PEARSON NEW INTERNATIONAL EDITION

Teaching Secondary and Middle School Mathematics

Daniel J. Brahier

Fourth Edition

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Mathematics as a Process

For a project in a Year I Integrated Mathematics class, Mr. James asked his students to think of an authentic example of a linear function that they have encountered in their lives. He asked his students to describe the function in words, to determine the independent and dependent variables, to generate a table of values, to write an equation based on the table, and to create a graph of the function. Finally, students were asked to present their functions to the class and were graded on their written papers and on the quality of their presentations. Mr. James listened to the presentations and read the papers enthusiastically because he was able to see his students applying their understanding of functions to their lives. However, the projects submitted by Francis and Joyce were different from those of the others in the class, and Mr. James immediately faced a decision about how to handle their examples. Francis's function was worded as follows:

> *The monthly fee for phone calls at my house is $10, which includes the first four outgoing calls. After that, every additional outgoing call costs another 50¢. The total cost for a month is a function of the number of outgoing calls.*

Joyce's function problem was the following:

> *My family belongs to a fitness club. The club has a flat rate of $55.00 a month, but in order to reserve a racquetball court you have to pay an additional $8.00 per hour. If you want a court for a fraction of an hour, it always rounds up to the nearest hour (1.25 or 1.50 hours = 2 hours = $16). The total cost of membership for a month is a function of the number of hours of court time reserved.*

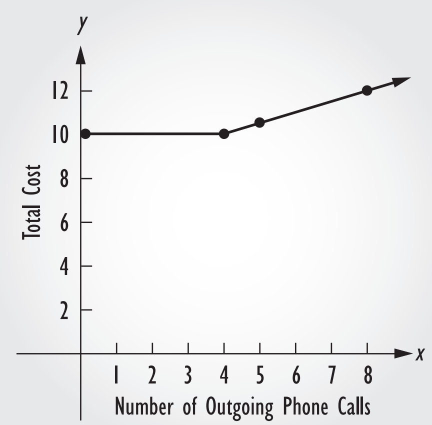
After reading this chapter, you should be able to answer the following questions:

- What do the results of national and international mathematics examinations tell us about current practices in mathematics education?
- What are the five process skills and the eight mathematical practices often associated with doing mathematics? How are they developed in the secondary and middle school mathematics programs?
- What should be the role of problem solving in the mathematics classroom?
- Can you list and illustrate several problem-solving strategies that can be promoted in the secondary and middle school mathematics classroom?
- What does it mean to "do," to "teach," and to "learn" mathematics?

When Mr. James read these papers, he recognized a "teachable moment"—an opportunity to use the two examples generated by the students as tools to get the class to consider functions that appeared to be simple and linear on the surface but were actually much more complex. So, the following day, he handed out a sheet with the two problems retyped and asked teams of four students to carefully draw a graph of the functions and to think of other functions they have encountered that had the same characteristics or behavior.

On careful inspection, students realized that the shape of Francis's graph depended on whether the number of outgoing calls was less than or greater than four. They modeled the problem by drawing the graph in Figure 1.

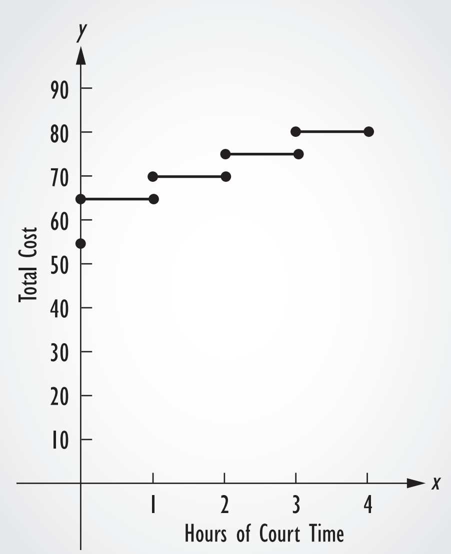
Figure 1 Graph of Francis's Function of Phone Service Costs



As students in the class presented their solutions, they shared similar examples, such as the cost of an on-line Internet service provider with a monthly access fee that includes a certain number of on-line hours, coupled with per-hour line charges after the number of free hours is exceeded. Mr. James recognized that, technically, the graph of Francis's function was not continuous; therefore, the individual points should not be connected to form a segment and a ray. He mentioned this point in passing but decided to save the discussion of "continuous" versus "discrete" for another day. More important, the students had discovered their first piecewise function with the rule:

$$f(x) = \begin{cases} 10 \text{ if } x \leq 4 \\ 0.5x + 10 \text{ if } x > 4 \end{cases}$$

Figure 2 Graph of Joyce's Function of Fitness Club Costs



While exploring Joyce's function, students noticed that the monthly cost would be the same whether, for example, they reserved 2.25, 2.50, 2.75, or 3.0 hours of court time in a month. After some class discussion, they came up with the graph presented in Figure 2.

Toni then raised her hand and said, "That's not exactly right. If you rent the court for one hour, you pay a total of $63, but the graph has a point above both $63 and $71. Only one of those points can actually be there, or it doesn't make any sense." Mr. James validated Toni's statement by showing the class how an "open point" can be placed at the left end of each segment to avoid the confusion. He went on to define Joyce's example as a *step function* because of its unique nature. Meanwhile, the class offered additional examples of step functions, such as postage cost that remains the same until a weight limit is reached, and then the price "steps up" to the next level.

Mr. James's classroom is not unusual; almost every day students raise important issues and ask questions that a teacher can use as a springboard for further discussion. In this sense, the students have the potential to steer the class and not simply be passive "sponges" that attempt to absorb mathematical content. Mr. James values the exploration of student ideas, and entered the teaching profession not only because he enjoyed mathematics but also because he was excited about having the chance to work with adolescents at a critical time in their development as students and young adults.

There are various reasons why people choose careers in mathematics education. In many cases, they experienced effective teaching in their own school endeavors and so they want to pass that same level of enthusiasm on to the next generation. Others were simply good at mathematics in school and, as a result of their interest in the subject area, decided to try teaching young adolescents. Still others have had unfortunate experiences with teachers of mathematics in school and want to try to improve the situation for future students. It is important for educators to reflect on the reasons for making a career choice and to discern whether their primary interest was mathematics, working with young students, or a combination of both. Although there is no formula for being an effective mathematics teacher, successful teaching requires a caring individual who is interested in both the field of mathematics and the development of students. This chapter introduces the discussion of mathematics teaching as a profession by examining trends in mathematics education over time and by evaluating various national and international assessments of student achievement.

National and International Assessment Data

In 1995, the most comprehensive international comparison of mathematics education in history was conducted. The Third International Mathematics and Science Study (TIMSS) report compared achievement, curriculum, and teaching practices in more than 50 countries around the world at the fourth, eighth, and twelfth grade levels. One of the questions[†] asked of seventh and eighth grade students was

If $3(x + 5) = 30$, then $x =$

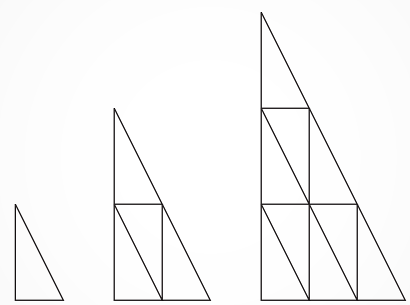
A. 2
B. 5
C. 10
D. 95

This equation, most would agree, should be fairly simple for a 13-year-old to answer. In fact, by placing a thumb over the $x + 5$ expression, even a fourth or fifth grader should be able to reason that 3 must be multiplied by 10 to get a result of 30. So, in the parentheses, x would have to be equal to 5. However, in the United States, only 63 percent of seventh graders and less than 75 percent of eighth graders were able to answer this question correctly. In Japan and Korea, more than 90 percent of the eighth graders obtained a correct answer.

[†]The International Association for the Evaluational Achievement (IEA) granted permission to reproduce exemplary items from A. E. Beaton et al. (1996), *Mathematics Achievement in the Middle School Years* (Chestnut Hill, MA: Center for the Study of Testing, Evaluation, and Educational Policy). Reprinted by permission.

On another item, students were given the sequence of triangles shown in Figure 3:

Figure 3 Sequence of Triangles in TIMSS Test Item

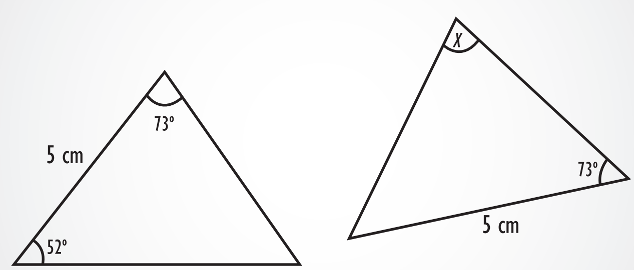


(Beaton et al., 1996. Reprinted with permission.)

The problem stated: "The sequence of similar triangles is extended to the eighth figure. How many small triangles would be needed for Figure 8?" An examination of the sequence reveals that the number of triangles required is always equal to the square of the figure number. Therefore, the eighth figure should need 64 (or 8^2) triangles. On this item, only 18 percent of the seventh graders and 25 percent of the eighth graders in the United States were able to give the correct answer. In Japan, the results were 43 percent and 52 percent, respectively.

On a geometry item for seventh and eighth graders, students were shown the diagram in Figure 4:

Figure 4 Congruent Triangles in a TIMSS Test Item



(Beaton et al., 1996. Reprinted with permission.)

The question read as follows: "These triangles are congruent. The measures of some of the sides and angles of the triangles are shown. What is the value of x?"

A. 52
B. 55
C. 65
D. 73
E. 75

Fifteen percent of the seventh graders and 17 percent of the eighth graders in the United States answered this item correctly with choice B (55°). In Japan, 40 percent of the seventh graders and 69 percent of the eighth graders found the correct answer. In fact, students in 25 out of 26 countries outscored the United States on this geometry question that involves fairly typical middle school mathematical content.

The results of the 1996 TIMSS achievement test report not only placed eighth graders in the United States well below the international average, but also showed that the U.S. middle school and secondary curricula were less rigorous than those in most other countries, with an overemphasis on number skills and a deficiency in algebra and geometry. Reports from the study described the mathematics curriculum in the United States as “an inch deep and a mile wide,” meaning that U.S. schools tend to address a great deal of content at a surface level that does not promote understanding of the underlying mathematics.

In 1999, the Third International Mathematics and Science Study–Repeat (TIMSS–R) was conducted. The major purpose of this study was to examine the performance of eighth graders on achievement tests 4 years after many of the same students were tested in the original TIMSS. Approximately one-third of the achievement test items used on the first TIMSS examination were repeated on the TIMSS–R. A total of 38 nations participated in the TIMSS–R study, 23 of which had also been involved in the first TIMSS.

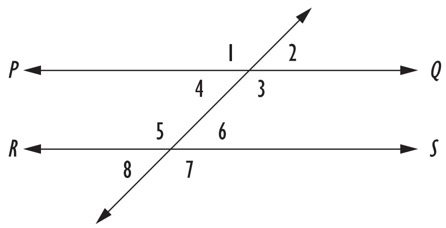
The TIMSS–R achievement test placed U.S. eighth graders slightly above the average. While 27 of 40 nations (68 percent) outscored the United States at the eighth grade level in 1995, only 18 of 37 nations (49 percent) outscored the United States in 1999 (Gonzales et al., 2000). This result appears promising, except that one must also consider that when those same U.S. students were in the fourth grade (in 1995), they were outperformed by only 11 of 25 nations (44 percent). Statistically, the average score of eighth graders in 1999 was about the same as the scores of eighth graders in 1995. However, the performance levels of eighth graders in 1999, relative to the same nations tested in 1995 when they were fourth graders, declined. In other words, U.S. students’ achievement levels dropped on an international scale as they progressed from fourth to eighth grade between 1995 and 1999. Also, the TIMSS–R results showed that students in the United States continued to have their greatest difficulty in the areas of geometry and measurement.

As the TIMSS–R data were being collected, a parallel Video Study was conducted (Hiebert et al., 2003). Eighth grade mathematics teachers from seven different countries were videotaped, and their lessons were analyzed. The results of this study showed that typical teaching strategies used in all of the countries included both small- and large-group work, review of previously studied content, and the use of textbooks and worksheets. However, major differences were identified when teachers’ presentations of new content, difficulty level of mathematics problems posed by teachers, and teachers’ handling of classwork and homework were considered. For example, whereas Japanese teachers spent approximately 60 percent of their class time introducing new mathematics content, only 23 percent of class time in the United States was used for this purpose, with more than half of the class time in the United States being spent on reviewing content. Similarly, when the complexity of problems posed to students was analyzed, only 17 percent of the Japanese problems were classified as “low-level” complexity, compared with 67 percent of the problems posed in the United States. Also, researchers noted that Japanese teachers were much more likely to make connections between a problem presented and another problem already explored than were teachers in any of the other six nations. Generally speaking, the authors of the report noted that teachers from nations with high-achieving students tend to use different teaching strategies than their counterparts.

By 2003, the project was renamed the Trends in International Mathematics and Science Study, maintaining the same acronym of TIMSS. A four-year cycle was established so that assessments could be compared from 1995 to 1999, 2003, 2007, 2011, and so on. In each instance, questionnaires were administered to students and teachers, and achievement tests were taken by students in the fourth, eighth, and/or twelfth grades. On the 2003 achievement test, 46 countries were involved at both the fourth and eighth grade levels (National Center for Education Statistics, 2005a). An example of a typical eighth grade geometry item on the 2003 test is shown in Figure 5.

Figure 5 Eighth Grade TIMSS Item from 2003 Assessment

In this figure *PQ* and *RS* are parallel.



Of the following, which pair of angles has the sum of 180°?

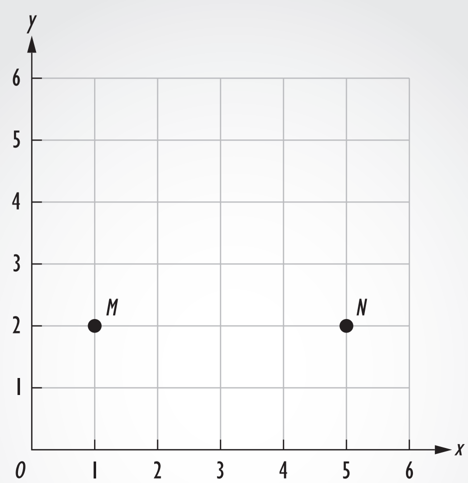
(A) ∠5 and ∠7
(B) ∠3 and ∠6
(C) ∠1 and ∠5
(D) ∠1 and ∠7
(E) ∠2 and ∠8

(National Center for Education Statistics, 2005b.)

On this item, the international average showed that 43 percent of the eighth graders answered the question correctly (choice B, which are same-side interior angles). In the United States, only 37 percent of the students got the item correct (significantly lower than the international average), whereas 83 percent of Japanese students answered it correctly. On the test as a whole, however, both fourth and eighth graders in the United States scored above the international average. The fourth graders performed lower than 11 countries but outperformed 13 of their peers. At the eighth grade level, students in the United States were outperformed by 9 countries but scored higher than 25 others (National Center for Education Statistics, 2005a).

On the 2007 TIMSS exam, eighth graders in the United States achieved a scaled score of 508, slightly above the international average of 500. Eighth graders in the United States outperformed 37 of 48 participating countries, with 5 Asian countries scoring significantly higher (Chinese Taipei, Korea, Singapore, Hong Kong, and Japan), and 5 countries scoring about the same (Hungary, England, Russia, Lithuania, and the Czech Republic) (International Association for the Evaluation of Educational Achievement, 2007). Interestingly, eighth graders in the United States in 2007 scored 16 scaled points higher than in 1995 when the average score was 492 (Gonzales, 2009). Still, on some items, students in the United States scored much lower than we might want or expect. Figure 6 illustrates a geometry item from the 2007 exam. On this question, only 45 percent of eighth graders in the United States correctly responded with choice A

Figure 6 Eighth Grade TIMSS Item from 2007 Assessment



Two points *M* and *N* are shown in the figure above. John is looking for a point *P* such that *MNP* is an isosceles triangle. Which of these points could be point *P*?

A) (3,5)
B) (3,2)
C) (1,5)
D) (5,1)

(Gonzales et al., 2009)

(showing their ability to create an isosceles triangle), while 81 percent of Japanese and 86 percent of Chinese students answered correctly.

Also, beginning in 2000, the Organization for Economic Co-operation and Development (OECD) began administering an international exam in mathematics, science, and reading literacy to 15-year-olds. The test is on a 3-year cycle, having also been given in 2003, 2006, 2009, and so on. The OECD is a collaborative of 34 countries whose main goal is to "foster prosperity and fight poverty through growth and financial stability" (OECD, 2011a). The test is called the Program for International Student Assessment (PISA). On the 2009 exam, students in the United States scored 487, whereas the international average was 496 (OECD, 2011b)—a performance that was statistically lower than that of 17 other countries and higher than that of only 5 other OECD countries (Greece, Israel, Turkey, Chile, and Mexico). While scores were higher in 2009 than in 2006, they were not significantly higher than in 2003 (Fleischman et al., 2010). So once again, in yet another assessment, we see the students in the United States scoring at or below the international average.

So, what does all of this international information tell us? In comparing performances over time, fourth graders in the United States scored higher in 2007 than in 1995 (fourth graders were not tested in 1999). Also, in 2007, there was a marked improvement at the eighth grade level over 1995, with the first test scoring 8 points below the scaled international average and the 2007 test being 8 points above the average (Gonzales et al., 2009). Twelfth graders were not tested on TIMSS after 1995 (National Center for Education Statistics, 2005a). Therefore, on an international scale, at middle school level (grades 4 and 8), we can detect an increase in mathematical content knowledge of students in the United States when compared with students in the rest of the world. Whether this trend will continue remains to be seen.

Results similar to those on TIMSS can be found when looking at reports from the National Assessment of Educational Progress (NAEP)—often called the Nation's Report Card—in the United States. Consider, for example, the following item that was featured on the 1996 NAEP for high school seniors (National Center for Education Statistics, 2007a):

> *This question requires you to show your work and explain your reasoning. You may use drawings, words, and numbers in your explanation. Your answer should be clear enough so that another person could read it and understand your thinking. It is important that you show all of your work.*
>
> *The table below shows the daily attendance at two movie theaters for 5 days and the mean (average) and the median attendance.*

	Theater A	*Theater B*
Day 1	*100*	*72*
Day 2	*87*	*97*
Day 3	*90*	*70*
Day 4	*10*	*71*
Day 5	*91*	*100*
Mean (Average)	*75.6*	*82*
Median	*90*	*72*

> *(a) Which statistic, the mean or the median, would you use to describe the typical daily attendance for the 5 days at Theater A? Justify your answer.*
>
> *(b) Which statistic, the mean or the median, would you use to describe the typical daily attendance for the 5 days at Theater B? Justify your answer.*

This item is particularly interesting because it does not ask students to compute the mean or the median but, instead, to consider the appropriate use of a statistic in a given situation. Twelfth graders are supposed to notice that the outlier of 10 people on Day 4 results in a mean that is lower than any of the other pieces of data, making the median the best measure of central tendency for Theater A. Likewise, the median for Theater B is an unrealistic "average," as it represents only one of the two clusters of data collected. At this point, you might want to ask yourself, "What percentage of high school seniors should be able to recognize the appropriate use of mean versus that of median?" Most people would agree that the vast majority of twelfth graders should be able to do that. After all, the concepts of mean and median are often introduced as early as the fourth or fifth grade.

This statistics item was scored on a **rubric**, a grading scale on which a student's response can fall into one of five categories—incorrect, minimal, partial, satisfactory, or extended. Papers were sorted into one of these five categories, depending on the correctness of the responses and the clarity of explanations. A satisfactory or an extended response is considered an acceptable answer. Only 4 percent of the twelfth graders tested in this national sample were able to perform at the top two levels. In fact, 56 percent of the students either left the item blank or wrote totally incorrect responses on their papers. It is possible that these students knew *how to calculate* a mean or a median but did not have the conceptual understanding of these statistics to know when to apply them. Moreover, although 27 percent of the white students left the item blank, 42 percent of African American and 48 percent of Hispanic students did not respond.

Likewise, results from NAEP tests between 2000 and 2009 showed that at all of the grade levels tested (grades 4, 8, and 12 in 2000, 2005, and 2009; grades 4 and 8 in 2003, 2005, and 2007), white students scored significantly higher than did African American, Hispanic, or Native American students (Braswell et al., 2001; Braswell, Daane, & Grigg, 2003; National Center for Education Statistics, 2007a, 2009, 2011). In addition, students at all grade levels who were at or near the poverty line in terms of family income tended to have, on average, lower scores than did their peers who came from wealthier families. Consequently, we perceive a wide gap between the performances of various socioeconomic groups—a trend that has been consistent throughout the NAEP reports. Such data prompted the National Council of Teachers of Mathematics (NCTM) to assert, "Expectations must be raised—mathematics can and must be learned by *all* students" (NCTM, 2000, p. 13). The optimistic news on the NAEP mathematics test is that between 1990 and 2009, the percentage of students who scored at or above the "basic" level increased from 50 to 82 percent for fourth graders and from 52 to 73 percent for eighth graders. Similarly, performance at or above the "proficient" level increased at both of these grade levels over the 19-year period as well (National Center for Education Statistics, 2007a, 2009). The percent of students at or above basic and proficient levels on the twelfth grade NAEP has shown an increase as well (National Center for Education Statistics, 2011). These results indicate a steady trend toward improvement in mathematics performance over time.

International assessment items explored in this chapter (involving a basic algebraic equation, a visual pattern, an angle measurement, a coordinate geometry question for eighth graders, and a straightforward statistics question for twelfth graders) should be fairly easy for most students to answer. Yet these items were missed by a large percentage of U.S. students who took these national and international exams. Why? What is it about the system that has made mathematics so inaccessible to so

How Would You React ?

SCENARIO

A senior comes to you during your planning period and asks for your signature on a form to drop your class. The student tells you it is because the class is too difficult and is not necessary, but the "word" in the halls is that the student, a starter on the basketball team, would rather drop your mathematics class than to jeopardize athletic eligibility. Your response is

- **a.** Sign the slip and let the student go; it's probably not worth an argument.
- **b.** Sign the slip but encourage the student to reconsider dropping the class because you are fairly certain that any college major will require some form of mathematics class.
- **c.** Refuse to sign the slip and call the student's parents immediately.
- **d.** Refuse to sign the slip, contact the student's guidance counselor, and arrange a meeting for the three of you to discuss the implications of this class change.
- **e.** Other.

DISCUSSION

When it comes to choosing mathematics courses, many students—as well as their parents and even their guidance counselors—often do not recognize the importance of having a significant background in mathematics. In a report from the Mathematical Sciences Education Board (1989), the authors pointed out that 75 percent of all jobs require at least some background in basic algebra and geometry. A decade later, the National Commission on Mathematics and Science Teaching for the Twenty-First Century (2000), headed by astronaut and senator John Glenn, reported that a firm background in mathematics and science is necessary to ensure that the workforce in the United States will continue to be able to compete globally, to "solve the unforeseen problems and dream the dreams that will define America's future" (p. 4). Similarly, the American Mathematical Society and the Mathematical Association of America jointly published a report entitled *The Mathematical Education of Teachers* (Kessel, Epstein, & Keynes, 2001) that emphasized that the nature of most jobs entails a background in mathematics. The authors described the influence of technology and noted that even those who work on a line in a factory are now expected to have a background in statistics to analyze their effectiveness.

The changing nature of our world and its workforce has made it more important today than ever for students to have a strong background in mathematics—a background that includes not only number sense but also a basic understanding of algebra, geometry, statistics, and probability. In many cases, secondary and middle school mathematics teachers are preparing students for professions that have yet to be created. Often students consider dropping (or not even enrolling in) mathematics classes, fearing that the courses will be too challenging as well as embracing a misconception that "my career area will never involve any math." But many students are mistaken, as mathematics will ultimately be necessary in most jobs, and one of the responsibilities of mathematics teachers is to encourage students to take—and to finish—courses that will give them the background they need to be successful in their careers.

many students over the years? Mathematics anxiety and a general fear of mathematics are quite common, but how did these fears evolve, and how are they perpetuated?

Perhaps one of the greatest myths about mathematics is that some people are natural "math people," and some are not. A common misconception is that mathematical ability is genetically inherited or predetermined. On the contrary, research does not support the idea of innate mathematical ability. Consequently, with the possible exception of individuals who are severely disabled, every student can become mathematically literate. Whether students will understand the content may depend not so much on the material itself but on the way in which teachers present it. Although a certain percentage of students will understand a mathematical concept despite a poor teacher, most students will thrive only in an atmosphere created by caring, knowledgeable teachers.

The Need for Reform

In the 1960s, students and their teachers experienced a mathematics reform movement known as the New Math, which was sparked by the launching of *Sputnik* satellites by the USSR in 1957 and 1958. The success of the *Sputnik* program fueled a national fear of falling behind the rest of the world and motivated U.S. educators to reconsider the topics explored in the curriculum. In her book *A Parent's Guide to the New Mathematics* (1964), Evelyn Sharp discussed the need for an updated mathematics curriculum, noting that a seventeenth-century teacher could readily walk into a classroom in the 1960s and teach mathematics because "the content of the courses hadn't changed in 300 years" (p. 11). Recognizing the need for mathematicians and scientists to be able to compete on a global level, the New Math exposed high school students to topics such as set theory and non-Euclidean geometry, which had not historically been explored until the college level. Sharp noted that the New Math moved mathematical topics down to lower grade levels to ensure that "all students" visited content that required much more rigor than had previously been the case. In his famous book, *Why Johnny Can't Add* (1973), Morris Kline assailed the New Math, stating that

> The new mathematics is taught to elementary and high school students who will ultimately enter into the full variety of professions, businesses, technical jobs, and trades or become primarily wives and mothers [sic]. Of the elementary school children, not one in a thousand will be a mathematician; and of the academic high school students, not one in a hundred will be a mathematician. Clearly then, a curriculum that might be ideal for the training of mathematicians would still not be right for these levels of education. (pp. 21–22)

In its attempt to expose all students to higher mathematics, the New Math movement catered more to the top students than the marginal or average students of mathematics. Furthermore, the public was confused about its intent, and the movement eventually fell to the wayside, only to be replaced by a "back-to-basics" movement in the 1970s.

The mathematics reform effort that began in the 1980s, however, was very different. In 1980, the NCTM countered the "back-to-basics" movement in the famous book *An Agenda for Action* (1980) by suggesting that problem solving should be the focal point of the curriculum. Three years later, the U.S. Department of Education's National Commission on Excellence in Education released a landmark document entitled *A Nation at Risk*. Recognizing that the rest of the world was catching up with the United States, the authors of the book called for a dramatic reform of mathematics education, including requiring three years of mathematics in high school to graduate from high school. The report stated, "The teaching of mathematics in high school should equip graduates to: (a) understand geometric and algebraic concepts; (b) understand elementary probability and statistics; (c) apply mathematics in everyday situations; and (d) estimate, approximate, measure, and test the accuracy of their calculations. In addition to the traditional sequence of studies available for college-bound students, new, equally demanding mathematics curricula need to be developed for those who do not plan to continue their formal education immediately" (U.S. Department of Education, 1983). A logical "next step" for the NCTM was to develop and promote a national vision of mathematics education.

As a result, the NCTM released a series of three Standards documents: *Curriculum and Evaluation Standards for School Mathematics* (1989), *Professional Standards for Teaching Mathematics* (1991), and *Assessment Standards for School Mathematics* (1995). The contents of these documents were then updated and refined into one volume, *Principles and Standards for School Mathematics,* in 2000 to set the tone for mathematics education in the third millennium. A **standard** is a benchmark that can be used by a school, a district, a state, or a country to determine the degree to which the educational program meets a list of recommendations. The Standards documents from the NCTM emphasized that mathematics should be for *all* students—regardless of gender, race, socioeconomic status, or any other factor that may have caused inequities in the past. This way of thinking was an invitation to stronger and weaker students alike to develop their mathematical abilities and a challenge for teachers to discern how to make the teaching and learning of mathematics accessible to all. By 2010, a more detailed (in terms of content to be taught at each grade level) national vision was completed by the National Governors Association Center for Best Practices and the Council of Chief State School Officers and released as a set of *Common Core State Standards* to be used by all states that chose to adopt them (Common Core State Standards Initiative, 2010).

As we think of mathematics for all students, we will need to turn our attention to research on effective teaching practices. Within that body of research lies a great deal of evidence as to how to structure classrooms; how to pose meaningful, motivational problems; and how to use technology and teaching strategies, such as cooperative learning, to appeal to the vastly different learning styles and confidence levels of students in the classroom. We now assume, as a premise, that all of the students in our secondary and middle school classes are capable of learning mathematics, and we can begin to decide how to structure learning experiences for students that will appeal to their curiosity and intellect simultaneously. If students can be actively engaged in "doing" mathematics, they may be motivated enough to perform their best in the classroom and on assignments. Let's explore what it means to "do" mathematics.

"Doing" Mathematics

Problem Solving

Suppose that you were asked to find the circumference of a circle with a radius of 5 centimeters. Easy, right? Sure, you simply double the radius and multiply it by π to get an answer of about 31.4 centimeters. But what if you didn't already know that the circumference of a circle can be found by multiplying $2\pi r$ or πd? You might have to resort to drawing a sketch of the circle, laying a piece of string on the sketch, and stretching it out along a meter stick to estimate the length. Can you think of another way to determine the circumference without knowing the formula? Here is another option: Draw a line segment with a ruler on a piece of paper, cut out the circle, roll it along the segment until the circle has completed one revolution, and then measure the length of the path. Finding a circumference is a routine task—an **exercise**—if you already know a formula and have encountered that type of question before. However, if the situation is new to you and you have no such formula, the question becomes a problem to be solved. A **problem**, then, can be defined as a task for which there is no immediate solution. The situation is generally unfamiliar to the person attempting to find the answer. When confronted with a problem, we have no choice but to dig deeply into a bag of tricks—a list of strategies—to attempt to solve it. It is important to distinguish between routine exercises that students do for practice and problem solving in the classroom. Also, keep in mind that an exercise for one student may be a problem for another. For example, the circumference question may be a problem for a sixth grader but an exercise for a high school sophomore.

Problem solving can be defined as the process by which an individual attempts to find a solution to a nonroutine mathematical question. Probably the most famous book on this topic was published in 1945 by George Polya, then at Stanford. Polya described problem solving as a four-step process: (1) understanding the problem, (2) devising a plan for finding a solution, (3) implementing the plan, and (4) looking back at the answer to ensure that it makes sense and to determine if another plan might have been more effective (Polya, 1945). Although much additional research and writing on problem solving have been conducted since his book appeared, Polya's four steps are still cited as being fundamental in solving problems and in teaching problem-solving skills in the classroom. In 1980, the NCTM document *An Agenda for Action* called for the 1980s to be a problem-solving decade. The NCTM *Curriculum and Evaluation Standards* (1989) listed problem solving as the first Standard at all grade levels, K–12, and *Principles and Standards for School Mathematics* (2000) also lists problem solving as the sixth Standard at all grade levels. Furthermore, the *Common Core State Standards* state that the first mathematical practice for emphasis in the classroom is to have students "make sense of problems and persevere in solving them" (2010, p. 6). But the vision of problem solving embodied in these documents goes beyond simple, routine tasks. Instead, they suggest rich, meaningful experiences through which students develop and refine strategies that can be used to solve other problems.

Consider the following problem:

> *A certain farmer in Florida has an orange grove. In his grove are 120 trees. Each tree ordinarily produces 650 oranges. He is interested in raising his orange production and knows that because of lost space and sunlight, every additional tree that he plants will cause a reduction of 5 oranges from each tree. What is the maximum number of oranges that he will be able to produce in his grove, and how many trees will he need to reach this maximum?*

It is unlikely that you have ever thought about this situation, and the problem does not have an obvious answer; therefore, it is probable that the statement constitutes a problem for you. How would you begin to solve it? Generally, people reach back and try to apply a strategy that they have used for similar problems in the past. Take a minute with a piece of paper, and think about how you would solve it. Let's look at several ways the problem can be approached.

A middle school child might analyze the problem by *guessing-and-checking* in some orderly fashion. If 120 trees produce 650 oranges per tree, the current production must be 78,000 oranges. However, an increase of 1 tree will result in 121 trees but only 645 oranges per tree for a total of 78,045 oranges, an increase of 45 oranges altogether. Similar calculations can be organized into a *table*. See Table 1.

Table 1 Orange Grove Problem Data

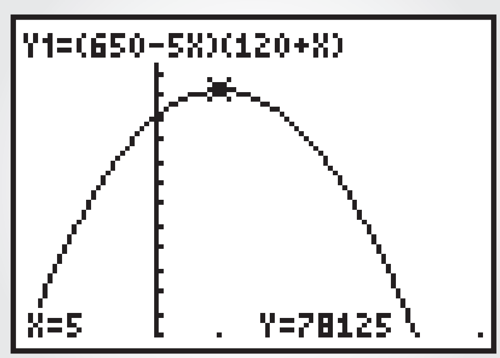
Total Trees	Oranges per Tree	Total Oranges
120	650	78,000
121	645	78,045
122	640	78,080
123	635	78,105
124	630	78,120
125	625	78,125
126	620	78,120
127	615	78,105
128	610	78,080
129	605	78,045
130	600	78,000

From Table 1 it is apparent that the maximum orange production occurs at 125 trees—the addition of 5 trees to the orange grove. However, students may notice some other things as well. For example, some may recognize that the orange production increases by 45 with the addition of one tree, 35 with the next tree, then 25, 15, and 5, decreasing beyond that point. The identification of this type of *pattern* can eliminate the need to generate the entire table, either by hand or on a computer spreadsheet. A seventh grader began by "guessing" what would happen if 10 trees were planted. When she realized that orange production was the same as the original amount, she immediately yelled out, "I think it goes up and back down, so if it's back to normal at 10, then it must reach its maximum with 5 new trees planted!" Because of a clever first guess (some might call it lucky) and a careful analysis of its result, she solved the problem before most of the others in the class could even write it down.

Suppose that the same problem was raised in a first-year algebra course in which students had been exposed to the use of variables for problem solving. In this case, a student might *write a variable expression* $(650 - 5x)(120 + x)$, where x stands for the number of trees added, to find the total production. The first binomial determines the number of oranges per tree, and the second binomial represents the total number of trees in the grove. At this point, the student can *graph* the function $y = (650 - 5x)(120 + x)$ on a graphing calculator (see Figure 7) and TRACE (TRACE is a common command on a graphing calculator) the curve to its vertex, finding that the parabola peaks at $x = 5$.

The student might also choose to solve the quadratic equation $(650 - 5x)(120 + x) = 0$ by setting the two factors equal to zero and find that the solutions are $x = 130$ or $x = 120$; therefore, the maximum must be the input value halfway between the x-intercepts, or when $x = 5$.

Figure 7 Orange Grove Parabolic Curve on a Graphing Calculator



Finally, a student with calculus background might choose to solve the problem by taking a first derivative to determine the maximum:

$$y = (650 - 5x)(120 + x)$$
$$y = 78{,}000 + 50x - 5x^2$$
$$y' = 50 - 10x$$

Setting the first derivative equal to 0,

$$0 = 50 - 10x, \text{ therefore}$$
$$x = 5$$

The orange grove problem is said to be *rich* in that it is nonroutine and can be solved in a variety of ways, featuring a number of different entry points. Depending on the grade level and experiences of the student, several problem-solving strategies can be applied, such as guess and check, make a table, look for a pattern, write a variable expression or equation, or draw a graph. Once students have effectively used a problem-solving strategy, they can apply the same technique to future problems as well. Ideally, a teacher would assign a problem such as this one, allow students to solve it individually or in small groups, and encourage students to share solutions and strategies so that students can reflect on their approaches when compared to strategies used by others. Then, if some students use a guess-and-check strategy but observe others writing an equation, they may choose to use an equation the next time a problem of this kind is posed. Students should not only seek accurate solutions to problems but also examine their problem-solving strategies to find the one that is most efficient.

Various resource books and materials provide students with examples of problems that use different strategies. Some of these resources are listed at the end of this chapter. Some of the other common problem-solving strategies developed in the secondary and middle school mathematics classroom include the following:

- Act out the problem.
- Make a drawing or diagram.
- Construct a physical model.
- Restate the problem in other words.
- Identify and verbalize the given, needed, and extraneous information.
- List all possibilities.
- Solve a simpler or similar problem.
- Work backwards.

Any single strategy or combination of these may be used for solving problems, and they constitute the "tool kit" that a student carries to a problem situation. Research has shown that in order for students to be effective problem solvers, they must have plenty of tools in their kits so that if one method is not working, they can move on to another one. There is an old joke that mathematicians tell:

Q. How do you kill a blue elephant?

A. With a blue elephant gun.

Q. How do you kill a white elephant?

A. With a white elephant gun?

No, you strangle it until it turns blue and then use the gun you already have!

Often, solving a problem boils down to nothing more than forcing it to look like a problem you've previously solved and using the same tools to find its solution. The *Common Core State Standards* call for the mathematical practice that students should "look for and express regularity in repeated reasoning" (Common Core State Standards Initiative, 2010, p. 8), which means that students should discover and use shortcuts and approaches that make them more effective problem solvers.

In *Principles and Standards for School Mathematics* (2000), the NCTM states that problem solving should enable all students to

- build new mathematical knowledge through problem solving;
- solve problems that arise in mathematics and in other contexts;
- apply and adapt a variety of appropriate strategies to solve problems;
- monitor and reflect on the process of mathematical problem solving. (p. 52)

Clearly, the emphasis of the Standards has been, and continues to be, on using problem solving both to develop strategies for solving future problems and as a context in which to learn or practice skills. Ideally, in the contemporary view of mathematics, every lesson should include some opportunity for students to refine their problem-solving skills. In addition, students should be required to reason mathematically—to think—in every lesson.

Reasoning and Proof

Try this trick on a friend: Write down the number of the month in which you were born (e.g., if you were born in October, write down a 10). Double this number. Add 6. Multiply this new number by 50. Then add on the day of your birth (e.g., if you were born on October 20, add 20). Finally, subtract 365. Now, ask your friend to give you the final number. On a calculator, secretly add 65 to that number. The result will tell you the day and month of your friend's birthday. This "trick" is particularly effective if you have several people do the calculation at once and ask for their answers, quickly telling each person the correct birthdate. But why does it work? Is it magic? No, it's mathematics.

Let m stand for the month in which the person is born and d for the day of the month. The steps of the problem for a person whose birthday is on October 20 are as follows, for that specific date and in general:

Instruction	**Specific**	**General**
Write down the month.	10	m
Double the number.	20	$2m$
Add 6.	26	$2m + 6$
Multiply by 50.	1300	$50(2m + 6) = 100m + 300$
Add the day of birth.	1320	$100m + 300 + d$
Subtract 365.	955	$100m - 65 + d$
Secretly add 65.	1020	$100m + d$

The answer 1020 represents the 10th month and 20th day, or October 20. Similarly, the final variable expression takes the birth month and moves it over two places to the left by multiplying by 100. When the day of birth is added, the result is a number from which the birthdate can be determined. With the power of algebra, this puzzle or trick can readily be analyzed.

In a mathematical situation, any time our students ask, "Why?" "How do we know that?" "What would happen if . . . ?" "Would it ever be true that . . . ?" they are asking questions that involve **reasoning** skills. In the orange grove problem, for example, the student might not be satisfied with seeing the production peak at five trees and might ask why this occurs. This can ignite a class discussion about how adding new trees may remove enough nutrients from the ground and shade enough sunlight from other trees so that eventually additional trees do more harm than good. Puzzles and other problems are generally worth exploring only if they engender discussions or discourse about why the problem works the way that it does. The pursuit of the question "why" in the mathematics classroom is critical. Students want to know, for example, why fractions are divided by inverting the last fraction and multiplying, why the formula for the area of a circle is $A = \pi r^2$, why the value of the constant e is irrational, and why the first derivative of the sine function is the cosine function. As they study mathematics, students should become inquisitive and inclined to seek proof and verification of conjectures raised in the classroom. And this is most likely to occur in classrooms in which mathematical reasoning is valued. Again, the mathematical practices in the *Common Core State Standards* call for helping all students to reason abstractly and to "construct viable arguments and critique the reasoning of others" (2010, p. 6).

Principles and Standards for School Mathematics (NCTM, 2000) lists reasoning and proof as Standard 7 and states that through emphasis on reasoning and proof in the classroom, all students will

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof. (p. 56)

Principles and Standards for School Mathematics states that "reasoning mathematically is a habit of mind, and like all habits, it must be developed through consistent use in many contexts" (p. 56). Every day that a child is in school, the student should be encouraged to reason mathematically by being challenged with "why" and "how" questions. In this way, students begin to recognize that it is not enough to be *able to* solve a problem; they must reason out the underlying mathematics, make conjectures or hypotheses, and communicate solutions and strategies to others.

Communication

Mrs. King teaches an Honors Geometry course in a small high school. In the spring, she divides her students into learning teams and assigns a famous mathematician to each team, such as Gauss, Newton, Pythagoras, Descartes, or Euclid. Each team is asked to research the life and contributions of its assigned mathematician. Then each student writes a term paper about the person, and the team creates a short skit about their mathematician and presents it to the class. The grade that students receive on the project is determined by a combination of individual written papers, self-assessments of how well the teams worked together, and the quality of their classroom presentations. (For details on this project, see the introductory chapter of the *Seventy-Third Yearbook of the National Council of Teachers of Mathematics* Brahier, 2011.)

SPOTlight on Technology

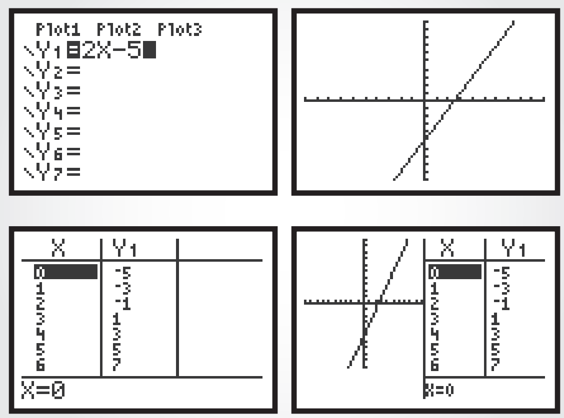
A common concern that teachers hear from parents and the community is that "when calculators are used in school, students become dependent on them"; parents and community members believe that calculators do more harm than good. This claim is certainly understandable because the parents of our secondary and middle school students often did not have access to calculators when they were in school. The first Texas Instruments graphing calculator was the TI-81, and it was released in 1990 at a cost of $110. So, the graphing calculator was not even invented until the parents of current secondary and middle school students were well out of high school. Consequently, many parents cannot relate to what it means for a student to learn mathematics using calculators. Research on calculator use, however, contradicts the general opinion that using technology hinders the learning of mathematics.

The most comprehensive meta-analysis (study of studies) on the use of basic, four-function calculators showed that regular use of calculators in the classroom improves student performance and attitudes in mathematics (Hembree & Dessart, 1992). In addition, the report from the 2000 National Assessment of Educational Progress (Braswell et al., 2001) showed that students at the eighth and twelfth grade levels who most frequently used calculators in their classes tended to outperform their peers who used calculators infrequently. The NAEP report also showed that 69 percent of eighth graders report that they use a calculator in class at least once a week. In fact, 44 percent of all eighth graders indicated that they use a calculator every day.

Another study synthesized the results of 43 other pieces of research on the use of graphing calculators. In this report, the authors noted that students who used graphing calculators in school have a better understanding of functions, their graphs, and the use of algebra in real-life contexts (Burrill et al., 2002). Interestingly, students who were taught with graphing calculators also showed no significant difference in their ability to do more traditional "paper-and-pencil" procedures than their peers. Finally, Dunham (2000), in reviewing research on graphing calculator use, pointed out that students who use graphing calculators are better able to connect various representations of the same function—algebraic, graphical, and tabular.

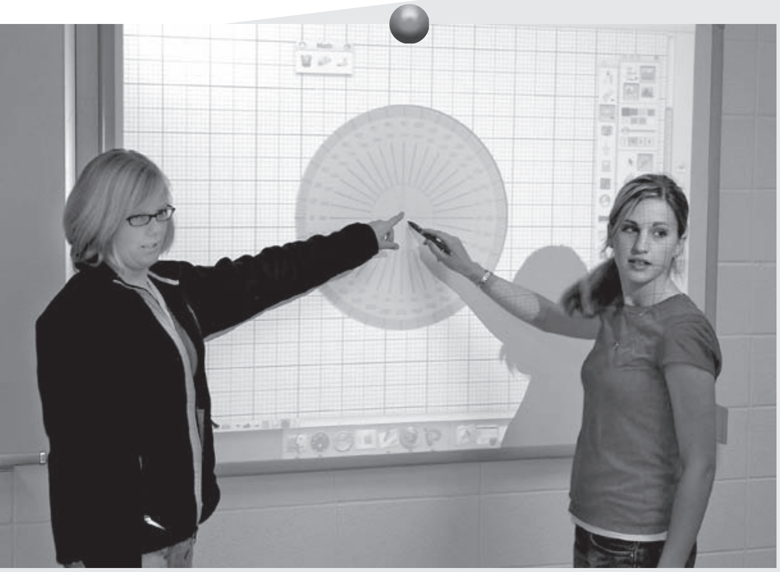
Using graphing calculators, students have opportunities to compare different representations of a function with the stroke of a key. The ability to switch back and forth from one representation to another, in turn, helps the student to think more deeply about the function and its meaning. Consider, for example, the linear function $y = 2x - 5$. On a graphing calculator, students can view the representations shown in Figure 8 within seconds.

Figure 8 Representations of the Linear Function $y = 2x - 5$



The student enters the algebraic equation of the function in the calculator and, with a few keystrokes, can view the function as a graph, as a table, or as a combination of a graph and a table in a single screen. Then, by making simple changes to the equation, such as using $y = 2x + 5$ or $y = 2x - 5$, students can readily explore the effects of these parameter changes on the graphs and tables. These types of explorations eliminate the tedious process of repeatedly drawing graphs on paper and place the emphasis on the behavior and meaning of the functions, rather than simply the skill of drawing them.

Mr. Shirley teaches in a middle school and requires his students to keep a mathematics journal. Students are regularly provided with prompts for their journal entries. Prompts include statements such as, "Identify the most difficult problem we solved this week and explain what made it difficult for you" and "Write a letter to a friend, explaining how to add numbers that include decimals. Be sure to



Students make presentations to the class and are assessed on their ability to communicate mathematically.

include a diagram." By collecting the journals every 2 weeks, Mr. Shirley learns much more about how his students are thinking than he can gather during class time. Also, as students write responses to the prompts, they are pushed to clarify their thinking and to explain it to another person. The authors of the *Common Core State Standards* stated that students should "make conjectures and build a logical progression of statements to explore the truth of their conjectures . . . They justify their conclusions, communicate them to others, and respond to the arguments of others" (2010, pp. 6–7). In addition, the document includes the mathematical practice of attending to precision, noting, "Mathematically proficient students try to communicate precisely to others" (p. 7). Often, it is not until we are asked to explain something to someone else that we realize there are gaps in our own understanding.

How often have you heard someone say, "Well, I know how to do it, but I can't really explain it"? Or they simply say, "I just did it." Learning mathematics effectively should exceed the ability to demonstrate skills; students should be able to explain, describe, and clearly communicate solutions and strategies that lead to the answers. The ability to communicate mathematically is a major goal of current reform efforts. In order to validate our thinking or to convince another person that our thinking is accurate, we need to communicate verbally and in writing. Consequently, when we think of "doing" mathematics, we should include the process of communicating with others as a critical component. Projects, written papers, presentations, and journals are examples of classroom strategies that promote mathematical communication.

In *Principles and Standards for School Mathematics* (2000), the authors point out in Standard 8 that **communication** should be stressed so that students

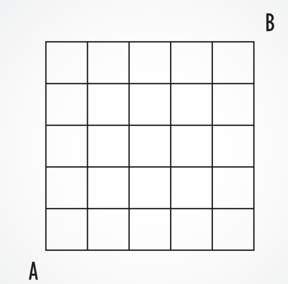
- organize and consolidate their mathematical thinking through communication;
- communicate their mathematical thinking coherently and clearly to peers, teachers, and others;
- analyze and evaluate the mathematical thinking and strategies of others;
- use the language of mathematics to express mathematical ideas precisely. (p. 60)

Since the release of the Standards, we have seen considerably more emphasis on communication in the classroom. For example, many teachers have begun to use cooperative learning teams in which students have specific roles and depend on the input of others. Also, teachers who consider communication an important goal frequently ask free-response questions, as in the example of the movie theater problem, in which the student is required to show work and to explain the thinking process in words or with diagrams. When a student provides a "correct answer" in the classroom, logical follow-up questions are "How do you know?" and "What were you thinking?" Whenever a teacher pushes students to explain their reasoning, the level of questioning is enhanced, and students are challenged to communicate mathematically.

Connections

Consider the following problem from an NCTM resource book (Phillips, Gardella, Kelly, & Stewart, 1991). The student is provided with the diagram shown in Figure 9:

Figure 9 Counting Paths from Oz



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The problem states that the city of Oz is located at point A and that a person wants to travel to point B, moving only right along horizontal lines or up along vertical lines. The question is to determine how many paths there are to move from point A to point B. Before reading on, take a few minutes with a piece of paper and a pencil, and think about how you would proceed to find a solution.

Often, students approach this problem by tracing possible routes on the grid while searching for a rule or pattern that can be generalized. They think about decisions that need to be made any time the pencil reaches an intersection point at which the "traveler" has a choice of directions to pursue. It is not unusual for students to struggle with solutions such as 5! or 2 raised to some power. These solutions, although incorrect, can help the students refine their thinking and lead them to a different way of viewing the problem. Often, a useful problem-solving strategy is to *solve a simpler problem* and then *look for a pattern,* and the teacher may choose to lead them in this direction. In Figure 10, there are three smaller grids that students might consider.

If the size of the map is 1×1, then there is exactly one way for the traveler to reach the vertices directly to the right and above A, the starting point. Consequently, there are two ways to reach the final destination. Using the same logic, students can construct a 2×2 grid and build on the previous map to find a total of six ways to get to the city. Can you trace all six possible paths in the second picture? Extending the idea to a 3×3 diagram, there are 20 paths that lead to the destination point. At this point, students might recognize the pattern and continue the process to find that there are 252 possible paths to get from point A to point B in the original problem. (Did you find this?) Turning the paper so that point A is at

Figure 10 Simpler Problems Used to Solve the Oz Paths Problem

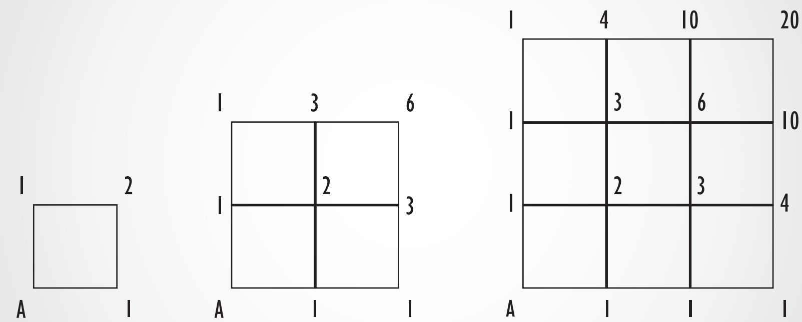


Figure 11 Pascal's Triangle

1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

the top of the page, if we look only at the numbers, we might recognize a familiar pattern. The numbers generated in this problem are located in Pascal's Triangle, as illustrated in Figure 11.

Most students associate the Triangle with the binomial theorem or with the determination of probabilities, so its connection to this path-counting problem may not be intuitively "obvious." However, it is important for students to make this type of connection across topics in mathematics. It may also be difficult to categorize this problem, because it falls into content topics such as coordinate graphing, patterning, and discrete mathematics while making important connections to the study of advanced algebra and probability.

In other words, this problem unites several mathematical concepts within a single investigation. It emphasizes the mathematical **connections** between a variety of topics. In the classroom, students should be encouraged to think of mathematics as the connected whole that it is rather than see a course as a chapter of this and a chapter of that. As such, even the idea that one can take an algebra class one year and a geometry class the following year as if they are not inherently connected can be very misleading to the young learner. Recent reform efforts have called for teachers to use activities and examples that help students to see how the various areas of mathematics are related. For example, the *Common Core State Standards'* mathematical practices include looking for and using patterns and structure, as well as making use of repeated reasoning strategies (2010).

NCTM's *Principles and Standards for School Mathematics* (2000) states in Standard 9 that connections should be made in the mathematics classroom to help students

- recognize and use connections among mathematical ideas;
- understand how mathematical ideas interconnect and build on one another to produce a coherent whole;
- recognize and apply mathematics in contexts outside of mathematics. (p. 64)

In the Standards, the idea of connecting mathematical topics to one another is joined with the notion of helping students see applications of mathematics in other subject areas so that they come to appreciate the utility of mathematics across the curriculum. When studying genetics in a biology course, for example, the student can explore some practical implications of probability theory. Matrices can be used to model production costs and profits for a business, and a discussion of symmetry or ratio and proportion can be rooted in the analysis of a piece of art. Again, the very fact that students take an Algebra II class during first period and a Chemistry class during sixth period almost suggests that there is a wall between these courses; but in reality, each subject depends on the other, and unless teachers help students make these connections by virtue of the problems posed, many of the applications are lost. Recent trends in the development of the middle school concept have addressed the issue of connections by providing teachers with common planning time during which they can discuss and arrange for experiences that help students make connections across the disciplines.

Representation

The last mathematical process described in *Principles and Standards for School Mathematics* (2000) is **representation**. The mathematics classroom frequently involves students attempting to represent problem situations in a variety of ways, deciding on which representation is the most helpful and appropriate in a given situation. For example, if you wanted to explore the orange grove problem in some depth, you might want to use an equation, as it is helpful in finding x-intercepts and calculating function values. However, it might be more useful to view a graph in such a way that a comparison could be made to related real-world phenomena. Similarly, the following three equations describe the same function:

(a) $y = x^2 + 2x - 15$
(b) $y = (x - 3)(x + 5)$
(c) $y + 16 = (x + 1)^2$

Although the first version is in Standard Form and communicates a y-intercept at $(0, -15)$, equation (b) is factored to make it much easier to determine that the parabola intersects the x-axis at the points $(3, 0)$ and $(-5, 0)$. And although equation (b) is useful for finding the roots of the function, equation (c) may be much more helpful if one needs to know that the vertex is at the point $(-1, -16)$ and that the minimum y-value is at -16. Equation (c) masks the coordinates of the x-intercepts, while making it easy to determine the vertex. So, although all three of these equations are acceptable and reasonable, we choose a particular representation of the function depending on the context of the problem and what information we need. Likewise, although the fraction $\frac{3}{5}$ is considered "simplified" and, therefore, a desirable way to express a quantity, we may choose to represent it as $\frac{12}{20}$ if we have been asked to add $\frac{3}{5} + \frac{1}{4}$ or as 0.6 if the object is to find $\frac{3}{5}$ of 75.

Specifically, *Principles and Standards for School Mathematics* (2000) stated that instructional programs from prekindergarten through grade 12 should enable all students to

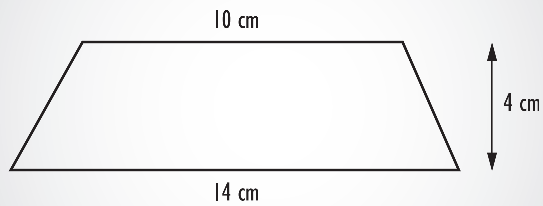
- create and use representations to organize, record, and communicate mathematical ideas;
- select, apply, and translate among mathematical representations to solve problems;
- use representations to model and interpret physical, social, and mathematical phenomena. (p. 67)

In the *Common Core State Standards,* one of the mathematical practices is to model with mathematics. The authors of the document asserted that students

Classroom Dialogues

The class in this scenario has been studying the formulas for finding the areas of various polygons. The diagram shown in Figure 12 is drawn on the whiteboard.

Figure 12 Trapezoid Drawn on the Whiteboard



Teacher: Can anyone tell me what this figure appears to be?
Student 1: It looks like a trapezoid.
Teacher: How do you know?
Student 2: A trapezoid has two parallel sides, and the sides that are 10 and 14 are parallel.
Teacher: What if the other two opposite sides were parallel?
Student 2: Then it would be a parallelogram.
Teacher: Good. So, how can we find the area of this trapezoid?
Student 1: You take one-half of the height times the sum of the bases. So, it would be like this (student comes up and writes on the whiteboard):

$$A = \frac{1}{2}(4)(10 + 14) = 2(24) = 48$$

So, the answer is 48.

Teacher: Okay, 48 what?
Student 1: Just 48.
Teacher: What about the units?
Student 2: It's 48 centimeters, not just 48.
Teacher: Are you sure it's centimeters?
Student 2: Square centimeters?
Student 1: I'm thinking cubic centimeters, but I'm not sure.
Teacher: Why cubic centimeters?
Student 1: Because you actually used three different numbers from the trapezoid, so wouldn't that make it cubed units?
Student 2: But I thought that cubed units were only for volume.

The confusion in this classroom discussion is very common in secondary as well as middle school mathematics classrooms. Often students know and can apply a formula to determine perimeter, area, or volume, but they do not understand how to label the answer properly so that the dimensions make sense. On the 2003 National Assessment of Educational Progress, eighth graders were asked whether the measure of the area of a triangle could be represented as 2 cm, 3 m, 5 cm^2, or 8 cm^3. In the United States, only 47 percent of the students answered this question correctly (National Center for Education Statistics, 2005a)! What might you say to these students to help them place the correct label on the answer? Do you consider it a "major" issue in your class if one student writes "48," while others write "48 cm," "48 sq cm," or "48 cu cm" on their papers as the answer to this question?

Although unit labeling often does not appear to be an important issue—after all, the student did the correct calculation and found the answer of 48—the science community and others have serious concerns about these types of errors. In a physics class, for example, velocity is often written in $\frac{\text{m}}{\text{sec}}$ (meters per second) because it represents distance traveled in a given amount of time. However, acceleration is a measure of how rapidly velocity changes over time, so it is expressed in $\frac{\text{m}}{\text{sec}}/\text{sec}$ (meters per second, per second) or $\frac{\text{m}}{\text{sec}^2}$. Although these units differ by "only" an exponent of 1 or 2 in the denominator, they represent two very different conceptual ideas. Particularly in the study of physics and chemistry, proper labeling of units is essential. So, the better students are prepared to think about the accuracy of units in a mathematics class, the more likely they will be to apply the ideas to science and other contexts.

The teacher in this scene needs to step back with the class and talk about how perimeter represents the distance around a region and is, therefore, measured in units such as feet or meters. However, area involves multiplying two dimensions—length times width, squaring a radius, and so on—so that a unit is multiplied by the same unit, resulting in a square unit. Students should be encouraged to think of area as the number of square units required to cover a region, so the answer to an area problem should be in "squares." Likewise, because volume involves determining the number of cubes required to fill a three-dimensional space, the answer to a volume problem should be expressed in cubic units.

The last question raised by Student 1 should not be taken lightly. The student recognizes that three dimensions were used in the area calculation and assumes, therefore, that the answer should be written in cubic units. However, what the student does not realize is that two of the dimensions were added together prior to multiplication, so that only two dimensions were actually multiplied (half of 4 cm is multiplied by 24 cm to result in an answer in square centimeters). It is important for teachers of mathematics to insist that students think carefully about the units they use to express answers to measurement problems, as the reasoning behind the units they choose can often reveal deeper misconceptions that can be addressed in the classroom.

should be able to "identify important quantities in a practical situation and map their relationships using tools such as diagrams, two-way tables, graphs, flowcharts and formulas" (2010, p. 7) as well as to use tools such as technology, rulers, or physical materials appropriately when solving a problem. Representation is an important part of the process of doing, teaching, and learning mathematics because students often find themselves trying to determine the best and the most appropriate way to model a problem situation.

In summary, the *Common Core State Standards* (2010) list eight **mathematical practices** that "mathematics educators at all levels should seek to develop in their students" (p. 6). These practices are as follows:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning. (pp. 6–8)

In *Principles and Standards for School Mathematics* (NCTM, 2000), there are five mathematical process skills that are described in detail. These skills are as follows:

1. Problem Solving
2. Reasoning and Proof
3. Connections
4. Communication
5. Representation

These two documents highlight many of the same skills, though they state them in different ways. A book, published by the NCTM and entitled *Making It Happen: A Guide to Interpreting and Implementing the Common Core State Standards for Mathematics* (NCTM, 2010), includes detailed comparisons of the "practices" of the Common Core and the "processes" of *Principles and Standards*.

Of course, leading a group of students through a problem-solving situation—as well as helping the students to reason, communicate, connect, and represent problems appropriately—requires a classroom teacher who is well-versed in mathematical content. The Conference Board of the Mathematical Sciences (2001) has called for deeper, higher-quality mathematics content backgrounds for prospective teachers. Likewise, in her book *Knowing and Teaching Elementary Mathematics* (1999), Liping Ma asserted the following:

> The teaching of a teacher with profound understanding of fundamental mathematics has connectedness, promotes multiple approaches to solving a given problem, revisits and reinforces basic ideas, and has longitudinal coherence. [Such a teacher] is able to reveal and represent connections among mathematical concepts and procedures to students. He or she appreciates different facets of an idea and various approaches to a solution, as well as their advantages and disadvantages—and is able to provide explanations for students of these various facets and approaches. [The teacher] is aware of the "simple but powerful" basic ideas of mathematics and tends to revisit and reinforce them. He or she has a fundamental understanding of the whole . . . curriculum, thus is ready to exploit an opportunity to review concepts that students have previously studied or to lay the groundwork for a concept to be studied later. (p. 124)

Conclusion

Think back to Mr. James's classroom as described in this chapter. His students were asked to select a real-life problem and find a function that modeled it. We can analyze the class project and student responses through the lens of the five mathematical processes described in this chapter. Students selected a function and were faced with the problem of expressing it in a variety of ways and determining how the function behaved so that they could explain it to others. As they explored the ways to write the function in a sentence, in a table, with an equation, and as a graph, they were faced with the task of representing their functions in a variety of ways and determining the strengths and weaknesses of each method of representation. When students made presentations to the class and submitted written papers about their functions, they were communicating the mathematics in written and oral form to their peers and teacher. Mr. James asked students to analyze two of the problems posed by others in the project; thus, he was promoting reasoning skills as students explored the question of "what about these functions makes them different?" Finally, the students were making connections among various representations of their functions as well as connecting algebra concepts to real-world phenomena in their daily lives.

Books have been written on the nature of mathematics and what it means to do mathematics. Although many people think of mathematics as a content area—algebra, geometry, number theory, and so on—the purpose of this chapter is to remind us that mathematics is also a process, something that one "does." NCTM has defined five **mathematical processes** that should permeate every lesson every day in the classroom—problem solving, reasoning and proof, communication, connections, and representation. These are often referred to as the *umbrella standards* because they constitute what students should be doing as they learn about specific content issues such as algebra or probability. Similarly, the *Common Core State Standards* include eight mathematical practices that parallel NCTM's process skills.

When a lesson is planned, mathematics teachers often consider the processes and practices and how they can be incorporated into the class period. The main intent of the class may be to learn to sketch a sine and cosine curve, but in the process, students are presented with problems and questions that develop their mathematical thinking skills along the way. As Mark Twain said, "Education is what's left over when you forget all the facts that your teachers made you memorize when you were in school." Students may forget how to factor a perfect cube six months after the method was taught, but if the lesson was effective, they will have learned to apply mathematical processes that will be useful long after the formulas have been forgotten. In addition, they should be able to reconstruct the method for themselves.

Surveys from industry consistently indicate that employers are looking for a workforce that has problem-solving and communication skills (see, for example, the famous *SCANS* [Secretary's Commission on Achieving Necessary Skills] *Report,* 1991). Businesses claim that they will undertake the necessary on-the-job training, but they need as their raw material employees who can think and work with others. Mathematics teachers can help produce reflective problem solvers by focusing lessons and activities on the mathematical processes and practices.

Glossary

Communication: Communication is the process by which students express their mathematical thinking to others in oral or written form. Students are encouraged to go beyond "being able to do" the mathematics and to communicate their thinking to others.

Connections: Making mathematical connections is the process by which students connect the mathematics they are studying to (1) other areas of the mathematics curriculum (such as knowing how Pascal's Triangle applies to the binomial theorem in algebra while it also is useful in determining sample spaces in the study of probability), (2) other areas of the curriculum, such as science or social studies, and to (3) the real world.

Exercise: An exercise is a task with which a person is already familiar, and doing an exercise is considered routine practice of a skill. This differs from a problem, in which there is no immediate solution.

Mathematical Practices: The *Common Core State Standards for Mathematics* describe eight practices that should be emphasized in the mathematics classroom at all grade levels, such as making sense of problems and persevering in solving them and reasoning abstractly and quantitatively.

Mathematical Processes: The National Council of Teachers of Mathematics has identified five processes that underlie the teaching and learning of mathematics.

These *umbrella* processes include problem solving, reasoning and proof, communication, connections, and representation.

Problem: A problem is a task that has no immediate solution, and the person solving it has to begin by defining the problem and identifying a strategy. This differs from an exercise, in which a person is already familiar with the task and merely needs to practice a skill.

Problem Solving: Problem solving is a mathematical process by which students attempt to identify what is needed, to set up a plan, to implement the plan, and to check the reasonableness of their answer (see Polya's 1945 book). As students engage in problem solving, they develop a set of problem-solving strategies that can be applied in other situations. Some of these strategies include writing an equation, making a physical model, working backward, drawing a graph, making a table, and using guess and check.

Reasoning: Reasoning is the mathematical process by which students seek to explain "why" something happens the way it does or "what would happen if . . ." something were different in a problem. Mathematical reasoning deals with constructing proof (either formally or informally) that conjectures are true or false.

Representation: Representation is the mathematical process by which students take a given problem situation and attempt to model it in a useful way that will enable them to solve the problem. Different representations of problems are appropriate at different times, depending on the context.

Rubric: A rubric is a grading scale that is often used for scoring open-ended (free-response) questions, essays, presentations, and projects within and outside of mathematics. Generally, each scale number in a rubric is attached to a description of student performance that is required to reach that particular level.

Standard: A standard is a benchmark that can be used by a school, a district, a state, or a country to determine the degree to which the educational program "measures up" to what is expected. The National Council of Teachers of Mathematics released three standards documents in the 1980s and 1990s and another comprehensive document entitled *Principles and Standards for School Mathematics* in 2000. The intent of these documents was to provide a direction and set of goals for those involved in planning mathematics curriculum, instruction, and assessment.

Discussion Questions and Activities

1. Why did you choose to enter the teaching profession and teach mathematics? In what ways might your reasons for becoming a mathematics teacher influence your beliefs about education?
2. Take some time to explore a TIMSS Web site. What are the implications of the results of TIMSS? What do the data tell us about mathematics education in the United States, including its apparent strengths and weaknesses?
3. Visit the Web site for the National Assessment of Educational Progress (NAEP) test. Explore recent results of the test, including examining several sample items. Are you surprised by the questions asked or the results? Explain.
4. Obtain the *Curriculum and Evaluation Standards for School Mathematics* (1989) and *Principles and Standards for School Mathematics* (2000) in hard copy or by visiting www.nctm.org. Compare the philosophy and contents of these two standards documents. How did the philosophy of mathematics education change during the 10 years between these documents, and what appears to have remained the same?
5. In a small group, select one of the process standards or mathematical practices. Obtain a copy of one of the source documents—*Principles and Standards for School Mathematics* (2000) or the *Common Core State Standards for Mathematics* (2010)—and research the process or practice. Present any new insights gained from the narratives or examples presented in the source documents.
6. Observe a secondary or middle school mathematics lesson (or view one on a videotape) and outline what the teacher does in the lesson to promote any or all of the mathematical processes or practices.
7. What does it mean if someone suggests that you "place problem solving at the focal point" of your mathematics classroom? How is this different from what we have traditionally experienced in the teaching and learning of mathematics?
8. Identify a rich problem in a resource book, a textbook, or on the Internet, and explain how it can be used to promote and develop a number of different problem-solving strategies.
9. Identify a routine algorithm such as "adding up the total number of decimal places and counting that many to the left" when you multiply decimals, "inverting the second fraction and multiplying" when you divide fractions, or "adding the opposite" when you subtract integers. Then think about and discuss how one might teach a class how to use the algorithm while you are promoting mathematical processes or practices in a lesson.
10. Discuss the degree to which this chapter changed your thinking on what it means to "do," to "teach," and to "learn" mathematics.

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Photo Credit

Daniel J. Brahier

Learning Theories and Psychology in Mathematics Education

Learning Theories and Psychology in Mathematics Education

After reading this chapter, you should be able to answer the following questions:

- How is research conducted in mathematics education, and how does it impact trends in curriculum, teaching, and assessment?
- What are the key components of learning theories, such as those of Vygotsky, Bruner, and the van Hieles? What are the principles underlying the constructivist model of teaching and learning?
- How do the use of inquiry and an inductive approach to teaching differ from the traditional model that emphasizes deductive methods? Explain why there has been a shift toward this inquiry-based approach.
- What is motivation, and what can teachers do to help "motivate" students?
- What does it mean to develop a positive mathematical disposition? Describe some strategies that might be used to promote positive dispositions and to counter mathematics anxiety in students.

Chelsea is a sophomore student in Mr. Metzger's third-period Geometry class. On standardized tests, she has demonstrated the potential to be a high-achieving student. Mr. Metzger knows her and is aware that she is an intelligent girl with a promising future, but she is failing his class. At the beginning of the school year, when many of the problems used equation solving from algebra and simple formulas from middle school, Chelsea performed quite well. However, as the year progressed and the class started to work with increasingly more difficult definitions and theorems and to write two-column proofs, she began to struggle and eventually gave up. She rarely does her homework and finds her mind drifting off during class, unable to focus on the problems and proofs being demonstrated by the teacher at the board. After class one day, Chelsea apologized to Mr. Metzger and said, "I really want to do well in your class, but I don't understand it. I'm just not as interested in this class as I have been with other math classes in the past." She urged him not to take it personally, but as he watched her walk out the door, he knew that it couldn't be entirely her fault.

Perhaps you have known a Chelsea in your mathematics career (or maybe you *were* a Chelsea at some point). Every class includes students who have the potential to be successful but are struggling nonetheless. A common reaction is to dismiss Chelsea's problem as a personal issue that only she can fix. After all, if she is not consistently paying attention in class and doing her homework, how can the teacher expect her to achieve acceptably in the course? However, the classroom is a two-way street that involves both the teacher *and* the learner, and there are ways in which a classroom teacher can help a student such as Chelsea. First of all, if she has been having problems with two-column proofs all year long and is one of the smartest students in the class, her peers are probably struggling as well. It is entirely possible that the content is too difficult for many students in the class—developmentally beyond what they are ready to handle at this point. Second, Mr. Metzger should examine how he spends his class time and what he expects his students to be able to do. It may be that his chalkboard-based teaching style is simply not interesting to his students and that they need the sort of problems and activities that evoke curiosity and, therefore, motivate the class to want to do the work. When it comes to the personality and family concerns of the student, much is beyond the realm of what a teacher can do. After all, the National Research Council (2000) estimates that only 14 percent of students' time is spent in school. The other 86 percent of their time is spent in family and community settings or sleeping. But after close examination and reflection, every teacher should be able to identify areas for improvement in the way that a class is run that may, in turn, help students such as Chelsea get back on track.

This chapter deals with the psychological concerns in the teaching and learning of mathematics. Because theories of educational psychology result from years of research in the classroom, the role of mathematics educational research is explored before some current theories about how students learn are discussed. Then a definition of motivation is discussed, as well as what the teacher can do to help motivate students to learn. Finally, the chapter discusses what the National Council of Teachers of Mathematics calls mathematical disposition—the attitudinal side of teaching mathematics—and how it develops over time.

Research in Mathematics Education

The announcer of a nationally televised professional football game stated that quarterbacks of ages 25 and under have a 35 percent winning percentage whereas quarterbacks over age 33 win 74 percent of their games. He used the data to argue that older players are better at what they do and that teams often "recycle" quarterbacks because their performances tend to improve significantly as they gain experience. What he failed to realize—or at least he never mentioned it—was that only the best quarterbacks in professional football are still playing when they are more than 33 years old. As a result, the data should be no surprise to the viewer—we would expect that the *reason* a quarterback would still be playing beyond age 33 is that he has been very successful and has a high winning percentage. Less-successful quarterbacks have already left the game by the time they are 30 years old and are no longer included in the data. In short, the statistics were mathematically accurate, but conclusions drawn from the data were questionable.

Similarly, a newspaper recently reported that "90 percent of all divorced adults blame the breakup on the other person," and another headline stated that at a famous university, 94 percent of the faculty considered themselves "above average" instructors. Not only do headlines such as these sometimes make us chuckle—after all, shouldn't 50 percent of the teachers be at or above average and 50 percent be at or below average?—but also they inevitably make us ask questions such as, "How do they know that? Who did they ask? Are these statistics reasonable for the whole population? Does the conclusion make sense?" Research is the process of gathering and analyzing data so that the results can be used to inform decision-makers. For example, if a state raises its maximum driving speed from 65 to 75 miles per hour, and research data show a significant increase in highway fatalities during the following year, the legislature may use the research to argue that the speed limit should be lowered again. Decisions are made based on mathematical information every day in our world.

However, we must use caution because some data and, therefore, the results of some research reports are flawed. Let's suppose you ask five of your best friends whether they generally vote Democrat or Republican, and four out of the five say that they are Democrats. Is it logical to assume that 80 percent of all adults are Democrats? Of course not! The sample size you chose was too small to generalize from, and you selected a unique and, therefore, biased part of the population—college-educated and, perhaps, all of the same gender and living in similar situations. If you really wanted to know what percentage of the population belonged to each party, you would need to take a large national survey that included a reasonable mix of geographic regions, socioeconomic groups, gender, and so forth. In addition, you might want to compare your results to data from similar studies to look for patterns, similarities, and inconsistencies. You hope that you would never have to make a significant life decision based on one, potentially flawed, study.

Just as the legislature makes speed limit decisions based, in part, on fatality-rate research, educators ideally should write curricula and select teaching and assessment methods with current research in mind. To this end you could begin to ask more questions that require reading, analysis, and a clearly communicated response in your classes in order to give students more experience with. In fact, a school district may use research such as the TIMSS, PISA, or NAEP reports to influence the topics taught in school and the type of expectations established for student progress.

Quantitative and Qualitative Methods

In general, two major types of research in education guide our decision-making—quantitative research and qualitative research. **Quantitative research** deals with gathering numerical data and analyzing them. For example, Slavin et al. (1990) described a quantitative study in which some high school students were placed in heterogeneous (mixed-ability) classes, and others were placed in homogeneous (ability-grouped) classes. At the end of a school year, the students in mixed classes significantly outperformed their peers. In short, they scored higher on achievement tests when they were members of heterogeneous classes. Similarly, a study conducted with junior high school students in Israel showed that although high achievers performed about the same in heterogeneous and homogeneous groupings, students at average or low-average levels achieved significantly higher while participating in heterogeneous groupings (Linchevski & Kutscher, 1998). In this article, the authors argued against ability grouping and tracking practices. On the other hand, quantitative research can also argue in favor of ability grouping. For example, a study by Rogers (1998) suggests that all students, particularly those who are gifted, benefit from homogeneous grouping. Therefore, decision-makers need to thoroughly explore a large base of research before reaching any conclusions and acting on the data.

Historically, educators first used quantitative methods. If we wanted to know, for example, whether calculators improve the learning of a particular concept, we could pretest two classes to establish that they are similar in background and proceed to teach them—one with calculators (the experimental group) and one without (the control group). Then, we could compare scores on a posttest to determine whether the calculator-based class actually outperformed the other and report those results to the education community. Recently, however, there has been a trend toward more qualitative research in education as researchers have become skeptical about the degree to which we can describe student performance based solely on numerical data. There has been a similar trend in the assessment of students in mathematics classes.

Qualitative research involves the collection and analysis of non-numerical data such as videotapes of classroom episodes, scripts of student–teacher conversations, audio recordings of interviews, or written summaries of student journal entries. For example, a qualitative study involving first- and second-year algebra classes in high school showed that teachers who prompted their students to write a five-minute response to a problem or question at the beginning of class several days per week tended to adjust their lesson plans accordingly, gaining more insight into student understanding than they would have without the prompts (Miller, 1992). The data for this study consisted of written student responses, written teacher reactions, and notes from meetings and interviews with participating teachers. In another study, a researcher concluded that students generally do not gain sufficient experience with justification and proof in mathematics at the middle school level. She presented two possible explanations for this conclusion: (1) Teachers often eliminate the discussion of students' reasoning due to lack of class time, and (2) teachers do not tend to provide students with adequate feedback when an answer is shared in the classroom (Bieda, 2010). In this middle school research, multiple case studies were conducted in seven schools that involved observation of classes, as well as written and interview reflections provided by classroom teachers. Although it is possible to attach numbers to qualitative raw data (e.g., one can count how many times students responded in a particular way), the research is primarily involved with "words" taken from observations and interviews, rather than "numbers" from tests.

On the other hand, a study does not have to be purely quantitative or qualitative, as some research employs both in a "mixed" method. By employing qualitative and quantitative methods in the same study, data from a variety of sources can be compared (a process researchers refer to as *triangulation* of the data). For example, a study conducted by Watson and Moritz (2003) studied beliefs about probability with students in grades 3 through 12. The students were interviewed about the fairness of dice and asked to verify their conjectures. Although interviewing is considered a qualitative research method, the researchers collected data from student responses and categorized the information to generate tables of quantitative data. Then, three or four years later, many of the same students were interviewed again. The researchers showed that only 45 percent of the students responded to the questions at a higher level than they had in the first interview. More than half of the students had not progressed in their understanding of fairness over the three- or four-year period of time. The authors concluded by suggesting that secondary and middle school students need more hands-on experiences with **manipulatives**, such as dice, to help them make sense of the need for significant sample sizes and fairness in the study of probability.

How Would You React?

SCENARIO

You have planned a blocked, 2-hour lesson in which students will explore geometric figures with dynamic geometry software in the computer lab. Your students gather in your classroom, and you escort them to the lab, only to discover that the room has accidentally been "double booked" with an English class, and the students in that class are using the computers. Your response is to

- **a.** Return to the classroom and teach the lesson anyway, but use a projector with a single computer to demonstrate the geometry, rather than having students manipulate the figures themselves.
- **b.** Return to the classroom and teach the lesson anyway, but omit the use of any technology. Instead, illustrate the geometric properties by drawing diagrams on the board rather than by using software.
- **c.** Scrap the lesson for today. Instead, teach an alternate lesson on a related topic that does not require the use of technology and reschedule the lab to do the original lesson at another time.
- **d.** Compromise with the English teacher to split the lab use time and attempt to teach your 2-hour lesson in 1 hour in the lab.
- **e.** Other.

DISCUSSION

One of the keys to effective teaching is learning to be *flexible*. Unexpected events, from fire drills to emergency assemblies to P.A. announcements, occur on a regular basis in schools. Particularly when it comes to teaching with technology, teachers should always be prepared with a backup plan. Many schools do not even have a computer lab, and any use of such technology may be restricted to small groups of students working on a limited number of classroom computers.

Current research points favorably toward the use of technology for exploration of numerical, algebraic, and geometric problems. For example, Laborde (1999) found that using dynamic geometry software allows students to solve problems that are impossible when only pencil and paper are used. The results of the study showed that students engaged in computer tasks that generated "intriguing visual phenomena that [were] not expected by students" (p. 300). Of course, when students are surprised by the results of an experiment, they are *motivated* due to interest and curiosity and will work harder on the task.

As a teacher who is the victim of a double-booked computer lab, you might find yourself asking, "Is the use of computers necessary to teach this lesson?" Although you will often hear some people emphasize the importance of cooperative learning and the use of technology, others do not agree. Some believe that manipulatives are always helpful, and others believe that lecture methods work best. The National Research Council (2000) provided an analogy in response to these arguments: It does not make sense to ask a carpenter whether a hammer or a screwdriver is the best tool because the choice of a tool depends on its purpose and context. A carpenter knows that it is appropriate to hit a nail with a hammer and to insert a screw with a screwdriver. In the same way, teaching strategies should be selected based on the needs of the students. Some geometry lessons are more effectively taught using a compass and straightedge, whereas others require a calculator or computer. No teaching method or tool used in a lesson is inherently "good" or "bad"; the choice depends on the intent of the lesson. So, whether the teacher in this scenario chooses to split the time in the lab, to teach another lesson, or to teach the lesson using a chalkboard will ultimately be decided by the context of the situation and what the teacher intended the students to gain from the experience.

Experimental and Descriptive Research

Some research in mathematics education is an attempt to prove that one teaching or assessment method is better than another, as was described earlier in the calculator example, and is referred to as **experimental research**. This type of research has its roots in agriculture: A farmer would apply a particular brand of fertilizer to one field and not to another to see whether the field with the treatment produced a heartier crop. In a study conducted by Whitman (1976), it was reported that students who learned to use the "cover-up" method of solving equations along with traditional symbol manipulation outperformed their peers who were only taught to manipulate the symbols with pencil and paper. A teacher can translate such a study into practice when deciding how to approach equation solving in a middle school classroom or in an algebra course.

In making educational decisions, it is often helpful to simply have a base of information. A study that is undertaken for the purpose of generating statistics and information for discussion, but not necessarily for comparison purposes, is **descriptive research**. The following are some examples of the results of descriptive studies: Stiggins (1988) found that teachers spend 20 to 30 percent of their work time designing and implementing assessments of student progress. In 1999, 91 percent of U.S. eighth graders' schools surveyed in the TIMSS–R study reported having access to the Internet, whereas only 41 percent of their international peers had network access (Gonzales et al., 2000). Another TIMSS report showed that 83 percent of the eighth graders in the United States were in classes of between 1 and 30 students, whereas 93 percent of the students in Korea were enrolled in classes with 41 or more students (Beaton et al., 1997). It is interesting to note, however, that Korean eighth graders significantly outscored U.S. eighth graders on the TIMSS achievement tests. The information presented in this paragraph is not intended as a foundation for arguing one position over another; it is purely a description of what is going on in the schools. Descriptive research frequently results from surveys or interviews and serves to inform the education community about the status of some program or situation.

Table 1 provides a generalized summary of the types of research that can be conducted in education and illustrates the way in which research can take on several different formats, depending on the intent of the researchers.

Table 1 Comparison of Research Methods

	Experimental	Descriptive
Quantitative	Experimental study conducted by comparing quantitative data	Descriptive study containing quantitative data
Qualitative	Experimental study conducted by comparing qualitative data	Descriptive study containing qualitative data

Most experimental studies are quantitative, and much of the descriptive research in mathematics education is qualitative. Any study can contain both quantitative and qualitative elements. In fact, some of the most powerful research conclusions can be drawn when, for example, test results of student performance can be extended through a series of open-ended interviews with students. Interviews often allow researchers to probe students' thinking more deeply than they could using a written test and to quote student comments in the research results.

Research in mathematics education serves as either a catalyst for change or an affirmation of current practices. After several studies have been conducted and patterns about teaching and learning begin to emerge, an educational theory is often formulated. We will now explore some of the theories on teaching and learning that have evolved over many years of educational research and address the question of how a student actually learns mathematics.

Learning Theories in Mathematics

Think back to some of the teachers you have had in your educational career. Were any of them extremely knowledgeable in the area of mathematics but ineffective in the classroom, leaving you or others in the class confused? Perhaps you spent the semester or the year feeling as though the teacher was teaching mathematics but not teaching the students. Clearly, there is a difference! It has been said that the best mathematics teachers are those who have not only an understanding of the content but also a firm grasp of how mathematics is learned—teachers who are knowledgeable about theories of child development and can appreciate how students learn and become able to do mathematics. Richard Skemp, who wrote a popular book entitled *The Psychology of Learning Mathematics* (1971), stated that "problems of learning and teaching are psychological problems, and before we can make much improvement in the teaching of mathematics we need to know more about how it is learned" (p. 14). His comments are backed by research that suggests that teaching teachers to reflect on how their students think can have a significant effect on student achievement (an example is the discussion on cognitively guided instruction by Fennema and Franke, 1992).

Skemp also stated that "the learning of mathematics (is) . . . very dependent on good teaching. Now, to know mathematics is one thing, and to be able to teach it—to communicate it to those at a lower conceptual level is quite another" (Skemp, 1971, p. 36). A study conducted by Ball (1990), for example, described the difficulty that elementary and secondary preservice teachers had in representing the problem $1\frac{3}{4} \div \frac{1}{2}$ in a form that promoted understanding of the process. Although the undergraduates could "do" the problem, only about half of the secondary mathematics majors (and none of the elementary preservice teachers) could put the problem into a form that helped students understand what they were doing. Using this same division problem, another study showed that 10 out of 23 elementary school mathematics teachers (most of whom were considered above average teachers) were able to find the correct answer. Furthermore, only 1 of these 23 teachers was able to generate a reasonable word problem that used the calculation (Ma, 1999). (An example of a word problem might be, "How many half-pound burgers can you make out of $1\frac{3}{4}$ pounds of hamburger?") So, as we think about teaching secondary and middle school students, it is important to focus on how they think and develop as learners of mathematics. Let's look at some current learning theories that are influencing reform in mathematics education. We begin by tracing the development of general learning psychology and then elaborate on the theories of Jerome Bruner and the van Hieles as well as the constructivist model, which have their most direct application in the teaching and learning of mathematics.

Development of Learning Theory

Traditional classroom teaching, which features the teacher providing examples and the students taking notes, is primarily driven by the **behaviorist theory** of psychology—the predominant learning theory in the United States from 1920 until 1970 (Hofstetter, 1997). In classical behavioral psychology, the belief is that learning can be controlled by the application of external rewards and punishments. B.F. Skinner (1938), probably the best known of the behaviorists, described, for example, how a dog could be taught to sit by giving it food treats. In a similar manner, people can learn to perform various tasks by the consequences of their actions. So, if we wanted students to learn to convert fractions to decimals, we would show them the procedure, have them practice it over and over, and then give them a test. A high score on the test (and perhaps a sticker next to the student's name) would constitute a reward, and a low score would be a form of punishment. As a result, students would work harder to perfect the skill in order to earn a reward. In this sense, the student will have been "conditioned" to act in a particular manner. One major shortcoming of this model is that Skinner paid much less attention to internal stimulation than does modern psychology. His model was more about modifying behavior in humans and animals than accounting for the complex mental (cognitive) processes that occur in the brain when people learn (Hofstetter, 1997).

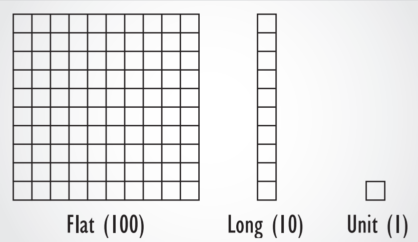
In a more contemporary view, psychologists such as Piaget (1969) held that people learn best when they can experiment and invent their own generalizations rather than simply being "told" what to do or how to think by a teacher. Similarly, Lev Vygotsky (1896–1934), a Russian-born psychologist, held that students actively construct their own knowledge (Santrock, 2004). Vygotsky is famous for defining the **zone of proximal development**, which is "the range of tasks that are too difficult for children to master alone but can be mastered with guidance and assistance from adults" (Santrock, 2004, p. 51). Vygotsky stated that the zone is between what the student already understands and what he or she is capable of comprehending through conversations with a more knowledgeable person (Steele, 2001; Vygotsky, 1978). As a result, learning depends on social interaction with others, whether that be the teacher or another peer. Riddle and Dabbagh (1999) noted that teaching strategies such as *scaffolding* (the teacher asking a series of structured questions that lead the students to higher levels of understanding) and *peer teaching* are natural applications of Vygotsky's theory.

Subsequently, Howard Gardner (1983, 1991) suggested a theory of multiple intelligences. Intelligence, he argued, can be defined among seven different areas: logical–mathematical, linguistic, spatial, musical, bodily–kinesthetic, and inter/intra-personal intelligences (Brualdi, 1996). In the traditional classroom, mathematics teachers have addressed primarily—if not exclusively—the logical–mathematical intelligence. But using Gardner's learning theory, we recognize that students have other strengths that can be incorporated in planning and teaching lessons. For example, the use of visual models in teaching can appeal to students' spatial intelligence, whereas the use of cooperative learning groups can tap into their interpersonal intelligence. Getting students to physically move around the room to collect data or to conduct a problem-solving activity may be a way to engage their bodily–kinesthetic intelligence. Gardner's theory spotlights what we have intuitively known for a long time—that to appeal to the needs of all students, teaching mathematics needs to move beyond lecturing and note taking.

Bruner's Stages of Representation

Another cognitive theory of learning, **Bruner's stages of representation**, was formulated by Jerome Bruner. Bruner, who was born in 1915 and received his doctorate in 1941 from Harvard, theorized that learning passes through three stages of representation—enactive, iconic, and symbolic. His theory has led to the extensive use of hands-on materials—manipulatives—in mathematics classrooms. We can illustrate these three stages of cognitive development with an example of combining similar terms in algebra.

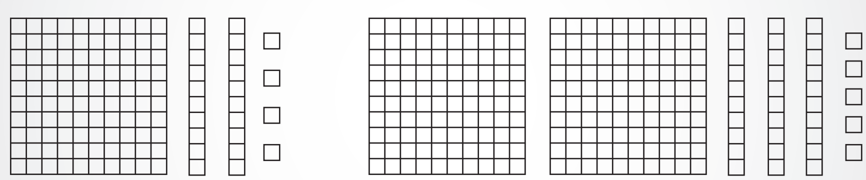
Figure 1 Base Ten Block Representations of Whole Numbers



In the primary grades, children often use a manipulative known as **base ten blocks** for exploring basic operations and place value. A set of base ten blocks (see Figure 1) is made up of cubes that measure 1 cm on a side; rods, often called *longs,* that are 10 cm × 1 cm × 1 cm; and *flats,* which are 10 cm × 10 cm squares with a depth of 1 cm. Each unit cube can represent a "1," whereas the long is a "10," and the flat is a "100."

When children are asked to add $124 + 235$, they can represent the numbers with base ten blocks, as shown in Figure 2:

Figure 2 Representation of $124 + 235$ with Base Ten Blocks



The child can readily see that there are 3 flats, 5 longs, and 9 unit cubes, so the sum of 124 and 235 must be 359. No pencil-and-paper computation is needed to do the problem (remember that for early elementary children this is still a problem, not an exercise!), and the design of the blocks makes it intuitively obvious to children which digits to add, as they simply combine the blocks that have the same shape and size.

Older students can allow the 10 × 10 flat to represent "1," which makes the long a representation for "0.1," and the smallest block, which is one-tenth of a long, becomes "0.01." Using this model, students can think about what it means to add $1.4 + 3.58$ without being taught rules, such as to "line up the decimal places"—a rule that many students (and adults) follow but do not understand. Figure 3 shows how easy it is for students to group together blocks of the same size to find that the sum of $1.4 + 3.58$ is 4.98.