Pearson New International Edition

A Problem Solving Approach to Mathematics For Elementary School Teachers

Eleventh Edition

R. Billstein S. Libeskind J. Lott

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A Problem Solving Approach to
Mathematics For Elementary School
Teachers Eleventh Edition
R. Billstein S. Libeskind J. Lott

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An Introduction to Problem Solving

1 Mathematics and Problem Solving
2 Explorations with Patterns
3 Reasoning and Logic: An Introduction

Preliminary Problem

A person removing 2 eggs at a time from a basket has 1 left; the person removing them 3 at a time has 2 left; and the person removing them 5 at a time has 3 left. What is the fewest number of eggs possible in the basket?

If needed, see Hint before the Chapter Summary.

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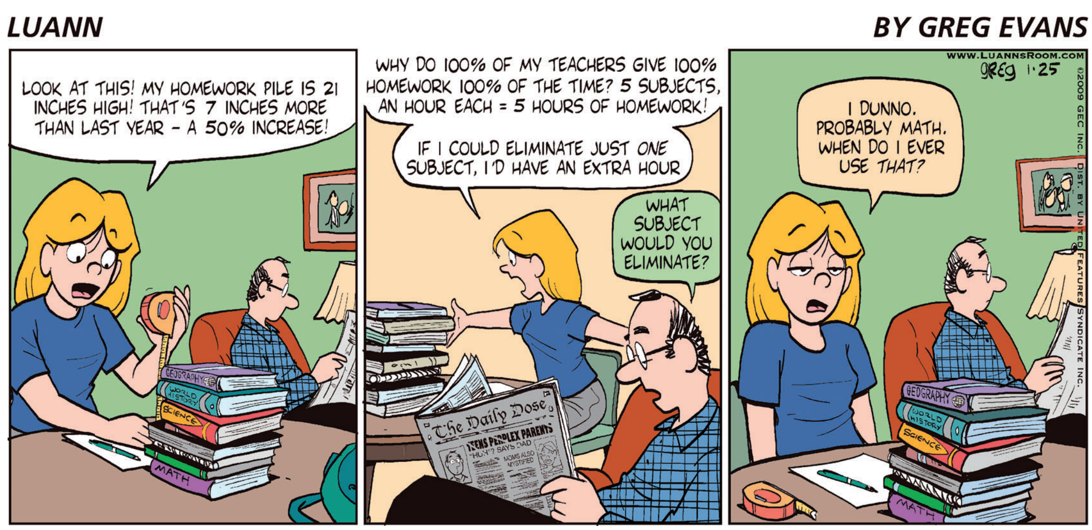
Problem solving has long been recognized as one of the hallmarks of mathematics. George Pólya (1887–1985), one of the great mathematicians and teachers of the twentieth century, pointed out that “solving a problem means finding a way out of difficulty, a way around an obstacle, attaining an aim which was not immediately attainable” (Pólya 1981, p. ix). In *Principles and Standards for School Mathematics (PSSM)*, (NCTM 2000), we find the following:

> Problem solving means engaging in a task for which the solution method is not known in advance. In order to find a solution, students must draw on their knowledge, and through this process, they will often develop new mathematical understandings. Solving problems is not only a goal of learning mathematics but also a major means of doing so. (p. 52)

Common Core State Standards for Mathematics (*CCSSM*) (2010) states: “Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for its solution.” (p. 6)

Students learn mathematics as a result of solving problems. *Exercises* are routine practice for skill building and serve a purpose in learning mathematics, but problem solving must be a focus of school mathematics. A reasonable amount of tension and discomfort improves problem-solving performance. Mathematical experience often determines whether situations are *problems* or *exercises*.

Worthwhile, interesting problems must be a part of elementary students’ mathematical experience. Otherwise, students may wind up with attitudes similar to Luann’s in the cartoon.



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Mathematical problem solving may occur when:

1. Students are presented with a situation that they understand but do not know how to proceed directly to a solution.
2. Students are interested in finding the solution and attempt to do so.
3. Students are required to use mathematical ideas to solve the problem.

In this text, we present many opportunities to solve problems. This chapter opens with a problem that can be solved by using the concepts developed in the chapter. We give a hint for the solution to the problem at the end of this chapter. Throughout the text, some problems are solved using a four-step process and others solved using different formats.

Working with other students to solve problems can enhance problem-solving ability and communication skills. We encourage *cooperative learning* and working in groups whenever possible. To encourage group work and help identify when cooperative learning could be useful, we identify activities and problems where group discussions might lead to strategies for solving the problem and learning mathematics.

---

1 Mathematics and Problem Solving

If problems are approached in only one way, a mind-set may be formed. For example, consider the following: Spell the word *spot* three times out loud. "S-P-O-T! S-P-O-T! S-P-O-T!" Now answer the question "What do we do when we come to a green light?" Write an answer.

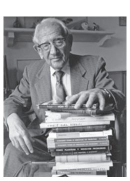
If we answer "Stop," we may be guilty of having formed a mind-set. We do not stop at a *green* light.

Consider the following problem: "A shepherd had 36 sheep. All but 10 died. How many lived?" If we answer "10," we are ready to try some problems. If not, we did not understand the question. *Understanding the problem* is the first step in the four-step problem-solving process developed by George Pólya. Using the four-step process does not guarantee a solution to a problem, but it does provide a systematic means of approaching it.

Four-Step Problem-Solving Process

1. **Understanding the problem**
   - **a.** Can the problem be stated differently?
   - **b.** What is to be found or what is needed?
   - **c.** What are the unknowns?
   - **d.** What information is obtained from the problem?
   - **e.** What information, if any, is missing or not needed?
2. **Devising a plan**
   The following list of strategies, although not exhaustive, is very useful:
   - **a.** Look for a pattern.
   - **b.** Examine related problems and determine whether the same techniques applied to them can be applied to the current problem.
   - **c.** Examine a simpler or special case of the problem to gain insight into the solution of the original problem.
   - **d.** Make a table or list.
   - **e.** Identify a subgoal.
   - **f.** Make a diagram.
   - **g.** Use guess and check.
   - **h.** Work backward.
   - **i.** Write an equation.
3. **Carrying out the plan**
   - **a.** Implement the strategy or strategies in step 2 and perform any necessary actions or computations.
   - **b.** Attend to precision in language and mathematics used.
   - **c.** Check each step of the plan along the way. This may be intuitive checking or a formal proof of each step.
   - **d.** Keep an accurate record of all work.

Historical Note



AP Images

George Pólya (1887–1985) was born in Hungary and received his Ph.D. from the University of Budapest. He moved to the United States in 1940, and after a brief stay at Brown University, joined the faculty at Stanford University. In addition to being a preeminent mathematician, he focused on mathematics education. At Stanford, he published 10 books, including *How To Solve It* (1945), which has been translated into 23 languages.

4. Looking back

a. Check the results in the original problem. (In some cases, this will require a proof.)
b. Interpret the solution in terms of the original problem. Does the answer make sense? Is it reasonable? Does it answer the question that was asked?
c. Determine whether there is another method of finding the solution.
d. If possible, determine other related or more general problems for which the techniques will work.

The role Pólya's problem-solving process plays in the teaching of elementary mathematics is discussed in *PSSM* in the following:

> An obvious question is, How should these strategies be taught? Should they receive explicit attention, and how should they be integrated with the mathematics curriculum? As with any other component of the mathematical tool kit, strategies must receive instructional attention if students are expected to learn them. (p. 54)

Strategies for Problem Solving

We next provide a variety of problems with different contexts to provide experience in problem solving. Frequently, a variety of problem-solving strategies is necessary to solve these and other problems. These strategies are used to discover or construct the means to achieve a solution. For each strategy described, we give an example that can be solved with that strategy. Often, problems can be solved in more than one way. There is no one best strategy to use.

In many of the examples, we use the set of **natural numbers**, 1, 2, 3, ... Note that the three dots, an *ellipsis*, are used to represent missing terms.

Strategy: Look for a Pattern

Problem Solving Gauss's Problem

As a student, Carl Gauss and his class were asked to find the sum of the first 100 natural numbers. The teacher expected to keep the class occupied for some time, but Gauss gave the answer almost immediately. How might he have done it?

Understanding the Problem The natural numbers are 1, 2, 3, 4, ... Thus, the problem is to find the sum $1 + 2 + 3 + 4 + \ldots + 100$.

Devising a Plan The strategy *look for a pattern* is useful here. One story about young Gauss reports that he listed the sum, and wrote the same sum backwards as in Figure 1. If $S = 1 + 2 + 3 + 4 + 5 + \ldots + 98 + 99 + 100$, then Gauss could have computed as follows using an identified pattern.

$$\begin{array}{rcl} S &=& 1 + 2 + 3 + 4 + 5 + \ldots + 98 + 99 + 100 \\ + \underline{S} &=& \underline{100 + 99 + 98 + 97 + 96 + \ldots + 3 + 2 + 1} \\ 2S &=& 101 + 101 + 101 + 101 + 101 + \ldots + 101 + 101 + 101 \end{array}$$

Figure 1

To discover the original sum from the last equation, Gauss could have divided the sum, $2S$, in Figure 1 by 2.

Historical Note

Omikron/ Colorization by Meredith Carlson/Photo Researchers, Inc.

Carl Gauss (1777–1855), one of the greatest mathematicians of all time, was born to humble parents in Brunswick, Germany. He was an infant prodigy who later made contributions in many areas of science as well as mathematics. After Gauss's death, the King of Hanover ordered a commemorative medal prepared in his honor. On the medal was an inscription referring to Gauss as the "Prince of Mathematics."

Carrying Out the Plan There are 100 sums of 101. Thus, $2S = 100 \cdot 101$ and $S = \frac{100 \cdot 101}{2}$, or 5050.

Looking Back Note that the sum in each pair $(1, 100), (2, 99), (3, 98), \ldots, (100, 1)$ is always 101 and there are 100 pairs with this sum. This technique can be used to solve a more general problem of finding the sum of the first n natural numbers $1 + 2 + 3 + 4 + 5 + 6 + \ldots + n$. We use the same plan as before and notice the relationship in Figure 2. There are n sums of $n + 1$ for a total of $n(n + 1)$. Therefore, $2S = n(n + 1)$ and $S = \frac{n(n + 1)}{2}$.

$$\begin{array}{rcccccccccccc} S = & 1 & + & 2 & + & 3 & + & 4 & + & \ldots & + & n \\ + \underline{\;S\;} = & n & + & (n-1) & + & (n-2) & + & (n-3) & + & \ldots & + & 1 \\ \hline 2S = & (n+1) & + & (n+1) & + & (n+1) & + & (n+1) & + & \ldots & + & (n+1) \end{array}$$

Figure 2

A different strategy for finding a sum of consecutive natural numbers involves the strategy of *making a diagram* and thinking of the sum geometrically as a stack of blocks. This alternative method is explored in exercise 2 of Assessment 1A.

NOW TRY THIS 1

One cut through a log produces two pieces, two cuts in the same direction as the first produce three pieces, and three similar cuts produce four pieces. How many pieces are produced by ten such cuts? Assume the cuts are made in the same manner as the first three cuts. How many pieces are produced by n such cuts?

Strategy: Examine a Related Problem

Problem Solving Sums of Even Natural Numbers

Find the sum of the even natural numbers less than or equal to 100. Find that sum and generalize the result.

Understanding the Problem Even natural numbers are 2, 4, 6, 8, 10, ... The problem is to find the sum of these numbers: $2 + 4 + 6 + 8 + \ldots + 100$.

Devising a Plan Recognizing that the sum can be *related to Gauss's original problem* helps us devise a plan. Consider the following:

$$\begin{aligned} 2 + 4 + 6 + 8 + \ldots + 100 &= 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 4 + \ldots + 2 \cdot 50 \\ &= 2(1 + 2 + 3 + 4 + \ldots + 50) \end{aligned}$$

Thus, we can use Gauss's method to find the sum of the first 50 natural numbers and then double that.

Carrying Out the Plan We carry out the plan as follows:

$$\begin{aligned} 2 + 4 + 6 + 8 + \ldots + 100 &= 2(1 + 2 + 3 + 4 + \ldots + 50) \\ &= 2[50(50 + 1)/2] \\ &= 2550 \end{aligned}$$

Thus, the sum is 2550.

Looking Back A different way to approach this problem is to realize that there are 25 sums of 102, as shown in Figure 3. (Why are there 25 sums to consider?)

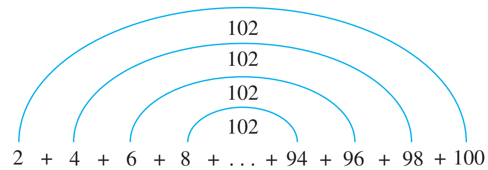


Figure 3

Thus, the sum is $25 \cdot 102$, or 2550.

NOW TRY THIS 2

a. Find the sum of the odd natural numbers less than 100.
b. Find the sum of consecutive natural numbers shown: $25 + 26 + 27 + \ldots + 120$.

Strategy: Examine a Simpler Case

One strategy for solving a complex problem is to *examine a simpler case* of the problem and then consider other parts of the original problem. An example is shown on the grade 7 student page on the next page, where 3^{50} means $\underbrace{3 \cdot 3 \cdot 3 \cdot \ldots \cdot 3}_{\text{Fifty 3s}}$. (Note that two lines after the table 4, 3^{49}should be 3^{48}.)

NOW TRY THIS 3

Each of 16 people in a round-robin handball tournament played each other person exactly once. How many games were played?

Strategy: Make a Table

An often-used strategy in problem solving is *making a table*. A table can be used to look for patterns that emerge in the problem, which in turn can lead to a solution. An example of this strategy is shown on the grade 6 student page on the following page.

NOW TRY THIS 4

Molly and Karly started a new job the same day. After they start work, Molly is to visit the home office every 15 days and Karly is to visit the home office every 18 days. How many days will it be before they both visit the home office the same day?

Strategy: Identify a Subgoal

In attempting to devise a plan for solving a problem, a solution to a somewhat easier or more familiar related problem could make it easier. In such a case, finding the solution to the easier problem may become a *subgoal*. The magic square problem on the following page shows an example of this.

School Book Page SOLVING A SIMPLER PROBLEM

Work a Simpler Problem

When to Use This Strategy If a problem seems to have many steps, you may be able to *Work a Simpler Problem* first. The result may give you a clue about the solution of the original problem.

When you simplify 3^{50}, what number is in the ones place?

Understand

You know that 3^{50} is a large number to calculate. You need to find the number in the ones place.

Plan

It is not easy to simplify 3^{50} with paper and pencil. Simplify easier expressions such as 3^2, 3^3, and 3^4, to see what number is in the ones place.

Carry Out

The table shows the values of the first 10 powers of 3.

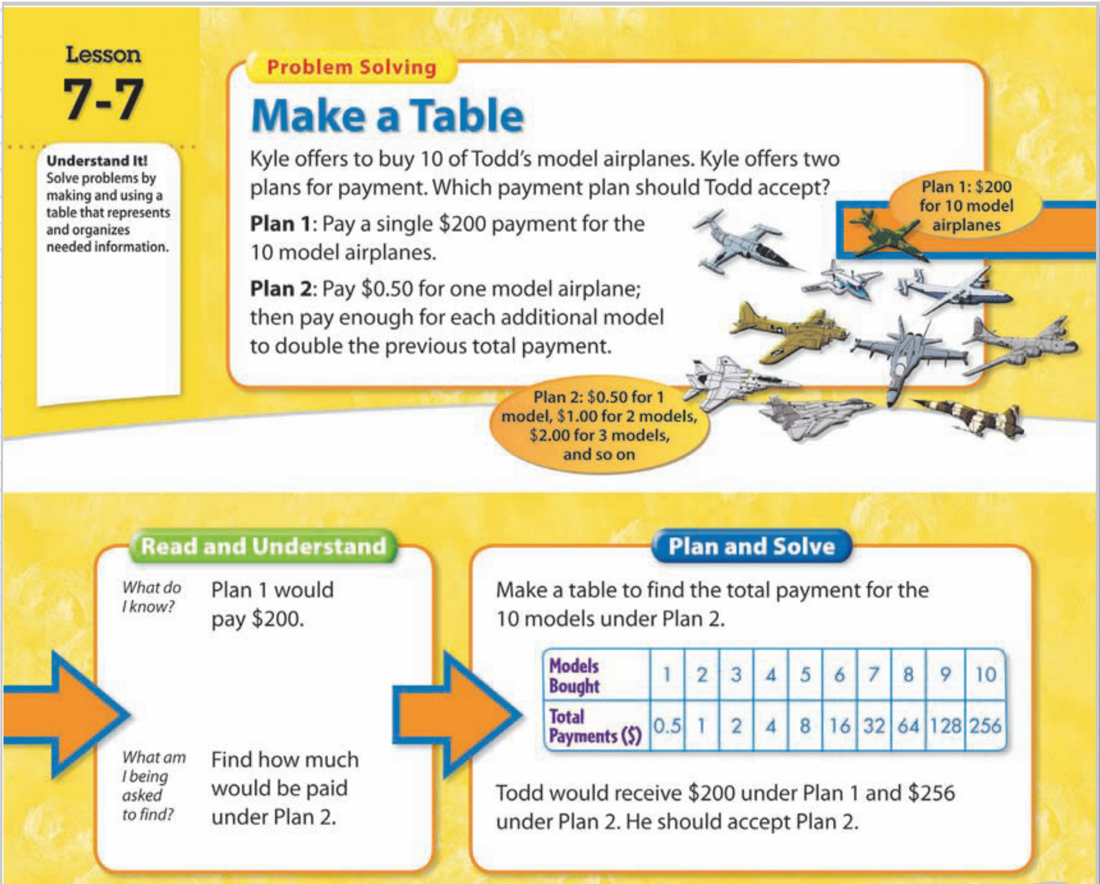
Power	Value
3^1	3
3^2	9
3^3	27
3^4	81
3^5	243
3^6	729
3^7	2,187
3^8	6,561
3^9	19,683
3^{10}	59,049

Notice that the ones digits in the value column repeat in the pattern 3, 9, 7, and 1. Every fourth power of 3 has a ones digit of 1. Since 4 is divisible by 4, 3^{48} has a ones digit of 1. Then the ones digit of 3^{49} is 3 and the ones digit of 3^{50} is 9.

Check

When the exponent of 3 is 2, 6, or 10, the ones digit is 9. The numbers 2, 6, and 10 are divisible by 2 but not by 4. Since 50 is divisible by 2 but not by 4, the ones digit of 3^{50} is 9.

School Book Page MAKING A TABLE



Lesson
7-7

Understand It! Solve problems by making and using a table that represents and organizes needed information.

Problem Solving

Make a Table

Kyle offers to buy 10 of Todd's model airplanes. Kyle offers two plans for payment. Which payment plan should Todd accept?

Plan 1: Pay a single $200 payment for the 10 model airplanes.

Plan 2: Pay $0.50 for one model airplane; then pay enough for each additional model to double the previous total payment.

Plan 1: $200 for 10 model airplanes

Plan 2: $0.50 for 1 model, $1.00 for 2 models, $2.00 for 3 models, and so on

Read and Understand

What do I know? Plan 1 would pay $200.

What am I being asked to find? Find how much would be paid under Plan 2.

Plan and Solve

Make a table to find the total payment for the 10 models under Plan 2.

Models Bought	1	2	3	4	5	6	7	8	9	10
Total Payments ($)	0.5	1	2	4	8	16	32	64	128	256

Todd would receive $200 under Plan 1 and $256 under Plan 2. He should accept Plan 2.

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Problem Solving A Magic Square

Arrange the numbers 1 through 9 into a square subdivided into nine smaller squares like the one shown in Figure 4 so that the sum of every row, column, and main diagonal is the same. The result is a *magic square*.

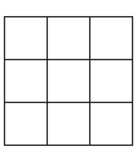


Figure 4

Understanding the Problem Each of the nine numbers 1, 2, 3, . . ., 9 must be placed in the small squares, a different number in each square, so that the sums of the numbers in each row, in each column, and in each of the two major diagonals are the same.

Devising a Plan If we knew the fixed sum of the numbers in each row, column, and diagonal, we would have a better idea of which numbers can appear together in a single row, column, or diagonal. Thus *the subgoal* is to find that fixed sum. The sum of the nine numbers, $1 + 2 + 3 + \ldots + 9$, equals 3 times the sum in one row. (Why?) Consequently, the fixed sum can be found using the process developed by Gauss. We have $(1 + 2 + 3 + \ldots + 9)/3 = \left(\frac{9 \cdot 10}{2}\right)/3$, or 15, so the sum in each row, column, and diagonal must be 15. Next, we need to decide what numbers could occupy the various squares. The number in the center space will appear in four sums, each adding to 15 (two diagonals, the second row, and the second column). Each number in the corners will appear in three sums of 15. (Why?) If we write 15 as a sum of three different numbers 1 through 9 in all possible ways, we could then count how many sums contain each of the numbers 1 through 9. The numbers that appear in at least four sums are candidates for placement in the center square, whereas the numbers that appear in at least three sums are candidates for the corner squares. Thus the new *subgoal* is to write 15 in as many ways as possible as a sum of three different numbers from 1, 2, 3, . . ., 9.

Carrying Out the Plan The sums of 15 can be written systematically as follows:

$$9 + 5 + 1$$
$$9 + 4 + 2$$
$$8 + 6 + 1$$
$$8 + 5 + 2$$
$$8 + 4 + 3$$
$$7 + 6 + 2$$
$$7 + 5 + 3$$
$$6 + 5 + 4$$

Note that the order of the numbers in sums like $9 + 5 + 1$ is irrelevant because the order in which additions are done does not matter. In the list, 1 appears in only two sums, 2 in three sums, 3 in two sums, and so on. Table 1 summarizes this information.

Table 1

Number	1	2	3	4	5	6	7	8	9
Number of sums containing the number	2	3	2	3	4	3	2	3	2

The only number that appears in four sums is 5; hence, 5 must be in the center of the square. (Why?) Because 2, 4, 6, and 8 appear 3 times each, they must go in the corners. Suppose we choose 2 for the upper left corner. Then 8 must be in the lower right corner. This is shown in Figure 5(a). Now we could place 6 in the lower left corner or upper right corner. If we choose the upper right corner, we obtain the result in Figure 5(b). The magic square can now be completed, as shown in Figure 5(c).

2		
	5	
		8

(a)

2		6
	5	
		8

(b)

2	7	6
9	5	1
4	3	8

(c)

Figure 5

Looking Back We have seen that 5 was the only number among the given numbers that could appear in the center. However, we had various choices for a corner, and so it seems that the magic square we found is not the only one possible. Can you find all the others?

Another way to see that 5 could be in the center square is to consider the sums $1 + 9, 2 + 8, 3 + 7, 4 + 6$, as shown in Figure 6. We could add 5 to each to obtain 15.

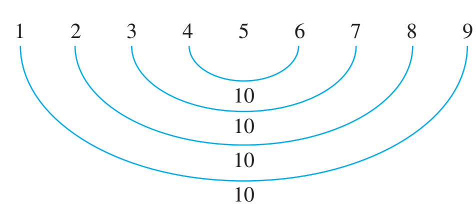


Figure 6

NOW TRY THIS 5

Five friends decided to give a party and split the costs equally. Al spent \$4.75 on invitations, Betty spent \$12.00 for drinks and \$5.25 on vegetables, Carl spent \$24.00 for pizza, Dani spent \$6.00 on paper plates and napkins, and Ellen spent \$13.00 on decorations. Determine who owes money to whom and how the money can be paid.

Strategy: Make a Diagram

In the following problem, *making a diagram* helps us to understand the problem and work toward a solution.

Problem Solving 50-m Race Problem

Bill and Jim ran a 50-m race three times. The speed of the runners did not vary. In the first race, Jim was at the 45-m mark when Bill crossed the finish line.

a. In the second race, Jim started 5 m ahead of Bill, who lined up at the starting line. Who won?
b. In the third race, Jim started at the starting line and Bill started 5 m behind. Who won?

Understanding the Problem When Bill and Jim ran a 50-m race, Bill won by 5 m; that is, whenever Bill covered 50 m, at the same time Jim covered only 45 m. If Bill started at the starting line and Jim had a 5-m head start or Jim started at the starting line and Bill started 5 m behind, we are to determine who would win in each case.

Devising a Plan A strategy to determine the winner under each condition is to *make a diagram*. A diagram for the first 50-m race is given in Figure 7(a). In this case, Bill won by 5 m. In the second race, Jim had a 5-m head start and hence when Bill ran 50 m to the finish line, Jim ran only 45 m. Because Jim is 45 m from the finish line, he reached the finish line at the same time as Bill did. This is shown in Figure 7(b). In the third race, because Bill started 5 m behind, we use Figure 7(a) but move Bill back 5 m, as shown in Figure 7(c). From the diagram we determine the results in each case.

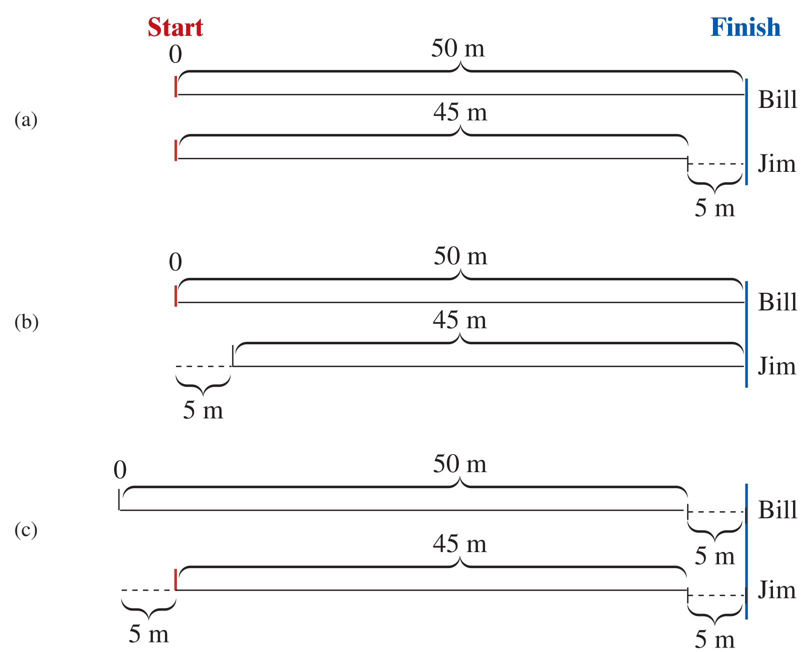


Figure 7

Carrying Out the Plan From Figure 7(b) we see that if Jim had a 5-m head start, then the race ends in a tie. If Bill started 5 m behind Jim, then at 45 m they would be tied. Because Bill is faster than Jim, Bill would cover the last 5 m faster than Jim and win the race.

Looking Back The diagrams show the solution makes sense and is appropriate. Other problems can be investigated involving racing and handicaps. For example, if Bill and Jim run on a 50-m oval track, how many laps will it take for Bill to lead Jim by one full lap? (Assume the same speeds as earlier.)

NOW TRY THIS 6

An elevator stopped at the middle floor of a building. It then moved up 4 floors, stopped, moved down 6 floors, stopped, and then moved up 10 floors and stopped. The elevator was now 3 floors from the top floor. How many floors does the building have?

Strategy: Use Guess and Check

In the strategy of *guess and check*, we first guess at a solution using as reasonable a guess as possible. Then we check to see whether the guess is correct. If not, the next step is to learn as much as possible about the solution based on the guess before making a next guess. This strategy can be regarded as a form of trial and error, whereby the information about the error helps us choose what to try next. The guess-and-check strategy is often used when a student does not know how to solve the problem more efficiently or if the student does not yet have the tools to solve the problem in a faster way. Frank Lester (1975)* suggested that students in grades 1–3 rely primarily on a *guess-and-check* strategy when faced with a mathematical problem. In grades 6–12 this tendency decreases. Older students benefit more from the observed "errors" after a guess when formulating a new "trial."

The grade 7 student page gives an example of this strategy identified as "*systematic guess and check*."

NOW TRY THIS 7

A cryptarithm is a collection of words in which each unique letter represents a unique digit. Find the digits that can be substituted in the following:

$$\begin{array}{r} \text{S U N} \\ +\ \text{F U N} \\ \hline \text{S W I M} \end{array}$$

Strategy: Work Backward

In some problems, it is easier to start with the result and to *work backward*. This is demonstrated as a test-taking strategy on the student page. Note that choice A can be eliminated with mental math and is not discussed.

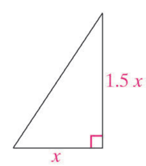
*Lester, F. "Developmental Aspects of Children's Ability to Understand Mathematical Proof." *Journal for Research in Mathematics Education* 6 (1975): 14–25.

School Book Page SYSTEMATIC GUESS AND CHECK

Systematic Guess and Check

When to Use This Strategy The strategy *Systematic Guess and Check* works well when you can start by making a reasonable estimate of the answer.

Construction A group of students is building a sailboat. The students have 48 ft^2 of material to make a sail. They design the sail in the shape of a right triangle as shown below. Find the length of the base and the height.



Understand

The diagram shows that the height is 1.5 times the length of the base.


Tomtsya/Shutterstock

Plan

Test possible dimensions of the triangle formed by the boom (base), the mast (height), and the sail. Check to see if they produce the desired area. Organize your results in a table.

Carry Out

Boom	Mast	Area	Conclusion
6	9	$\frac{1}{2} \cdot 6 \cdot 9 = 27$	Too low
10	15	$\frac{1}{2} \cdot 10 \cdot 15 = 75$	Too high
8	12	$\frac{1}{2} \cdot 8 \cdot 12 = 48$	✔

Check

A triangle with a base length of 8 ft and a height of 12 ft has an area of 48 ft^2.

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School Book Page WORK BACKWARD

Working Backward

The problem-solving strategy *Work Backward* is useful when taking multiple-choice tests. Work backward by testing each choice in the original problem. You will eliminate incorrect answers. Eventually you will find the correct answer.

EXAMPLE

A fruit stand is selling 8 bananas for $1.00. At this rate, how much will 20 bananas cost?

Ⓐ $1.50 Ⓑ $2.00 Ⓒ $2.50 Ⓓ $3.00

Use mental math to test the choices that are easy to use.

$2.00 is twice $1.00. Twice 8 is only 16, so choice B is not the answer.

$3.00 is three times $1.00. Three times 8 is 24, so choice D is not the correct answer.

Since 20 is between 16 and 24, the cost must be between $2.00 and $3.00. The correct answer is choice C.

NOW TRY THIS 8

When Linda added all her test scores and divided by 11 (the number of tests), she found her average to be 80. Her teacher tells her she can drop her single low score of 50. What is her new average?

Strategy: Write an Equation

Even though algebraic thinking is involved in the strategy *writing an equation* and may evoke thoughts of traditional algebra, a closer look reveals that algebraic thinking starts very early in students' school lives. For example, finding the missing addend in a problem like

$$\begin{array}{r} 14 \\ -\square \\ \hline 3 \end{array}$$

could be thought of algebraically as $14 - \square = 3$, or as $3 + \square = 14$. In a traditional algebra course, this might be seen as $14 - x = 3$ or $3 + x = 14$ with 11 as a solution. We use such algebraic thinking long before formal algebra is taught.

A student example of *writing an equation* to solve a problem is seen on the grade 6 student page.

School Book Page WRITE AN EQUATION

Write an Equation

When to Use This Strategy To *write an equation* is one way of organizing the information needed to solve a problem.

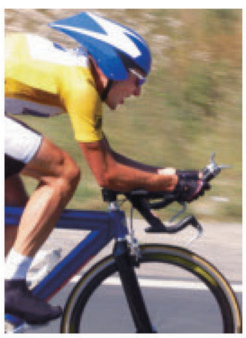
Discount A bicycle is on sale for \$139.93. This is 30% off the regular price. What is the regular price of the bicycle?

Understand

The sale price of the bicycle, \$139.93, is 30% off the regular price. You need to find the regular price.

Plan

Translate the words into an equation. You will pay $100\% - 30\% = 70\%$ of the regular price.



Richard Haynes/Pearson Education, Inc.

Carry Out

The percent you pay times the regular price equals the sale price.

Words percent you pay times regular price equals sale price

Let r = the regular price.

Equation $70\% \times r = \$139.93$

$0.7r = 139.93$ ← Write 70% as a decimal: 0.7.

$0.7r \div 0.7 = 139.93 \div 0.7$ ← Divide each side by 0.7 to find r.

$r = \$199.90$ ← Simplify.

The regular price of the bicycle is \$199.90.

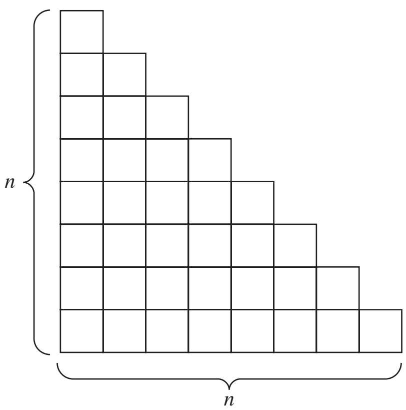
Check

The regular price is about \$200. The sale price is about 70% of \$200, or \$140. This is close to the sale price.

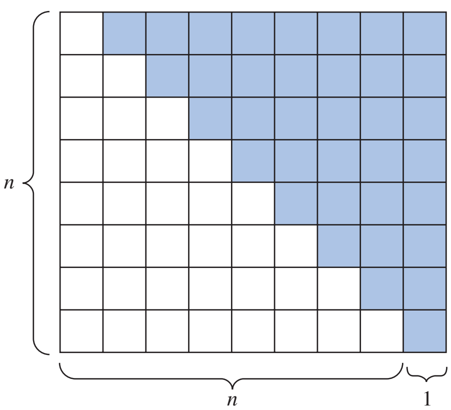
xlviii Problem Solving Handbook

Assessment 1A

1. Use the approach in Gauss's Problem to find the following sums (do not use formulas):
   a. $1 + 2 + 3 + 4 + \ldots + 99$
   b. $1 + 3 + 5 + 7 + \ldots + 1001$
2. Use the ideas behind the drawings in (a) and (b) to find the solution to Gauss's problem. Explain your reasoning.



(a)

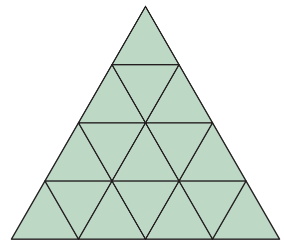


(b)

3. Find the sum of $36 + 37 + 38 + 39 + \ldots + 146 + 147$.
4. Cookies are sold singly or in packages of 2 or 6. With this packaging, how many ways can you buy a dozen cookies?
5. A nursery rhyme states:

   As I was going to St. Ives
   I met a man with seven wives.
   Every wife had seven sacks,
   Every sack had seven cats,
   Every cat had seven kits.
   Kits, cats, sacks, and wives,
   How many were going to St. Ives?

   Explain how many are going to St. Ives.
6. How many triangles are in the following figure?



7. Without computing each sum, find which is greater, O or E, and by how much.

$$O = 1 + 3 + 5 + 7 + \ldots + 97$$
$$E = 2 + 4 + 6 + 8 + \ldots + 98$$

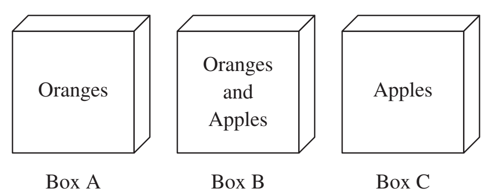
8. Alababa, Bubba, Cory, and Dandy are in a horse race. Bubba is the slowest, Cory is faster than Alababa but slower than Dandy. Name the finishing order of the horses.
9. How many ways can you make change for a \$50 bill using \$5, \$10, and \$20 bills?
10. The following is a magic square (all rows, columns, and diagonals sum to the same number). Find the values of each letter.

17	a	7
12	22	b
c	d	27

11. Frankie and Johnny began reading a novel on the same day. Frankie reads 8 pages a day and Johnny reads 5 pages a day. If Frankie, what page is Johnny on?
12. The 14 digits of a credit card are written in the boxes shown. If the sum of any three consecutive digits is 20, what is the value of A?

A		7									7		4

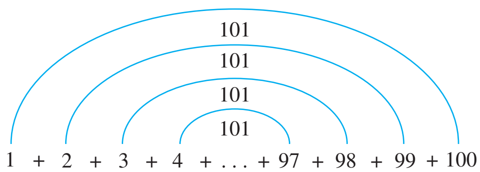
13. Three closed boxes (A, B, and C) of fruit arrive as a gift from a friend. Each box is mislabeled. How could you choose only one fruit from one box to decide how the boxes should be labeled?



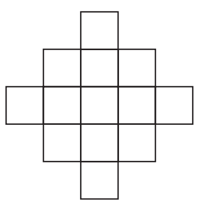
14. A compass and a ruler together cost \$4. The compass costs 90¢ more than the ruler. How much does the compass cost?
15. Kathy stood on the middle rung of a ladder. She climbed up three rungs, moved down five rungs, and then climbed up seven rungs. Then she climbed up the remaining six rungs to the top of the ladder. How many rungs are there in the whole ladder?

Assessment 1B

1. Use the approach in Gauss's Problem to find the following sums (do not use formulas):
   a. $1 + 2 + 3 + 4 + \ldots + 49$
   b. $1 + 3 + 5 + 7 + \ldots + 2009$
2. Use the diagram below to explain how to find the sum of the first 100 natural numbers.



3. Find the sum of $58 + 59 + 60 + 61 + \ldots + 203$.
4. Eve Merriam* entitled her children's book *12 Ways to Get to 11* (1993). Using only addition and natural numbers, describe 12 ways that one can arrive at the sum of 11.
5. Explain why in a drawer containing only two different colors of socks one must draw only three socks to find a matching pair.
6. How many squares are in the following figure?



7. If $E = 2 + 4 + 6 + 8 + \ldots + 98$ and $P = 1 + 3 + 5 + 7 + \ldots + 99$, which is greater, E or P, and by how much?
8. The sign says that you are leaving Missoula, Butte is 120 mi away, and Bozeman is 200 mi away. There is a rest stop halfway between Butte and Bozeman. How far is the rest stop from Missoula if both Butte and Bozeman are in the same direction?
9. Marc goes to the store with exactly \$1.00 in change. He has at least one of each coin less than a half-dollar coin, but he does not have a half-dollar coin.
   a. What is the least number of coins he could have?
   b. What is the greatest number of coins he could have?
10. Find a 3-by-3 magic square using the numbers 3, 5, 7, 9, 11, 13, 15, 17, and 19.
11. Eight marbles look alike, but one is slightly heavier than the others. Using a balance scale, explain how you can determine the heavier one in exactly three weighings.
12. Recall the song "The Twelve Days of Christmas":

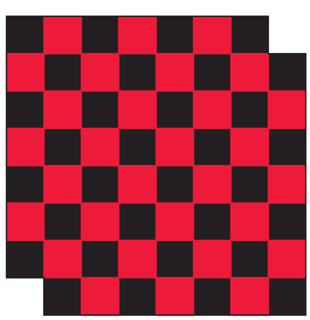
    On the first day of Christmas my true love gave to me a partridge in a pear tree.
    On the second day of Christmas my true love gave to me two turtle doves and a partridge in a pear tree.
    On the third day of Christmas my true love gave to me three French hens, two turtle doves, and a partridge in a pear tree.

    This pattern continues for 9 more days. After 12 days,
    a. which gifts did my true love give the most? (Yes, you will have to remember the song.)
    b. how many total gifts did my true love give to me?
13. **a.** Suppose you have quarters, dimes, and pennies with a total value of \$1.19. How many of each coin can you have without being able to make change for a dollar?
    b. Tell why the combination of coins you have in part (a) is the greatest amount of money that you can have without being able to make change for a dollar.
14. Suppose you buy lunch for the math club. You have enough money to buy 20 salads or 15 sandwiches. The group wants 12 sandwiches. How many salads can you buy?
15. One winter night the temperature fell 15 degrees between midnight and 5 A.M. By 9 A.M., the temperature had doubled from what it was at 5 A.M. By noon, it had risen another 10 degrees to 32 degrees. What was the temperature at midnight?

Mathematical Connections 1

Communication

1. Why is teaching problem solving an important part of mathematics?
2. In the checkerboard, two squares on opposite corners have been removed. A domino can cover two adjacent squares on the board. Can dominoes be arranged in such a way that all the remaining squares on the board can be coverd with no dominoes overlapping or hanging off the board? If not, why not? (Hint: Each domino must cover one black and one red square. Compare this with the number of each color of squares on the board.)



3. **a.** If eight people shake hands with one another exactly once, how many handshakes take place?
   b. Compare strategies for working the problem. How are they the same? How are they different?
   c. Find as many ways as possible to do the problem.
   d. Generalize the solution for n people.

Open-Ended

4. Choose a problem-solving strategy and make up a problem that would use this strategy. Write the solution using Pólya's four-step approach.
5. The distance around the world is approximately 40,000 km. Approximately how many people of average size would it take to stretch around the world if they were holding hands?

*Merriam, E. *12 Ways to Get to 11*. New York: Aladdin Paperbacks, 1993.

Cooperative Learning

6. Work in pairs on the following version of a game called NIM. A calculator is needed for each pair.
   a. Player 1 presses $\boxed{1}$ and $\boxed{+}$ or $\boxed{2}$ and $\boxed{+}$. Player 2 does the same. The players take turns until the target number of 21 is reached. The first player to make the display read 21 is the winner. Determine a strategy for deciding who always wins.
   b. Try a game of NIM using the digits 1, 2, 3, and 4, with a target number of 104. The first player to reach 104 wins. What is the winning strategy?
   c. Try a game of NIM using the digits 3, 5, and 7, with a target number of 73. The first player to exceed 73 loses. What is the winning strategy?
   d. Now play Reverse NIM with the keys $\boxed{1}$ and $\boxed{2}$. Instead of $\boxed{+}$, use $\boxed{-}$. Put 21 on the display. Let the target number be 0. Determine a strategy for winning Reverse NIM.
   e. Try Reverse NIM using the digits 1, 2, and 3 and starting with 24 on the display. The target number is 0. What is the winning strategy?
   f. Try Reverse NIM using the digits 3, 5, and 7 and starting with 73 on the display. The first player to display a negative number loses. What is the winning strategy?

Questions from the Classroom

7. John asks why the last step of Pólya's four-step problem-solving process, *looking back*, is necessary since he has already given the answer. What could you tell him?
8. A student asks why he just can't make "random guesses" rather than "intelligent guesses" when using the guess-and-check problem-solving strategy. How do you respond?
9. Rob says that it is possible to create a magic square with the numbers 1, 3, 4, 5, 6, 7, 8, 9, and 10. How do you respond?

Trends in Mathematics and Science Study (TIMSS) Question

4	11	6
9		5
8	3	10

The rule for the table is that numbers in each row and column must add up to the same number. What number goes in the center of the table?

a. 1 **b.** 2
c. 7 **d.** 12

TIMSS, Grade 4, 2003

Joe knows that a pen costs 1 zed more than a pencil.
His friend bought 2 pens and 3 pencils for 17 zeds.
How many zeds will Joe need to buy 1 pen and 2 pencils?

TIMSS, Grade 8, 2007

National Assessment of Educational Progress (NAEP) Question

There will be 58 people at a breakfast and each person will eat 2 eggs. There are 12 eggs in each carton. How many cartons of eggs will be needed for the breakfast?

a. 9 **b.** 10
c. 72 **d.** 116

NAEP, Grade 4, 2007

Source: Trends in International Mathematics and Science Study (TIMSS), IES National Center for Education Statistics, 2011.
Source: National Assessment of Education Progress (NAEP), IES National Center for Education Statistics, 2011.

BRAIN TEASER

Ten women are fishing all in a row in a boat. One seat in the center of the boat is empty. The five women in the front of the boat want to change seats with the five women in the back of the boat. A person can move from her seat to the next empty seat or she can step over one person without capsizing the boat. What is the minimum number of moves needed for the five women in front to change places with the five in back?

BRAIN TEASER

Place a half-dollar, a quarter, and a nickel in position *A* as shown in Figure 8. Try to move these coins, one at a time, to position *C*. At no time may a larger coin be placed on a smaller coin. Coins may be placed in position *B*. How many moves does it take to get them to position *C*? Now add a penny to the pile and see how many moves are required. This is a simple case of the famous Tower of Hanoi problem, in which ancient Brahman priests were required to move a pile of 64 disks of decreasing size, after which the world would end. How long would it take at a rate of one move per second?

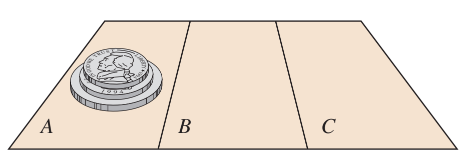


Figure 8

2 Explorations with Patterns

Mathematics has been described as the study of patterns. Patterns are everywhere—in wallpaper, tiles, traffic, and even television schedules. Police investigators study case files to find the *modus operandi*, or pattern of operation, when a series of crimes is committed. Scientists look for patterns in order to isolate variables so that they can reach valid conclusions in their research.

Non-numerical patterns abound. For young children, a pattern could appear as shown in Now Try This 9.

NOW TRY THIS 9

a. Find three more terms to continue a pattern:

$$\bigcirc, \triangle, \triangle, \bigcirc, \triangle, \triangle, \bigcirc, \_\_, \_\_, \_\_$$

b. Describe in words the pattern found in part (a).

Patterns can be surprising. Consider Example 1.

EXAMPLE 1

a. Describe any patterns seen in the following:

$$\begin{aligned} 1 + 0 \cdot 9 &= 1 \\ 2 + 1 \cdot 9 &= 11 \\ 3 + 12 \cdot 9 &= 111 \\ 4 + 123 \cdot 9 &= 1111 \\ 5 + 1234 \cdot 9 &= 11111 \end{aligned}$$

b. Do the patterns continue? Why or why not?

Solution

a. There are several possible patterns. For example, the numbers on the far left are natural numbers. The pattern starts with 1 and continues to the next greater natural number in each successive line. The numbers "in the middle" are products of two numbers, the second of which is 9; the left-most number in the first product is 0; after that the left-most number in each product is formed using natural numbers and including an additional natural number in each successive line. The numbers after the "=" sign are formed using 1s and include an additional 1 in each successive line.

b. The pattern in the complete equation appears to continue for a number of cases, but it does not continue in general; for example,

$$13 + 123456789101112 \cdot 9 = 1{,}111{,}111{,}101{,}910{,}021$$

This pattern breaks down when the pattern of digits in the number being multiplied by 9 contains previously used digits.

As seen in Example 1, determining a pattern on the basis of a few cases is not reliable. For all patterns found, we should either show the pattern does not hold in general or justify that the pattern always works. Reasoning is used in both cases.

Reasoning

Some books list various types of reasoning as a problem-solving strategy. However, we think that reasoning underlies problem solving. *PSSM* states that students at all grade levels should be enabled to

- recognize reasoning and proof as fundamental aspects of mathematics
- make and evaluate mathematical conjectures
- develop and evaluate mathematical arguments and proofs
- select and use various types of reasoning and methods of proof. (p. 402)

For students to recognize reasoning and proof as fundamental aspects of mathematics, it is necessary that they use both reasoning and proof in their studies. However, it must be recognized that the level of use depends on the level of the students and their understanding of mathematics. For example, from very early ages, students use inductive reasoning to look for regularities in patterns based on a very few cases and to develop **conjectures** statements or conclusions that have not been proven. **Inductive reasoning** is the method of making generalizations based on observations and patterns. Such reasoning may or may not be accurate, and conjectures based on inductive reasoning may or may not be true. The validity, or truth, of conjectures in mathematics relies on **deductive reasoning**—the use of mathematical axioms, theorems, definitions, undefined terms assumed to be true, and logic for proof.

Throughout mathematics, there is a fine interweaving of inductive reasoning and conjecturing to develop conclusions thought to be true. Deductive reasoning is required to prove those conclusions. We show how inductive reasoning may lead to false conclusions or false conjectures. We show how deductive reasoning is used to prove true conjectures.

Inductive and Deductive Reasoning

Scientists make observations and propose general laws based on patterns. Statisticians use patterns when they form conclusions based on collected data. This process of *inductive reasoning* may lead to new discoveries; its weakness is that conclusions are drawn only from the collected evidence. If not all cases have been checked, another case may prove the conclusion false. For example, considering only that $0^2 = 0$ and that $1^2 = 1$, we might conjecture that *every number squared is equal to itself.* When we find an example ($2^2 = 4$) that contradicts the conjecture, that **counterexample** proves the conjecture false. Students frequently experience difficulty with the concept of a counterexample. Sometimes finding a counterexample is difficult, but not finding one immediately does not make a conjecture true.

Next, consider a pattern that does work and helps solve a problem. How can you find the sum of three consecutive natural numbers without performing the addition? Three examples are given below.

$$14 + 15 + 16 \qquad (\mathbf{45})$$
$$19 + 20 + 21 \qquad (\mathbf{60})$$
$$99 + 100 + 101 \qquad (\mathbf{300})$$

After studying the sums, a pattern of multiplying the middle number by 3 emerges. The pattern suggests other mathematical questions to consider. For example,

1. Does this work for any three consecutive natural numbers?
2. How can we find the sum of any odd number of consecutive natural numbers?
3. What happens if there is an even number of consecutive natural numbers?

To answer question (1), we give a proof showing that the sum of three consecutive natural numbers is equal to 3 times the middle number. This proof is an example of *deductive reasoning*.

Proof

Let n be the first of three consecutive natural numbers. Then the three numbers are n, $n + 1$, and $n + 2$. The sum of these three numbers is $n + (n + 1) + (n + 2) = 3n + 3 = 3(n + 1)$. Therefore, the sum of the three consecutive natural numbers is 3 times the middle number.

The Danger of Making Conjectures Based on a Few Cases

In grade 5, *PSSM*, we find the following:

> Students should move toward reasoning that depends on relationships and properties. Students need to be challenged with questions such as, What if I gave you twenty more problems like this to do—would they all work the same way? How do you know? (p. 190)

The following discussion illustrates the danger of making a conjecture based on a few cases. In Figure 9, we choose points on a circle and connect them to form distinct, nonoverlapping regions. In this figure, 2 points determine 2 regions, 3 points determine 4 regions, and 4 points determine 8 regions. What is the maximum number of regions that would be determined by 10 points?

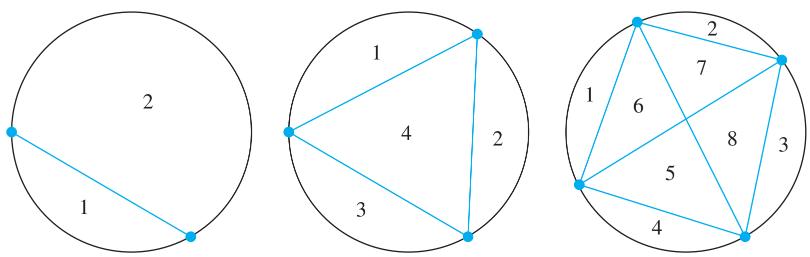


Figure 9

The data from Figure 9 are recorded in Table 2. It appears that each time the number of points increases by 1, the number of regions doubles. If this were true, then for 5 points we would have 2 times the number of regions with 4 points, or $2 \cdot 8 = 16 = 2^4$, and so on. If we base our conjecture on this pattern, we might believe that for 10 points, we would have 2^9, or 512 regions. (Why?)

Table 2

Number of points	2	3	4	5	6	...	10
Maximum number of regions	2	4	8				?

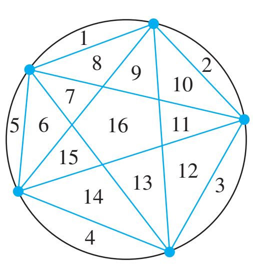


Figure 10

An initial check for this conjecture is to see whether we obtain 16 regions for 5 points. We obtain a diagram similar to that in Figure 10, and our guess of 16 regions is confirmed. For 6 points, the pattern predicts that the number of regions will be 32. Choose the points so that they are neither symmetrically arranged nor equally spaced and count the regions carefully. You should obtain 31 regions and not 32 regions as predicted. No matter how the points are located on the circle, the guess of 32 regions is not correct. The counterexample tells us that the doubling pattern is not correct; note that it does not tell us whether or not there are 512 regions with 10 points, but only that the pattern is not what we conjectured. Thus, in the context of counting the number of regions of a circle, the pattern is incorrect.

NOW TRY THIS 10

A *prime number* is a natural number with exactly two distinct positive numbers, 1 and the number itself, that divide it with 0 remainder; for example, 2, 3, 5, 7, 11, 13 are primes. One day Amy makes a *conjecture* that the formula, $y = x^2 + x + 11$ will produce only prime numbers if she substitutes the natural numbers, 1, 2, 3, 4, 5, . . . for x. She shows her work so far in Table 3 for $x = 1, 2, 3, 4$.

Table 3

x	1	2	3	4
y	13	17	23	31

a. What type of reasoning is Amy using?
b. Try the next several natural numbers and see whether they seem to work.
c. Can you find a counterexample to show that Amy's conjecture is false?

Arithmetic Sequences

A **sequence** is an ordered arrangement of numbers, figures, or objects. A sequence has items or terms identified as *1st*, *2nd*, *3rd*, and so on. Often, sequences can be classified by their properties. For example, what property do the following first three sequences have that the fourth does not?

a. 1, 2, 3, 4, 5, 6, . . .
b. 0, 5, 10, 15, 20, 25, . . .
c. 2, 6, 10, 14, 18, 22, . . .
d. 1, 11, 111, 1111, 11111, 111111, . . .

In each of the first three sequences, each term—starting from the second term—is obtained from the preceding term by adding a fixed number, the **common difference** or **difference**. In part (a) the difference is 1, in part (b) the difference is 5, and in part (c) the difference is 4. Sequences such as the first three are arithmetic sequences. An **arithmetic sequence** is one in which each successive term from the second term on is obtained from the previous term by the addition or subtraction of a fixed number. The sequence in part (d) is not arithmetic because there is no single fixed number that can be added to or subtract from the previous term to obtain the next term.

Arithmetic sequences can be generated from objects, as shown in Example 2.

EXAMPLE 2

Find a numerical pattern in the number of matchsticks required to continue the sequence shown in Figure 11.

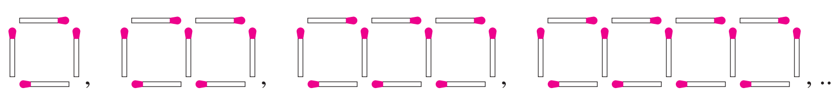


Figure 11

Solution Assume the matchsticks are arranged so that each figure has one more square on the right than the preceding figure. Note that the addition of a square to an arrangement requires the addition of three matchsticks each time. Thus, with this assumption, the numerical pattern obtained is 4, 7, 10, 13, 16, 19, . . ., an arithmetic sequence starting at 4 and having a difference of 3.

An informal description of an arithmetic sequence is one that can be described as an "add d" pattern, where d is the common difference. In Example 2, $d = 3$. In the language of children, the pattern in Example is "add 3." This is an example of a **recursive pattern**. In a recursive pattern, after one or more consecutive terms are given to start, each successive term of the sequence is obtained from the previous term(s). For example, 3, 6, 9, . . . is another "add 3" sequence starting with 3.

A recursive pattern is typically used in a spreadsheet, as seen in Table 4 where the index column tracks the order of the terms. The headers for the columns are A, B, etc. The first entry in the A column (in the A1 cell) is 4; and to find the term in the A2 cell, we use the number in the A1 cell and add 3. The pattern is continued using the *Fill Down* command. In spreadsheet language, the formula $= \text{A1} + 3$ finds any term after the first by adding 3 to the previous term. A formula based on a recursive pattern is a **recursive formula**. (For more explicit directions on using a spreadsheet, see the Technology Manual, which can be found online at www.pearson-highered.com/Billstein11einfo.)

Table 4

Index Column

	A	B
1	4	
2	7	
3	10	
4	13	
5	16	
6	19	
7	22	
8	25	
9	28	
10		
11		
12		
13		

If we want to find the number of matchsticks in the 100th figure in Example 2, we can use the spreadsheet or we can find an explicit formula or a general rule for finding the number of matchsticks when given the number of the term. The problem-solving strategy of *making a table* is again helpful here.

The spreadsheet in Table 4 provides an easy way to *make a table*. The index column gives the numbers of the terms and column A gives the terms of the sequence. If we are building such a table without a spreadsheet, it might look like Table 5. Notice that each term is a sum of 4 and a certain number of 3s. We see that the number of 3s is 1 less than the number of the term. This pattern should continue, since the first term is $4 + 0 \cdot 3$ and each time we increase the number of the term by 1, we add one *more* 3. Thus, it seems that the 100th term is $4 + (100 - 1)3$; and, in general, the ***n*th term** is $4 + (n - 1)3$. Note that $4 + (n - 1)3$ could be written as $3n + 1$.

Table 5

Number of Term	Term
1	4
2	$7 = 4 + 3 = 4 + 1 \cdot 3$
3	$10 = (4 + 1 \cdot 3) + 3 = 4 + 2 \cdot 3$
4	$13 = (4 + 2 \cdot 3) + 3 = 4 + 3 \cdot 3$
.	.
.	.
.	.
n	$4 + (n - 1)3 = 3n + 1$

Still a different approach to finding the number of matchsticks in the 100th term of Figure 11 might be as follows: If the matchstick figure has 100 squares, we could find the total number of matchsticks by adding the number of horizontal and vertical sticks. There are $2 \cdot 100$ placed horizontally. (Why?) Notice that in the first figure, there are 2 matchsticks placed vertically; in the second, 3; and in the third, 4. In the 100th figure, there should be $100 + 1$ vertical matchsticks.

Altogether there will be $2 \cdot 100 + (100 + 1)$, or 301, matchsticks in the 100th figure. Similarly, in the nth figure, there would be $2n$ horizontal and $n + 1$ vertical matchsticks, for a total of $3n + 1$. This discussion is summarized in Table 6.

Table 6

Number of Term	Number of Matchsticks Horizontally	Number of Matchsticks Vertically	Total
1	2	2	4
2	4	3	7
3	6	4	10
4	8	5	13
.	.	.	.
.	.	.	.
.	.	.	.
100	200	101	301
.	.	.	.
.	.	.	.
.	.	.	.
n	$2n$	$n + 1$	$2n + (n + 1) = 3n + 1$

If we are given the value of the term, we can use the formula for the nth term in Table 6 to *work backward* to find the number of the term. For example, given the term 1798, we can write an equation: $3n + 1 = 1798$. Therefore, $3n = 1797$ and $n = 599$. Consequently, 1798 is the 599th term. We could obtain the same answer by solving $4 + (n - 1)3 = 1798$ for n.

In the matchstick problem, we found the nth term of a sequence. If the nth term of a sequence is given, we can find any term of the sequence, as shown in Example 3.

EXAMPLE 3

Find the first four terms of a sequence, the nth term of which is given by the following, and determine whether the sequence seems to be arithmetic:

a. $4n + 3$ **b.** $n^2 - 1$

Solution

a.

Number of Term	*Term*
1	$4 \cdot 1 + 3 = 7$
2	$4 \cdot 2 + 3 = 11$
3	$4 \cdot 3 + 3 = 15$
4	$4 \cdot 4 + 3 = 19$

Hence, the first four terms of the sequence are 7, 11, 15, 19. This sequence seems arithmetic with difference 4.

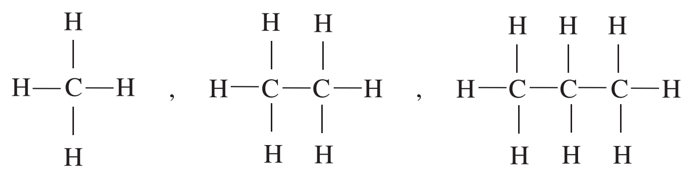
b.

Number of Term	*Term*
1	$1^2 - 1 = 0$
2	$2^2 - 1 = 3$
3	$3^2 - 1 = 8$
4	$4^2 - 1 = 15$

Thus, the first four terms of the sequence are 0, 3, 8, 15. This sequence is not arithmetic because it has no common difference.

EXAMPLE 4

The diagrams in Figure 12 show the molecular structure of alkanes, a class of hydrocarbons. C represents a carbon atom and H a hydrogen atom. A connecting segment shows a chemical bond.



methane (C_1H_4) ethane (C_2H_6) propane (C_3H_8)

Figure 12

a. Hectane is an alkane with 100 carbon atoms. How many hydrogen atoms does it have?
b. Write a general rule for alkanes C_nH_m showing the relationship between m and n.

Solution

a. To determine the relationship between the number of carbon and hydrogen atoms, we first study the drawing of the alkanes and disregard the extreme left and right hydrogen atoms in each. With this restriction, we see that for every carbon atom, there are two hydrogen atoms. Therefore, there are twice as many hydrogen atoms as carbon atoms plus the two hydrogen atoms at the extremes. For example, when there are 3 carbon atoms, there are $(2 \cdot 3) + 2$, or 8, hydrogen atoms. This notion is summarized in Table 7. If we extend the table for 4 carbon atoms, we get $(2 \cdot 4) + 2$, or 10, hydrogen atoms. For 100 carbon atoms, there are $(2 \cdot 100) + 2$, or 202, hydrogen atoms.

Table 7

No. of Carbon Atoms	No. of Hydrogen Atoms
1	4
2	6
3	8
.	.
.	.
.	.
100	202
.	.
.	.
.	.
n	m

b. In general, for n carbon atoms there would be n hydrogen atoms attached above, n attached below, and 2 attached on the sides. Hence, the total number of hydrogen atoms m would be $2n + 2$. It follows that the number of hydrogen atoms is $m = 2n + 2$.

EXAMPLE 5

A theater is set up so that there are 20 seats in the first row, and 4 additional seats in each consecutive row to the back of the theater where there are 144 seats. How many rows are there in the theater?

Solution

Two strategies lend themselves to this problem. One is to *build a table* and to consider the entries as seen in Table 8.

Table 8

Row Number	Number of Seats
1	20
2	$20 + 4$
3	$20 + 2 \cdot 4$
4	$20 + 3 \cdot 4$
5	$20 + 4 \cdot 4$
.	.
.	.
.	.
n	$20 + (n - 1)4$

Observe that in Table 8 when we write the number of seats as 20 plus the number of additional 4 seats in consecutive rows, the number of 4s added is one less than the number of the row. We know that in the last row there are 144 seats. Thus, we have the following:

$144 = 20 + (n - 1)4$, but 20 added to 124 is 144 so $124 = (n - 1)4$.

Now $4 \cdot 31$ is 124 giving us $31 = n - 1$.

Therefore, $n = 32$, and there are 32 rows in the theater.

A different way to solve the problem is to use a spreadsheet as seen in Table 9, where the number of the row is seen in the index column and the entry in cell A1 indicates 20 seats in that row. Filling down the A column using the recursive formula $= \text{A1} + 4$, we find 144 seats in row 32. Thus, there are 32 rows in the theater.

Table 9

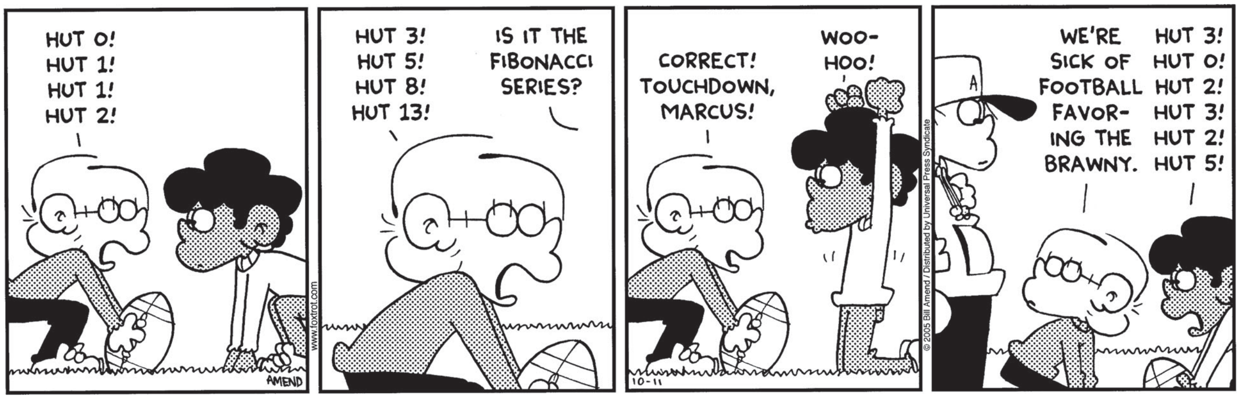
	A	B
1	20	
2	24	
3	28	
4	32	
5	36	
6	40	
7	44	
8	48	
9	52	
10	56	
11	60	
12	64	
13	68	
14	72	
15	76	
16	80	
17	84	
18	88	

Spreadsheet continued.

19	92	
20	96	
21	100	
22	104	
23	108	
24	112	
25	116	
26	120	
27	124	
28	128	
29	132	
30	136	
31	140	
32	144	
33	148	
34	152	
35	156	
36	160	
37	164	

Fibonacci Sequence

Dan Brown's popular book *The Da Vinci Code* brought renewed interest to one of the most famous sequences of all time, the ***Fibonacci sequence***. The Fibonacci sequence is hinted at in the following Foxtrot cartoon.



The Fibonacci series in the cartoon is actually a sequence with 0 as a starting term. More typically, the sequence is seen as follows:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

This sequence is not *arithmetic* as there is no fixed difference, d. The first two terms of the Fibonacci sequence are 1, 1 and each subsequent term is the sum of the previous two.

NOW TRY THIS 11

Starting with the first two terms, the seeds, as 1, 1,

a. Add the first three Fibonacci numbers.
b. Add the first four Fibonacci numbers.
c. Add the first five Fibonacci numbers.
d. Add the first six Fibonacci numbers.
e. Add the first seven Fibonacci numbers.
f. What pattern is there in the sums in parts (a)–(e) and any of the remaining numbers in the Fibonacci sequence?

Geometric Sequences

A child has 2 biological parents, 4 grandparents, 8 great grandparents, 16 great-great grandparents, and so on. The number of generational ancestors form the **geometric sequence** 2, 4, 8, 16, 32, ... Each successive term of a geometric sequence is obtained from its predecessor by multiplying by a fixed nonzero number, the **ratio**. In this example, both the first term and

Historical Note

SPL/Photo Researchers, Inc.

Leonardo de Pisa was born around 1170. His real family name was Bonaccio but he preferred the nickname Fibonacci, derived from *filius Bonacci*, meaning "son of Bonacci." In his book *Liber Abaci* (1202) he described the now-famous rabbit problem, whose solution, the sequence 1, 1, 2, 3, 5, 8, 13, 21, . . ., became known as the *Fibonacci sequence* with terms called *Fibonacci numbers*.

the ratio are 2. (The ratio is 2 because each person has two parents.) To find the nth term examine the pattern in Table 10.

Table 10

Number of Term	Term
1	$2 = 2^1$
2	$4 = 2 \cdot 2 = 2^2$
3	$8 = (2 \cdot 2) \cdot 2 = 2^3$
4	$16 = (2 \cdot 2 \cdot 2) \cdot 2 = 2^4$
5	$32 = (2 \cdot 2 \cdot 2 \cdot 2) \cdot 2 = 2^5$
.	.
.	.
.	.

In Table 10, when the given term is written as a power of 2, the number of the term is the **exponent**. Following this pattern, the 10th term is 2^{10}, or 1024, the 100th term is 2^{100}, and the nth term is 2^n. Thus, the number of ancestors in the nth previous generation is 2^n. The notation used in Table 10 can be generalized as follows.

Definition of a^n

If n is a natural number, then $a^n = \overbrace{a \cdot a \cdot a \cdot \ldots \cdot a}^{n \text{ factors}}$.

If $n = 0$ and $a \neq 0$, then $a^0 = 1$.

Geometric sequences play an important role in everyday life. For example, suppose we have \$1000 in a bank that pays 5% interest annually. (Note that $5\% = 0.05$.) If no money is added or taken out, then at the end of the first year we have all of the money we started with plus 5% more.

$$\textbf{Year 1: } \$1000 + 0.05(\$1000) = \$1000(1 + 0.05) = \$1000(1.05) = \$1050$$

If no money is added or taken out, then at the end of the second year we would have 5% more money than the previous year.

$$\textbf{Year 2: } \$1050 + 0.05(\$1050) = \$1050(1 + 0.05) = \$1050(1.05) = \$1102.50$$

The amount of money in the account after any number of years can be found by noting that every dollar invested for one year becomes $1 + 0.05 \cdot 1$, or 1.05 dollars. Therefore, the amount in each year is obtained by multiplying the amount from the previous year by 1.05. The amounts in the bank after each year form a geometric sequence because the amount in each year (starting from year 2) is obtained by multiplying the amount in the previous year by the same number, 1.05. This is summarized in Table 11. ($\approx$ means approximately equal to.)

Table 11

Number of Term (Year)	Term (Amount at the Beginning of Each Year)
1	\$1000
2	$\$1000(1.05)^1 = \$1050.00$
3	$\$1000(1.05)^2 = \$1102.50$
4	$\$1000(1.05)^3 \approx \$1157.63$
.	.
.	.
.	.
n	$\$1000(1.05)^{n-1}$

NOW TRY THIS 12

a. Two bacteria are in a dish. The number of bacteria triples every hour. Following this pattern, find the number of bacteria in the dish after 10 hours and after n hours.

b. Suppose that instead of increasing geometrically as in part (a), the number of bacteria increases arithmetically by 3 each hour. Compare the growth after 10 hours and after n hours. Comment on the difference in growth of a geometric sequence versus an arithmetic sequence.

Other Sequences

Figurate numbers, based on geometrical patterns, provide examples of sequences that are neither arithmetic nor geometric. Such numbers can be represented by dots arranged in the shape of certain geometric figures. The number 1 is the beginning of most patterns involving figurate numbers. The arrays in Figure 13 represents the first four terms of the sequence of **triangular numbers**.

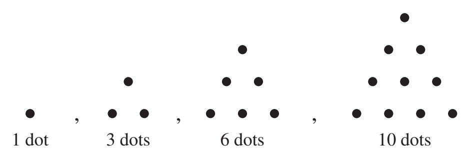


Figure 13

The triangular numbers can be written numerically as 1, 3, 6, 10, 15, ... The sequence 1, 3, 6, 10, 15, . . . is not an arithmetic sequence because there is no *common difference*, as Figure 14 shows. It is not a geometric sequence because there is no common ratio. It is not a Fibonacci sequence.

$$\begin{array}{lccccccccc} & 1 & & 3 & & 6 & & 10 & & 15 \\ \text{(First difference)} & & 2 & & 3 & & 4 & & 5 & \end{array}$$

Figure 14

However, the sequence of differences, 2, 3, 4, 5, . . ., appears to form an arithmetic sequence with difference 1, as Figure 15 shows. The next successive terms for the original sequence are shown in color in Figure 15.

$$\begin{array}{lccccccccccccc} & 1 & & 3 & & 6 & & 10 & & 15 & & 21 & & 28 \\ \text{(First difference)} & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & \\ \text{(Second difference)} & & & 1 & & 1 & & 1 & & 1 & & 1 & & \end{array}$$

Figure 15

Table 12 suggests a pattern for finding the next terms and the nth term for the triangular numbers. The second term is obtained from the first term by adding 2; the third term is obtained from the second term by adding 3; and so on.

Table 12

Number of Term	Term
1	1
2	$3 = 1 + 2$
3	$6 = 1 + 2 + 3$
4	$10 = 1 + 2 + 3 + 4$
5	$15 = 1 + 2 + 3 + 4 + 5$
.	.
.	.
.	.
10	$55 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$

In general, because the nth triangular number has n dots in the nth row, it is equal to the sum of the dots in the previous triangular number (the $(n - 1)$st one) plus the n dots in the nth row. Following this pattern, the 10th term is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$, or 55, and the nth term is $1 + 2 + 3 + 4 + 5 + \ldots + (n - 1) + n$. This problem is similar to Gauss's Problem in Section 1. Because of the work done in Section 1, we know that this sum can be expressed as

$$\frac{n(n + 1)}{2}.$$

Next consider the first four *square numbers* in Figure 16. These square numbers, 1, 4, 9, 16 can be written as $1^2, 2^2, 3^2, 4^2$. Continuing, the number of dots in the 10th array would be 10^2, the number of dots in the 100th array would be 100^2, and the number of dots in the nth array would be n^2. The sequence of square numbers is neither arithmetic nor geometric. Investigate whether the sequence of first differences is an arithmetic sequence and tell why.

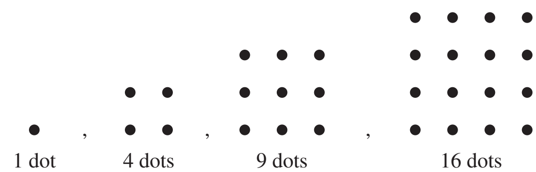


Figure 16

EXAMPLE 6

Use differences to find a pattern. Then assuming that the pattern discovered continues, find the seventh term in each of the following sequences:

a. 5, 6, 14, 29, 51, 80, . . . **b.** 2, 3, 9, 23, 48, 87, . . .

Solution

a. Figure 17 shows the sequence of first differences.

	5		6		14		29		51		80
(First difference)		1		8		15		22		29	

Figure 17

To discover a pattern for the original sequence, we try to find a pattern for the sequence of differences 1, 8, 15, 22, 29, ... This sequence is an arithmetic sequence with fixed difference 7 as seen in Figure 18.

	5		6		14		29		51		80
(First difference)		1		8		15		22		29	
(Second difference)			7		7		7		7		

Figure 18

Thus, the sixth term in the first difference row is $29 + 7$, or 36, and the seventh term in the original sequence is $80 + 36$, or 116. What number follows 116?

b. Because the second difference is not a fixed number, we go on to the third difference as in Figure 19.

	2		3		9		23		48		87
(First difference)		1		6		14		25		39	
(Second difference)			5		8		11		14		
(Third difference)				3		3		3			

Figure 19

The third difference is a fixed number; therefore, the second difference is an arithmetic sequence. The fifth term in the second-difference sequence is $14 + 3$, or 17; the sixth term in the first-difference sequence is $39 + 17$, or 56; and the seventh term in the original sequence is $87 + 56$, or 143.

NOW TRY THIS 13

Figure 20 shows the first three figures of arrays of sticks with the number of sticks written below the figures.

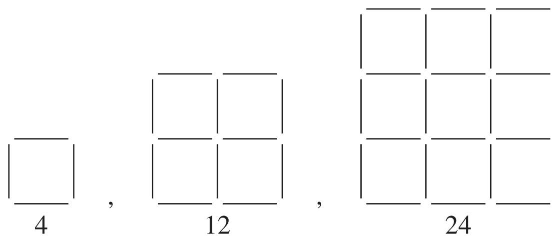


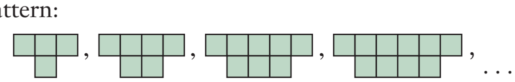
Figure 20

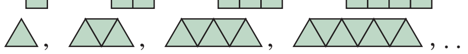
a. Draw the next array of sticks.
b. Build a table showing the term number and the number of sticks for $n = 1, 2, 3, 4$.
c. Use differences to predict the number of sticks for $n = 5, 6, 7$.
d. Is finding differences the best way to determine how many sticks there are in the 100th term? Tell how you would find that term.

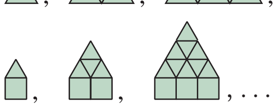
When asked to find a pattern for a given sequence, we first look for some easily recognizable pattern and determine whether the sequence is arithmetic or geometric. If a pattern is unclear, taking successive differences may help. *It is possible that none of the methods described reveal a pattern.*

Assessment 2 A

1. For each of the following sequences of figures, determine a possible pattern and draw the next figure according to that pattern:

a. 

b. 

c. 

2. In each of the following, list three terms that continue the arithmetic or geometric sequences. Identify the sequences as arithmetic or geometric.
a. 1, 3, 5, 7, 9
b. 0, 50, 100, 150, 200
c. 3, 6, 12, 24, 48
d. 10, 100, 1,000, 10,000, 100,000
e. 9, 13, 17, 21, 25, 29

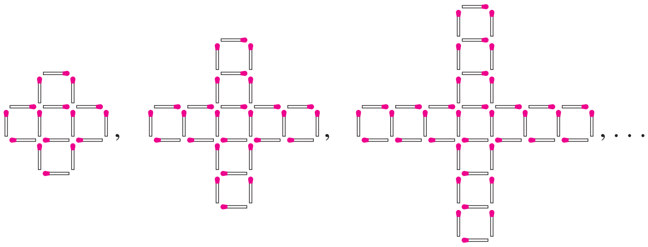
3. Find the 100th term and the nth term for each of the sequences in exercise 2.

4. Use a traditional clock face to determine the next three terms in the following sequence:

$$1, 6, 11, 4, 9, \ldots$$

5. The pattern 1, 8, 27, 64, 125, . . . is a cubic pattern named because $1 = 1 \cdot 1 \cdot 1$ or 1^3, $8 = 2 \cdot 2 \cdot 2$ or 2^3, and so on.
a. What is the least 4-digit number greater than 1000 in this pattern?
b. What is the greatest 3-digit number in this pattern?
c. If this pattern was produced in a normal spreadsheet, what is the number in cell A14?

6. The first windmill has 5 matchstick squares, the second has 9, and the third has 13, as shown. How many matchstick squares are in **(a)** the 10th windmill? **(b)** the nth windmill? **(c)** How many matchsticks will it take to build the nth windmill?



7. In the following sequence, the figures are made of cubes that are glued together. If the exposed surface needs to be painted, how many squares will be painted in **(a)** the 10th figure? **(b)** the nth figure?

8. The school population for a certain school is predicted to increase by 50 students per year for the next 10 years. If the current enrollment is 700 students, what will the enrollment be after 10 years?

9. Joe's annual income has been increasing each year by the same dollar amount. The first year his income was $24,000, and the ninth year his income was $31,680. In which year was his income $45,120?

10. The first difference of a sequence is the arithmetic sequence 2, 4, 6, 8,10, ... Find the first six terms of the original sequence in each of the following cases:

a. The first term of the original sequence is 3.
b. The sum of the first two terms of the original sequence is 10.
c. The fifth term of the original sequence is 35.

11. List the next three terms to continue a pattern in each of the following. (Finding differences may be helpful.)

a. 5, 6, 14, 32, 64, 115, 191
b. 0, 2, 6, 12, 20, 30, 42

12. How many terms are there in each of the following sequences?

a. 51, 52, 53, 54, . . ., 151
b. $1, 2, 2^2, 2^3, \ldots, 2^{60}$
c. 10, 20, 30, 40, . . ., 2000
d. 1, 2, 4, 8, 16, 32, . . ., 1024

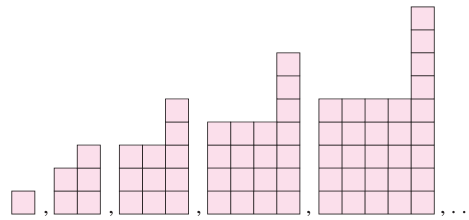
13. Find the first five terms in sequences with the following nth terms.

a. $n^2 + 2$
b. $5n - 1$
c. $10^n - 1$
d. $3n + 2$

14. Find a counterexample for each of the following:

a. If n is a natural number, then $(n + 5)/5 = n + 1$.
b. If n is a natural number, then $(n + 4)^2 = n + 16$.

15. Assume that the following patterns are built of square tile and the pattern continues. Answer the questions that follow.



a. How many square tiles are there in the sixth figure?
b. How many square tiles are in the nth figure?
c. Is there a figure that has exactly 1259 square tiles? If so, which one? (*Hint*: To determine if there is a figure in the sequence containing exactly 1259 square tiles, first think about the greatest square number less than 1259.)

16. Consider the sequences given in the table below. Find the least number, n, such that the nth term of the geometric sequence is greater than the corresponding term in the arithmetic sequence.

Number of term	1	2	3	4	5	6	. . .	n
Arithmetic	300	500	700	900	1100	1300	. . .	
Geometric	2	4	8	16	32	64	. . .	

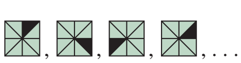
17. Start out with a piece of paper. Next, cut the piece of paper into five pieces. Take any one of the pieces and cut it into five pieces, and so on.

a. What sequence can be obtained through this process?
b. What is the total number of pieces obtained after n cuts?

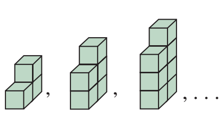
Assessment 2 B

1. In each of the following, determine a possible pattern and draw the next figure according to that pattern if the sequence continues.

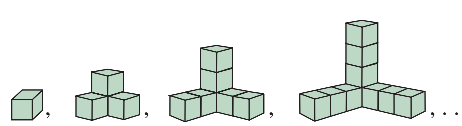
a.



b.



c.



2. In each of the following, list three terms that continue the arithmetic or geometric sequences. Identify the sequences as arithmetic or geometric.

a. 8, 11, 14, 17, 20, . . .
b. 5, 15, 45, 135, 405, . . .

c. 2, 7, 12, 17, 22, . . .
d. 1, 1, 1, 1, 1, . . .
e. 2, 10, 50, 250, 1250, . . .

3. Find the 100th term and the nth term for each of the sequences in exercise 2.

4. Use a traditional clock face to determine the next three terms in the following sequence:

$$1, 9, 5, 1, \ldots$$

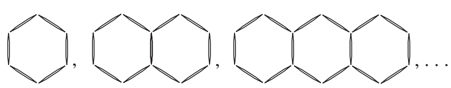
5. Observe the following pattern:

$$1 + 3 = 2^2,$$
$$1 + 3 + 5 = 3^2,$$
$$1 + 3 + 5 + 7 = 4^2$$

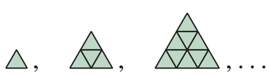
a. State an inductive generalization based on this pattern.
b. Based on the generalization in (a), find

$$1 + 3 + 5 + 7 + \ldots + 35$$

6. In the following pattern, one hexagon takes 6 toothpicks to build, two hexagons take 11 toothpicks to build, and so on. How many toothpicks would it take to build **(a)** 10 hexagons? **(b)** n hexagons?



7. Each successive figure below is made of small triangles like the first one in the sequence. Conjecture the number of small triangles needed to make **(a)** the 100th figure and **(b)** the nth figure.



8. A tank contains 15,360 L of water. At the end of each subsequent day, half of the water is removed and not replaced. How much water is left in the tank after 10 days?

9. The Washington Middle School schedule is an arithmetic sequence. Each period is the same length and includes a 4th period lunch. The first three periods begin at 8:10 A.M., 9:00 A.M., and 9:50 A.M., respectively. At what time does the eighth period begin?

10. The first difference of a sequence is 3, 6, 9, 12, 15, . . . Find the first six terms of the original sequence in each of the following cases:
a. The first term of the original sequence is 3.
b. The sum of the first two terms of the original sequence is 7.
c. The fifth term of the original sequence is 34.

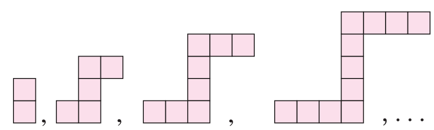
11. List the next three terms to continue a pattern in each of the following. (Finding differences may be helpful.)
a. 3, 8, 15, 24, 35, 48, . . .
b. 1, 7, 18, 37, 67, 111, . . .

12. How many terms are there in each of the following sequences?
a. $1, 3, 3^2, 3^3, \ldots, 3^{60}$
b. 9, 13, 17, 21, 25, . . . , 353
c. 38, 39, 40, 41, . . . , 198

13. Find the first five terms in the following sequences with the nth term.
a. $5n + 6$
b. $6n - 2$
c. $5n + 1$
d. $n^2 - 1$

14. Find a counterexample for each of the following:
a. If n is a natural number, then $(3 + n)/3 = n$.
b. If n is a natural number, then $(n - 2)^2 = n^2 - 2^2$.

15. Assume the following pattern with terms built of square tile figures continues and answer the questions that follow.



a. How many square tiles are there in the sixth figure?
b. How many square tiles are in the nth figure?
c. Is there a figure that has exactly 449 square tiles? If so, which one?

16. Consider the sequences given in the table below. Find the least number, n, such that the nth term of the geometric sequence is greater than the corresponding term in the arithmetic sequence.

Number of term	1	2	3	4	5	6	. . .	n
Arithmetic	200	500	800	1100	1400	1700	. . .	
Geometric	1	3	9	27	81	243	. . .	

17. Female bees are born from fertilized eggs, and male bees are born from unfertilized eggs. This means that a male bee has only a mother, whereas a female bee has a mother and a father. If the ancestry of a male bee is traced 10 generations including the generation of the male bee, how many bees are there in all 10 generations? (*Hint:* The Fibonacci sequence might be helpful.)

Mathematical Connections 2

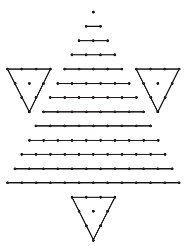
Communication

1. a. If a fixed number is added to each term of an arithmetic sequence, is the resulting sequence an arithmetic sequence? Justify the answer.
b. If each term of an arithmetic sequence is multiplied by a fixed number, will the resulting sequence always be an arithmetic sequence? Justify the answer.
c. If the corresponding terms of two arithmetic sequences are added, is the resulting sequence arithmetic?

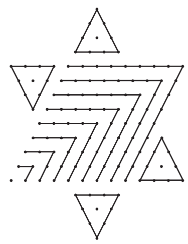
2. A student says she read that Thomas Robert Malthus (1766–1834), a renowned British economist and demographer, claimed that the increase of population will take place, if unchecked, in a geometric sequence, whereas the supply of food will increase in only an arithmetic sequence. This theory implies that population increases faster than food production. The student is wondering why. How do you respond?

Open-Ended

3. Patterns can be used to count the number of dots on the Chinese checkerboard; two patterns are shown here. Determine several other patterns to count the dots.



$1 + 2 + 3 + \ldots + 13 + 3(10)$



$1 + 3 + 5 + 7 + \ldots + 17 + 4(10)$

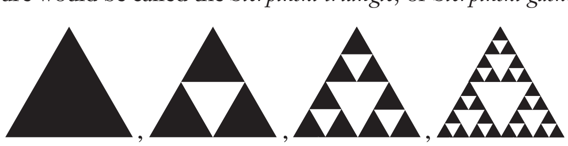
4. Make up a pattern involving figurate numbers and find a formula for the 100th term. Describe the pattern and how to find the 100th term.
5. A sequence that follows the same pattern as the Fibonacci sequence but in which the first two terms are any numbers except 1s is called a *Fibonacci type sequence*. Choose a few such sequences and answer the questions in Now Try This 11. Do these sequences behave in the same way?

Cooperative Learning

6. The following pattern is called *Pascal's triangle*. It was named for the mathematician Blaise Pascal (1623–1662).

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

   a. Have each person in the group find four different patterns in the triangle and then share them with the rest of the group.
   b. Add the numbers in each horizontal row. Discuss the pattern that occurs.
   c. Use part (b) to find the sum in the 16th row.
   d. What is the sum of the numbers in the nth row?
7. If the following pattern continued indefinitely, the resulting figure would be called the *Sierpinski triangle*, or *Sierpinski gasket*.



In a group, determine each of the following. Discuss different counting strategies.
   a. How many black triangles would be in the fifth figure?
   b. How many white triangles would be in the fifth figure?
   c. If the pattern is continued for n figures, how many black triangles will there be?

Questions from the Classroom

8. Joey said that 4, 24, 44, and 64 all have remainder 0 when divided by 4, so all numbers that end in 4 must have 0 remainder when divided by 4. How do you respond?
9. Al and Betty were asked to extend the sequence 2, 4, 8, ... Al said his answer of 2, 4, 8, 16, 32, 64, . . . was the correct one. Betty said Al was wrong and it should be 2, 4, 8, 14, 22, 32, 44, ... What do you tell these students?
10. A student claims the sequence 6, 6, 6, 6, 6, . . . never changes, so it is neither arithmetic nor geometric. How do you respond?

Review Problems

11. In a baseball league consisting of 10 teams, each team plays each of the other teams twice. How may games will be played?
12. How many ways can you make change for 40¢ using only nickels, dimes, and quarters?
13. Tents hold 2, 3, 5, 6, or 12 people. What combinations of tents are possible to sleep 26 people if all tents are fully occupied and only one 12-person tent is used?

Trends in Mathematics and Science Study (TIMSS) Questions

The numbers in the sequence 7, 11, 15, 19, 23, . . . increase by four. The numbers in the sequence 1, 10, 19, 28, 37, . . . increase by nine. The number 19 is in both sequences. If the two sequences are continued, what is the next number that is in BOTH the first and the second sequences?

TIMSS, Grade 8, 2003

Matchsticks are arranged as shown in the figures.

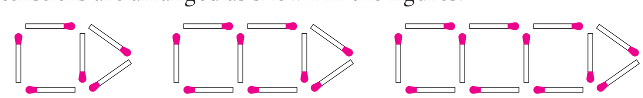


Figure 1 Figure 2 Figure 3

If the pattern is continued, how many matchsticks would be used to make Figure 10?

a. 30 b. 33 c. 36
d. 39 e. 42

TIMSS, Grade 8, 2003

The three figures below are divided into small congruent triangles.

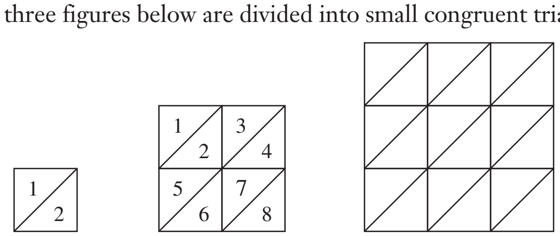


Figure 1 Figure 2 Figure 3

Source: Trends in International Mathematics and Science Study (TIMSS), IES National Center for Education Statistics, 2011.

a. Complete the table below. First, fill in how many small triangles make up Figure 3. Then, find the number of small triangles that would be needed for the fourth figure if the sequence of figures is extended.

Figure	Numbers of Small Triangles
1	2
2	8
3	
4	

b. The sequence of figures is extended to the seventh figure. How many small triangles would be needed for Figure 7?

c. The sequence of figures is extended to the 50th figure. Explain a way to find the number of small triangles in the 50th figure that does not involve drawing it and counting the number of triangles.

TIMSS, Grade 8, 2003

BRAIN TEASER

Find the next row in the following pattern and explain your pattern:

1
1 1
2 1
1 2 1 1
1 1 1 2 2 1

3 Reasoning and Logic: An Introduction

Logic is a tool used in mathematical thinking and problem solving. It is essential for reasoning and, although it cannot be taught in a single unit, we present a few "basics" of logic in this section. In logic, a **statement** *is a sentence that is either true or false, but not both*. The following expressions are not statements because their truth values cannot be determined:

1. She has blue eyes.
2. $x + 7 = 18$
3. $2y + 7 > 1$
4. $2 + 3$
5. How are you?
6. Look out!
7. Lincoln was the best president.

Expressions (**1**), (**2**), and (**3**) become statements if, for (**1**), "she" is identified, and for (**2**) and (**3**), values are assigned to x and y, respectively. However, an expression involving he or she or x or y may already be a statement. For example, "If he is over 210 cm tall, then he is over 2 m tall" and "$2(x + y) = 2x + 2y$" are both statements because they are true no matter who *he* is or what the numerical values of x and y are.

Negation and Quantifiers

From a given statement, p, it is possible to create the **negation** of p denoted by $\sim p$ and meaning not p. *If a statement is true, its negation is false, and if the statement is false, its negation is true.* Consider the statement "Snow is falling." The negation of this statement may be stated simply as "Snow is not falling."

EXAMPLE 7

Negate each of the following statements:

a. $2 + 3 = 5$
b. A hexagon has six sides.

Solution

a. $2 + 3 \neq 5$
b. A hexagon does not have six sides.

At a given time, sentences such as "The shirt I have on is blue" and "The shirt I have on is green" are statements. However, they are not negations of each other. A statement and its negation must have opposite truth values in all possible cases. If the shirt I have on is actually red, then both of the statements are false and, hence, cannot be negations of each other. However, at a given time the statements "The shirt I have on is blue" and "The shirt I have on is not blue" are negations of each other because they have opposite truth values no matter what color the shirt really is.

Some statements involve **quantifiers** and are more complicated to negate. Quantifiers include words such as *all*, *some*, *every*, and *there exists*.

- The quantifiers *all*, *every*, and *no* refer to each and every element in a set and are called *universal quantifiers*.
- The quantifiers *some* and *there exists at least one* refer to one or more, or possibly all, of the elements in a set and are called *existential quantifiers*.
- *All*, *every*, and *each* have the same mathematical meaning. Similarly, *some* and *there exists at least one* have the same meaning.

Assume the following is true: "Some professors at Paxson University have blue eyes." This means that at least one professor at Paxson University has blue eyes. It does not rule out the possibilities that *all* the Paxson professors have blue eyes or that some of the Paxson professors do not have blue eyes. Because the negation of a true statement is false, neither "Some professors at Paxson University do not have blue eyes" nor "All professors at Paxson have blue eyes" are negations of the original statement. One possible negation of the original statement is "No professors at Paxson University have blue eyes."

To discover if one statement is a negation of another, we use arguments similar to the preceding one to determine whether they have opposite truth values in all possible cases.

General forms of quantified statements with their negations follow.

Statement	Negation
Some a are b.	No a is b.
Some a are not b.	All a are b.
All a are b.	Some a are not b.
No a is b.	Some a are b.

EXAMPLE 8

Negate each of the following regardless of its truth value:

a. All students like hamburgers.
b. Some people like mathematics.
c. There exists a natural number n such that $3n = 6$.
d. For all natural numbers n, $3n = 3n$.

Solution

a. Some students do not like hamburgers.
b. No people like mathematics.
c. For all natural numbers n, $3n \neq 6$.
d. There exists a natural number n such that $3n \neq 3n$.

Truth Tables and Compound Statements

To investigate the truth of statements, consider the following puzzle by a foremost writer of logic puzzles, Raymond Smullyan. One of Smullyan's books, *The Lady or the Tiger?*, has a puzzle about a prisoner who must make a choice between two rooms. Each room has a sign on the door and the prisoner knows that exactly *one* of the signs is true. The signs are shown in Figure 21.

IN THIS ROOM THERE IS A LADY AND IN THE OTHER ROOM THERE IS A TIGER.	IN ONE OF THESE ROOMS THERE IS A LADY AND IN ONE OF THESE ROOMS THERE IS A TIGER.
Door Sign for Room 1	Door Sign for Room 2

Figure 21

With the information on the signs in Figure 21, the prisoner can choose the correct room. Discuss Smullyan's puzzle and try to find a solution before reading on.

Consider that if the sign on Room 1 is true, then the sign on Room 2 must be true. Since this cannot happen, the sign on Room 2 must be true, making the sign on Room 1 false. Because the sign on Room 1 is false, the lady can't be in Room 1 and must be in Room 2.

A symbolic system can help in the study of logic. **Truth tables** are often used to show all possible true-false patterns for statements. Table 13 summarizes the truth tables for p and $\sim p$.

Table 13

Statement p	Negation $\sim p$
T	F
F	T

From two given statements, it is possible to create a new, **compound statement** by using a connective such as *and.* A compound statement may be formed by combining two or more statements. For example, "Snow is falling" and "The ski run is open" together with *and* give "Snow is falling and the ski run is open." Other compound statements can be obtained by using the connective *or.* For example, "Snow is falling or the ski run is open." The symbols $\wedge$ and $\vee$ are used to represent the connectives *and* and *or*, respectively. For example, if p represents "Snow is falling" and q represents "The ski run is open," then "Snow is falling and the ski run is open" is denoted by $p \wedge q$. Similarly, "Snow is falling or the ski run is open" is denoted by $p \vee q$.

Table 14

p	q	Conjunction $p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The truth value of any compound statement, such as $p \wedge q$, is defined using the truth value of each of the simple statements. Because each of the statements p and q may be either true or false, there are four distinct possibilities for the truth value of $p \wedge q$, as shown in Table 14. The compound statement is the **conjunction** of p and q and is defined to be true if, and only if, both p and q are true. Otherwise, it is false.

The compound statement $p \vee q$ (p *or* q) is a **disjunction**. In everyday language, *or* is not always interpreted in the same way. In logic, we use an *inclusive or*. The statement "I will go to a movie or I will read a book" means I will either go to a movie, or read a book, or do both. Hence, in logic, p or q, symbolized $p \vee q$, is defined to be false if both p and q are false and true in all other cases. This is summarized in Table 15.

Table 15

p	q	Disjunction $p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

EXAMPLE 9

Classify each of the following as true or false:

p: $2 + 3 = 5$ q: $2 \cdot 3 = 6$ r: $5 + 3 = 9$

a. $p \wedge q$
b. $q \vee r$
c. $\sim p \vee r$
d. $\sim p \wedge \sim q$
e. $\sim(p \wedge q)$
f. $(p \wedge q) \vee \sim r$

Solution

a. p is true and q is true, so $p \wedge q$ is true.
b. q is true and r is false, so $q \vee r$ is true.
c. $\sim p$ is false and r is false, so $\sim p \vee r$ is false.
d. $\sim p$ is false and $\sim q$ is false, so $\sim p \wedge \sim q$ is false.
e. $p \wedge q$ is true so $\sim(p \wedge q)$ is false.
f. $p \wedge q$ is true and $\sim r$ is true, so $(p \wedge q) \vee \sim r$ is true.

Truth tables are used not only to summarize the truth values of compound statements; they also are used to determine if two statements are **logically equivalent**. *Two statements are **logically equivalent** if, and only if, they have the same truth values in every possible situation.* If p and q are logically equivalent, we write $p \equiv q$.

EXAMPLE 10

Show that $\sim(p \wedge q) \equiv \sim p \vee \sim q$ [First of De Morgan's Laws].

Solution Two statements are logically equivalent if they have the same truth values. Truth tables for these statements are given in Table 16.

Table 16

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Because the two statements have the same truth values, we know that $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

Example 10 shows that $\sim(p \wedge q) \equiv \sim p \vee \sim q$. In the same way we can show that $\sim(p \vee q) \equiv \sim p \wedge \sim q$. We call these equivalencies **De Morgan's Laws** and state them as Theorem 1.

Historical Note


World History/ Topham/The Image Works

George Boole (1815–1864), born in Lincoln, England, is called "the father of logic." As a professor at Queens College in Ireland, he used symbols to represent concepts and developed a system of algebraic manipulations to accompany the symbols. His work, a marriage of logic and mathematics, known as Boolean algebra, has applications in computer science.

Theorem 1: De Morgan's Laws

a. $\sim(p \wedge q) \equiv \sim p \vee \sim q$

b. $\sim(p \vee q) \equiv \sim p \wedge \sim q$

NOW TRY THIS 14

Use truth tables to confirm the second De Morgan Law.

Conditionals and Biconditionals

Statements expressed in the form "if p, then q" are **conditionals**, or **implications**, and are denoted by $p \rightarrow q$. Such statements also can be read "p implies q." The "if" part of a conditional is the **hypothesis** of the implication and the "then" part is the **conclusion**. Many types of statements can be put in "if-then" form. An example follows:

Statement: All equilateral triangles have acute angles.

If-then form: $\underbrace{\text{If a triangle is equilateral,}}_{\text{Hypothesis}}$ $\underbrace{\text{then it has acute angles.}}_{\text{Conclusion}}$

An implication may also be thought of as a promise. Suppose Betty makes the promise "If I get a raise, then I will take you to dinner." If Betty keeps her promise, the implication is true; if Betty breaks her promise, the implication is false. Consider the four possibilities in Table 17.

Table 17

Case	p	q	Translation of Symbols	Result
(1)	T	T	Betty gets the raise; she takes you to dinner.	Promise Kept
(2)	T	F	Betty gets the raise; she does not take you to dinner.	Promise Broken
(3)	F	T	Betty does not get the raise; she takes you to dinner.	Promise Kept
(4)	F	F	Betty does not get the raise; she does not take you to dinner.	Promise Kept

Table 18

p	q	Implication $p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The only case in which Betty breaks her promise is when she gets her raise and fails to take you to dinner, case (2). If she does not get the raise, she can either take you to dinner or not without breaking her promise. The definition of implication is summarized in Table 18. Observe that the only case for which the implication is false is when p is true and q is false.

An implication can be worded in several equivalent ways, as follows:

1. If the sun is shining, then the swimming pool is open. (If p, then q.)
2. If the sun is shining, the swimming pool is open. (If p, q.)
3. The swimming pool is open if the sun is shining. (q if p.)
4. The sun is shining implies the swimming pool is open. (p implies q.)
5. The sun is shining only if the pool is open. (p only if q.)
6. The sun's shining is a sufficient condition for the swimming pool to be open. (p is a sufficient condition for q.)
7. The swimming pool's being open is a necessary condition for the sun to be shining. (q is a necessary condition for p.)

A statement in the form $p \rightarrow q$ has three related implication statements, as follows:

Statement:	If p, then q.	$p \rightarrow q$
Converse:	If q, then p.	$q \rightarrow p$
Inverse:	If not p, then not q.	$\sim p \rightarrow \sim q$
Contrapositive:	If not q, then not p.	$\sim q \rightarrow \sim p$

EXAMPLE 11

Write the converse, the inverse, and the contrapositive for the following statement:

If I am in San Francisco, then I am in California.

Solution *Converse:* If I am in California, then I am in San Francisco.
Inverse: If I am not in San Francisco, then I am not in California.
Contrapositive: If I am not in California, then I am not in San Francisco.

Example 11 illustrates that if an implication is true, its converse and inverse are not necessarily true. However, the contrapositive is true. We check these observations on the following true statement: *If a number is a natural number, the number is not 0.* The natural numbers are 1, 2, 3, 4, 5, 6, ... We check the truth of the converse, inverse, and contrapositive.

Inverse: *If a number is not a natural number, then it is 0.* This is false, since $^{-}6$ is not a natural number but it also is not 0.

Converse: *If a number is not 0, then it is a natural number.* This is false, since $^{-}6$ is not 0 but neither is it a natural number.

Contrapositive: *If a number is 0, then it is not a natural number.* This is true because the natural numbers are 1, 2, 3, 4, 5, 6 . . . .

The contrapositive of the last statement is the original statement. Hence, the preceding discussion suggests that a statement and its contrapositive are logically equivalent. It follows that a statement and its contrapositive cannot have opposite truth values. We summarize this in the following theorem.

Theorem 2: Equivalence of a statement and its contrapositive

The implication, $p \rightarrow q$, and its contrapositive, $\sim q \rightarrow \sim p$, are logically equivalent; that is, $p \rightarrow q \equiv \sim q \rightarrow \sim p$.

EXAMPLE 12

Use truth tables to prove that $p \rightarrow q \equiv \sim q \rightarrow \sim p$.

Solution Truth tables for these statements are given in Table 19.

Table 19

p	q	$p \rightarrow q$	$\sim q$	$\sim p$	$\sim q \rightarrow \sim p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Because the two statements have the same truth values, $p \rightarrow q \equiv \sim q \rightarrow \sim p$.

NOW TRY THIS 15

The implication "If Betty gets a raise (p), then she will take you to dinner (q)" motivated the truth table for $p \rightarrow q$. The only case where Betty broke her promise is when she gets her raise and then fails to take you to dinner: that is, $p \wedge \sim q$. Therefore, $p \wedge \sim q$ is a candidate for the negation of $p \rightarrow q$. To investigate whether $p \wedge \sim q$ is the negation of $p \rightarrow q$, use truth tables to determine whether $\sim(p \rightarrow q) \equiv p \wedge \sim q$.

Connecting a statement and its converse with the connective *and* gives $(p \rightarrow q) \wedge (q \rightarrow p)$. This compound statement can be written as $p \leftrightarrow q$ and usually is read "***p* if, and only if, *q*.**" The statement "p if, and only if, q" is a **biconditional**.

NOW TRY THIS 16

Build a truth table to determine when a biconditional is true.

Valid Reasoning

In problem solving, the reasoning is said to be **valid** if the conclusion follows unavoidably from the hypotheses. Consider the following examples:

Hypotheses: All dogs are animals.
Goofy is a dog.

Conclusion: Goofy is an animal.

The statement "All dogs are animals" can be pictured with the **Euler diagram** in Figure 22(a).

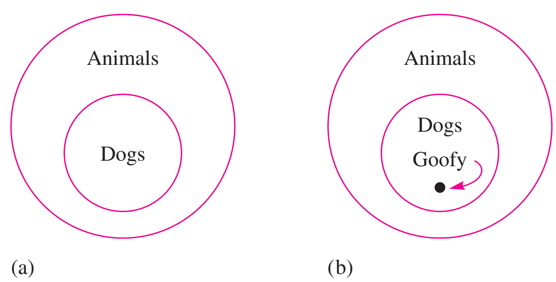


Figure 22

The information "Goofy is a dog" implies that Goofy now also belongs to the circle containing dogs, as pictured in Figure 22(b). Goofy must also belong to the circle containing animals. Thus, the reasoning is valid because it is impossible to draw a picture that satisfies the hypotheses and contradicts the conclusion.

Consider the following argument:

Hypotheses: All elementary schoolteachers are mathematically literate.
Some mathematically literate people are not children.

Conclusion: No elementary schoolteacher is a child.

Let E be the set of elementary schoolteachers, M be the set of mathematically literate people, and C be the set of children. Then the statement "All elementary schoolteachers are mathematically literate" can be pictured as in Figure 23(a). The statement "Some mathematically literate people are not children" can be pictured in several ways. Three of these are illustrated in Figure 23(b) through (d). The argument appears to be true with the drawings in Figure 23(b) and (c).

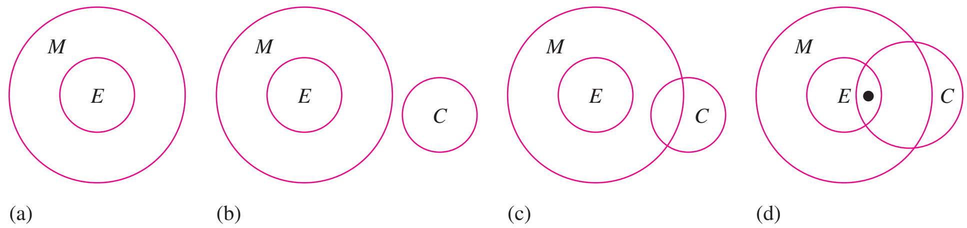


Figure 23

However, according to Figure 23(d), it is possible that some elementary schoolteachers are children, as noted by the dot placed in the figure and yet the given statements are satisfied. Therefore, the conclusion that "No elementary schoolteacher is a child" does not follow from the given hypotheses. Hence, the reasoning is not valid.

If even one picture can be drawn to satisfy the hypotheses of an argument and contradict the conclusion, the argument is not valid. However, *to show that an argument is valid, all possible pictures must show that there are no contradictions.* There must be no way to satisfy the hypotheses and contradict the conclusion if the argument is valid.

EXAMPLE 13

Determine whether the following argument is valid:

Hypotheses: In Washington, D.C., all lobbyists wear suits.
No one in Washington, D.C., over 6 ft tall wears a suit.

Conclusion: Persons over 6 ft tall are not lobbyists in Washington, D.C.

Solution If L represents the lobbyists in Washington, D.C., and S the people who wear suits, the first hypothesis is pictured as shown in Figure 24 (a). If W represents the people in Washington, D.C., over 6 ft tall, the second hypothesis is pictured in Figure 24 (b). Because people over 6 ft tall are outside the circle representing suit wearers and lobbyists are in the circle S, the conclusion is valid and no person over 6 ft tall is a lobbyist in Washington, D.C.

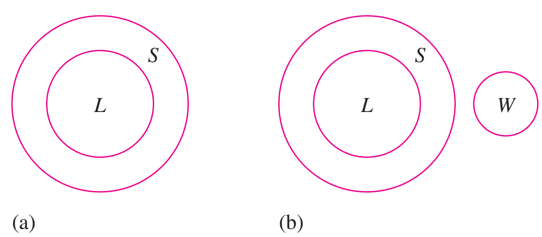


Figure 24

A different method for determining whether an argument is valid uses **direct reasoning** and a form of argument called the **law of detachment** (or ***modus ponens***). For example, consider the following:

Hypotheses: If the sun is shining, then we shall go to the beach.
The sun is shining.

Conclusion: We shall go to the beach.

In general, the law of detachment (or *modus ponens*) is stated as follows:

LAW OF DETACHMENT (*MODUS PONENS*)

If the statement "if p, then q" is true, and p is true, then q must be true.

EXAMPLE 14

Show that $[(p \to q) \wedge p] \to q$ is always true.

Solution A truth table for this implication is given in Table 20.

Table 20

p	q	$p \to q$	$(p \to q) \wedge p$	$[(p \to q) \wedge p] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

The statement $[(p \to q) \wedge p] \to q$ is a **tautology;** that is, a statement that is true all the time.

EXAMPLE 15

Determine whether the following argument is valid if the hypotheses are true and x is a natural number:

Hypotheses: If $x > 2$, then $x^2 > 4$.
$x > 2$
Conclusion: $x^2 > 4$

Solution Using the law of detachment, *modus ponens*, we see that the conclusion is valid.

A different type of reasoning, **indirect reasoning**, uses a form of argument called ***modus tollens***. For example, consider the following:

Hypotheses: If a figure is a square, then it is a rectangle.
The figure is not a rectangle.
Conclusion: The figure cannot be a square.

Modus tollens can be interpreted as follows:

MODUS TOLLENS

With a conditional accepted as true but having a false conclusion, the hypothesis must be false. Symbolically if $p \to q$ is true, and q is not true, then p is not true.

Modus tollens follows from the fact that an implication and its contrapositive are logically equivalent.

EXAMPLE 16

Determine conclusions for each of the following pairs of true statements:

a. If a person lives in Boston, then the person lives in Massachusetts. Jessica does not live in Massachusetts.
b. If $x = 3$, then $2x \neq 7$. And we know that $2x = 7$.

Solution

a. Jessica does not live in Boston (*modus tollens*).
b. $x \neq 3$ (*modus tollens*)

The final reasoning argument we consider here involves the **chain rule (*transitivity*)**. Consider the following:

Hypotheses: If I save, I will retire early.
If I retire early, I will play golf.

Conclusion: If I save, I will play golf.

In general, the chain rule is stated as follows:

CHAIN RULE (TRANSITIVITY)

If "if p, then q" and "if q, then r" are true, then "if p, then r" is true.

People often make invalid conclusions based on advertising or other information. Assume, for example, the statement "Healthy people eat Super-Bran cereal" is true. Are the following conclusions valid?

If a person eats Super-Bran cereal, then the person is healthy.

If a person is not healthy, the person does not eat Super-Bran cereal.

If the original statement is denoted by $p \rightarrow q$, where p is "a person is healthy" and q is "a person eats Super-Bran cereal," then the first conclusion is the converse of $p \rightarrow q$; that is, $q \rightarrow p$, and the second conclusion is the inverse of $p \rightarrow q$; that is, $\sim p \rightarrow \sim q$. Hence, neither is valid.

EXAMPLE 17

Determine valid conclusions for the following true statements:

a. If a triangle is equilateral, then it is isosceles. If a triangle is isosceles, it has at least two congruent sides.
b. If a number is a whole number, then the number is an integer. If a number is an integer, then the number is a rational number. If a number is a rational number, then the number is a real number.

Solution

a. If a triangle is equilateral, then it has at least two congruent sides.
b. If a number is a whole number, then it is a real number.

Assessment 3A

1. Determine which of the following are statements and then classify each statement as true or false:
   a. $2 + 4 = 8$
   b. Los Angeles is a state in the United States.
   c. What time is it?
   d. $3 \cdot 2 = 6$
   e. This statement is false.
2. Use quantifiers to make each of the following true, where n is a natural number:
   a. $n + 8 = 11$ **b.** $n^2 = 4$
   c. $n + 3 = 3 + n$ **d.** $5n + 4n = 9n$
3. Use quantifiers to make each equation in exercise 2 false.
4. Write the negation of each of the following statements:
   a. This book has 500 pages.
   b. $3 \cdot 5 = 15$
   c. All dogs have four legs.
   d. Some rectangles are squares.
   e. Not all rectangles are squares.
   f. No dogs have fleas.
5. Identify the following as true or false:
   a. For some natural numbers n, $n < 6$ and $n > 3$.
   b. For all natural numbers n, $n > 0$ or $n < 5$.

6. Complete each of the following truth tables:

a.

p	$\sim p$	$\sim(\sim p)$
T		
F		

b.

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T			
F			

c. Based on part (a), is p logically equivalent to $\sim(\sim p)$?
d. Based on part (b), is $p \vee \sim p$ logically equivalent to $p \wedge \sim p$?

7. If q stands for "This course is easy" and r stands for "Lazy students do not study," write each of the following in symbolic form:
a. This course is easy, and lazy students do not study.
b. Lazy students do not study, or this course is not easy.
c. It is false that both this course is easy and lazy students do not study.
d. This course is not easy.

8. If p is false and q is true, find the truth values for each of the following:
a. $p \wedge q$
b. $\sim p$
c. $\sim(\sim p)$
d. $p \wedge \sim q$
e. $\sim(\sim p \wedge q)$

9. Find the truth value for each statement in exercise 8 if p is false and q is false.

10. For each of the following, is the pair of statements logically equivalent?
a. $\sim(p \vee q)$ and $\sim p \vee \sim q$
b. $\sim(p \wedge q)$ and $\sim p \wedge \sim q$

11. Complete the following truth table:

p	q	$\sim p$	$\sim p \wedge q$
T	T		
T	F		
F	T		
F	F		

12. Write each of the following in symbolic form if p is the statement "It is raining" and q is the statement "The grass is wet."
a. If it is raining, then the grass is wet.
b. If it is not raining, then the grass is wet.
c. If it is raining, then the grass is not wet.
d. The grass is wet if it is raining.
e. The grass is not wet implies that it is not raining.
f. The grass is wet if, and only if, it is raining.

13. For each of the following implications, state the converse, inverse, and contrapositive:
a. If $x = 5$, then $2x = 10$.
b. If you do not like this book, then you do not like mathematics.
c. If you do not use Ultra Brush toothpaste, then you have cavities.
d. If you are good at logic, then your grades are high.

14. Write a statement logically equivalent to the statement "If a number is a multiple of 8, then it is a multiple of 4."

15. Investigate the validity of each of the following arguments:
a. All squares are quadrilaterals.
All quadrilaterals are polygons.
Conclusion: All squares are polygons.
b. All teachers are intelligent.
Some teachers are rich.
Conclusion: Some intelligent people are rich.
c. If a student is a freshman, then the student takes mathematics.
Jane is a sophomore.
Conclusion: Jane does not take mathematics.

16. For each of the following, form a conclusion that follows logically from the given statements:
a. Some freshmen like mathematics.
All people who like mathematics are intelligent.
b. If I study for the final, then I will pass the final.
If I pass the final, then I will pass the course.
If I pass the course, then I will look for a teaching job.
c. Every equilateral triangle is isosceles.
There exist triangles that are equilateral.

17. Write the following in if-then form:
a. Every figure that is a square is a rectangle.
b. All integers are rational numbers.
c. Polygons with exactly three sides are triangles.

18. Use De Morgan's Laws to write a negation of each of the following:
a. $3 \cdot 2 = 6$ and $1 + 1 \neq 3$.
b. You can pay me now or you can pay me later.

Assessment 3B

1. Determine which of the following are statements and then classify each statement as true or false:
a. Shut the window.
b. He is in town.
c. $2 \cdot 2 = 2 + 2$
d. $2 + 3 = 8$
e. Stay put!

2. Use quantifiers to make each of the following true, where n is a natural number:
a. $n + 0 = n$
b. $n + 1 = n + 2$
c. $3(n + 2) = 12$
d. $n^3 = 8$

3. Use quantifiers to make each equation in exercise 2 false.
4. Write the negation of each of the following statements:
   a. $6 < 8$
   b. Some cats do not have nine lives.
   c. All squares are rectangles.
   d. Not all numbers are positive.
   e. Some people have blond hair.
5. Identify the following as true or false:
   a. For some natural numbers n, $n > 5$ and $n > 2$.
   b. For all natural numbers n, $n > 5$ or $n < 5$.
6. a. If you know that p is true, what can you conclude about the truth value of $p \vee q$, even if you don't know the truth value of q?
   b. If you know that p is false, what can you conclude about the truth value of $p \rightarrow q$, even if you don't know the truth value of q?
7. If q stands for "You said goodbye" and r stands for "I said hello," write each of the following in symbolic form:
   a. You said goodbye and I said hello.
   b. You said goodbye and I did not say hello.
   c. I did not say hello or you did not say goodbye.
   d. It is false that both you said goodbye and I said hello.
8. If p is false and q is true, find the truth values for each of the following:
   a. $p \vee q$
   b. $\sim q$
   c. $\sim p \vee q$
   d. $\sim(p \vee q)$
   e. $\sim q \wedge \sim p$
9. Find the truth value for each statement in exercise 8 if p is false and q is false.
10. For each of the following, is the pair of statements logically equivalent?
   a. $\sim(p \vee q)$ and $\sim p \wedge \sim q$
   b. $\sim(p \wedge q)$ and $\sim p \vee \sim q$
11. Complete the following truth table:

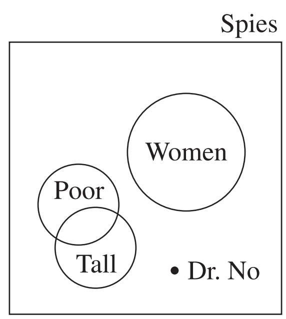
p	q	$\sim q$	$p \vee \sim q$
T	T		
T	F		
F	T		
F	F		

12. Write each of the following in symbolic form if p is the statement "You build it" and q is the statement "They will come:"
   a. If you build it, they will come.
   b. If you do not build it, then they will come.
   c. If you build it, they will not come.
   d. They will come if you build it.
   e. If you do not build it, then they will not come.
   f. If they will not come, then you do not build it.
13. For each of the following implications, state the converse, inverse, and contrapositive:
   a. If $x = 3$, then $x^2 = 9$.
   b. If it snows, then classes are canceled.
14. Iris makes the true statement "If it rains, then I am going to the movies." Does it follow logically that if it does not rain, then Iris does not go to the movies?
15. Investigate the validity of each of the following arguments:
   a. All women are mortal.
      Hypatia was a woman.
      Conclusion: Hypatia was mortal.
   b. All rainy days are cloudy.
      Today is not cloudy.
      Conclusion: Today is not rainy.
   c. Some students like skiing.
      Al is a student.
      Conclusion: Al likes skiing.
16. For each of the following, form a conclusion that follows logically from the given statements:
   a. All college students are poor.
      Helen is a college student.
   b. All engineers need mathematics.
      Ron does not need mathematics.
   c. All bicycles have tires.
      All tires use rubber.
17. Write each of the following in if-then form:
   a. All natural numbers are real numbers.
   b. Every circle is a closed figure.
18. Use De Morgan's Laws to write a negation of each of the following:
   a. $3 + 5 \neq 9$ and $3 \cdot 5 = 15$.
   b. I am going or she is going.

Mathematical Connections 3

Communication

1. Explain why commands and questions are not statements.
2. Explain how to write the negation of a quantified statement in the form "Some As are Bs." Give an example.
3. a. Describe under what conditions a disjunction is true.
   b. Describe under what conditions an implication is true.
4. What does the use of an "inclusive" *or* mean?
5. Describe Dr. No as completely as possible.



6. Consider the nursery rhyme:

For want of a nail, the shoe was lost.
For want of a shoe, the horse was lost.
For want of a horse, the rider was lost.
For want of a rider, the battle was lost.
For want of a battle, the war was lost.
Therefore, for want of a nail, the war was lost.

- **a.** Write each line as an *if-then* statement.
- **b.** Does the conclusion (the *therefore* statement) follow logically. Why?

7. In an e-mail address line, a comma or a semicolon is used to separate addresses. From an e-mail sender's standpoint, explain the logical meaning of the punctuation mark.

Open-Ended

8. Give two examples from mathematics for each of the following:
- **a.** A statement and its converse are true.
- **b.** A statement is true, but its converse is false.
- **c.** An "if, and only if," true statement.
- **d.** An "if, and only if," false statement.

Cooperative Learning

9. Discuss the paradox arising from the following:
- **a.** This textbook is 2000 pages long.
- **b.** The author of this textbook is Dante.
- **c.** The statements (a), (b), and (c) are all false.

Questions from the Classroom

10. A student says that she does not see how a compound statement consisting of two simple sentences that are false can be true. How do you respond?

11. A student says that if the hypothesis is false, an argument cannot be valid. How do you respond?

Hint for Solving the Preliminary Problem

The strategy of *looking for a pattern* might be useful here. For example, we could make three lists based on the number of eggs being removed and left over in each case and then search the lists until matching numbers are found in all lists. The least of these should be the minimum number of eggs.

Chapter Summary

1 Mathematics and Problem Solving

Problem Solving Guided by the Four-step Process:
- Understanding the problem
- Devising a plan
- Carrying out the plan
- Looking back

Important Problem-solving Strategies:
- Look for a pattern
- Examine a related problem
- Examine a simpler case
- Make a table
- Identify a subgoal
- Make a diagram
- Use guess and check
- Work backward
- Write an equation

The natural numbers are 1, 2, 3, 4, 5, ...

Ellipsis—three dots that indicate the continuation of a sequence

2 Explorations with Patterns

Reasoning, both **inductive** and **deductive**, is used in problem solving.

Inductive reasoning—a method of making generalizations based on observations and patterns

Conjecture—an statement thought to be true but not yet proved

Counterexample—an example contradicting a conjecture

Sequence—a ordered arrangment of terms that may be numbers, figures, or objects

- **Arithmetic sequence**—a sequence with each successive term from the second on obtained from the previous term by the addition or subtraction of a fixed number, the difference
- **Geometric sequence**—a sequence with each successive term from the second on obtained from the previous term by the multiplication of a fixed nonzero number, the ratio
- **Fibonacci sequence**—1, 1, 2, 3, 5, 8, 13, . . .
- **Fibonacci-type sequence**—$a, b, a + b, a + 2b, 2a + 3b, \ldots$
- **Figurate numbers**—sequences of numbers formed by counting the dots used to form geometric figures
- **Recursive sequence**—a sequence in which one or more consecutive terms are given to start and each successive term is obtained from previous terms

Finding **common differences**—a technique used to discover patterns

Exponentiation—If n is a natural number, then $a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}$. If $n = 0$ and $a \neq 0$, then $a^0 = 1$.

3 Reasoning and Logic: An Introduction

Statement—a sentence that is either true or false but not both

Negation of a statement, p—a statement, not p, or $\sim p$ having the opposite truth value of p

Universal quantifiers—words such as *all* and *every*

Existential quantifiers—words or phrases such as *some* and *there exists at least one*

Compound statement—a statement formed from combinations of simple statements

Conjunction—a compound statement formed using "and" (in symbols: $p \wedge q$) and defined to be true if, and only if both p and q are true

Disjunction—a compound statement formed using "or" (in symbols: $p \vee q$) and defined to be true if, and only if, either p, or q, or both are true

Logically equivalent statements—statements with the same truth value

Conditional statement or implication—a statement in the form "if p, then q" (in symbols $p \rightarrow q$) and defined to be true unless p is true and q is false

Theorem: (Equivalence of a statement and its contrapositive) The implication, $p \rightarrow q$, and its Contrapositive, $\sim q \rightarrow \sim p$, are logically equivalent

Biconditional statement—$p \rightarrow q$ and $q \rightarrow p$ (in symbols $p \leftrightarrow q$) and referred to as "p if, and only if, q"

Other forms of statements—**contrapositive**, **inverse**, and **converse**

Laws to determine the validity of arguments—**law of detachment** (***modus ponens***), ***modus tollens***, and the **chain rule**

- **Law of Detachment** (***modus ponens***): If the statement "if p, then q" is true, then q must be true.
- **Modus tollens**: If $p \rightarrow q$ is true, and q is not true, then p is not true.
- **Chain Rule** (**Transivity**): If "if p, then q" and "if q, then r" are true, then "if p, then r" is true.

Tautology—a statement that is always true

Theorem: De Morgan's Laws

a. $\sim(p \wedge q) \equiv \sim p \vee \sim q$
b. $\sim(p \vee q) \equiv \sim p \wedge \sim q$

Chapter Review

1. If today is Sunday, July 4, and next year is not a leap year, what day of the week will July 4 be on next year?

2. A coded message was written as 19-5-3-18-5-20 3-15-4-5-19. What did the message say?

3. A nursery rhyme states:

A diller, a dollar, a ten o'clock scholar!
What makes you come so soon?
You used to come at ten o'clock,
But now you come at noon.

Explain whether the rhyme makes sense mathematically.

4. List three more terms that complete a pattern in each of the following; explain your reasoning, and tell whether each sequence is arithmetic or geometric, or neither.

a. 0, 1, 3, 6, 10, ____, ____, ____,
b. 52, 47, 42, 37, ____, ____, ____,
c. 6400, 3200, 1600, 800, ____, ____, ____,
d. 1, 2, 3, 5, 8, 13, ____, ____, ____,
e. 2, 5, 8, 11, 14, ____, ____, ____,
f. 1, 4, 16, 64, ____, ____, ____,
g. 0, 4, 8, 12, ____, ____, ____,
h. 1, 8, 27, 64, ____, ____, ____,

5. Find a possible nth term in each of the following and explain your reasoning.

a. 5, 8, 11, 14, . . .
b. 3, 9, 27, 81, 243, . . .
c. 0, 7, 26, 63, . . .

6. Find the first five terms of the sequences whose nth term is given as follows:

a. $3n - 2$
b. $n^2 + n$
c. $4n - 1$

7. Find the following sums:

a. $2 + 4 + 6 + 8 + 10 + \ldots + 200$
b. $51 + 52 + 53 + 54 + \ldots + 151$

8. Produce a counterexample, if possible, to disprove each of the following:

a. If two odd numbers are added, then the sum is odd.
b. If a number is odd, then it ends in a 1 or a 3.
c. If two even numbers are added, then the sum is even.

9. Complete the following magic square; that is, complete the square so that the sum in each row, column, and diagonal is the same.

16	3	2	13
	10		
9		7	12
4		14	

10. How many people can be seated at 12 square tables lined up end to end if each table individually holds four persons?

11. A shirt and a tie sell for \$9.50. The shirt costs \$5.50 more than the tie. What is the cost of the tie?

12. If fence posts are to be placed in a row 5 m apart, how many posts are needed for 100 m of fence?

13. If a complete rotation of a car tire moves a car forward 6 ft, how many rotations of the tire occur before the tire goes off its 50,000 mi warranty?

14. The members of Mrs. Grant's class are standing in a circle; they are evenly spaced and are numbered in order. The student with number 7 is standing directly across from the student with number 17. How many students are in the class?

15. A carpenter has three large boxes. Inside each large box are two medium-sized boxes. Inside each medium-sized box are five small boxes. How many boxes are there altogether?

16. Use differences to find the next term in the following sequence:

$$5, 15, 37, 77, 141, \_\_\_\_,$$

17. An ant farm can hold 100,000 ants. If the farm held 1500 ants on the first day, 3000 ants on the second day, 6000 ants on the third day, and so on forming a geometric sequence, in how many days will the farm be full?

18. Toma's team entered a mathematics contest where teams of students compete by answering questions that are worth either 3 points or 5 points. No partial credit was given. Toma's team scored 44 points on 12 questions. How many 5-point questions did the team answer correctly?

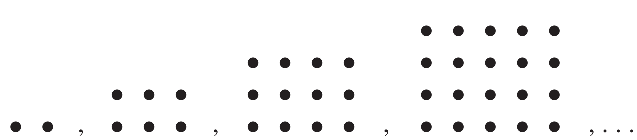
19. Three pieces of wood are needed for a project. They are to be cut from a 90-cm-long piece of wood. The longest piece is to be 3 times as long as the middle-sized piece and the shortest piece is to be 10 cm shorter than the middle-sized piece. How long are the pieces?

20. How many four-digit numbers have the same digits as 1993?

21. Al, Betty, Carl, and Dan were each born in a different season. Al was born in February; Betty was born in the fall; Carl was born in the spring. Determine the season in which each was born.

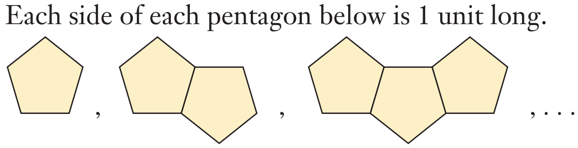
22. We have two containers, one of which holds 7 cups and the other holds 4 cups. How can we measure exactly 5 cups of water if we have an unlimited amount of water with which to start?

23. The following geometric arrays suggest a sequence of numbers: 2, 6, 12, 20, . . .



a. Find the next three terms
b. Find the 100th term
c. Find the nth term

24. Each side of each pentagon below is 1 unit long.



a. Draw a possible next figure in the sequence.
b. What is the perimeter (distance around) of each of the first four figures?
c. What is the perimeter of the 100th figure?
d. What is the perimeter of the nth figure?

25. Explain the difference between the following two statements: (i) All students passed the final. (ii) Some students passed the final.

26. Which of the following are statements?
a. The moon is inhabited.
b. $3 + 5 = 8$
c. $n + 7 = 15$
d. Some women have Ph.D.'s in mathematics.

27. Negate each of the following:
a. Some women smoke.
b. $3 + 5 = 8$
c. Beethoven wrote only classical music.

28. Write the converse, inverse, and contrapositive of the following: If we have a rock concert, someone will faint.

29. Use truth tables to show that $p \rightarrow \sim q \equiv q \rightarrow \sim p$.

30. Construct truth tables for each of the following:
a. $(p \wedge \sim q) \vee (p \wedge q)$
b. $[(p \vee q) \wedge \sim p] \rightarrow q$

31. Find valid conclusions for the following hypotheses:
a. All Americans love Mom and apple pie.
Joe Czernyu is an American.
b. Steel eventually rusts.
The Statue of Liberty has a steel structure.
c. Albertina passed Math 100 or Albertina dropped out.
Albertina did not drop out.

32. Write the following argument symbolically and then determine its validity:
If you are fair-skinned, you will sunburn.
If you sunburn, you will not go to the dance.
If you do not go to the dance, your parents will want to know why you didn't go to the dance.
Your parents do not want to know why you didn't go to the dance.
Conclusion: You are not fair-skinned.

33. State whether the conclusion is valid and tell why.
If Bob scores at least 80 on the final, he will pass the course.
Bob did not pass the course.
Conclusion: Bob did not score at least 80 on the final.

Answers to Problems

Assessment 1A

1. (a) 4950 **(b)** 251,001 **2.** Building a staircase as seen in **(a)** gives a visual graphic of the sum $1 + 2 + \ldots + n$. Copying the staircase as in **(b)** and placing it as shown demonstrates that an array that is n units high and $n + 1$ units long is produced. There are $n(n + 1)$ units in **(b)** which is twice the number desired. So the sum $1 + 2 + \ldots + n$ must be $n(n + 1)/2$. Gauss's sum when $n = 100$ would be $100(100 + 1)/2$ or 5050. **3.** 10,248 **4.** 12 **5.** I was the only one going, so there is 1. **6.** 27 **7.** E is greater by 49. **8.** Dandy, Cory, Alababa, Bubba **9.** 12 **10.** $a = 42$; $b = 32$; $c = 37$; $d = 2$ **11.** 45 **12.** 9 **13.** If we choose Box B and pull out a fruit, we know exactly what the label should be. That box does not contain the apples and oranges but is either a box of only apples or of only oranges. If we pull out an orange, the box should be labeled Oranges. If we pull out an apple, the box should be labeled Apples. Once we label Box B correctly, we do know the labels for the other two boxes. **14.** \$2.45 **15.** 23 rungs

Mathematical Connections 1

Communication

1. Answers vary. For example, problem-solving skills can help students to meet future challenges in work, life, and school. Problem-solving skills allow students to take on new tasks and problems and have the confidence to do so. If a first approach to a problem fails, good problem solvers can come up with alternative approaches. Much of the mathematics that students are taught is introduced through interesting problems. Students need to know how to problem solve to make progress on these problems and in turn learn the mathematics. **3. (a)** 28 **(b)** Answers vary depending on the strategies used. **(c)** Answers vary. **(d)** $\dfrac{n(n-1)}{2}$

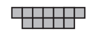
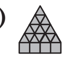
Open-Ended

5. Answers vary. For example, if the average reach of people in your group was 1.8 m, then it would take approximately 22,000,000 people.

Cooperative Learning

7. Answers vary. For example, it is in the last step where students examine whether the answer is reasonable and whether it checks given the original conditions in the problem. Many times students discover wrong answers at this point since they have never bothered to check whether the answer they arrived at makes sense. It is also at this step that students reflect on the mathematics used and determine whether there might be different ways of solving the problem. Also at this stage students reflect on any connections to other problems or generalizations. **9.** Answers vary. For example, if these prime numbers are to be used in a magic square, then the sum in each of the three columns must be the same and must be a natural number. The natural number must be 1/3 of the sum of all nine numbers. However, $1 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 53$ and $53/3 = 17\frac{2}{3}$, which is not a natural number. Therefore these numbers cannot be used for a magic square.

Assessment 2A

1. (a)  **(b)**  **(c)** 

2. (a) 11, 13, 15; arithmetic **(b)** 250, 300, 350; arithmetic **(c)** 96, 192, 384; geometric **(d)** $10^6, 10^7, 10^8$; geometric **(e)** 33, 37, 41; arithmetic **3. (a)** 199; $2n - 1$ **(b)** 4950; $50(n - 1)$ **(c)** $3 \cdot 2^{99}$; $3 \cdot 2^{n-1}$ **(d)** 10^{100}; 10^n **(e)** 405; $5 + 4n$ or $9 + 4(n - 1)$ **4.** 2, 7, 12 **5. (a)** 1331 **(b)** 729 **(c)** 2744 **6. (a)** 41 **(b)** $4n + 1$, or $5 + (n - 1)4$ **(c)** $12n + 4$ **7. (a)** 42 **(b)** $4n + 2$ or $6 + (n - 1)4$ **8.** 1200 students **9.** 23rd year **10. (a)** 3, 5, 9, 15, 23, 33 **(b)** 4, 6, 10, 16, 24, 34 **(c)** 15, 17, 21, 27, 35, 45 **11. (a)** 299, 447, 644 **(b)** 56, 72, 90 **12. (a)** 101 **(b)** 61 **(c)** 200 **(d)** 11 **13. (a)** 3, 6, 11, 18, 27 **(b)** 4, 9, 14, 19, 24 **(c)** 9, 99, 999, 9999, 99999 **(d)** 5, 8, 11, 14, 17 **14. (a)** Answers vary. For example, if $n = 5$, then $\dfrac{5+5}{5} \neq 5 + 1$. **(b)** Answers vary. For example, if $n = 2$, then $(2 + 4)^2 \neq 2 + 16$. **15. (a)** 41 **(b)** $n^2 + (n - 1)$ **(c)** Yes, the 35th figure **16.** 12 **17. (a)** 1, 5, 9, 13, 17, 21, . . . **(b)** Answers vary. The sequences may be either 1, 5, 9, . . . or 5, 9, 13, . . . , depending on whether you start counting before or after the first cut. The total number of pieces after n cuts is $4n + 1$.

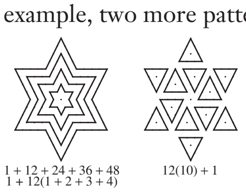
Mathematical Connections 2

Communication

1. (a) Yes. The difference between terms in the new sequence is the same as in the old sequence because a fixed number was added to each number in the sequence. **(b)** Yes. If the fixed number is k, the difference between terms of the second sequence is k times the difference between terms of the first sequence. **(c)** Yes. The difference of the new sequence is the sum of the original differences.

Open-Ended

3. Answers vary. For example, two more patterns follow:



5. Answers vary depending on sequence.

Cooperative Learning

7. (a) 81 **(b)** 40 **(c)** 3^{n-1}

Questions from the Classroom

9. Al and Betty should be told that both answers could be correct as long as each person's rule works for all the given terms. Al is thinking of a geometric sequence in which each term is multiplied by 2 to get the next term. His sequence is correct and he should be asked to explain his rule. Betty is thinking of a different sequence. She starts with 2, then adds 4 to get the 2nd term, then adds 6 to get the next term, then 8, then 10, then 12, and so on. Both students are correct because each rule works for all the given terms.

Review Problems

11. 90 **12.** 7 **13.** We need one 12-person tent and a combination of tents to hold 14 people. There are 10 ways: 662, 653, 6332, 62222, 5522, 5333, 53222, 33332, 332222, and 2222222.

Assessment 3A

1. (a) False statement **(b)** False statement **(c)** Not a statement **(d)** True statement **(e)** Not a statement **2.** Answers vary. **(a)** There exists a natural number n such that $n + 8 = 11$. **(b)** There exists a natural number n such that $n^2 = 4$. **(c)** For all natural numbers n, $n + 3 = 3 + n$. **(d)** For all natural numbers n, $5n + 4n = 9n$. **3.** Answers vary. **(a)** For every natural number, n, $n + 8 = 11$. **(b)** Every natural number n satisfies $n^2 = 4$. **(c)** There is no natural number n such that $n + 3 = 3 + n$. **(d)** There is no natural number n such that $5n + 4n = 9n$. **4. (a)** This book does not have 500 pages. **(b)** $3 \cdot 5 \neq 15$ **(c)** Some dogs do not have four legs. **(d)** No rectangles are squares. **(e)** All rectangles are squares. **(f)** Some dogs have fleas. **5. (a)** True **(b)** True

6. (a)

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

(b)

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	F	T	F
F	T	T	F

(c) Yes **(d)** No **7. (a)** $q \wedge r$ **(b)** $r \vee \sim q$ **(c)** $\sim(q \wedge r)$ **(d)** $\sim q$ **8. (a)** False **(b)** True **(c)** False **(d)** False **(e)** False **9. (a)** False **(b)** True **(c)** False **(d)** False **(e)** True **10. (a)** No **(b)** No

11.

p	q	$\sim p$	$\sim p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

12. (a) $p \rightarrow q$ **(b)** $\sim p \rightarrow q$ **(c)** $p \rightarrow \sim q$ **(d)** $p \rightarrow q$ **(e)** $\sim q \rightarrow \sim p$ **(f)** $q \leftrightarrow p$ **13. (a)** Converse: If $2x = 10$, then $x = 5$. Inverse: If $x \neq 5$, then $2x \neq 10$. Contrapositive: If $2x \neq 10$, then $x \neq 5$. **(b)** Converse: If you do not like mathematics, then you do not like this book. Inverse: If you like this book, then you like mathematics. Contrapositive: If you like mathematics, then you like this book. **(c)** Converse: If you have cavities, then you do not use Ultra Brush toothpaste. Inverse: If you use Ultra Brush toothpaste, then you do not have cavities. Contrapositive: If you do not have cavities, then you use Ultra Brush toothpaste. **(d)** Converse: If your grades are high, then you are good at logic. Inverse: If you are not good at logic, then your grades are not high. Contrapositive: If your grades are not high, then you are not good at logic. **14.** Answers vary. If a number is not a multiple of 4, then it is not a multiple of 8. (Contrapositive) **15. (a)** Valid **(b)** Valid **(c)** Invalid **16.** Answers vary. **(a)** Some freshmen are intelligent. **(b)** If I study for the final, then I will look for a teaching job. **(c)** There exist triangles that are isosceles. **17. (a)** If a figure is a square, then it is a rectangle. **(b)** If a number is an integer, then it is a rational number. **(c)** If a polygon has exactly three sides, then it is a triangle. **18. (a)** $3 \cdot 2 \neq 6$ or $1 + 1 = 3$. **(b)** You cannot pay me now *and* you cannot pay me later.

Mathematical Connections 3

Communication

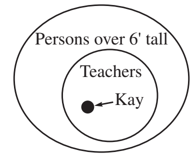
1. Commands and questions are not statements because they can't be classified as true or false. **3. (a)** A disjunction is in the form p or q and it is true if either p or q or both are true. The only time a disjunction is false is when both p and q are false. **(b)** An implication in the form if p, then q is false if, and only if, p is true and q is false. **5.** Dr. No is a male spy who is not poor and not tall. **7.** When a comma or a semicolon is used in an e-mail address (depending on the server), the logical meaning is "and" so that all addresses will receive the e-mail.

Cooperative Learning

9. Answers vary. Statements (a) and (b) are false. Statement (c) causes the problem. If statement (c) is also false, then that makes statement (c) a true assertion which is a contradiction. On the other hand, if statement (c) is true, then it wrongly asserts that it is false.

Questions from the Classroom

11. Consider the example.
Hypotheses: All teachers are over 6 ft tall.
Kay is a teacher.
Conclusion: Kay is over 6 ft tall.



An Euler diagram can be drawn to show that all teachers belong to the set of people over 6 ft tall. Kay belongs to the set of teachers. Thus, the argument is valid even though the hypothesis is false.

Chapter Review

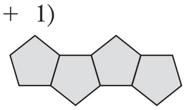
1. Monday **2.** Answers vary. One solution is to pair a number with its corresponding letter in the English alphabet giving the message: SECRET CODES. **3.** The question is asked "What makes you come so soon?" when the scholar arrives two hours later than usual. This makes no sense *unless* the "ten o'clock" is at night. **4. (a)** 15, 21, 28; neither **(b)** 32, 27, 22; arithmetic **(c)** 400, 200, 100; geometric **(d)** 21, 34, 55; neither **(e)** 17, 20, 23; arithmetic **(f)** 256, 1024, 4096; geometric **(g)** 16, 20, 24; arithmetic **(h)** 125, 216, 343; neither **5. (a)** $3n + 2$ if the sequence is arithmetic with difference of 3. **(b)** 3^n if the sequence is geometric with ratio 3. **(c)** $n^3 - 1$ if the sequence is based on the sequence of cubes, 1, 8, 27, 64, . . .

6. **(a)** 1, 4, 7, 10, 13 **(b)** 2, 6, 12, 20, 30 **(c)** 3, 7, 11, 15, 19 **7.** **(a)** 10,100 **(b)** 10,201 **8.** **(a)** False; for example, $3 + 3 = 6$ and 6 is not odd. **(b)** False; for example, 19 is odd and it ends in 9. **(c)** True; the sum of any two even numbers, $2m$ and $2n$, is even because $2m + 2n = 2(m + n)$

9.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

10. 26 **11.** \$2.00 **12.** 21 posts **13.** 44, 000, 000 **14.** 20 **15.** 39 **16.** 235 **17.** 8 **18.** 4 **19.** 10 cm, 20 cm, 60 cm **20.** 12 **21.** Al-winter; Betty-fall; Carl-spring; Dan-summer **22.** Answers vary. For example, fill the 4-cup pot and empty it into the 7-cup pot. Repeat. There is now 1 cup in the 4-cup pot. Empty the 7-cup pot and pour the 1 cup into the 7-cup pot. Fill the 4-cup pot and empty it into the 7-cup pot. It will now contain 5 cups. **23.** **(a)** 30, 42, 56 **(b)** 10,100 **(c)** $n(n + 1)$

24. Answers vary. **(a)**  **(b)** 5, 8, 11, 14 **(c)** 302 **(d)** $3n + 2$ **25.** In statement (i) each and every student passed the final. In statement (ii) at least one student passed the final and possibly all the students passed. **26.** **(a)** Yes **(b)** Yes **(c)** No **(d)** Yes **27.** **(a)** No women smoke. **(b)** $3 + 5 \neq 8$ **(c)** Beethoven wrote some music that is not classical. **28.** Converse: If someone will faint, we will have a rock concert. Inverse: If we do not have a rock concert, then no one will faint. Contrapositive: If no one will faint, then we will not have a rock concert.

29.

p	q	$\sim p$	$\sim q$	$p \rightarrow \sim q$	$q \rightarrow \sim p$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

Therefore, $p \rightarrow \sim q \equiv q \rightarrow \sim p$.

30. **(a)**

p	q	$\sim q$	$p \wedge \sim q$	$p \wedge q$	$(p \wedge \sim q) \vee (p \wedge q)$
T	T	F	F	T	T
T	F	T	T	F	T
F	T	F	F	F	F
F	F	T	F	F	F

(b)

p	q	$\sim p$	$p \vee q$	$(p \vee q) \wedge \sim p$	$[(p \vee q) \wedge \sim p] \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

31. Answers vary. **(a)** Joe Czernyu loves Mom and apple pie. **(b)** The structure of the Statue of Liberty will eventually rust. **(c)** Albertina passed Math 100.

32. Let the following letters represent the given sentences:
p: You are fair-skinned.
q: You will sunburn.
r: You do not go to the dance.
s: Your parents want to know why you didn't go to the dance.
Symbolically, $p \rightarrow q$, $q \rightarrow r$, $r \rightarrow s$. Using contrapositives we have: $\sim s \rightarrow \sim r$, $\sim r \rightarrow \sim q$, $\sim q \rightarrow \sim p$. By the chain rule, $\sim s \rightarrow \sim p$; that is, if your parents do not want to know why you didn't go to the dance, then you are not fair-skinned.
33. Valid, *modus tollens*

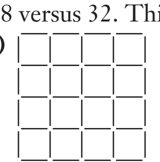
Answers to Now Try This

1. 11 pieces for 10 cuts; $(n + 1)$ pieces for n cuts **2.** **(a)** 2500 **(b)** 6960 **3.** 120 games **4.** 90 days **5.** Answers vary. For example, because each person owes \$13, Al could pay \$4.25 to Betty and \$4.00 to Carl, Dani could pay \$7.00 to Carl, and everyone would be even. **6.** 23 floors

7. Answers vary. For example,

$$\begin{array}{r} 132 \\ +\ 932 \\ \hline 1064 \end{array} \quad \text{or} \quad \begin{array}{r} 173 \\ +\ 873 \\ \hline 1046 \end{array}$$

8. 83 **9.** **(a)** Answers vary. For example, the next three terms could be △, △, ○. **(b)** The pattern could be one circle, two triangles, one circle, two triangles, and so on.
10. **(a)** Inductive reasoning **(b)** The next sev-eral numbers also work. **(c)** Yes, if $x = 11$, then $11^2 + 11 + 11$ is not prime because it is divisible by 11. **11.** **(a)** 4 **(b)** 7 **(c)** 12 **(d)** 20 **(e)** 33 **(f)** The sum of the first n Fibonacci numbers is one less than the Fibonacci number two numbers later in the sequence.
12. **(a)** 118,098 bacteria, $2 \cdot 3^n$ bacteria. **(b)** After 10 hours, there are $2 + 10 \cdot 3 = 32$ bacteria, and after n hours, there are $2 + n(3)$ bacteria. We can see that after only 10 hours geometric growth is much faster than arithmetic growth. In this case, 118,098 versus 32. This is true in general when $n > 1$.

13. **(a)**  **(b)**

1	2	3	4
4	12	24	40

(c)

$$\begin{array}{ccccccccccccc} 4 & & 12 & & 24 & & 40 & & 60 & & 84 & & 112 \\ & 8 & & 12 & & 16 & & 20 & & 24 & & 28 & \\ & & 4 & & 4 & & 4 & & 4 & & 4 & & \end{array}$$

(d) No, finding differences for the 100th term is very hard. It is easier to find a pattern involving the number of horizontal sticks and vertical sticks; that is, the 100th term is $= 101 \cdot 100 + 101 \cdot 100 = 20{,}200$.

14.

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

$\sim(p \vee q) \equiv \sim p \wedge \sim q$

15.

p	q	$p \rightarrow q$	$\sim(p \rightarrow q)$	$\sim q$	$p \wedge \sim q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

$\sim(p \rightarrow q) \equiv p \wedge \sim q$

16.

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Answers to Brain Teasers

Section 1

35 moves. This can be solved using the strategy of examining simpler cases and looking for a pattern. If one person is on each side, 3 moves are necessary. If two people are on each side, 8 moves are necessary. With 3 people on each side, 15 moves are necessary. If n people are on each side, $(n + 1)^2 - 1$ moves are required.

Section 1

The number of moves is $2^n - 1$ for n coins. This can be solved using the strategy of examining a simpler problem. If there is one coin, 1 move is necessary. If there are two coins, 3 moves are necessary. For three coins, the number of moves is 7. For four coins, the number of moves is 15.

At the rate of one move per second, it would take approximately 585 billion years to move 64 coins.

Section 2

3 1 2 2 11; the pattern counts the number of times a number occurs in the previous row. For example, to find the sixth row we examine the fifth row. There are three 1s, two 2s, and one 1, so the sixth row is 3 1 2 2 11. The pattern continues using this rule.

Answer to Preliminary Problem

If eggs are removed 2 at a time, there is 1 left, revealing us that the number of eggs must be an odd number. Also, the number could not be 1 because one cannot remove eggs 2 at a time if there is only 1 in the basket. So there must be at least 3 and the least number must be in arithmetic sequence 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, ...

Similarly we can deduce from the second statement that there are 2 eggs left when they are removed 3 at a time that the number of eggs must be in the pattern 5, 8, 11, 14, 17, 20, 23, 26, . . . and that there must be at least 5 eggs in the basket.

Finally, because there are 3 left when the eggs are removed 5 at a time, the number of eggs has to be in the pattern 8, 13, 18, 23, . . . so there must be at least 8 in the basket.

But more importantly for the problem, there is one number—23—that appears in all sequences and is the least number of eggs that satisfy the conditions. Thus, 23 is the minimum number of eggs that must be in the basket.

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Numeration Systems and Sets

1 Numeration Systems

2 Describing Sets

3 Other Set Operations and Their Properties

Bill Brooks/Alamy

Preliminary Problem

John has applied for the registrar's job at a small college. He submitted the following report to the hiring committee on a survey he did of 100 students: 45 take mathematics; 40 take chemistry; 47 take physics; 20 take mathematics and chemistry; 15 take chemistry and physics; 10 take mathematics and physics; 8 take all three of these subjects; and 10 students take none of these three subjects. Do you think John should be hired on the basis of his report? Explain why.

If needed, see Hint before the Chapter Summary.

In the NCTM document *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence*, the Council suggested specific topics that must be taught in grades pre-K through 8. In that document, as early as pre-K we find the following:

> Children develop an understanding of the meanings of whole numbers and recognize the number of objects in small groups without counting—the first and most basic mathematical algorithm. They understand that number words refer to quantity. They use one-to-one correspondence to solve problems by matching sets and comparing number amounts and in counting objects to 10 and beyond. (p. 11)

In this chapter, we introduce several counting systems. Next, we discuss set theory and the ways in which it adds structure to our number system.

1 Numeration Systems

In this section, we introduce various number systems and compare them to the system of numbers used today in the United States. Comparing our current system with ancient systems helps to develop a clearer appreciation of our system. The commonly used base-ten system relies on 10 digits—0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. The written symbols for the digits, such as 2 or 5, are **numerals**. Different cultures developed different numerals over the years to represent numbers. Table 1 shows other representations along with how they relate to the digits 0 through 9 and the number 10.

Table 1

Babylonian		▼	▼▼	▼▼▼	▼▼▼▼	▼▼▼ ▼▼	▼▼▼ ▼▼▼	▼▼▼▼ ▼▼▼	▼▼▼▼ ▼▼▼▼	▼▼▼▼▼ ▼▼▼▼	<
Egyptian		|	||	|||	||||	||| ||	||| |||	|||| |||	|||| ||||	||| ||| |||	∩
Mayan	𝋠	•	••	•••	••••	—	• —	•• —	••• —	•••• —	=
Greek		α	β	γ	δ	∈	ϕ	ζ	η	υ	ι
Roman		I	II	III	IV	V	VI	VII	VIII	IX	X
Hindu	०	१	२	३	४	५	६	७	८	९	
Arabic	٠	١	٢	٣	٤	٥	٦	٧	٨	٩	
Hindu-Arabic	0	1	2	3	4	5	6	7	8	9	10

Table 1 shows rudiments of different sets of numbers. A **numeration system** is a collection of properties and symbols agreed upon to represent numbers systematically. Through the study of various numeration systems, we explore the evolution of our current system, the Hindu-Arabic system.

Hindu-Arabic Numeration System

The **Hindu-Arabic numeration system** that we use today was developed by the Hindus and transported to Europe by the Arabs—hence, the name *Hindu-Arabic*. The Hindu-Arabic system relies on the following properties:

1. All numerals are constructed from the 10 digits—0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.
2. Place value is based on powers of 10, the number base of the system.

Because the Hindu-Arabic system is based on powers of 10, the system is a base ten, or a decimal, system. **Place value** assigns a value to a digit depending on its placement in a numeral. To find the value of a digit in this system, we multiply the place value of the digit by its **face value**, where the face value is a digit. For example, in the numeral 5984, the 5 has place value "thousands," the 9 has place value "hundreds," the 8 has place value "tens," and the 4 has place value "ones" or "units," as seen in Figure 1. The values of the respective digits are $5 \cdot 1000$ or 5000, $9 \cdot 100$ or 900, $8 \cdot 10$ or 80, and $4 \cdot 1$ or 4.

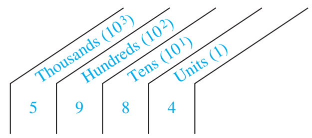


Figure 1

We could write 5984 in **expanded form** as $5 \cdot 10^3 + 9 \cdot 10^2 + 8 \cdot 10 + 4 \cdot 1$. In the expanded form of 5984, exponents are used. For example, 1000, or $10 \cdot 10 \cdot 10$, is written as 10^3. In this case, 10 is a **factor** of the product. In general, we have the following:

Definition of a^n

If a is any number and n is any natural number, then

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}.$$

If $n = 0$, $a \neq 0$, then $a^0 = 1$.

A set of base-ten blocks (or Dienes blocks), shown in Figure 2, can be used to better understand place value. The blocks consist of *units*, *longs*, *flats*, and *blocks*, representing 1, 10, 100, and 1000, respectively.

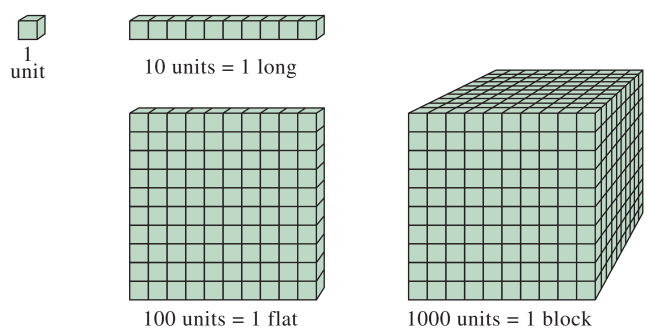


Figure 2

$$1 \text{ long} \rightarrow 10^1 = 1 \text{ row of 10 units}$$
$$1 \text{ flat} \rightarrow 10^2 = 1 \text{ row of 10 longs, or 100 units}$$
$$1 \text{ block} \rightarrow 10^3 = 1 \text{ row of 10 flats, or 100 longs, or 1000 units}$$

Students trade blocks by regrouping. That is, they take a set of base-ten blocks representing a number and trade them until they have the fewest possible pieces representing the same number. For example, suppose you have 58 units and want to trade them for other base-ten blocks. You start trading the units for as many longs as possible. Five sets of 10 units each can be traded for 5 longs. Thus, 58 units can be traded so that you now have 5 longs and 8 units. In terms of numbers, this is analogous to rewriting 58 as $5 \cdot 10 + 8$.

EXAMPLE 1

What is the fewest number of pieces you can receive in a fair exchange for 11 flats, 17 longs, and 16 units?

Solution

	11 flats	17 longs	~~16 units~~	(16 units = 1 long and 6 units)
		1 long	6 units	(Trade)
	11 flats	18 longs	6 units	(After the first trade)
	11 flats	~~18 longs~~	6 units	(18 longs = 1 flat and 8 longs)
	1 flat	8 longs		(Trade)
	12 flats	8 longs	6 units	(After the second trade)
	~~12 flats~~	8 longs	6 units	(12 flats = 1 block and 2 flats)
1 block	2 flats			(Trade)
1 block	2 flats	8 longs	6 units	(After the third trade)

Therefore, the fewest number of pieces is $1 + 2 + 8 + 6 = 17$. Notice that as a result of the trading we obtain 1 block, 2 flats, 8 longs, and 6 units that can be written as $11 \cdot 100 + 17 \cdot 10 + 16$ or $1 \cdot 10^3 + 2 \cdot 10^2 + 8 \cdot 10 + 6$, which implies that there are 1286 units.

NOW TRY THIS 1

a. Use trading with base-ten blocks (as shown in Figure 2) to re-write 3 blocks, 12 flats, 11 longs, and 17 units with the fewest number of blocks. Write a Hindu-Arabic number to represent this fewest number of blocks.

b. Write 3282 in expanded form.

Next, we discuss other numeration systems. The study of such systems provides a historical perspective on the development of numeration systems and helps us better understand our own system.

Tally Numeration System

The **tally numeration system** used single strokes, or tally marks, to represent each object that was counted; for example, the first 10 counting numbers are

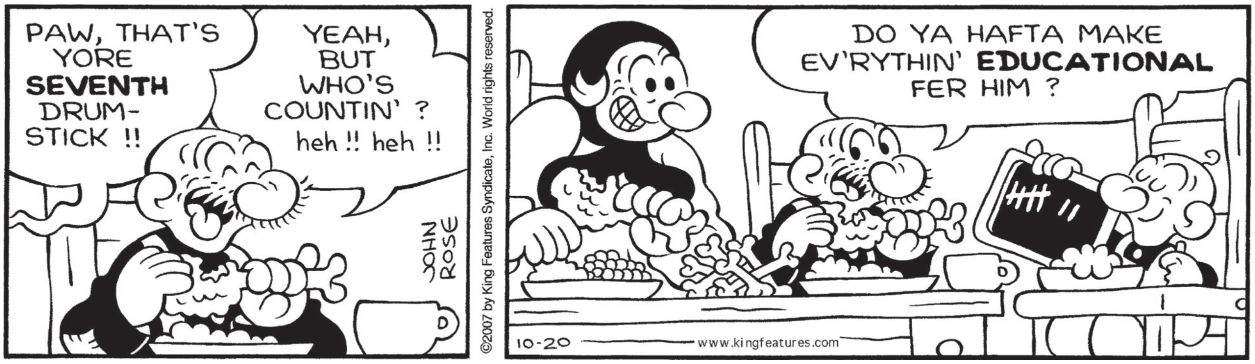
|, ||, |||, ||||, |||||, ||||||, |||||||, ||||||||, |||||||||, ||||||||||

A tally system has a correspondence between the marks and the items being counted. The system is simple, but requires many symbols, when numbers are great. Also as numbers become greater, the tally marks for them are harder to read.

As we see in the *Barney Google and Snuffy Smith* cartoon, the tally system can be improved by *grouping*. We see that the tallies are grouped into fives by placing a diagonal across four tallies to make a group of five. Grouping makes it easier to read the numeral.

Historical Note

The invention of the Hindu-Arabic numeration system is considered one of the most important developments in mathematics. The system was introduced in India and then transmitted by the Arabs to North Africa and Spain and then to the rest of Europe. Historians trace the use of zero as a placeholder to the fourth century BCE (Before the Common Era). Arab mathematicians extended the decimal system to include fractions. The Italian mathematician Fibonacci, also known as Leonardo de Pisa (1170–1250), studied in Algeria and brought back with him the new numeration system, which he described and used in a book he published in 1202.



Barney Google copyright ©2005 King Features Syndicate

Egyptian Numeration System

The Egyptian numeration system, dating to about 3400 BCE, used tally marks for the first nine numerals. The Egyptians improved on the system based only on tally marks by developing a *grouping system* to represent certain sets of numbers. This makes the numbers easier to record. For example, the Egyptians used a heel bone symbol, ∩, to stand for a grouping of 10 tally marks.

$$||||||||||\ \rightarrow \cap$$

Table 2 shows other numerals that the Egyptians used in their system. Some of the symbols from the Karnak temple in Luxor are depicted in Figure 3.

Table 2

Egyptian Numeral	Description	Hindu-Arabic Equivalent
𓏺	Vertical staff	1
∩	Heel bone	10
𓍢	Scroll	100
𓆼	Lotus flower	1000
𓂭	Pointing finger	10,000
𓆐	Polliwog or burbot	100,000
𓁨	Astonished man	1,000,000

Johnny Lott/Shutterstock

Figure 3

In its simplest form, the Egyptian system involved an **additive property**; that is, the value of a number was the sum of the face values of the numerals. The Egyptians customarily wrote the numerals in decreasing order from left to right, as in 𓆐𓍢𓍢𓍢𓎆𓎆𓏺𓏺. The number can be converted to base ten as shown below:

𓆐	represents	100,000	
𓍢𓍢𓍢	represents	300	(100 + 100 + 100)
𓎆𓎆	represents	20	(10 + 10)
𓏺𓏺	represents	2	(1 + 1)
𓆐𓍢𓍢𓍢𓎆𓎆𓏺𓏺	represents	100,322	

NOW TRY THIS 2

a. Use the Egyptian system to represent 1,312,322.
b. Use the Hindu-Arabic system to represent 𓆐𓆐𓆼𓆼𓆼𓎆𓎆𓎆𓏺𓏺𓏺𓏺𓏺.
c. What disadvantages do you see in the Egyptian system compared to the Hindu-Arabic system?

Babylonian Numeration System

The Babylonian numeration system was developed at about the same time as the Egyptian system. The symbols in Table 3 were made using a stylus either vertically or horizontally on clay tablets.

Table 3

Babylonian Numeral	Hindu-Arabic Equivalent
▼	1
<	10

The clay tablets were heated and dried to preserve a permanent record. Babylonian symbols on a clay tablet are pictured in Figure 4.

Collection of Louvre Museum, Paris/Photo by Erich Lessing/Art Resource, New York

Figure 4

The Babylonian numerals 1 through 59 were similar to the Egyptian numerals, but the vertical staff and the heel bone were replaced by the symbols shown in Table 3. For example, <<▼▼▼ represented 23.

The Babylonian numeration system used a *place-value system*. Numbers greater than 59 were represented by repeated groupings of 60, much as we use groupings of 10 today. For example, ▼▼ << represents $2 \cdot 60 + 20$, or 140. The space indicates that ▼▼ represents $2 \cdot 60$ rather than 2. Numerals immediately to the left of a second space have a value $60 \cdot 60$ or 60^2 times their face value, and so on.

<< ▼	represents	$20 \cdot 60 + 1$, or 1201
<▼ <▼ ▼	represents	$11 \cdot 60 \cdot 60 + 11 \cdot 60 + 1$, or $11 \cdot 60^2 + 11 \cdot 60 + 1$, or 40,261
▼ <▼ <▼ ▼	represents	$1 \cdot 60 \cdot 60 \cdot 60 + 11 \cdot 60 \cdot 60 + 11 \cdot 60 + 1$, or $1 \cdot 60^3 + 11 \cdot 60^2 + 11 \cdot 60 + 1$, or 256,261

The initial Babylonian system was inadequate by today's standards. For example, the symbol ▼▼ could have represented 2 or $2 \cdot 60$. Later, the Babylonians introduced the symbol ▲ as a placeholder for missing position values. Using this symbol, < <<▼ represented $10 \cdot 60 + 21$ and < ▲ <<▼ represented $10 \cdot 60^2 + 0 \cdot 60 + 21$. In this sense, ▲ represented 0.

NOW TRY THIS 3

a. Use the Babylonian system to represent 12,321.

b. Use the Hindu-Arabic system to represent ▼▼ <▼ ▼.

c. What advantages does the Hindu-Arabic system have over the Babylonian system?

Mayan Numeration System

In the early development of numeration systems, people frequently used parts of their bodies to count. Fingers could be matched to objects to stand for one, two, three, four, or five objects. Two hands could then stand for a set of ten objects. In warmer climates where people went barefoot, people may have used their toes as well as their fingers for counting. The Mayans introduced an attribute that was not present in the Egyptian or early Babylonian systems, namely, a symbol for zero. The Mayan system used only three symbols, which Table 4 shows, and based their system primarily on 20 with vertical groupings.

Table 4

Mayan Numeral	Hindu-Arabic Equivalent
•	1
—	5
𝋠	0

The symbols for the first eleven numers (beginning with zero) in the Mayan system are shown in Table 1. Notice the groupings of five, where each horizontal bar represents a group of five. Thus, the symbol for 19 was 𝋳, or three 5s and four 1s. The symbol for 20 was •𝋠, which represents one group of 20 plus zero 1s. In Figure 5(a), we have $2 \cdot 5 + 3 \cdot 1$ (or 13) groups of 20 plus $2 \cdot 5 + 1 \cdot 1$ (or 11), for a total of 271. In Figure 5(b), we have $3 \cdot 5 + 1 \cdot 1$ (or 16) groups of 20 and zero 1s, for a total of 320.

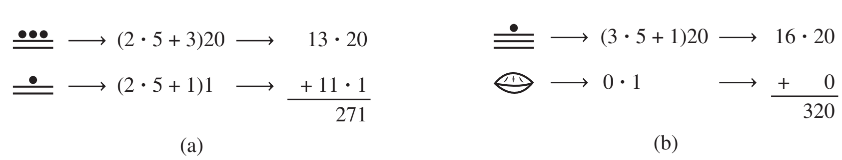


Figure 5

In a true base-twenty system, the place value of the symbols in the third position vertically from the bottom should be 20^2, or 400. However, the Mayans used $20 \cdot 18$, or 360, instead of 400. (The number 360 is an approximation of the length of a calendar year, which consisted of 18 months of 20 days each, plus 5 "unlucky" days.) Thus, instead of place values of $1, 20, 20^2, 20^3, 20^4$, and so on, the Mayans used $1, 20, 20 \cdot 18, 20^2 \cdot 18, 20^3 \cdot 18$, and so on. For example, in Figure 6(a), we have $5 + 1$ (or 6) groups of 360, plus $2 \cdot 5 + 2$ (or 12) groups of 20, plus $5 + 4$ (or 9) groups of 1, for a total of 2409. In Figure 6(b), we have $2 \cdot 5$ (or 10) groups of 360, plus 0 groups of 20, plus two 1s, for a total of 3602. Spacing is important in the Mayan system. For example, if two horizontal bars are placed close together, as in ═, the symbols represent $5 + 5 = 10$. If the bars are spaced apart, as in ═ , then the value is $5 \cdot 20 + 5 \cdot 1 = 105$.

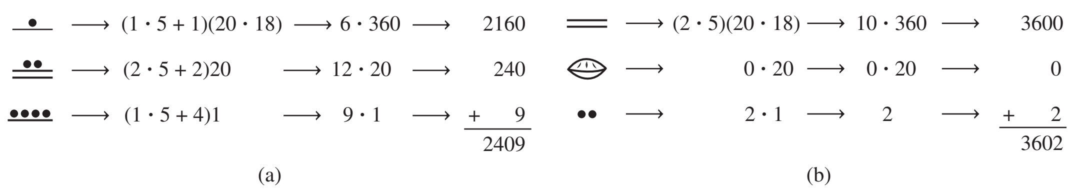


Figure 6

Roman Numeration System

The Roman numeration system was used in Europe in its early form from the third century BCE. It remains in use today, as seen on cornerstones, on the opening pages of books, and on the faces of some clocks. The Roman system uses only the symbols shown in Table 5.

Table 5

Roman Numeral	Hindu-Arabic Equivalent
I	1
V	5
X	10
L	50
C	100
D	500
M	1000

Roman numerals can be combined by using an additive property. For example, MDCLXVI represents $1000 + 500 + 100 + 50 + 10 + 5 + 1 = 1666$, CCCXXVIII represents 328, and VI represents 6. Romans wrote their symbols in decreasing order.

To avoid repeating a symbol more than three times, as in IIII, a **subtractive property** was introduced in the Middle Ages. For example, I is less than V, so if it immediately is to the left of V, it is subtracted. Thus, IV has a value of $5 - 1$, or 4, and XC represents $100 - 10$, or 90. Some extensions of the subtractive property could lead to ambiguous results. For example, IXC could be 91 or 89. By custom, 91 is written XCI and 89 is written LXXXIX. In general, only one smaller number symbol can be to the left of a larger number symbol and the pair must be one of those listed in Table 6.

Table 6

Roman Numeral	Hindu-Arabic Equivalent
IV	$5 - 1$, or 4
IX	$10 - 1$, or 9
XL	$50 - 10$, or 40
XC	$100 - 10$, or 90
CD	$500 - 100$, or 400
CM	$1000 - 100$, or 900

In the Middle Ages, a bar was placed over a Roman numeral to multiply it by 1000. The use of bars is based on a **multiplicative property**. For example, $\overline{\text{V}}$ represents $5 \cdot 1000$, or 5000, and $\overline{\text{CDX}}$ represents $410 \cdot 1000$, or 410,000. To indicate even greater numbers, more bars appear. For example, $\overline{\overline{\text{V}}}$ represents $(5 \cdot 1000)1000$, or 5,000,000; $\overline{\overline{\overline{\text{CXI}}}}$ represents $111 \cdot 1000^3$, or 111,000,000,000; and $\overline{\text{CX}}\text{I}$ represents $110 \cdot 1000 + 1$, or 110,001.

Several properties might be used to represent some numbers, for example:

$$\overline{\text{D}}\text{CLIX} = \underbrace{\underbrace{(500 \cdot 1000)}_{\text{Multiplicative}} + \underbrace{(100 + 50)}_{\text{Additive}} + \underbrace{(10 - 1)}_{\text{Subtractive}}}_{\text{Additive}} = 500{,}159$$

NOW TRY THIS 4

a. Write CCXLIX as a Hindu-Arabic numeral.
b. Use Roman numerals to represent each of the following
(i) 1634 (ii) 5280 (iii) 88
c. Use Mayan numerals to represent each of the following
(i) 684 (ii) 164

Other Number-base Systems

To better understand our system and to investigate some of the problems that students might have when learning it, we investigate similar systems that have different number bases.

Base Five

The Luo peoples of Kenya used a *quinary*, or base-five, system. A system of this type can be modeled by counting with only one hand. The digits available for counting are 0, 1, 2, 3, and 4. In the "one-hand system," or base-five system, we count 1, 2, 3, 4, 10, where *10 represents one hand and no fingers*. Counting in base five proceeds as shown in Figure 7. We write the small "five" below the numeral as a reminder that the numeral is written in base five. **If no base is written, a numeral is assumed to be in base ten**. Also note that 1, 2, 3, 4 are the same, and have the same meaning, in both base five and base ten.

One-Hand System	Base-Five Symbol	Base-Five Blocks
0 fingers	0_{five}	
1 finger	1_{five}	
2 fingers	2_{five}	
3 fingers	3_{five}	
4 fingers	4_{five}	
1 hand and 0 fingers	10_{five}	
1 hand and 1 finger	11_{five}	
1 hand and 2 fingers	12_{five}	
1 hand and 3 fingers	13_{five}	
1 hand and 4 fingers	14_{five}	
2 hands and 0 fingers	20_{five}	
2 hands and 1 finger	21_{five}	

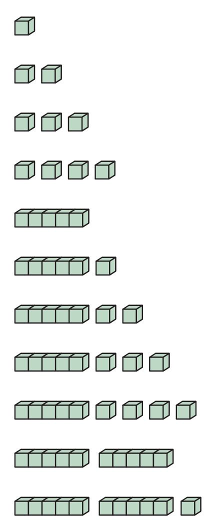


Figure 7

Counting in base five is similar to counting in base ten. Because we have only five digits (0_{five}, 1_{five}, 2_{five}, 3_{five}, and 4_{five}), 4_{five} plays the role of 9 in base ten. Figure 8 shows how we can find the number that comes after 34_{five} by using base five blocks. We see that if we add one more unit block to 34_{five} and perform a trade as shown in Figure 8, we obtain 40_{five}. Note that this is read "four zero base five" and not "forty base five."

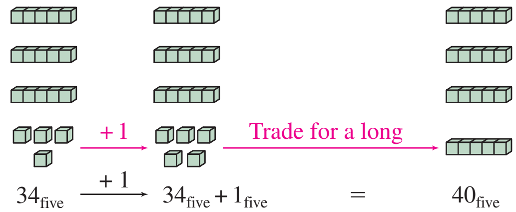


Figure 8

What number follows 44_{five}? There are no more two-digit numerals in the system after 44_{five}. In base-ten, the same situation occurs at 99. We use 100 to represent ten 10s, or one 100. In the base-five system, we need a symbol to represent five 5s. To continue the analogy with base ten, we use 100_{five} to represent one group of five 5s or 5^2, zero groups of five, and zero units. The name for 100_{five} is read "one zero zero base five." The number 100 means $1 \cdot 10^2 + 0 \cdot 10^1 + 0$, whereas the numeral 100_{five} means $(1 \cdot 10^2 + 0 \cdot 10^1 + 0)_{\text{five}}$, or $1 \cdot 5^2 + 0 \cdot 5^1 + 0$, or 25.

Examples of base-five numerals along with their base-five block representations and conversions to base ten are given in Figure 9. Multibase blocks will be used to illustrate various concepts.

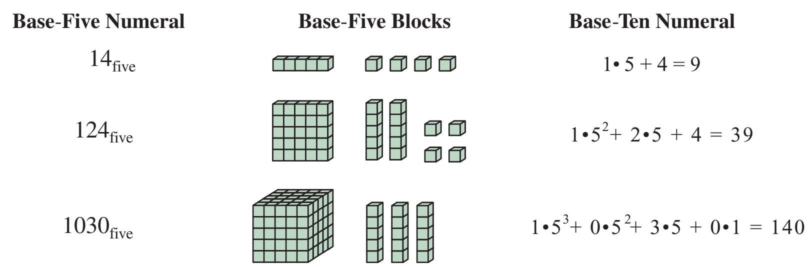


Figure 9

EXAMPLE 2

Convert 11244_{five} to base ten.

Solution

$$
\begin{aligned}
11244_{\text{five}} &= 1 \cdot 5^4 + 1 \cdot 5^3 + 2 \cdot 5^2 + 4 \cdot 5^1 + 4 \cdot 1 \\
&= 1 \cdot 625 + 1 \cdot 125 + 2 \cdot 25 + 4 \cdot 5 + 4 \cdot 1 \\
&= 625 + 125 + 50 + 20 + 4 \\
&= 824
\end{aligned}
$$

Example 2 suggests a method for changing a base-five numeral to a base-ten numeral using powers of 5. To convert 824 to base five, we consider how to write 824 using powers of 5. We first determine the greatest power of 5 less than or equal to 824. Because $5^4 = 625$ and $5^5 = 3125$, the greatest power is 5^4. How many 5^4s are contained in 824? There is only one and so we have $824 = 1 \cdot 5^4 + 199$. Next we find the greatest power of 5 in 199. How many 5^3s are in 199? There is only one giving us $824 = 1 \cdot 5^4 + 1 \cdot 5^3 + 74$. Similarly, there are two 5^2s in 74 with 24 left over, giving $824 = 1 \cdot 5^4 + 1 \cdot 5^3 + 2 \cdot 5^2 + 24$. Because there are four 5s in 24 with 4 left over, we have $824 = 1 \cdot 5^4 + 1 \cdot 5^3 + 2 \cdot 5^2 + 4 \cdot 5^1 + 4 \cdot 1$. Therefore, $824 = 11244_{\text{five}}$. A shorthand method for illustrating this conversion follows.

$$\begin{array}{llrl|ll}
5^4 = 625 & \rightarrow & 625 & \overline{)\,824} & 1 & \text{How many groups of 625 in 824?}\\
 & & & -625 & & \\
5^3 = 125 & \rightarrow & 125 & \overline{)\,199} & 1 & \text{How many groups of 125 in 199?}\\
 & & & -125 & & \\
5^2 = 25 & \rightarrow & 25 & \overline{)\,74} & 2 & \text{How many groups of 25 in 74?}\\
 & & & -50 & & \\
5^1 = 5 & \rightarrow & 5 & \overline{)\,24} & 4 & \text{How many groups of 5 in 24?}\\
 & & & -20 & & \\
5^0 = 1 & \rightarrow & 1 & \overline{)\,4} & 4 & \text{How many 1s in 4?}\\
 & & & -4 & & \\
 & & & 0 & &
\end{array}$$

Thus, $824 = 11244_{\text{five}}$.

NOW TRY THIS 5

A different method of converting 824 to base five is shown using successive divisions by 5. The quotient in each case is placed below the dividend and the remainder is placed on the right, on the same line with the quotient. The answer is read from bottom to top, that is, as 11244_{five}. Use this method to convert 728 to base five.

$$\begin{array}{r|ll}
5 & 824 & \\
5 & 164 & 4\\
5 & 32 & 4\\
5 & 6 & 2\\
 & 1 & 1
\end{array}$$

Base Two

Historians tell of early tribes that used base two. Some aboriginal tribes still count "one, two, two and one, two twos, two twos and one, ..." Because base two has only two digits, it is called the **binary system**. Base two is especially important because of its use in computers. One of the two digits is represented by the presence of an electrical signal and the other by the absence of an electrical signal. Although base two works well for some purposes, it is inefficient for everyday use because multidigit numbers are reached very rapidly in counting in this system.

Conversions from base two to base ten, and vice versa, can be accomplished in a manner similar to that used for base five conversions.

EXAMPLE 3

a. Convert 10111_{two} to base ten.
b. Convert 27 to base two.

Solution

a.
$$\begin{aligned}
10111_{\text{two}} &= 1\cdot 2^4 + 0\cdot 2^3 + 1\cdot 2^2 + 1\cdot 2^1 + 1\\
&= 16 + 0 + 4 + 2 + 1\\
&= 23
\end{aligned}$$

b.

$$\begin{array}{rrll}
16 & 27 & 1 & \text{How many groups of 16 in 27?} \\
 & -16 & & \\
8 & 11 & 1 & \text{How many groups of 8 in 11?} \\
 & -8 & & \\
4 & 3 & 0 & \text{How many groups of 4 in 3?} \\
 & -0 & & \\
2 & 3 & 1 & \text{How many groups of 2 in 3?} \\
 & -2 & & \\
1 & 1 & 1 & \text{How many 1s in 1?} \\
 & -1 & & \\
 & 0 & &
\end{array}$$

Alternative Solution:

$$\begin{array}{r|rr}
2 & 27 & \\
2 & 13 & 1 \\
2 & 6 & 1 \\
2 & 3 & 0 \\
 & 1 & 1
\end{array}$$

Thus, 27 is equivalent to 11011_{two}.

Base Twelve

Another commonly used number-base system is the base twelve, or the duodecimal ("dozens"), system. Eggs are bought by the dozen, and pencils are bought by the *gross* (a dozen dozen). In base twelve, there are 12 digits, just as there are 10 digits in base ten, 5 digits in base five, and 2 digits in base two. In base twelve, new symbols are needed to represent the following groups of *x*'s:

$$\overbrace{xxxxxxxxxx}^{10\ x\text{'s}} \quad \text{and} \quad \overbrace{xxxxxxxxxxx}^{11\ x\text{'s}}$$

The new symbols chosen are T and E, respectively, so that the base twelve digits are $(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E)_{\text{twelve}}$. Thus, in base twelve we count $(1, 2, 3, 4, 5, 6, 7, 8, 9, T, E, 10, 11, 12, \ldots, 17, 18, 19, 1T, 1E, 20, 21, 22, \ldots, 28, 29, 2T, 2E, 30, \ldots)_{\text{twelve}}$.

EXAMPLE 4

a. Convert $E2T_{\text{twelve}}$ to base ten.
b. Convert 1277 to base twelve.

Solution

a.
$$\begin{aligned}
E2T_{\text{twelve}} &= 11 \cdot 12^2 + 2 \cdot 12^1 + 10 \cdot 1 \\
&= 11 \cdot 144 + 24 + 10 \\
&= 1584 + 24 + 10 \\
&= 1618
\end{aligned}$$

b.

$$\begin{array}{rrll}
144 & 1277 & 8 & \text{How many groups of 144 in 1277?} \\
 & -1152 & & \\
12 & 125 & T & \text{How many groups of 12 in 125?} \\
 & -120 & & \\
1 & 5 & 5 & \text{How many 1s in 5?} \\
 & -5 & & \\
 & 0 & &
\end{array}$$

Thus, $1277 = 8T5_{\text{twelve}}$.

EXAMPLE 5

Rob used base twelve to write the following:

$$g36_{\text{twelve}} = 1050_{\text{ten}}$$

What is the value of g?

Solution We could write the following equations:

$$\begin{aligned} g \cdot 12^2 + 3 \cdot 12 + 6 \cdot 1 &= 1050 \\ 144g + 36 + 6 &= 1050 \\ 144g + 42 &= 1050 \\ 144g &= 1008 \\ g &= 7 \end{aligned}$$

Check $736_{\text{twelve}} = 7 \cdot 12^2 + 3 \cdot 12 + 6 \cdot 1 = 1050$

Assessment 1A

1. For each of the following, tell which numeral represents the greater number and why:
   a. $\overline{\text{M}}$CDXXIV and $\overline{\overline{\text{M}}}$CDXXIV
   b. 4632 and 46,032
   c. <▼▼ and < ▼▼
   d. 999∩∩|| and 𓆼∩|
   e.  and 
2. For each of the following, write both the succeeding and the preceding numerals (one more and one less):
   a. MCMXLIX
   b. << <▼
   c. 𓆼99
   d. 
3. If the cornerstone represents when a building was built and it reads MCMXXII, when was this building built?
4. Write each of the following in Roman symbols:
   a. 121
   b. 42
5. Complete the following table, which compares symbols for numbers in different numeration systems:

	Hindu-Arabic	Babylonian	Egyptian	Roman	Mayan
a.	72				
b.		< ▼▼			
c.					

6. For each of the following base ten numerals, give the place value of the underlined digit:
   a. 827,<u>3</u>67
   b. 8,421,0<u>0</u>0
7. Rewrite each of the following as a base ten numeral:
   a. $3 \cdot 10^6 + 4 \cdot 10^3 + 5$
   b. $2 \cdot 10^4 + 1$
8. A certain three-digit natural number has the following properties: The hundreds digit is greater than 7; the tens digit is an odd number; and the sum of the three digits is 10. What could the number be?
9. Study the following counting frame. In the frame, the value of each dot is represented by the number in the box below the dot. For example, the following figure represents the number 154:

••	•••	••
64	8	1

   What numbers are represented in the frames in (a) and (b)?

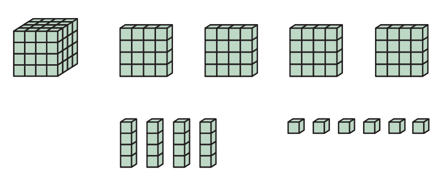
   a.

•••	••	•
25	5	1

   b.

•		•	•
8	4	2	1

10. Write the base-four numeral for the base-four representation shown.



11. Write the first 15 counting numerals for each of the following bases:
    a. base two
    b. base four
12. How many different digits are needed for base twenty?
13. Write 2032_{four} in expanded notation and then convert it to base ten.

14. Determine the greatest three-digit number in each of the following bases:
a. Base two
b. Base twelve

15. Find the numeral preceding and succeeding each of the following:
a. $EE0_{\text{twelve}}$
b. 100000_{two}
c. 555_{six}

16. What, if anything, is wrong with the following numerals?
a. 204_{four}
b. 607_{five}

17. What is the fewest number of base four blocks needed to represent 214?

18. Draw base-five blocks to represent 231_{five}.

19. An introduction to base five is especially suitable for early learning in elementary school, as children can think of making change using quarters, nickels, and pennies. Use only these coins to answer the following:
a. What is the fewest number of quarters, nickels, and pennies you can receive in a fair exchange for two quarters, nine nickels, and eight pennies?
b. How could you use the approach in (a) to write 73 in base five?

20. Without converting to base ten, tell which is the lesser for each of the following pairs.
a. 3030_{four} or 3100_{four}
b. $EOTE_{\text{twelve}}$ or $EOET_{\text{twelve}}$

21. Recall that with base-ten blocks, 1 long = 10 units, 1 flat = 10 longs, and 1 block = 10 flats. Make all possible exchanges to obtain the fewest number of pieces and write the corresponding numeral in the given base in the following.
a. Ten flats in base ten
b. Twenty flats in base twelve

22. Convert each of the following base ten numerals to a numeral in the indicated bases.
a. 456 in base five
b. 1782 in base twelve
c. 32 in base two

23. Write each of the following numerals in base ten:
a. 432_{five}
b. 101101_{two}
c. $92E_{\text{twelve}}$

24. You are asked to distribute $900 in prize money. The dollar amounts for the prizes are $625, $125, $25, $5, and $1. How should this $900 be distributed in order to give the fewest number of prizes?

25. Convert each of the following:
a. 58 days to weeks and days
b. 29 hours to days and hours

26. For each of the following, find b if possible. If not possible, tell why.
a. $b2_{\text{seven}} = 44_{\text{ten}}$
b. $5b2_{\text{twelve}} = 734_{\text{ten}}$

27. Write the following in the indicated base without multiplying out the powers:
a. $3 \cdot 5^4 + 3 \cdot 5^2$ in base five
b. $2 \cdot 12^5 + 8 \cdot 12^3 + 12$ in base twelve

28. In a game called WIPEOUT, we are to "wipe out" digits from a calculator's display without changing any of the other digits. "Wipeout" in this case means to replace the chosen digit(s) with a 0. For example, if the initial number is 54,321 and we are to wipe out the 4, we could subtract 4000 to obtain 50,321. Complete the following two problems and then try other numbers or challenge another person to wipe out a number from the number you have placed on the screen:
a. Wipe out the 2s from 32,420.
b. Wipe out the 5 from 67,357.

Assessment 1B

1. For each of the following, tell which numeral represents the greater number and why:
a. $\overline{\text{M}}$DCXXIV and $\overline{\text{M}}$CDXXIV
b. 3456 and 30,456
c. ◁▼ and ◁ ▼▼
d. 𓍢𓍢𓎆𓏺 and 𓍢𓍢𓍢
e. 𝋭 and 𝋢𝋠
2. For each of the following, name both the preceding and the succeeding numbers (one more and one less):
a. $\overline{\text{M}}$I
b. CMXCIX
c. ◁ ◁▼
d. 𓆼𓍢
e. 𝋦𝋨
3. On the United States one dollar bill, the number MDCCLXXVI is written on the base of the pyramid. What year does this represent?
4. Write each of the following in Roman symbols:
a. 89
b. 5202

5. Complete the following table, which compares symbols for numbers in different numeration systems:

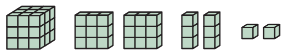
	Hindu-Arabic	Babylonian	Egyptian	Roman	Mayan
a.	78				
b.		◁ ▼			
c.			𓆼𓍢𓎆𓏺		

6. For each of the following base ten numerals, give the place value of the underlined digit:
a. 9<u>7</u>, 998
b. <u>8</u>10, 485
7. Rewrite each of the following as a base ten numeral:
a. $3 \cdot 10^3 + 5 \cdot 10^2 + 6 \cdot 10$
b. $9 \cdot 10^6 + 9 \cdot 10 + 9$
8. A two-digit number has the property that the units digit is 4 less than the tens digit and the tens digit is twice the units digit. What is the number?

9. On a counting frame, the following number is represented. What might the number be? Explain your reasoning.

•	••	••
27	9	1

10. Write the base-three numeral for the base-three representation shown.



11. Write the first 10 counting numerals for each of the following bases:
   a. Base three
   b. Base eight
12. How many different digits are needed for base eighteen?
13. Write 2022_{three} in expanded form and then convert it to base ten.
14. Determine the greatest three-digit numeral in each of the following bases:
   a. Base three
   b. Base twelve
15. Find the numeral preceding and succeeding each of the following:
   a. 100_{seven}
   b. 10000_{two}
   c. 101_{two}
16. What, if anything, is wrong with the following numerals?
   a. 306_{four}
   b. 1023_{two}
17. What is the fewest number of base-three blocks needed to represent 79?
18. Draw base-two blocks to represent 1001_{two}.
19. Using a number system based on dozen and gross, how would you describe the representation for 277?
20. Without converting to base ten, tell which is the lesser for each of the following pairs and explain why?
   a. $EET9E_{\text{twelve}}$ or $E0T9E_{\text{twelve}}$
   b. 1011011_{two} or 101011_{two}
   c. 50555_{six} or 51000_{six}
21. What is the fewest number of multibase blocks that can be used to write the corresponding numeral in the given base?
   a. 10 longs in base four
   b. 10 longs in base three
22. Convert each of the following base-ten numerals to a numeral in the indicated base:
   a. 234 in base four
   b. 1876 in base twelve
   c. 303 in base three
   d. 22 in base two
23. Write each of the following numerals in base ten:
   a. 432_{six}
   b. 11011_{two}
   c. $E29_{\text{twelve}}$
24. *Who Wants the Money*, a game show, distributes prizes that are powers of 2. What is the minimum number of prizes that could be distributed from $900?
25. A coffee shop sold 1 cup, 1 pint, and 1 quart of coffee. Express the number of cups sold in base two.
26. For each of the following, find b, if possible. If not possible, tell why.
   a. $b3_{\text{four}} = 31_{\text{ten}}$
   b. $1b2_{\text{twelve}} = 1534_{\text{six}}$
27. Using only the number keys on a calculator, fill the display to show the greatest four-digit number if each key can be used only once.

Mathematical Connections 1

Communication

1. Ben claims that zero is the same as nothing. Explain how you as a teacher would respond to Ben's statement.
2. What are the major drawbacks to each of the following systems?
   a. Egyptian
   b. Babylonian
   c. Roman
3. **a.** Why are large numbers in the United States written with commas separating groups of three digits?
   b. Find examples from other countries that do not use commas to separate groups of three digits.
4. In the Roman numeral system explain **(a)** when you add values, **(b)** when you subtract values, and **(c)** when you multiply values. Give an example to illustrate each case.

Open-Ended

5. An inspector of weights and measures uses a special set of weights to check the accuracy of scales. Various weights are placed on a scale to check accuracy of any amount from 1 oz through 15 oz. What is the fewest number of weights the inspector needs? What weights are needed to check the accuracy of scales from 1 oz through 15 oz? From 1 oz through 31 oz?

Cooperative Learning

6. **a.** Create a numeration system with unique symbols and write a paragraph explaining the properties of the system.
   b. Complete the following table using the system:

Hindu-Arabic Numeral	Your System Numeral
1	
5	
10	
50	
100	
5000	
10,000	
115,280	

Questions from the Classroom

7. A student claims that the Roman system is a base-ten system since it has symbols for 10, 100, and 1000. How do you respond?
8. When using Roman numerals, a student asks whether it is correct to write $\overline{\text{I}}$I, as well as MI for 1001. How do you reply?

Trends in Mathematics and Science Study (TIMSS) Questions

Which digit is in the hundreds place in 2345?

a. 2
b. 3
c. 4
d. 5

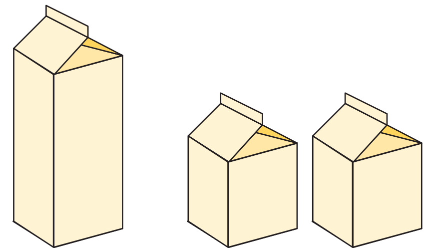
TIMSS, Grade 4, 2003

Which number equals 3 ones + 2 tens + 4 hundreds.

a. 432
b. 423
c. 324
d. 234

TIMSS, Grade 4, 2007

National Assessment of Educational Progress (NAEP) Question



1 quart = 2 pints

Mr. Harper bought 6 pints of milk. How many quarts of milk is this equal to?

a. 3
b. 4
c. 6
d. 12

NAEP, Grade 4, 2007

Source: Trends in International Mathematics and Science Study (TIMSS), IES National Center for Education Statistics, 2011.

Source: National Assessment of Education Progress (NAEP), IES National Center for Education Statistics, 2011.

2 Describing Sets

In the years from 1871 through 1884, Georg Cantor created *set theory*, which had a profound effect on research and mathematics teaching. Sets, and relations between sets, form a basis for teaching children the *concept* of a whole number, $\{0, 1, 2, 3, \ldots\}$, and the concept of "less than" as well as addition, subtraction, and multiplication of whole numbers. Understanding whole numbers and operations on whole numbers can be enhanced by the notion of sets. In this section we introduce set notation, relations between sets, set operations, and their properties.

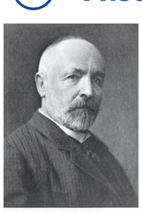
The Language of Sets

A **set** is understood to be any collection of objects. Individual objects in a set are **elements**, or **members**, of the set. For example, each letter is an element of the set of letters in the English language. Capital letters are generally used to name sets. The elements of the set are listed inside a pair of braces, { }. The set A of lowercase letters of the English alphabet can be written in set notation as follows:

$$A = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$$

The order in which the elements are written makes no difference, and *each element is listed only once*. Thus, $\{b, o, k\}$ and $\{k, o, b\}$ are considered to be the same set.

Historical Note



Interfoto/Alamy

Georg Cantor (1845–1918) pursued a career in mathematics and obtained his doctorate in Berlin at age 22. Most of his academic career was spent at the University of Halle. His hope of becoming a professor at the University of Berlin did not materialize because his work gained little recognition during his lifetime. However, after his death Cantor's work in set theory was praised as an astonishing product of mathematical thought, one of the most beautiful realizations of human activity.

We show that an element belongs to a set by using the symbol $\in$. For example, $b \in A$. If an element does not belong to a set, we use the symbol $\notin$. For example, $3 \notin A$. In mathematics, the same letter, one lowercase and the other uppercase, cannot be freely interchanged. For example, in the set $A = \{a, b, c\}$ we have $b \in A$ but $B \notin A$.

A set must be **well defined**; that is, if we are given a set and some particular object, then we must be able to tell whether the object does or does not belong to the set. For example, the set of all citizens of Pasadena, California, who ate rice on January 1, 2011, is well defined. We personally may not know if a particular resident of Pasadena ate rice or not, but that resident either belongs or does not belong to the set. On the other hand, the set of all tall people is not well defined because there is no clear meaning of "tall."

We may use sets to define mathematical terms. For example, the set N of ***natural numbers*** is defined by the following:

$$N = \{1, 2, 3, 4, \ldots\}$$

An ***ellipsis*** (three dots) indicates that the sequence continues in the same manner.

Two common methods of describing sets are the **listing** or **roster method** and **set-builder notation**, as seen in these examples:

Listing or roster method:	$C = \{1, 2, 3, 4\}$
Set-builder notation:	$C = \{x \mid x \in N, x < 5\}$

The latter notation is read as follows:

C	$=$	$\{$	x	$\mid$	$x \in N,$	$x < 5\}$
Set C	is equal to	the set of	all elements x	such that	x is an element of the natural numbers, and	x is less than 5

In the set-builder notation $C = \{x \mid x \in N, x < 5\}$, the comma is a place holder for "and." With this notation both conditions, $x \in N$ and $x < 5$, must be true.

In set-builder notation any lowercase letter can be used to represent a general element. Set-builder notation is useful when the individual elements of a set are not known or they are too numerous to list. For example, the set of decimals between 0 and 1 can be written as

$$D = \{x \mid x \text{ is a decimal between 0 and 1}\}.$$

This is read "D is the set of all elements x such that x is a decimal between 0 and 1." It would be impossible to list all the elements of D. Hence the set-builder notation is indispensable here.

EXAMPLE 6

Write the following sets consisting of terms in arithmetic sequences using set-builder notation:

a. $\{2, 4, 6, 8, 10, \ldots\}$ **b.** $\{1, 3, 5, 7, \ldots\}$

Solution

a. $\{x \mid x \text{ is an even natural number}\}$. Or because every even natural number can be written as 2 times some natural number, this set can be written as $\{x \mid x = 2n, n \in N\}$ or, in a somewhat simpler form, as $\{2n \mid n \in N\}$.

b. $\{x \mid x \text{ is an odd natural number}\}$. Or because every odd natural number can be written as some even number minus 1, this set can be written as $\{x \mid x = 2n - 1, n \in N\}$ or $\{2n - 1 \mid n \in N\}$.

EXAMPLE 7

Each of the following sets is described in set-builder notation. Write each of the sets by listing its elements.

a. $C = \{2k + 1 \mid k = 3, 4, 5\}$
b. $D = \{x \mid x \text{ is a positive even natural number less than } 8\}$

Solution

a. We substitute $k = 3, 4, 5$ in $2k + 1$ and obtain the corresponding values shown in Table 7. Thus, $C = \{7, 9, 11\}$.

Table 7

k	$2k + 1$
3	$2 \cdot 3 + 1 = 7$
4	$2 \cdot 4 + 1 = 9$
5	$2 \cdot 5 + 1 = 11$

b. $D = \{2, 4, 6\}$

As noted earlier, the order in which the elements are listed does not matter. If sets A and B are equal, written $A = B$, then every element of A is an element of B, and every element of B is an element of A. If A does not equal B, we write $A \neq B$.

Definition of Equal Sets
Two sets are **equal** if, and only if, they contain exactly the same elements.

One-to-One Correspondence

One of the most useful concepts in set theory is a **one-to-one correspondence** between two sets. For example, consider the set of people $P = \{\text{Tomas, Dick, Mari}\}$ and the set of swimming lanes $S = \{1, 2, 3\}$. Suppose each person in P is to swim in a lane numbered 1, 2, or 3 so that no two people swim in the same lane. Such a person-lane pairing is a one-to-one correspondence. One way to exhibit a one-to-one correspondence is shown in Figure 10 with arrows connecting corresponding elements.

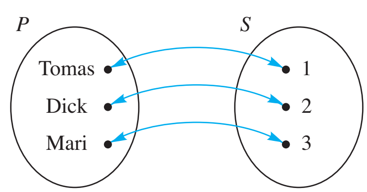


Figure 10

Other possible one-to-one correspondences exist between the sets P and S. All six possible one-to-one correspondences between sets P and S can be listed as follows:

1. Tomas $\leftrightarrow$ 1	**2.** Tomas $\leftrightarrow$ 1	**3.** Tomas $\leftrightarrow$ 2
Dick $\leftrightarrow$ 2	Dick $\leftrightarrow$ 3	Dick $\leftrightarrow$ 1
Mari $\leftrightarrow$ 3	Mari $\leftrightarrow$ 2	Mari $\leftrightarrow$ 3

4. Tomas $\leftrightarrow$ 2
Dick $\leftrightarrow$ 3
Mari $\leftrightarrow$ 1

5. Tomas $\leftrightarrow$ 3
Dick $\leftrightarrow$ 1
Mari $\leftrightarrow$ 2

6. Tomas $\leftrightarrow$ 3
Dick $\leftrightarrow$ 2
Mari $\leftrightarrow$ 1

Notice that each listing 1–6 above represents a single one-to-one correspondence between the sets P and S. A complete set of one-to-one correspondences between sets P and S can also be listed using a table as in Table 8.

Table 8

Pairings \ Lanes	1	2	3
1.	Tomas	Dick	Mari
2.	Tomas	Mari	Dick
3.	Dick	Tomas	Mari
4.	Dick	Mari	Tomas
5.	Mari	Tomas	Dick
6.	Mari	Dick	Tomas

The general definition of one-to-one correspondence follows.

Definition of One-to-One Correspondence

If the elements of sets P and S can be paired so that for each element of P there is exactly one element of S and for each element of S there is exactly one element of P, then the two sets P and S are said to be in **one-to-one correspondence**.

NOW TRY THIS 6

Consider a set of four people $\{A, B, C, D\}$ and a set of four swimming lanes $\{1, 2, 3, 4\}$.

a. Exhibit all the one-to-one correspondences between the two sets.
b. How many such one-to-one correspondences are there?
c. Find the number of one-to-one correspondences between two sets with five elements each and explain your reasoning.

A tree diagram also lists the possible one-to-one correspondences in Figure 11. To read the tree diagram and see the one-to-one correspondence, we follow each branch. The person occupying a specific lane in a correspondence is listed below the lane number. For example, the top branch gives the pairing (Tomas, 1), (Dick, 2), and (Mari, 3).

Observe in Figure 11 when assigning a swimmer to lane 1 we have a choice of three people: Tomas, Dick, or Mari. If we put Tomas in lane 1, then he cannot be in lane 2, and hence the second lane must be occupied by either Dick or Mari. In the same way, we see that if Dick is in lane 1, then there are two choices for lane 2: Tomas or Mari. Similarly, if Mari is in lane 1, then again there are two choices for the second lane: Tomas or Dick. Thus, for each of the three ways we can fill the first lane, there are two subsequent ways to fill the second lane, and hence there are $2 + 2 + 2$, or $3 \cdot 2$, or 6 ways to arrange the swimmers in the first two lanes. Notice that for each arrangement of the swimmers in the first two lanes, there remains only one possible swimmer to fill the third lane. For example, if Mari fills the first lane and Dick fills the second, then Tomas must be in the third. Thus, the total number of arrangements for the three swimmers is equal to $3 \cdot 2 \cdot 1$, or 6.

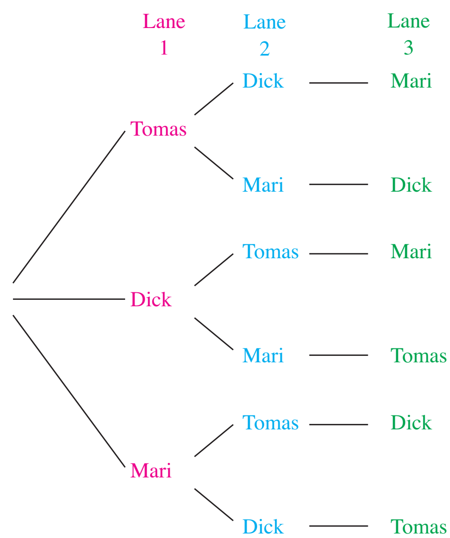


Figure 11

Similar reasoning can be used to find how many ice-cream arrangements are possible on a two-scoop cone if 10 flavors are offered. If we count chocolate and vanilla (chocolate on bottom and vanilla on top) different from vanilla and chocolate (vanilla on bottom and chocolate on top) and allow two scoops to be of the same flavor, we can proceed as follows. There are 10 choices for the first scoop, and for each of these 10 choices there are 10 subsequent choices for the second scoop. Thus, the total number of arrangements is $10 \cdot 10$, or 100.

The counting argument used to find the number of possible one-to-one correspondences between the set of swimmers and the set of lanes and the previous problem about ice-cream-scoop arrangements are examples of the Fundamental Counting Principle.

Theorem 1: Fundamental Counting Principle

If event M can occur in m ways and, after it has occurred, event N can occur in n ways, then event M followed by event N can occur in mn ways.

NOW TRY THIS 7

How many one-to-one correspondences are there between two sets with n elements each?

Equivalent Sets

Closely associated with one-to-one correspondences is the concept of **equivalent sets**. For example, suppose a room contains 20 chairs and one student is sitting in each chair with no one standing. There is a one-to-one correspondence between the set of chairs and the set of students in the room. In this case, the set of chairs and the set of students are equivalent sets.

Definition of Equivalent Sets

Two sets A and B are **equivalent**, written $A \sim B$, if, and only if, there exists a one-to-one correspondence between the sets.

The term *equivalent* should not be confused with *equal*. The difference should be made clear by Example 8.

EXAMPLE 8

Let

$$A = \{p, q, r, s\}, \qquad B = \{a, b, c\}, \qquad C = \{x, y, z\}, \qquad \text{and} \qquad D = \{b, a, c\}.$$

Compare the sets, using the terms *equal* and *equivalent*.

Solution

Each set is both equivalent to and equal to itself.

Sets A and B are not equivalent ($A \nsim B$) and not equal ($A \neq B$).

Sets A and C are not equivalent ($A \nsim C$) and not equal ($A \neq C$).

Sets A and D are not equivalent ($A \nsim D$) and not equal ($A \neq D$).

Sets B and C are equivalent ($B \sim C$) but not equal ($B \neq C$).

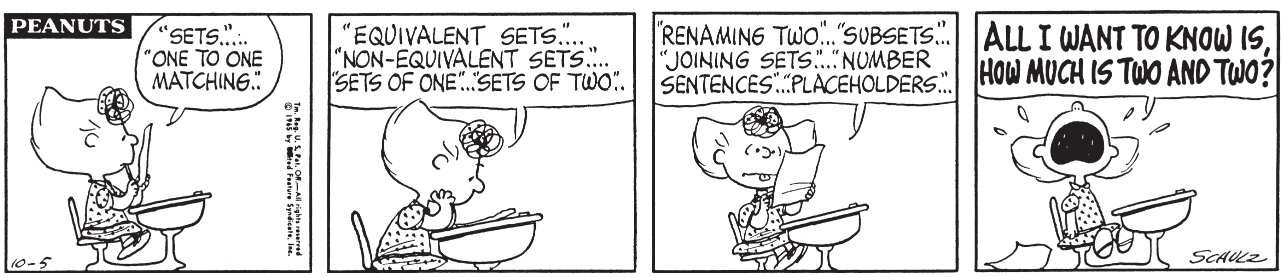
Sets B and D are equivalent ($B \sim D$) and equal ($B = D$).

Sets C and D are equivalent ($C \sim D$) but not equal ($C \neq D$).

NOW TRY THIS 8

a. If two sets are equivalent, are they necessarily equal? Explain why or why not.
b. If two sets are equal, are they necessarily equivalent? Explain why or why not.

The following *Peanuts* cartoon demonstrates some set theory concepts related to addition, though a child would not be expected to know all these concepts to add 2 and 2.



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Cardinal Numbers

The concept of one-to-one correspondence can be used to consider the notion of two sets having the same number of elements. Without knowing how to count, a child might tell that there are as many fingers on the left hand as on the right hand by matching the fingers on one hand with the fingers on the other hand, as in Figure 12. Naturally placing the fingers so that the left thumb touches the right thumb, the left index finger touches the right index finger, and so on, exhibits a one-to-one correspondence between the fingers of the two hands. Similarly, without counting, children realize that if every student in a class sits in a chair and no chairs are empty, there are as many chairs as students, and vice versa.

A one-to-one correspondence between sets helps explain the concept of a number. Consider the five sets $\{a, b\}$, $\{p, q\}$, $\{x, y\}$, $\{b, a\}$, and $\{*, \#\}$; the sets are equivalent to one another and share the property of "twoness"; that is, these sets have the same cardinal number, namely, 2. The **cardinal number** of a set S, denoted $n(S)$, indicates the number of elements in the set S. If $S = \{a, b\}$, the cardinal number of S is 2, and we write $n(S) = 2$. *If two sets, A and B, are equivalent, then A and B have the same cardinal number; that is,* $n(A) = n(B)$.

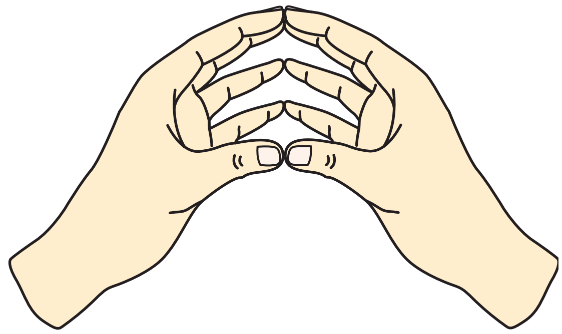


Figure 12

The Empty Set

A set that contains no elements has cardinal number 0 and is an **empty**, or **null, set**. The empty set is designated by the symbol $\varnothing$ or $\{\ \}$. Two examples of sets with no elements are the following:

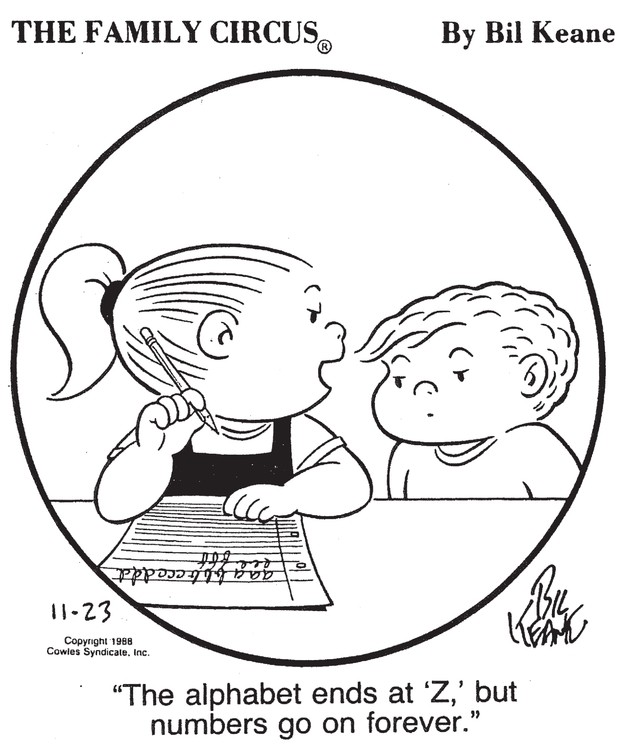
$$C = \{x \mid x \text{ was a state of the United States before 1200 CE}\}$$
$$D = \{x \mid x \text{ is a natural number such that } x^2 = 17\}$$

The empty set is often incorrectly recorded as $\{\varnothing\}$. This set is not empty but contains one element. Likewise, $\{0\}$ does not represent the empty set. Why?

A set is a **finite set** if the cardinal number of the set is zero or a natural number. The set of natural numbers N is an **infinite set**; it is not finite. The set W, containing all the natural numbers and 0, is the set of **whole numbers**: $W = \{0, 1, 2, 3, \ldots\}$. W is an infinite set.

NOW TRY THIS 9

Use *reasoning* to explain why there can be no greatest natural number. That is, explain why the set of natural numbers is not a finite set as Dolly implies in the cartoon below.



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More About Sets

The **universal set**, or the **universe**, denoted U, is the set that contains all elements being considered in a given discussion. Suppose $U = \{x \mid x \text{ is a person living in California}\}$ and $F = \{x \mid x \text{ is a female living in California}\}$. The universal set, U, and set F can be represented by a diagram, as in Figure 13(a). The universal set is represented by a large rectangle, and F is indicated by the circle inside the rectangle, as shown in Figure 13(a). This figure is an example of a **Venn diagram**, named after the Englishman John Venn (1834–1923), who used such diagrams to illustrate ideas in logic. The set of elements in the universe that are not in F, denoted by $\overline{F}$, is the set of males living in California and is the **complement** of F. It is represented by the shaded region in Figure 13(b).

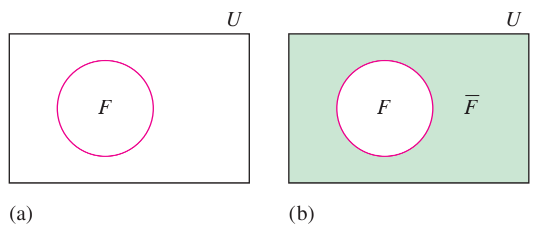


Figure 13

Definition of Set Complement

The **complement** of a set A, written $\overline{A}$, is the set of all elements in the universal set U that are not in A; that is, $\overline{A} = \{x \mid x \in U \text{ and } x \notin A\}$.

EXAMPLE 9

a. If $U = \{a, b, c, d\}$ and $B = \{c, d\}$, find (i) $\overline{B}$; (ii) $\overline{U}$; (iii) $\overline{\varnothing}$.
b. If $U = \{x \mid x \text{ is an animal in the zoo}\}$ and $S = \{x \mid x \text{ is a snake in the zoo}\}$, describe $\overline{S}$.
c. If $U = N$, $E = \{2, 4, 6, 8, \ldots\}$, and $O = \{1, 3, 5, 7, \ldots\}$, find (i) $\overline{E}$; (ii) $\overline{O}$.

Solution

a. (i) $\overline{B} = \{a, b\}$; (ii) $\overline{U} = \varnothing$; (iii) $\overline{\varnothing} = U$
b. Because the individual animals in the zoo are not known, $\overline{S}$ must be described using set-builder notation:

$$\overline{S} = \{x \mid x \text{ is an animal in the zoo and } x \text{ is not a snake}\}$$

c. (i) $\overline{E} = O$; (ii) $\overline{O} = E$

Subsets

Consider the sets $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6\}$. All the elements of B are contained in A and we say that B is a **subset** of A. We write $B \subseteq A$. In general, we have the following:

Definition of Subset

For all sets A and B, B is a **subset** of A, written $B \subseteq A$, if, and only if, every element of B is an element of A.

This definition allows B to be equal to A. The definition is written with the phrase "if, and only if," which means "if B is a subset of A, then every element of B is an element of A, and if every element of B is an element of A, then B is a subset of A." *If both* $A \subseteq B$ *and* $B \subseteq A$, *then* $A = B$.

When a set A is not a subset of another set B, we write $A \nsubseteq B$. To show that $A \nsubseteq B$, we must find at least one element of A that is not in B. If $A = \{1, 3, 5\}$ and $B = \{1, 2, 3\}$, then A is not a subset of B because 5 is an element of A but not of B. Likewise, $B \nsubseteq A$ because 2 belongs to B but not to A.

The Empty Set as a Subset of Every Set

It is not obvious how the empty set fits the definition of a subset because no elements in the empty set are elements of another set. To investigate this problem, we use the strategies of *indirect reasoning* and *looking at a special case*.

For the set $\{1, 2\}$, either $\varnothing \subseteq \{1, 2\}$ or $\varnothing \nsubseteq \{1, 2\}$. Suppose $\varnothing \nsubseteq \{1, 2\}$. Then there must be some element in $\varnothing$ that is not in $\{1, 2\}$. Because the empty set has no elements, there cannot be an element in the empty set that is not in $\{1, 2\}$. Consequently, $\varnothing \nsubseteq \{1, 2\}$ is false. Therefore, the only other possibility, $\varnothing \subseteq \{1, 2\}$, is true. The same reasoning can be applied in the case of the empty set and any other set. Therefore, *the empty set is a subset of every set*.

Proper Subsets

For sets $A = \{a, b, c\}$ and $B = \{c, b, a\}$, we have $B \subseteq A$, $A \subseteq B$, and $B = A$. If B is a subset of A and B is not equal to A, then B is a **proper subset** of A, written $B \subset A$. This means that every element of B is contained in A and there is at least one element of A that is not in B.

Definition of Proper Subset

For all sets A and B, B is a **proper subset** of A, written $B \subset A$, if, and only if, $B \subseteq A$ and $B \neq A$; that is, every element of B is an element of A, and there is at least one element of A that is not an element of B.

To indicate a proper subset, sometimes a Venn diagram like the one shown in Figure 14 is used, showing a dot (an element) in A that is not in B. An argument similar to the one given above can be used to show that the empty set is a proper subset of every non-empty set.

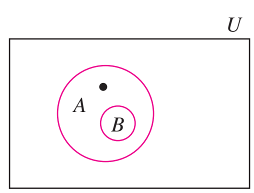


Figure 14

EXAMPLE 10

Given $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3\}$, $P = \{x \mid x = 2^n - 1, n \in N\}$:

a. Identify all the subset relationships that occur among these sets.
b. Identify all the proper subset relationships that occur among these sets.
c. If $C = \{2k \mid k \in N\}$ and $D = \{4k \mid k \in N\}$, show that one of the sets is a subset of the other.

Solution

a. Because $2^1 - 1 = 1$, $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^4 - 1 = 15$, $2^5 - 1 = 31$, and so on, $P = \{1, 3, 7, 15, 31, \ldots\}$. Thus, $B \subseteq P$. Also $B \subseteq A$, $A \subseteq A$, $B \subseteq B$ and $P \subseteq P$.

b. $B \subset A$ and $B \subset P$.

c. $C = \{2 \cdot 1, 2 \cdot 2, 2 \cdot 3, 2 \cdot 4, \ldots\} = \{2, 4, 6, 8, \ldots\}$ and $D = \{4 \cdot 1, 4 \cdot 2, 4 \cdot 3, 4 \cdot 4, \ldots\} = \{4, 8, 12, 16, \ldots\}$. Each element in D appears in C. This is true because $4k = 2(2k)$. Therefore, every element of D is an element of C and $D \subseteq C$.

NOW TRY THIS 10

a. Suppose $A \subset B$. Can we always conclude that $A \subseteq B$?

b. If $A \subseteq B$, does it follow that $A \subset B$?

c. Is the empty set a proper subset of itself? Why?

Subsets and elements of sets are often confused. We say that $2 \in \{1, 2, 3\}$. But because 2 is not a set, we cannot substitute the symbol $\subseteq$ for $\in$. However, $\{2\} \subseteq \{1, 2, 3\}$ and $\{2\} \subset \{1, 2, 3\}$. Notice that $\{2\} \notin \{1, 2, 3\}$.

Inequalities: An Application of Set Concepts

The notion of proper subset and the concept of one-to-one correspondence can be used to define the concept of “less than” among natural numbers. The set $\{a, b, c\}$ has fewer elements than the set $\{w, x, y, z\}$ because when we try to pair the elements of the two sets, as in

$$\begin{array}{cccc} \{a, & b, & c\} & \\ \updownarrow & \updownarrow & \updownarrow & \\ \{x, & y, & z, & w\} \end{array}$$

we see that there is at least one element of the second set that is not paired with an element of the first set. The set $\{a, b, c\}$ is equivalent to a proper subset of the set $\{x, y, z, w\}$.

In general, *if A and B are finite sets, A has fewer elements than B if A is equivalent to a proper subset of B*. We say that $n(A)$ is **less than** $n(B)$ and write $n(A) < n(B)$. We say that b is **greater than** a, written $b > a$, if, and only if, $a < b$.

So, if A and B are finite sets and $A \subset B$, then A has fewer elements than B and it is not possible to find a one-to-one correspondence between the sets. Consequently, A and B are not equivalent. However, when both sets are infinite and $A \subset B$, the sets could be equivalent. For example, consider the set N of natural numbers and the set E of even natural numbers. $E \subset N$, but it is still possible to find a one-to-one correspondence between the sets. To do so, we correspond each number in set N to a number in set E that is twice as great. That is, $n \in N$ corresponds to $2n \in E$, as shown next.

$$\begin{array}{lcccccccc} N = \{1, & 2, & 3, & 4, & 5, & \ldots, & n, & \ldots\} \\ \quad\ \ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ E = \{2, & 4, & 6, & 8, & 10, & \ldots, & 2n, & \ldots\} \end{array}$$

Notice that in the correspondence, every element of N corresponds to a unique element in E and, conversely, every element of E corresponds to a unique element in N. For example, 11 in N corresponds to $2 \cdot 11$, or 22, in E. And 100 in E corresponds to $100 \div 2$, or 50, in N. Thus, $N \sim E$; that is, N and E are equivalent.

Students sometimes have difficulty with infinite sets and especially with their cardinal numbers, called **transfinite numbers**. As shown, E is a proper subset of N, but because they can be placed in a one-to-one correspondence, they are equivalent and have the same cardinal number. Georg Cantor was the first to introduce the concept of a transfinite number.

Problem Solving Passing a Senate Measure

A committee of senators consists of Abel, Baro, Carni, and Davis. Suppose each member of the committee has one vote and a simple majority is needed to either pass or reject any measure. A measure that is neither passed nor rejected is considered to be blocked and will be voted on again. Determine the number of ways a measure could be passed or rejected and the number of ways a measure could be blocked.

Understanding the Problem We are asked to determine how many ways the committee of four could pass or reject a measure and how many ways the committee of four could block a measure. To pass or reject a measure requires a winning coalition, that is, a group of senators who can pass or reject the measure, regardless of what the others do. To block a proposal, there must be a blocking coalition, that is, a group who can prevent any measure from passing but who cannot reject the measure.

Devising a Plan To solve the problem, we can *make a list* of subsets of the set of senators. Any subset of the set of senators with three or four members will form a winning coalition. Any subset of the set of senators with exactly two members will form a blocking coalition.

Carrying Out the Plan We list all subsets of the set $S = \{\text{Abel, Baro, Carni, Davis}\}$ that have at least three elements and all subsets that have exactly two elements. For ease, we identify the members as follows: A—Abel, B—Baro, C—Carni, D—Davis. All the subsets are given next:

$\varnothing$	$\{A\}$	$\{A, B\}$	$\{A, B, C\}$	$\{A, B, C, D\}$
	$\{B\}$	$\{A, C\}$	$\{A, B, D\}$	
	$\{C\}$	$\{A, D\}$	$\{A, C, D\}$	
	$\{D\}$	$\{B, C\}$	$\{B, C, D\}$	
		$\{B, D\}$		
		$\{C, D\}$		

There are five subsets with at least three members that can form a winning coalition to pass or reject a measure and six subsets with exactly two members that can block a measure.

Looking Back Other questions that might be considered include:

1. How many minimal winning coalitions are there? In other words, how many subsets are there of which no proper subset could pass a measure?
2. Devise a method to solve this problem without listing all subsets.
3. In "Carrying Out the Plan," 16 subsets of $\{A, B, C, D\}$ are listed. Use that result to systematically list all the subsets of a committee of five senators. Can you find the number of subsets of the 5-member committee without actually counting the subsets?

NOW TRY THIS 11

Suppose a committee of U.S. senators consists of five members.

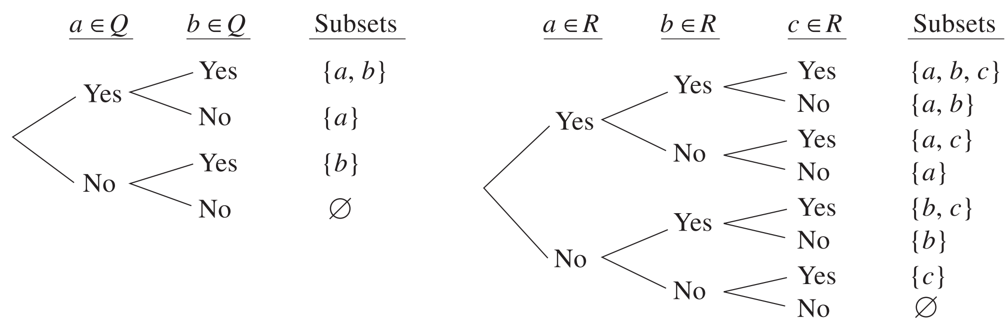
a. Compare the number of winning coalitions having exactly four members with the number of senators on the committee. What is the reason for the result?
b. Compare the number of winning coalitions having exactly three members with the number of subsets of the committee having exactly two members. What is the reason for the result?

Number of Subsets of a Finite Set

How many subsets can be made from a set containing n elements? To obtain a general formula, we use the strategy of *trying simpler cases* first.

1. If $P = \{a\}$, then P has two subsets, $\varnothing$ and $\{a\}$.
2. If $Q = \{a, b\}$, then Q has four subsets, $\varnothing$, $\{a\}$, $\{b\}$, and $\{a, b\}$.
3. If $R = \{a, b, c\}$, then R has eight subsets, $\varnothing$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, and $\{a, b, c\}$.

Methodically list all the subsets of a given set by using a tree diagram. For example, tree diagrams for the subsets of $Q = \{a, b\}$ and $R = \{a, b, c\}$ are given in Figure 15(a) and (b) respectively.



(a) (b)

Figure 15

Using the information from these cases, we *make a table* and *search for a pattern*, as in Table 9.

Table 9

Number of Elements	Number of Subsets
1	2, or 2^1
2	4, or 2^2
3	8, or 2^3
.	.
.	.
.	.

Table 9 suggests that for four elements, there might be 2^4, or 16, subsets. Is this correct? If $S = \{a, b, c, d\}$, then all the subsets of $R = \{a, b, c\}$ are also subsets of S. Eight new subsets are also formed by including the element d in each of the eight subsets of R. The eight new subsets are $\{d\}$, $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, and $\{a, b, c, d\}$. Thus, there are twice as many subsets of set S (with four elements) as there are of set R (with three elements). Consequently, there are $2 \cdot 8$, or 2^4, subsets of a set with four elements. Because including one more element in a finite set doubles the number of possible subsets of the new set, a set with five elements will have $2 \cdot 2^4$, or 2^5, subsets, and so on. In each case, the number of elements and the power of 2 used to obtain the number of subsets are equal. *Therefore, if there are n elements in a set, 2^n subsets can be formed.* If we apply this formula to the empty set—that is, when $n = 0$—then we have $2^0 = 1$. The pattern is meaningful because the empty set has only one subset—itself.

NOW TRY THIS 12

a. How many subsets does a set with five elements have?
b. How many proper subsets does a set with four elements have?
c. How many proper subsets does a set with n elements have?
d. How many subsets that include the element a does the set $\{a, b, c, d, e\}$ have?

Assessment 2A

1. Write the following sets using the listing (roster) method or using set-builder notation:
   a. The set of letters in the word *mathematics*
   b. The set of natural numbers greater than 20

2. Rewrite the following using mathematical symbols:
   a. P is equal to the set containing a, b, c, and d.
   b. The set consisting of the elements 1 and 2 is a proper subset of $\{1, 2, 3, 4\}$.
   c. The set consisting of the elements 0 and 1 is not a subset of $\{1, 2, 3, 4\}$.
   d. 0 is not an element of the empty set.

3. Which of the following pairs of sets can be placed in one-to-one correspondence?
   a. $\{1, 2, 3, 4, 5\}$ and $\{m, n, o, p, q\}$
   b. $\{a, b, c, d, e, f, \ldots, m\}$ and $\{1, 2, 3, 4, 5, 6, \ldots, 13\}$
   c. $\{x \mid x$ is a letter in the word *mathematics*$\}$ and $\{1, 2, 3, 4, \ldots, 11\}$

4. How many one-to-one correspondences are there between two sets with
   a. 6 elements each?
   b. n elements each?

5. How many one-to-one correspondences are there between the sets $\{x, y, z, u, v\}$ and $\{1, 2, 3, 4, 5\}$ if in each correspondence
   a. x must correspond to 5?
   b. x must correspond to 5 and y to 1?
   c. x, y, and z must correspond to odd numbers?

6. Which of the following represent equal sets?

$A = \{a, b, c, d\}$ $\quad B = \{x, y, z, w\}$
$C = \{c, d, a, b\}$ $\quad D = \{x \mid 1 \leq x \leq 4,\ x \in N\}$
$E = \varnothing$ $\quad F = \{\varnothing\}$
$G = \{0\}$ $\quad H = \{\ \}$
$I = \{x \mid x = 2n + 1, \text{ and } n \in \{0, 1, 2, 3, \ldots\}$
$L = \{x \mid x = 2n - 1, n \in N\}$

7. Find the cardinal number of each of the following sets. Assume the pattern continues in each part:
   a. $\{101, 102, 103, \ldots, 1100\}$
   b. $\{1, 3, 5, \ldots, 1001\}$
   c. $\{1, 2, 4, 8, 16, \ldots, 1024\}$
   d. $\{x \mid x = k^2, k = 1, 2, 3, \ldots, 100\}$
   e. $\{i + j \mid i \in \{1, 2, 3\} \text{ and } j \in \{1, 2, 3\}\}$

8. If U is the set of all college students and A is the set of all college students with a straight-A average, describe $\overline{A}$.

9. Suppose B is a proper subset of C.
   a. If $n(C) = 8$, what is the maximum number of elements in B?
   b. What is the least possible number of elements in B?

10. Suppose C is a subset of D and D is a subset of C.
   a. If $n(C) = 5$, find $n(D)$.
   b. What other relationship exists between sets C and D?

11. If $A = \{a, b, c, d, e\}$,
   a. how many subsets does A have?
   b. how many proper subsets does A have?
   c. how many subsets does A have that include the elements a and e?

12. If a set has 255 proper subsets, how many elements are in the set?

13. Identify all the possible proper subset relationships that occur among the following sets:
$A = \{3n \mid n \in N\}$, $B = \{6n \mid n \in N\}$,
$C = \{12n \mid n \in N\}$.

14. Indicate which symbol, $\in$ or $\notin$, makes each of the following statements true:
   a. 0 ______ $\varnothing$
   b. $\{1\}$ ______ $\{1, 2\}$
   c. 1024 ______ $\{x \mid x = 2^n, n \in N\}$
   d. 3002 ______ $\{x \mid x = 3n - 1, n \in N\}$

15. Indicate which symbol, $\subseteq$ or $\nsubseteq$, makes each part of problem 14 true.

16. Answer each of the following. If your answer is *no*, tell why.
   a. If $A = B$, can we always conclude that $A \subseteq B$?
   b. If $A \subseteq B$, can we always conclude that $A \subset B$?
   c. If $A \subset B$, can we always conclude that $A \subseteq B$?
   d. If $A \subseteq B$, can we always conclude that $A = B$?

17. Use the definition of *less than* to show each of the following:
   a. $3 < 100$ $\quad$ **b.** $0 < 3$

18. On a certain senate committee there are seven senators: Abel, Brooke, Cox, Dean, Eggers, Funk, and Gage. Three of these members are to be appointed to a subcommittee. How many possible subcommittees are there?

19. How many two-digit numbers in base ten can be formed if the tens digit cannot be 0 and no digit can be repeated?

Assessment 2B

1. Write the following sets using the listing (roster) method or set-builder notation:
   a. the set of letters in the word *geometry*
   b. the set of natural numbers greater than 7
2. Rewrite the following using mathematical symbols:
   a. Q is equal to the set whose elements are a, b, and c.
   b. The set containing 1 and 3 is equal to the set containing 3 and 1.
   c. The set containing 1 and 3 only is not a proper subset of $\{1, 4, 6, 7\}$
   d. The empty set does not contain 0 as an element.
3. Which of the following pairs of sets can be placed in a one-to-one correspondence?
   a. $\{1, 2, 3, 4\}$ and $\{w, c, y, z\}$
   b. $\{1, 2, 3, \ldots, 25\}$ and $\{a, b, c, d, \ldots, x, y\}$
   c. $\{x \mid x$ is a letter in the word *geometry*$\}$ and $\{1, 2, 3, 4, 5, 6, 7, 8\}$
4. How many one-to-one correspondences exist between two sets with
   a. 8 elements each?
   b. $n - 1$ elements each?
5. How many one-to-one correspondences are there between the sets $\{a, b, c, d\}$ and $\{1, 2, 3, 4\}$ if in each correspondence
   a. b must correspond to 3?
   b. b must correspond to 3 and d to 4?
   c. a and c must correspond to even numbers?
6. Which of the following represent equal sets?
   $A = \{a, b, c\}$ $\quad B = \{x, y\}$
   $C = \{c, a, b\}$ $\quad D = \{x \mid 1 \leq x \leq 3, x \in N\}$
   $I = \{x \mid x = 2n$, and $n \in \{0, 1, 2, 3, \ldots\}$,
   $K = \{2, 4, 6, 8, 10, 12, \ldots\}$
   $L = \{x \mid x = 2n - 1, n \in N\}$
7. Find the cardinal number of each of the following sets. Assume the pattern continues in each part:
   a. $\{9, 10, 11, \ldots, 99\}$
   b. $\{2, 4, 6, 8, \ldots, 2002\}$
   c. $\{x^2 \mid x = 1, 3, 5, 7, \ldots, 99\}$
   d. $\{x \mid x = x + 1, x \in N\}$
8. If U is the set of all women and G is the set of alumnae of Georgia State University, describe $\overline{G}$.
9. Suppose $A \subseteq B$.
   a. What is the minimum number of elements in set A?
   b. Is it possible for set B to be the empty set? If so, give an example of sets A and B satisfying this. If not, explain why not.
10. If two sets are subsets of each other, what other relationships must they have?
11. If $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$,
    a. how many subsets does A have?
    b. how many proper subsets does A have?
12. If a set has 16 subsets, how many elements are in the set?
13. Identify all possible proper subset relationships that occur among the following sets:
    $A = \{3n + 1 \mid n \in N\}$, $B = \{6n + 1 \mid n \in N\}$,
    $C = \{12n + 1 \mid n \in N\}$
14. Indicate which symbol, $\in$ or $\notin$, makes each of the following statements true:
    a. $\varnothing$ ______ $\varnothing$
    b. $\{2\}$ ______ $\{3, 2, 1\}$
    c. 1022 ______ $\{s \mid s = 2^n - 2, n \in N\}$
    d. 3004 ______ $\{x \mid x = 3n + 1, n \in N\}$
15. Indicate which symbol, $\subseteq$ or $\nsubseteq$, makes each part of problem 14 true.
16. Answer each of the following. If your answer is *no*, tell why.
    a. If $A \subseteq B$, can we always conclude that $A = B$?
    b. If $A \subset B$, can we conclude that $A = B$?
    c. If A and B can be placed in a one-to-one correspondence, must $A = B$?
    d. If A and B can be placed in a one-to-one correspondence, must $A \subseteq B$?
17. Use the definition of *less than* to show each of the following:
    a. $0 < 2$
    b. $99 < 100$
18. How many ways are there to stack an ice-cream cone with 4 scoops if the choices are
    a. vanilla, chocolate, rhubarb, and strawberry and each scoop must be different?
    b. vanilla, chocolate, rhubarb, and strawberry and there are no restrictions on different scoops?
19. How many seven-digit numbers are there when 0 and 1 cannot be the leading number?

Mathematical Connections 2

Communication

1. Explain the difference between a well-defined set and one that is not. Give examples.
2. Which of the following sets are not well defined? Explain.
   a. The set of wealthy schoolteachers
   b. The set of great books
   c. The set of natural numbers greater than 100
   d. The set of subsets of $\{1, 2, 3, 4, 5, 6\}$
   e. The set $\{x \mid x \neq x, x \in N\}$
3. Is $\varnothing$ a proper subset of every non-empty set? Explain your reasoning.
4. Explain why $\{\varnothing\}$ has $\varnothing$ as an element and also as a subset.
5. Tell how you would show that $A \nsubseteq B$.
6. Explain why every set is a subset of itself.
7. Define *less than or equal to* in a way similar to the definition of *less than*.

Open-Ended

8. a. Give three examples of sets A and B and a universal set U such that $A \subset B$; find $\overline{A}$ and $\overline{B}$.
   b. Based on your observations, conjecture a relationship between $\overline{B}$ and $\overline{A}$.
   c. Demonstrate your conjecture in (b) using a Venn diagram.
9. Find an infinite set A such that
   a. $\overline{A}$ is finite.
   b. $\overline{A}$ is infinite.
10. Describe two sets from real-life situations such that it is clear from using one-to-one correspondence, and not from counting, that one set has fewer elements than the other.

Cooperative Learning

11. Assume the fastest computer can list one subset in approximately 1 microsecond (one-millionth of a second).
   a. Use a calculator if necessary to estimate the time in years it would take a computer to list all the subsets of $\{1, 2, 3, \ldots, 64\}$.
   b. Estimate the time in years it would take the computer to exhibit all the one-to-one correspondences between the sets $\{1, 2, 3, \ldots, 64\}$ and $\{65, 66, 67, \ldots, 128\}$.

Questions from the Classroom

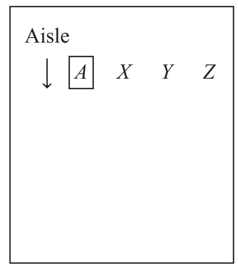
12. A student argues that $\{\varnothing\}$ is the proper notation for the empty set. What is your response?
13. A student asks if $A \subseteq B$ and $B \subseteq C$, do we know $A \subseteq C$? How do you answer?
14. A student argues that $A = \{1, \{1\}\}$ has only one element. How do you respond?
15. A student states that either $A \subseteq B$ or $B \subseteq A$. Is the student correct?

Review Problems

16. Write 5280 in expanded form.
17. What is the value of MCDX in Hindu-Arabic numerals?
18. Convert each of the following to base ten:
   a. $E0T_{\text{twelve}}$
   b. 1011_{two}
   c. 43_{five}
19. Write $12^4 + 12^2 + 13$ in base twelve.

National Assessment of Educational Progress (NAEP) Question

Four people—A, X, Y, and Z—go to a movie and sit in adjacent seats. If A sits in the aisle seat, list all possible arrangements of the other three people. One of the arrangements is shown below.



NAEP, Grade 12, 1996

Source: National Assessment of Education Progress (NAEP), IES National Center for Education Statistics, 2011.

BRAIN TEASER

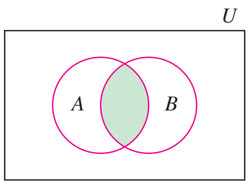
Mr. Gonzales's and Ms. Chan's seventh-grade classes in Paxson Middle School have 24 and 25 students, respectively. Linda, a student in Mr. Gonzales's class, claims that the number of school committees that could be formed to contain at least one student from each class is greater than the number of people in the world. Assuming that a committee can have up to 49 students, find the number of committees and determine whether Linda is right.

3 Other Set Operations and Their Properties

Finding the complement of a set is an operation that acts on only one set at a time. In this section, we consider operations that act on two sets at a time.

Set Intersection

Suppose that during the fall quarter, a college wants to mail a survey to all its students who are enrolled in both art and biology classes. To do this, the school officials must identify those students who are taking both classes. If A and B are the set of students taking art courses and the set of students taking biology courses, respectively, during the fall quarter, then the desired set



$A \cap B$

Figure 16

of students includes those common to A and B, or the **intersection** of A and B. The intersection of sets A and B is the shaded region in Figure 16. Figure 16 depicts the possibility of A and B containing common elements. The intersection might contain no elements. Notice that for all sets A and B, $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

Definition of Set Intersection

The **intersection** of two sets A and B, written $A \cap B$, is the set of all elements common to both A and B; $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

The key word in the definition of *intersection* is *and*. In everyday language, as in mathematics, *and* implies that both conditions must be met. In the example, the desired set is the set of those students enrolled in both art and biology.

If sets such as A and B have no elements in common, they are **disjoint sets**. In other words, two sets A and B are disjoint if, and only if, $A \cap B = \varnothing$. For example, if there are no students that are taking both art (A) and biology (B), then the sets are disjoint. Note that the Venn diagram for this situation could be drawn as in Figure 17.

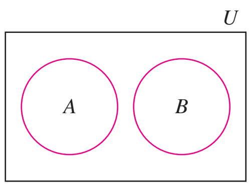


Figure 17

EXAMPLE 11

Find $A \cap B$ in each of the following:

a. $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$
b. $A = \{0, 2, 4, 6, \ldots\}$, $B = \{1, 3, 5, 7, \ldots\}$
c. $A = \{2, 4, 6, 8, \ldots\}$, $B = \{1, 2, 3, 4, \ldots\}$

Solution

a. $A \cap B = \{3, 4\}$.
b. $A \cap B = \varnothing$; therefore A and B are disjoint.
c. $A \cap B = A$ because all the elements of A are also in B.

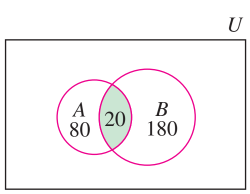
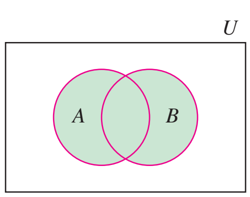


Figure 18

If A represents all students enrolled in art classes and B all students enrolled in biology classes, we may use a Venn diagram, taking into account that some students are enrolled in both subjects. If we know that 100 students are enrolled in art and 200 in biology and that 20 of these students are enrolled in both art and biology, then $100 - 20$, or 80, students are enrolled in art but not in biology and $200 - 20$, or 180, are enrolled in biology but not art. We can record this information as in Figure 18. Notice that the total number of students in set A is 100 and the total in set B is 200.

Set Union

If A is the set of students taking art courses during the fall quarter and B is the set of students taking biology courses during the fall quarter, then the set of students taking art or biology or both during the fall quarter is the **union** of sets A and B. The union of sets A and B is pictured in Figure 19. Notice that for all sets A and B, $A \subseteq A \cup B$, $B \subseteq A \cup B$, and $A \cap B \subseteq A \cup B$.



$A \cup B$

Figure 19

Definition of Set Union

The **union** of two sets A and B, written $A \cup B$, is the set of all elements in A or in B; $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

The key word in the definition of *union* is *or*. In mathematics, *or* usually means "one or the other or both." This is known as the *inclusive or*

EXAMPLE 12

Find $A \cup B$ for each of the following:

a. $A = \{1, 2, 3, 4\},\ B = \{3, 4, 5, 6\}$
b. $A = \{0, 2, 4, 6, \ldots\},\ B = \{1, 3, 5, 7, \ldots\}$
c. $A = \{2, 4, 6, 8, \ldots\},\ B = \{1, 2, 3, 4, \ldots\}$

Solution

a. $A \cup B = \{1, 2, 3, 4, 5, 6\}$. **b.** $A \cup B = \{0, 1, 2, 3, 4, \ldots\}$.
c. Because every element of A is already in B, we have $A \cup B = B$.

EXAMPLE 13

Find each of the following if $A = \{a, b, c\}$.

a. $A \cap \varnothing$ **b.** $A \cup \varnothing$ **c.** $\varnothing \cap \varnothing$ **d.** $\varnothing \cup \varnothing$

Solution

a. $\varnothing$ **b.** A **c.** $\varnothing$ **d.** $\varnothing$

NOW TRY THIS 13

In Figure 18, $n(A \cup B) = 80 + 20 + 180 = 280$, but $n(A) + n(B) = 100 + 200 = 300$; hence in general, $n(A \cup B) \neq n(A) + n(B)$. Use the concept of intersection of sets to write a formula for $n(A \cup B)$.

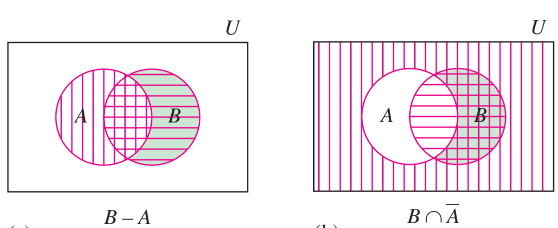
Set Difference

During the fall quarter if A is the set of students taking art classes and B is the set of students taking biology classes, then the set of all students taking biology but not art is called the **complement of A relative to B**, or the **set difference** of B and A.

Definition of Relative Complement

The **complement of A relative to B**, written $B - A$, is the set of all elements in B that are not in A; $B - A = \{x \mid x \in B \text{ and } x \notin A\}$.

Note that $B - A$ is not read as "B minus A." The *minus* sign indicates the subtraction operation on numbers and *set difference* is an operation on sets. A Venn diagram representing $B - A$ is shown in Figure 20(a). The shaded region represents all the elements that are in B but not in A. A Venn diagram for $B \cap \overline{A}$ is given in Figure 20(b). The shaded region represents all the elements that are in B and in $\overline{A}$. Notice that $B \cap \overline{A} = B - A$ because $B \cap \overline{A}$ is, by definitions of intersection and complement, the set of all elements in B and not in A.



(a) (b)

Figure 20

EXAMPLE 14

If $A = \{d, e, f\}$, $B = \{a, b, c, d, e, f\}$, and $C = \{a, b, c\}$, find or answer each of the following:

a. $A - B$
b. $B - A$
c. $B - C$
d. $C - B$
e. To answer parts (a)–(d), does it matter what the universal set is?

Solution

a. $A - B = \varnothing$
b. $B - A = \{a, b, c\}$
c. $B - C = \{d, e, f\}$
d. $C - B = \varnothing$
e. Parts (a)–(d) can be answered independently of the universal set. The definition of set difference relates one set to another, independent of the universal set.

Properties of Set Operations

Because the order of elements in a set is not important, $A \cup B$ is equal to $B \cup A$. It does not matter in which order we write the sets when the union of two sets is involved. Similarly, $A \cap B = B \cap A$. These properties are stated formally next.

Theorem 2: Commutative Property of Set Intersection and Commutative Property of Set Union

For all sets A and B, $A \cap B = B \cap A$ is the **commutative property of set intersection**. Similarly, $A \cup B = B \cup A$ is the **commutative property of set union**.

NOW TRY THIS 14

Use Venn diagrams and other means to find whether grouping is important when the same operation is involved. For example, is it always true that $A \cap (B \cap C) = (A \cap B) \cap C$? Similar questions should be investigated involving union and set difference.

In answering Now Try This 14, the following properties become evident:

Theorem 3: Associative Property of Set Intersection and Associative Property of Set Union

For all sets A, B, and C, $A \cap (B \cap C) = (A \cap B) \cap C$ is the **associative property of set intersection.** Similarly, $A \cup (B \cup C) = (A \cup B) \cup C$ is the **associative property of set union**.

EXAMPLE 15

Is grouping important when two different set operations are involved? For example, is it true that $A \cap (B \cup C) = (A \cap B) \cup C$?

Solution

To investigate this, let $A = \{a, b, c\}$, $B = \{c, d\}$, and $C = \{d, e, f\}$. Then

$$\begin{aligned} A \cap (B \cup C) &= \{a, b, c\} \cap (\{c, d\} \cup \{d, e, f\}) \\ &= \{a, b, c\} \cap \{c, d, e, f\} \\ &= \{c\} \end{aligned}$$