Pearson New International Edition

Discrete-Time Signal Processing

Alan V. Oppenheim   Ronald W. Schafer

Third Edition

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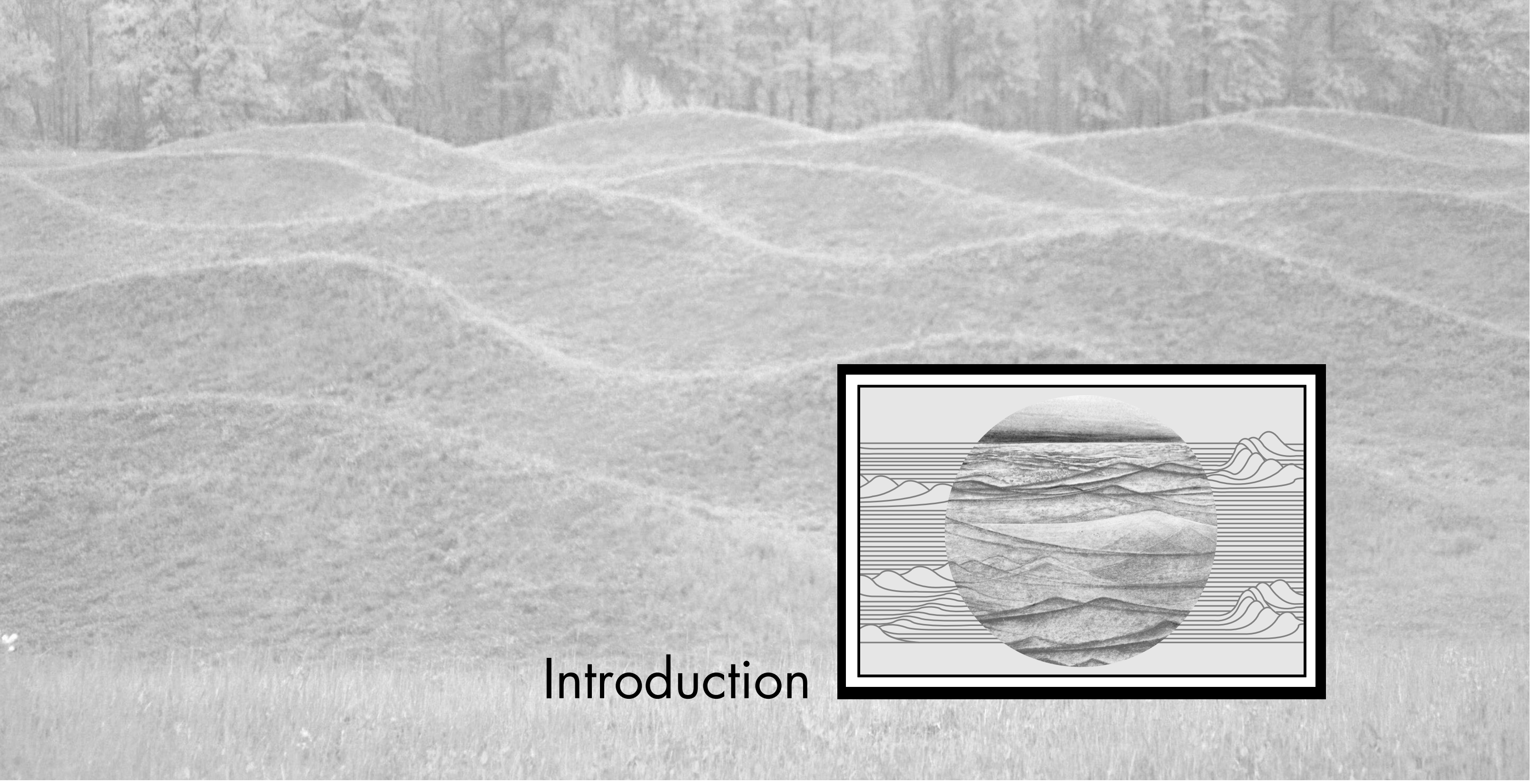

Introduction

The rich history and future promise of signal processing derive from a strong synergy between increasingly sophisticated applications, new theoretical developments and constantly emerging new hardware architectures and platforms. Signal processing applications span an immense set of disciplines that include entertainment, communications, space exploration, medicine, archaeology, geophysics, just to name a few. Signal processing algorithms and hardware are prevalent in a wide range of systems, from highly specialized military systems and industrial applications to low-cost, high-volume consumer electronics. Although we routinely take for granted the extraordinary performance of multimedia systems, such as high definition video, high fidelity audio, and interactive games, these systems have always relied heavily on state-of-the-art signal processing. Sophisticated digital signal processors are at the core of all modern cell phones. MPEG audio and video and JPEG[1] image data compression standards rely heavily on signal processing principles and techniques. High-density data storage devices and new solid-state memories rely increasingly on the use of signal processing to provide consistency and robustness to otherwise fragile technologies. As we look to the future, it is clear that the role of signal processing is expanding, driven in part by the convergence of communications, computers, and signal processing in both the consumer arena and in advanced industrial and government applications.

The growing number of applications and demand for increasingly sophisticated algorithms go hand-in-hand with the rapid development of device technology for implementing signal processing systems. By some estimates, even with impending limitations

[1]The acronyms MPEG and JPEG are the terms used in even casual conversation for referring to the standards developed by the "Moving Picture Expert Group (MPEG)" and the "Joint Photographic Expert Group (JPEG)" of the "International Organization for Standardization (ISO)."

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on Moore's Law, the processing capability of both special-purpose signal processing microprocessors and personal computers is likely to increase by several orders of magnitude over the next 10 years. Clearly, the importance and role of signal processing will continue to expand at an accelerating rate well into the future.

Signal processing deals with the representation, transformation, and manipulation of signals and the information the signals contain. For example, we may wish to separate two or more signals that have been combined by some operation, such as addition, multiplication, or convolution, or we may want to enhance some signal component or estimate some parameter of a signal model. In communications systems, it is generally necessary to do preprocessing such as modulation, signal conditioning, and compression prior to transmission over a communications channel, and then to carry out postprocessing at the receiver to recover a facsimile of the original signal. Prior to the 1960s, the technology for such signal processing was almost exclusively continuous-time analog technology.[2] A continual and major shift to digital technologies has resulted from the rapid evolution of digital computers and microprocessors and low-cost chips for analog to digital (A/D) and digital to analog (D/A) conversion. These developments in technology have been reinforced by many important theoretical developments, such as the fast Fourier transform (FFT) algorithm, parametric signal modeling, multirate techniques, polyphase filter implementation, and new ways of representing signals, such as with wavelet expansions. As just one example of this shift, analog radio communication systems are evolving into reconfigurable "software radios" that are implemented almost exclusively with digital computation.

Discrete-time signal processing is based on processing of numeric sequences indexed on integer variables rather than functions of a continuous independent variable. In digital signal processing (DSP), signals are represented by sequences of finite-precision numbers, and processing is implemented using digital computation. The more general term *discrete-time signal processing* includes digital signal processing as a special case but also includes the possibility that sequences of samples (sampled data) could be processed with other discrete-time technologies. Often the distinction between the terms discrete-time signal processing and digital signal processing is of minor importance, since both are concerned with discrete-time signals. This is particularly true when high-precision computation is employed. Although there are many examples in which signals to be processed are inherently discrete-time sequences, most applications involve the use of discrete-time technology for processing signals that originate as continuous-time signals. In this case, a continuous-time signal is typically converted into a sequence of samples, i.e., a discrete-time signal. Indeed, one of the most important spurs to widespread application of digital signal processing was the development of low-cost A/D, D/A conversion chips based on differential quantization with noise shaping. After discrete-time processing, the output sequence is converted back to a continuous-time signal. Real-time operation is often required or desirable for such systems. As computer speeds have increased, discrete-time processing of continuous-time signals in real time has become commonplace in communication systems, radar and sonar, speech and video coding and enhancement, biomedical engineering, and many

[2]In a general context, we shall refer to the independent variable as "time," even though in specific contexts, the independent variable may take on any of a broad range of possible dimensions. Consequently, continuous time and discrete time should be thought of as generic terms referring to a continuous independent variable and a discrete independent variable, respectively.

other areas of application. Non-real-time applications are also common. The compact disc player and MP3 player are examples of asymmetric systems in which an input signal is processed only once. The initial processing may occur in real time, slower than real time, or even faster than real time. The processed form of the input is stored (on the compact disc or in a solid state memory), and final processing for reconstructing the audio signal is carried out in real time when the output is played back for listening. The compact disc and MP3 recording and playback systems rely on many signal processing concepts.

Financial Engineering represents another rapidly emerging field which incorporates many signal processing concepts and techniques. Effective modeling, prediction and filtering of economic data can result in significant gains in economic performance and stability. Portfolio investment managers, for example, are relying increasingly on using sophisticated signal processing since even a very small increase in signal predictability or signal-to-noise ratio (SNR) can result in significant gain in performance.

Another important area of signal processing is *signal interpretation*. In such contexts, the objective of the processing is to obtain a characterization of the input signal. For example, in a speech recognition or understanding system, the objective is to interpret the input signal or extract information from it. Typically, such a system will apply digital pre-processing (filtering, parameter estimation, and so on) followed by a pattern recognition system to produce a symbolic representation, such as a phonemic transcription of the speech. This symbolic output can, in turn, be the input to a symbolic processing system, such as a rules-based expert system, to provide the final signal interpretation.

Still another relatively new category of signal processing involves the symbolic manipulation of signal processing expressions. This type of processing is potentially useful in signal processing workstations and for the computer-aided design of signal processing systems. In this class of processing, signals and systems are represented and manipulated as abstract data objects. Object-oriented programming languages provide a convenient environment for manipulating signals, systems, and signal processing expressions without explicitly evaluating the data sequences. The sophistication of systems designed to do signal expression processing is directly influenced by the incorporation of fundamental signal processing concepts, theorems, and properties, such as those that form the basis for this book. For example, a signal processing environment that incorporates the property that convolution in the time domain corresponds to multiplication in the frequency domain can explore a variety of rearrangements of filtering structures, including those involving the direct use of the discrete Fourier transform (DFT) and the FFT algorithm. Similarly, environments that incorporate the relationship between sampling rate and aliasing can make effective use of decimation and interpolation strategies for filter implementation. Similar ideas are currently being explored for implementing signal processing in network environments. In this type of environment, data can potentially be tagged with a high-level description of the processing to be done, and the details of the implementation can be based dynamically on the resources available on the network.

Many concepts and design techniques are now incorporated into the structure of sophisticated software systems such as MATLAB, Simulink, Mathematica, and LabVIEW. In many cases where discrete-time signals are acquired and stored in computers, these tools allow extremely sophisticated signal processing operations to be formed from basic functions. In such cases, it is not generally necessary to know the details of the underlying algorithm that implements the computation of an operation like the FFT, but nevertheless it is essential to understand what is computed and how it should be interpreted. In other words, a good understanding of the concepts considered in this text is essential for intelligent use of the signal processing software tools that are now widely available.

Signal processing problems are not confined, of course, to one-dimensional signals. Although there are some fundamental differences in the theories for one-dimensional and multidimensional signal processing, much of the material that we discuss in this text has a direct counterpart in multidimensional systems. The theory of multidimensional digital signal processing is presented in detail in a variety of references including Dudgeon and Mersereau (1984), Lim (1989), and Bracewell (1994).[3] Many image processing applications require the use of two-dimensional signal processing techniques. This is the case in such areas as video coding, medical imaging, enhancement and analysis of aerial photographs, analysis of satellite weather photos, and enhancement of video transmissions from lunar and deep-space probes. Applications of multidimensional digital signal processing to image processing are discussed, for example, in Macovski (1983), Castleman (1996), Jain (1989), Bovic (ed.) (2005), Woods (2006), Gonzalez and Woods (2007), and Pratt (2007). Seismic data analysis as required in oil exploration, earthquake measurement, and nuclear test monitoring also uses multidimensional signal processing techniques. Seismic applications are discussed in, for example, Robinson and Treitel (1980) and Robinson and Durrani (1985).

Multidimensional signal processing is only one of many advanced and specialized topics that build on signal-processing fundamentals. Spectral analysis based on the use of the DFT and the use of signal modeling is another particularly rich and important aspect of signal processing. High resolution spectrum analysis methods also are based on representing the signal to be analyzed as the response of a discrete-time linear time-invariant (LTI) filter to either an impulse or to white noise. Spectral analysis is achieved by estimating the parameters (e.g., the difference equation coefficients) of the system and then evaluating the magnitude squared of the frequency response of the model filter. Detailed discussions of spectrum analysis can be found in the texts by Kay (1988), Marple (1987), Therrien (1992), Hayes (1996) and Stoica and Moses (2005).

Signal modeling also plays an important role in data compression and coding, and here again, the fundamentals of difference equations provide the basis for understanding many of these techniques. For example, one class of signal coding techniques, referred to as linear predictive coding (LPC), exploits the notion that if a signal is the response of a certain class of discrete-time filters, the signal value at any time index is a linear function of (and thus linearly predictable from) previous values. Consequently, efficient signal representations can be obtained by estimating these prediction parameters and using them along with the prediction error to represent the signal. The signal can then be regenerated when needed from the model parameters. This class of signal

[3]Authors names and dates are used to refer to books and papers listed in the Bibliography at the end of this chapter.

coding techniques has been particularly effective in speech coding and is described in considerable detail in Jayant and Noll (1984), Markel and Gray (1976), Rabiner and Schafer (1978) and Quatieri (2002).

Another advanced topic of considerable importance is adaptive signal processing. Adaptive systems represent a particular class of time-varying and, in some sense, non-linear systems with broad application and with established and effective techniques for their design and analysis. Again, many of these techniques build from the fundamentals of discrete-time signal processing. Details of adaptive signal processing are given by Widrow and Stearns (1985), Haykin (2002) and Sayed (2008).

These represent only a few of the many advanced topics that extend from the content covered in this text. Others include advanced and specialized filter design procedures, a variety of specialized algorithms for evaluation of the Fourier transform, specialized filter structures, and various advanced multirate signal processing techniques, including wavelet transforms. (See Burrus, Gopinath, and Guo (1997), Vaidyanathan (1993) and Vetterli and Kovačević (1995) for introductions to these topics.)

It has often been said that the purpose of a fundamental textbook should be to uncover, rather than cover, a subject. We have been guided by this philosophy. There is a rich variety of both challenging theory and compelling applications to be uncovered by those who diligently prepare themselves with a study of the fundamentals of DSP.

HISTORIC PERSPECTIVE

Discrete-time signal processing has advanced in uneven steps over time. Looking back at the development of the field of discrete-time signal processing provides a valuable perspective on fundamentals that will remain central to the field for a long time to come. Since the invention of calculus in the 17th century, scientists and engineers have developed models to represent physical phenomena in terms of functions of continuous variables and differential equations. However, numeric techniques have been used to solve these equations when analytical solutions are not possible. Indeed, Newton used finite-difference methods that are special cases of some of the discrete-time systems that we present in this text. Mathematicians of the 18th century, such as Euler, Bernoulli, and Lagrange, developed methods for numeric integration and interpolation of functions of a continuous variable. Interesting historic research by Heideman, Johnson, and Burrus (1984) showed that Gauss discovered the fundamental principle of the FFT as early as 1805—even before the publication of Fourier's treatise on harmonic series representation of functions.

Until the early 1950s, signal processing as we have defined it was typically carried out with analog systems implemented with electronic circuits or even with mechanical devices. Even though digital computers were becoming available in business environments and in scientific laboratories, they were expensive and had relatively limited capabilities. About that time, the need for more sophisticated signal processing in some application areas created considerable interest in discrete-time signal processing. One of the first uses of digital computers in DSP was in geophysical exploration, where relatively low frequency seismic signals could be digitized and recorded on magnetic tape for later processing. This type of signal processing could not generally be done in real time; minutes or even hours of computer time were often required to process only seconds of data. Even so, the flexibility of the digital computer and the potential payoffs made this alternative extremely inviting.

Also in the 1950s, the use of digital computers in signal processing arose in a different way. Because of the flexibility of digital computers, it was often useful to simulate a signal processing system on a digital computer before implementing it in analog hardware. In this way, a new signal processing algorithm or system could be studied in a flexible experimental environment before committing economic and engineering resources to constructing it. Typical examples of such simulations were the vocoder simulations carried out at Massachusetts Institute of Technology (MIT) Lincoln Laboratory and Bell Telephone Laboratories. In the implementation of an analog channel vocoder, for example, the filter characteristics affected the perceived quality of the coded speech signal in ways that were difficult to quantify objectively. Through computer simulations, these filter characteristics could be adjusted and the perceived quality of a speech coding system evaluated prior to construction of the analog equipment.

In all of these examples of signal processing using digital computers, the computer offered tremendous advantages in flexibility. However, the processing could not be done in real time. Consequently, the prevalent attitude up to the late 1960s was that the digital computer was being used to *approximate*, or *simulate*, an analog signal processing system. In keeping with that style, early work on digital filtering concentrated on ways in which a filter could be programmed on a digital computer so that with A/D conversion of the signal, followed by digital filtering, followed by D/A conversion, the overall system approximated a good analog filter. The notion that digital systems might, in fact, be practical for the actual real-time implementation of signal processing in speech communication, radar processing, or any of a variety of other applications seemed, even at the most optimistic times, to be highly speculative. Speed, cost, and size were, of course, three of the important factors in favor of the use of analog components.

As signals were being processed on digital computers, researchers had a natural tendency to experiment with increasingly sophisticated signal processing algorithms. Some of these algorithms grew out of the flexibility of the digital computer and had no apparent practical implementation in analog equipment. Thus, many of these algorithms were treated as interesting, but somewhat impractical, ideas. However, the development of such signal processing algorithms made the notion of all-digital implementation of signal processing systems even more tempting. Active work began on the investigation of digital vocoders, digital spectrum analyzers, and other all-digital systems, with the hope that eventually, such systems would become practical.

The evolution of a new point of view toward discrete-time signal processing was further accelerated by the disclosure by Cooley and Tukey (1965) of an efficient class of algorithms for computation of Fourier transforms known collectively as the FFT. The FFT was significant for several reasons. Many signal processing algorithms that had been developed on digital computers required processing times several orders of magnitude greater than real time. Often, this was because spectrum analysis was an important component of the signal processing and no efficient means were available for implementing it. The FFT reduced the computation time of the Fourier transform by orders of magnitude, permitting the implementation of increasingly sophisticated signal processing algorithms with processing times that allowed interactive experimentation with the system. Furthermore, with the realization that the FFT algorithms might, in fact, be implementable with special-purpose digital hardware, many signal processing algorithms that previously had appeared to be impractical began to appear feasible.

Another important implication of the FFT was that it was an inherently discrete-time concept. It was directed toward the computation of the Fourier transform of a discrete-time signal or sequence and involved a set of properties and mathematics that was exact in the discrete-time domain—it was not simply an approximation to a continuous-time Fourier transform. This had the effect of stimulating a reformulation of many signal processing concepts and algorithms in terms of discrete-time mathematics, and these techniques then formed an exact set of relationships in the discrete-time domain. Following this shift away from the notion that signal processing on a digital computer was merely an approximation to analog signal processing techniques, there emerged the current view that discrete-time signal processing is an important field of investigation in its own right.

Another major development in the history of discrete-time signal processing occurred in the field of microelectronics. The invention and subsequent proliferation of the microprocessor paved the way for low-cost implementations of discrete-time signal processing systems. Although the first microprocessors were too slow to implement most discrete-time systems in real time except at very low sampling rates, by the mid-1980s, integrated circuit technology had advanced to a level that permitted the implementation of very fast fixed-point and floating-point microcomputers with architectures specially designed for implementing discrete-time signal processing algorithms. With this technology came, for the first time, the possibility of widespread application of discrete-time signal processing techniques. The rapid pace of development in microelectronics also significantly impacted the development of signal processing algorithms in other ways. For example, in the early days of real-time digital signal processing devices, memory was relatively costly and one of the important metrics in developing signal processing algorithms was the efficient use of memory. Digital memory is now so inexpensive that many algorithms purposely incorporate more memory than is absolutely required so that the power requirements of the processor are reduced. Another area in which technology limitations posed a significant barrier to widespread deployment of DSP was in conversion of signals from analog to discrete-time (digital) form. The first widely available A/D and D/A converters were stand-alone devices costing thousands of dollars. By combining digital signal processing theory with microelectronic technology, oversampled A/D and D/A converters costing a few dollars or less have enabled a myriad of real-time applications.

In a similar way, minimizing the number of arithmetic operations, such as multiplies or floating point additions, is now less essential, since multicore processors often have several multipliers available and it becomes increasingly important to reduce communication between cores, even if it then requires more multiplications. In a multicore environment, for example, direct computation of the DFT (or the use of the Goertzel algorithm) is more "efficient" than the use of an FFT algorithm since, although many more multiplications are required, communication requirements are significantly reduced because the processing can be more efficiently distributed among multiple processors or cores. More broadly, the restructuring of algorithms and the development of new ones to exploit the opportunity for more parallel and distributed processing is becoming a significant new direction in the development of signal processing algorithms.

FUTURE PROMISE

Microelectronics engineers continue to strive for increased circuit densities and production yields, and as a result, the complexity and sophistication of microelectronic systems continually increase. The complexity, speed, and capability of DSP chips have grown exponentially since the early 1980s and show no sign of slowing down. As wafer-scale integration techniques become highly developed, very complex discrete-time signal processing systems will be implemented with low cost, miniature size, and low power consumption. Furthermore, technologies such as microelectronic mechanical systems (MEMS) promise to produce many types of tiny sensors whose outputs will need to be processed using DSP techniques that operate on distributed arrays of sensor inputs. Consequently, the importance of discrete-time signal processing will continue to increase, and the future development of the field promises to be even more dramatic than the course of development that we have just described.

Discrete-time signal processing techniques have already promoted revolutionary advances in some fields of application. A notable example is in the area of telecommunications, where discrete-time signal processing techniques, microelectronic technology, and fiber optic transmission have combined to change the nature of communication systems in truly revolutionary ways. A similar impact can be expected in many other areas. Indeed, signal processing has always been, and will always be, a field that thrives on new applications. The needs of a new field of application can sometimes be filled by knowledge adapted from other applications, but frequently, new application needs stimulate new algorithms and new hardware systems to implement those algorithms. Early on, applications to seismology, radar, and communication provided the context for developing many of the core signal processing techniques. Certainly, signal processing will remain at the heart of applications in national defense, entertainment, communication, and medical care and diagnosis. Recently, we have seen applications of signal processing techniques in new areas as disparate as finance and DNA sequence analysis.

Although it is difficult to predict where other new applications will arise, there is no doubt that they will be obvious to those who are prepared to recognize them. The key to being ready to solve new signal processing problems is, and has always been, a thorough grounding in the fundamental mathematics of signals and systems and in the associated design and processing algorithms. While discrete-time signal processing is a dynamic, steadily growing field, its fundamentals are well formulated, and it is extremely valuable to learn them well. Our goal is to uncover the fundamentals of the field by providing a coherent treatment of the theory of discrete-time linear systems, filtering, sampling, discrete-time Fourier analysis, and signal modeling. We hope to provide the reader with the knowledge necessary for an appreciation of the wide scope of applications for discrete-time signal processing and a foundation for contributing to future developments in this exciting field.

BIBLIOGRAPHY

Bovic, A., ed., *Handbook of Image and Video Processing*, 2nd ed., Academic Press, Burlington, MA, 2005.

Bracewell, R. N., *Two-Dimensional Imaging*, Prentice Hall, New York, NY, 1994.

Burrus, C. S., Gopinath, R. A., and Guo, H., *Introduction to Wavelets and Wavelet Transforms: A Primer*, Prentice Hall, 1997.

Castleman, K. R., *Digital Image Processing*, 2nd ed., Prentice Hall, Upper Saddle River, NJ, 1996.

Cooley, J. W., and Tukey, J. W., "An Algorithm for the Machine Computation of Complex Fourier Series," *Mathematics of Computation*, Vol. 19, pp. 297–301, Apr. 1965.

Dudgeon, D. E., and Mersereau, R. M., *Two-Dimensional Digital Signal Processing*, Prentice Hall, Englewood Cliffs, NJ, 1984.

Gonzalez, R. C., and Woods, R. E., *Digital Image Processing*, Wiley, 2007.

Hayes, M., *Statistical Digital Signal Processing and Modeling*, Wiley, New York, NY, 1996.

Haykin, S., *Adaptive Filter Theory*, 4th ed., Prentice Hall, 2002.

Heideman, M. T., Johnson, D. H., and Burrus, C. S., "Gauss and the History of the Fast Fourier Transform," *IEEE ASSP Magazine*, Vol. 1, No. 4, pp. 14–21, Oct. 1984.

Jain, A. K., *Fundamentals of Digital Image Processing*, Prentice Hall, Englewood Cliffs, NJ, 1989.

Jayant, N. S., and Noll, P., *Digital Coding of Waveforms*, Prentice Hall, Englewood Cliffs, NJ, 1984.

Kay, S. M., *Modern Spectral Estimation Theory and Application*, Prentice Hall, Englewood Cliffs, NJ, 1988.

Lim, J. S., *Two-Dimensional Digital Signal Processing*, Prentice Hall, Englewood Cliffs, NJ, 1989.

Macovski, A., *Medical Image Processing*, Prentice Hall, Englewood Cliffs, NJ, 1983.

Markel, J. D., and Gray, A. H., Jr., *Linear Prediction of Speech*, Springer-Verlag, New York, NY, 1976.

Marple, S. L., *Digital Spectral Analysis with Applications*, Prentice Hall, Englewood Cliffs, NJ, 1987.

Pratt, W., *Digital Image Processing*, 4th ed., Wiley, New York, NY, 2007.

Quatieri, T. F., *Discrete-Time Speech Signal Processing: Principles and Practice*, Prentice Hall, Englewood Cliffs, NJ, 2002.

Rabiner, L. R., and Schafer, R. W., *Digital Processing of Speech Signals*, Prentice Hall, Englewood Cliffs, NJ, 1978.

Robinson, E. A., and Durrani, T. S., *Geophysical Signal Processing*, Prentice Hall, Englewood Cliffs, NJ, 1985.

Robinson, E. A., and Treitel, S., *Geophysical Signal Analysis*, Prentice Hall, Englewood Cliffs, NJ, 1980.

Sayed, A., *Adaptive Filters*, Wiley, Hoboken, NJ, 2008.

Stoica, P., and Moses, R., *Spectral Analysis of Signals*, Pearson Prentice Hall, Upper Saddle River, NJ, 2005.

Therrien, C. W., *Discrete Random Signals and Statistical Signal Processing*, Prentice Hall, Englewood Cliffs, NJ, 1992.

Vaidyanathan, P. P., *Multirate Systems and Filter Banks*, Prentice Hall, Englewood Cliffs, NJ, 1993.

Vetterli, M., and Kovačević, J., *Wavelets and Subband Coding*, Prentice Hall, Englewood Cliffs, NJ, 1995.

Widrow, B., and Stearns, S. D., *Adaptive Signal Processing*, Prentice Hall, Englewood Cliffs, NJ, 1985.

Woods, J. W., *Multidimensional Signal, Image, and Video Processing and Coding*, Academic Press, 2006.

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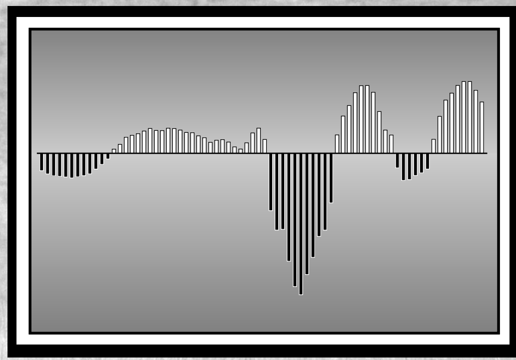
Discrete-Time Signals and Systems



0 INTRODUCTION

The term *signal* is generally applied to something that conveys information. Signals may, for example, convey information about the state or behavior of a physical system. As another class of examples, signals are synthesized for the purpose of communicating information between humans or between humans and machines. Although signals can be represented in many ways, in all cases, the information is contained in some pattern of variations. Signals are represented mathematically as functions of one or more independent variables. For example, a speech signal is represented mathematically as a function of time, and a photographic image is represented as a brightness function of two spatial variables. A common convention is to refer to the independent variable of the mathematical representation of a signal as time, although in specific examples, the independent variable may not in fact correspond to time.

The independent variable in the mathematical representation of a signal may be either continuous or discrete. *Continuous-time signals* are defined along a continuum of time and are thus represented by a continuous independent variable. Continuous-time signals are often referred to as *analog signals*. *Discrete-time signals* are defined at discrete times, and thus, the independent variable has discrete values; that is, discrete-time signals are represented as sequences of numbers. Signals such as speech or images may have either a continuous- or a discrete-variable representation, and if certain conditions hold, these representations are entirely equivalent. Besides the independent variables being either continuous or discrete, the signal amplitude may be either continuous or discrete. *Digital signals* are those for which both time and amplitude are discrete.

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Signal-processing systems may be classified along the same lines as signals. That is, continuous-time systems are systems for which both the input and the output are continuous-time signals, and discrete-time systems are those for which both the input and the output are discrete-time signals. Similarly, a digital system is a system for which both the input and the output are digital signals. Digital signal processing, then, deals with the transformation of signals that are discrete in both amplitude and time. The principal focus of this book is on discrete-time—rather than digital—signals and systems. However, the theory of discrete-time signals and systems is also exceedingly useful for digital signals and systems, particularly if the signal amplitudes are finely quantized.

In this chapter, we present the basic definitions, establish notation, and develop and review the basic concepts associated with discrete-time signals and systems. The presentation of this material assumes that the reader has had previous exposure to some of this material, perhaps with a different emphasis and notation. Thus, this chapter is primarily intended to provide a common foundation for more advanced material.

In Section 1, we discuss the representation of discrete-time signals as sequences and describe the basic sequences such as the unit impulse, the unit step, and complex exponential, which play a central role in characterizing discrete-time systems and form building blocks for more general sequences. In Section 2, the representation, basic properties, and simple examples of discrete-time systems are presented. Sections 3 and 4 focus on the important class of linear time-invariant (LTI) systems and their time-domain representation through the convolution sum, with Section 5 considering the specific class of LTI systems represented by linear, constant–coefficient difference equations. Section 6 develops the frequency domain representation of discrete-time systems through the concept of complex exponentials as eigenfunctions, and Sections 7, 8, and 9 develop and explore the Fourier transform representation of discrete-time signals as a linear combination of complex exponentials. Section 10 provides a brief introduction to discrete-time random signals.

1 DISCRETE-TIME SIGNALS

Discrete-time signals are represented mathematically as sequences of numbers. A sequence of numbers x, in which the n^{th} number in the sequence is denoted $x[n]$,[1] is formally written as

$$x = \{x[n]\}, \qquad -\infty < n < \infty, \tag{1}$$

where n is an integer. In a practical setting, such sequences can often arise from periodic sampling of an analog (i.e., continuous-time) signal $x_a(t)$. In that case, the numeric value of the n^{th} number in the sequence is equal to the value of the analog signal, $x_a(t)$, at time nT: i.e.,

$$x[n] = x_a(nT), \qquad -\infty < n < \infty. \tag{2}$$

The quantity T is the *sampling period,* and its reciprocal is the *sampling frequency*. Although sequences do not always arise from sampling analog waveforms, it is convenient to refer to $x[n]$ as the "n^{th} sample" of the sequence. Also, although, strictly

[1] Note that we use [ ] to enclose the independent variable of discrete-variable functions, and we use ( ) to enclose the independent variable of continuous-variable functions.

speaking, $x[n]$ denotes the n^{th} number in the sequence, the notation of Eq. (1) is often unnecessarily cumbersome, and it is convenient and unambiguous to refer to "the sequence $x[n]$" when we mean the entire sequence, just as we referred to "the analog signal $x_a(t)$." We depict discrete-time signals (i.e., sequences) graphically, as shown in Figure 1. Although the abscissa is drawn as a continuous line, it is important to recognize that $x[n]$ is defined only for integer values of n. It is not correct to think of $x[n]$ as being zero when n is not an integer; $x[n]$ is simply undefined for noninteger values of n.

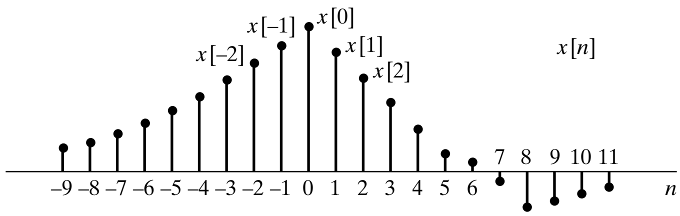


Figure 1 Graphic representation of a discrete-time signal.

As an example of a sequence obtained by sampling, Figure 2(a) shows a segment of a speech signal corresponding to acoustic pressure variation as a function of time, and Figure 2(b) presents a sequence of samples of the speech signal. Although the original speech signal is defined at all values of time t, the sequence contains information about the signal only at discrete instants. The sampling theorem guarantees that the original

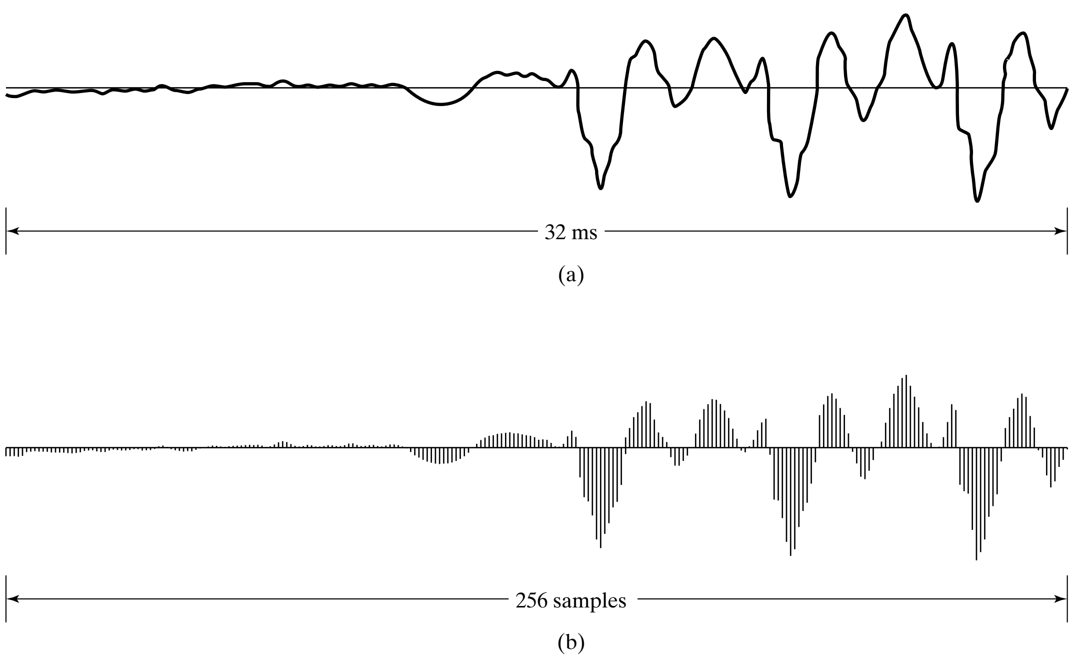


Figure 2 (a) Segment of a continuous-time speech signal $x_a(t)$. (b) Sequence of samples $x[n] = x_a(nT)$ obtained from the signal in part (a) with $T = 125\ \mu$s.

signal can be reconstructed as accurately as desired from a corresponding sequence of samples if the samples are taken frequently enough.

In discussing the theory of discrete-time signals and systems, several basic sequences are of particular importance. These sequences are shown in Figure 3 and will be discussed next.

The *unit sample sequence* (Figure 3a) is defined as the sequence

$$\delta[n] = \begin{cases} 0, & n \neq 0, \\ 1, & n = 0. \end{cases} \tag{3}$$

The unit sample sequence plays the same role for discrete-time signals and systems that the unit impulse function (Dirac delta function) does for continuous-time signals and systems. For convenience, we often refer to the unit sample sequence as a discrete-time impulse or simply as an impulse. It is important to note that a discrete-time impulse does not suffer from the mathematic complications of the continuous-time impulse; its definition in Eq. (3) is simple and precise.

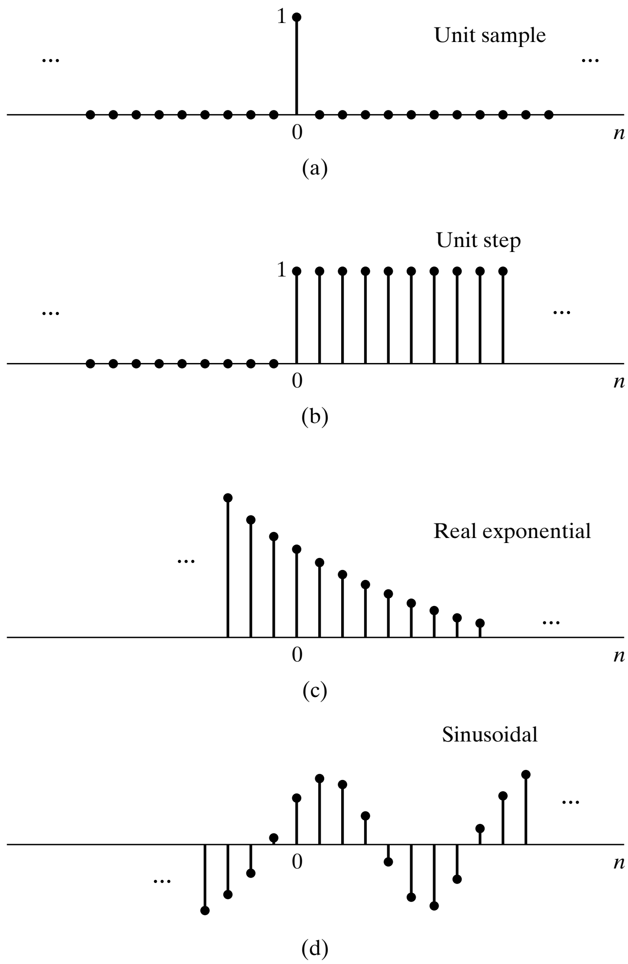


Figure 3 Some basic sequences. The sequences shown play important roles in the analysis and representation of discrete-time signals and systems.

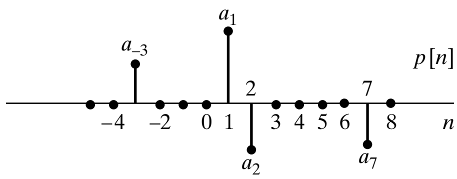


Figure 4 Example of a sequence to be represented as a sum of scaled, delayed impulses.

One of the important aspects of the impulse sequence is that an arbitrary sequence can be represented as a sum of scaled, delayed impulses. For example, the sequence $p[n]$ in Figure 4 can be expressed as

$$p[n] = a_{-3}\delta[n+3] + a_1\delta[n-1] + a_2\delta[n-2] + a_7\delta[n-7]. \tag{4}$$

More generally, any sequence can be expressed as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]. \tag{5}$$

We will make specific use of Eq. (5) in discussing the representation of discrete-time linear systems.

The unit step sequence (Figure 3b) is defined as

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases} \tag{6}$$

The unit step is related to the unit impulse by

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]; \tag{7}$$

that is, the value of the unit step sequence at (time) index n is equal to the accumulated sum of the value at index n and all previous values of the impulse sequence. An alternative representation of the unit step in terms of the impulse is obtained by interpreting the unit step in Figure 3(b) in terms of a sum of delayed impulses, as in Eq. (5). In this case, the nonzero values are all unity, so

$$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \cdots \tag{8a}$$

or

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]. \tag{8b}$$

As yet another alternative, the impulse sequence can be expressed as the first backward difference of the unit step sequence, i.e.,

$$\delta[n] = u[n] - u[n-1]. \tag{9}$$

Exponential sequences are another important class of basic signals. The general form of an exponential sequence is

$$x[n] = A\alpha^n. \tag{10}$$

If A and α are real numbers, then the sequence is real. If $0 < \alpha < 1$ and A is positive, then the sequence values are positive and decrease with increasing n, as in Figure 3(c).

For $-1 < \alpha < 0$, the sequence values alternate in sign but again decrease in magnitude with increasing n. If $|\alpha| > 1$, then the sequence grows in magnitude as n increases.

The exponential sequence $A\,\alpha^n$ with α complex has real and imaginary parts that are exponentially weighted sinusoids. Specifically, if $\alpha = |\alpha|e^{j\omega_0}$ and $A = |A|e^{j\phi}$, the sequence $A\,\alpha^n$ can be expressed in any of the following ways:

$$\begin{aligned} x[n] = A\,\alpha^n &= |A|e^{j\phi}|\alpha|^n e^{j\omega_0 n} \\ &= |A|\,|\alpha|^n e^{j(\omega_0 n+\phi)} \\ &= |A|\,|\alpha|^n \cos(\omega_0 n + \phi) + j|A|\,|\alpha|^n \sin(\omega_0 n + \phi). \end{aligned} \tag{11}$$

The sequence oscillates with an exponentially growing envelope if $|\alpha| > 1$ or with an exponentially decaying envelope if $|\alpha| < 1$. (As a simple example, consider the case $\omega_0 = \pi$.)

When $|\alpha| = 1$, the sequence has the form

$$x[n] = |A|e^{j(\omega_0 n+\phi)} = |A|\cos(\omega_0 n + \phi) + j|A|\sin(\omega_0 n + \phi); \tag{12}$$

that is, the real and imaginary parts of $e^{j\omega_0 n}$ vary sinusoidally with n. By analogy with the continuous-time case, the quantity ω_0 is called the *frequency* of the complex sinusoid or complex exponential, and ϕ is called the *phase*. However, since n is a dimensionless integer, the dimension of ω_0 is radians. If we wish to maintain a closer analogy with the continuous-time case, we can specify the units of ω_0 to be radians per sample and the units of n to be samples.

The fact that n is always an integer in Eq. (12) leads to some important differences between the properties of discrete-time and continuous-time complex exponential sequences and sinusoidal sequences. Consider, for example, a frequency $(\omega_0 + 2\pi)$. In this case,

$$\begin{aligned} x[n] &= A\,e^{j(\omega_0+2\pi)n} \\ &= A\,e^{j\omega_0 n}e^{j2\pi n} = A\,e^{j\omega_0 n}. \end{aligned} \tag{13}$$

Generally, complex exponential sequences with frequencies $(\omega_0 + 2\pi r)$, where r is an integer, are indistinguishable from one another. An identical statement holds for sinusoidal sequences. Specifically, it is easily verified that

$$\begin{aligned} x[n] &= A\,\cos[(\omega_0 + 2\pi r)n + \phi] \\ &= A\,\cos(\omega_0 n + \phi). \end{aligned} \tag{14}$$

When discussing complex exponential signals of the form $x[n] = A\,e^{j\omega_0 n}$ or real sinusoidal signals of the form $x[n] = A\,\cos(\omega_0 n + \phi)$, we need only consider frequencies in an interval of length 2π. Typically, we will choose either $-\pi < \omega_0 \le \pi$ or $0 \le \omega_0 < 2\pi$.

Another important difference between continuous-time and discrete-time complex exponentials and sinusoids concerns their periodicity in n. In the continuous-time case, a sinusoidal signal and a complex exponential signal are both periodic in time with the period equal to 2π divided by the frequency. In the discrete-time case, a periodic sequence is a sequence for which

$$x[n] = x[n + N], \qquad \text{for all } n, \tag{15}$$

where the period N is necessarily an integer. If this condition for periodicity is tested for the discrete-time sinusoid, then

$$A\cos(\omega_0 n + \phi) = A\cos(\omega_0 n + \omega_0 N + \phi), \tag{16}$$

which requires that

$$\omega_0 N = 2\pi k, \tag{17}$$

where k is an integer. A similar statement holds for the complex exponential sequence $Ce^{j\omega_0 n}$; that is, periodicity with period N requires that

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n}, \tag{18}$$

which is true only for $\omega_0 N = 2\pi k$, as in Eq. (17). Consequently, complex exponential and sinusoidal sequences are not necessarily periodic in n with period $(2\pi/\omega_0)$ and, depending on the value of ω_0, may not be periodic at all.

Example 1 Periodic and Aperiodic Discrete-Time Sinusoids

Consider the signal $x_1[n] = \cos(\pi n/4)$. This signal has a period of $N = 8$. To show this, note that $x[n+8] = \cos(\pi(n+8)/4) = \cos(\pi n/4 + 2\pi) = \cos(\pi n/4) = x[n]$, satisfying the definition of a discrete-time periodic signal. Contrary to continuous-time sinusoids, increasing the value of ω_0 for a discrete-time sinusoid does not necessarily decrease the period of the signal. Consider the discrete-time sinusoid $x_2[n] = \cos(3\pi n/8)$, which has a higher frequency than $x_1[n]$. However, $x_2[n]$ is not periodic with period 8, since $x_2[n+8] = \cos(3\pi(n+8)/8) = \cos(3\pi n/8 + 3\pi) = -x_2[n]$. Using an argument analogous to the one for $x_1[n]$, we can show that $x_2[n]$ has a period of $N = 16$. Thus, increasing the value of $\omega_0 = 2\pi/8$ to $\omega_0 = 3\pi/8$ also increases the period of the signal. This occurs because discrete-time signals are defined only for integer indices n.

The integer restriction on n results in some sinusoidal signals not being periodic at all. For example, there is no integer N such that the signal $x_3[n] = \cos(n)$ satisfies the condition $x_3[n+N] = x_3[n]$ for all n. These and other properties of discrete-time sinusoids that run counter to their continuous-time counterparts are caused by the limitation of the time index n to integers for discrete-time signals and systems.

When we combine the condition of Eq. (17) with our previous observation that ω_0 and $(\omega_0 + 2\pi r)$ are indistinguishable frequencies, it becomes clear that there are N distinguishable frequencies for which the corresponding sequences are periodic with period N. One set of frequencies is $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$. These properties of complex exponential and sinusoidal sequences are basic to both the theory and the design of computational algorithms for discrete-time Fourier analysis.

Related to the preceding discussion is the fact that the interpretation of high and low frequencies is somewhat different for continuous-time and discrete-time sinusoidal and complex exponential signals. For a continuous-time sinusoidal signal $x(t) = A\cos(\Omega_0 t + \phi)$, as Ω_0 increases, $x(t)$ oscillates progressively more rapidly. For the discrete-time sinusoidal signal $x[n] = A\cos(\omega_0 n + \phi)$, as ω_0 increases from $\omega_0 = 0$ toward $\omega_0 = \pi$, $x[n]$ oscillates progressively more rapidly. However, as ω_0 increases from $\omega_0 = \pi$ to $\omega_0 = 2\pi$, the oscillations become slower. This is illustrated in Figure 5. In

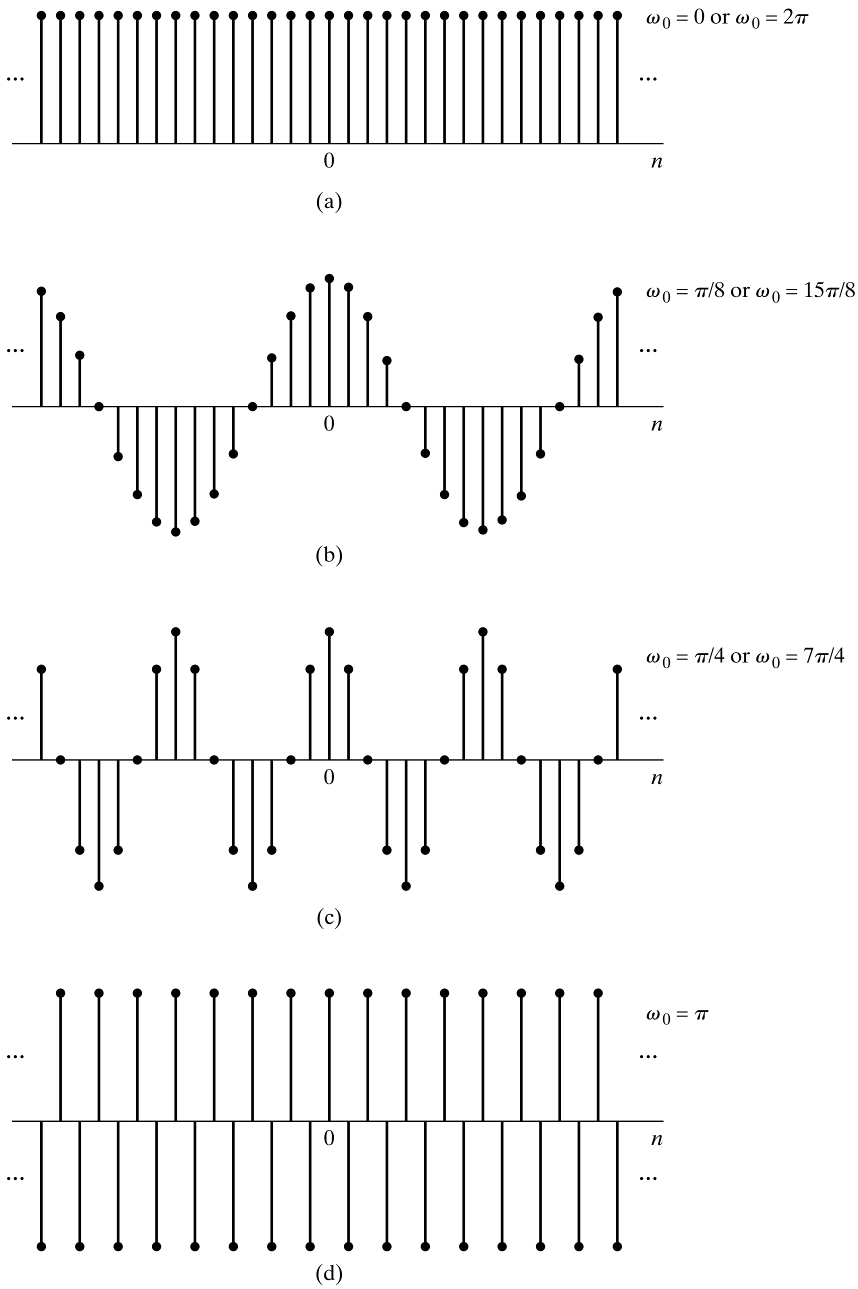


Figure 5 $\cos \omega_0 n$ for several different values of ω_0. As ω_0 increases from zero toward π (parts a–d), the sequence oscillates more rapidly. As ω_0 increases from π to 2π (parts d–a), the oscillations become slower.

fact, because of the periodicity in ω_0 of sinusoidal and complex exponential sequences, $\omega_0 = 2\pi$ is indistinguishable from $\omega_0 = 0$, and, more generally, frequencies around $\omega_0 = 2\pi$ are indistinguishable from frequencies around $\omega_0 = 0$. As a consequence, for sinusoidal and complex exponential signals, values of ω_0 in the vicinity of $\omega_0 = 2\pi k$ for any integer value of k are typically referred to as low frequencies (relatively slow oscillations), whereas values of ω_0 in the vicinity of $\omega_0 = (\pi + 2\pi k)$ for any integer value of k are typically referred to as high frequencies (relatively rapid oscillations).

2 DISCRETE-TIME SYSTEMS

A discrete-time system is defined mathematically as a transformation or operator that maps an input sequence with values $x[n]$ into an output sequence with values $y[n]$. This can be denoted as

$$y[n] = T\{x[n]\} \tag{19}$$

and is indicated pictorially in Figure 6. Equation (19) represents a rule or formula for computing the output sequence values from the input sequence values. It should be emphasized that the value of the output sequence at each value of the index n may depend on input samples $x[n]$ for all values of n, i.e., y at time n can depend on all or part of the entire sequence x. The following examples illustrate some simple and useful systems.

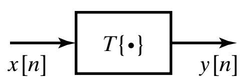


Figure 6 Representation of a discrete-time system, i.e., a transformation that maps an input sequence $x[n]$ into a unique output sequence $y[n]$.

Example 2 The Ideal Delay System

The ideal delay system is defined by the equation

$$y[n] = x[n - n_d], \qquad -\infty < n < \infty, \tag{20}$$

where n_d is a fixed positive integer representing the delay of the system. In other words, the ideal delay system shifts the input sequence to the right by n_d samples to form the output. If, in Eq. (20), n_d is a fixed negative integer, then the system would shift the input to the left by $|n_d|$ samples, corresponding to a time advance.

In the system of Example 2, only one sample of the input sequence is involved in determining a certain output sample. In the following example, this is not the case.

Example 3 Moving Average

The general moving-average system is defined by the equation

$$\begin{aligned} y[n] &= \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k] \\ &= \frac{1}{M_1 + M_2 + 1} \{x[n+M_1] + x[n+M_1-1] + \cdots + x[n] \\ &\quad + x[n-1] + \cdots + x[n-M_2]\}. \end{aligned} \tag{21}$$

This system computes the n^{th} sample of the output sequence as the average of $(M_1 + M_2 + 1)$ samples of the input sequence around the n^{th} sample. Figure 7 shows an input sequence plotted as a function of a dummy index k and the samples (solid dots) involved in the computation of the output sample $y[n]$ for $n = 7, M_1 = 0$, and $M_2 = 5$. The output sample $y[7]$ is equal to one-sixth of the sum of all the samples between the vertical dotted lines. To compute $y[8]$, both dotted lines would move one sample to the right.

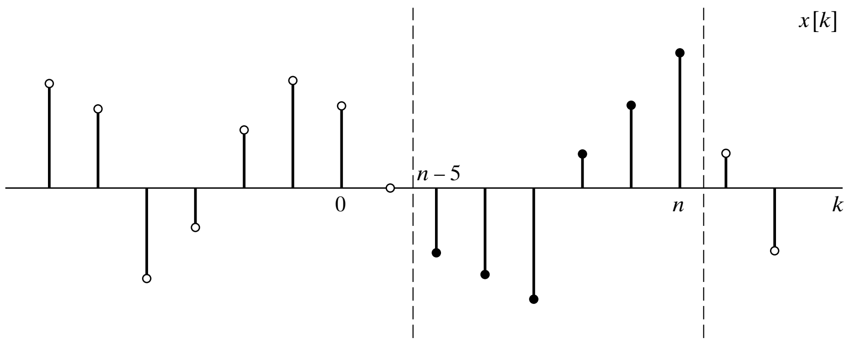


Figure 7 Sequence values involved in computing a moving average with $M_1 = 0$ and $M_2 = 5$.

Classes of systems are defined by placing constraints on the properties of the transformation $T\{\cdot\}$. Doing so often leads to very general mathematical representations, as we will see. Of particular importance are the system constraints and properties, discussed in Sections 2.1–2.5.

2.1 Memoryless Systems

A system is referred to as memoryless if the output $y[n]$ at every value of n depends only on the input $x[n]$ at the same value of n.

Example 4 A Memoryless System

An example of a memoryless system is a system for which $x[n]$ and $y[n]$ are related by

$$y[n] = (x[n])^2, \qquad \text{for each value of } n. \tag{22}$$

The system in Example 2 is not memoryless unless $n_d = 0$; in particular, that system is referred to as having "memory" whether n_d is positive (a time delay) or negative (a time advance). The moving average system in Example 3 is not memoryless unless $M_1 = M_2 = 0$.

2.2 Linear Systems

The class of linear systems is defined by the principle of superposition. If $y_1[n]$ and $y_2[n]$ are the responses of a system when $x_1[n]$ and $x_2[n]$ are the respective inputs, then the system is linear if and only if

$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n] \tag{23a}$$

and

$$T\{ax[n]\} = aT\{x[n]\} = ay[n], \tag{23b}$$

where a is an arbitrary constant. The first property is the *additivity property*, and the second the *homogeneity* or *scaling property*. These two properties together comprise the principle of superposition, stated as

$$T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\} \tag{24}$$

for arbitrary constants a and b. This equation can be generalized to the superposition of many inputs. Specifically, if

$$x[n] = \sum_k a_k x_k[n], \tag{25a}$$

then the output of a linear system will be

$$y[n] = \sum_k a_k y_k[n], \tag{25b}$$

where $y_k[n]$ is the system response to the input $x_k[n]$.

By using the definition of the principle of superposition, it is easily shown that the systems of Examples 2 and 3 are linear systems. (See Problem 39.) An example of a nonlinear system is the system in Example 4.

Example 5 The Accumulator System

The system defined by the input–output equation

$$y[n] = \sum_{k=-\infty}^{n} x[k] \tag{26}$$

is called the accumulator system, since the output at time n is the accumulation or sum of the present and all previous input samples. The accumulator system is a linear system. Since this may not be intuitively obvious, it is a useful exercise to go through the steps of more formally showing this. We begin by defining two arbitrary inputs $x_1[n]$ and $x_2[n]$ and their corresponding outputs

$$y_1[n] = \sum_{k=-\infty}^{n} x_1[k], \tag{27}$$

$$y_2[n] = \sum_{k=-\infty}^{n} x_2[k]. \tag{28}$$

When the input is $x_3[n] = ax_1[n] + bx_2[n]$, the superposition principle requires the output $y_3[n] = ay_1[n] + by_2[n]$ for all possible choices of a and b. We can show this by starting from Eq. (26):

$$y_3[n] = \sum_{k=-\infty}^{n} x_3[k], \tag{29}$$

$$= \sum_{k=-\infty}^{n} (ax_1[k] + bx_2[k]), \tag{30}$$

$$= a \sum_{k=-\infty}^{n} x_1[k] + b \sum_{k=-\infty}^{n} x_2[k], \tag{31}$$

$$= ay_1[n] + by_2[n]. \tag{32}$$

Thus, the accumulator system of Eq. (26) satisfies the superposition principle for all inputs and is therefore linear.

Example 6 A Nonlinear System

Consider the system defined by

$$w[n] = \log_{10}(|x[n]|). \tag{33}$$

This system is not linear. To prove this, we only need to find one counterexample—that is, one set of inputs and outputs which demonstrates that the system violates the superposition principle, Eq. (24). The inputs $x_1[n] = 1$ and $x_2[n] = 10$ are a counterexample. However, the output for $x_1[n] + x_2[n] = 11$ is

$$\log_{10}(1 + 10) = \log_{10}(11) \neq \log_{10}(1) + \log_{10}(10) = 1.$$

Also, the output for the first signal is $w_1[n] = 0$, whereas for the second, $w_2[n] = 1$. The scaling property of linear systems requires that, since $x_2[n] = 10x_1[n]$, if the system is linear, it must be true that $w_2[n] = 10w_1[n]$. Since this is not so for Eq. (33) for this set of inputs and outputs, the system is *not* linear.

2.3 Time-Invariant Systems

A time-invariant system (often referred to equivalently as a shift-invariant system) is a system for which a time shift or delay of the input sequence causes a corresponding shift in the output sequence. Specifically, suppose that a system transforms the input sequence with values $x[n]$ into the output sequence with values $y[n]$. Then, the system is said to be time invariant if, for all n_0, the input sequence with values $x_1[n] = x[n - n_0]$ produces the output sequence with values $y_1[n] = y[n - n_0]$.

As in the case of linearity, proving that a system is time invariant requires a general proof making no specific assumptions about the input signals. On the other hand, proving non-time invariance only requires a counter example to time invariance. All of the systems in Examples 2–6 are time invariant. The style of proof for time invariance is illustrated in Examples 7 and 8.

Example 7 The Accumulator as a Time-Invariant System

Consider the accumulator from Example 5. We define $x_1[n] = x[n - n_0]$. To show time invariance, we solve for both $y[n - n_0]$ and $y_1[n]$ and compare them to see whether they are equal. First,

$$y[n - n_0] = \sum_{k=-\infty}^{n-n_0} x[k]. \tag{34}$$

Next, we find

$$y_1[n] = \sum_{k=-\infty}^{n} x_1[k] \tag{35}$$

$$= \sum_{k=-\infty}^{n} x[k - n_0]. \tag{36}$$

Substituting the change of variables $k_1 = k - n_0$ into the summation gives

$$y_1[n] = \sum_{k_1=-\infty}^{n-n_0} x[k_1]. \tag{37}$$

Since the index k in Eq. (34) and the index k_1 in Eq. (37) are dummy indices of summation, and can have any label, Eqs. (34) and (37) are equal and therefore $y_1[n] = y[n - n_0]$. The accumulator is a time-invariant system.

The following example illustrates a system that is not time invariant.

Example 8 The Compressor System

The system defined by the relation

$$y[n] = x[Mn], \qquad -\infty < n < \infty, \tag{38}$$

with M a positive integer, is called a compressor. Specifically, it discards $(M - 1)$ samples out of M; i.e., it creates the output sequence by selecting every M^{th} sample. This system is not time invariant. We can show that it is not by considering the response $y_1[n]$ to the input $x_1[n] = x[n - n_0]$. For the system to be time invariant, the output of the system when the input is $x_1[n]$ must be equal to $y[n - n_0]$. The output $y_1[n]$ that results from the input $x_1[n]$ can be directly computed from Eq. (38) to be

$$y_1[n] = x_1[Mn] = x[Mn - n_0]. \tag{39}$$

Delaying the output $y[n]$ by n_0 samples yields

$$y[n - n_0] = x[M(n - n_0)]. \tag{40}$$

Comparing these two outputs, we see that $y[n - n_0]$ is not equal to $y_1[n]$ for all M and n_0, and therefore, the system is not time invariant.

It is also possible to prove that a system is not time invariant by finding a single counterexample that violates the time-invariance property. For instance, a counterexample for the compressor is the case when $M = 2$, $x[n] = \delta[n]$, and $x_1[n] = \delta[n - 1]$. For this choice of inputs and M, $y[n] = \delta[n]$, but $y_1[n] = 0$; thus, it is clear that $y_1[n] \neq y[n - 1]$ for this system.

2.4 Causality

A system is causal if, for every choice of n_0, the output sequence value at the index $n = n_0$ depends only on the input sequence values for $n \leq n_0$. This implies that if $x_1[n] = x_2[n]$ for $n \leq n_0$, then $y_1[n] = y_2[n]$ for $n \leq n_0$. That is, the system is *nonanticipative*. The system of Example 2 is causal for $n_d \geq 0$ and is noncausal for $n_d < 0$. The system of Example 3 is causal if $-M_1 \geq 0$ and $M_2 \geq 0$; otherwise it is noncausal. The system of Example 4 is causal, as is the accumulator of Example 5 and the nonlinear system in Example 6. However, the system of Example 8 is noncausal if $M > 1$, since $y[1] = x[M]$. Another noncausal system is given in the following example.

Example 9 The Forward and Backward Difference Systems

The system defined by the relationship

$$y[n] = x[n+1] - x[n] \tag{41}$$

is referred to as the *forward difference system*. This system is not causal, since the current value of the output depends on a future value of the input. The violation of causality can be demonstrated by considering the two inputs $x_1[n] = \delta[n-1]$ and $x_2[n] = 0$ and their corresponding outputs $y_1[n] = \delta[n] - \delta[n-1]$ and $y_2[n] = 0$ for all n. Note that $x_1[n] = x_2[n]$ for $n \leq 0$, so the definition of causality requires that $y_1[n] = y_2[n]$ for $n \leq 0$, which is clearly not the case for $n = 0$. Thus, by this counterexample, we have shown that the system is not causal.

The *backward difference system,* defined as

$$y[n] = x[n] - x[n-1], \tag{42}$$

has an output that depends only on the present and past values of the input. Because $y[n_0]$ depends only on $x[n_0]$ and $x[n_0 - 1]$, the system is causal by definition.

2.5 Stability

A number of somewhat different definitions are commonly used for stability of a system. Throughout this text, we specifically use bounded-input bounded-output stability.

A system is stable in the bounded-input, bounded-output (BIBO) sense if and only if every bounded input sequence produces a bounded output sequence. The input $x[n]$ is bounded if there exists a fixed positive finite value B_x such that

$$|x[n]| \leq B_x < \infty, \qquad \text{for all } n. \tag{43}$$

Stability requires that, for every bounded input, there exists a fixed positive finite value B_y such that

$$|y[n]| \leq B_y < \infty, \qquad \text{for all } n. \tag{44}$$

It is important to emphasize that the properties we have defined in this section are properties of *systems,* not of the inputs to a system. That is, we may be able to find inputs for which the properties hold, but the existence of the property for some inputs does not mean that the system has the property. For the system to have the property, it must hold for *all* inputs. For example, an unstable system may have some bounded inputs for which the output is bounded, but for the system to have the property of stability, it must be true that for *all* bounded inputs, the output is bounded. If we can find just one input for which the system property does not hold, then we have shown that the system does *not* have that property. The following example illustrates the testing of stability for several of the systems that we have defined.

Example 10 Testing for Stability or Instability

The system of Example 4 is stable. To see this, assume that the input $x[n]$ is bounded such that $|x[n]| \leq B_x$ for all n. Then $|y[n]| = |x[n]|^2 \leq B_x^2$. Thus, we can choose $B_y = B_x^2$ and prove that $y[n]$ is bounded.

Likewise, we can see that the system defined in Example 6 is unstable, since $y[n] = \log_{10}(|x[n]|) = -\infty$ for any values of the time index n at which $x[n] = 0$, even though the output will be bounded for any input samples that are not equal to zero.

The accumulator, as defined in Example 5 by Eq. (26), is also not stable. For example, consider the case when $x[n] = u[n]$, which is clearly bounded by $B_x = 1$. For this input, the output of the accumulator is

$$y[n] = \sum_{k=-\infty}^{n} u[k] \tag{45}$$

$$= \begin{cases} 0, & n < 0, \\ (n+1), & n \geq 0. \end{cases} \tag{46}$$

There is no finite choice for B_y such that $(n+1) \leq B_y < \infty$ for all n; thus, the system is unstable.

Using similar arguments, it can be shown that the systems in Examples 2, 3, 8, and 9 are all stable.

3 LTI SYSTEMS

As in continuous time, a particularly important class of discrete-time systems consists of those that are both linear and time invariant. These two properties in combination lead to especially convenient representations for such systems. Most important, this class of systems has significant signal-processing applications. The class of linear systems is defined by the principle of superposition in Eq. (24). If the linearity property is combined with the representation of a general sequence as a linear combination of delayed impulses as in Eq. (5), it follows that a linear system can be completely characterized by its impulse response. Specifically, let $h_k[n]$ be the response of the system to the input $\delta[n-k]$, an impulse occurring at $n = k$. Then, using Eq. (5) to represent the input, it follows that

$$y[n] = T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}, \tag{47}$$

and the principle of superposition in Eq. (24), we can write

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]h_k[n]. \tag{48}$$

According to Eq. (48), the system response to any input can be expressed in terms of the responses of the system to the sequences $\delta[n-k]$. If only linearity is imposed, then $h_k[n]$ will depend on both n and k, in which case the computational usefulness of Eq. (48) is somewhat limited. We obtain a more useful result if we impose the additional constraint of time invariance.

The property of time invariance implies that if $h[n]$ is the response to $\delta[n]$, then the response to $\delta[n-k]$ is $h[n-k]$. With this additional constraint, Eq. (48) becomes

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k], \qquad \text{for all } n. \tag{49}$$

As a consequence of Eq. (49), an LTI system is completely characterized by its impulse response $h[n]$ in the sense that, given the sequences $x[n]$ and $h[n]$ for all n, it is possible to use Eq. (49) to compute each sample of the output sequence $y[n]$.

Equation (49) is referred to as the *convolution sum*, and we represent this by the operator notation

$$y[n] = x[n] * h[n]. \tag{50}$$

The operation of discrete-time convolution takes two sequences $x[n]$ and $h[n]$ and produces a third sequence $y[n]$. Equation (49) expresses each sample of the output sequence in terms of all of the samples of the input and impulse response sequences.

The notation of Eq. (50) for the operation of convolution as shorthand for Eq. (49) is convenient and compact but needs to be used with caution. The basic definition of the convolution of two sequences is embodied in Eq. (49) and any use of the shorthand form in Eq. (50) should always be referred back to Eq. (49). For example, consider $y[n-n_0]$. From Eq. (49) we see that

$$y[n-n_0] = \sum_{k=-\infty}^{\infty} x[k]h[n-n_0-k] \tag{51}$$

or in short hand notation

$$y[n-n_0] = x[n] * h[n-n_0] \tag{52}$$

Substituting $(n-n_0)$ for n in Eq. (49) leads to the correct result and conclusion, but blindly trying the same substitution in Eq. (50) does not. In fact, $x[n-n_0] * h[n-n_0]$ results in $y[n-2n_0]$.

The derivation of Eq. (49) suggests the interpretation that the input sample at $n = k$, represented as $x[k]\delta[n-k]$, is transformed by the system into an output sequence $x[k]h[n-k]$, for $-\infty < n < \infty$, and that, for each k, these sequences are superimposed (summed) to form the overall output sequence. This interpretation is illustrated in Figure 8, which shows an impulse response, a simple input sequence having three nonzero samples, the individual outputs due to each sample, and the composite output due to all

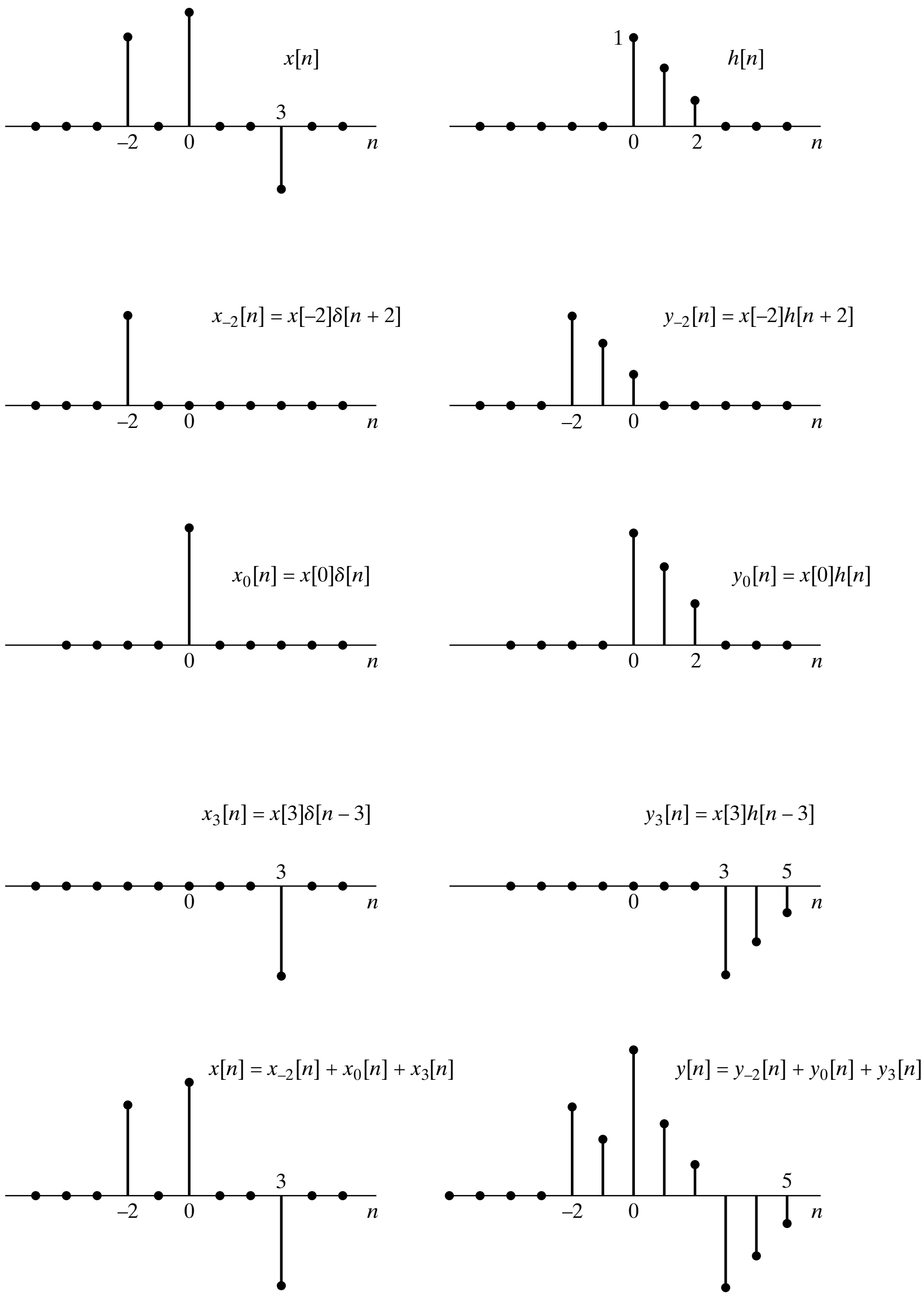


Figure 8 Representation of the output of an LTI system as the superposition of responses to individual samples of the input.

the samples in the input sequence. Specifically, $x[n]$ can be decomposed as the sum of the three sequences $x[-2]\delta[n+2]$, $x[0]\delta[n]$, and $x[3]\delta[n-3]$ representing the three nonzero values in the sequence $x[n]$. The sequences $x[-2]h[n+2]$, $x[0]h[n]$, and $x[3]h[n-3]$ are the system responses to $x[-2]\delta[n+2]$, $x[0]\delta[n]$, and $x[3]\delta[n-3]$, respectively. The response to $x[n]$ is then the sum of these three individual responses.

Although the convolution-sum expression is analogous to the convolution integral of continuous-time linear system theory, the convolution sum should not be thought of as an approximation to the convolution integral. The convolution integral is mainly a tool of mathematical analysis in continuous-time linear system theory; we will see that the convolution sum, in addition to its analytical importance, often serves as an explicit realization of a discrete-time linear system. Thus, it is important to gain some insight into the properties of the convolution sum in actual calculations.

The preceding interpretation of Eq. (49) emphasizes that the convolution sum is a direct result of linearity and time invariance. However, a slightly different way of looking at Eq. (49) leads to a particularly useful computational interpretation. When viewed as a formula for computing a single value of the output sequence, Eq. (49) dictates that $y[n]$ (i.e., the n^{th} value of the output) is obtained by multiplying the input sequence (expressed as a function of k) by the sequence whose values are $h[n-k]$, $-\infty < k < \infty$ for any fixed value of n, and then summing all the values of the products $x[k]h[n-k]$, with k a counting index in the summation process. Therefore, the operation of convolving two sequences involves doing the computation specified by Eq. (49) for each value of n, thus generating the complete output sequence $y[n]$, $-\infty < n < \infty$. The key to carrying out the computations of Eq. (49) to obtain $y[n]$ is understanding how to form the sequence $h[n-k]$, $-\infty < k < \infty$, for all values of n that are of interest. To this end, it is useful to note that

$$h[n-k] = h[-(k-n)]. \tag{53}$$

To illustrate the interpretation of Eq. (53), suppose $h[k]$ is the sequence shown in Figure 9(a) and we wish to find $h[n-k] = h[-(k-n)]$. Define $h_1[k]$ to be $h[-k]$, which is shown in Figure 9(b). Next, define $h_2[k]$ to be $h_1[k]$, delayed, by n samples on the k axis, i.e., $h_2[k] = h_1[k-n]$. Figure 9(c) shows the sequence that results from delaying the sequence in Figure 9(b) by n samples. Using the relationship between $h_1[k]$ and $h[k]$, we can show that $h_2[k] = h_1[k-n] = h[-(k-n)] = h[n-k]$, and thus, the bottom figure is the desired signal. To summarize, to compute $h[n-k]$ from $h[k]$, we first reverse $h[k]$ in time about $k=0$ and then delay the time-reversed signal by n samples.

To implement discrete-time convolution, the two sequences $x[k]$ and $h[n-k]$ are multiplied together sample by sample for $-\infty < k < \infty$, and the products are summed to compute the output sample $y[n]$. To obtain another output sample, the origin of the sequence $h[-k]$ is shifted to the new sample position, and the process is repeated. This computational procedure applies whether the computations are carried out numerically on sampled data or analytically with sequences for which the sample values have simple formulas. The following example illustrates discrete-time convolution for the latter case.

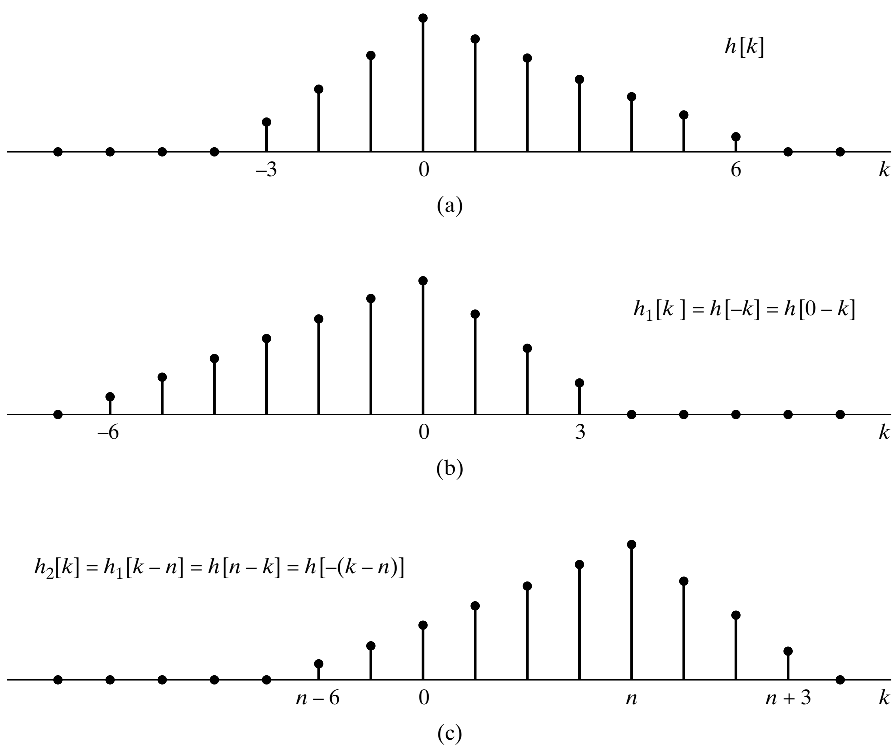


Figure 9 Forming the sequence $h[n-k]$. (a) The sequence $h[k]$ as a function of k. (b) The sequence $h[-k]$ as a function of k. (c) The sequence $h[n-k] = h[-(k-n)]$ as a function of k for $n = 4$.

Example 11 Analytical Evaluation of the Convolution Sum

Consider a system with impulse response

$$h[n] = u[n] - u[n-N]$$
$$= \begin{cases} 1, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$$

The input is

$$x[n] = \begin{cases} a^n, & n \ge 0, \\ 0, & n < 0, \end{cases}$$

or equivalently,

$$x[n] = a^n u[n].$$

To find the output at a particular index n, we must form the sums over all k of the product $x[k]h[n-k]$. In this case, we can find formulas for $y[n]$ for different sets of values of n. To do this, it is helpful to sketch the sequences $x[k]$ and $h[n-k]$ as functions of k for different representative values of n. For example, Figure 10(a) shows the sequences $x[k]$ and $h[n-k]$, plotted for n a negative integer. Clearly, all

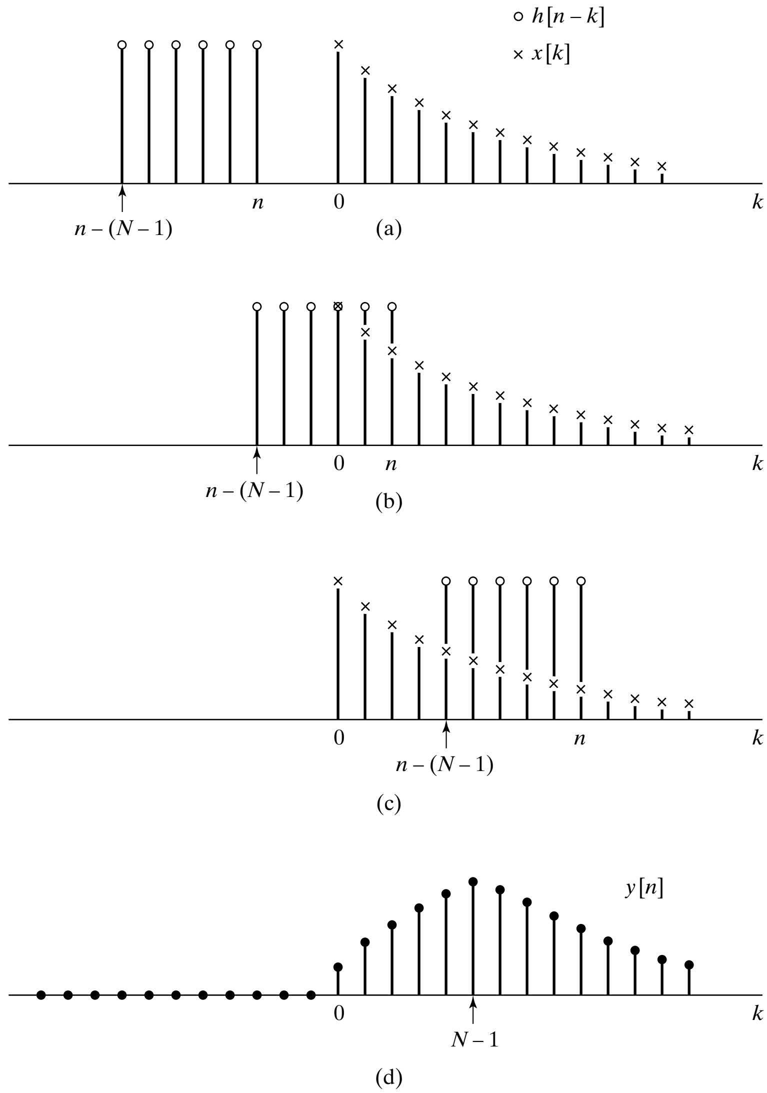


Figure 10 Sequence involved in computing a discrete convolution. (a)–(c) The sequences $x[k]$ and $h[n-k]$ as a function of k for different values of n. (Only nonzero samples are shown.) (d) Corresponding output sequence as a function of n.

negative values of n give a similar picture; i.e., the nonzero portions of the sequences $x[k]$ and $h[n-k]$ do not overlap, so

$$y[n] = 0, \qquad n < 0.$$

Figure 10(b) illustrates the two sequences when $0 \le n$ and $n - N + 1 \le 0$. These two conditions can be combined into the single condition $0 \le n \le N - 1$. By considering Figure 10(b), we see that since

$$x[k]h[n-k] = a^k, \qquad \text{for } 0 \le k \le n$$

when $0 \le n \le N - 1$.

it follows that

$$y[n] = \sum_{k=0}^{n} a^k, \qquad \text{for } 0 \le n \le N-1. \tag{54}$$

The limits on the sum can be seen directly from Figure 10(b). Equation (54) shows that $y[n]$ is the sum of $n+1$ terms of a geometric series in which the ratio of terms is a. This sum can be expressed in closed form using the general formula

$$\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}, \qquad N_2 \ge N_1. \tag{55}$$

Applying this formula to Eq. (54), we obtain

$$y[n] = \frac{1-a^{n+1}}{1-a}, \qquad 0 \le n \le N-1. \tag{56}$$

Finally, Figure 10(c) shows the two sequences when $0 < n - N + 1$ or $N - 1 < n$. As before,

$$x[k]h[n-k] = a^k, \qquad n - N + 1 \le k \le n,$$

but now the lower limit on the sum is $n - N + 1$, as seen in Figure 10(c). Thus,

$$y[n] = \sum_{k=n-N+1}^{n} a^k, \qquad \text{for } N-1 < n. \tag{57}$$

Using Eq. (55), we obtain

$$y[n] = \frac{a^{n-N+1} - a^{n+1}}{1-a},$$

or

$$y[n] = a^{n-N+1}\left(\frac{1-a^N}{1-a}\right). \tag{58}$$

Thus, because of the piecewise-exponential nature of both the input and the unit sample response, we have been able to obtain the following closed-form expression for $y[n]$ as a function of the index n:

$$y[n] = \begin{cases} 0, & n < 0, \\ \dfrac{1-a^{n+1}}{1-a}, & 0 \le n \le N-1, \\ a^{n-N+1}\left(\dfrac{1-a^N}{1-a}\right), & N-1 < n. \end{cases} \tag{59}$$

This sequence is shown in Figure 10(d).

Example 11 illustrates how the convolution sum can be computed analytically when the input and the impulse response are given by simple formulas. In such cases, the sums may have a compact form that may be derived using the formula for the sum of a geometric series or other "closed-form" formulas.[2] When no simple form is available,

[2]Such results are discussed, for example, in Grossman (1992) and Jolley (2004).

the convolution sum can still be evaluated numerically using the technique illustrated in Example 11 whenever the sums are finite, which will be the case if either the input sequence or the impulse response is of finite length, i.e., has a finite number of nonzero samples.

4 PROPERTIES OF LINEAR TIME-INVARIANT SYSTEMS

Since all LTI systems are described by the convolution sum of Eq. (49), the properties of this class of systems are defined by the properties of discrete-time convolution. Therefore, the impulse response is a complete characterization of the properties of a specific LTI system.

Some general properties of the class of LTI systems can be found by considering properties of the convolution operation.[3] For example, the convolution operation is commutative:

$$x[n] * h[n] = h[n] * x[n]. \tag{60}$$

This can be shown by applying a substitution of variables to the summation index in Eq. (49). Specifically, with $m = n - k$,

$$y[n] = \sum_{m=\infty}^{-\infty} x[n-m]h[m] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] = h[n] * x[n], \tag{61}$$

so the roles of $x[n]$ and $h[n]$ in the summation are interchanged. That is, the order of the sequences in a convolution operator is unimportant; hence, the system output is the same if the roles of the input and impulse response are reversed. Accordingly, an LTI system with input $x[n]$ and impulse response $h[n]$ will have the same output as an LTI system with input $h[n]$ and impulse response $x[n]$. The convolution operation also distributes over addition; i.e.,

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]. \tag{62}$$

This follows in a straightforward way from Eq. (49) and is a direct result of the linearity and commutativity of convolution. Equation (62) is represented pictorially in Figure 11, where Figure 11(a) represents the right-hand side of Eq. (62) and Figure 11(b) the left-hand side.

The convolution operation also satisfies the associative property, i.e.,

$$y[n] = (x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n]). \tag{63}$$

Also since the convolution operation is commutative, Eq. (63) is equivalent to

$$y[n] = x[n] * (h_2[n] * h_1[n]) = (x[n] * h_2[n]) * h_1[n]. \tag{64}$$

These equivalences are represented pictorially in Figure 12. Also, Eqs. (63) and (64) clearly imply that if two LTI systems with impulse responses $h_1[n]$ and $h_2[n]$ are cascaded in either order, the equivalent overall impulse response $h[n]$ is

$$h[n] = h_1[n] * h_2[n] = h_2[n] * h_1[n]. \tag{65}$$

[3]In our discussion below, we use the shorthand notation of Eq. (50) for the operation of convolution, but again emphasize that the properties of convolution are derived from the definition of Eq. (49).

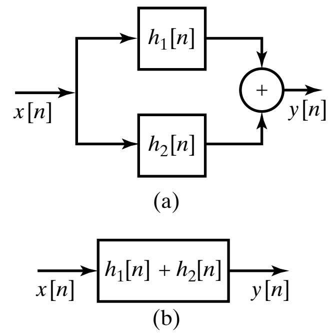


Figure 11 (a) Parallel combination of LTI systems. (b) An equivalent system.

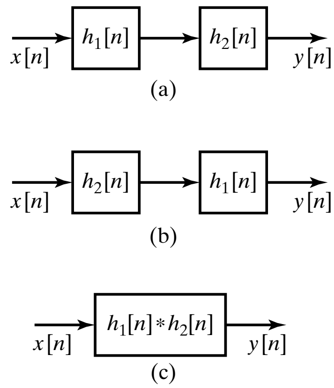


Figure 12 (a) Cascade combination of two LTI systems. (b) Equivalent cascade. (c) Single equivalent system.

In a parallel combination, the systems have the same input, and their outputs are summed to produce an overall output. It follows from the distributive property of convolution that the connection of two LTI systems in parallel is equivalent to a single system whose impulse response is the sum of the individual impulse responses; i.e.,

$$h[n] = h_1[n] + h_2[n]. \tag{66}$$

The constraints of linearity and time invariance define a class of systems with very special properties. Stability and causality represent additional properties, and it is often important to know whether an LTI system is stable and whether it is causal. Recall from Section 2.5 that a stable system is a system for which every bounded input produces a bounded output. LTI systems are stable if and only if the impulse response is absolutely summable, i.e., if

$$B_h = \sum_{k=-\infty}^{\infty} |h[k]| < \infty. \tag{67}$$

This can be shown as follows. From Eq. (61),

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \leq \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]|. \tag{68}$$

If $x[n]$ is bounded, so that

$$|x[n]| \leq B_x,$$

then substituting B_x for $|x[n-k]|$ can only strengthen the inequality. Hence,

$$|y[n]| \leq B_x B_h. \tag{69}$$

Thus, $y[n]$ is bounded if Eq. (67) holds; in other words, Eq. (67) is a sufficient condition for stability. To show that it is also a necessary condition, we must show that if $B_h = \infty$, then a bounded input can be found that will cause an unbounded output. Such an input is the sequence with values

$$x[n] = \begin{cases} \dfrac{h^*[-n]}{|h[-n]|}, & h[n] \neq 0, \\ 0, & h[n] = 0, \end{cases} \tag{70}$$

where $h^*[n]$ is the complex conjugate of $h[n]$. The sequence $x[n]$ is clearly bounded by unity. However, the value of the output at $n = 0$ is

$$y[0] = \sum_{k=-\infty}^{\infty} x[-k]h[k] = \sum_{k=-\infty}^{\infty} \frac{|h[k]|^2}{|h[k]|} = B_h. \tag{71}$$

Therefore, if $B_h = \infty$, it is possible for a bounded input sequence to produce an unbounded output sequence.

The class of causal systems was defined in Section 2.4 as comprising those systems for which the output $y[n_0]$ depends only on the input samples $x[n]$, for $n \leq n_0$. It follows from Eq. (49) or Eq. (61) that this definition implies the condition

$$h[n] = 0, \qquad n < 0, \tag{72}$$

for causality of LTI systems. (See Problem 69.) For this reason, it is sometimes convenient to refer to a sequence that is zero for $n < 0$ as a *causal sequence*, meaning that it could be the impulse response of a causal system.

To illustrate how the properties of LTI systems are reflected in the impulse response, let us consider again some of the systems defined in Examples 2–9. First, note that only the systems of Examples 2, 3, 5, and 9 are linear and time invariant. Although the impulse response of nonlinear or time-varying systems can be found by simply using an impulse input, it is generally of limited interest, since the convolution-sum formula and Eqs. (67) and (72), expressing stability and causality, do not apply to such systems.

First, let us determine the impulse responses of the systems in Examples 2, 3, 5, and 9. We can do this by simply computing the response of each system to $\delta[n]$, using the defining relationship for the system. The resulting impulse responses are as follows:

Ideal Delay (Example 2)

$$h[n] = \delta[n - n_d], \qquad n_d \text{ a positive fixed integer.} \tag{73}$$

Moving Average (Example 3)

$$\begin{aligned} h[n] &= \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} \delta[n-k] \\ &= \begin{cases} \dfrac{1}{M_1 + M_2 + 1}, & -M_1 \leq n \leq M_2, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{74}$$

Accumulator (Example 5)

$$h[n] = \sum_{k=-\infty}^{n} \delta[k] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0, \end{cases} = u[n]. \tag{75}$$

Forward Difference (Example 9)

$$h[n] = \delta[n+1] - \delta[n]. \tag{76}$$

Backward Difference (Example 9)

$$h[n] = \delta[n] - \delta[n-1]. \tag{77}$$

Given the impulse responses of these basic systems [Eqs. (73)–(77)], we can test the stability of each one by computing the sum

$$B_h = \sum_{n=-\infty}^{\infty} |h[n]|.$$

For the ideal delay, moving-average, forward difference, and backward difference examples, it is clear that $B_h < \infty$, since the impulse response has only a finite number of nonzero samples. In general, a system with a finite-duration impulse response (henceforth referred to as an FIR system) will always be stable, as long as each of the impulse response values is finite in magnitude. The accumulator, however, is unstable because

$$B_h = \sum_{n=0}^{\infty} u[n] = \infty.$$

In Section 2.5, we also demonstrated the instability of the accumulator by giving an example of a bounded input (the unit step) for which the output is unbounded.

The impulse response of the accumulator has infinite duration. This is an example of the class of systems referred to as *infinite-duration impulse response* (IIR) systems. An example of an IIR system that is stable is a system whose impulse response is $h[n] = a^n u[n]$ with $|a| < 1$. In this case,

$$B_h = \sum_{n=0}^{\infty} |a|^n. \tag{78}$$

If $|a| < 1$, the formula for the sum of the terms of an infinite geometric series gives

$$B_h = \frac{1}{1 - |a|} < \infty. \tag{79}$$

If, on the other hand, $|a| \geq 1$, then the sum is infinite and the system is unstable.

To test causality of the LTI systems in Examples 2, 3, 5, and 9, we can check to see whether $h[n] = 0$ for $n < 0$. As discussed in Section 2.4, the ideal delay [$n_d \geq 0$ in Eq. (20)] is causal. If $n_d < 0$, then the system is noncausal. For the moving average, causality requires that $-M_1 \geq 0$ and $M_2 \geq 0$. The accumulator and backward difference systems are causal, and the forward difference system is noncausal.

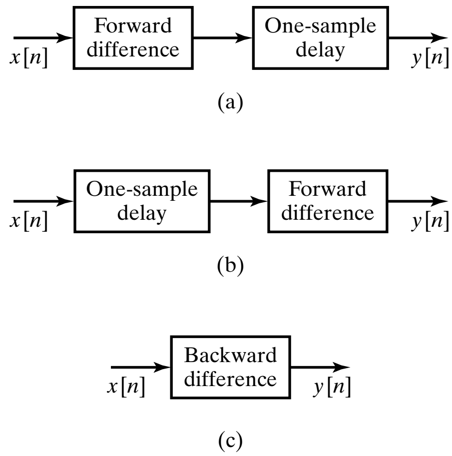


Figure 13 Equivalent systems found by using the commutative property of convolution.

The concept of convolution as an operation between two sequences leads to the simplification of many problems involving systems. A particularly useful result can be stated for the ideal delay system. Since the output of the delay system is $y[n] = x[n-n_d]$, and since the delay system has impulse response $h[n] = \delta[n - n_d]$, it follows that

$$x[n] * \delta[n - n_d] = \delta[n - n_d] * x[n] = x[n - n_d]. \tag{80}$$

That is, the convolution of a shifted impulse sequence with any signal $x[n]$ is easily evaluated by simply shifting $x[n]$ by the displacement of the impulse.

Since delay is a fundamental operation in the implementation of linear systems, the preceding result is often useful in the analysis and simplification of interconnections of LTI systems. As an example, consider the system of Figure 13(a), which consists of a forward difference system cascaded with an ideal delay of one sample. According to the commutative property of convolution, the order in which systems are cascaded does not matter, as long as they are linear and time invariant. Therefore, we obtain the same result when we compute the forward difference of a sequence and delay the result (Figure 13a) as when we delay the sequence first and then compute the forward difference (Figure 13b). Also, as indicated in Eq. (65) and in Figure 12, the overall impulse response of each cascade system is the convolution of the individual impulse responses. Consequently,

$$\begin{aligned} h[n] &= (\delta[n + 1] - \delta[n]) * \delta[n - 1] \\ &= \delta[n - 1] * (\delta[n + 1] - \delta[n]) \\ &= \delta[n] - \delta[n - 1]. \end{aligned} \tag{81}$$

Thus, $h[n]$ is identical to the impulse response of the backward difference system; that is, the cascaded systems of Figures 13(a) and 13(b) can be replaced by a backward difference system, as shown in Figure 13(c).

Note that the noncausal forward difference systems in Figures 13(a) and (b) have been converted to causal systems by cascading them with a delay. In general, any noncausal FIR system can be made causal by cascading it with a sufficiently long delay.

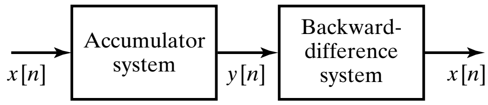


Figure 14 An accumulator in cascade with a backward difference. Since the backward difference is the inverse system for the accumulator, the cascade combination is equivalent to the identity system.

Another example of cascaded systems introduces the concept of an *inverse system*. Consider the cascade of systems in Figure 14. The impulse response of the cascade system is

$$\begin{aligned} h[n] &= u[n] * (\delta[n] - \delta[n-1]) \\ &= u[n] - u[n-1] \\ &= \delta[n]. \end{aligned} \tag{82}$$

That is, the cascade combination of an accumulator followed by a backward difference (or vice versa) yields a system whose overall impulse response is the impulse. Thus, the output of the cascade combination will always be equal to the input, since $x[n] * \delta[n] = x[n]$. In this case, the backward difference system compensates exactly for (or inverts) the effect of the accumulator; that is, the backward difference system is the *inverse system* for the accumulator. From the commutative property of convolution, the accumulator is likewise the inverse system for the backward difference system. Note that this example provides a system interpretation of Eqs. (7) and (9). In general, if an LTI system has impulse response $h[n]$, then its inverse system, if it exists, has impulse response $h_i[n]$ defined by the relation

$$h[n] * h_i[n] = h_i[n] * h[n] = \delta[n]. \tag{83}$$

Inverse systems are useful in many situations where it is necessary to compensate for the effects of a system. In general, it is difficult to solve Eq. (83) directly for $h_i[n]$, given $h[n]$. However, the z-transform provides a straightforward method of finding the inverse of an LTI system.

5 LINEAR CONSTANT-COEFFICIENT DIFFERENCE EQUATIONS

An important class of LTI systems consists of those systems for which the input $x[n]$ and the output $y[n]$ satisfy an N^{th}-order linear constant-coefficient difference equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]. \tag{84}$$

The properties discussed in Section 4 and some of the analysis techniques introduced there can be used to find difference equation representations for some of the LTI systems that we have defined.

Example 12 Difference Equation Representation of the Accumulator

The accumulator system is defined by

$$y[n] = \sum_{k=-\infty}^{n} x[k]. \tag{85}$$

To show that the input and output satisfy a difference equation of the form of Eq. (84), we rewrite Eq. (85) as

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k] \tag{86}$$

Also, from Eq. (85)

$$y[n-1] = \sum_{k=-\infty}^{n-1} x[k]. \tag{87}$$

Substituting Eq. (87) into Eq. (86) yields

$$y[n] = x[n] + y[n-1], \tag{88}$$

and equivalently,

$$y[n] - y[n-1] = x[n]. \tag{89}$$

Thus, in addition to satisfying the defining relationship of Eq. (85), the input and output of an accumulator satisfy a linear constant-coefficient difference equation of the form Eq. (84), with $N = 1, a_0 = 1, a_1 = -1, M = 0$, and $b_0 = 1$.

The difference equation in the form of Eq. (88) suggests a simple implementation of the accumulator system. According to Eq. (88), for each value of n, we add the current input value $x[n]$ to the previously accumulated sum $y[n-1]$. This interpretation of the accumulator is represented in block diagram form in Figure 15.

Equation (88) and the block diagram in Figure 15 are referred to as a recursive representation of the system, since each value is computed using previously computed values. This general notion will be explored in more detail later in this section.

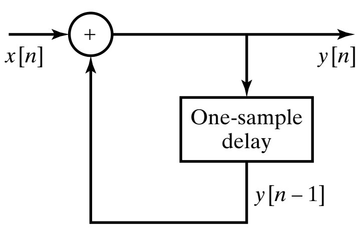


Figure 15 Block diagram of a recursive difference equation representing an accumulator.

Example 13 Difference Equation Representation of the Moving-Average System

Consider the moving-average system of Example 3, with $M_1 = 0$ so that the system is causal. In this case, from Eq. (74), the impulse response is

$$h[n] = \frac{1}{(M_2+1)}(u[n] - u[n - M_2 - 1]), \tag{90}$$

from which it follows that

$$y[n] = \frac{1}{(M_2+1)} \sum_{k=0}^{M_2} x[n-k], \tag{91}$$

which is a special case of Eq. (84), with $N = 0, a_0 = 1, M = M_2$, and $b_k = 1/(M_2+1)$ for $0 \le k \le M_2$.

Also, the impulse response can be expressed as

$$h[n] = \frac{1}{(M_2+1)} (\delta[n] - \delta[n - M_2 - 1]) * u[n], \tag{92}$$

which suggests that the causal moving-average system can be represented as the cascade system of Figure 16. We can obtain a difference equation for this block diagram by noting first that

$$x_1[n] = \frac{1}{(M_2+1)}(x[n] - x[n - M_2 - 1]). \tag{93}$$

From Eq. (89) of Example 12, the output of the accumulator satisfies the difference equation

$$y[n] - y[n-1] = x_1[n],$$

so that

$$y[n] - y[n-1] = \frac{1}{(M_2+1)}(x[n] - x[n - M_2 - 1]). \tag{94}$$

Again, we have a difference equation in the form of Eq. (84), but this time $N = 1$, $a_0 = 1, a_1 = -1, M = M_2 + 1$ and $b_0 = -b_{M_2+1} = 1/(M_2+1)$, and $b_k = 0$ otherwise.

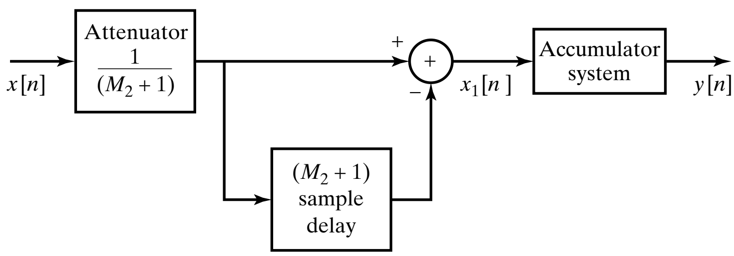


Figure 16 Block diagram of the recursive form of a moving-average system.

In Example 13, we showed two different difference-equation representations of the moving-average system. Many distinct difference equations can be used to represent a given LTI input–output relation.

Just as in the case of linear constant-coefficient differential equations for continuous-time systems, without additional constraints or other information, a linear constant-coefficient difference equation for discrete-time systems does not provide a unique specification of the output for a given input. Specifically, suppose that, for a given input $x_p[n]$, we have determined by some means one output sequence $y_p[n]$, so that an equation of the form of Eq. (84) is satisfied. Then, the same equation with the same input is satisfied by any output of the form

$$y[n] = y_p[n] + y_h[n], \tag{95}$$

where $y_h[n]$ is any solution to Eq. (84) with $x[n] = 0$, i.e., a solution to the equation

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0. \tag{96}$$

Equation (96) is called the *homogeneous difference equation* and $y_h[n]$ the homogeneous solution. The sequence $y_h[n]$ is in fact a member of a family of solutions of the form

$$y_h[n] = \sum_{m=1}^{N} A_m z_m^n, \tag{97}$$

where the coefficients A_m can be chosen to satisfy a set of auxiliary conditions on $y[n]$. Substituting Eq. (97) into Eq. (96) shows that the complex numbers z_m must be roots of the polynomial

$$A(z) = \sum_{k=0}^{N} a_k z^{-k}. \tag{98}$$

i.e., $A(z_m) = 0$ for $m = 1, 2, \ldots, N$. Equation (97) assumes that all N roots of the polynomial in Eq. (98) are distinct. The form of terms associated with multiple roots is slightly different, but there are always N undetermined coefficients. An example of the homogeneous solution with multiple roots is considered in Problem 50.

Since $y_h[n]$ has N undetermined coefficients, a set of N auxiliary conditions is required for the unique specification of $y[n]$ for a given $x[n]$. These auxiliary conditions might consist of specifying fixed values of $y[n]$ at specific values of n, such as $y[-1]$, $y[-2], \ldots, y[-N]$, and then solving a set of N linear equations for the N undetermined coefficients.

Alternatively, if the auxiliary conditions are a set of auxiliary values of $y[n]$, the other values of $y[n]$ can be generated by rewriting Eq. (84) as a recurrence formula, i.e., in the form

$$y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k]. \tag{99}$$

If the input $x[n]$ for all n, together with a set of auxiliary values, say, $y[-1]$, $y[-2], \ldots,$ $y[-N]$, is specified, then $y[0]$ can be determined from Eq. (99). With $y[0]$, $y[-1], \ldots,$ $y[-N+1]$ now available, $y[1]$ can then be calculated, and so on. When this procedure is used, $y[n]$ is said to be computed *recursively;* i.e., the output computation involves not only the input sequence, but also previous values of the output sequence.

To generate values of $y[n]$ for $n < -N$ (again assuming that the values $y[-1]$, $y[-2], \ldots, y[-N]$ are given as auxiliary conditions), we can rearrange Eq. (84) in the form

$$y[n-N] = -\sum_{k=0}^{N-1} \frac{a_k}{a_N} y[n-k] + \sum_{k=0}^{M} \frac{b_k}{a_N} x[n-k], \tag{100}$$

from which $y[-N-1], y[-N-2], \ldots$ can be computed recursively in the backward direction.

Our principal interest is in systems that are linear and time invariant, in which case the auxiliary conditions must be consistent with these additional requirements. When solving difference equations using the z-transform, we implicitly incorporate conditions of linearity and time invariance. Even with the additional constraints of linearity and time invariance, the solution to the difference equation, and therefore the system, is not uniquely specified. In particular, there are, in general, both causal and noncausal LTI systems consistent with a given difference equation.

If a system is characterized by a linear constant-coefficient difference equation and is further specified to be linear, time invariant, and causal, then the solution is unique. In this case, the auxiliary conditions are often stated as *initial-rest conditions*. In other words, the auxiliary information is that if the input $x[n]$ is zero for n less than some time n_0, then the output $y[n]$ is constrained to be zero for n less than n_0. This then provides sufficient initial conditions to obtain $y[n]$ for $n \geq n_0$ recursively using Eq. (99).

To summarize, for a system for which the input and output satisfy a linear constant-coefficient difference equation:

- The output for a given input is not uniquely specified. Auxiliary information or conditions are required.
- If the auxiliary information is in the form of N sequential values of the output, later values can be obtained by rearranging the difference equation as a recursive relation running forward in n, and prior values can be obtained by rearranging the difference equation as a recursive relation running backward in n.
- Linearity, time invariance, and causality of the system will depend on the auxiliary conditions. If an additional condition is that the system is initially at rest, then the system will be linear, time invariant, and causal.

The preceding discussion assumed that $N \geq 1$ in Eq. (84). If, instead, $N = 0$, no recursion is required to use the difference equation to compute the output, and therefore, no auxiliary conditions are required. That is,

$$y[n] = \sum_{k=0}^{M} \left(\frac{b_k}{a_0}\right) x[n-k]. \tag{101}$$

Equation (101) is in the form of a convolution, and by setting $x[n] = \delta[n]$, we see that the corresponding impulse response is

$$h[n] = \sum_{k=0}^{M} \left(\frac{b_k}{a_0}\right) \delta[n-k],$$

or

$$h[n] = \begin{cases} \left(\dfrac{b_n}{a_0}\right), & 0 \le n \le M, \\ 0, & \text{otherwise.} \end{cases} \tag{102}$$

The impulse response is obviously finite in duration. Indeed, the output of any FIR system can be computed nonrecursively where the coefficients are the values of the impulse response sequence. The moving-average system of Example 13 with $M_1 = 0$ is an example of a causal FIR system. An interesting feature of that system was that we also found a recursive equation for the output. There are many possible ways of implementing a desired signal transformation. Advantages of one method over another depend on practical considerations, such as numerical accuracy, data storage, and the number of multiplications and additions required to compute each sample of the output.

6 FREQUENCY-DOMAIN REPRESENTATION OF DISCRETE-TIME SIGNALS AND SYSTEMS

In the previous sections, we summarized some of the fundamental concepts of the theory of discrete-time signals and systems. For LTI systems, we saw that a representation of the input sequence as a weighted sum of delayed impulses leads to a representation of the output as a weighted sum of delayed impulse responses. As with continuous-time signals, discrete-time signals may be represented in a number of different ways. For example, sinusoidal and complex exponential sequences play a particularly important role in representing discrete-time signals. This is because complex exponential sequences are eigenfunctions of LTI systems, and the response to a sinusoidal input is sinusoidal with the same frequency as the input and with amplitude and phase determined by the system. These fundamental properties of LTI systems make representations of signals in terms of sinusoids or complex exponentials (i.e., Fourier representations) very useful in linear system theory.

6.1 Eigenfunctions for Linear Time-Invariant Systems

The eigenfunction property of complex exponentials for discrete-time systems follows directly from substitution into Eq. (61). Specifically, with input $x[n] = e^{j\omega n}$ for $-\infty < n < \infty$, the corresponding output of an LTI system with impulse response $h[n]$ is easily shown to be

$$y[n] = H(e^{j\omega})e^{j\omega n}, \tag{103}$$

where

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}. \tag{104}$$

Consequently, $e^{j\omega n}$ is an eigenfunction of the system, and the associated eigenvalue is $H(e^{j\omega})$. From Eq. (103), we see that $H(e^{j\omega})$ describes the change in complex amplitude of a complex exponential input signal as a function of the frequency ω. The eigenvalue $H(e^{j\omega})$ is the *frequency response* of the system. In general, $H(e^{j\omega})$ is complex and can be expressed in terms of its real and imaginary parts as

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega}) \tag{105}$$

or in terms of magnitude and phase as

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\angle H(e^{j\omega})}. \tag{106}$$

Example 14 Frequency Response of the Ideal Delay System

As a simple and important example, consider the ideal delay system defined by

$$y[n] = x[n - n_d], \tag{107}$$

where n_d is a fixed integer. With input $x[n] = e^{j\omega n}$ from Eq. (107), we have

$$y[n] = e^{j\omega(n-n_d)} = e^{-j\omega n_d}e^{j\omega n}.$$

The frequency response of the ideal delay is therefore

$$H(e^{j\omega}) = e^{-j\omega n_d}. \tag{108}$$

As an alternative method of obtaining the frequency response, recall that the impulse response for the ideal delay system is $h[n] = \delta[n - n_d]$. Using Eq. (104), we obtain

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_d]e^{-j\omega n} = e^{-j\omega n_d}.$$

The real and imaginary parts of the frequency response are

$$H_R(e^{j\omega}) = \cos(\omega n_d), \tag{109a}$$

$$H_I(e^{j\omega}) = -\sin(\omega n_d). \tag{109b}$$

The magnitude and phase are

$$|H(e^{j\omega})| = 1, \tag{110a}$$

$$\angle H(e^{j\omega}) = -\omega n_d. \tag{110b}$$

In Section 7, we will show that a broad class of signals can be represented as a linear combination of complex exponentials in the form

$$x[n] = \sum_k \alpha_k e^{j\omega_k n}. \tag{111}$$

From the principle of superposition and Eq. (103), the corresponding output of an LTI system is

$$y[n] = \sum_k \alpha_k H(e^{j\omega_k})e^{j\omega_k n}. \tag{112}$$

Thus, if we can find a representation of $x[n]$ as a superposition of complex exponential sequences, as in Eq. (111), we can then find the output using Eq. (112) if we know the frequency response of the system at all frequencies ω_k. The following simple example illustrates this fundamental property of LTI systems.

Example 15 Sinusoidal Response of LTI Systems

Let us consider a sinusoidal input

$$x[n] = A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}. \tag{113}$$

From Eq. (103), the response to $x_1[n] = (A/2)e^{j\phi}e^{j\omega_0 n}$ is

$$y_1[n] = H(e^{j\omega_0})\frac{A}{2}e^{j\phi}e^{j\omega_0 n}. \tag{114a}$$

The response to $x_2[n] = (A/2)e^{-j\phi}e^{-j\omega_0 n}$ is

$$y_2[n] = H(e^{-j\omega_0})\frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}. \tag{114b}$$

Thus, the total response is

$$y[n] = \frac{A}{2}[H(e^{j\omega_0})e^{j\phi}e^{j\omega_0 n} + H(e^{-j\omega_0})e^{-j\phi}e^{-j\omega_0 n}]. \tag{115}$$

If $h[n]$ is real, it can be shown (see Problem 78) that $H(e^{-j\omega_0}) = H^*(e^{j\omega_0})$. Consequently,

$$y[n] = A\,|H(e^{j\omega_0})|\cos(\omega_0 n + \phi + \theta), \tag{116}$$

where $\theta = \angle H(e^{j\omega_0})$ is the phase of the system function at frequency ω_0.

For the simple example of the ideal delay, $|H(e^{j\omega_0})| = 1$ and $\theta = -\omega_0 n_d$, as we determined in Example 14. Therefore,

$$\begin{aligned} y[n] &= A\cos(\omega_0 n + \phi - \omega_0 n_d) \\ &= A\cos[\omega_0(n - n_d) + \phi], \end{aligned} \tag{117}$$

which is identical to what we would obtain directly using the definition of the ideal delay system.

The concept of the frequency response of LTI systems is essentially the same for continuous-time and discrete-time systems. However, an important distinction arises because the frequency response of discrete-time LTI systems is always a periodic function of the frequency variable ω with period 2π. To show this, we substitute $\omega + 2\pi$ into Eq. (104) to obtain

$$H(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j(\omega+2\pi)n}. \tag{118}$$

Using the fact that $e^{\pm j2\pi n} = 1$ for n an integer, we have

$$e^{-j(\omega+2\pi)n} = e^{-j\omega n}e^{-j2\pi n} = e^{-j\omega n}.$$

Therefore,

$$H(e^{j(\omega+2\pi)}) = H(e^{j\omega}), \qquad \text{for all } \omega, \tag{119}$$

and, more generally,

$$H(e^{j(\omega+2\pi r)}) = H(e^{j\omega}), \qquad \text{for } r \text{ an integer.} \tag{120}$$

That is, $H(e^{j\omega})$ is periodic with period 2π. Note that this is obviously true for the ideal delay system, since $e^{-j(\omega+2\pi)n_d} = e^{-j\omega n_d}$ when n_d is an integer.

The reason for this periodicity is related directly to our earlier observation that the sequence

$$\{e^{j\omega n}\}, \qquad -\infty < n < \infty,$$

is indistinguishable from the sequence

$$\{e^{j(\omega+2\pi)n}\}, \qquad -\infty < n < \infty.$$

Because these two sequences have identical values for all n, the system must respond identically to both input sequences. This condition requires that Eq. (119) hold.

Since $H(e^{j\omega})$ is periodic with period 2π, and since the frequencies ω and $\omega+2\pi$ are indistinguishable, it follows that we need only specify $H(e^{j\omega})$ over an interval of length 2π, e.g., $0 \le \omega \le 2\pi$ or $-\pi < \omega \le \pi$. The inherent periodicity defines the frequency response everywhere outside the chosen interval. For simplicity and for consistency with the continuous-time case, it is generally convenient to specify $H(e^{j\omega})$ over the interval $-\pi < \omega \le \pi$. With respect to this interval, the "low frequencies" are frequencies close to zero, whereas the "high frequencies" are frequencies close to $\pm\pi$. Recalling that frequencies differing by an integer multiple of 2π are indistinguishable, we might generalize the preceding statement as follows: The "low frequencies" are those that are close to an even multiple of π, while the "high frequencies" are those that are close to an odd multiple of π, consistent with our earlier discussion in Section 1.

An important class of LTI systems includes those systems for which the frequency response is unity over a certain range of frequencies and is zero at the remaining frequencies, corresponding to ideal frequency-selective filters. The frequency response of an ideal lowpass filter is shown in Figure 17(a). Because of the inherent periodicity of the discrete-time frequency response, it has the appearance of a multiband filter, since frequencies around $\omega = 2\pi$ are indistinguishable from frequencies around $\omega = 0$. In effect, however, the frequency response passes only low frequencies and rejects high frequencies. Since the frequency response is completely specified by its behavior over the interval $-\pi < \omega \le \pi$, the ideal lowpass filter frequency response is more typically shown only in the interval $-\pi < \omega \le \pi$, as in Figure 17(b). It is understood that the frequency response repeats periodically with period 2π outside the plotted interval. With this implicit assumption, the frequency responses for ideal highpass, bandstop, and bandpass filters are as shown in Figures 18(a), (b), and (c), respectively.

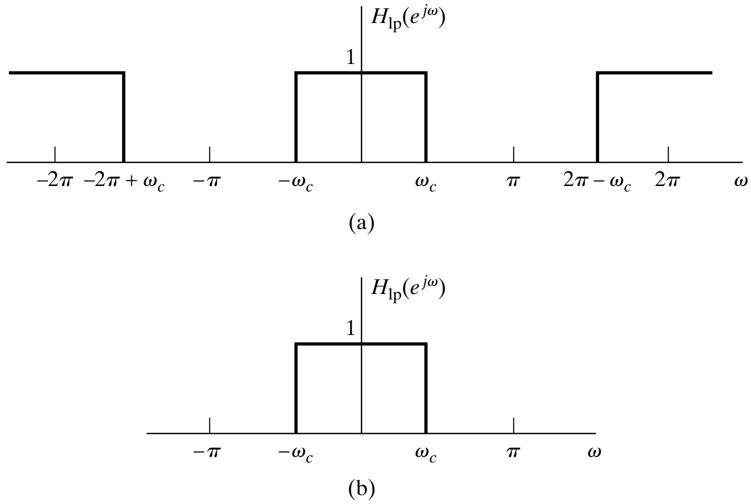


Figure 17 Ideal lowpass filter showing (a) periodicity of the frequency response and (b) one period of the periodic frequency response.

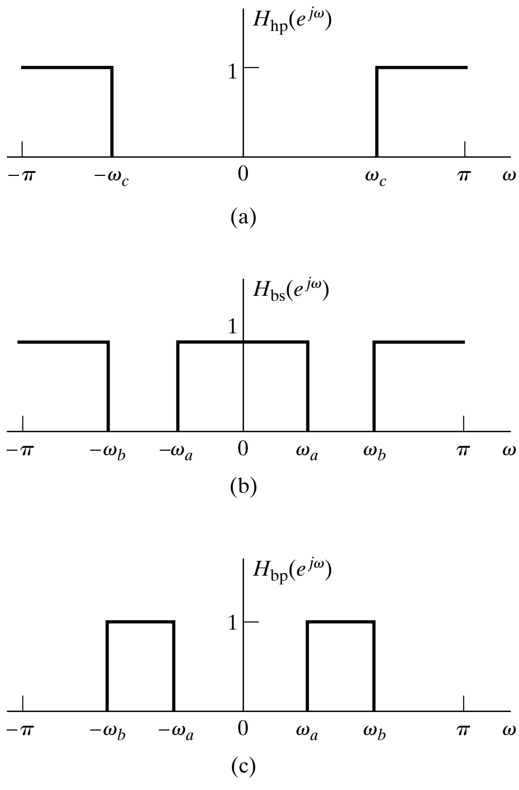


Figure 18 Ideal frequency-selective filters. (a) Highpass filter. (b) Bandstop filter. (c) Bandpass filter. In each case, the frequency response is periodic with period 2π. Only one period is shown.

Example 16 Frequency Response of the Moving-Average System

The impulse response of the moving-average system of Example 3 is

$$h[n] = \begin{cases} \dfrac{1}{M_1 + M_2 + 1}, & -M_1 \le n \le M_2, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the frequency response is

$$H(e^{j\omega}) = \frac{1}{M_1 + M_2 + 1} \sum_{n=-M_1}^{M_2} e^{-j\omega n}. \tag{121}$$

For the causal moving average system, $M_1 = 0$ and Eq. (121) can be expressed as

$$H(e^{j\omega}) = \frac{1}{M_2 + 1} \sum_{n=0}^{M_2} e^{-j\omega n}. \tag{122}$$

Using Eq. (55), Eq. (122) becomes

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{M_2+1}\left(\frac{1 - e^{-j\omega(M_2+1)}}{1 - e^{-j\omega}}\right) \\ &= \frac{1}{M_2+1} \frac{(e^{j\omega(M_2+1)/2} - e^{-j\omega(M_2+1)/2})e^{-j\omega(M_2+1)/2}}{(e^{j\omega/2} - e^{-j\omega/2})e^{-j\omega/2}} \\ &= \frac{1}{M_2+1} \frac{\sin[\omega(M_2+1)/2]}{\sin \omega/2} e^{-j\omega M_2/2}. \end{aligned} \tag{123}$$

The magnitude and phase of $H(e^{j\omega})$ for this case, with $M_2 = 4$, are shown in Figure 19.

If the moving-average filter is symmetric, i.e., if $M_1 = M_2$, then Eq. (123) is replaced by

$$H(e^{j\omega}) = \frac{1}{2M_2 + 1} \frac{\sin[\omega(2M_2 + 1)/2]}{\sin(\omega/2)}. \tag{124}$$

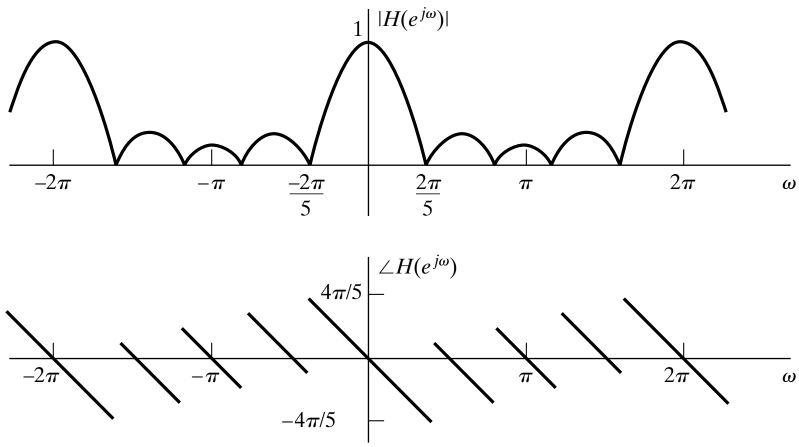


Figure 19 (a) Magnitude and (b) phase of the frequency response of the moving-average system for the case $M_1 = 0$ and $M_2 = 4$.

Note that in both cases $H(e^{j\omega})$ is periodic, as is required of the frequency response of a discrete-time system. Note also that $|H(e^{j\omega})|$ falls off at "high frequencies" and $\angle H(e^{j\omega})$, i.e., the phase of $H(e^{j\omega})$, varies linearly with ω. This attenuation of the high frequencies suggests that the system will smooth out rapid variations in the input sequence; in other words, the system is a rough approximation to a lowpass filter. This is consistent with what we would intuitively expect about the behavior of the moving-average system.

6.2 Suddenly Applied Complex Exponential Inputs

We have seen that complex exponential inputs of the form $e^{j\omega n}$ for $-\infty < n < \infty$ produce outputs of the form $H(e^{j\omega})e^{j\omega n}$ for LTI systems. Models of this kind are important in the mathematical representation of a wide range of signals, even those that exist only over a finite domain. We can also gain additional insight into LTI systems by considering inputs of the form

$$x[n] = e^{j\omega n}u[n], \tag{125}$$

i.e., complex exponentials that are suddenly applied at an arbitrary time, which for convenience here we choose as $n = 0$. Using the convolution sum in Eq. (61), the corresponding output of a causal LTI system with impulse response $h[n]$ is

$$y[n] = \begin{cases} 0, & n < 0, \\ \left(\displaystyle\sum_{k=0}^{n} h[k]e^{-j\omega k}\right) e^{j\omega n}, & n \geq 0. \end{cases}$$

If we consider the output for $n \geq 0$, we can write

$$y[n] = \left(\sum_{k=0}^{\infty} h[k]e^{-j\omega k}\right) e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right) e^{j\omega n} \tag{126}$$

$$= H(e^{j\omega})e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right) e^{j\omega n}. \tag{127}$$

From Eq. (127), we see that the output consists of the sum of two terms, i.e., $y[n] = y_{ss}[n] + y_t[n]$. The first term,

$$y_{ss}[n] = H(e^{j\omega})e^{j\omega n},$$

is the steady-state response. It is identical to the response of the system when the input is $e^{j\omega n}$ for all n. In a sense, the second term,

$$y_t[n] = -\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}e^{j\omega n},$$

is the amount by which the output differs from the eigenfunction result. This part corresponds to the transient response, because it is clear that in some cases it may approach zero. To see the conditions for which this is true, let us consider the size of the second term. Its magnitude is bounded as follows:

$$|y_t[n]| = \left|\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}e^{j\omega n}\right| \leq \sum_{k=n+1}^{\infty} |h[k]|. \tag{128}$$

From Eq. (128), it should be clear that if the impulse response has finite length, so that $h[n] = 0$ except for $0 \le n \le M$, then the term $y_t[n] = 0$ for $n+1 > M$, or $n > M-1$. In this case,

$$y[n] = y_{ss}[n] = H(e^{j\omega})e^{j\omega n}, \qquad \text{for } n > M-1.$$

When the impulse response has infinite duration, the transient response does not disappear abruptly, but if the samples of the impulse response approach zero with increasing n, then $y_t[n]$ will approach zero. Note that Eq. (128) can be written

$$|y_t[n]| = \left| \sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}e^{j\omega n} \right| \le \sum_{k=n+1}^{\infty} |h[k]| \le \sum_{k=0}^{\infty} |h[k]|. \tag{129}$$

That is, the transient response is bounded by the sum of the absolute values of *all* of the impulse response samples. If the right-hand side of Eq. (129) is bounded, i.e., if

$$\sum_{k=0}^{\infty} |h[k]| < \infty,$$

then the system is stable. From Eq. (129), it follows that, for stable systems, the transient response must become increasingly smaller as $n \to \infty$. Thus, a sufficient condition for the transient response to decay asymptotically is that the system be stable.

Figure 20 shows the real part of a complex exponential signal with frequency $\omega = 2\pi/10$. The solid dots indicate the samples $x[k]$ of the suddenly applied complex

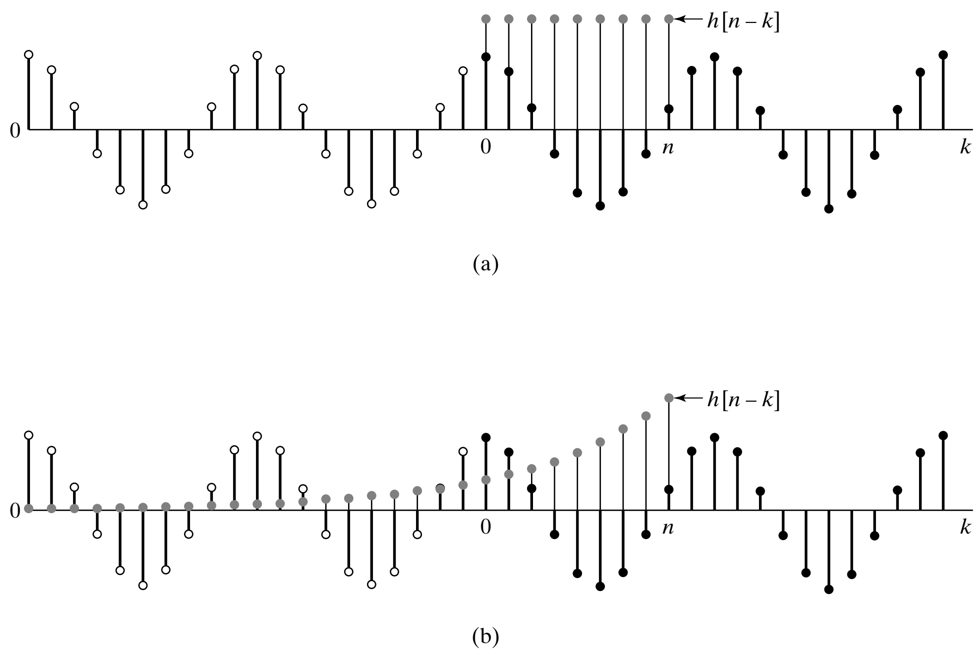


Figure 20 Illustration of a real part of suddenly applied complex exponential input with (a) FIR and (b) IIR.

exponential, while the open circles indicate the samples of the complex exponential that are "missing," i.e., that would be nonzero if the input were of the form $e^{j\omega n}$ for all n. The shaded dots indicate the samples of the impulse response $h[n-k]$ as a function of k for $n = 8$. In the finite-length case shown in Figure 20(a), it is clear that the output would consist only of the steady-state component for $n \geq 8$, whereas in the infinite-length case, it is clear that the "missing" samples have less and less effect as n increases, owing to the decaying nature of the impulse response.

The condition for stability is also a sufficient condition for the existence of the frequency response function. To see this, note that, in general,

$$|H(e^{j\omega})| = \left| \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \right| \leq \sum_{k=-\infty}^{\infty} |h[k]e^{-j\omega k}| \leq \sum_{k=-\infty}^{\infty} |h[k]|,$$

so the general condition

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

ensures that $H(e^{j\omega})$ exists. It is no surprise that the condition for existence of the frequency response is the same as the condition for dominance of the steady-state solution. Indeed, a complex exponential that exists for all n can be thought of as one that is applied at $n = -\infty$. The eigenfunction property of complex exponentials depends on stability of the system, since at finite n, the transient response must have become zero, so that we only see the steady-state response $H(e^{j\omega})e^{j\omega n}$ for all finite n.

7 REPRESENTATION OF SEQUENCES BY FOURIER TRANSFORMS

One of the advantages of the frequency-response representation of an LTI system is that interpretations of system behavior such as the one we made in Example 16 often follow easily. At this point, let us return to the question of how we may find representations of the form of Eq. (111) for an arbitrary input sequence.

Many sequences can be represented by a Fourier integral of the form

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega, \tag{130}$$

where

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}. \tag{131}$$

Equations (130) and (131) together form a Fourier representation for the sequence. Equation (130), the *inverse Fourier transform,* is a synthesis formula. That is, it represents $x[n]$ as a superposition of infinitesimally small complex sinusoids of the form

$$\frac{1}{2\pi} X(e^{j\omega})e^{j\omega n} d\omega,$$

with ω ranging over an interval of length 2π and with $X(e^{j\omega})$ determining the relative amount of each complex sinusoidal component. Although, in writing Eq. (130), we have chosen the range of values for ω between $-\pi$ and $+\pi$, any interval of length 2π can be used. Equation (131), the *Fourier transform,*[4] is an expression for computing $X(e^{j\omega})$ from the sequence $x[n]$, i.e., for analyzing the sequence $x[n]$ to determine how much of each frequency component is required to synthesize $x[n]$ using Eq. (130).

In general, the Fourier transform is a complex-valued function of ω. As with the frequency response, we may either express $X(e^{j\omega})$ in rectangular form as

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega}) \tag{132a}$$

or in polar form as

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\angle X(e^{j\omega})}. \tag{132b}$$

With $|X(e^{j\omega})|$ representing the magnitude and $\angle X(e^{j\omega})$ the phase.

The phase $\angle X(e^{j\omega})$ is not uniquely specified by Eq. (132b), since any integer multiple of 2π may be added to $\angle X(e^{j\omega})$ at any value of ω without affecting the result of the complex exponentiation. When we specifically want to refer to the principal value, i.e., $\angle X(e^{j\omega})$ restricted to the range of values between $-\pi$ and $+\pi$, we denote this as $\text{ARG}[X(e^{j\omega})]$. If we want to refer to a phase function that is a continuous function of ω for $0 < \omega < \pi$, i.e., not evaluated modulo 2π, we use the notation $\arg[X(e^{j\omega})]$.

As is clear from comparing Eqs. (104) and (131), the frequency response of an LTI system is the Fourier transform of the impulse response. The impulse response can be obtained from the frequency response by applying the inverse Fourier transform integral; i.e.,

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\,d\omega. \tag{133}$$

As discussed previously, the frequency response is a periodic function of ω. Likewise, the Fourier transform is periodic in ω with period 2π. A Fourier series is commonly used to represent periodic signals, and it is worth noting that indeed, Eq. (131) is of the form of a Fourier series for the periodic function $X(e^{j\omega})$. Eq. (130), which expresses the sequence values $x[n]$ in terms of the periodic function $X(e^{j\omega})$, is of the form of the integral that would be used to obtain the coefficients in the Fourier series. Our use of Eqs. (130) and (131) focuses on the representation of the sequence $x[n]$. Nevertheless, it is useful to be aware of the equivalence between the Fourier series representation of continuous-variable periodic functions and the Fourier transform representation of discrete-time signals, since all the familiar properties of Fourier series can be applied, with appropriate interpretation of variables, to the Fourier transform representation of a sequence. (Oppenheim and Willsky (1997), McClellan, Schafer and Yoder (2003).)

Determining the class of signals that can be represented by Eq. (130) is equivalent to considering the convergence of the infinite sum in Eq. (131). That is, we are concerned with the conditions that must be satisfied by the terms in the sum in Eq. (131) such that

$$|X(e^{j\omega})| < \infty \qquad \text{for all } \omega,$$

[4]Eq. (131) is sometimes more explicitly referred to as the discrete-time Fourier transform, or DTFT, particularly when it is important to distinguish it from the continuous-time Fourier transform.

where $X(e^{j\omega})$ is the limit as $M \to \infty$ of the finite sum

$$X_M(e^{j\omega}) = \sum_{n=-M}^{M} x[n]e^{-j\omega n}. \tag{134}$$

A sufficient condition for convergence can be found as follows:

$$\begin{aligned} |X(e^{j\omega})| &= \left| \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right| \\ &\le \sum_{n=-\infty}^{\infty} |x[n]|\,|e^{-j\omega n}| \\ &\le \sum_{n=-\infty}^{\infty} |x[n]| < \infty. \end{aligned}$$

Thus, if $x[n]$ is *absolutely summable,* then $X(e^{j\omega})$ exists. Furthermore, in this case, the series can be shown to converge uniformly to a continuous function of ω (Körner (1988), Kammler (2000)). Since a stable sequence is, by definition, absolutely summable, all stable sequences have Fourier transforms. It also follows, then, that any stable *system*, i.e., one having an absolutely summable impulse response, will have a finite and continuous frequency response.

Absolute summability is a sufficient condition for the existence of a Fourier transform representation. In Examples 14 and 16, we computed the Fourier transforms of the impulse response of the delay system and the moving average system. The impulse responses are absolutely summable, since they are finite in length. Clearly, any finite-length sequence is absolutely summable and thus will have a Fourier transform representation. In the context of LTI systems, any FIR system will be stable and therefore will have a finite, continuous frequency response. However, when a sequence has infinite length, we must be concerned about convergence of the infinite sum. The following example illustrates this case.

Example 17 Absolute Summability for a Suddenly-Applied Exponential

Consider $x[n] = a^n u[n]$. The Fourier transform of this sequence is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n \\ &= \frac{1}{1 - ae^{-j\omega}} \qquad \text{if } |ae^{-j\omega}| < 1 \quad \text{or} \quad |a| < 1. \end{aligned}$$

Clearly, the condition $|a| < 1$ is the condition for the absolute summability of $x[n]$; i.e.,

$$\sum_{n=0}^{\infty} |a|^n = \frac{1}{1 - |a|} < \infty \qquad \text{if } |a| < 1. \tag{135}$$

Absolute summability is a *sufficient* condition for the existence of a Fourier transform representation, and it also guarantees uniform convergence. Some sequences are not absolutely summable, but are square summable, i.e.,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty. \tag{136}$$

Such sequences can be represented by a Fourier transform if we are willing to relax the condition of uniform convergence of the infinite sum defining $X(e^{j\omega})$. Specifically, in this case, we have mean-square convergence; that is, with

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{137a}$$

and

$$X_M(e^{j\omega}) = \sum_{n=-M}^{M} x[n]e^{-j\omega n}, \tag{137b}$$

it follows that

$$\lim_{M\to\infty} \int_{-\pi}^{\pi} |X(e^{j\omega}) - X_M(e^{j\omega})|^2 d\omega = 0. \tag{138}$$

In other words, the error $|X(e^{j\omega}) - X_M(e^{j\omega})|$ may not approach zero at each value of ω as $M \to \infty$, but the total "energy" in the error does. Example 18 illustrates this case.

Example 18 Square-Summability for the Ideal Lowpass Filter

In this example we determine the impulse response of the ideal lowpass filter discussed in Section 6. The frequency response is

$$H_{\mathrm{lp}}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi, \end{cases} \tag{139}$$

with periodicity 2π also understood. The impulse response $h_{\mathrm{lp}}[n]$ can be found using the Fourier transform synthesis equation (130):

$$\begin{aligned} h_{\mathrm{lp}}[n] &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi jn}\left[e^{j\omega n}\right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi jn}(e^{j\omega_c n} - e^{-j\omega_c n}) \\ &= \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty. \end{aligned} \tag{140}$$

We note that, since $h_{\mathrm{lp}}[n]$ is nonzero for $n < 0$, the ideal lowpass filter is noncausal. Also, $h_{\mathrm{lp}}[n]$ is not absolutely summable. The sequence values approach zero as $n \to \infty$, but only as $1/n$. This is because $H_{\mathrm{lp}}(e^{j\omega})$ is discontinuous at $\omega = \omega_c$. Since $h_{\mathrm{lp}}[n]$ is not absolutely summable, the infinite sum

$$\sum_{n=-\infty}^{\infty} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

does not converge uniformly for all values of ω. To obtain an intuitive feeling for this, let us consider $H_M(e^{j\omega})$ as the sum of a finite number of terms:

$$H_M(e^{j\omega}) = \sum_{n=-M}^{M} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}. \tag{141}$$

The function $H_M(e^{j\omega})$ is evaluated in Figure 21 for several values of M. Note that as M increases, the oscillatory behavior at $\omega = \omega_c$ (often referred to as the Gibbs phenomenon) is more rapid, but the size of the ripples does not decrease. In fact, it can be shown that as $M \to \infty$, the maximum amplitude of the oscillations does not approach zero, but the oscillations converge in location toward the points $\omega = \pm\omega_c$. Thus, the infinite sum does not converge uniformly to the discontinuous function $H_{\text{lp}}(e^{j\omega})$ of Eq. (139). However, $h_{\text{lp}}[n]$, as given in Eq. (140), is square summable, and correspondingly, $H_M(e^{j\omega})$ converges in the mean-square sense to $H_{\text{lp}}(e^{j\omega})$; i.e.,

$$\lim_{M\to\infty} \int_{-\pi}^{\pi} |H_{\text{lp}}(e^{j\omega}) - H_M(e^{j\omega})|^2 d\omega = 0.$$

Although the error between $H_M(e^{j\omega})$ and $H_{\text{lp}}(e^{j\omega})$ as $M \to \infty$ might seem unimportant because the two functions differ only at $\omega = \omega_c$, the behavior of finite sums such as Eq. (141) has important implications in the design of discrete-time systems for filtering.

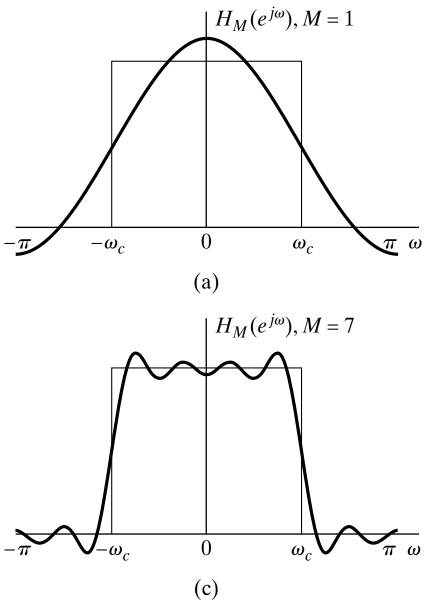


Figure 21 Convergence of the Fourier transform. The oscillatory behavior at $\omega = \omega_c$ is often called the Gibbs phenomenon.

It is sometimes useful to have a Fourier transform representation for certain sequences that are neither absolutely summable nor square summable. We illustrate several of these in the following examples.

Example 19 Fourier Transform of a Constant

Consider the sequence $x[n] = 1$ for all n. This sequence is neither absolutely summable nor square summable, and Eq. (131) does not converge in either the uniform or mean-square sense for this case. However, it is possible and useful to define the Fourier transform of the sequence $x[n]$ to be the periodic impulse train

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r). \tag{142}$$

The impulses in this case are functions of a continuous variable and therefore are of "infinite height, zero width, and unit area," consistent with the fact that Eq. (131) does not converge in any regular sense. (See Oppenheim and Willsky (1997) for a discussion of the definition and properties of the impulse function.) The use of Eq. (142) as a Fourier representation of the sequence $x[n] = 1$ is justified principally because formal substitution of Eq. (142) into Eq. (130) leads to the correct result. Example 20 represents a generalization of this example.

Example 20 Fourier Transform of Complex Exponential Sequences

Consider a sequence $x[n]$ whose Fourier transform is the periodic impulse train

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi r). \tag{143}$$

We show in this example that $x[n]$ is the complex exponential sequence $e^{j\omega_0 n}$, with $-\pi < \omega_0 \leq \pi$.

We can determine $x[n]$ by substituting $X(e^{j\omega})$ into the inverse Fourier transform integral of Eq. (130). Because the integration of $X(e^{j\omega})$ extends only over one period, from $-\pi < \omega < \pi$, we need include only the $r = 0$ term from Eq. (143). Consequently, we can write

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\omega - \omega_0)e^{j\omega n}d\omega. \tag{144}$$

From the definition of the impulse function, it follows that

$$x[n] = e^{j\omega_0 n} \qquad \text{for any } n.$$

For $\omega_0 = 0$, this reduces to the sequence considered in Example 19.

Clearly, $x[n]$ in Example 20 is not absolutely summable, nor is it square summable, and $|X(e^{j\omega})|$ is not finite for all ω. Thus, the mathematical statement

$$\sum_{n=-\infty}^{\infty} e^{j\omega_0 n}e^{-j\omega n} = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi r) \tag{145}$$

must be interpreted in the context of generalized functions (Lighthill, 1958). Using that theory, the concept of a Fourier transform representation can be extended to the class of sequences that can be expressed as a sum of discrete frequency components, such as

$$x[n] = \sum_k a_k e^{j\omega_k n}, \qquad -\infty < n < \infty. \tag{146}$$

From the result of Example 20, it follows that

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty}\sum_k 2\pi a_k\delta(\omega - \omega_k + 2\pi r) \tag{147}$$

is a consistent Fourier transform representation of $x[n]$ in Eq. (146).

Another sequence that is neither absolutely summable nor square summable is the unit step sequence $u[n]$. Although it is not completely straightforward to show, this sequence can be represented by the following Fourier transform:

$$U(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} + \sum_{r=-\infty}^{\infty} \pi\delta(\omega + 2\pi r). \tag{148}$$

8 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

In using Fourier transforms, it is useful to have a detailed knowledge of the way that properties of the sequence manifest themselves in the Fourier transform and vice versa. In this section and Section 9, we discuss and summarize a number of such properties.

Symmetry properties of the Fourier transform are often very useful for simplifying the solution of problems. The following discussion presents these properties. The proofs are considered in Problems 79 and 80. Before presenting the properties, however, we begin with some definitions.

A *conjugate-symmetric sequence* $x_e[n]$ is defined as a sequence for which $x_e[n] = x_e^*[-n]$, and a *conjugate-antisymmetric sequence* $x_o[n]$ is defined as a sequence for which $x_o[n] = -x_o^*[-n]$, where * denotes complex conjugation. Any sequence $x[n]$ can be expressed as a sum of a conjugate-symmetric and conjugate-antisymmetric sequence. Specifically,

$$x[n] = x_e[n] + x_o[n], \tag{149a}$$

where

$$x_e[n] = \tfrac{1}{2}(x[n] + x^*[-n]) = x_e^*[-n] \tag{149b}$$

and

$$x_o[n] = \tfrac{1}{2}(x[n] - x^*[-n]) = -x_o^*[-n]. \tag{149c}$$

Adding Eqs. (149b) and (149c) confirms that Eq. (149a) holds. A real sequence that is conjugate symmetric such that $x_e[n] = x_e[-n]$ is referred to as an *even sequence*, and a real sequence that is conjugate antisymmetric such that $x_o[n] = -x_o[-n]$ is referred to as an *odd sequence*.

A Fourier transform $X(e^{j\omega})$ can be decomposed into a sum of conjugate-symmetric and conjugate-antisymmetric functions as

$$X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega}), \tag{150a}$$

where

$$X_e(e^{j\omega}) = \tfrac{1}{2}[X(e^{j\omega}) + X^*(e^{-j\omega})] \tag{150b}$$

and

$$X_o(e^{j\omega}) = \tfrac{1}{2}[X(e^{j\omega}) - X^*(e^{-j\omega})]. \tag{150c}$$

By substituting $-\omega$ for ω in Eqs. (150b) and (150c), it follows that $X_e(e^{j\omega})$ is conjugate symmetric and $X_o(e^{j\omega})$ is conjugate antisymmetric; i.e.,

$$X_e(e^{j\omega}) = X_e^*(e^{-j\omega}) \tag{151a}$$

and

$$X_o(e^{j\omega}) = -X_o^*(e^{-j\omega}). \tag{151b}$$

If a real function of a continuous variable is conjugate symmetric, it is referred to as an *even function*, and a real conjugate-antisymmetric function of a continuous variable is referred to as an *odd function*.

The symmetry properties of the Fourier transform are summarized in Table 1. The first six properties apply for a general complex sequence $x[n]$ with Fourier transform $X(e^{j\omega})$. Properties 1 and 2 are considered in Problem 79. Property 3 follows from properties 1 and 2, together with the fact that the Fourier transform of the sum of two sequences is the sum of their Fourier transforms. Specifically, the Fourier transform of $\mathcal{R}e\{x[n]\} = \frac{1}{2}(x[n] + x^*[n])$ is the conjugate-symmetric part of $X(e^{j\omega})$, or $X_e(e^{j\omega})$. Similarly, $j\mathcal{I}m\{x[n]\} = \frac{1}{2}(x[n] - x^*[n])$, or equivalently, $j\mathcal{I}m\{x[n]\}$ has a Fourier transform that is the conjugate-antisymmetric component $X_o(e^{j\omega})$ corresponding to property 4. By considering the Fourier transform of $x_e[n]$ and $x_o[n]$, the conjugate-symmetric and conjugate-antisymmetric components, respectively, of $x[n]$, it can be shown that properties 5 and 6 follow.

If $x[n]$ is a real sequence, these symmetry properties become particularly straightforward and useful. Specifically, for a real sequence, the Fourier transform is conjugate symmetric; i.e., $X(e^{j\omega}) = X^*(e^{-j\omega})$ (property 7). Expressing $X(e^{j\omega})$ in terms of its real and imaginary parts as

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega}), \tag{152}$$

TABLE 1 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
11. Any real $x[n]$	$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

we can derive properties 8 and 9—specifically,

$$X_R(e^{j\omega}) = X_R(e^{-j\omega}) \tag{153a}$$

and

$$X_I(e^{j\omega}) = -X_I(e^{-j\omega}). \tag{153b}$$

In other words, the real part of the Fourier transform is an even function, and the imaginary part is an odd function, if the sequence is real. In a similar manner, by expressing $X(e^{j\omega})$ in polar form as

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\angle X(e^{j\omega})}, \tag{154}$$

we can show that, for a real sequence $x[n]$, the magnitude of the Fourier transform, $|X(e^{j\omega})|$, is an even function of ω and the phase, $\angle X(e^{j\omega})$, can be chosen to be an odd function of ω (properties 10 and 11). Also, for a real sequence, the even part of $x[n]$ transforms to $X_R(e^{j\omega})$, and the odd part of $x[n]$ transforms to $jX_I(e^{j\omega})$ (properties 12 and 13).

Example 21 Illustration of Symmetry Properties

Let us return to the sequence of Example 17, where we showed that the Fourier transform of the real sequence $x[n] = a^n u[n]$ is

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} \qquad \text{if } |a| < 1. \tag{155}$$

Then, from the properties of complex numbers, it follows that

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} = X^*(e^{-j\omega}) \qquad \text{(property 7)},$$

$$X_R(e^{j\omega}) = \frac{1 - a\cos\omega}{1 + a^2 - 2a\cos\omega} = X_R(e^{-j\omega}) \qquad \text{(property 8)},$$

$$X_I(e^{j\omega}) = \frac{-a\sin\omega}{1 + a^2 - 2a\cos\omega} = -X_I(e^{-j\omega}) \qquad \text{(property 9)},$$

$$|X(e^{j\omega})| = \frac{1}{(1 + a^2 - 2a\cos\omega)^{1/2}} = |X(e^{-j\omega})| \qquad \text{(property 10)},$$

$$\angle X(e^{j\omega}) = \tan^{-1}\left(\frac{-a\sin\omega}{1 - a\cos\omega}\right) = -\angle X(e^{-j\omega}) \qquad \text{(property 11)}.$$

These functions are plotted in Figure 22 for $a > 0$, specifically, $a = 0.75$ (solid curve) and $a = 0.5$ (dashed curve). In Problem 32, we consider the corresponding plots for $a < 0$.

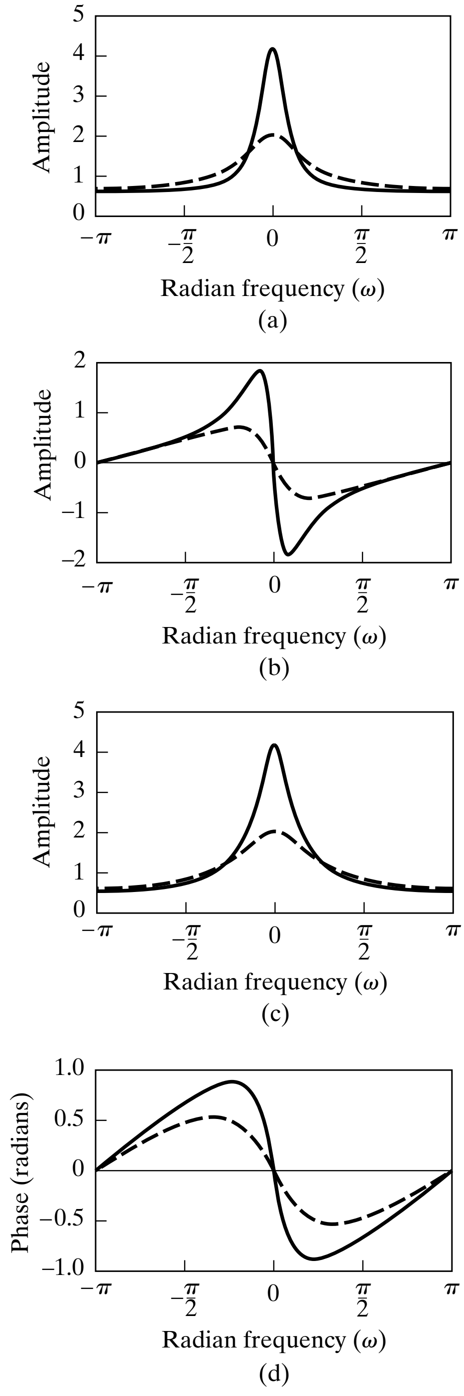


Figure 22 Frequency response for a system with impulse response $h[n] = a^n u[n]$. (a) Real part. $a > 0$; $a = 0.75$ (solid curve) and $a = 0.5$ (dashed curve). (b) Imaginary part. (c) Magnitude. $a > 0$; $a = 0.75$ (solid curve) and $a = 0.5$ (dashed curve). (d) Phase.

9 FOURIER TRANSFORM THEOREMS

In addition to the symmetry properties, a variety of theorems (presented in Sections 9.1–9.7) relate operations on the sequence to operations on the Fourier transform. We will see that these theorems are quite similar in most cases to corresponding theorems for continuous-time signals and their Fourier transforms. To facilitate the statement of the theorems, we introduce the following operator notation:

$$\begin{aligned} X(e^{j\omega}) &= \mathcal{F}\{x[n]\}, \\ x[n] &= \mathcal{F}^{-1}\{X(e^{j\omega})\}, \\ x[n] &\stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}). \end{aligned}$$

That is, $\mathcal{F}$ denotes the operation of "taking the Fourier transform of $x[n]$," and $\mathcal{F}^{-1}$ is the inverse of that operation. Most of the theorems will be stated without proof. The proofs, which are left as exercises (Problem 81), generally involve only simple manipulations of variables of summation or integration. The theorems in this section are summarized in Table 2.

TABLE 2 FOURIER TRANSFORM THEOREMS

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Parseval's theorem:	
8. $\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	
9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$	

9.1 Linearity of the Fourier Transform

If

$$x_1[n] \overset{\mathcal{F}}{\longleftrightarrow} X_1(e^{j\omega})$$

and

$$x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} X_2(e^{j\omega}),$$

then it follows by substitution into the definition of the DTFT that

$$ax_1[n] + bx_2[n] \overset{\mathcal{F}}{\longleftrightarrow} aX_1(e^{j\omega}) + bX_2(e^{j\omega}). \tag{156}$$

9.2 Time Shifting and Frequency Shifting Theorem

If

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then, for the time-shifted sequence $x[n - n_d]$, a simple transformation of the index of summation in the DTFT yields

$$x[n - n_d] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_d} X(e^{j\omega}). \tag{157}$$

Direct substitution proves the following result for the frequency-shifted Fourier transform:

$$e^{j\omega_0 n} x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)}). \tag{158}$$

9.3 Time Reversal Theorem

If

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then if the sequence is time reversed,

$$x[-n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{-j\omega}). \tag{159}$$

If $x[n]$ is real, this theorem becomes

$$x[-n] \overset{\mathcal{F}}{\longleftrightarrow} X^*(e^{j\omega}). \tag{160}$$

9.4 Differentiation in Frequency Theorem

If

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then, by differentiating the DTFT, it is seen that

$$nx[n] \overset{\mathcal{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}. \tag{161}$$

9.5 Parseval's Theorem

If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega. \tag{162}$$

The function $|X(e^{j\omega})|^2$ is called the *energy density spectrum,* since it determines how the energy is distributed in the frequency domain. Necessarily, the energy density spectrum is defined only for finite-energy signals. A more general form of Parseval's theorem is shown in Problem 84.

9.6 The Convolution Theorem

If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

and

$$h[n] \stackrel{\mathcal{F}}{\longleftrightarrow} H(e^{j\omega}),$$

and if

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n], \tag{163}$$

then

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}). \tag{164}$$

Thus, convolution of sequences implies multiplication of the corresponding Fourier transforms. Note that the time-shifting property is a special case of the convolution property, since

$$\delta[n-n_d] \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_d} \tag{165}$$

and if $h[n] = \delta[n-n_d]$, then $y[n] = x[n] * \delta[n-n_d] = x[n-n_d]$. Therefore,

$$H(e^{j\omega}) = e^{-j\omega n_d} \qquad \text{and} \qquad Y(e^{j\omega}) = e^{-j\omega n_d}X(e^{j\omega}).$$

A formal derivation of the convolution theorem is easily achieved by applying the definition of the Fourier transform to $y[n]$ as expressed in Eq. (163). This theorem can also be interpreted as a direct consequence of the eigenfunction property of complex exponentials for LTI systems. Recall that $H(e^{j\omega})$ is the frequency response of the LTI system whose impulse response is $h[n]$. Also, if

$$x[n] = e^{j\omega n},$$

then

$$y[n] = H(e^{j\omega})e^{j\omega n}.$$

That is, complex exponentials are *eigenfunctions* of LTI systems, where $H(e^{j\omega})$, the Fourier transform of $h[n]$, is the eigenvalue. From the definition of integration, the Fourier transform synthesis equation corresponds to the representation of a sequence $x[n]$ as a superposition of complex exponentials of infinitesimal size; that is,

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega = \lim_{\Delta\omega\to 0}\frac{1}{2\pi}\sum_k X(e^{jk\Delta\omega})e^{jk\Delta\omega n}\Delta\omega.$$

By the eigenfunction property of linear systems and by the principle of superposition, the corresponding output will be

$$y[n] = \lim_{\Delta\omega\to 0}\frac{1}{2\pi}\sum_k H(e^{jk\Delta\omega})X(e^{jk\Delta\omega})e^{jk\Delta\omega n}\Delta\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})X(e^{j\omega})e^{j\omega n}d\omega.$$

Thus, we conclude that

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}),$$

as in Eq. (164).

9.7 The Modulation or Windowing Theorem

If

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

and

$$w[n] \overset{\mathcal{F}}{\longleftrightarrow} W(e^{j\omega}),$$

and if

$$y[n] = x[n]w[n], \tag{166}$$

then

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta. \tag{167}$$

Equation (167) is a periodic convolution, i.e., a convolution of two periodic functions with the limits of integration extending over only one period. The duality inherent in most Fourier transform theorems is evident when we compare the convolution and modulation theorems. However, in contrast to the continuous-time case, where this duality is complete, in the discrete-time case fundamental differences arise because the Fourier transform is a sum, whereas the inverse transform is an integral with a periodic integrand. Although for continuous time, we can state that convolution in the time domain is represented by multiplication in the frequency domain and vice versa; in discrete time, this statement must be modified somewhat. Specifically, discrete-time convolution of sequences (the convolution sum) is equivalent to multiplication of corresponding periodic Fourier transforms, and multiplication of sequences is equivalent to *periodic* convolution of corresponding Fourier transforms.

The theorems of this section and a number of fundamental Fourier transform pairs are summarized in Tables 2 and 3, respectively. One of the ways that knowledge of Fourier transform theorems and properties is useful is in determining Fourier transforms

TABLE 3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\|r\| < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2 e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \|\omega\| < \omega_c, \\ 0, & \omega_c < \|\omega\| \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

or inverse transforms. Often, by using the theorems and known transform pairs, it is possible to represent a sequence in terms of operations on other sequences for which the transform is known, thereby simplifying an otherwise difficult or tedious problem. Examples 22–25 illustrate this approach.

Example 22 Determining a Fourier Transform Using Tables 2 and 3

Suppose we wish to find the Fourier transform of the sequence $x[n] = a^n u[n-5]$. This transform can be computed by exploiting Theorems 1 and 2 of Table 2 and transform pair 4 of Table 3. Let $x_1[n] = a^n u[n]$. We start with this signal because it is the most similar signal to $x[n]$ in Table 3. The table states that

$$X_1(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}. \tag{168}$$

To obtain $x[n]$ from $x_1[n]$, we first delay $x_1[n]$ by five samples, i.e., $x_2[n] = x_1[n-5]$. Theorem 2 of Table 2 gives the corresponding frequency-domain relationship, $X_2(e^{j\omega}) = e^{-j5\omega} X_1(e^{j\omega})$, so

$$X_2(e^{j\omega}) = \frac{e^{-j5\omega}}{1-ae^{-j\omega}}. \tag{169}$$

To get from $x_2[n]$ to the desired $x[n]$, we need only multiply by the constant a^5, i.e., $x[n] = a^5 x_2[n]$. The linearity property of the Fourier transform, Theorem 1 of Table 2, then yields the desired Fourier transform,

$$X(e^{j\omega}) = \frac{a^5 e^{-j5\omega}}{1 - ae^{-j\omega}}. \tag{170}$$

Example 23 Determining an Inverse Fourier Transform Using Tables 2 and 3

Suppose that

$$X(e^{j\omega}) = \frac{1}{(1 - ae^{-j\omega})(1 - be^{-j\omega})}. \tag{171}$$

Direct substitution of $X(e^{j\omega})$ into Eq. (130) leads to an integral that is difficult to evaluate by ordinary real integration techniques. However, using the technique of partial fraction expansion, we can expand $X(e^{j\omega})$ into the form

$$X(e^{j\omega}) = \frac{a/(a-b)}{1 - ae^{-j\omega}} - \frac{b/(a-b)}{1 - be^{-j\omega}}. \tag{172}$$

From Theorem 1 of Table 2 and transform pair 4 of Table 3, it follows that

$$x[n] = \left(\frac{a}{a-b}\right) a^n u[n] - \left(\frac{b}{a-b}\right) b^n u[n]. \tag{173}$$

Example 24 Determining the Impulse Response from the Frequency Response

The frequency response of a highpass filter with linear phase is

$$H(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & \omega_c < |\omega| < \pi, \\ 0, & |\omega| < \omega_c, \end{cases} \tag{174}$$

where a period of 2π is understood. This frequency response can be expressed as

$$H(e^{j\omega}) = e^{-j\omega n_d}(1 - H_{\text{lp}}(e^{j\omega})) = e^{-j\omega n_d} - e^{-j\omega n_d} H_{\text{lp}}(e^{j\omega}),$$

where $H_{\text{lp}}(e^{j\omega})$ is periodic with period 2π and

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| < \pi. \end{cases}$$

Using the result of Example 18 to obtain the inverse transform of $H_{\text{lp}}(e^{j\omega})$, together with properties 1 and 2 of Table 2, we have

$$\begin{aligned} h[n] &= \delta[n - n_d] - h_{\text{lp}}[n - n_d] \\ &= \delta[n - n_d] - \frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)}. \end{aligned}$$

Example 25 Determining the Impulse Response for a Difference Equation

In this example, we determine the impulse response for a stable LTI system for which the input $x[n]$ and output $y[n]$ satisfy the linear constant-coefficient difference equation

$$y[n] - \tfrac{1}{2}y[n-1] = x[n] - \tfrac{1}{4}x[n-1]. \tag{175}$$

The z-transform is more useful than the Fourier transform for dealing with difference equations. However, this example offers a hint of the utility of transform methods in the analysis of linear systems. To find the impulse response, we set $x[n] = \delta[n]$; with $h[n]$ denoting the impulse response, Eq. (175) becomes

$$h[n] - \tfrac{1}{2}h[n-1] = \delta[n] - \tfrac{1}{4}\delta[n-1]. \tag{176}$$

Applying the Fourier transform to both sides of Eq. (176) and using properties 1 and 2 of Table 2, we obtain

$$H(e^{j\omega}) - \tfrac{1}{2}e^{-j\omega}H(e^{j\omega}) = 1 - \tfrac{1}{4}e^{-j\omega}, \tag{177}$$

or

$$H(e^{j\omega}) = \frac{1 - \frac{1}{4}e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}. \tag{178}$$

To obtain $h[n]$, we want to determine the inverse Fourier transform of $H(e^{j\omega})$. Toward this end, we rewrite Eq. (178) as

$$H(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} - \frac{\frac{1}{4}e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}. \tag{179}$$

From transform 4 of Table 3,

$$\left(\tfrac{1}{2}\right)^n u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - \frac{1}{2}e^{-j\omega}}.$$

Combining this transform with property 2 of Table 2, we obtain

$$-\left(\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right)^{n-1} u[n-1] \stackrel{\mathcal{F}}{\longleftrightarrow} -\frac{\frac{1}{4}e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}. \tag{180}$$

Based on property 1 of Table 2, then,

$$h[n] = \left(\tfrac{1}{2}\right)^n u[n] - \left(\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right)^{n-1} u[n-1]. \tag{181}$$

10 DISCRETE-TIME RANDOM SIGNALS

The preceding sections have focused on mathematical representations of discrete-time signals and systems and the insights that derive from such mathematical representations. Discrete-time signals and systems have both a time-domain and a frequency-domain representation, each with an important place in the theory and design of discrete-time signal-processing systems. Until now, we have assumed that the signals are deterministic, i.e., that each value of a sequence is uniquely determined by a mathematical expression, a table of data, or a rule of some type.

In many situations, the processes that generate signals are so complex as to make precise description of a signal extremely difficult or undesirable, if not impossible. In such cases, modeling the signal as a random process is analytically useful.[5] As an example, many of the effects encountered in implementing digital signal-processing algorithms with finite register length can be represented by additive noise, i.e., a random sequence. Many mechanical systems generate acoustic or vibratory signals that can be processed to diagnose potential failure; again, signals of this type are often best modeled in terms of random signals. Speech signals to be processed for automatic recognition or bandwidth compression and music to be processed for quality enhancement are two more of many examples.

A random signal is considered to be a member of an ensemble of discrete-time signals that is characterized by a set of probability density functions. More specifically, for a particular signal at a particular time, the amplitude of the signal sample at that time is assumed to have been determined by an underlying scheme of probabilities. That is, each individual sample $x[n]$ of a particular signal is assumed to be an outcome of some underlying random variable x_n. The entire signal is represented by a collection of such random variables, one for each sample time, $-\infty < n < \infty$. This collection of random variables is referred to as a *random process*, and we assume that a particular sequence of samples $x[n]$ for $-\infty < n < \infty$ has been generated by the random process that underlies the signal. To completely describe the random process, we need to specify the individual and joint probability distributions of all the random variables.

The key to obtaining useful results from such models of signals lies in their description in terms of averages that can be computed from assumed probability laws or estimated from specific signals. While random signals are not absolutely summable or square summable and, consequently, do not directly have Fourier transforms, many (but not all) of the properties of such signals can be summarized in terms of averages such as the autocorrelation or autocovariance sequence, for which the Fourier transform often exists. As we will discuss in this section, the Fourier transform of the autocorrelation sequence has a useful interpretation in terms of the frequency distribution of the power in the signal. The use of the autocorrelation sequence and its transform has another important advantage: The effect of processing random signals with a discrete-time linear system can be conveniently described in terms of the effect of the system on the autocorrelation sequence.

In the following discussion, we assume that the reader is familiar with the basic concepts of random processes, such as averages, correlation and covariance functions, and the power spectrum. A detailed presentation of the theory of random signals can be found in a variety of excellent texts, such as Davenport (1970), and Papoulis (2002), Gray and Davidson (2004), Kay (2006) and Bertsekas and Tsitsiklis (2008).

Our primary objective in this section is to present a specific set of results that will be useful. Therefore, we focus on wide-sense stationary random signals and their representation in the context of processing with LTI systems. Although, for simplicity,

[5]It is common in the signal processing literature to use the terms "random" and "stochastic" interchangeably. We primarily refer to this class of signals as random signals or random processes.

we assume that $x[n]$ and $h[n]$ are real valued, the results can be generalized to the complex case.

Consider a stable LTI system with real impulse response $h[n]$. Let $x[n]$ be a real-valued sequence that is a sample sequence of a wide-sense stationary discrete-time random process. Then, the output of the linear system is also a sample sequence of a discrete-time random process related to the input process by the linear transformation

$$y[n] = \sum_{k=-\infty}^{\infty} h[n-k]x[k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k].$$

As we have shown, since the system is stable, $y[n]$ will be bounded if $x[n]$ is bounded. We will see shortly that if the input is stationary,[6] then so is the output. The input signal may be characterized by its mean m_x and its autocorrelation function $\phi_{xx}[m]$, or we may also have additional information about 1st- or even 2nd-order probability distributions. In characterizing the output random process $y[n]$ we desire similar information. For many applications, it is sufficient to characterize both the input and output in terms of simple averages, such as the mean, variance, and autocorrelation. Therefore, we will derive input–output relationships for these quantities.

The means of the input and output processes are, respectively,

$$m_{x_n} = \mathcal{E}\{\mathrm{x}_n\}, \qquad m_{y_n} = \mathcal{E}\{\mathrm{y}_n\}, \tag{182}$$

where $\mathcal{E}\{\cdot\}$ denotes the expected value of a random variable. In most of our discussion, it will not be necessary to carefully distinguish between the random variables x_n and y_n and their specific values $x[n]$ and $y[n]$. This will simplify the mathematical notation significantly. For example, Eqs. (182) will alternatively be written

$$m_x[n] = \mathcal{E}\{x[n]\}, \qquad m_y[n] = \mathcal{E}\{y[n]\}. \tag{183}$$

If $x[n]$ is stationary, then $m_x[n]$ is independent of n and will be written as m_x, with similar notation for $m_y[n]$ if $y[n]$ is stationary.

The mean of the output process is

$$m_y[n] = \mathcal{E}\{y[n]\} = \sum_{k=-\infty}^{\infty} h[k]\mathcal{E}\{x[n-k]\},$$

where we have used the fact that the expected value of a sum is the sum of the expected values. Since the input is stationary, $m_x[n-k] = m_x$, and consequently,

$$m_y[n] = m_x \sum_{k=-\infty}^{\infty} h[k]. \tag{184}$$

From Eq. (184), we see that the mean of the output is also constant. An equivalent expression to Eq. (184) in terms of the frequency response is

$$m_y = H(e^{j0})m_x. \tag{185}$$

[6]We will use the term *stationary* to mean "wide-sense stationary," i.e., that $E\{x[n_1]x[n_2]\}$ for all n_1, n_2 depends only on the difference $(n_1 - n_2)$. Equivalently, the autocorrelation is only a function of the time difference $(n_1 - n_2)$.

Assuming temporarily that the output is nonstationary, the autocorrelation function of the output process for a real input is

$$\begin{aligned}\phi_{yy}[n, n+m] &= \mathcal{E}\{y[n]y[n+m]\}\\ &= \mathcal{E}\left\{\sum_{k=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} h[k]h[r]x[n-k]x[n+m-r]\right\}\\ &= \sum_{k=-\infty}^{\infty} h[k]\sum_{r=-\infty}^{\infty} h[r]\mathcal{E}\{x[n-k]x[n+m-r]\}.\end{aligned}$$

Since $x[n]$ is assumed to be stationary, $\mathcal{E}\{x[n-k]x[n+m-r]\}$ depends only on the time difference $m+k-r$. Therefore,

$$\phi_{yy}[n, n+m] = \sum_{k=-\infty}^{\infty} h[k]\sum_{r=-\infty}^{\infty} h[r]\phi_{xx}[m+k-r] = \phi_{yy}[m]. \tag{186}$$

That is, the output autocorrelation sequence also depends only on the time difference m. Thus, for an LTI system having a wide-sense stationary input, the output is also wide-sense stationary.

By making the substitution $\ell = r-k$, we can express Eq. (186) as

$$\begin{aligned}\phi_{yy}[m] &= \sum_{\ell=-\infty}^{\infty} \phi_{xx}[m-\ell]\sum_{k=-\infty}^{\infty} h[k]h[\ell+k]\\ &= \sum_{\ell=-\infty}^{\infty} \phi_{xx}[m-\ell]c_{hh}[\ell],\end{aligned} \tag{187}$$

where we have defined

$$c_{hh}[\ell] = \sum_{k=-\infty}^{\infty} h[k]h[\ell+k]. \tag{188}$$

The sequence $c_{hh}[\ell]$ is referred to as the *deterministic autocorrelation sequence* or, simply, the *autocorrelation sequence of* $h[n]$. It should be emphasized that $c_{hh}[\ell]$ is the autocorrelation of an aperiodic—i.e., finite-energy—sequence and should not be confused with the autocorrelation of an infinite-energy random sequence. Indeed, it can be seen that $c_{hh}[\ell]$ is simply the discrete convolution of $h[n]$ with $h[-n]$. Equation (187), then, can be interpreted to mean that the autocorrelation of the output of a linear system is the convolution of the autocorrelation of the input with the aperiodic autocorrelation of the system impulse response.

Equation (187) suggests that Fourier transforms may be useful in characterizing the response of an LTI system to a random input. Assume, for convenience, that $m_x = 0$; i.e., the autocorrelation and autocovariance sequences are identical. Then, with $\Phi_{xx}(e^{j\omega})$, $\Phi_{yy}(e^{j\omega})$, and $C_{hh}(e^{j\omega})$ denoting the Fourier transforms of $\phi_{xx}[m]$, $\phi_{yy}[m]$, and $c_{hh}[\ell]$, respectively, from Eq. (187),

$$\Phi_{yy}(e^{j\omega}) = C_{hh}(e^{j\omega})\Phi_{xx}(e^{j\omega}). \tag{189}$$

Also, from Eq. (188),

$$\begin{aligned}C_{hh}(e^{j\omega}) &= H(e^{j\omega})H^*(e^{j\omega})\\ &= |H(e^{j\omega})|^2,\end{aligned}$$

so

$$\Phi_{yy}(e^{j\omega}) = |H(e^{j\omega})|^2 \Phi_{xx}(e^{j\omega}). \tag{190}$$

Equation (190) provides the motivation for the term *power density spectrum*. Specifically,

$$\begin{aligned}\mathcal{E}\{y^2[n]\} = \phi_{yy}[0] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{yy}(e^{j\omega})\,d\omega \\ &= \text{total average power in output.}\end{aligned} \tag{191}$$

Substituting Eq. (190) into Eq. (191), we have

$$\mathcal{E}\{y^2[n]\} = \phi_{yy}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 \Phi_{xx}(e^{j\omega})\,d\omega. \tag{192}$$

Suppose that $H(e^{j\omega})$ is an ideal bandpass filter, as shown in Figure 18(c). Since $\phi_{xx}[m]$ is a real, even sequence, its Fourier transform is also real and even, i.e.,

$$\Phi_{xx}(e^{j\omega}) = \Phi_{xx}(e^{-j\omega}).$$

Likewise, $|H(e^{j\omega})|^2$ is an even function of ω. Therefore, we can write

$$\begin{aligned}\phi_{yy}[0] &= \text{average power in output} \\ &= \frac{1}{2\pi}\int_{\omega_a}^{\omega_b} \Phi_{xx}(e^{j\omega})\,d\omega + \frac{1}{2\pi}\int_{-\omega_b}^{-\omega_a} \Phi_{xx}(e^{j\omega})\,d\omega.\end{aligned} \tag{193}$$

Thus, the area under $\Phi_{xx}(e^{j\omega})$ for $\omega_a \le |\omega| \le \omega_b$ can be taken to represent the mean-square value of the input in that frequency band. We observe that the output power must remain nonnegative, so

$$\lim_{(\omega_b-\omega_a)\to 0} \phi_{yy}[0] \ge 0.$$

This result, together with Eq. (193) and the fact that the band $\omega_a \le \omega \le \omega_b$ can be arbitrarily small, implies that

$$\Phi_{xx}(e^{j\omega}) \ge 0 \qquad \text{for all } \omega. \tag{194}$$

Hence, we note that the power density function of a real signal is real, even, and nonnegative.

Example 26 White Noise

The concept of white noise is exceedingly useful in a wide variety of contexts in the design and analysis of signal processing and communications systems. A white-noise signal is a signal for which $\phi_{xx}[m] = \sigma_x^2\delta[m]$. We assume in this example that the signal has zero mean. The power spectrum of a white-noise signal is a constant, i.e.,

$$\Phi_{xx}(e^{j\omega}) = \sigma_x^2 \qquad \text{for all } \omega.$$

The average power of a white-noise signal is therefore

$$\phi_{xx}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_{xx}(e^{j\omega})\,d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sigma_x^2\,d\omega = \sigma_x^2.$$

The concept of white noise is also useful in the representation of random signals whose power spectra are not constant with frequency. For example, a random signal $y[n]$ with power spectrum $\Phi_{yy}(e^{j\omega})$ can be assumed to be the output of an LTI system with a white-noise input. That is, we use Eq. (190) to define a system with frequency response $H(e^{j\omega})$ to satisfy the equation

$$\Phi_{yy}(e^{j\omega}) = |H(e^{j\omega})|^2\sigma_x^2,$$

where σ_x^2 is the average power of the assumed white-noise input signal. We adjust the average power of this input signal to give the correct average power for $y[n]$. For example, suppose that $h[n] = a^n u[n]$. Then,

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}},$$

and we can represent all random signals whose power spectra are of the form

$$\Phi_{yy}(e^{j\omega}) = \left|\frac{1}{1 - ae^{-j\omega}}\right|^2\sigma_x^2 = \frac{\sigma_x^2}{1 + a^2 - 2a\cos\omega}.$$

Another important result concerns the cross-correlation between the input and output of an LTI system:

$$\begin{aligned}\phi_{yx}[m] &= \mathcal{E}\{x[n]y[n+m]\}\\ &= \mathcal{E}\left\{x[n]\sum_{k=-\infty}^{\infty}h[k]x[n+m-k]\right\}\\ &= \sum_{k=-\infty}^{\infty}h[k]\phi_{xx}[m-k].\end{aligned} \tag{195}$$

In this case, we note that the cross-correlation between input and output is the convolution of the impulse response with the input autocorrelation sequence.

The Fourier transform of Eq. (195) is

$$\Phi_{yx}(e^{j\omega}) = H(e^{j\omega})\Phi_{xx}(e^{j\omega}). \tag{196}$$

This result has a useful application when the input is white noise, i.e., when $\phi_{xx}[m] = \sigma_x^2\delta[m]$. Substituting into Eq. (195), we note that

$$\phi_{yx}[m] = \sigma_x^2 h[m]. \tag{197}$$

That is, for a zero-mean white-noise input, the cross-correlation between input and output of a linear system is proportional to the impulse response of the system. Similarly, the power spectrum of a white-noise input is

$$\Phi_{xx}(e^{j\omega}) = \sigma_x^2, \qquad -\pi \le \omega \le \pi. \tag{198}$$

Thus, from Eq. (196),

$$\Phi_{yx}(e^{j\omega}) = \sigma_x^2 H(e^{j\omega}). \tag{199}$$

In other words, the cross power spectrum is in this case proportional to the frequency response of the system. Equations (197) and (199) may serve as the basis for estimating the impulse response or frequency response of an LTI system if it is possible to observe the output of the system in response to a white-noise input. An example application is in the measurement of the acoustic impulse response of a room or concert hall.

11 SUMMARY

In this chapter, we have reviewed and discussed a number of basic definitions relating to discrete-time signals and systems. We considered the definition of a set of basic sequences, the definition and representation of LTI systems in terms of the convolution sum, and some implications of stability and causality. The class of systems for which the input and output satisfy a linear constant-coefficient difference equation with initial rest conditions was shown to be an important subclass of LTI systems. The recursive solution of such difference equations was discussed and the classes of FIR and IIR systems defined.

An important means for the analysis and representation of LTI systems lies in their frequency-domain representation. The response of a system to a complex exponential input was considered, leading to the definition of the frequency response. The relation between impulse response and frequency response was then interpreted as a Fourier transform pair.

We called attention to many properties of Fourier transform representations and discussed a variety of useful Fourier transform pairs. Tables 1 and 2 summarize the properties and theorems, and Table 3 contains some useful Fourier transform pairs.

The chapter concluded with an introduction to discrete-time random signals.

Problems

Basic Problems with Answers

1. For each of the following systems, determine whether the system is (1) stable, (2) causal, (3) linear, (4) time invariant, and (5) memoryless:

(a) $T(x[n]) = g[n]x[n]$ with $g[n]$ given

(b) $T(x[n]) = \sum_{k=n_0}^{n} x[k] \qquad n \neq 0$

(c) $T(x[n]) = \sum_{k=n-n_0}^{n+n_0} x[k]$

(d) $T(x[n]) = x[n - n_0]$

(e) $T(x[n]) = e^{x[n]}$
(f) $T(x[n]) = ax[n] + b$
(g) $T(x[n]) = x[-n]$
(h) $T(x[n]) = x[n] + 3u[n+1]$.

2. **(a)** The impulse response $h[n]$ of an LTI system is known to be zero, except in the interval $N_0 \leq n \leq N_1$. The input $x[n]$ is known to be zero, except in the interval $N_2 \leq n \leq N_3$. As a result, the output is constrained to be zero, except in some interval $N_4 \leq n \leq N_5$. Determine N_4 and N_5 in terms of N_0, N_1, N_2, and N_3.
(b) If $x[n]$ is zero, except for N consecutive points, and $h[n]$ is zero, except for M consecutive points, what is the maximum number of consecutive points for which $y[n]$ can be nonzero?

3. By direct evaluation of the convolution sum, determine the unit step response ($x[n] = u[n]$) of an LTI system whose impulse response is

$$h[n] = a^{-n}u[-n], \qquad 0 < a < 1.$$

4. Consider the linear constant-coefficient difference equation

$$y[n] - \tfrac{3}{4}y[n-1] + \tfrac{1}{8}y[n-2] = 2x[n-1].$$

Determine $y[n]$ for $n \geq 0$ when $x[n] = \delta[n]$ and $y[n] = 0, n < 0$.

5. A causal LTI system is described by the difference equation

$$y[n] - 5y[n-1] + 6y[n-2] = 2x[n-1].$$

(a) Determine the homogeneous response of the system, i.e., the possible outputs if $x[n] = 0$ for all n.
(b) Determine the impulse response of the system.
(c) Determine the step response of the system.

6. **(a)** Determine the frequency response $H(e^{j\omega})$ of the LTI system whose input and output satisfy the difference equation

$$y[n] - \tfrac{1}{2}y[n-1] = x[n] + 2x[n-1] + x[n-2].$$

(b) Write a difference equation that characterizes a system whose frequency response is

$$H(e^{j\omega}) = \frac{1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}}{1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}}.$$

7. Determine whether each of the following signals is periodic. If the signal is periodic, state its period.

(a) $x[n] = e^{j(\pi n/6)}$
(b) $x[n] = e^{j(3\pi n/4)}$
(c) $x[n] = [\sin(\pi n/5)]/(\pi n)$
(d) $x[n] = e^{j\pi n/\sqrt{2}}$.

8. An LTI system has impulse response $h[n] = 5(-1/2)^n u[n]$. Use the Fourier transform to find the output of this system when the input is $x[n] = (1/3)^n u[n]$.

9. Consider the difference equation

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = \frac{1}{3}x[n-1].$$

(a) What are the impulse response, frequency response, and step response for the causal LTI system satisfying this difference equation?
(b) What is the general form of the homogeneous solution of the difference equation?
(c) Consider a different system satisfying the difference equation that is neither causal nor LTI, but that has $y[0] = y[1] = 1$. Find the response of this system to $x[n] = \delta[n]$.

10. Determine the output of an LTI system if the impulse response $h[n]$ and the input $x[n]$ are as follows:

(a) $x[n] = u[n]$ and $h[n] = a^n u[-n-1]$, with $a > 1$.
(b) $x[n] = u[n-4]$ and $h[n] = 2^n u[-n-1]$.
(c) $x[n] = u[n]$ and $h[n] = (0.5)2^n u[-n]$.
(d) $h[n] = 2^n u[-n-1]$ and $x[n] = u[n] - u[n-10]$.

Use your knowledge of linearity and time invariance to minimize the work in parts (b)–(d).

11. Consider an LTI system with frequency response

$$H(e^{j\omega}) = \frac{1 - e^{-j2\omega}}{1 + \frac{1}{2}e^{-j4\omega}}, \qquad -\pi < \omega \leq \pi.$$

Determine the output $y[n]$ for all n if the input $x[n]$ for all n is

$$x[n] = \sin\left(\frac{\pi n}{4}\right).$$

12. Consider a system with input $x[n]$ and output $y[n]$ that satisfy the difference equation

$$y[n] = ny[n-1] + x[n].$$

The system is causal and satisfies initial-rest conditions; i.e., if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$.

(a) If $x[n] = \delta[n]$, determine $y[n]$ for all n.
(b) Is the system linear? Justify your answer.
(c) Is the system time invariant? Justify your answer.

13. Indicate which of the following discrete-time signals are eigenfunctions of stable, LTI discrete-time systems:

(a) $e^{j2\pi n/3}$
(b) 3^n
(c) $2^n u[-n-1]$
(d) $\cos(\omega_0 n)$
(e) $(1/4)^n$
(f) $(1/4)^n u[n] + 4^n u[-n-1]$.

14. A single input–output relationship is given for each of the following three systems:

(a) System A: $x[n] = (1/3)^n, \quad y[n] = 2(1/3)^n$.
(b) System B: $x[n] = (1/2)^n, \quad y[n] = (1/4)^n$.
(c) System C: $x[n] = (2/3)^n u[n], \quad y[n] = 4(2/3)^n u[n] - 3(1/2)^n u[n]$.

Based on this information, pick the strongest possible conclusion that you can make about each system from the following list of statements:

(i) The system cannot possibly be LTI.
(ii) The system must be LTI.

(iii) The system can be LTI, and there is only one LTI system that satisfies this input–output constraint.
(iv) The system can be LTI, but cannot be uniquely determined from the information in this input–output constraint.

If you chose option (iii) from this list, specify either the impulse response $h[n]$ or the frequency response $H(e^{j\omega})$ for the LTI system.

15. Consider the system illustrated in Figure P15. The output of an LTI system with an impulse response $h[n] = \left(\frac{1}{4}\right)^n u[n + 10]$ is multiplied by a unit step function $u[n]$ to yield the output of the overall system. Answer each of the following questions, and briefly justify your answers:

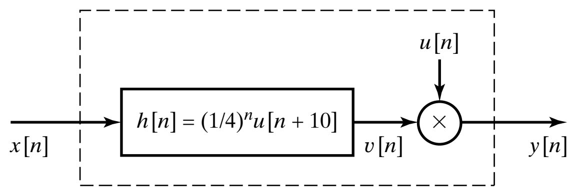


Figure P15

(a) Is the overall system LTI?
(b) Is the overall system causal?
(c) Is the overall system stable in the BIBO sense?

16. Consider the following difference equation:

$$y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = 3x[n].$$

(a) Determine the general form of the homogeneous solution to this difference equation.
(b) Both a causal and an anticausal LTI system are characterized by this difference equation. Find the impulse responses of the two systems.
(c) Show that the causal LTI system is stable and the anticausal LTI system is unstable.
(d) Find a particular solution to the difference equation when $x[n] = (1/2)^n u[n]$.

17. (a) Determine the Fourier transform of the sequence

$$r[n] = \begin{cases} 1, & 0 \le n \le M, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Consider the sequence

$$w[n] = \begin{cases} \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{M}\right)\right], & 0 \le n \le M, \\ 0, & \text{otherwise.} \end{cases}$$

Sketch $w[n]$ and express $W(e^{j\omega})$, the Fourier transform of $w[n]$, in terms of $R(e^{j\omega})$, the Fourier transform of $r[n]$. (*Hint*: First express $w[n]$ in terms of $r[n]$ and the complex exponentials $e^{j(2\pi n/M)}$ and $e^{-j(2\pi n/M)}$.)

(c) Sketch the magnitude of $R(e^{j\omega})$ and $W(e^{j\omega})$ for the case when $M = 4$.

18. For each of the following impulse responses of LTI systems, indicate whether or not the system is causal:

(a) $h[n] = (1/2)^n u[n]$
(b) $h[n] = (1/2)^n u[n-1]$
(c) $h[n] = (1/2)^{|n|}$
(d) $h[n] = u[n+2] - u[n-2]$
(e) $h[n] = (1/3)^n u[n] + 3^n u[-n-1]$.

19. For each of the following impulse responses of LTI systems, indicate whether or not the system is stable:

(a) $h[n] = 4^n u[n]$
(b) $h[n] = u[n] - u[n-10]$
(c) $h[n] = 3^n u[-n-1]$
(d) $h[n] = \sin(\pi n/3)u[n]$
(e) $h[n] = (3/4)^{|n|} \cos(\pi n/4 + \pi/4)$
(f) $h[n] = 2u[n+5] - u[n] - u[n-5]$.

20. Consider the difference equation representing a causal LTI system

$$y[n] + (1/a)y[n-1] = x[n-1].$$

(a) Find the impulse response of the system, $h[n]$, as a function of the constant a.
(b) For what range of values of a will the system be stable?

Basic Problems

21. A discrete-time signal $x[n]$ is shown in Figure P21.

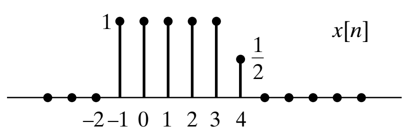


Figure P21

Sketch and label carefully each of the following signals:

(a) $x[n-2]$
(b) $x[4-n]$
(c) $x[2n]$
(d) $x[n]u[2-n]$
(e) $x[n-1]\delta[n-3]$.

22. Consider a discrete-time LTI system with impulse response $h[n]$. If the input $x[n]$ is a periodic sequence with period N (i.e., if $x[n] = x[n+N]$), show that the output $y[n]$ is also a periodic sequence with period N.

23. For each of the following systems, determine whether the system is (1) stable, (2) causal, (3) linear, and (4) time invariant.

(a) $T(x[n]) = (\cos \pi n)x[n]$
(b) $T(x[n]) = x[n^2]$
(c) $T(x[n]) = x[n]\displaystyle\sum_{k=0}^{\infty} \delta[n-k]$
(d) $T(x[n]) = \displaystyle\sum_{k=n-1}^{\infty} x[k]$.

24. Consider an arbitrary linear system with input $x[n]$ and output $y[n]$. Show that if $x[n] = 0$ for all n, then $y[n]$ must also be zero for all n.

25. Consider a system for which the input $x[n]$ and the output $y[n]$ satisfy the following relation.

$$8y[n] + 2y[n-1] - 3y[n-2] = x[n] \qquad \text{(P25-1)}$$

(a) For $x[n] = \delta[n]$, show that a *particular* sequence satisfying the difference equation is $y_p[n] = \frac{3}{40}\left(-\frac{3}{4}\right)^n u[n] + \frac{1}{20}\left(\frac{1}{2}\right)^n u[n]$.
(b) Determine the homogeneous solution(s) to the difference equation specified in Eq. (P25-1).
(c) Determine $y[n]$ for $-2 \le n \le 2$ when $x[n]$ is equal to $\delta[n]$ in Eq. (P25-1) and the *initial rest condition* is assumed in solving the difference equation. Note that the initial rest condition implies the system described by Eq. (P25-1) is causal.

26. For each of the systems in Figure P26, pick the strongest valid conclusion that you can make about each system from the following list of statements:

(i) The system must be LTI and is uniquely specified by the information given.
(ii) The system must be LTI, but cannot be uniquely determined from the information given.
(iii) The system could be LTI and if it is, the information given uniquely specifies the system.
(iv) The system could be LTI, but cannot be uniquely determined from the information given.
(v) The system could not possibly be LTI.

For each system for which you choose option (i) or (iii), give the impulse response $h[n]$ for the uniquely specified LTI system. One example of an input and its corresponding output are shown for each system.

System A:

$$\left(\frac{1}{2}\right)^n \longrightarrow \boxed{\text{System A}} \longrightarrow \left(\frac{1}{4}\right)^n$$

System B:

$$\cos\left(\frac{\pi}{3}n\right) \longrightarrow \boxed{\text{System B}} \longrightarrow 3j\sin\left(\frac{\pi}{3}n\right)$$

System C:

$$\frac{1}{5}\left(\frac{1}{5}\right)^n u[n] \longrightarrow \boxed{\text{System C}} \longrightarrow -6\left(\frac{1}{2}\right)^n u[-n-1] - 6\left(\frac{1}{3}\right)^n u[n]$$

Figure P26

27. For each of the systems in Figure P27, pick the strongest valid conclusion that you can make about each system from the following list of statements:

(i) The system must be LTI and is uniquely specified by the information given.
(ii) The system must be LTI, but cannot be uniquely determined from the information given.
(iii) The system could be LTI, and if it is, the information given uniquely specifies the system.
(iv) The system could be LTI, but cannot be uniquely determined from the information given.
(v) The system could not possibly be LTI.

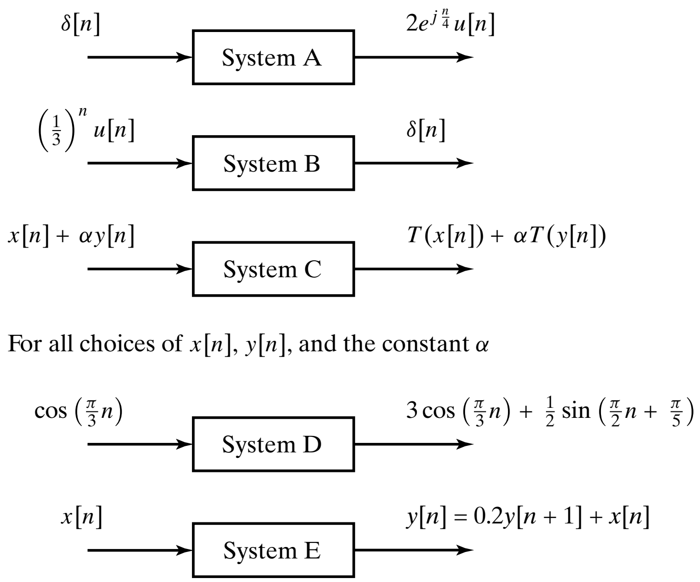


Figure P27

28. Four input–output pairs of a particular system S are specified in Figure P28-1:

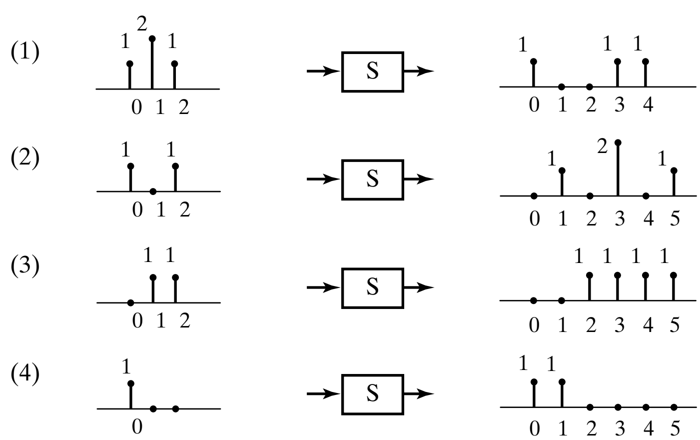


Figure P28–1

(a) Can system S be time-invariant? Explain.
(b) Can system S be linear? Explain.
(c) Suppose (2) and (3) are input–output pairs of a particular system S_2, and the system is known to be LTI. What is $h[n]$, the impulse response of the system?
(d) Suppose (1) is the input–output pair of an LTI system S_3. What is the output of this system for the input in Figure P28-2:

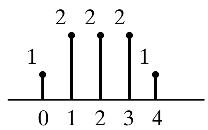


Figure P28–2

29. An LTI system has impulse response defined by

$$h[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0, 1, 2, 3 \\ -2 & n = 4, 5 \\ 0 & n > 5 \end{cases}$$

Determine and plot the output $y[n]$ when the input $x[n]$ is:

(a) $u[n]$
(b) $u[n-4]$
(c) $u[n] - u[n-4]$.

30. Consider the cascade connection of two LTI systems in Figure P30:

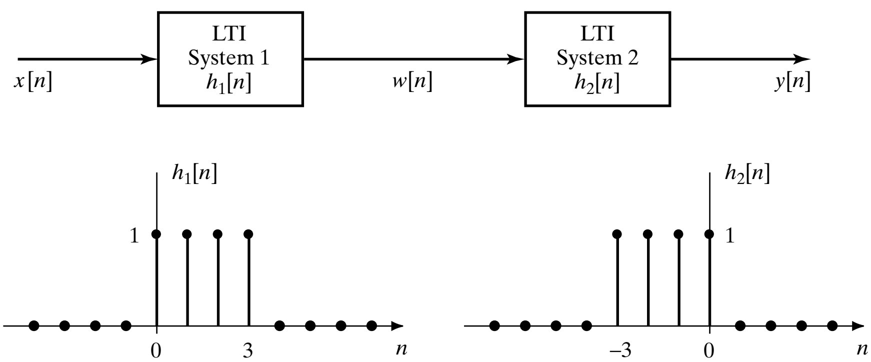


Figure P30

(a) Determine and sketch $w[n]$ if $x[n] = (-1)^n u[n]$. Also, determine the overall output $y[n]$.
(b) Determine and sketch the overall impulse response of the cascade system; i.e., plot the output $y[n] = h[n]$ when $x[n] = \delta[n]$.
(c) Now consider the input $x[n] = 2\delta[n] + 4\delta[n-4] - 2\delta[n-12]$. Sketch $w[n]$.
(d) For the input of part (c), write an expression for the output $y[n]$ in terms of the overall impulse response $h[n]$ as defined in part (b). Make a carefully labeled sketch of your answer.

31. If the input and output of a causal LTI system satisfy the difference equation

$$y[n] = ay[n-1] + x[n],$$

then the impulse response of the system must be $h[n] = a^n u[n]$.

(a) For what values of a is this system stable?

(b) Consider a causal LTI system for which the input and output are related by the difference equation

$$y[n] = ay[n-1] + x[n] - a^N x[n-N],$$

where N is a positive integer. Determine and sketch the impulse response of this system. *Hint*: Use linearity and time-invariance to simplify the solution.

(c) Is the system in part (b) an FIR or an IIR system? Explain.

(d) For what values of a is the system in part (b) stable? Explain.

32. For $X(e^{j\omega}) = 1/(1 - ae^{-j\omega})$, with $-1 < a < 0$, determine and sketch the following as a function of ω:

(a) $\mathcal{R}e\{X(e^{j\omega})\}$

(b) $\mathcal{I}m\{X(e^{j\omega})\}$

(c) $|X(e^{j\omega})|$

(d) $\angle X(e^{j\omega})$.

33. Consider an LTI system defined by the difference equation

$$y[n] = -2x[n] + 4x[n-1] - 2x[n-2].$$

(a) Determine the impulse response of this system.

(b) Determine the frequency response of this system. Express your answer in the form

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\omega n_d},$$

where $A(e^{j\omega})$ is a real function of ω. Explicitly specify $A(e^{j\omega})$ and the delay n_d of this system.

(c) Sketch a plot of the magnitude $|H(e^{j\omega})|$ and a plot of the phase $\angle H(e^{j\omega})$.

(d) Suppose that the input to the system is

$$x_1[n] = 1 + e^{j0.5\pi n} \qquad -\infty < n < \infty.$$

Use the frequency response function to determine the corresponding output $y_1[n]$.

(e) Now suppose that the input to the system is

$$x_2[n] = (1 + e^{j0.5\pi n})u[n] \qquad -\infty < n < \infty.$$

Use the defining difference equation or discrete convolution to determine the corresponding output $y_2[n]$ for $-\infty < n < \infty$. Compare $y_1[n]$ and $y_2[n]$. They should be equal for certain values of n. Over what range of values of n are they equal?

34. An LTI system has the frequency response

$$H(e^{j\omega}) = \frac{1 - 1.25e^{-j\omega}}{1 - 0.8e^{-j\omega}} = 1 - \frac{0.45e^{-j\omega}}{1 - 0.8e^{-j\omega}}.$$

(a) Specify the difference equation that is satisfied by the input $x[n]$ and the output $y[n]$.

(b) Use one of the above forms of the frequency response to determine the impulse response $h[n]$.

(c) Show that $|H(e^{j\omega})|^2 = G^2$, where G is a constant. Determine the constant G. (This is an example of an *allpass filter*.)

(d) If the input to the above system is $x[n] = \cos(0.2\pi n)$, the output should be of the form $y[n] = A\cos(0.2\pi n + \theta)$. What are A and θ?

35. An LTI system has impulse response given by the following plot:

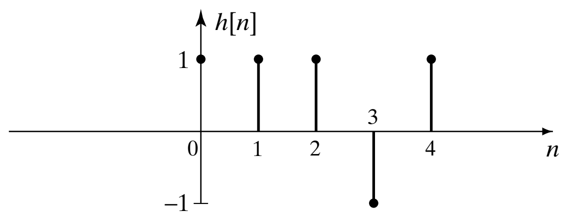


Figure P35-1

The input to the system, $x[n]$, is plotted below as a function of n.

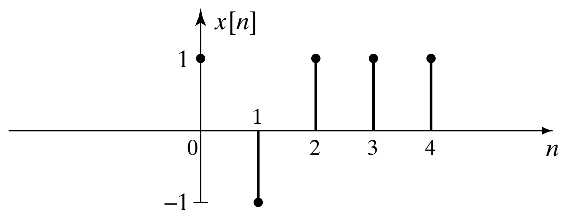


Figure P35-2

(a) Use discrete convolution to determine the output of the system $y[n] = x[n] * h[n]$ for the above input. Give your answer as a carefully labeled sketch of $y[n]$ over a range sufficient to define it completely.

(b) The deterministic autocorrelation of a signal $x[n]$ is defined in Eq. (188) as $c_{xx}[n] = x[n] * x[-n]$. The system defined by Figure P35-1 is a *matched filter* for the input in Figure P35-2. Noting that $h[n] = x[-(n-4)]$, express the output in part (a) in terms of $c_{xx}[n]$.

(c) Determine the output of the system whose impulse response is $h[n]$ when the input is $x[n] = u[n+2]$. Sketch your answer.

36. An LTI discrete-time system has frequency response given by

$$H(e^{j\omega}) = \frac{(1 - je^{-j\omega})(1 + je^{-j\omega})}{1 - 0.8e^{-j\omega}} = \frac{1 + e^{-j2\omega}}{1 - 0.8e^{-j\omega}} = \frac{1}{1 - 0.8e^{-j\omega}} + \frac{e^{-j2\omega}}{1 - 0.8e^{-j\omega}}.$$

(a) Use one of the above forms of the frequency response to obtain an equation for the impulse response $h[n]$ of the system.

(b) From the frequency response, determine the difference equation that is satisfied by the input $x[n]$ and the output $y[n]$ of the system.

(c) If the input to this system is

$$x[n] = 4 + 2\cos(\omega_0 n) \quad \text{for} - \infty < n < \infty,$$

for what value of ω_0 will the output be of the form

$$y[n] = A = \text{constant}$$

for $-\infty < n < \infty$? What is the constant A?

37. Consider the cascade of LTI discrete-time systems shown in Figure P37.

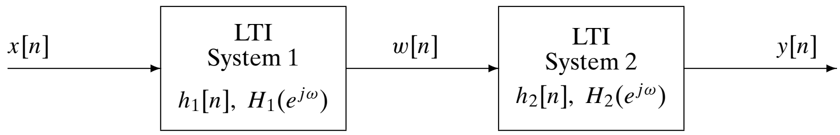


Figure P37

The first system is described by the frequency response

$$H_1(e^{j\omega}) = e^{-j\omega}\begin{cases} 0 & |\omega| \le 0.25\pi \\ 1 & 0.25\pi < |\omega| \le \pi \end{cases}$$

and the second system is described by

$$h_2[n] = 2\frac{\sin(0.5\pi n)}{\pi n}$$

(a) Determine an equation that defines the frequency response, $H(e^{j\omega})$, of the overall system over the range $-\pi \le \omega \le \pi$.
(b) Sketch the magnitude, $|H(e^{j\omega})|$, and the phase, $\angle H(e^{j\omega})$, of the overall frequency response over the range $-\pi \le \omega \le \pi$.
(c) Use any convenient means to determine the impulse response $h[n]$ of the overall cascade system.

38. Consider the cascade of two LTI systems shown in Figure P38.

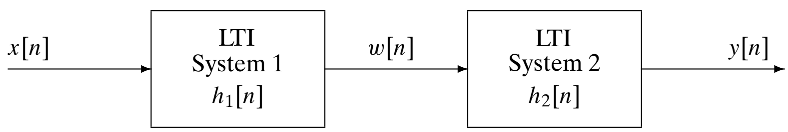


Figure P38

The impulse responses of the two systems are:

$$h_1[n] = u[n-5] \qquad \text{and} \qquad h_2[n] = \begin{cases} 1 & 0 \le n \le 4 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Make a sketch showing both $h_2[k]$ and $h_1[n-k]$ (for some arbitrary $n < 0$) as functions of k.
(b) Determine $h[n] = h_1[n] * h_2[n]$, the impulse response of the overall system. Give your answer as an equation (or set of equations) that define $h[n]$ for $-\infty < n < \infty$ or as a carefully labelled plot of $h[n]$ over a range sufficient to define it completely.

39. Using the definition of linearity (Eqs. (23a)–(23b)), show that the ideal delay system (Example 2) and the moving-average system (Example 3) are both linear systems.

40. Determine which of the following signals is periodic. If a signal is periodic, determine its period.

(a) $x[n] = e^{j(2\pi n/5)}$
(b) $x[n] = \sin(\pi n/19)$
(c) $x[n] = ne^{j\pi n}$
(d) $x[n] = e^{jn}$.

41. Consider an LTI system with $|H(e^{j\omega})| = 1$, and let $\arg[H(e^{j\omega})]$ be as shown in Figure P41. If the input is

$$x[n] = \cos\left(\frac{3\pi}{2}n + \frac{\pi}{4}\right),$$

determine the output $y[n]$.

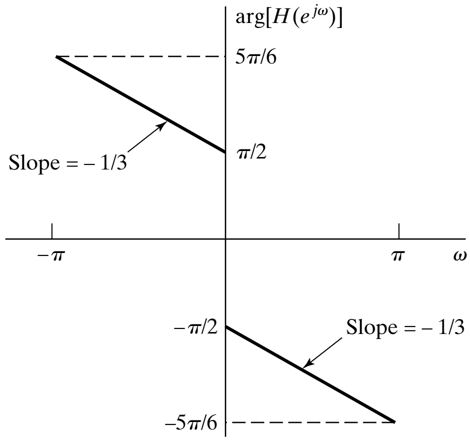


Figure P41

42. The sequences $s[n]$, $x[n]$, and $w[n]$ are sample sequences of wide-sense stationary random processes where

$$s[n] = x[n]w[n].$$

The sequences $x[n]$ and $w[n]$ are zero-mean and statistically independent. The autocorrelation function of $w[n]$ is

$$E\{w[n]w[n+m]\} = \sigma_w^2\delta[m],$$

and the variance of $x[n]$ is σ_x^2.

Show that $s[n]$ is white, with variance $\sigma_x^2\sigma_w^2$.

Advanced Problems

43. The operator T represents an LTI system. As shown in the following figures, if the input to the system T is $(\frac{1}{3})^n u[n]$, the output of the system is $g[n]$. If the input is $x[n]$, the output is $y[n]$.

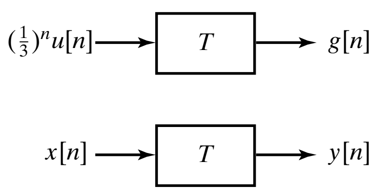


Figure P43

Express $y[n]$ in terms of $g[n]$ and $x[n]$.

44. $X(e^{j\omega})$ denotes the Fourier transform of the complex-valued signal $x[n]$, where the real and imaginary parts of $x[n]$ are given in Figure P44. (*Note*: The sequence is zero outside the interval shown.)

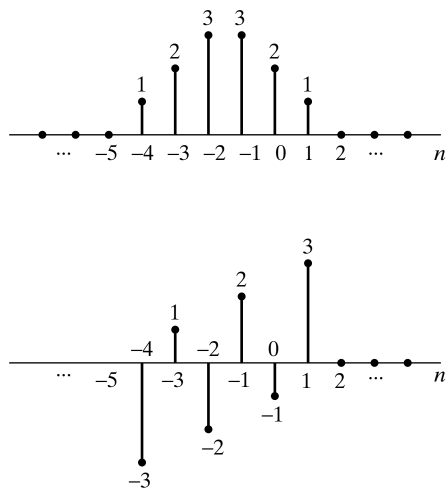


Figure P44

Perform the following calculations without explicitly evaluating $X(e^{j\omega})$.

(a) Evaluate $X(e^{j\omega})|_{\omega=0}$.
(b) Evaluate $X(e^{j\omega})|_{\omega=\pi}$.
(c) Evaluate $\int_{-\pi}^{\pi} X(e^{j\omega})\, d\omega$.
(d) Determine and sketch the signal (in the time domain) whose Fourier transform is $X(e^{-j\omega})$.
(e) Determine and sketch the signal (in the time domain) whose Fourier transform is $j\mathrm{Im}\{X(e^{j\omega})\}$.

45. Consider the cascade of LTI discrete-time systems shown in Figure P45.

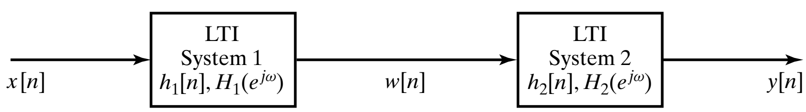


Figure P45

System 1 is described by the difference equation

$$w[n] = x[n] - x[n-1],$$

and System 2 is described by

$$h_2[n] = \frac{\sin(0.5\pi n)}{\pi n} \Longleftrightarrow H_2(e^{j\omega}) = \begin{cases} 1 & |\omega| < 0.5\pi \\ 0 & 0.5\pi < |\omega| < \pi. \end{cases}$$

The input $x[n]$ is

$$x[n] = \cos(0.4\pi n) + \sin(.6\pi n) + 5\delta[n-2] + 2u[n].$$

Determine the overall output $y[n]$.

(*With careful thought, you will be able to use the properties of LTI systems to write down the answer by inspection.*)

46. The DTFT pair

$$a^n u[n] \Longleftrightarrow \frac{1}{1 - ae^{-j\omega}} \qquad |a| < 1 \tag{P46-1}$$

is given.

(a) Using Eq. (P46-1), determine the DTFT, $X(e^{j\omega})$, of the sequence

$$x[n] = -b^n u[-n-1] = \begin{cases} -b^n & n \le -1 \\ 0 & n \ge 0. \end{cases}$$

What restriction on b is necessary for the DTFT of $x[n]$ to exist?

(b) Determine the sequence $y[n]$ whose DTFT is

$$Y(e^{j\omega}) = \frac{2e^{-j\omega}}{1 + 2e^{-j\omega}}.$$

47. Consider a "windowed cosine signal"

$$x[n] = w[n]\cos(\omega_0 n).$$

(a) Determine an expression for $X(e^{j\omega})$ in terms of $W(e^{j\omega})$.

(b) Suppose that the sequence $w[n]$ is the finite-length sequence

$$w[n] = \begin{cases} 1 & -L \le n \le L \\ 0 & \text{otherwise.} \end{cases}$$

Determine the DTFT $W(e^{j\omega})$. *Hint*: Use Tables 2 and 3 to obtain a "closed form" solution. You should find that $W(e^{j\omega})$ is a real function of ω.

(c) Sketch the DTFT $X(e^{j\omega})$ for the window in (b). For a given ω_0, how should L be chosen so that your sketch shows two distinct peaks?

48. The system T in Figure P48 is known to be *time invariant*. When the inputs to the system are $x_1[n]$, $x_2[n]$, and $x_3[n]$, the responses of the system are $y_1[n]$, $y_2[n]$, and $y_3[n]$, as shown.

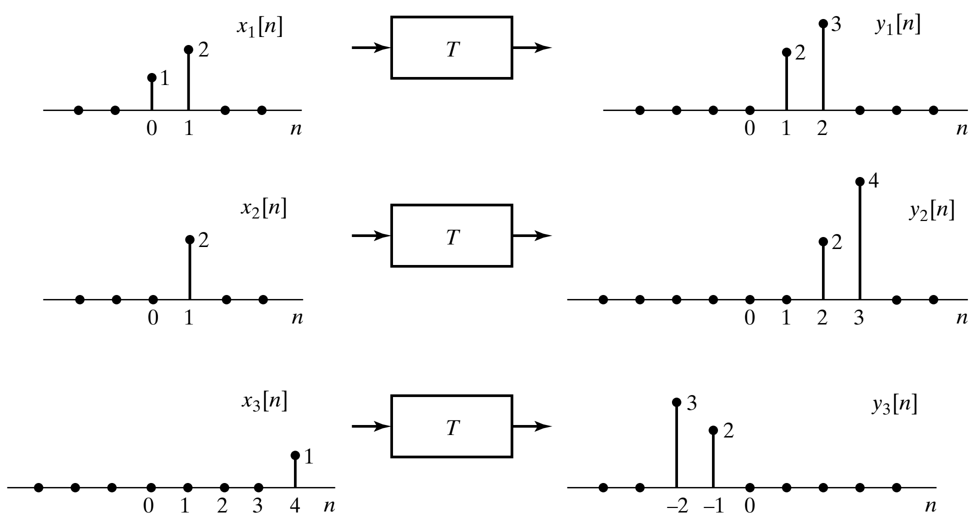


Figure P48

(a) Determine whether the system T could be linear.
(b) If the input $x[n]$ to the system T is $\delta[n]$, what is the system response $y[n]$?
(c) What are all possible inputs $x[n]$ for which the response of the system T can be determined from the given information alone?

49. The system L in Figure P49 is known to be *linear*. Shown are three output signals $y_1[n]$, $y_2[n]$, and $y_3[n]$ in response to the input signals $x_1[n]$, $x_2[n]$, and $x_3[n]$, respectively.

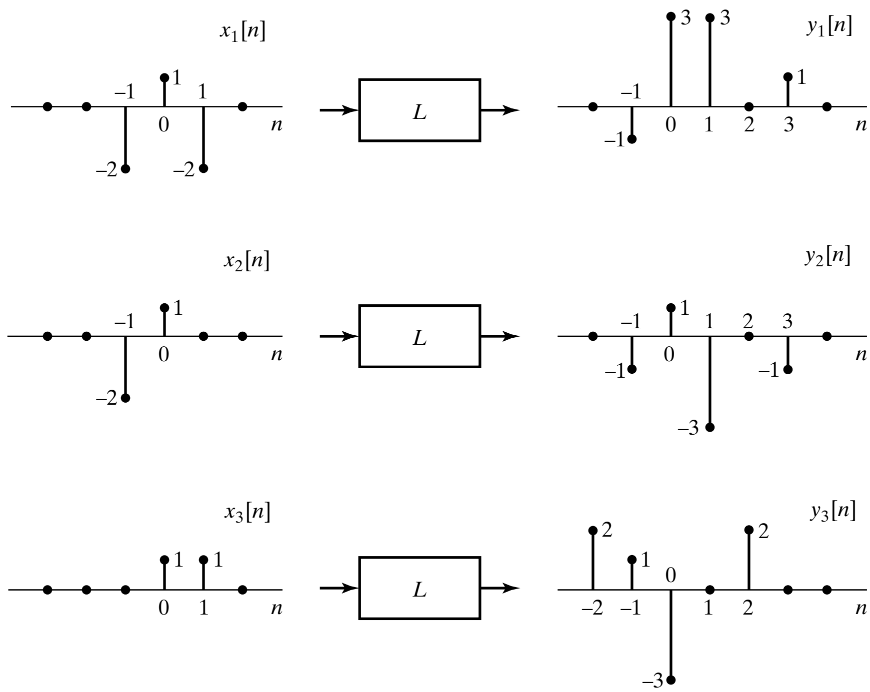


Figure P49

(a) Determine whether the system L could be time invariant.
(b) If the input $x[n]$ to the system L is $\delta[n]$, what is the system response $y[n]$?

50. In Section 5, we stated that the solution to the homogeneous difference equation

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0$$

is of the form

$$y_h[n] = \sum_{m=1}^{N} A_m z_m^n, \tag{P50-1}$$

with the A_m's arbitrary and the z_m's the N roots of the polynomial

$$A(z) = \sum_{k=0}^{N} a_k z^{-k}; \tag{P50-2}$$

i.e.,

$$A(z) = \sum_{k=0}^{N} a_k z^{-k} = \prod_{m=1}^{N} (1 - z_m z^{-1}).$$

(a) Determine the general form of the homogeneous solution to the difference equation

$$y[n] - \tfrac{3}{4}y[n-1] + \tfrac{1}{8}y[n-2] = 2x[n-1].$$

(b) Determine the coefficients A_m in the homogeneous solution if $y[-1] = 1$ and $y[0] = 0$.

(c) Now consider the difference equation

$$y[n] - y[n-1] + \tfrac{1}{4}y[n-2] = 2x[n-1]. \tag{P50-3}$$

If the homogeneous solution contains only terms of the form of Eq. (P50-1), show that the initial conditions $y[-1] = 1$ and $y[0] = 0$ cannot be satisfied.

(d) If Eq. (P50-2) has two roots that are identical, then, in place of Eq. (P50-1), $y_h[n]$ will take the form

$$y_h[n] = \sum_{m=1}^{N-1} A_m z_m^n + nB_1 z_1^n, \tag{P50-4}$$

where we have assumed that the double root is z_1. Using Eq. (P50-4), determine the general form of $y_h[n]$ for Eq. (P50-3). Verify explicitly that your answer satisfies Eq. (P50-3) with $x[n] = 0$.

(e) Determine the coefficients A_1 and B_1 in the homogeneous solution obtained in part (d) if $y[-1] = 1$ and $y[0] = 0$.

51. Consider a system with input $x[n]$ and output $y[n]$. The input–output relation for the system is defined by the following two properties:

1. $y[n] - ay[n-1] = x[n]$,
2. $y[0] = 1$.

(a) Determine whether the system is time invariant.
(b) Determine whether the system is linear.
(c) Assume that the difference equation (property 1) remains the same, but the value $y[0]$ is specified to be zero. Does this change your answer to either part (a) or part (b)?

52. Consider the LTI system with impulse response

$$h[n] = \left(\frac{j}{2}\right)^n u[n], \qquad \text{where } j = \sqrt{-1}.$$

Determine the steady-state response, i.e., the response for large n, to the excitation

$$x[n] = \cos(\pi n)u[n].$$

53. An LTI system has frequency response

$$H(e^{j\omega}) = \begin{cases} e^{-j\omega 3}, & |\omega| < \dfrac{2\pi}{16}\left(\dfrac{3}{2}\right), \\ 0, & \dfrac{2\pi}{16}\left(\dfrac{3}{2}\right) \le |\omega| \le \pi. \end{cases}$$

The input to the system is a periodic unit-impulse train with period $N = 16$; i.e.,

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n + 16k].$$

Determine the output of the system.

54. Consider the system in Figure P54.

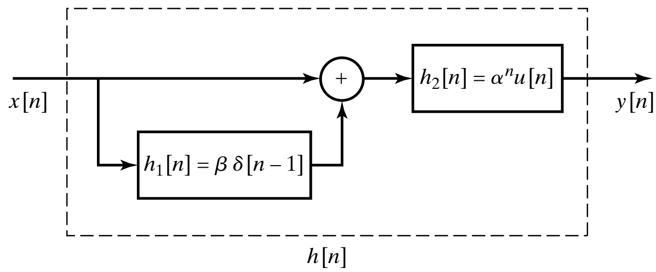


Figure P54

(a) Determine the impulse response $h[n]$ of the overall system.
(b) Determine the frequency response of the overall system.
(c) Specify a difference equation that relates the output $y[n]$ to the input $x[n]$.
(d) Is this system causal? Under what condition would the system be stable?

55. Let $X(e^{j\omega})$ denote the Fourier transform of the signal $x[n]$ shown in Figure P55. Perform the following calculations without explicitly evaluating $X(e^{j\omega})$:

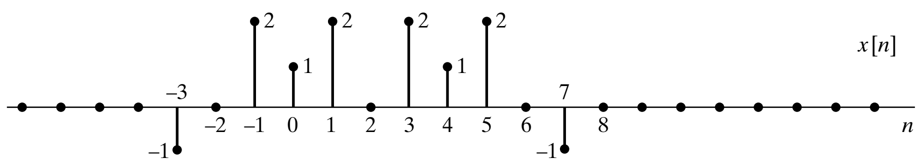


Figure P55

(a) Evaluate $X(e^{j\omega})|_{\omega=0}$.
(b) Evaluate $X(e^{j\omega})|_{\omega=\pi}$.
(c) Find $\angle X(e^{j\omega})$.
(d) Evaluate $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega$.
(e) Determine and sketch the signal whose Fourier transform is $X(e^{-j\omega})$.
(f) Determine and sketch the signal whose Fourier transform is $\mathcal{R}e\{X(e^{j\omega})\}$.

56. For the system in Figure P56, determine the output $y[n]$ when the input $x[n]$ is $\delta[n]$ and $H(e^{j\omega})$ is an ideal lowpass filter as indicated, i.e.,

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \pi/2, \\ 0, & \pi/2 < |\omega| \le \pi. \end{cases}$$

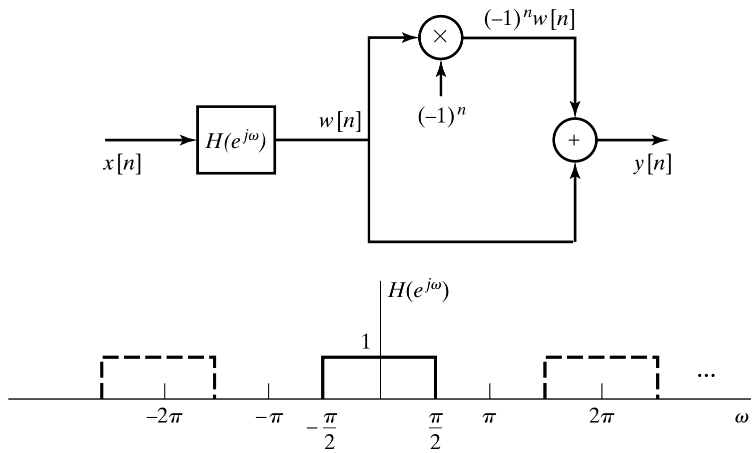


Figure P56

57. A sequence has the DTFT

$$X(e^{j\omega}) = \frac{1 - a^2}{(1 - ae^{-j\omega})(1 - ae^{j\omega})}, \qquad |a| < 1.$$

(a) Find the sequence $x[n]$.
(b) Calculate $1/2\pi \int_{-\pi}^{\pi} X(e^{j\omega}) \cos(\omega) d\omega$.

58. An LTI system is described by the input–output relation

$$y[n] = x[n] + 2x[n-1] + x[n-2].$$

(a) Determine $h[n]$, the impulse response of the system.
(b) Is this a stable system?
(c) Determine $H(e^{j\omega})$, the frequency response of the system. Use trigonometric identities to obtain a simple expression for $H(e^{j\omega})$.
(d) Plot the magnitude and phase of the frequency response.
(e) Now consider a new system whose frequency response is $H_1(e^{j\omega}) = H(e^{j(\omega+\pi)})$. Determine $h_1[n]$, the impulse response of the new system.

59. Let the real discrete-time signal $x[n]$ with Fourier transform $X(e^{j\omega})$ be the input to a system with the output defined by

$$y[n] = \begin{cases} x[n], & \text{if } n \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

(a) Sketch the discrete-time signal $s[n] = 1 + \cos(\pi n)$ and its (generalized) Fourier transform $S(e^{j\omega})$.
(b) Express $Y(e^{j\omega})$, the Fourier transform of the output, as a function of $X(e^{j\omega})$ and $S(e^{j\omega})$.
(c) Suppose that it is of interest to approximate $x[n]$ by the interpolated signal $w[n] = y[n] + (1/2)(y[n+1] + y[n-1])$. Determine the Fourier transform $W(e^{j\omega})$ as a function of $Y(e^{j\omega})$.
(d) Sketch $X(e^{j\omega})$, $Y(e^{j\omega})$, and $W(e^{j\omega})$ for the case when $x[n] = \sin(\pi n/a)/(\pi n/a)$ and $a > 1$. Under what conditions is the proposed interpolated signal $w[n]$ a good approximation for the original $x[n]$?

60. Consider a discrete-time LTI system with frequency response $H(e^{j\omega})$ and corresponding impulse response $h[n]$.

(a) We are first given the following three clues about the system:

(i) The system is causal.
(ii) $H(e^{j\omega}) = H^*(e^{-j\omega})$.
(iii) The DTFT of the sequence $h[n+1]$ is real.

Using these three clues, show that the system has an impulse response of finite duration.

(b) In addition to the preceding three clues, we are now given two more clues:

(iv) $\frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})d\omega = 2.$
(v) $H(e^{j\pi}) = 0.$

Is there enough information to identify the system uniquely? If so, determine the impulse response $h[n]$. If not, specify as much as you can about the sequence $h[n]$.

61. Consider the three sequences

$$v[n] = u[n] - u[n-6],$$
$$w[n] = \delta[n] + 2\delta[n-2] + \delta[n-4],$$
$$q[n] = v[n] * w[n].$$

(a) Find and sketch the sequence $q[n]$.

(b) Find and sketch the sequence $r[n]$ such that $r[n] * v[n] = \sum_{k=-\infty}^{n-1} q[k]$.

(c) Is $q[-n] = v[-n] * w[-n]$? Justify your answer.

62. Consider an LTI system with frequency response

$$H(e^{j\omega}) = e^{-j[(\omega/2)+(\pi/4)]}, \qquad -\pi < \omega \le \pi.$$

Determine $y[n]$, the output of this system, if the input is

$$x[n] = \cos\left(\frac{15\pi n}{4} - \frac{\pi}{3}\right)$$

for all n.

63. Consider a system S with input $x[n]$ and output $y[n]$ related according to the block diagram in Figure P63-1.

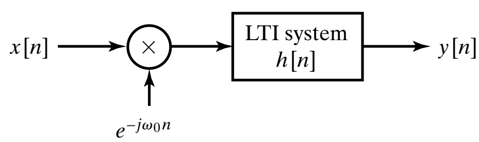


Figure P63–1

The input $x[n]$ is multiplied by $e^{-j\omega_0 n}$, and the product is passed through a stable LTI system with impulse response $h[n]$.

(a) Is the system S linear? Justify your answer.
(b) Is the system S time invariant? Justify your answer.
(c) Is the system S stable? Justify your answer.
(d) Specify a system C such that the block diagram in Figure P63-2 represents an alternative way of expressing the input–output relationship of the system S. (*Note:* The system C does not have to be an LTI system.)

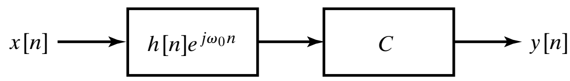


Figure P63–2

64. Consider an ideal lowpass filter with impulse response $h_{\text{lp}}[n]$ and frequency response

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} 1, & |\omega| < 0.2\pi, \\ 0, & 0.2\pi \leq |\omega| \leq \pi. \end{cases}$$

(a) A new filter is defined by the equation $h_1[n] = (-1)^n h_{\text{lp}}[n] = e^{j\pi n} h_{\text{lp}}[n]$. Determine an equation for the frequency response of $H_1(e^{j\omega})$, and plot the equation for $|\omega| < \pi$. What kind of filter is this?

(b) A second filter is defined by the equation $h_2[n] = 2h_{\text{lp}}[n]\cos(0.5\pi n)$. Determine the equation for the frequency response $H_2(e^{j\omega})$, and plot the equation for $|\omega| < \pi$. What kind of filter is this?

(c) A third filter is defined by the equation

$$h_3[n] = \frac{\sin(0.1\pi n)}{\pi n} h_{\text{lp}}[n].$$

Determine the equation for the frequency response $H_3(e^{j\omega})$, and plot the equation for $|\omega| < \pi$. What kind of filter is this?

65. The LTI system

$$H(e^{j\omega}) = \begin{cases} -j, & 0 < \omega < \pi, \\ j, & -\pi < \omega < 0, \end{cases}$$

is referred to as a 90° phase shifter and is used to generate what is referred to as an analytic signal $w[n]$ as shown in Figure P65-1. Specifically, the analytic signal $w[n]$ is a complex-valued signal for which

$$\mathcal{R}e\{w[n]\} = x[n],$$
$$\mathcal{I}m\{w[n]\} = y[n].$$

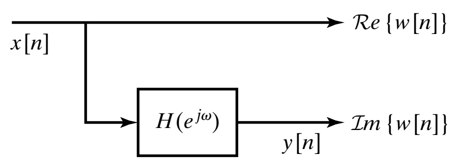


Figure P65–1

If $\mathcal{R}e\{X(e^{j\omega})\}$ is as shown in Figure P65-2 and $\mathcal{I}m\{X(e^{j\omega})\} = 0$, determine and sketch $W(e^{j\omega})$, the Fourier transform of the analytic signal $w[n] = x[n] + jy[n]$.

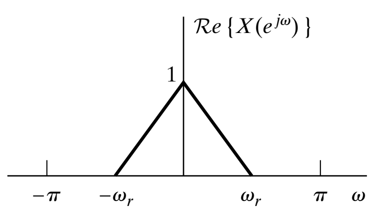


Figure P65–2

66. The autocorrelation sequence of a signal $x[n]$ is defined as

$$R_x[n] = \sum_{k=-\infty}^{\infty} x^*[k]x[n+k].$$

(a) Show that for an appropriate choice of the signal $g[n]$, $R_x[n] = x[n] * g[n]$, and identify the proper choice for $g[n]$.
(b) Show that the Fourier transform of $R_x[n]$ is equal to $|X(e^{j\omega})|^2$.

67. The signals $x[n]$ and $y[n]$ shown in Figure P67-1 are the input and corresponding output for an LTI system.

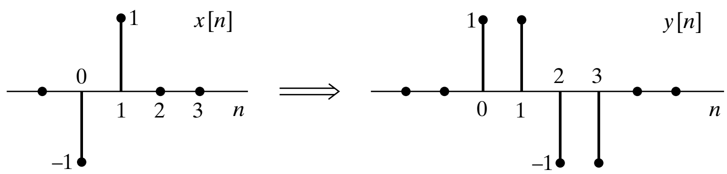


Figure P67-1

(a) Find the response of the system to the sequence $x_2[n]$ in Figure P67-2.

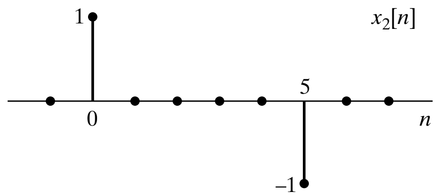


Figure P67-2

(b) Find the impulse response $h[n]$ for this LTI system.

68. Consider a system for which the input $x[n]$ and output $y[n]$ satisfy the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

and for which $y[-1]$ is constrained to be zero for every input. Determine whether or not the system is stable. If you conclude that the system is stable, show your reasoning. If you conclude that the system is not stable, give an example of a bounded input that results in an unbounded output.

Extension Problems

69. The causality of a system was defined in Section 2.4. From this definition, show that, for an LTI system, causality implies that the impulse response $h[n]$ is zero for $n < 0$. One approach is to show that if $h[n]$ is not zero for $n < 0$, then the system cannot be causal. Show also that if the impulse response is zero for $n < 0$, then the system will necessarily be causal.

70. Consider a discrete-time system with input $x[n]$ and output $y[n]$. When the input is

$$x[n] = \left(\frac{1}{4}\right)^n u[n],$$

the output is

$$y[n] = \left(\frac{1}{2}\right)^n \qquad \text{for all } n.$$

Determine which of the following statements is correct:

- The system must be LTI.
- The system could be LTI.
- The system cannot be LTI.

If your answer is that the system must or could be LTI, give a possible impulse response. If your answer is that the system could not be LTI, explain clearly why not.

71. Consider an LTI system whose frequency response is

$$H(e^{j\omega}) = e^{-j\omega/2}, \quad |\omega| < \pi.$$

Determine whether or not the system is causal. Show your reasoning.

72. In Figure P72, two sequences $x_1[n]$ and $x_2[n]$ are shown. Both sequences are zero for all n outside the regions shown. The Fourier transforms of these sequences are $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$, which, in general, can be expected to be complex and can be written in the form

$$X_1(e^{j\omega}) = A_1(\omega)e^{j\theta_1(\omega)},$$
$$X_2(e^{j\omega}) = A_2(\omega)e^{j\theta_2(\omega)},$$

where $A_1(\omega), \theta_1(\omega), A_2(\omega)$, and $\theta_2(\omega)$ are all real functions chosen so that both $A_1(\omega)$ and $A_2(\omega)$ are nonnegative at $\omega = 0$, but otherwise can take on both positive and negative values. Determine appropriate choices for $\theta_1(\omega)$ and $\theta_2(\omega)$, and sketch these two phase functions in the range $0 < \omega < 2\pi$.

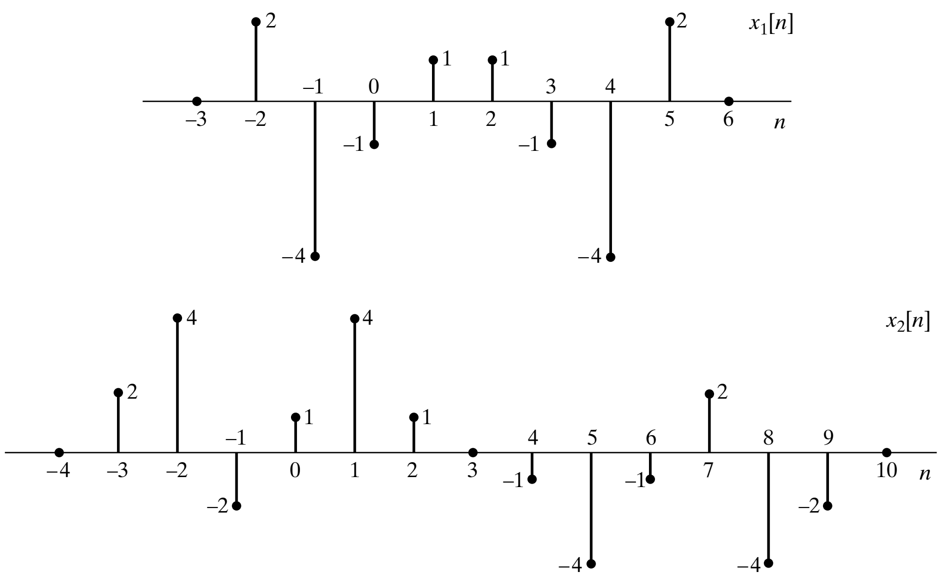


Figure P72

73. Consider the cascade of discrete-time systems in Figure P73. The time-reversal systems are defined by the equations $f[n] = e[-n]$ and $y[n] = g[-n]$. Assume throughout the problem that $x[n]$ and $h_1[n]$ are real sequences.

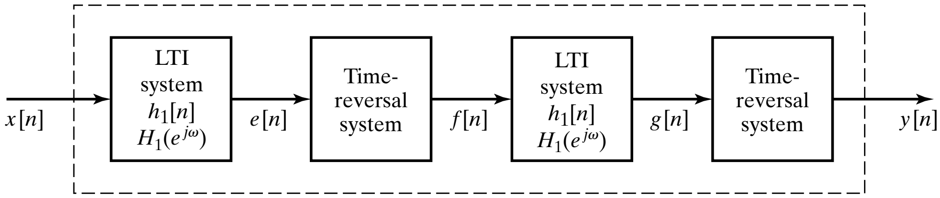


Figure P73

(a) Express $E(e^{j\omega})$, $F(e^{j\omega})$, $G(e^{j\omega})$, and $Y(e^{j\omega})$ in terms of $X(e^{j\omega})$ and $H_1(e^{j\omega})$.
(b) The result from part (a) should convince you that the overall system is LTI. Find the frequency response $H(e^{j\omega})$ of the overall system.
(c) Determine an expression for the impulse response $h[n]$ of the overall system in terms of $h_1[n]$.

74. The overall system in the dotted box in Figure P74 can be shown to be linear and time invariant.

(a) Determine an expression for $H(e^{j\omega})$, the frequency response of the overall system from the input $x[n]$ to the output $y[n]$, in terms of $H_1(e^{j\omega})$, the frequency response of the internal LTI system. Remember that $(-1)^n = e^{j\pi n}$.
(b) Plot $H(e^{j\omega})$ for the case when the frequency response of the internal LTI system is

$$H_1(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi. \end{cases}$$

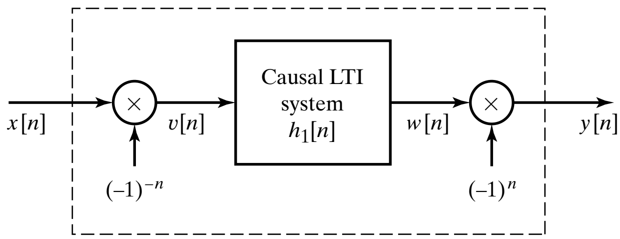


Figure P74

75. Figure P75-1 shows the input–output relationships of Systems A and B, while Figure P75-2 contains two possible cascade combinations of these systems.

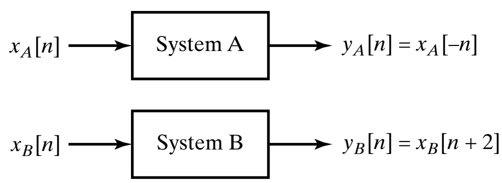


Figure P75-1

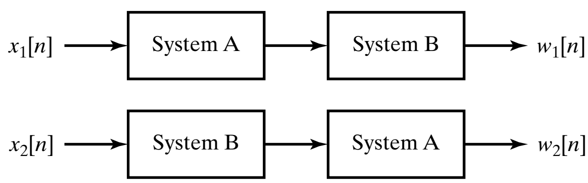


Figure P75-2

If $x_1[n] = x_2[n]$, will $w_1[n]$ and $w_2[n]$ necessarily be equal? If your answer is *yes,* clearly and concisely explain why and demonstrate with an example. If your answer is *not necessarily,* demonstrate with a counterexample.

76. Consider the system in Figure P76, where the subsystems S_1 and S_2 are LTI.

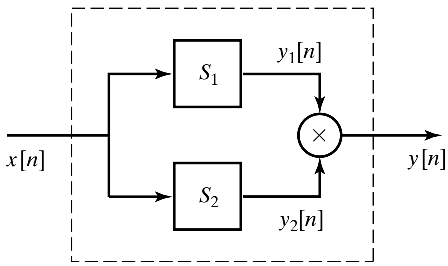


Figure P76

(a) Is the overall system enclosed by the dashed box, with input $x[n]$ and output $y[n]$ equal to the product of $y_1[n]$ and $y_2[n]$, guaranteed to be an LTI system? If so, explain your reasoning. If not, provide a counterexample.

(b) Suppose S_1 and S_2 have frequency responses $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$ that are known to be zero over certain regions. Let

$$H_1(e^{j\omega}) = \begin{cases} 0, & |\omega| \le 0.2\pi, \\ \text{unspecified}, & 0.2\pi < |\omega| \le \pi, \end{cases}$$

$$H_2(e^{j\omega}) = \begin{cases} \text{unspecified}, & |\omega| \le 0.4\pi, \\ 0, & 0.4\pi < |\omega| \le \pi. \end{cases}$$

Suppose also that the input $x[n]$ is known to be bandlimited to 0.3π, i.e.,

$$X(e^{j\omega}) = \begin{cases} \text{unspecified}, & |\omega| < 0.3\pi, \\ 0, & 0.3\pi \le |\omega| \le \pi. \end{cases}$$

Over what region of $-\pi \le \omega < \pi$ is $Y(e^{j\omega})$, the DTFT of $y[n]$, guaranteed to be zero?

77. A commonly used numerical operation called the first backward difference is defined as

$$y[n] = \nabla(x[n]) = x[n] - x[n-1],$$

where $x[n]$ is the input and $y[n]$ is the output of the first-backward-difference system.

(a) Show that this system is linear and time invariant.

(b) Find the impulse response of the system.

(c) Find and sketch the frequency response (magnitude and phase).

(d) Show that if

$$x[n] = f[n] * g[n],$$

then

$$\nabla(x[n]) = \nabla(f[n]) * g[n] = f[n] * \nabla(g[n]).$$

(e) Find the impulse response of a system that could be cascaded with the first-difference system to recover the input; i.e., find $h_i[n]$, where

$$h_i[n] * \nabla(x[n]) = x[n].$$

78. Let $H(e^{j\omega})$ denote the frequency response of an LTI system with impulse response $h[n]$, where $h[n]$ is, in general, complex.

(a) Using Eq. (104), show that $H^*(e^{-j\omega})$ is the frequency response of a system with impulse response $h^*[n]$.

(b) Show that if $h[n]$ is real, the frequency response is conjugate symmetric, i.e., $H(e^{-j\omega}) = H^*(e^{j\omega})$.

79. Let $X(e^{j\omega})$ denote the Fourier transform of $x[n]$. Using the Fourier transform synthesis or analysis equations (Eqs. (130) and (131)), show that

(a) the Fourier transform of $x^*[n]$ is $X^*(e^{-j\omega})$,

(b) the Fourier transform of $x^*[-n]$ is $X^*(e^{j\omega})$.

80. Show that for $x[n]$ real, property 7 in Table 1 follows from property 1 and that properties 8–11 follow from property 7.

81. In Section 9, we stated a number of Fourier transform theorems without proof. Using the Fourier synthesis or analysis equations (Eqs. (130) and (131)), demonstrate the validity of Theorems 1–5 in Table 2.

82. In Section 9.6, it was argued intuitively that

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}), \tag{P82-1}$$

when $Y(e^{j\omega})$, $H(e^{j\omega})$, and $X(e^{j\omega})$ are, respectively, the Fourier transforms of the output $y[n]$, impulse response $h[n]$, and input $x[n]$ of an LTI system; i.e.,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]. \tag{P82-2}$$

Verify Eq. (P82-1) by applying the Fourier transform to the convolution sum given in Eq. (P82-2).

83. By applying the Fourier synthesis equation (Eq. (130)) to Eq. (167) and using Theorem 3 in Table 2, demonstrate the validity of the modulation theorem (Theorem 7, Table 2).

84. Let $x[n]$ and $y[n]$ denote complex sequences and $X(e^{j\omega})$ and $Y(e^{j\omega})$ their respective Fourier transforms.

(a) By using the convolution theorem (Theorem 6 in Table 2) and appropriate properties from Table 2, determine, in terms of $x[n]$ and $y[n]$, the sequence whose Fourier transform is $X(e^{j\omega})Y^*(e^{j\omega})$.

(b) Using the result in part (a), show that

$$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega. \tag{P84-1}$$

Equation (P84-1) is a more general form of Parseval's theorem, as given in Section 9.5.

(c) Using Eq. (P84-1), determine the numerical value of the sum

$$\sum_{n=-\infty}^{\infty} \frac{\sin(\pi n/4)}{2\pi n}\frac{\sin(\pi n/6)}{5\pi n}.$$

85. Let $x[n]$ and $X(e^{j\omega})$ represent a sequence and its Fourier transform, respectively. Determine, in terms of $X(e^{j\omega})$, the transforms of $y_s[n]$, $y_d[n]$, and $y_e[n]$ as defined below. In each case, sketch the corresponding output Fourier transform $Y_s(e^{j\omega})$, $Y_d(e^{j\omega})$, and $Y_e(e^{j\omega})$, respectively for $X(e^{j\omega})$ as shown in Figure P85.

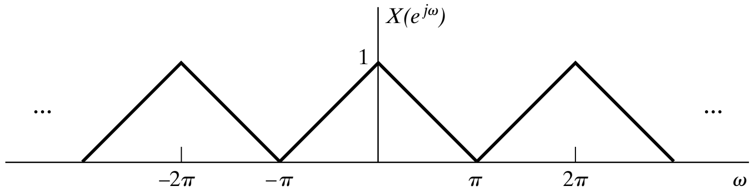


Figure P85

(a) Sampler:

$$y_s[n] = \begin{cases} x[n], & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

Note that $y_s[n] = \frac{1}{2}\{x[n] + (-1)^n x[n]\}$ and $-1 = e^{j\pi}$.

(b) Compressor:

$$y_d[n] = x[2n].$$

(c) Expander:

$$y_e[n] = \begin{cases} x[n/2], & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

86. The two-frequency correlation function $\Phi_x(N, \omega)$ is often used in radar and sonar to evaluate the frequency and travel-time resolution of a signal. For discrete-time signals, we define

$$\Phi_x(N, \omega) = \sum_{n=-\infty}^{\infty} x[n + N]x^*[n - N]e^{-j\omega n}.$$

(a) Show that

$$\Phi_x(-N, -\omega) = \Phi_x^*(N, \omega).$$

(b) If

$$x[n] = A\,a^n u[n], \qquad 0 < a < 1,$$

find $\Phi_x(N, \omega)$. (Assume that $N \geq 0$.)

(c) The function $\Phi_x(N, \omega)$ has a frequency domain dual. Show that

$$\Phi_x(N, \omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j[v+(\omega/2)]}\right)X^*\left(e^{j[v-(\omega/2)]}\right)e^{j2vN}\,dv.$$

87. Let $x[n]$ and $y[n]$ be stationary, uncorrelated random signals. Show that if

$$w[n] = x[n] + y[n],$$

then

$$m_w = m_x + m_y \quad \text{and} \quad \sigma_w^2 = \sigma_x^2 + \sigma_y^2.$$

88. Let $e[n]$ denote a white-noise sequence, and let $s[n]$ denote a sequence that is uncorrelated with $e[n]$. Show that the sequence

$$y[n] = s[n]e[n]$$

is white, i.e., that

$$E\{y[n]y[n + m]\} = A\,\delta[m],$$

where A is a constant.

89. Consider a random signal $x[n] = s[n] + e[n]$, where both $s[n]$ and $e[n]$ are independent zero-mean stationary random signals with autocorrelation functions $\phi_{ss}[m]$ and $\phi_{ee}[m]$, respectively.

(a) Determine expressions for $\phi_{xx}[m]$ and $\Phi_{xx}(e^{j\omega})$.
(b) Determine expressions for $\phi_{xe}[m]$ and $\Phi_{xe}(e^{j\omega})$.
(c) Determine expressions for $\phi_{xs}[m]$ and $\Phi_{xs}(e^{j\omega})$.

90. Consider an LTI system with impulse response $h[n] = a^n u[n]$ with $|a| < 1$.

(a) Compute the deterministic autocorrelation function $\phi_{hh}[m]$ for this impulse response.
(b) Determine the magnitude-squared function $|H(e^{j\omega})|^2$ for the system.
(c) Use Parseval's theorem to evaluate the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 d\omega$$

for the system.

91. The input to the first-backward-difference system (Example 9) is a zero-mean white-noise signal whose autocorrelation function is $\phi_{xx}[m] = \sigma_x^2 \delta[m]$.

(a) Determine and plot the autocorrelation function and the power spectrum of the corresponding output of the system.
(b) What is the average power of the output of the system?
(c) What does this problem tell you about the first backward difference of a noisy signal?

92. Let $x[n]$ be a real, stationary, white-noise process, with zero mean and variance σ_x^2. Let $y[n]$ be the corresponding output when $x[n]$ is the input to an LTI system with impulse response $h[n]$. Show that

(a) $E\{x[n]y[n]\} = h[0]\sigma_x^2$,
(b) $\sigma_y^2 = \sigma_x^2 \sum_{n=-\infty}^{\infty} h^2[n]$.

93. Let $x[n]$ be a real stationary white-noise sequence, with zero mean and variance σ_x^2. Let $x[n]$ be the input to the cascade of two causal LTI discrete-time systems, as shown in Figure P93.

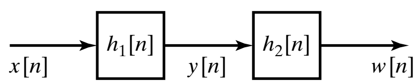
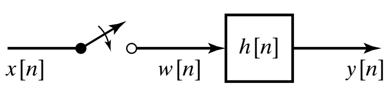


Figure P93

(a) Is $\sigma_y^2 = \sigma_x^2 \sum_{k=0}^{\infty} h_1^2[k]$?
(b) Is $\sigma_w^2 = \sigma_y^2 \sum_{k=0}^{\infty} h_2^2[k]$?
(c) Let $h_1[n] = a^n u[n]$ and $h_2[n] = b^n u[n]$. Determine the impulse response of the overall system in Figure P93, and, from this, determine σ_w^2. Are your answers to parts (b) and (c) consistent?

94. Sometimes we are interested in the statistical behavior of an LTI system when the input is a suddenly applied random signal. Such a situation is depicted in Figure P94.



(switch closed at $n = 0$)

Figure P94

Let $x[n]$ be a stationary white-noise process. The input to the system, $w[n]$, given by

$$w[n] = \begin{cases} x[n], & n \geq 0, \\ 0, & n < 0, \end{cases}$$

is a nonstationary process, as is the output $y[n]$.

(a) Derive an expression for the mean of the output in terms of the mean of the input.
(b) Derive an expression for the autocorrelation sequence $\phi_{yy}[n_1, n_2]$ of the output.
(c) Show that, for large n, the formulas derived in parts (a) and (b) approach the results for stationary inputs.
(d) Assume that $h[n] = a^n u[n]$. Find the mean and mean-square values of the output in terms of the mean and mean-square values of the input. Sketch these parameters as a function of n.

95. Let $x[n]$ and $y[n]$ respectively denote the input and output of a system. The input–output relation of a system sometimes used for the purpose of noise reduction in images is given by

$$y[n] = \frac{\sigma_s^2[n]}{\sigma_x^2[n]}(x[n] - m_x[n]) + m_x[n],$$

where

$$\sigma_x^2[n] = \frac{1}{3}\sum_{k=n-1}^{n+1}(x[k] - m_x[n])^2,$$

$$m_x[n] = \frac{1}{3}\sum_{k=n-1}^{n+1} x[k],$$

$$\sigma_s^2[n] = \begin{cases} \sigma_x^2[n] - \sigma_w^2, & \sigma_x^2[n] \geq \sigma_w^2, \\ 0, & \text{otherwise}, \end{cases}$$

and σ_w^2 is a known constant proportional to the noise power.

(a) Is the system linear?
(b) Is the system shift invariant?
(c) Is the system stable?
(d) Is the system causal?
(e) For a fixed $x[n]$, determine $y[n]$ when σ_w^2 is very large (large noise power) and when σ_w^2 is very small (small noise power). Does $y[n]$ make sense for these extreme cases?

96. Consider a random process $x[n]$ that is the response of the LTI system shown in Figure P96. In the figure, $w[n]$ represents a real zero-mean stationary white-noise process with $E\{w^2[n]\} = \sigma_w^2$.

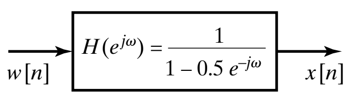


Figure P96

(a) Express $\mathcal{E}\{x^2[n]\}$ in terms of $\phi_{xx}[n]$ or $\Phi_{xx}(e^{j\omega})$.
(b) Determine $\Phi_{xx}(e^{j\omega})$, the power density spectrum of $x[n]$.
(c) Determine $\phi_{xx}[n]$, the correlation function of $x[n]$.

97. Consider an LTI system whose impulse response is real and is given by $h[n]$. Suppose the responses of the system to the two inputs $x[n]$ and $v[n]$ are, respectively, $y[n]$ and $z[n]$, as shown in Figure P97.

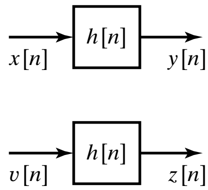


Figure P97

The inputs $x[n]$ and $v[n]$ in the figure represent real zero-mean stationary random processes with autocorrelation functions $\phi_{xx}[n]$ and $\phi_{vv}[n]$, cross-correlation function $\phi_{xv}[n]$, power spectra $\Phi_{xx}(e^{j\omega})$ and $\Phi_{vv}(e^{j\omega})$, and cross power spectrum $\Phi_{xv}(e^{j\omega})$.

(a) Given $\phi_{xx}[n]$, $\phi_{vv}[n]$, $\phi_{xv}[n]$, $\Phi_{xx}(e^{j\omega})$, $\Phi_{vv}(e^{j\omega})$, and $\Phi_{xv}(e^{j\omega})$, determine $\Phi_{yz}(e^{j\omega})$, the cross power spectrum of $y[n]$ and $z[n]$, where $\Phi_{yz}(e^{j\omega})$ is defined by

$$\phi_{yz}[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \Phi_{yz}(e^{j\omega}),$$

with $\phi_{yz}[n] = E\{y[k]z[k-n]\}$.

(b) Is the cross power spectrum $\Phi_{xv}(e^{j\omega})$ always nonnegative; i.e., is $\Phi_{xv}(e^{j\omega}) \geq 0$ for all ω? Justify your answer.

98. Consider the LTI system shown in Figure P98. The input to this system, $e[n]$, is a stationary zero-mean white-noise signal with average power σ_e^2. The first system is a backward-difference system as defined by $f[n] = e[n] - e[n-1]$. The second system is an ideal lowpass filter with frequency response

$$H_2(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \leq \pi. \end{cases}$$

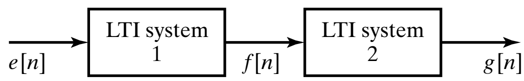


Figure P98

(a) Determine an expression for $\Phi_{ff}(e^{j\omega})$, the power spectrum of $f[n]$, and plot this expression for $-2\pi < \omega < 2\pi$.

(b) Determine an expression for $\phi_{ff}[m]$, the autocorrelation function of $f[n]$.

(c) Determine an expression for $\Phi_{gg}(e^{j\omega})$, the power spectrum of $g[n]$, and plot this expression for $-2\pi < \omega < 2\pi$.

(d) Determine an expression for σ_g^2, the average power of the output.

Answers to Selected Basic Problems

1. **(a)** Always (2), (3), (5). If $g[n]$ is bounded, (1).

(b) (3).

(c) Always (1), (3), (4). If $n_0 = 0$, (2) and (5).

(d) Always (1), (3), (4). If $n_0 = 0$, (5). If $n_0 \geq 0$, (2).

(e) (1), (2), (4), (5).

(f) Always (1), (2), (4), (5). If $b = 0$, (3).
(g) (1), (3).
(h) (1), (5).

2. (a) $N_4 = N_0 + N_2, N_5 = N_1 + N_3$.
(b) At most $N + M - 1$ nonzero points.

3.

$$y[n] = \begin{cases} \dfrac{a^{-n}}{1-a}, & n < 0, \\ \dfrac{1}{1-a}, & n \geq 0. \end{cases}$$

4. $y[n] = 8[(1/2)^n - (1/4)^n]u[n]$.

5. (a) $y_h[n] = A_1(2)^n + A_2(3)^n$.
(b) $h[n] = 2(3^n - 2^n)u[n]$.
(c) $s[n] = [-8(2)^{(n-1)} + 9(3)^{(n-1)} + 1]u[n]$.

6. (a)

$$H(e^{j\omega}) = \frac{1 + 2e^{-j\omega} + e^{-j2\omega}}{1 - \frac{1}{2}e^{-j\omega}}.$$

(b) $y[n] + \frac{1}{2}y[n-1] + \frac{3}{4}y[n-2] = x[n] - \frac{1}{2}x[n-1] + x[n-3]$.

7. (a) Periodic, $N = 12$.
(b) Periodic, $N = 8$.
(c) Not periodic.
(d) Not periodic.

8. $y[n] = 3(-1/2)^n u[n] + 2(1/3)^n u[n]$.

9. (a)

$$h[n] = 2\left[\left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n\right]u[n],$$

$$H(e^{j\omega}) = \frac{\frac{1}{3}e^{-j\omega}}{1 - \frac{5}{6}e^{-j\omega} + \frac{1}{6}e^{-j2\omega}},$$

$$s[n] = \left[-2\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n + 1\right]u[n].$$

(b) $y_h[n] = A_1(1/2)^n + A_2(1/3)^n$.
(c) $y[n] = 4(1/2)^n - 3(1/3)^n - 2(1/2)^n u[-n-1] + 2(1/3)^n u[-n-1]$. Other answers are possible.

10. (a)

$$y[n] = \begin{cases} a^{-1}/(1 - a^{-1}), & n \geq -1, \\ a^n/(1 - a^{-1}), & n \leq -2. \end{cases}$$

(b)

$$y[n] = \begin{cases} 1, & n \geq 3, \\ 2^{(n-3)}, & n \leq 2. \end{cases}$$

(c)

$$y[n] = \begin{cases} 1, & n \geq 0, \\ 2^n, & n \leq -1. \end{cases}$$