have something to say.

Scholem Asch, author

Either write something worth reading or do something worth writing.

Benjamin Franklin, statesman, writer, inventor

What is written without effort is in general read without pleasure.

Samuel Johnson, writer

Easy reading is damned hard writing.

Nathaniel Hawthorne, novelist (*The Scarlet Letter*)

Everything that is written merely to please the author is worthless.

The last thing one knows when writing a book is what to put first.

I have made this letter longer because I lack the time to make it short.

Blaise Pascal, mathematician and physicist

The best way to become acquainted with a subject is to write a book about it.

Benjamin Disraeli, prime minister of England

In a very real sense, the writer writes in order to teach himself, to understand himself, to satisfy himself; the publishing of his ideas, though it brings gratification, is a curious anticlimax.

Alfred Kazin, literary critic

The skill of writing is to create a context in which other people can think.

Edwin Schlossberg, exhibit designer

A writer needs three things, experience, observation, and imagination, any two of which, at times any one of which, can supply the lack of the other.

William Faulkner, writer (*The Sound and the Fury*)

If confusion runs rampant in the passage just read,
It may very well be that too much has been said.

So that's what he meant! Then why didn't he say so?

Frank Harary, mathematician

A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street.

David Hilbert, mathematician

Everything should be made as simple as possible, but not simpler.

Albert Einstein, physicist

Never let anything you write be published without having had others critique it.

Donald E. Knuth, computer scientist and writer

Some books are to be tasted, others to be swallowed, and some few to be chewed and digested.

Reading maketh a full man, conference a ready man, and writing an exact man.

Francis Bacon, writer and philosopher

Judge an article not by the quality of what is framed and hanging on the wall, but by the quality of what's in the wastebasket.

Anonymous (Quote by Leslie Lamport)

We are all apprentices in a craft where no-one ever becomes a master.

Ernest Hemingway, author (*For Whom the Bell Tolls*)

There are three rules for writing a novel. Unfortunately, no one knows what they are.

W. Somerset Maugham, author (*Of Human Bondage*)

Mathematical Writing

Most of the quotes given above pertain to writing in general, not to mathematical writing in particular. However these suggestions for writing apply as well to writing mathematics. For us, mathematical writing means writing assignments for a mathematics course (particularly a course with proofs). Such an assignment might consist of writing a single proof, writing solutions to a number of problems or perhaps writing a term paper which, more than likely, includes definitions, examples, background *and* proofs. We'll refer to any of these as an *assignment*. Your goal should be to write correctly, clearly and in an interesting manner.

Before you even begin to write, you should have already thought about a number of things. First, you should know what examples and proofs you plan to include if this is appropriate for your assignment. You should not be overly concerned about writing good proofs on your first attempt—but be certain that you do have *proofs*.

As you're writing your assignment, you must be aware of your audience. What is the target group for your assignment? Of course, it should be written for your instructor. But it should be written so that a classmate would understand it. As you grow mathematically, your audience will grow with you and you will adapt your writing to this new audience.

Give yourself enough time to write your assignment. Don't try to put it together just a few minutes before it's due. The disappointing result will be obvious to your instructor. And to you! Find a place to write that is comfortable for you: your room, an office, a study room, the library and sitting at a desk, at a table, in a chair. Do what works best for you. Perhaps you write best when it's quiet or when there is background music.

Now that you're comfortably settled and have allowed enough time to do a good job, let's put a plan together. If the assignment is fairly lengthy, you may need an outline, which, most likely, will include one or more of the following:

1. Background and motivation
2. The definitions to be presented and possibly the notation to be used
3. The examples to include
4. The results to be presented (whose proofs have already been written, probably in rough form)
5. References to other results you intend to use
6. The order of everything mentioned above.

If the assignment is a term paper, it may include extensive background material and may need to be carefully motivated. The subject of the paper should be placed in perspective. Where does it fit in with what we already know?

Many writers write in *spirals*. Even though you have a plan for your assignment which includes an ordered list of things you want to say, it is likely that you will reach some point (perhaps sooner than you think) when you realize that you should have included something earlier—perhaps a definition, a theorem, an example, some notation. (This happened to us many times while writing this textbook.) Insert the missing material, start over again and write until once again you realize that something is missing. It is important, as you reread, that you start at the beginning each time. Then repeat the steps listed above.

We are about to give you some advice, some *pointers*, about writing mathematics. Such advice is necessarily subjective. Not everyone subscribes to these suggestions on writing. Indeed, writing *experts* don't agree on all issues. For the present, your instructor will be your best guide. But writing does not follow a list of rules. As you mature mathematically, perhaps the best advice about your writing is the same advice given by Jiminy Cricket to Disney's Pinocchio: *Always let your conscience be your guide*. You must be yourself. And one additional piece of advice: Be careful about accepting advice on writing. Originality and creativity don't follow rules. Until you reach the stage of being comfortable and confident with your own writing, however, we believe that it is useful to consider a few writing tips.

Since a number of these writing tips may not make sense (since, after all, we don't even have anything to write as yet), it will probably be most useful to return to this chapter periodically.

Using Symbols

Since mathematics is a symbol-oriented subject, mathematical writing involves a mixture of words and symbols. Here are several guidelines to which a number of mathematicians subscribe.

1. *Never start a sentence with a symbol.*
Writing mathematics follows the same practice as writing all sentences, namely that the first word should be capitalized. This is confusing if the sentence were to begin with a symbol since the sentence appears to be incomplete. Also, in general, a sentence sounds better if it starts with a word. Instead of writing:

$x^2 - 6x + 8 = 0$ has two distinct roots.

write:

The equation $x^2 - 6x + 8 = 0$ has two distinct roots.

2. *Separate symbols not in a list by words if possible.*
Separating symbols by words makes the sentence easier to read and therefore easier to understand. The sentence:

Except for a, b is the only root of $(x - a)(x - b) = 0$.

would be clearer if it were written as:

Except for a, the number b is the only root of $(x - a)(x - b) = 0$.

3. *Except when discussing logic, avoid writing the following symbols in your assignment:*

$$\Rightarrow,\ \forall,\ \exists,\ \ni,\ etc.$$

The first four symbols stand for "implies", "for every", "there exists" and "such that", respectively. You may have already seen these symbols and know what they mean. If so, this is good. It is useful when taking notes or writing early drafts of an assignment to use shorthand symbols but many mathematicians avoid such symbols in their professional writing.

4. *Be careful about using i.e. and e.g.*
These stand for *that is* and *for example,* respectively. There are situations when writing the words is preferable to writing the abbreviations as there may be confusion with nearby symbols. For example, $\sqrt{-1}$ and $\lim\limits_{n\to\infty}\left(1 + \dfrac{1}{n}\right)^n$ are not rational numbers, that is, i and e are not rational numbers.

5. *Write out integers as words when they are used as adjectives and when the numbers are relatively small or are easy to describe in words. Write out numbers numerically when they specify the value of something.*

There are exactly two groups of order 4.
Fifty million Frenchmen can't be wrong.
There are one million positive integers less than 1,000,001.

6. *Don't mix words and symbols improperly.*
Instead of writing:

Every integer ≥ 2 is a prime or is composite.

it is preferable to write:

Every integer exceeding 1 is a prime or is composite.

or

If $n \geq 2$ is an integer, then n is prime or composite.

Although

Since $(x-2)(x-3)=0$, it follows that $x=2$ or 3.

sounds correct, it is not written correctly. It should be:

Since $(x-2)(x-3)=0$, it follows that $x=2$ or $x=3$.

7. *Avoid using a symbol in the statement of a theorem when it's not needed.*
Don't write:

Theorem *Every bijective function f has an inverse.*

Delete "f". It serves no useful purpose. The theorem does not depend on what the function is called. A symbol should not be used in the statement of a theorem (or in its proof) exactly once. If it is useful to have a name for an arbitrary bijective function in the proof (as it probably will be), then "f" can be introduced there.

8. *Explain the meaning of every symbol that you introduce.*
Although what you intended may seem clear, don't assume this. For example, if you write $n = 2k + 1$ and k has never appeared before, then say that k is an integer (if indeed k *is* an integer).

9. *Use "frozen symbols" properly.*
If m and n are typically used for integers (as they probably are), then don't use them for real numbers. If A and B are used for sets, then don't use these as typical elements of a set. If f is used for a function, then don't use this as an integer. Write symbols that the reader would expect. To do otherwise could very well confuse the reader.

10. *Use consistent symbols.*
Unless there is some special reason to the contrary, use symbols that "fit" together. Otherwise, it is distracting to the reader.
Instead of writing

If x and y are even integers, then $x = 2a$
and $y = 2r$ for some integers a and r.

replace $2r$ by $2b$ (where then, of course, we write "for some integers a and b"). On the other hand, you might prefer to write $x = 2r$ and $y = 2s$.

Writing Mathematical Expressions

There will be numerous occasions when you will want to write mathematical expressions in your assignment, such as algebraic equations, inequalities, and formulas. If these expressions are relatively short, then they should probably be written within the text of the proof or discussion. (We'll explain this in a moment.) If the expressions are rather lengthy, then it is probably preferred for these expressions to be written as *displays*.

For example, suppose that we are discussing the Binomial Theorem. (It's not important if you don't recall what this theorem is.) It's possible that what we are writing includes the following passage:

> For example, if we expand $(a+b)^4$, then we obtain $(a+b)^4 = a^4 + 4a^3b +$
> $6a^2b^2 + 4ab^3 + b^4$.

It would probably be better to write the expansion of $(a+b)^4$ as a **display**, where the mathematical expression is placed on a line or lines by itself and is centered. This is illustrated below.

> For example, if we expand $(a+b)^4$, then we obtain
> $$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

If there are several mathematical expressions that are linked by equal signs and inequality symbols, then we would almost certainly write this as a display. For example, suppose that we wanted to write $n^3 + 3n^2 - n + 4$ in terms of k, where $n = 2k+1$. A possible display is given next:

> Since $n = 2k+1$, it follows that
> $$\begin{aligned} n^3 + 3n^2 - n + 4 &= (2k+1)^3 + 3(2k+1)^2 - (2k+1) + 4 \\ &= (8k^3 + 12k^2 + 6k + 1) + 3(4k^2 + 4k + 1) - 2k - 1 + 4 \\ &= 8k^3 + 24k^2 + 16k + 7 = 8k^3 + 24k^2 + 16k + 6 + 1 \\ &= 2(4k^3 + 12k^2 + 8k + 3) + 1. \end{aligned}$$

Notice how the equal signs are lined up. (We wrote two equal signs on one line since that line would have contained very little material otherwise, as well as to balance the lengths of the lines better.)

Let's return to the expression $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ for the moment. If we were to write this expression in the text of a paragraph (as we are doing) and if we find it necessary to write portions of this expression on two separate lines, then this expression should be broken so that the first line ends with an operation or comparative symbol such as $+$, $-$, $<$, $\geq$ or $=$. In other words, the second line should *not* begin with one of these symbols. The reason for doing this is that ending the line with one of these symbols alerts the reader that more will follow; otherwise, the reader might conclude (incorrectly) that the portion of the expression appearing on the first line is the entire expression. Consequently, write

> For example, if we expand $(a+b)^4$, then we obtain $(a+b)^4 = a^4 + 4a^3b +$
> $6a^2b^2 + 4ab^3 + b^4$.

and not

> For example, if we expand $(a+b)^4$, then we obtain $(a+b)^4 = a^4 + 4a^3b$
> $+ 6a^2b^2 + 4ab^3 + b^4$.

If there is an occasion to refer to an expression that has already appeared, then this expression should have been written as a display and labeled as below:

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 6ab^2 + b^4. \qquad \textbf{(1)}$$

Then we can simply refer to expression (1) rather than writing it out each time.

Common Words and Phrases in Mathematics

There are some words and phrases that appear so often in mathematical writing that it is useful to discuss them.

1. *I We One Let's*

I will now show that n is even.
We will now show that n is even.
One now shows that n is even.
Let's now show that n is even.

These are four ways that we might write a sentence in a proof. Which of these sounds the best to you? It is not considered good practice to use "I" unless you are writing a personal account of something. Otherwise, "I" sounds egotistical and can be annoying. Using "one" is often awkward. Using "we" is standard practice in mathematics. This word also brings the reader into the discussion with the author and gives the impression of a team effort. The word "let's" accomplishes this as well but is much less formal. There is a danger of being *too* casual, however. In general, your writing should be balanced, maintaining a professional style. Of course, there is the possibility of avoiding all of these words:

The integer n is now shown to be even.

2. *Clearly Obviously Of course Certainly*

These and similar words can turn a reader off if what's written is not clear to the reader. It can give the impression that the author is putting the reader down. These words should be used sparingly and with caution. If they *are* used, then at least be certain that what you say is true. There is also the possibility that the writer (a student?) has a lack of understanding of the mathematics or is not being careful and is using these words as a cover-up. This gives us even more reasons to avoid these words.

3. *Any Each Every*

This statement is true for any integer n.

Does this mean that the statement is true for *some* integer n or *all* integers n? Since the word *any* can be vague, perhaps it is best to avoid it. If by *any*, we mean *each* or *every*, then use one of these two words instead. When the word *any* is encountered, most of the time the author means *each* or *every*.

4. *Since* $\cdots$, *then* $\cdots$

A number of people connect these two words. You should use either "If $\cdots$, then $\cdots$" (should this be the intended meaning) or "Since $\cdots$, it follows that $\cdots$" or, possibly, "Since $\cdots$, we have $\cdots$". For example, it is correct to write

If n^2 is even, then n is even.

or

Since n^2 is even, it follows that n is even.

or perhaps

Since n^2 is even, n is even.

but avoid

Since n^2 is even, then n is even.

In this context, the word *since* can be replaced by *because*.

5. *Therefore Thus Hence Consequently So It follows that This implies that*

This is tricky. Mathematicians cannot survive without these words. Often within a proof, we proceed from something we've just learned to something else that can be concluded from it. There are many (many!) openings to sentences which attempt to say this. Although each of the words or phrases

Therefore Thus Hence Consequently So It follows that This implies that

is suitable, it is good to introduce some variety into your writing and not use the same words or phrases any more often than necessary.

6. *That Which*

These words are often confused with each other. Sometimes they are interchangeable; more often they are not.

The solution to the equation is the number less than 5 that is positive. **(2)**

The solution to the equation is the number less than 5 which is positive. **(3)**

Which of these two sentences is correct? The simple answer is: Both are correct—or, at least, both might be correct.

For example, sentence (2) could be the response to the question: Which of the numbers 2, 3, and 5 is the solution of the equation? Sentence (3) could be the response to the question: Which of the numbers 4.9 and 5.0 is the solution of the equation?

The word *that* introduces a *restrictive clause* and, as such, the clause is essential to the meaning of the sentence. That is, if sentence (2) were written only as "The solution to the equation is the number less than 5" then the entire meaning is changed. Indeed, we no longer know what the solution of the equation is.

On the other hand, the word *which* does *not* introduce a restrictive clause. It introduces a nonrestrictive (or parenthetical) clause. A *nonrestrictive clause* only provides additional information that is not essential to the meaning of the sentence. In sentence (3) the phrase "which is positive" simply provides more information about the solution. This clause may have been added because the solution to an earlier equation is negative. In fact, it would be more appropriate to add a comma:

The solution to the equation is the number less than 5, which is positive.

For another illustration, consider the following two statements:

I always keep the math text that I like with me. **(4)**

I always keep the math text which I like with me. **(5)**

What is the difference between these two sentences? In (4), the writer of the sentence clearly has more than one math text and is referring to the one that he/she likes. In (5), the writer has only one math text and is providing the added information that he/she likes it. The nonrestrictive clause in (5) should be set off by commas:

I always keep the math text, which I like, with me.

A possible guideline to follow as you seek to determine whether *that* or *which* is the proper word to use is to ask yourself: Does it sound right if it reads "which, by the way"? In general, *that* is normally used considerably more often than *which*. Hence the advice here is: Beware of wicked which's!

While we are discussing the word *that*, we mention that the words *assume* and *suppose* often precede restrictive clauses and, as such, the word *that* should immediately follow one of these words. Omitting *that* leaves us with an implied *that*. Many mathematicians prefer to include it rather than omit it.

In other words, instead of writing:

Assume N is a normal subgroup.

many would write

Assume that N is a normal subgroup.

Some Closing Comments about Writing

1. Use good English. Write in complete sentences, ending each sentence with a period (or a question mark when appropriate) and capitalize the first word of each sentence. (Remember: No sentence begins with a symbol!)
2. Capitalize theorem and lemma as in Theorem 1 and Lemma 4.
3. Many mathematicians do not hyphenate words containing the prefix *non*, such as

   nonempty, nonnegative, nondecreasing, nonzero.
4. Many words that occur often in mathematical writing are commonly misspelled. Among these are:

   commutative (independent of order)
   complement (supplement, balance, remainder)
   consistent (conforming, agreeing)

feasible (suitable, attainable)
its (possessive, not "it is")
occurrence (incident)
parallel (non-intersecting)
preceding (foregoing, former)
principle (postulate, regulation, rule)
proceed (continue, move on)

and, of course,

corollary, lemma, theorem.

5. There are many pairs of words that fit together in mathematics (while interchanging words among the pairs do not). For example,

We ask questions.
We pose problems.
We present solutions.
We prove theorems.
We solve problems.
and
We conclude this chapter.

Logic

In mathematics our goal is to seek the truth. Are there connections between two given mathematical concepts? If so, what are they? Under what conditions does an object possess a particular property? Finding answers to questions such as these is important, but we cannot be satisfied only with this. We must be certain that we are right and that our explanation for why we believe we are correct is convincing to others. The reasoning we use as we proceed from what we know to what we wish to show must be logical. It must make sense to others, not just to ourselves.

There is joint responsibility here, however. It is the writer's responsibility to use the rules of logic to give a valid and clear argument with enough details provided to allow the reader to understand what we have written and to be convinced. It is the reader's responsibility to know the basics of logic and to study the concepts involved sufficiently well so that he or she will not only be able to understand a well-presented argument but can decide as well whether it is valid. Consequently, both writer and reader must be familiar with logic.

Although it is possible to spend a great deal of time studying logic, we will present only what we actually need and will instead use the majority of our time putting what we learn into practice.

1 Statements

In mathematics we are constantly dealing with statements. By a **statement** we mean a declarative sentence or assertion that is true or false (but not both). Statements therefore declare or assert the truth of something. Of course, the statements in which we will be primarily interested deal with mathematics. For example, the sentences

The integer 3 is odd.
The integer 57 is prime.

are statements (only the first of which is true).

Every statement has a **truth value**, namely **true** (denoted by T) or **false** (denoted by F). We often use P, Q and R to denote statements, or perhaps $P_1, P_2, \ldots, P_n$ if there are several statements involved. We have seen that

$$P_1 : \text{The integer 3 is odd.}$$

and

$$P_2 : \text{The integer 57 is prime.}$$

are statements, where P_1 has truth value T and P_2 has truth value F.

Sentences that are imperative (commands) such as

Substitute the number 2 for x.
Find the derivative of $f(x) = e^{-x} \cos 2x$.

or are interrogative (questions) such as

Are these sets disjoint?
What is the derivative of $f(x) = e^{-x} \cos 2x$?

or are exclamatory such as

What an interesting question!
How difficult this problem is!

are not statements since these sentences are not declarative.

It may not be immediately clear whether a statement is true or false. For example, the sentence "The 100th digit in the decimal expansion of π is 7." is a statement, but it may be necessary to find this information in a Web site on the Internet to determine whether this statement is true. Indeed, for a sentence to be a statement, it is not a requirement that we be able to determine its truth value.

The sentence "The real number r is rational." is a statement *provided* we know what real number r is being referred to. Without this additional information, however, it is impossible to assign a truth value to it. This is an example of what is often referred to as an open sentence. In general, an **open sentence** is a declarative sentence that contains one or more variables, each variable representing a value in some prescribed set, called the **domain** of the variable, and which becomes a statement when values from their respective domains are substituted for these variables. For example, the open sentence "$3x = 12$" where the domain of x is the set of integers is a true statement only when $x = 4$.

An open sentence that contains a variable x is typically represented by $P(x)$, $Q(x)$ or $R(x)$. If $P(x)$ is an open sentence, where the domain of x is S, then we say $P(x)$ is an **open sentence over the domain** S. Also, $P(x)$ is a statement for each $x \in S$. For example, the open sentence

$$P(x): \ (x-3)^2 \le 1$$

over the domain $\mathbf{Z}$ is a true statement when $x \in \{2, 3, 4\}$ and is a false statement otherwise.

Example 1 *For the open sentence*

$$P(x, y): \ |x+1| + |y| = 1$$

P
T
F

Q
T
F

P	Q
T	T
T	F
F	T
F	F

P	Q	R
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Figure 1 Truth tables for one, two and three statements

in two variables, suppose that the domain of the variable x *is* $S = \{-2, -1, 0, 1\}$ *and the domain of the variable* y *is* $T = \{-1, 0, 1\}$. *Then*

$$P(-1, 1): \ |(-1) + 1| + |1| = 1$$

is a true statement, while

$$P(1, -1): \ |1 + 1| + |-1| = 1$$

is a false statement. In fact, $P(x, y)$ *is a true statement when*

$$(x, y) \in \{(-2, 0), (-1, -1), (-1, 1), (0, 0)\},$$

while $P(x, y)$ *is a false statement for all other elements* $(x, y) \in S \times T$. ♦

The possible truth values of a statement are often listed in a table, called a **truth table.** The truth tables for two statements P and Q are given in Figure 1. Since there are two possible truth values for each of P and Q, there are four possible combinations of truth values for P and Q. The truth table showing all these combinations is also given in Figure 1. If a third statement R is involved, then there are eight possible combinations of truth values for P, Q and R. This is displayed in Figure 1 as well. In general, a truth table involving n statements $P_1, P_2, \cdots, P_n$ contains 2^n possible combinations of truth values for these statements and a truth table showing these combinations would have n columns and 2^n rows. Much of the time, we will be dealing with two statements, usually denoted by P and Q; so the associated truth table will have four rows with the first two columns headed by P and Q. In this case, it is customary to consider the four combinations of the truth values in the order TT, TF, FT, FF, from top to bottom.

2 The Negation of a Statement

Much of the interest in integers and other familiar sets of numbers comes not only from the numbers themselves but from properties of the numbers that result by performing operations on them (such as taking their negatives, adding or multiplying them or combinations of these). Similarly, much of our interest in statements comes from investigating the truth or falseness of new statements that can be produced from one or more given statements by performing certain operations on them. Our first example concerns producing a new statement from a single given statement.

The **negation** of a statement P is the statement:

not P.

and is denoted by $\sim P$. Although $\sim P$ could always be expressed as

It is not the case that P**.**

there are usually better ways to express the statement $\sim P$.

Example 2 *For the statement*

P_1 : *The integer* 3 *is odd.*

described above, we have

$\sim P_1$: *It is not the case that the integer* 3 *is odd.*

but it would be much preferred to write

$\sim P_1$: *The integer* 3 *is not odd.*

or better yet to write

$\sim P_1$: *The integer* 3 *is even.*

Similarly, the negation of the statement

P_2 : *The integer* 57 *is prime.*

considered above is

$\sim P_2$: *The integer* 57 *is not prime.*

Note that $\sim P_1$ *is false, while* $\sim P_2$ *is true.* ♦

Indeed, the negation of a true statement is always false and the negation of a false statement is always true; that is, the truth value of $\sim P$ is opposite to that of P. This is summarized in Figure 2, which gives the truth table for $\sim P$ (in terms of the possible truth values of P).

P	$\sim P$
T	F
F	T

Figure 2 The truth table for negation

3 The Disjunction and Conjunction of Statements

For two given statements P and Q, a common way to produce a new statement from them is by inserting the word "or" or "and" between P and Q. The **disjunction** of the statements P and Q is the statement

$$P \textbf{ or } Q$$

and is denoted by $P \vee Q$. The disjunction $P \vee Q$ is true if at least one of P and Q is true; otherwise, $P \vee Q$ is false. Therefore, $P \vee Q$ is true if exactly one of P and Q is true or if both P and Q are true.

Example 3 *For the statements*

$$P_1 : \textit{The integer } 3 \textit{ is odd. and } P_2 : \textit{The integer } 57 \textit{ is prime.}$$

described earlier, the disjunction is the new statement

$$P_1 \vee P_2\text{: } \textit{Either } 3 \textit{ is odd or } 57 \textit{ is prime.}$$

which is true since at least one of P_1 and P_2 is true (namely, P_1 is true). Of course, in this case exactly one of P_1 and P_2 is true. ♦

For two statements P and Q, the truth table for $P \vee Q$ is shown in Figure 3. This truth table then describes precisely when $P \vee Q$ is true (or false).

Although the truth of "P or Q" allows for both P and Q to be true, there are instances when the use of "or" does not allow that possibility. For example, for an integer n, if we were to say "n is even or n is odd," then surely it is not possible for both "n is even" and "n is odd" to be true. When "or" is used in this manner, it is called the **exclusive or**. Suppose, for example, that $\mathcal{P} = \{S_1, S_2, \ldots, S_k\}$, where $k \geq 2$, is a partition of a set S and x is some element of S. If

$$x \in S_1 \text{ or } x \in S_2$$

is true, then it is impossible for both $x \in S_1$ and $x \in S_2$ to be true.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Figure 3 The truth table for disjunction

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Figure 4 The truth table for conjunction

The **conjunction** of the statements P and Q is the statement:

P **and** Q

and is denoted by $P \wedge Q$. The conjunction $P \wedge Q$ is true only when both P and Q are true; otherwise, $P \wedge Q$ is false.

Example 4 *For P_1 : The integer 3 is odd. and P_2 : The integer 57 is prime., the statement*

$P_1 \wedge P_2$: 3 *is odd and* 57 *is prime.*

is false since P_2 is false and so not both P_1 and P_2 are true. ♦

The truth table for the conjunction of two statements is shown in Figure 4.

4 The Implication

A statement formed from two given statements in which we will be most interested is the implication (also called the conditional). For statements P and Q, the **implication** (or **conditional**) is the statement

If P, then Q.

and is denoted by $P \Rightarrow Q$. In addition to the wording "If P, then Q," we also express $P \Rightarrow Q$ in words as

P **implies** Q**.**

The truth table for $P \Rightarrow Q$ is given in Figure 5.

Notice that $P \Rightarrow Q$ is false only when P is true and Q is false ($P \Rightarrow Q$ is true otherwise).

Example 5 *For P_1 : The integer 3 is odd. and P_2 : The integer 57 is prime., the implication*

$P_1 \Rightarrow P_2$: *If* 3 *is an odd integer, then* 57 *is prime.*

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Figure 5 The truth table for implication

is a false statement. The implication

$$P_2 \Rightarrow P_1 : \textit{If } 57 \textit{ is prime, then } 3 \textit{ is odd.}$$

is true, however. ♦

While the truth tables for the negation $\sim P$, the disjunction $P \vee Q$ and the conjunction $P \wedge Q$ are probably what one would expect, this may not be so for the implication $P \Rightarrow Q$. There is ample justification, however, for the truth values in the truth table of $P \Rightarrow Q$. We illustrate this with an example.

Example 6 *A student is taking a math class (let's say this one) and is currently receiving a* $B+$*. He visits his instructor a few days before the final examination and asks her, "Is there any chance that I can get an A in this course?" His instructor looks through her grade book and says, "If you earn an A on the final exam, then you will receive an A for your final grade." We now check the truth or falseness of this implication based on the various combinations of truth values of the statements*

$$P : \textit{You earn an A on the final exam.}$$

and

$$Q : \textit{You receive an A for your final grade.}$$

which make up the implication.

Analysis Suppose first that P and Q are both true. That is, the student receives an A on his final exam and later learns that he got an A for his final grade in the course. Did his instructor tell the truth? I think we would all agree that she did. So if P and Q are both true, then so too is $P \Rightarrow Q$, which agrees with the first row of the truth table of Figure 5.

Second, suppose that P is true and Q is false. So the student got an A on his final exam but did not receive an A as a final grade, say he received a B. Certainly, his instructor did not do as she promised (as she will soon be reminded by her student). What she said was false, which agrees with the second row of the table in Figure 5.

Third, suppose that P is false and Q is true. In this case, the student did not get an A on his final exam (say he earned a B) *but* when he received his final grades, he learned (and was pleasantly surprised) that his final grade was an A. How could this happen? Perhaps his instructor was lenient. Perhaps the final exam was unusually difficult and a grade of B on it indicated an exceptionally good performance. Perhaps the instructor made a mistake. In any case, the instructor did not lie; so she told the truth. Indeed, she never promised anything if the student did not get an A on his final exam. This agrees with the third row of the table in Figure 5.

Finally, suppose that P and Q are both false. That is, suppose the student did not get an A on his final exam and he also did not get an A for a final grade. The instructor did not lie here either. She only promised the student an A *if* he got an A on the final exam. Once again, she did not promise anything if the student did not get an A on the final exam. So the instructor told the truth and this agrees with the fourth and final row of the table. ♦

In summary then, the only situation for which $P \Rightarrow Q$ is false is when P is true and Q is false (so $\sim Q$ is true). That is, the truth tables for

$$\sim(P \Rightarrow Q) \text{ and } P \wedge (\sim Q)$$

are the same. We'll revisit this observation again soon.

We have already mentioned that the implication $P \Rightarrow Q$ can be expressed as both "If P, then Q" and "P implies Q." In fact, there are several ways of expressing $P \Rightarrow Q$ in words, namely:

If P, then Q.
Q if P.
P implies Q.
P only if Q.
P is sufficient for Q.
Q is necessary for P.

It is probably not surprising that the first three of these say the same thing, but perhaps not at all obvious that the last three say the same thing as the first three. Consider the statement "P only if Q." This says that P is true only under the condition that Q is true; in other words, it cannot be the case that P is true and Q is false. Thus it says that if P is true, then necessarily Q must be true. We can also see from this that the statement "Q is necessary for P" has the same meaning as "P only if Q." The statement "P is sufficient for Q" states that the truth of P is sufficient for the truth of Q. In other words, the truth of P implies the truth of Q; that is, "P implies Q."

5 More on Implications

We have just discussed four ways to create new statements from one or two given statements. In mathematics, however, we are often interested in declarative sentences containing variables and whose truth or falseness is only known once we have assigned values to the variables. The values assigned to the variables come from their respective domains. These sentences are, of course, precisely the sentences we have referred to as open sentences. Just as new statements can be formed from statements P and Q by negation, disjunction, conjunction or implication, new open sentences can be constructed from open sentences in the same manner.

Example 7 *Consider the open sentences*

$$P_1(x) : x = -3. \textit{ and } P_2(x) : |x| = 3,$$

where $x \in \mathbf{R}$, that is, where the domain of x is $\mathbf{R}$ in each case. We can then form the following open sentences:

$$\sim P_1(x) : \ x \neq -3.$$
$$P_1(x) \vee P_2(x) : \ x = -3 \textit{ or } |x| = 3.$$
$$P_1(x) \wedge P_2(x) : \ x = -3 \textit{ and } |x| = 3.$$
$$P_1(x) \Rightarrow P_2(x) : \ \textit{If } x = -3, \textit{ then } |x| = 3.$$

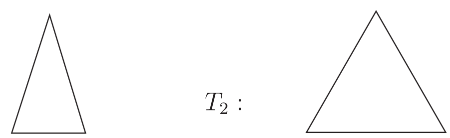

Figure 6 Isosceles and equilateral triangles

For a specific real number x, the truth value of each resulting statement can be determined. For example, $\sim P_1(-3)$ is a false statement, while each of the remaining sentences above results in a true statement when $x = -3$. Both $P_1(2) \vee P_2(2)$ and $P_1(2) \wedge P_2(2)$ are false statements. On the other hand, both $\sim P_1(2)$ and $P_1(2) \Rightarrow P_2(2)$ are true statements. In fact, for each real number $x \neq -3$, the implication $P_1(x) \Rightarrow P_2(x)$ is a true statement since $P_1(x) : x = -3$ is a false statement. Thus $P_1(x) \Rightarrow P_2(x)$ is true for all $x \in \mathbf{R}$. We will see that open sentences which result in true statements for all values of the domain will be especially interesting to us.

Listed below are various ways of wording the implication $P_1(x) \Rightarrow P_2(x)$:

If $x = -3$, then $|x| = 3$.
$|x| = 3$ if $x = -3$.
$x = -3$ implies that $|x| = 3$.
$x = -3$ only if $|x| = 3$.
$x = -3$ is sufficient for $|x| = 3$.
$|x| = 3$ is necessary for $x = -3$. ♦

We now consider another example, this time from geometry. You may recall that a triangle is called **equilateral** if the lengths of its three sides are the same, while a triangle is **isosceles** if the lengths of any two of its three sides are the same. Figure 6 shows an isosceles triangle T_1 and an equilateral triangle T_2. Actually, since the lengths of any two of the three sides of T_2 are the same, T_2 is isosceles as well. Indeed, this is precisely the fact we wish to discuss.

Example 8 *For a triangle T, let*

$$P(T) : T \text{ is equilateral. and } Q(T) : T \text{ is isosceles.}$$

Thus, $P(T)$ and $Q(T)$ are open sentences over the domain S of all triangles. Consider the implication $P(T) \Rightarrow Q(T)$, where the domain then of the variable T is the set S. For an equilateral triangle T_1, both $P(T_1)$ and $Q(T_1)$ are true statements and so $P(T_1) \Rightarrow Q(T_1)$ is a true statement as well. If T_2 is not an equilateral triangle, then $P(T_2)$ is a false statement and so $P(T_2) \Rightarrow Q(T_2)$ is true. Therefore, $P(T) \Rightarrow Q(T)$ is a true statement for all $T \in S$. We now state $P(T) \Rightarrow Q(T)$ in a variety of ways:

If T is an equilateral triangle, then T is isosceles.
A triangle T is isosceles if T is equilateral.
A triangle T being equilateral implies that T is isosceles.
A triangle T is equilateral only if T is isosceles.
For a triangle T to be isosceles, it is sufficient that T be equilateral.
For a triangle T to be equilateral, it is necessary that T be isosceles. ♦

Notice that at times we change the wording to make the sentence sound better. In general, the sentence P in the implication $P \Rightarrow Q$ is commonly referred to as the **hypothesis** or **premise** of $P \Rightarrow Q$, while Q is called the **conclusion** of $P \Rightarrow Q$.

It is often easier to deal with an implication when expressed in an "if, then" form. This allows us to identify the hypothesis and conclusion more easily. Indeed, since implications can be stated in a wide variety of ways (even in addition to those mentioned above), being able to reword an implication as "if, then" is especially useful. For example, the implication $P(T) \Rightarrow Q(T)$ described in Example 8 can be encountered in many ways, including the following:

- Let T be an equilateral triangle. Then T is isosceles.
- Suppose that T is an equilateral triangle. Then T is isosceles.
- Every equilateral triangle is isosceles.
- Whenever a triangle is equilateral, it is isosceles.

We now investigate the truth or falseness of implications involving open sentences for values of their variables.

Example 9 *Let $S = \{2, 3, 5\}$ and let*

$$P(n):\ n^2 - n + 1 \text{ is prime. and } Q(n):\ n^3 - n + 1 \text{ is prime.}$$

be open sentences over the domain S. Determine the truth or falseness of the implication $P(n) \Rightarrow Q(n)$ for each $n \in S$.

Solution In this case, we have the following:

$$\begin{array}{lll} P(2):\ 3 \text{ is prime.} & P(3):\ 7 \text{ is prime.} & P(5):\ 21 \text{ is prime.} \\ Q(2):\ 7 \text{ is prime.} & Q(3):\ 25 \text{ is prime.} & Q(5):\ 121 \text{ is prime.} \end{array}$$

Consequently, $P(2) \Rightarrow Q(2)$ and $P(5) \Rightarrow Q(5)$ are true, while $P(3) \Rightarrow Q(3)$ is false. ♦

Example 10 *Let $S = \{1, 2\}$ and let $T = \{-1, 4\}$. Also, let*

$$P(x, y):\ ||x + y| - |x - y|| = 2. \text{ and } Q(x, y):\ x^{y+1} = y^x.$$

be open sentences, where the domain of the variable x is S and the domain of y is T. Determine the truth or falseness of the implication $P(x, y) \Rightarrow Q(x, y)$ for all $(x, y) \in S \times T$.

Solution For $(x, y) = (1, -1)$, we have

$$P(1, -1) \Rightarrow Q(1, -1):\ \text{If } 2 = 2, \text{ then } 1 = -1.$$

which is false. For $(x, y) = (1, 4)$, we have

$$P(1, 4) \Rightarrow Q(1, 4):\ \text{If } 2 = 2, \text{ then } 1 = 4.$$

which is also false. For $(x, y) = (2, -1)$, we have

$$P(2, -1) \Rightarrow Q(2, -1):\ \text{If } 2 = 2, \text{ then } 1 = 1.$$

which is true; while for $(x, y) = (2, 4)$, we have

$$P(2, 4) \Rightarrow Q(2, 4):\ \text{If } 2 = 4, \text{ then } 32 = 16.$$

which is true. ♦

6 The Biconditional

For statements (or open sentences) P and Q, the implication $Q \Rightarrow P$ is called the **converse** of $P \Rightarrow Q$. The converse of an implication will often be of interest to us, either by itself or in conjunction with the original implication.

Example 11 *For the statements*

$$P_1: \text{ 3 } \textit{is an odd integer.} \qquad P_2: \text{ 57 } \textit{is prime.}$$

the converse of the implication

$$P_1 \Rightarrow P_2: \textit{ If } 3 \textit{ is an odd integer, then } 57 \textit{ is prime.}$$

is the implication

$$P_2 \Rightarrow P_1: \textit{ If } 57 \textit{ is prime, then } 3 \textit{ is an odd integer.}$$ ♦

For statements (or open sentences) P and Q, the conjunction

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

of the implication $P \Rightarrow Q$ and its converse is called the **biconditional** of P and Q and is denoted by $P \Leftrightarrow Q$. For statements P and Q, the truth table for $P \Leftrightarrow Q$ can therefore be determined. This is given in Figure 7. From this table, we see that $P \Leftrightarrow Q$ is true whenever the statements P and Q are both true or are both false, while $P \Leftrightarrow Q$ is false otherwise. That is, $P \Leftrightarrow Q$ is true precisely when P and Q have the same truth values.

The biconditional $P \Leftrightarrow Q$ is often stated as

P is equivalent to Q.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T	T	***T***
T	F	F	T	***F***
F	T	T	F	***F***
F	F	T	T	***T***

P	$\star$	$Q P \Leftrightarrow Q$
T	T	***T***
T	F	***F***
F	T	***F***
F	F	***T***

Figure 7 The truth table for a biconditional

or

$$P \textbf{ if and only if } Q.$$

or as

$$P \textbf{ is a necessary and sufficient condition for } Q.$$

For statements P and Q, it then follows that the biconditional "P if and only if Q" is true only when P and Q have the same truth values.

Example 12 *The biconditional*

$$3 \textit{ is an odd integer if and only if } 57 \textit{ is prime.}$$

is false; while the biconditional

$$100 \textit{ is even if and only if } 101 \textit{ is prime.}$$

is true. Furthermore, the biconditional

$$5 \textit{ is even if and only if } 4 \textit{ is odd.}$$

is also true. ♦

The phrase "if and only if" occurs often in mathematics and we shall discuss this at greater length later. For the present, we consider two examples involving statements containing the phrase "if and only if."

Example 13 *We noted in Example 7 that for the open sentences*

$$P_1(x) : x = -3. \textit{ and } P_2(x) : |x| = 3.$$

over the domain $\mathbf{R}$*, the implication*

$$P_1(x) \Rightarrow P_2(x) : \textit{If } x = -3, \textit{ then } |x| = 3.$$

is a true statement for each $x \in \mathbf{R}$*. However, the converse*

$$P_2(x) \Rightarrow P_1(x) : \textit{If } |x| = 3, \textit{ then } x = -3.$$

is a false statement when $x = 3$ *since* $P_2(3)$ *is true and* $P_1(3)$ *is false. For all other real numbers* x*, the implication* $P_2(x) \Rightarrow P_1(x)$ *is true. Therefore, the biconditional*

$$P_1(x) \Leftrightarrow P_2(x) : \ x = -3 \textit{ if and only if } |x| = 3.$$

is false when $x = 3$ *and is true for all other real numbers* x*.* ♦

Example 14 *For the open sentences*

$$P(T) : \ T \textit{ is equilateral. and } Q(T) : \ T \textit{ is isosceles.}$$

over the domain S of all triangles, the converse of the implication

$$P(T) \Rightarrow Q(T): \textit{If } T \textit{ is equilateral, then } T \textit{ is isosceles.}$$

is the implication

$$Q(T) \Rightarrow P(T): \textit{If } T \textit{ is isosceles, then } T \textit{ is equilateral.}$$

We noted that $P(T) \Rightarrow Q(T)$ is a true statement for all triangles T, while $Q(T) \Rightarrow P(T)$ is a false statement when T is an isosceles triangle that is not equilateral. On the other hand, the second implication becomes a true statement for all other triangles T. Therefore, the biconditional

$$P(T) \Leftrightarrow Q(T): T \textit{ is equilateral if and only if } T \textit{ is isosceles.}$$

is false for all triangles that are isosceles and not equilateral, while it is true for all other triangles T. ♦

We now investigate the truth or falseness of biconditionals obtained by assigning to a variable each value in its domain.

Example 15 *Let $S = \{0, 1, 4\}$. Consider the following open sentences over the domain S:*

$$P(n): \frac{n(n+1)(2n+1)}{6} \textit{ is odd.}$$
$$Q(n): (n+1)^3 = n^3 + 1.$$

Determine three distinct elements a, b, c in S such that $P(a) \Rightarrow Q(a)$ is false, $Q(b) \Rightarrow P(b)$ is false, and $P(c) \Leftrightarrow Q(c)$ is true.

Solution Observe that

$$P(0): \text{ 0 is odd.} \quad P(1): \text{ 1 is odd.} \quad P(4): \text{ 30 is odd.}$$

$$Q(0): 1 = 1. \quad Q(1): 8 = 2. \quad Q(4): 125 = 65.$$

Thus $P(0)$ and $P(4)$ are false, while $P(1)$ is true. Also, $Q(1)$ and $Q(4)$ are false, while $Q(0)$ is true. Thus $P(1) \Rightarrow Q(1)$ and $Q(0) \Rightarrow P(0)$ are false, while $P(4) \Leftrightarrow Q(4)$ is true. Hence we may take $a = 1$, $b = 0$ and $c = 4$. ♦

Analysis Notice in Example 15 that both $P(0) \Leftrightarrow Q(0)$ and $P(1) \Leftrightarrow Q(1)$ are false biconditionals. Hence the value 4 in S is the only choice for c. ♦

7 Tautologies and Contradictions

The symbols $\sim$, $\vee$, $\wedge$, $\Rightarrow$ and $\Leftrightarrow$ are sometimes referred to as **logical connectives**. From given statements, we can use these logical connectives to form more intricate statements. For example, the statement $(P \vee Q) \wedge (P \vee R)$ is a statement formed from the given statements P, Q and R and the logical connectives $\vee$ and $\wedge$. We call $(P \vee Q) \wedge (P \vee R)$ a compound statement. More generally, a **compound statement** is a statement composed of one or more given statements (called **component statements** in this context) and at least one logical connective. For example, for a given component statement P, its negation $\sim P$ is a compound statement.

The compound statement $P \vee (\sim P)$, whose truth table is given in Figure 8, has the feature that it is true regardless of the truth value of P.

A compound statement S is called a **tautology** if it is true for all possible combinations of truth values of the component statements that comprise S. Hence $P \vee (\sim P)$ is a tautology, as is $(\sim Q) \vee (P \Rightarrow Q)$. This latter fact is verified in the truth table shown in Figure 9.

Letting

$$P_1 : \text{3 is odd. and } P_2 : \text{57 is prime.}$$

we see that not only is

57 is not prime, or 57 is prime if 3 is odd.

a true statement, but $(\sim P_2) \vee (P_1 \Rightarrow P_2)$ is true regardless of which statements P_1 and P_2 are being considered.

On the other hand, a compound statement S is called a **contradiction** if it is false for all possible combinations of truth values of the component statements that are used to form S. The statement $P \wedge (\sim P)$ is a contradiction, as is shown in Figure 10. Hence the statement

3 is odd and 3 is not odd.

is false.

Another example of a contradiction is $(P \wedge Q) \wedge (Q \Rightarrow (\sim P))$, which is verified in the truth table shown in Figure 11.

Indeed, if a compound statement S is a tautology, then its negation $\sim S$ is a contradiction.

P	$\sim P$	$P \vee (\sim P)$
T	F	$\boldsymbol{T}$
F	T	$\boldsymbol{T}$

Figure 8 An example of a tautology

P	Q	$\sim Q$	$P \Rightarrow Q$	$(\sim Q) \vee (P \Rightarrow Q)$
T	T	F	T	$\boldsymbol{T}$
T	F	T	F	$\boldsymbol{T}$
F	T	F	T	$\boldsymbol{T}$
F	F	T	T	$\boldsymbol{T}$

Figure 9 Another tautology

P	$\sim P$	$P \wedge (\sim P)$
T	F	***F***
F	T	***F***

Figure 10 An example of a contradiction

P	Q	$\sim P$	$P \wedge Q$	$Q \Rightarrow (\sim P)$	$(P \wedge Q) \wedge (Q \Rightarrow (\sim P))$
T	T	F	T	F	***F***
T	F	F	F	T	***F***
F	T	T	F	T	***F***
F	F	T	F	T	***F***

Figure 11 Another contradiction

8 Logical Equivalence

Figure 12 shows a truth table for the two statements $P \Rightarrow Q$ and $(\sim P) \vee Q$. The corresponding columns of these compound statements are identical; in other words, these two compound statements have exactly the same truth value for every combination of truth values of the statements P and Q. Let R and S be two compound statements involving the same component statements. Then R and S are called **logically equivalent** if R and S have the same truth values for all combinations of truth values of their component statements. If R and S are logically equivalent, then this is denoted by $R \equiv S$. Hence $P \Rightarrow Q$ and $(\sim P) \vee Q$ are logically equivalent and so $P \Rightarrow Q \equiv (\sim P) \vee Q$.

Another, even simpler, example of logical equivalence concerns $P \wedge Q$ and $Q \wedge P$. That $P \wedge Q \equiv Q \wedge P$ is verified in the truth table shown in Figure 13.

What is the practical significance of logical equivalence? Suppose that R and S are logically equivalent compound statements. Then we know that R and S have the same truth values for all possible combinations of truth values of their component statements. But this means that the biconditional $R \Leftrightarrow S$ is true for all possible combinations of truth values of their component statements and hence $R \Leftrightarrow S$ is a tautology. Conversely, if $R \Leftrightarrow S$ is a tautology, then R and S are logically equivalent.

Let R be a mathematical statement that we would like to show is true and suppose that R and some statement S are logically equivalent. If we can show that S is true, then R is true as well. For example, suppose that we want to verify the truth of an

P	Q	$\sim P$	$P \Rightarrow Q$	$(\sim P) \vee Q$
T	T	F	$\boldsymbol{T}$	$\boldsymbol{T}$
T	F	F	$\boldsymbol{F}$	$\boldsymbol{F}$
F	T	T	$\boldsymbol{T}$	$\boldsymbol{T}$
F	F	T	$\boldsymbol{T}$	$\boldsymbol{T}$

Figure 12 Verification of $P \Rightarrow Q \equiv (\sim P) \vee Q$

P	Q	$P \wedge Q$	$Q \wedge P$
T	T	***T***	***T***
T	F	***F***	***F***
F	T	***F***	***F***
F	F	***F***	***F***

Figure 13 Verification of $P \wedge Q \equiv Q \wedge P$

implication $P \Rightarrow Q$. If we can establish the truth of the statement $(\sim P) \vee Q$, then the logical equivalence of $P \Rightarrow Q$ and $(\sim P) \vee Q$ guarantees that $P \Rightarrow Q$ is true as well.

Example 16 *Returning to the mathematics instructor in Example 6 and whether she kept her promise that*

If you earn an A on the final exam, then you will receive an A for the final grade.

we need only know that the student did not receive an A on the final exam or the student received an A as a final grade to see that she kept her promise. ♦

Since the logical equivalence of $P \Rightarrow Q$ and $(\sim P) \vee Q$, verified in Figure 12, is especially important and we will have occasion to use this fact often, we state it as a theorem.

Theorem 17 *Let P and Q be two statements. Then*

$$P \Rightarrow Q \text{ and } (\sim P) \vee Q$$

are logically equivalent.

Let's return to the truth table in Figure 13, where we showed that $P \wedge Q$ and $Q \wedge P$ are logically equivalent for any two statements P and Q. In particular, this says that

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P) \text{ and } (Q \Rightarrow P) \wedge (P \Rightarrow Q)$$

are logically equivalent. Of course, $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ is precisely what is called the biconditional of P and Q. Since $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ and $(Q \Rightarrow P) \wedge (P \Rightarrow Q)$ are logically equivalent, $(Q \Rightarrow P) \wedge (P \Rightarrow Q)$ represents the biconditional of P and Q as well. Since $Q \Rightarrow P$ can be written as "P if Q" and $P \Rightarrow Q$ can be expressed as "P only if Q," their conjunction can be written as "P if Q and P only if Q" or, more simply, as

$$P \text{ if and only if } Q.$$

Consequently, expressing $P \Leftrightarrow Q$ as "P if and only if Q" is justified. Furthermore, since $Q \Rightarrow P$ can be phrased as "P is necessary for Q" and $P \Rightarrow Q$ can be expressed as "P is sufficient for Q," writing $P \Leftrightarrow Q$ as "P is necessary and sufficient for Q" is likewise justified.

9 Some Fundamental Properties of Logical Equivalence

It probably comes as no surprise that the statements P and $\sim(\sim P)$ are logically equivalent. This fact is verified in Figure 14.

We mentioned in Figure 13 that, for two statements P and Q, the statements $P \wedge Q$ and $Q \wedge P$ are logically equivalent. There are other fundamental logical equivalences that we often encounter as well.

Theorem 18 *For statements P, Q and R,*

(1) *Commutative Laws*
  (a) $P \vee Q \equiv Q \vee P$
  (b) $P \wedge Q \equiv Q \wedge P$
(2) *Associative Laws*
  (a) $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$
  (b) $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$
(3) *Distributive Laws*
  (a) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
  (b) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
(4) *De Morgan's Laws*
  (a) $\sim (P \vee Q) \equiv (\sim P) \wedge (\sim Q)$
  (b) $\sim (P \wedge Q) \equiv (\sim P) \vee (\sim Q)$.

Each part of Theorem 18 is verified by means of a truth table. We have already established the commutative law for conjunction (namely $P \wedge Q \equiv Q \wedge P$) in Figure 13. In Figure 15 $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ is verified by observing that the columns corresponding to the statements $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge (P \vee R)$ are identical.

The laws given in Theorem 18, together with other known logical equivalences, can be used to good advantage at times to prove other logical equivalences (without introducing a truth table).

Example 19 *Suppose we are asked to verify that*

$$\sim (P \Rightarrow Q) \equiv P \wedge (\sim Q)$$

for every two statements P and Q. Using the logical equivalence of $P \Rightarrow Q$ and $(\sim P) \vee Q$ from Theorem 17 and Theorem 18(4a), we see that

$$\sim (P \Rightarrow Q) \equiv \sim((\sim P) \vee Q) \equiv (\sim(\sim P)) \wedge (\sim Q) \equiv P \wedge (\sim Q), \qquad \textbf{(1)}$$

P	$\sim P$	$\sim(\sim P)$
T	F	T
F	T	F

Figure 14 Verification of $P \equiv \sim(\sim P)$

P	Q	R	$Q \wedge R$	$P \vee Q$	$P \vee R$	$P \vee (Q \wedge R)$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	T	T	$\boldsymbol{T}$	$\boldsymbol{T}$
T	T	F	F	T	T	$\boldsymbol{T}$	$\boldsymbol{T}$
T	F	T	F	T	T	$\boldsymbol{T}$	$\boldsymbol{T}$
T	F	F	F	T	T	$\boldsymbol{T}$	$\boldsymbol{T}$
F	T	T	T	T	T	$\boldsymbol{T}$	$\boldsymbol{T}$
F	T	F	F	T	F	$\boldsymbol{F}$	$\boldsymbol{F}$
F	F	T	F	F	T	$\boldsymbol{F}$	$\boldsymbol{F}$
F	F	F	F	F	F	$\boldsymbol{F}$	$\boldsymbol{F}$

Figure 15 Verification of the distributive law $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

implying that the statements $\sim (P \Rightarrow Q)$ *and* $P \wedge (\sim Q)$ *are logically equivalent, which we alluded to earlier.* ♦

It is important to keep in mind what we have said about logical equivalence. For example, the logical equivalence of $P \wedge Q$ and $Q \wedge P$ allows us to replace a statement of the type $P \wedge Q$ by $Q \wedge P$ without changing its truth value. As an additional example, according to De Morgan's Laws in Theorem 18, if it is not the case that an integer a is even or an integer b is even, then it follows that a and b are both odd.

Example 20 *Using the second of De Morgan's Laws and statement (1), we can establish a useful logically equivalent form of the negation of* $P \Leftrightarrow Q$ *by the following string of logical equivalences:*

$$\begin{aligned} \sim(P \Leftrightarrow Q) &\equiv \sim((P \Rightarrow Q) \wedge (Q \Rightarrow P)) \\ &\equiv (\sim(P \Rightarrow Q)) \vee (\sim(Q \Rightarrow P)) \\ &\equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P)). \end{aligned}$$ ♦

What we have observed about the negation of an implication and a biconditional is repeated in the following theorem.

Theorem 21 *For statements* P *and* Q*,*

(*a*) $\sim (P \Rightarrow Q) \equiv P \wedge (\sim Q)$
(*b*) $\sim (P \Leftrightarrow Q) \equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P))$.

Example 22 *Once again, let's return to what the mathematics instructor in Example 6 said:*

> *If you earn an A on the final exam, then you will receive an A for your final grade.*

If this instructor was not truthful, then it follows by Theorem 21(a) that

> *You earned an A on the final exam and did not receive A as your final grade.*

Suppose, on the other hand, that the mathematics instructor had said:

> *If you earn an A on the final exam, then you will receive an A for the final grade—and that's the only way that you will get an A for a final grade.*

If this instructor was not truthful, then it follows by Theorem 21(b) that

> *Either you earned an A on the final exam and didn't receive A as your final grade or you received an A for your final grade and you didn't get an A on the final exam.* ♦

10 Quantified Statements

We have mentioned that if $P(x)$ is an open sentence over a domain S, then $P(x)$ is a statement for each $x \in S$. We illustrate this again.

Example 23 *If* $S = \{1, 2, \cdots, 7\}$*, then*

$$P(n): \ \frac{2n^2 + 5 + (-1)^n}{2} \text{ is prime.}$$

is a statement for each $n \in S$*. Therefore,*

$$\begin{aligned} P(1)&:\ 3 \text{ is prime.}\\ P(2)&:\ 7 \text{ is prime.}\\ P(3)&:\ 11 \text{ is prime.}\\ P(4)&:\ 19 \text{ is prime.}\end{aligned}$$

are true statements; while

$$\begin{aligned} P(5)&:\ 27 \text{ is prime.}\\ P(6)&:\ 39 \text{ is prime.}\\ P(7)&:\ 51 \text{ is prime.}\end{aligned}$$

are false statements. ♦

There are other ways that an open sentence can be converted into a statement, namely by a method called **quantification**. Let $P(x)$ be an open sentence over a domain S. Adding the phrase "For every $x \in S$" to $P(x)$ produces a statement called a **quantified statement**. The phrase "for every" is referred to as the **universal quantifier** and is denoted by the symbol ∀. Other ways to express the universal quantifier are "for each" and "for all." This quantified statement is expressed in symbols by

$$\forall x \in S,\ P(x) \tag{2}$$

and is expressed in words by

$$\text{For every } x \in S,\ P(x). \tag{3}$$

The quantified statement (2) (or (3)) is true if $P(x)$ is true for every $x \in S$, while the quantified statement (2) is false if $P(x)$ is false for at least one element $x \in S$.

Another way to convert an open sentence $P(x)$ over a domain S into a statement through quantification is by the introduction of a quantifier called an existential quantifier. Each of the phrases *there exists*, *there is*, *for some* and *for at least one* is referred to as an **existential quantifier** and is denoted by the symbol $\exists$. The quantified statement

$$\exists x \in S,\ P(x) \tag{4}$$

can be expressed in words by

$$\text{There exists } x \in S \text{ such that } P(x). \tag{5}$$

The quantified statement (4) (or (5)) is true if $P(x)$ is true for at least one element $x \in S$, while the quantified statement (4) is false if $P(x)$ is false for all $x \in S$.

We now consider two quantified statements constructed from the open sentence we saw in Example 23.

Example 24 *For the open sentence*

$$P(n): \ \frac{2n^2+5+(-1)^n}{2} \textit{ is prime.}$$

over the domain $S = \{1, 2, \cdots, 7\}$*, the quantified statement*

$$\forall n \in S,\ P(n): \ \textit{For every } n \in S,\ \frac{2n^2+5+(-1)^n}{2} \ \textit{ is prime.}$$

is false since $P(5)$ *is false, for example; while the quantified statement*

$$\exists n \in S,\ P(n): \ \textit{There exists } n \in S \textit{ such that } \frac{2n^2+5+(-1)^n}{2} \ \textit{ is prime.}$$

is true since $P(1)$ *is true, for example.* ♦

The quantified statement $\forall x \in S,\ P(x)$ can also be expressed as

If $x \in S$, then $P(x)$.

Consider the open sentence $P(x):\ x^2 \geq 0$. over the set $\mathbf{R}$ of real numbers. Then

$$\forall x \in \mathbf{R},\ P(x)$$

or, equivalently,

$$\forall x \in \mathbf{R},\ x^2 \geq 0$$

can be expressed as

For every real number x, $x^2 \geq 0$.

or

If x is a real number, then $x^2 \geq 0$.

as well as

The square of every real number is nonnegative.

In general, the universal quantifier is used to claim that the statement resulting from a given open sentence is true when each value of the domain of the variable is assigned to the variable. Consequently, the statement $\forall x \in \mathbf{R}, x^2 \geq 0$ is true since $x^2 \geq 0$ is true for every real number x.

Suppose now that we were to consider the open sentence $Q(x) : x^2 \leq 0$. The statement $\forall x \in \mathbf{R}, Q(x)$ (that is, for every real number x, we have $x^2 \leq 0$) is false since, for example, $Q(1)$ is false. Of course, this means that its negation is true. If it were not the case that for every real number x, we have $x^2 \leq 0$, then there must exist some real number x such that $x^2 > 0$. This negation

There exists a real number x such that $x^2 > 0$.

can be written in symbols as

$$\exists x \in \mathbf{R}, x^2 > 0 \;\text{ or }\; \exists x \in \mathbf{R}, \sim Q(x).$$

More generally, if we are considering an open sentence $P(x)$ over a domain S, then

$$\sim (\forall x \in S, P(x)) \equiv \exists x \in S, \sim P(x).$$

Example 25 *Suppose that we are considering the set $A = \{1, 2, 3\}$ and its power set $\mathcal{P}(A)$, the set of all subsets of A. Then the quantified statement*

For every set $B \in \mathcal{P}(A)$, $A - B \neq \emptyset$. **(6)**

is false since for the subset $B = A = \{1, 2, 3\}$, we have $A - B = \emptyset$. The negation of the statement (6) is

There exists $B \in \mathcal{P}(A)$ such that $A - B = \emptyset$. **(7)**

The statement (7) is therefore true since for $B = A \in \mathcal{P}(A)$, we have $A - B = \emptyset$. The statement (6) can also be written as

If $B \subseteq A$, then $A - B \neq \emptyset$. **(8)**

Consequently, the negation of (8) can be expressed as

There exists some subset B of A such that $A - B = \emptyset$. ♦

The existential quantifier is used to claim that at least one statement resulting from a given open sentence is true when the values of a variable are assigned from its domain. We know that for an open sentence $P(x)$ over a domain S, the quantified statement $\exists x \in S, P(x)$ is true provided $P(x)$ is a true statement for at least one element $x \in S$. Thus the statement $\exists x \in \mathbf{R}, x^2 > 0$ is true since, for example, $x^2 > 0$ is true for $x = 1$.

The quantified statement

$$\exists x \in \mathbf{R},\; 3x = 12$$

is therefore true since there is some real number x for which $3x = 12$, namely $x = 4$ has this property. (Indeed, $x = 4$ is the *only* real number for which $3x = 12$.) On the other hand, the quantified statement

$$\exists n \in \mathbf{Z},\; 4n - 1 = 0$$

is false as there is no integer n for which $4n - 1 = 0$. (Of course, $4n - 1 = 0$ when $n = 1/4$, but $1/4$ is not an integer.)

Suppose that $Q(x)$ is an open sentence over a domain S. If the statement $\exists x \in S$, $Q(x)$ is *not* true, then it must be the case that for every $x \in S$, $Q(x)$ is false. That is,

$$\sim (\exists x \in S,\, Q(x)) \equiv \forall x \in S, {\sim} Q(x)$$

is true. We illustrate this with a specific example.

Example 26 *The following statement contains the existential quantifier:*

$$\textit{There exists a real number } x \textit{ such that } x^2 = 3. \tag{9}$$

If we let $P(x) : x^2 = 3$*, then (9) can be rewritten as* $\exists x \in \mathbf{R}, P(x)$*. The statement (9) is true since* $P(x)$ *is true when* $x = \sqrt{3}$ *(or when* $x = -\sqrt{3}$*). Hence the negation of (9) is:*

$$\textit{For every real number } x,\ x^2 \neq 3. \tag{10}$$

The statement (10) is therefore false. ♦

Let $P(x, y)$ be an open sentence, where the domain of the variable x is S and the domain of the variable y is T. Then the quantified statement

For all $x \in S$ and $y \in T$, $P(x, y)$.

can be expressed symbolically as

$$\forall x \in S, \forall y \in T,\ P(x, y). \tag{11}$$

The negation of the statement (11) is

$$\begin{aligned} \sim (\forall x \in S, \forall y \in T,\ P(x, y)) &\equiv \exists x \in S,\ \sim (\forall y \in T,\ P(x, y)) \\ &\equiv \exists x \in S, \exists y \in T,\ {\sim} P(x, y). \end{aligned} \tag{12}$$

We now consider examples of quantified statements involving two variables.

Example 27 *Consider the statement*

$$\textit{For every two real numbers } x \textit{ and } y,\ x^2 + y^2 \geq 0. \tag{13}$$

If we let

$$P(x, y) :\ x^2 + y^2 \geq 0$$

where the domain of both x *and* y *is* $\mathbf{R}$*, then statement (13) can be expressed as*

$$\forall x \in \mathbf{R}, \forall y \in \mathbf{R},\ P(x, y) \tag{14}$$

or as

$$\forall x, y \in \mathbf{R},\ P(x, y).$$

Since $x^2 \geq 0$ *and* $y^2 \geq 0$ *for all real numbers* x *and* y*, it follows that* $x^2 + y^2 \geq 0$ *and so* $P(x, y)$ *is true for all real numbers* x *and* y*. Thus the quantified statement (14) is true.*

The negation of statement (14) is therefore

$$\sim (\forall x \in \mathbf{R}, \forall y \in \mathbf{R},\ P(x, y)) \equiv \exists x \in \mathbf{R}, \exists y \in \mathbf{R},\ \sim P(x, y) \equiv \exists x, y \in \mathbf{R}, \sim P(x, y), \tag{15}$$

which, in words, is

$$\text{There exist real numbers } x \text{ and } y \text{ such that } x^2 + y^2 < 0. \tag{16}$$

The statement (16) is therefore false. ♦

For an open sentence containing two variables, the domains of the variables need not be the same.

Example 28 *Consider the statement*

$$\text{For every } s \in S \text{ and } t \in T,\ st + 2 \text{ is a prime.} \tag{17}$$

where the domain of the variable s is $S = \{1, 3, 5\}$ and the domain of the variable t is $T = \{3, 9\}$. If we let

$$Q(s, t):\ st + 2 \text{ is a prime.}$$

then the statement (17) can be expressed as

$$\forall s \in S, \forall t \in T,\ Q(s, t). \tag{18}$$

Since all of the statements

$Q(1, 3)$: $1 \cdot 3 + 2$ *is a prime.* $Q(3, 3)$: $3 \cdot 3 + 2$ *is a prime.*
$Q(5, 3)$: $5 \cdot 3 + 2$ *is a prime.*

$Q(1, 9)$: $1 \cdot 9 + 2$ *is a prime.* $Q(3, 9)$: $3 \cdot 9 + 2$ *is a prime.*
$Q(5, 9)$: $5 \cdot 9 + 2$ *is a prime.*

are true, the quantified statement (18) is true.

As we saw in (12), the negation of the quantified statement (18) is

$$\sim(\forall s \in S, \forall t \in T,\ Q(s, t)) \equiv \exists s \in S, \exists t \in T,\ \sim Q(s, t)$$

and so the negation of (17) is

$$\text{There exist } s \in S \text{ and } t \in T \text{ such that } st + 2 \text{ is not a prime.} \tag{19}$$

The statement (19) is therefore false. ♦

Again, let $P(x, y)$ be an open sentence, where the domain of the variable x is S and the domain of the variable y is T. The quantified statement

There exist $x \in S$ and $y \in T$ such that $P(x, y)$

can be expressed in symbols as

$$\exists x \in S, \exists y \in T,\ P(x, y). \tag{20}$$

The negation of the statement (20) is then

$$\begin{aligned}\sim(\exists x \in S, \exists y \in T,\ P(x, y)) &\equiv \forall x \in S,\ \sim(\exists y \in T,\ P(x, y))\\ &\equiv \forall x \in S, \forall y \in T,\ \sim P(x, y).\end{aligned} \qquad \textbf{(21)}$$

We now illustrate this situation.

Example 29 *Consider the open sentence*

$$R(s, t):\ |s-1| + |t-2| \leq 2.$$

where the domain of the variable s is the set S of even integers and the domain of the variable t is the set T of odd integers. Then the quantified statement

$$\exists s \in S, \exists t \in T, R(s, t). \qquad \textbf{(22)}$$

can be expressed in words as

There exist an even integer s and an odd integer t such that $|s-1| + |t-2| \leq 2$. **(23)**

Since $R(2, 3):\ 1 + 1 \leq 2$ is true, the quantified statement (23) is true.

The negation of (22) is therefore

$$\sim(\exists s \in S, \exists t \in T,\ R(s, t)) \equiv \forall s \in S, \forall t \in T,\ \sim R(s, t). \qquad \textbf{(24)}$$

and so the negation of (22), in words, is

For every even integer s and every odd integer t, $|s-1| + |t-2| > 2$. **(25)**

The quantified statement (25) *is therefore false.* ♦

In the next two examples of negations of quantified statements, De Morgan's laws are also used.

Example 30 *The negation of*

For all integers a and b, if ab is even, then a is even and b is even.

is

There exist integers a and b such that ab is even and a or b is odd. ♦

Example 31 *The negation of*

There exists a rational number r such that $r \in A = \{\sqrt{2}, \pi\}$ or $r \in B = \{-\sqrt{2}, \sqrt{3}, e\}$.

is

For every rational number r, both $r \notin A$ and $r \notin B$. ♦

Quantified statements may contain both universal and existential quantifiers. Some examples are presented here.