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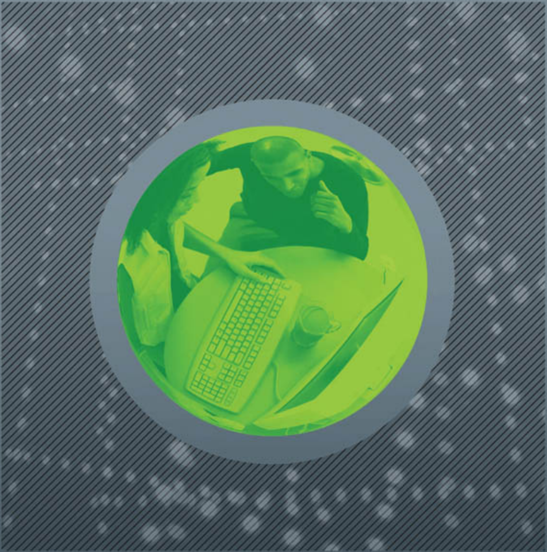

Discrete and Combinatorial Mathematics
An Applied Introduction

Ralph Grimaldi

Fifth Edition

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Fundamental Principles of Counting

Enumeration, or counting, may strike one as an obvious process that a student learns when first studying arithmetic. But then, it seems, very little attention is paid to further development in counting as the student turns to “more difficult” areas in mathematics, such as algebra, geometry, trigonometry, and calculus. Consequently, this first chapter should provide some warning about the seriousness and difficulty of “mere” counting.

Enumeration does not end with arithmetic. It also has applications in such areas as coding theory, probability and statistics, and in the analysis of algorithms.

As we enter this fascinating field of mathematics, we shall come upon many problems that are very simple to state but somewhat “sticky” to solve. Thus, be sure to learn and understand the basic formulas — but do *not* rely on them too heavily. For without an analysis of each problem, a mere knowledge of formulas is next to useless. Instead, welcome the challenge to solve unusual problems or those that are different from problems you have encountered in the past. Seek solutions based on your own scrutiny, regardless of whether it reproduces what the author provides. There are often several ways to solve a given problem.

1 The Rules of Sum and Product

Our study of discrete and combinatorial mathematics begins with two basic principles of counting: the rules of sum and product. The statements and initial applications of these rules appear quite simple. In analyzing more complicated problems, one is often able to break down such problems into parts that can be solved using these basic principles. We want to develop the ability to “decompose” such problems and piece together our partial solutions in order to arrive at the final answer. A good way to do this is to analyze and solve many diverse enumeration problems, taking note of the principles being used. This is the approach we shall follow here.

Our first principle of counting can be stated as follows:

The Rule of Sum: If a first task can be performed in m ways, while a second task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any one of $m + n$ ways.

Note that when we say that a particular occurrence, such as a first task, can come about in m ways, these m ways are assumed to be distinct, unless a statement is made to the contrary. This will be true throughout the entire text.

EXAMPLE 1

A college library has 40 textbooks on sociology and 50 textbooks dealing with anthropology. By the rule of sum, a student at this college can select among $40 + 50 = 90$ textbooks in order to learn more about one or the other of these two subjects.

EXAMPLE 2

The rule can be extended beyond two tasks as long as no pair of tasks can occur simultaneously. For instance, a computer science instructor who has, say, seven different introductory books each on C++, Java, and Perl can recommend any one of these 21 books to a student who is interested in learning a first programming language.

EXAMPLE 3

The computer science instructor of Example 2 has two colleagues. One of these colleagues has three textbooks on the analysis of algorithms, and the other has five such textbooks. If n denotes the maximum number of different books on this topic that this instructor can borrow from them, then $5 \leq n \leq 8$, for here both colleagues *may* own copies of the same textbook(s).

The following example introduces our second principle of counting.

EXAMPLE 4

In trying to reach a decision on plant expansion, an administrator assigns 12 of her employees to two committees. Committee A consists of five members and is to investigate possible favorable results from such an expansion. The other seven employees, committee B, will scrutinize possible unfavorable repercussions. Should the administrator decide to speak to just one committee member before making her decision, then by the rule of sum there are 12 employees she can call upon for input. However, to be a bit more unbiased, she decides to speak with a member of committee A on Monday, and then with a member of committee B on Tuesday, before reaching a decision. Using the following principle, we find that she can select two such employees to speak with in $5 \times 7 = 35$ ways.

The Rule of Product: If a procedure can be broken down into first and second stages, and if there are m possible outcomes for the first stage and if, for each of these outcomes, there are n possible outcomes for the second stage, then the total procedure can be carried out, in the designated order, in mn ways.

EXAMPLE 5

The drama club of Central University is holding tryouts for a spring play. With six men and eight women auditioning for the leading male and female roles, by the rule of product the director can cast his leading couple in $6 \times 8 = 48$ ways.

EXAMPLE 6

Here various extensions of the rule are illustrated by considering the manufacture of license plates consisting of two letters followed by four digits.

a) If no letter or digit can be repeated, there are $26 \times 25 \times 10 \times 9 \times 8 \times 7 = 3{,}276{,}000$ different possible plates.

b) With repetitions of letters and digits allowed, $26 \times 26 \times 10 \times 10 \times 10 \times 10 = 6{,}760{,}000$ different license plates are possible.

c) If repetitions are allowed, as in part (b), how many of the plates have only vowels (A, E, I, O, U) and even digits? (0 is an even integer.)

EXAMPLE 7

In order to store data, a computer's main memory contains a large collection of circuits, each of which is capable of storing a *bit* — that is, one of the *bi*nary dig*its* 0 or 1. These storage circuits are arranged in units called (memory) cells. To identify the cells in a computer's main memory, each is assigned a unique name called its *address*. For some computers, such as embedded microcontrollers (as found in the ignition system for an automobile), an address is represented by an ordered list of eight bits, collectively referred to as a *byte*. Using the rule of product, there are $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8 = 256$ such bytes. So we have 256 addresses that may be used for cells where certain information may be stored.

A kitchen appliance, such as a microwave oven, incorporates an embedded microcontroller. These "small computers" (such as the PICmicro microcontroller) contain thousands of memory cells and use two-byte addresses to identify these cells in their main memory. Such addresses are made up of two consecutive bytes, or 16 consecutive bits. Thus there are $256 \times 256 = 2^8 \times 2^8 = 2^{16} = 65{,}536$ available addresses that could be used to identify cells in the main memory. Other computers use addressing systems of four bytes. This 32-bit architecture is presently used in the Pentium[†] processor, where there are as many as $2^8 \times 2^8 \times 2^8 \times 2^8 = 2^{32} = 4{,}294{,}967{,}296$ addresses for use in identifying the cells in main memory. When a programmer deals with the UltraSPARC[‡] or Itanium[§] processors, he or she considers memory cells with eight-byte addresses. Each of these addresses comprises $8 \times 8 = 64$ bits, and there are $2^{64} = 18{,}446{,}744{,}073{,}709{,}551{,}616$ possible addresses for this architecture. (Of course, not all of these possibilities are actually used.)

EXAMPLE 8

At times it is necessary to combine several different counting principles in the solution of one problem. Here we find that the rules of both sum and product are needed to attain the answer.

At the AWL corporation Mrs. Foster operates the Quick Snack Coffee Shop. The menu at her shop is limited: six kinds of muffins, eight kinds of sandwiches, and five beverages (hot coffee, hot tea, iced tea, cola, and orange juice). Ms. Dodd, an editor at AWL, sends her assistant Carl to the shop to get her lunch — either a muffin and a hot beverage or a sandwich and a cold beverage.

By the rule of product, there are $6 \times 2 = 12$ ways in which Carl can purchase a muffin and hot beverage. A second application of this rule shows that there are $8 \times 3 = 24$ possibilities for a sandwich and cold beverage. So by the rule of sum, there are $12 + 24 = 36$ ways in which Carl can purchase Ms. Dodd's lunch.

[†] Pentium (R) is a registered trademark of the Intel Corporation.
[‡] The UltraSPARC processor is manufactured by Sun (R) Microsystems, Inc.
[§] Itanium (TM) is a trademark of the Intel Corporation.

2
Permutations

Continuing to examine applications of the rule of product, we turn now to counting linear arrangements of objects. These arrangements are often called *permutations* when the objects are distinct. We shall develop some systematic methods for dealing with linear arrangements, starting with a typical example.

EXAMPLE 9

In a class of 10 students, five are to be chosen and seated in a row for a picture. How many such linear arrangements are possible?

The key word here is *arrangement*, which designates the importance of *order*. If A, B, C, . . . , I, J denote the 10 students, then BCEFI, CEFIB, and ABCFG are three such different arrangements, even though the first two involve the same five students.

To answer this question, we consider the positions and possible numbers of students we can choose from in order to fill each position. The filling of a position is a stage of our procedure.

$$\underset{\text{1st position}}{10} \times \underset{\text{2nd position}}{9} \times \underset{\text{3rd position}}{8} \times \underset{\text{4th position}}{7} \times \underset{\text{5th position}}{6}$$

Each of the 10 students can occupy the 1st position in the row. Because repetitions are not possible here, we can select only one of the nine remaining students to fill the 2nd position. Continuing in this way, we find only six students to select from in order to fill the 5th and final position. This yields a total of 30,240 possible arrangements of five students selected from the class of 10.

Exactly the same answer is obtained if the positions are filled from right to left—namely, $6 \times 7 \times 8 \times 9 \times 10$. If the 3rd position is filled first, the 1st position second, the 4th position third, the 5th position fourth, and the 2nd position fifth, then the answer is $9 \times 6 \times 10 \times 8 \times 7$, still the same value, 30,240.

As in Example 9, the product of certain consecutive positive integers often comes into play in enumeration problems. Consequently, the following notation proves to be quite useful when we are dealing with such counting problems. It will frequently allow us to express our answers in a more convenient form.

Definition 1

For an integer $n \geq 0$, *n factorial* (denoted $n!$) is defined by

$$0! = 1,$$
$$n! = (n)(n-1)(n-2)\cdots(3)(2)(1), \quad \text{for} \quad n \geq 1.$$

One finds that $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, and $5! = 120$. In addition, for each $n \geq 0$, $(n+1)! = (n+1)(n!)$.

Before we proceed any further, let us try to get a somewhat better appreciation for how fast $n!$ grows. We can calculate that $10! = 3{,}628{,}800$, and it just so happens that this is exactly the number of *seconds* in six *weeks*. Consequently, 11! exceeds the number of seconds in one *year*, 12! exceeds the number in 12 years, and 13! surpasses the number of seconds in a *century*.

If we make use of the factorial notation, the answer in Example 9 can be expressed in the following more compact form:

$$10 \times 9 \times 8 \times 7 \times 6 = 10 \times 9 \times 8 \times 7 \times 6 \times \frac{5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = \frac{10!}{5!}.$$

Definition 2 Given a collection of n distinct objects, any (linear) arrangement of these objects is called a *permutation* of the collection.

Starting with the letters a, b, c, there are six ways to arrange, or permute, all of the letters: abc, acb, bac, bca, cab, cba. If we are interested in arranging only two of the letters at a time, there are six such size-2 permutations: ab, ba, ac, ca, bc, cb.

If there are n distinct objects and r is an integer, with $1 \leq r \leq n$, then by the rule of product, the number of permutations of size r for the n objects is

$$\begin{aligned} P(n, r) &= \underset{\text{1st position}}{n} \times \underset{\text{2nd position}}{(n-1)} \times \underset{\text{3rd position}}{(n-2)} \times \cdots \times \underset{r\text{th position}}{(n-r+1)} \\ &= (n)(n-1)(n-2)\cdots(n-r+1) \times \frac{(n-r)(n-r-1)\cdots(3)(2)(1)}{(n-r)(n-r-1)\cdots(3)(2)(1)} \\ &= \frac{n!}{(n-r)!}. \end{aligned}$$

For $r = 0$, $P(n, 0) = 1 = n!/(n-0)!$, so $P(n, r) = n!/(n-r)!$ holds for all $0 \leq r \leq n$. A special case of this result is Example 9, where $n = 10$, $r = 5$, and $P(10, 5) = 30{,}240$. When permuting all of the n objects in the collection, we have $r = n$ and find that $P(n, n) = n!/0! = n!$.

Note, for example, that if $n \geq 2$, then $P(n, 2) = n!/(n-2)! = n(n-1)$. When $n > 3$ one finds that $P(n, n-3) = n!/[n-(n-3)]! = n!/3! = (n)(n-1)(n-2)\cdots(5)(4)$.

The number of permutations of size r, where $0 \leq r \leq n$, from a collection of n objects, is $P(n, r) = n!/(n-r)!$. (Remember that $P(n, r)$ counts (linear) arrangements in which the objects can*not* be repeated.) However, if repetitions are allowed, then by the rule of product there are n^r possible arrangements, with $r \geq 0$.

EXAMPLE 10 The number of permutations of the letters in the word COMPUTER is 8!. If only five of the letters are used, the number of permutations (of size 5) is $P(8, 5) = 8!/(8-5)! = 8!/3! = 6720$. If repetitions of letters are allowed, the number of possible 12-letter sequences is $8^{12} \doteq 6.872 \times 10^{10}$.†

EXAMPLE 11 Unlike Example 10, the number of (linear) arrangements of the four letters in BALL is 12, not 4! (= 24). The reason is that we do not have four distinct letters to arrange. To get the 12 arrangements, we can list them as in Table 1(a).

†The symbol "$\doteq$" is read "is approximately equal to."

Table 1

A	B	L	L	A	B	L_1	L_2	A	B	L_2	L_1
A	L	B	L	A	L_1	B	L_2	A	L_2	B	L_1
A	L	L	B	A	L_1	L_2	B	A	L_2	L_1	B
B	A	L	L	B	A	L_1	L_2	B	A	L_2	L_1
B	L	A	L	B	L_1	A	L_2	B	L_2	A	L_1
B	L	L	A	B	L_1	L_2	A	B	L_2	L_1	A
L	A	B	L	L_1	A	B	L_2	L_2	A	B	L_1
L	A	L	B	L_1	A	L_2	B	L_2	A	L_1	B
L	B	A	L	L_1	B	A	L_2	L_2	B	A	L_1
L	B	L	A	L_1	B	L_2	A	L_2	B	L_1	A
L	L	A	B	L_1	L_2	A	B	L_2	L_1	A	B
L	L	B	A	L_1	L_2	B	A	L_2	L_1	B	A

(a) **(b)**

If the two L's are distinguished as L_1, L_2, then we can use our previous ideas on permutations of distinct objects; with the four distinct symbols B, A, L_1, L_2, we have $4! = 24$ permutations. These are listed in Table 1(b). Table 1 reveals that for each arrangement in which the L's are indistinguishable there corresponds a *pair* of permutations with distinct L's. Consequently,

$$2 \times (\text{Number of arrangements of the letters B, A, L, L})$$
$$= (\text{Number of permutations of the symbols B, A, } L_1, L_2),$$

and the answer to the original problem of finding all the arrangements of the four letters in BALL is $4!/2 = 12$.

EXAMPLE 12

Using the idea developed in Example 11, we now consider the arrangements of all nine letters in DATABASES.

There are $3! = 6$ arrangements with the A's distinguished for each arrangement in which the A's are not distinguished. For example, $DA_1TA_2BA_3SES$, $DA_1TA_3BA_2SES$, $DA_2TA_1BA_3SES$, $DA_2TA_3BA_1SES$, $DA_3TA_1BA_2SES$, and $DA_3TA_2BA_1SES$ all correspond to DATABASES, when we remove the subscripts on the A's. In addition, to the arrangement $DA_1TA_2BA_3SES$ there corresponds the pair of permutations $DA_1TA_2BA_3S_1ES_2$ and $DA_1TA_2BA_3S_2ES_1$, when the S's are distinguished. Consequently,

$$(2!)(3!)(\text{Number of arrangements of the letters in DATABASES})$$
$$= (\text{Number of permutations of the symbols D, } A_1, \text{T}, A_2, \text{B}, A_3, S_1, \text{E}, S_2),$$

so the number of arrangements of the nine letters in DATABASES is $9!/(2!\,3!) = 30{,}240$.

Before stating a general principle for arrangements with repeated symbols, note that in our prior two examples we solved a new type of problem by relating it to previous enumeration principles. This practice is common in mathematics in general, and often occurs in the derivations of discrete and combinatorial formulas.

If there are n objects with n_1 indistinguishable objects of a first type, n_2 indistinguishable objects of a second type, . . . , and n_r indistinguishable objects of an rth type, where $n_1 + n_2 + \cdots + n_r = n$, then there are $\dfrac{n!}{n_1!\, n_2! \cdots n_r!}$ (linear) arrangements of the given n objects.

EXAMPLE 13

The MASSASAUGA is a brown and white venomous snake indigenous to North America. Arranging all of the letters in MASSASAUGA, we find that there are

$$\frac{10!}{4!\,3!\,1!\,1!\,1!} = 25{,}200$$

possible arrangements. Among these are

$$\frac{7!}{3!\,1!\,1!\,1!\,1!} = 840$$

in which all four A's are together. To get this last result, we considered all arrangements of the seven symbols AAAA (one symbol), S, S, S, M, U, G.

EXAMPLE 14

Determine the number of (staircase) paths in the xy-plane from (2, 1) to (7, 4), where each such path is made up of individual steps going one unit to the right (R) or one unit upward (U). The blue lines in Fig. 1 show two of these paths.

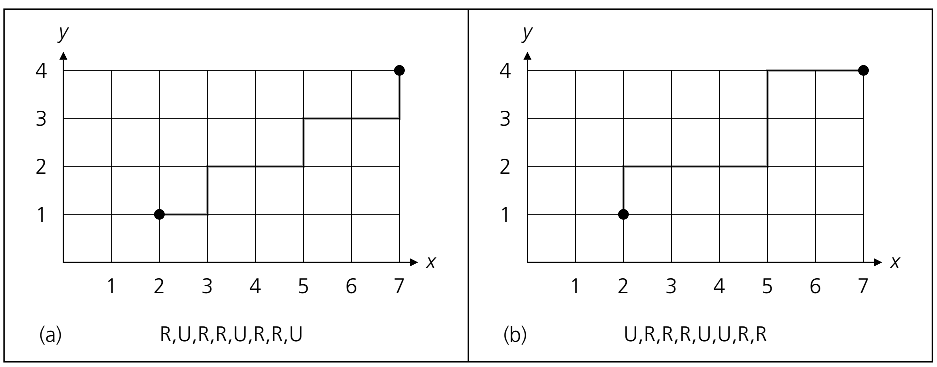

Figure 1

Beneath each path in Fig. 1 we have listed the individual steps. For example, in part (a) the list R, U, R, R, U, R, R, U indicates that starting at the point (2, 1), we first move one unit to the right [to (3, 1)], then one unit upward [to (3, 2)], followed by two units to the right [to (5, 2)], and so on, until we reach the point (7, 4). The path consists of five R's for moves to the right and three U's for moves upward.

The path in part (b) of the figure is also made up of five R's and three U's. In general, the overall trip from (2, 1) to (7, 4) requires $7 - 2 = 5$ horizontal moves to the right and $4 - 1 = 3$ vertical moves upward. Consequently, each path corresponds to a list of five R's and three U's, and the solution for the number of paths emerges as the number of arrangements of the five R's and three U's, which is $8!/(5!\,3!) = 56$.

EXAMPLE 15

We now do something a bit more abstract and prove that if n and k are positive integers with $n = 2k$, then $n!/2^k$ is an integer. Because our argument relies on counting, it is an example of a *combinatorial proof*.

Consider the n symbols $x_1, x_1, x_2, x_2, \ldots, x_k, x_k$. The number of ways in which we can arrange all of these $n = 2k$ symbols is an integer that equals

$$\frac{n!}{\underbrace{2!\,2!\cdots 2!}_{k \text{ factors of } 2!}} = \frac{n!}{2^k}.$$

Finally, we will apply what has been developed so far to a situation in which the arrangements are no longer linear.

EXAMPLE 16

If six people, designated as A, B, . . . , F, are seated about a round table, how many different circular arrangements are possible, if arrangements are considered the same when one can be obtained from the other by rotation? [In Fig. 2, arrangements (a) and (b) are considered identical, whereas (b), (c), and (d) are three distinct arrangements.]

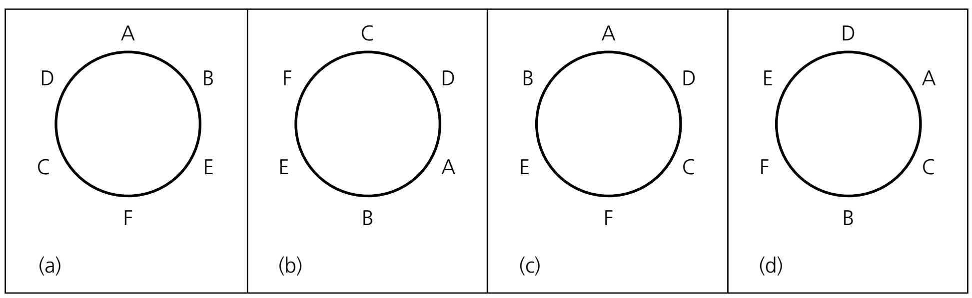

Figure 2

We shall try to relate this problem to previous ones we have already encountered. Consider Figs. 2(a) and (b). Starting at the top of the circle and moving clockwise, we list the distinct linear arrangements ABEFCD and CDABEF, which correspond to the same circular arrangement. In addition to these two, four other linear arrangements — BEFCDA, DABEFC, EFCDAB, and FCDABE — are found to correspond to the same circular arrangement as in (a) or (b). So inasmuch as each circular arrangement corresponds to six linear arrangements, we have $6 \times$ (Number of circular arrangements of A, B, . . . , F) $=$ (Number of linear arrangements of A, B, . . . , F) $= 6!$.

Consequently, there are $6!/6 = 5! = 120$ arrangements of A, B, . . . , F around the circular table.

EXAMPLE 17

Suppose now that the six people of Example 16 are three married couples and that A, B, and C are the females. We want to arrange the six people around the table so that the sexes alternate. (Once again, arrangements are considered identical if one can be obtained from the other by rotation.)

Before we solve this problem, let us solve Example 16 by an alternative method, which will assist us in solving our present problem. If we place A at the table as shown in Fig. 3(a), five locations (clockwise from A) remain to be filled. Using B, C, . . . , F to fill

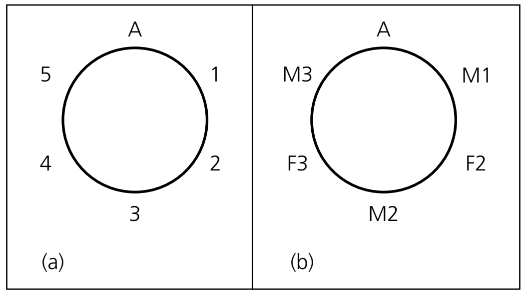

Figure 3

these five positions is the problem of permuting B, C, . . . , F in a linear manner, and this can be done in $5! = 120$ ways.

To solve the new problem of alternating the sexes, consider the method shown in Fig. 3(b). A (a female) is placed as before. The next position, clockwise from A, is marked M1 (Male 1) and can be filled in three ways. Continuing clockwise from A, position F2 (Female 2) can be filled in two ways. Proceeding in this manner, by the rule of product, there are $3 \times 2 \times 2 \times 1 \times 1 = 12$ ways in which these six people can be arranged with no two men or women seated next to each other.

EXERCISES 1 AND 2

1. During a local campaign, eight Republican and five Democratic candidates are nominated for president of the school board.

a) If the president is to be one of these candidates, how many possibilities are there for the eventual winner?

b) How many possibilities exist for a pair of candidates (one from each party) to oppose each other for the eventual election?

c) Which counting principle is used in part (a)? in part (b)?

2. Answer part (c) of Example 6.

3. Buick automobiles come in four models, 12 colors, three engine sizes, and two transmission types. (a) How many distinct Buicks can be manufactured? (b) If one of the available colors is blue, how many different blue Buicks can be manufactured?

4. The board of directors of a pharmaceutical corporation has 10 members. An upcoming stockholders' meeting is scheduled to approve a new slate of company officers (chosen from the 10 board members).

a) How many different slates consisting of a president, vice president, secretary, and treasurer can the board present to the stockholders for their approval?

b) Three members of the board of directors are physicians. How many slates from part (a) have (i) a physician nominated for the presidency? (ii) exactly one physician appearing on the slate? (iii) at least one physician appearing on the slate?

5. While on a Saturday shopping spree Jennifer and Tiffany witnessed two men driving away from the front of a jewelry shop, just before a burglar alarm started to sound. Although everything happened rather quickly, when the two young ladies were questioned they were able to give the police the following information about the license plate (which consisted of two letters followed by four digits) on the get-away car. Tiffany was sure that the second letter on the plate was either an O or a Q and the last digit was either a 3 or an 8. Jennifer told the investigator that the first letter on the plate was either a C or a G and that the first digit was definitely a 7. How many different license plates will the police have to check out?

6. To raise money for a new municipal pool, the chamber of commerce in a certain city sponsors a race. Each participant pays a \$5 entrance fee and has a chance to win one of the different-sized trophies that are to be awarded to the first eight runners who finish.

a) If 30 people enter the race, in how many ways will it be possible to award the trophies?

b) If Roberta and Candice are two participants in the race, in how many ways can the trophies be awarded with these two runners among the top three?

7. A certain "Burger Joint" advertises that a customer can have his or her hamburger with or without any or all of the following: catsup, mustard, mayonnaise, lettuce, tomato, onion, pickle, cheese, or mushrooms. How many different kinds of hamburger orders are possible?

8. Matthew works as a computer operator at a small university. One evening he finds that 12 computer programs have been submitted earlier that day for batch processing. In how many ways can Matthew order the processing of these programs if (a) there are no restrictions? (b) he considers four of the programs higher in priority than the other eight and wants to process those four first? (c) he first separates the programs into four of top priority, five of lesser priority, and three of least priority, and he wishes to process the 12 programs in such a way that the top-priority programs are processed first and the three programs of least priority are processed last?

9. Patter's Pastry Parlor offers eight different kinds of pastry and six different kinds of muffins. In addition to bakery items one can purchase small, medium, or large containers of the following beverages: coffee (black, with cream, with sugar, or with cream and sugar), tea (plain, with cream, with sugar, with cream and sugar, with lemon, or with lemon and sugar), hot cocoa, and orange juice. When Carol comes to Patter's, in how many ways can she order

a) one bakery item and one medium-sized beverage for herself?

b) one bakery item and one container of coffee for herself and one muffin and one container of tea for her boss, Ms. Didio?

c) one piece of pastry and one container of tea for herself, one muffin and a container of orange juice for Ms. Didio, and one bakery item and one container of coffee for each of her two assistants, Mr. Talbot and Mrs. Gillis?

10. Pamela has 15 different books. In how many ways can she place her books on two shelves so that there is at least one book on each shelf? (Consider the books in each arrangement to be stacked one next to the other, with the first book on each shelf at the left of the shelf.)

11. Three small towns, designated by A, B, and C, are interconnected by a system of two-way roads, as shown in Fig. 4.

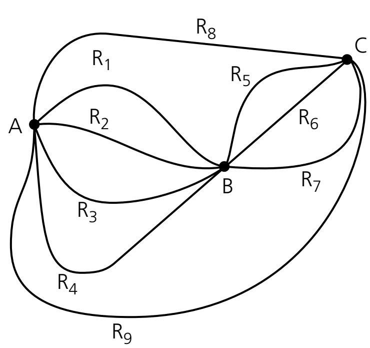

Figure 4

a) In how many ways can Linda travel from town A to town C?

b) How many different round trips can Linda travel from town A to town C and back to town A?

c) How many of the round trips in part (b) are such that the return trip (from town C to town A) is at least partially different from the route Linda takes from town A to town C? (For example, if Linda travels from town A to town C along roads R_1 and R_6, then on her return she might take roads R_6 and R_3, or roads R_7 and R_2, or road R_9, among other possibilities, but she does *not* travel on roads R_6 *and* R_1.)

12. List all the permutations for the letters a, c, t.

13. **a)** How many permutations are there for the eight letters a, c, f, g, i, t, w, x?

b) Consider the permutations in part (a). How many start with the letter t? How many start with the letter t and end with the letter c?

14. Evaluate each of the following.

a) $P(7, 2)$ **b)** $P(8, 4)$ **c)** $P(10, 7)$ **d)** $P(12, 3)$

15. In how many ways can the symbols a, b, c, d, e, e, e, e, e be arranged so that no e is adjacent to another e?

16. An alphabet of 40 symbols is used for transmitting messages in a communication system. How many distinct messages (lists of symbols) of 25 symbols can the transmitter generate if symbols can be repeated in the message? How many if 10 of the 40 symbols can appear only as the first and/or last symbols of the message, the other 30 symbols can appear anywhere, and repetitions of all symbols are allowed?

17. In the Internet each network interface of a computer is assigned one, or more, Internet addresses. The nature of these Internet addresses is dependent on network size. For the Internet Standard regarding reserved network numbers (STD 2), each address is a 32-bit string which falls into one of the following three classes: (1) A class A address, used for the largest networks, begins with a 0 which is then followed by a seven-bit *network number*, and then a 24-bit *local address*. However, one is restricted from using the network numbers of all 0's or all 1's and the local addresses of all 0's or all 1's. (2) The class B address is meant for an intermediate-sized network. This address starts with the two-bit string 10, which is followed by a 14-bit network number and then a 16-bit local address. But the local addresses of all 0's or all 1's are not permitted. (3) Class C addresses are used for the smallest networks. These addresses consist of the three-bit string 110, followed by a 21-bit network number, and then an eight-bit local address. Once again the local addresses of all 0's or all 1's are excluded. How many different addresses of each class are available on the Internet, for this Internet Standard?

18. Morgan is considering the purchase of a low-end computer system. After some careful investigating, she finds that there are seven basic systems (each consisting of a monitor, CPU, keyboard, and mouse) that meet her requirements. Furthermore, she also plans to buy one of four modems, one of three CD ROM drives, and one of six printers. (Here each peripheral device of a given type, such as the modem, is compatible with all seven basic systems.) In how many ways can Morgan configure her low-end computer system?

19. A computer science professor has seven different programming books on a bookshelf. Three of the books deal with C++, the other four with Java. In how many ways can the professor arrange these books on the shelf (a) if there are no restrictions? (b) if the languages should alternate? (c) if all the C++ books must be next to each other? (d) if all the C++ books must be next to each other and all the Java books must be next to each other?

20. Over the Internet, data are transmitted in structured blocks of bits called *datagrams*.

a) In how many ways can the letters in DATAGRAM be arranged?

b) For the arrangements of part (a), how many have all three A's together?

21. a) How many arrangements are there of all the letters in SOCIOLOGICAL?

b) In how many of the arrangements in part (a) are A and G adjacent?

c) In how many of the arrangements in part (a) are all the vowels adjacent?

22. How many positive integers n can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want n to exceed 5,000,000?

23. Twelve clay targets (identical in shape) are arranged in four hanging columns, as shown in Fig. 5. There are four red targets in the first column, three white ones in the second column, two green targets in the third column, and three blue ones in the fourth column. To join her college drill team, Deborah must break all 12 of these targets (using her pistol and only 12 bullets) and in so doing must always break the existing target at the bottom of a column. Under these conditions, in how many different orders can Deborah shoot down (and break) the 12 targets?

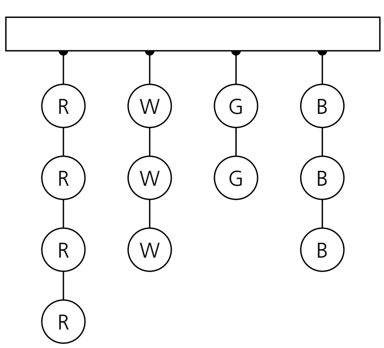

Figure 5

24. Show that for all integers $n, r \geq 0$, if $n + 1 > r$, then

$$P(n+1, r) = \left(\frac{n+1}{n+1-r}\right) P(n, r).$$

25. Find the value(s) of n in each of the following:
(a) $P(n, 2) = 90$, (b) $P(n, 3) = 3P(n, 2)$, and
(c) $2P(n, 2) + 50 = P(2n, 2)$.

26. How many different paths in the xy-plane are there from (0, 0) to (7, 7) if a path proceeds one step at a time by going either one space to the right (R) or one space upward (U)? How many such paths are there from (2, 7) to (9, 14)? Can any general statement be made that incorporates these two results?

27. a) How many distinct paths are there from $(-1, 2, 0)$ to $(1, 3, 7)$ in Euclidean three-space if each move is one of the following types?

$$(H): (x, y, z) \rightarrow (x+1, y, z);$$
$$(V): (x, y, z) \rightarrow (x, y+1, z);$$
$$(A): (x, y, z) \rightarrow (x, y, z+1)$$

b) How many such paths are there from (1, 0, 5) to (8, 1, 7)?

c) Generalize the results in parts (a) and (b).

28. a) Determine the value of the integer variable *counter* after execution of the following program segment. (Here i, j, and k are integer variables.)

```
counter := 0
for i := 1 to 12 do
  counter := counter + 1
for j := 5 to 10 do
  counter := counter + 2
for k := 15 downto 8 do
  counter := counter + 3
```

b) Which counting principle is at play in part (a)?

29. Consider the following program segment where i, j, and k are integer variables.

```
for i := 1 to 12 do
  for j := 5 to 10 do
    for k := 15 downto 8 do
      print (i - j)*k
```

a) How many times is the **print** statement executed?

b) Which counting principle is used in part (a)?

30. A sequence of letters of the form abcba, where the expression is unchanged upon reversing order, is an example of a *palindrome* (of five letters). (a) If a letter may appear more than twice, how many palindromes of five letters are there? of six letters? (b) Repeat part (a) under the condition that no letter appears more than twice.

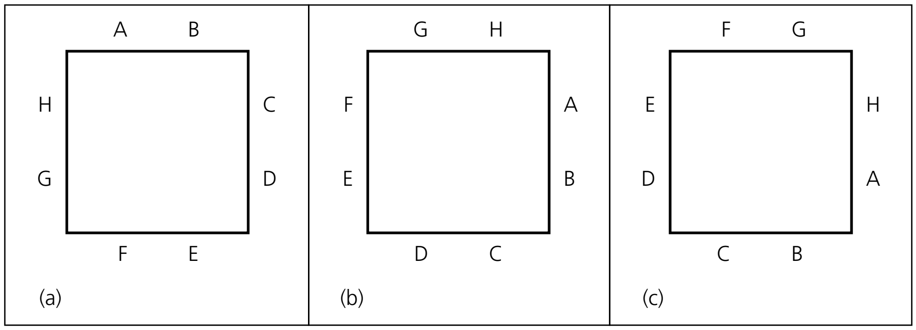

Figure 6

31. Determine the number of six-digit integers (no leading zeros) in which (a) no digit may be repeated; (b) digits may be repeated. Answer parts (a) and (b) with the extra condition that the six-digit integer is (i) even; (ii) divisible by 5; (iii) divisible by 4.

32. **a)** Provide a combinatorial argument to show that if n and k are positive integers with $n = 3k$, then $n!/(3!)^k$ is an integer.

b) Generalize the result of part (a).

33. **a)** In how many possible ways could a student answer a 10-question true-false test?

b) In how many ways can the student answer the test in part (a) if it is possible to leave a question unanswered in order to avoid an extra penalty for a wrong answer?

34. How many distinct four-digit integers can one make from the digits 1, 3, 3, 7, 7, and 8?

35. **a)** In how many ways can seven people be arranged about a circular table?

b) If two of the people insist on sitting next to each other, how many arrangements are possible?

36. **a)** In how many ways can eight people, denoted A, B, . . . , H be seated about the square table shown in Fig. 6, where Figs. 6(a) and 6(b) are considered the same but are distinct from Fig. 6(c)?

b) If two of the eight people, say A and B, do not get along well, how many different seatings are possible with A and B not sitting next to each other?

37. Sixteen people are to be seated at two circular tables, one of which seats 10 while the other seats six. How many different seating arrangements are possible?

38. A committee of 15 — nine women and six men — is to be seated at a circular table (with 15 seats). In how many ways can the seats be assigned so that no two men are seated next to each other?

39. Write a computer program (or develop an algorithm) to determine whether there is a three-digit integer abc $(= 100a + 10b + c)$ where $abc = a! + b! + c!$.

3 Combinations: The Binomial Theorem

The standard deck of playing cards consists of 52 cards comprising four suits: clubs, diamonds, hearts, and spades. Each suit has 13 cards: ace, 2, 3, . . . , 9, 10, jack, queen, king. If we are asked to draw three cards from a standard deck, in succession and without replacement, then by the rule of product there are

$$52 \times 51 \times 50 = \frac{52!}{49!} = P(52, 3)$$

possibilities, one of which is AH (ace of hearts), 9C (nine of clubs), KD (king of diamonds). If instead we simply select three cards at one time from the deck so that the order of selection of the cards is no longer important, then the six permutations AH–9C–KD, AH–KD–9C, 9C–AH–KD, 9C–KD–AH, KD–9C–AH, and KD–AH–9C all correspond to just one (unordered) selection. Consequently, each selection, or combination, of three cards, *with no reference to order*, corresponds to 3! permutations of three cards. In equation form this translates into

$$\begin{aligned}(3!) \times (&\text{Number of selections of size 3 from a deck of 52}) \\ &= \text{Number of permutations of size 3 for the 52 cards} \\ &= P(52, 3) = \frac{52!}{49!}.\end{aligned}$$

Consequently, three cards can be drawn, without replacement, from a standard deck in $52!/(3!\,49!) = 22{,}100$ ways.

If we start with n distinct objects, each *selection*, or *combination*, of r of these objects, with no reference to order, corresponds to $r!$ permutations of size r from the n objects. Thus the number of combinations of size r from a collection of size n is

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}, \qquad 0 \le r \le n.$$

In addition to $C(n, r)$ the symbol $\binom{n}{r}$ is also frequently used. Both $C(n, r)$ and $\binom{n}{r}$ are sometimes read "n choose r." Note that for all $n \ge 0$, $C(n, 0) = C(n, n) = 1$. Further, for all $n \ge 1$, $C(n, 1) = C(n, n-1) = n$. When $0 \le n < r$, then $C(n, r) = \binom{n}{r} = 0$.

A word to the wise! When dealing with any counting problem, we should ask ourselves about the importance of order in the problem. When order is relevant, we think in terms of permutations and arrangements and the rule of product. When order is not relevant, combinations could play a key role in solving the problem.

EXAMPLE 18

A hostess is having a dinner party for some members of her charity committee. Because of the size of her home, she can invite only 11 of the 20 committee members. Order is not important, so she can invite "the lucky 11" in $C(20, 11) = \binom{20}{11} = 20!/(11!\,9!) = 167{,}960$ ways. However, once the 11 arrive, how she arranges them around her rectangular dining table is an arrangement problem. Unfortunately, no part of the theory of combinations and permutations can help our hostess deal with "the offended nine" who were not invited.

EXAMPLE 19

Lynn and Patti decide to buy a PowerBall ticket. To win the grand prize for PowerBall one must match five numbers selected from 1 to 49 inclusive and then must also match the powerball, an integer from 1 to 42 inclusive. Lynn selects the five numbers (between 1 and 49 inclusive). This she can do in $\binom{49}{5}$ ways (since matching does *not* involve order). Meanwhile Patti selects the powerball — here there are $\binom{42}{1}$ possibilities. Consequently, by the rule of product, Lynn and Patti can select the six numbers for their PowerBall ticket in $\binom{49}{5}\binom{42}{1} = 80{,}089{,}128$ ways.

EXAMPLE 20

a) A student taking a history examination is directed to answer any seven of 10 essay questions. There is no concern about order here, so the student can answer the examination in

$$\binom{10}{7} = \frac{10!}{7!\,3!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120 \text{ ways.}$$

b) If the student must answer three questions from the first five and four questions from the last five, three questions can be selected from the first five in $\binom{5}{3} = 10$ ways, and the other four questions can be selected in $\binom{5}{4} = 5$ ways. Hence, by the rule of product, the student can complete the examination in $\binom{5}{3}\binom{5}{4} = 10 \times 5 = 50$ ways.

c) Finally, should the directions on this examination indicate that the student must answer seven of the 10 questions where at least three are selected from the first five, then there are three cases to consider:

- **i)** The student answers three of the first five questions and four of the last five: By the rule of product this can happen in $\binom{5}{3}\binom{5}{4} = 10 \times 5 = 50$ ways, as in part (b).
- **ii)** Four of the first five questions and three of the last five questions are selected by the student: This can come about in $\binom{5}{4}\binom{5}{3} = 5 \times 10 = 50$ ways—again by the rule of product.
- **iii)** The student decides to answer all five of the first five questions and two of the last five: The rule of product tells us that this last case can occur in $\binom{5}{5}\binom{5}{2} = 1 \times 10 = 10$ ways.

Combining the results for cases (i), (ii), and (iii), by the rule of sum we find that the student can make $\binom{5}{3}\binom{5}{4} + \binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2} = 50 + 50 + 10 = 110$ selections of seven (out of 10) questions where each selection includes at least three of the first five questions.

EXAMPLE 21

a) At Rydell High School, the gym teacher must select nine girls from the junior and senior classes for a volleyball team. If there are 28 juniors and 25 seniors, she can make the selection in $\binom{53}{9} = 4{,}431{,}613{,}550$ ways.

b) If two juniors and one senior are the best spikers and must be on the team, then the rest of the team can be chosen in $\binom{50}{6} = 15{,}890{,}700$ ways.

c) For a certain tournament the team must comprise four juniors and five seniors. The teacher can select the four juniors in $\binom{28}{4}$ ways. For each of these selections she has $\binom{25}{5}$ ways to choose the five seniors. Consequently, by the rule of product, she can select her team in $\binom{28}{4}\binom{25}{5} = 1{,}087{,}836{,}750$ ways for this particular tournament.

Some problems can be treated from the viewpoint of either arrangements or combinations, depending on how one analyzes the situation. The following example demonstrates this.

EXAMPLE 22

The gym teacher of Example 21 must make up four volleyball teams of nine girls each from the 36 freshman girls in her P.E. class. In how many ways can she select these four teams? Call the teams A, B, C, and D.

a) To form team A, she can select any nine girls from the 36 enrolled in $\binom{36}{9}$ ways. For team B the selection process yields $\binom{27}{9}$ possibilities. This leaves $\binom{18}{9}$ and $\binom{9}{9}$ possible ways to select teams C and D, respectively. So by the rule of product, the four teams can be chosen in

$$\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9} = \left(\frac{36!}{9!\,27!}\right)\left(\frac{27!}{9!\,18!}\right)\left(\frac{18!}{9!\,9!}\right)\left(\frac{9!}{9!\,0!}\right)$$
$$= \frac{36!}{9!\,9!\,9!\,9!} \doteq 2.145 \times 10^{19} \text{ ways.}$$

b) For an alternative solution, consider the 36 students lined up as follows:

1st student	2nd student	3rd student	. . .	35th student	36th student

To select the four teams, we must distribute nine A's, nine B's, nine C's, and nine D's in the 36 spaces. The number of ways in which this can be done is the number of arrangements of 36 letters comprising nine each of A, B, C, and D. This is now the familiar problem of arrangements of nondistinct objects, and the answer is

$$\frac{36!}{9!\,9!\,9!\,9!}, \quad \text{as in part (a).}$$

Our next example points out how some problems require the concepts of both arrangements and combinations for their solutions.

EXAMPLE 23

The number of arrangements of the letters in TALLAHASSEE is

$$\frac{11!}{3!\,2!\,2!\,2!\,1!\,1!} = 831{,}600.$$

How many of these arrangements have no adjacent A's?

When we disregard the A's, there are

$$\frac{8!}{2!\,2!\,2!\,1!\,1!} = 5040$$

ways to arrange the remaining letters. One of these 5040 ways is shown in the following figure, where the arrows indicate nine possible locations for the three A's.

$$\uparrow \text{E} \uparrow \text{E} \uparrow \text{S} \uparrow \text{T} \uparrow \text{L} \uparrow \text{L} \uparrow \text{S} \uparrow \text{H} \uparrow$$

Three of these locations can be selected in $\binom{9}{3} = 84$ ways, and because this is also possible for all the other 5039 arrangements of E, E, S, T, L, L, S, H, by the rule of product there are $5040 \times 84 = 423{,}360$ arrangements of the letters in TALLAHASSEE with no consecutive A's.

Before proceeding we need to introduce a concise way of writing the sum of a list of $n+1$ terms like $a_m, a_{m+1}, a_{m+2}, \ldots, a_{m+n}$, where m and n are integers and $n \geq 0$. This notation is called the *Sigma notation* because it involves the capital Greek letter Σ; we use it to represent a summation by writing

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_{m+n} = \sum_{i=m}^{m+n} a_i.$$

Here, the letter i is called the *index* of the summation, and this index accounts for all integers starting with the *lower limit* m and continuing on up to (and including) the *upper limit* $m+n$.

We may use this notation as follows.

1) $\sum_{i=3}^{7} a_i = a_3 + a_4 + a_5 + a_6 + a_7 = \sum_{j=3}^{7} a_j$, for there is nothing special about the letter i.

2) $\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30 = \sum_{k=0}^{4} k^2$, because $0^2 = 0$.

3) $\sum_{i=11}^{100} i^3 = 11^3 + 12^3 + 13^3 + \cdots + 100^3 = \sum_{j=12}^{101} (j-1)^3 = \sum_{k=10}^{99} (k+1)^3$.

4) $\sum_{i=7}^{10} 2i = 2(7) + 2(8) + 2(9) + 2(10) = 68 = 2(34) = 2(7+8+9+10) = 2\sum_{i=7}^{10} i$.

5) $\sum_{i=3}^{3} a_i = a_3 = \sum_{i=4}^{4} a_{i-1} = \sum_{i=2}^{2} a_{i+1}$.

6) $\sum_{i=1}^{5} a = a + a + a + a + a = 5a$.

Furthermore, using this summation notation, we see that one can express the answer to part (c) of Example 20 as

$$\binom{5}{3}\binom{5}{4} + \binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2} = \sum_{i=3}^{5}\binom{5}{i}\binom{5}{7-i} = \sum_{j=2}^{4}\binom{5}{7-j}\binom{5}{j}.$$

We shall find use for this new notation in the following example and in many other places throughout the remainder of this text.

EXAMPLE 24

In the studies of algebraic coding theory and the theory of computer languages, we consider certain arrangements, called *strings*, made up from a prescribed *alphabet* of symbols. If the prescribed alphabet consists of the symbols 0, 1, and 2, for example, then 01, 11, 21, 12, and 20 are five of the nine strings of *length* 2. Among the 27 strings of length 3 are 000, 012, 202, and 110.

In general, if n is any positive integer, then by the rule of product there are 3^n strings of length n for the alphabet 0, 1, and 2. If $x = x_1x_2x_3 \cdots x_n$ is one of these strings, we define the *weight* of x, denoted $\text{wt}(x)$, by $\text{wt}(x) = x_1 + x_2 + x_3 + \cdots + x_n$. For example, $\text{wt}(12) = 3$ and $\text{wt}(22) = 4$ for the case where $n = 2$; $\text{wt}(101) = 2$, $\text{wt}(210) = 3$, and $\text{wt}(222) = 6$ for $n = 3$.

Among the 3^{10} strings of length 10, we wish to determine how many have even weight. Such a string has even weight precisely when the number of 1's in the string is even.

There are six different cases to consider. If the string x contains no 1's, then each of the 10 locations in x can be filled with either 0 or 2, and by the rule of product there are 2^{10} such strings. When the string contains two 1's, the locations for these two 1's can be selected in $\binom{10}{2}$ ways. Once these two locations have been specified, there are 2^8 ways to place either 0 or 2 in the other eight positions. Hence there are $\binom{10}{2}2^8$ strings of even weight that contain two 1's. The numbers of strings for the other four cases are given in Table 2.

Table 2

Number of 1's	Number of Strings	Number of 1's	Number of Strings
4	$\binom{10}{4}2^6$	8	$\binom{10}{8}2^2$
6	$\binom{10}{6}2^4$	10	$\binom{10}{10}$

Consequently, by the rule of sum, the number of strings of length 10 that have even weight is $2^{10} + \binom{10}{2}2^8 + \binom{10}{4}2^6 + \binom{10}{6}2^4 + \binom{10}{8}2^2 + \binom{10}{10} = \sum_{n=0}^{5} \binom{10}{2n}2^{10-2n}$.

Often we must be careful of *overcounting* — a situation that seems to arise in what may appear to be rather easy enumeration problems. The next example demonstrates how overcounting may come about.

EXAMPLE 25

a) Suppose that Ellen draws five cards from a standard deck of 52 cards. In how many ways can her selection result in a hand with no clubs? Here we are interested in counting all five-card selections such as

- **i)** ace of hearts, three of spades, four of spades, six of diamonds, and the jack of diamonds.
- **ii)** five of spades, seven of spades, ten of spades, seven of diamonds, and the king of diamonds.
- **iii)** two of diamonds, three of diamonds, six of diamonds, ten of diamonds, and the jack of diamonds.

If we examine this more closely we see that Ellen is restricted to selecting her five cards from the 39 cards in the deck that are not clubs. Consequently, she can make her selection in $\binom{39}{5}$ ways.

b) Now suppose we want to count the number of Ellen's five-card selections that contain at least one club. These are precisely the selections that were *not* counted in part (a). And since there are $\binom{52}{5}$ possible five-card hands in total, we find that

$$\binom{52}{5} - \binom{39}{5} = 2{,}598{,}960 - 575{,}757 = 2{,}023{,}203$$

of all five-card hands contain at least one club.

c) Can we obtain the result in part (b) in another way? For example, since Ellen wants to have at least one club in the five-card hand, let her first select a club. This she can do in $\binom{13}{1}$ ways. And now she doesn't care what comes up for the other four cards. So after she eliminates the one club chosen from her standard deck, she can then select the other four cards in $\binom{51}{4}$ ways. Therefore, by the rule of product, we count the number of selections here as

$$\binom{13}{1}\binom{51}{4} = 13 \times 249{,}900 = 3{,}248{,}700.$$

Something here is definitely *wrong*! This answer is larger than that in part (b) by more than one million hands. Did we make a mistake in part (b)? Or is something wrong with our present reasoning?

For example, suppose that Ellen first selects

the three of clubs

and then selects

the five of clubs,
king of clubs,
seven of hearts, and
jack of spades.

If, however, she first selects

the five of clubs

and then selects

the three of clubs,
king of clubs,
seven of hearts, and
jack of spades,

is her selection here really different from the prior selection we mentioned? Unfortunately, no! And the case where she first selects

the king of clubs

and then follows this by selecting

the three of clubs,
five of clubs,
seven of hearts, and
jack of spades

is not different from the other two selections mentioned earlier.

Consequently, this approach is *wrong* because we are overcounting — by considering like selections as if they were distinct.

d) But is there any other way to arrive at the answer in part (b)? Yes! Since the five-card hands must each contain at least one club, there are five cases to consider. These are given in Table 3. From the results in Table 3 we see, for example, that there are $\binom{13}{2}\binom{39}{3}$ five-card hands that contain exactly two clubs. If we are interested in having exactly three clubs in the hand, then the results in the table indicate that there are $\binom{13}{3}\binom{39}{2}$ such hands.

Table 3

Number of Clubs	Number of Ways to Select This Number of Clubs	Number of Cards That Are Not Clubs	Number of Ways to Select This Number of Nonclubs
1	$\binom{13}{1}$	4	$\binom{39}{4}$
2	$\binom{13}{2}$	3	$\binom{39}{3}$
3	$\binom{13}{3}$	2	$\binom{39}{2}$
4	$\binom{13}{4}$	1	$\binom{39}{1}$
5	$\binom{13}{5}$	0	$\binom{39}{0}$

Since no two of the cases in Table 3 have any five-card hand in common, the number of hands that Ellen can select with at least one club is

$$\binom{13}{1}\binom{39}{4}+\binom{13}{2}\binom{39}{3}+\binom{13}{3}\binom{39}{2}+\binom{13}{4}\binom{39}{1}+\binom{13}{5}\binom{39}{0}$$
$$=\sum_{i=1}^{5}\binom{13}{i}\binom{39}{5-i}$$
$$=(13)(82{,}251)+(78)(9139)+(286)(741)+(715)(39)+(1287)(1)$$
$$=2{,}023{,}203.$$

We shall close this section with three results related to the concept of combinations.

First we note that for integers n, r, with $n \geq r \geq 0$, $\binom{n}{r}=\binom{n}{n-r}$. This can be established algebraically from the formula for $\binom{n}{r}$, but we prefer to observe that when dealing with a selection of size r from a collection of n distinct objects, the selection process leaves behind $n-r$ objects. Consequently, $\binom{n}{r}=\binom{n}{n-r}$ affirms the existence of a correspondence between the selections of size r (objects chosen) and the selections of size $n-r$ (objects left behind). An example of this correspondence is shown in Table 4, where $n=5$, $r=2$, and the distinct objects are 1, 2, 3, 4, and 5.

Table 4

Selections of Size $r = 2$ (Objects Chosen)		Selections of Size $n - r = 3$ (Objects Left Behind)	
1. 1, 2	6. 2, 4	1. 3, 4, 5	6. 1, 3, 5
2. 1, 3	7. 2, 5	2. 2, 4, 5	7. 1, 3, 4
3. 1, 4	8. 3, 4	3. 2, 3, 5	8. 1, 2, 5
4. 1, 5	9. 3, 5	4. 2, 3, 4	9. 1, 2, 4
5. 2, 3	10. 4, 5	5. 1, 4, 5	10. 1, 2, 3

Our second result is a theorem from our past experience in algebra.

THEOREM 1 *The Binomial Theorem.* If x and y are variables and n is a positive integer, then

$$(x+y)^n=\binom{n}{0}x^0y^n+\binom{n}{1}x^1y^{n-1}+\binom{n}{2}x^2y^{n-2}+\cdots$$
$$+\binom{n}{n-1}x^{n-1}y^1+\binom{n}{n}x^ny^0=\sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}.$$

Before considering the general proof, we examine a special case. If $n=4$, the coefficient of x^2y^2 in the expansion of the product

$$\underset{\text{1st factor}}{(x+y)}\;\underset{\text{2nd factor}}{(x+y)}\;\underset{\text{3rd factor}}{(x+y)}\;\underset{\text{4th factor}}{(x+y)}$$

is the number of ways in which we can select two x's from the four x's, one of which is available in each factor. (Although the x's are the same in appearance, we distinguish them as the x in the first factor, the x in the second factor, ..., and the x in the fourth factor. Also, we note that when we select two x's, we use two factors, leaving us with two other factors from which we can select the two y's that are needed.) For example, among the possibilities, we can select (1) x from the first two factors and y from the last two or (2) x from the first and third factors and y from the second and fourth. Table 5 summarizes the six possible selections.

Table 5

Factors Selected for x		Factors Selected for y	
(1)	1, 2	(1)	3, 4
(2)	1, 3	(2)	2, 4
(3)	1, 4	(3)	2, 3
(4)	2, 3	(4)	1, 4
(5)	2, 4	(5)	1, 3
(6)	3, 4	(6)	1, 2

Consequently, the coefficient of x^2y^2 in the expansion of $(x+y)^4$ is $\binom{4}{2}=6$, the number of ways to select two distinct objects from a collection of four distinct objects.

Now we turn to the proof of the general case.

Proof: In the expansion of the product

$$\underbrace{(x+y)}_{\text{1st factor}}\ \underbrace{(x+y)}_{\text{2nd factor}}\ \underbrace{(x+y)}_{\text{3rd factor}}\cdots\underbrace{(x+y)}_{n\text{th factor}}$$

the coefficient of $x^k y^{n-k}$, where $0 \le k \le n$, is the number of different ways in which we can select k x's [and consequently $(n-k)$ y's] from the n available factors. (One way, for example, is to choose x from the first k factors and y from the last $n-k$ factors.) The total number of such selections of size k from a collection of size n is $C(n,k)=\binom{n}{k}$, and from this the binomial theorem follows.

In view of this theorem, $\binom{n}{k}$ is often referred to as a *binomial coefficient*. Notice that it is also possible to express the result of Theorem 1 as

$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{n-k}x^k y^{n-k}.$$

EXAMPLE 26

a) From the binomial theorem it follows that the coefficient of x^5y^2 in the expansion of $(x+y)^7$ is $\binom{7}{5}=\binom{7}{2}=21$.

b) To obtain the coefficient of a^5b^2 in the expansion of $(2a-3b)^7$, replace $2a$ by x and $-3b$ by y. From the binomial theorem the coefficient of x^5y^2 in $(x+y)^7$ is $\binom{7}{5}$, and $\binom{7}{5}x^5y^2=\binom{7}{5}(2a)^5(-3b)^2=\binom{7}{5}(2)^5(-3)^2a^5b^2=6048a^5b^2$.

COROLLARY 1 For each integer $n > 0$,

a) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$, and
b) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0$.

Proof: Part (a) follows from the binomial theorem when we set $x = y = 1$. When $x = -1$ and $y = 1$, part (b) results.

Our third and final result generalizes the binomial theorem and is called the *multinomial theorem.*

THEOREM 2 For positive integers n, t, the coefficient of $x_1^{n_1}x_2^{n_2}x_3^{n_3}\cdots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + x_3 + \cdots + x_t)^n$ is

$$\frac{n!}{n_1!\,n_2!\,n_3!\cdots n_t!},$$

where each n_i is an integer with $0 \le n_i \le n$, for all $1 \le i \le t$, and $n_1 + n_2 + n_3 + \cdots + n_t = n$.

Proof: As in the proof of the binomial theorem, the coefficient of $x_1^{n_1}x_2^{n_2}x_3^{n_3}\cdots x_t^{n_t}$ is the number of ways we can select x_1 from n_1 of the n factors, x_2 from n_2 of the $n - n_1$ remaining factors, x_3 from n_3 of the $n - n_1 - n_2$ now remaining factors, ..., and x_t from n_t of the last $n - n_1 - n_2 - n_3 - \cdots - n_{t-1} = n_t$ remaining factors. This can be carried out, as in part (a) of Example 22, in

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-n_3-\cdots-n_{t-1}}{n_t}$$

ways. We leave to the reader the details of showing that this product is equal to

$$\frac{n!}{n_1!\,n_2!\,n_3!\cdots n_t!},$$

which is also written as

$$\binom{n}{n_1, n_2, n_3, \ldots, n_t}$$

and is called a *multinomial coefficient.* (When $t = 2$ this reduces to a binomial coefficient.)

EXAMPLE 27

a) In the expansion of $(x + y + z)^7$ it follows from the multinomial theorem that the coefficient of $x^2y^2z^3$ is $\binom{7}{2,2,3} = \frac{7!}{2!\,2!\,3!} = 210$, while the coefficient of xyz^5 is $\binom{7}{1,1,5} = 42$ and that of x^3z^4 is $\binom{7}{3,0,4} = \frac{7!}{3!\,0!\,4!} = 35$.

b) Suppose we need to know the coefficient of $a^2b^3c^2d^5$ in the expansion of $(a + 2b - 3c + 2d + 5)^{16}$. If we replace a by v, $2b$ by w, $-3c$ by x, $2d$ by y, and 5 by z, then we can apply the multinomial theorem to $(v + w + x + y + z)^{16}$ and determine the coefficient of $v^2w^3x^2y^5z^4$ as $\binom{16}{2,3,2,5,4} = 302{,}702{,}400$. But $\binom{16}{2,3,2,5,4}(a)^2(2b)^3(-3c)^2(2d)^5(5)^4 = \binom{16}{2,3,2,5,4}(1)^2(2)^3(-3)^2(2)^5(5)^4(a^2b^3c^2d^5) = 435{,}891{,}456{,}000{,}000\, a^2b^3c^2d^5$.

EXERCISES 3

1. Calculate $\binom{6}{2}$ and check your answer by listing all the selections of size 2 that can be made from the letters a, b, c, d, e, and f.

2. Facing a four-hour bus trip back to college, Diane decides to take along five magazines from the 12 that her sister Ann Marie has recently acquired. In how many ways can Diane make her selection?

3. Evaluate each of the following.

a) $C(10, 4)$ **b)** $\binom{12}{7}$ **c)** $C(14, 12)$ **d)** $\binom{15}{10}$

4. In the Braille system a symbol, such as a lowercase letter, punctuation mark, suffix, and so on, is given by raising at least one of the dots in the six-dot arrangement shown in part (a) of Fig. 7. (The six Braille positions are labeled in this part of the figure.) For example, in part (b) of the figure the dots in positions 1 and 4 are raised and this six-dot arrangement represents the letter c. In parts (c) and (d) of the figure we have the representations for the letters m and t, respectively. The definite article "the" is shown in part (e) of the figure, while part (f) contains the form for the suffix "ow." Finally, the semicolon, ; , is given by the six-dot arrangement in part (g), where the dots at positions 2 and 3 are raised.

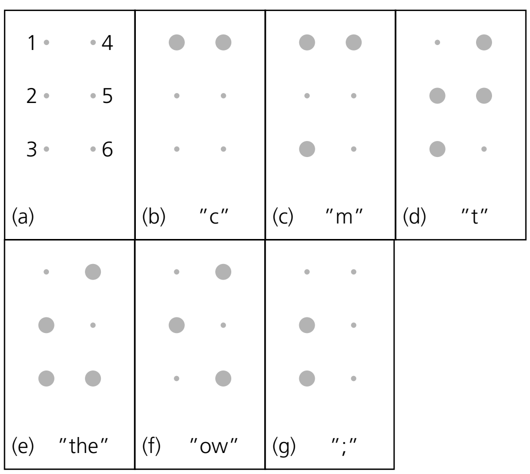

Figure 7

a) How many different symbols can we represent in the Braille system?

b) How many symbols have exactly three raised dots?

c) How many symbols have an even number of raised dots?

5. **a)** How many *permutations* of size 3 can one produce with the letters m, r, a, f, and t?

b) List all the *combinations* of size 3 that result for the letters m, r, a, f, and t.

6. If n is a positive integer and $n > 1$, prove that $\binom{n}{2} + \binom{n-1}{2}$ is a perfect square.

7. A committee of 12 is to be selected from 10 men and 10 women. In how many ways can the selection be carried out if (a) there are no restrictions? (b) there must be six men and six women? (c) there must be an even number of women? (d) there must be more women than men? (e) there must be at least eight men?

8. In how many ways can a gambler draw five cards from a standard deck and get (a) a flush (five cards of the same suit)? (b) four aces? (c) four of a kind? (d) three aces and two jacks? (e) three aces and a pair? (f) a full house (three of a kind and a pair)? (g) three of a kind? (h) two pairs?

9. How many bytes contain (a) exactly two 1's; (b) exactly four 1's; (c) exactly six 1's; (d) at least six 1's?

10. How many ways are there to pick a five-person basketball team from 12 possible players? How many selections include the weakest and the strongest players?

11. A student is to answer seven out of 10 questions on an examination. In how many ways can he make his selection if (a) there are no restrictions? (b) he must answer the first two questions? (c) he must answer at least four of the first six questions?

12. In how many ways can 12 different books be distributed among four children so that (a) each child gets three books? (b) the two oldest children get four books each and the two youngest get two books each?

13. How many arrangements of the letters in MISSISSIPPI have no consecutive S's?

14. A gym coach must select 11 seniors to play on a football team. If he can make his selection in 12,376 ways, how many seniors are eligible to play?

15. **a)** Fifteen points, no three of which are collinear, are given on a plane. How many lines do they determine?

b) Twenty-five points, no four of which are coplanar, are given in space. How many triangles do they determine? How many planes? How many tetrahedra (pyramidlike solids with four triangular faces)?

16. Determine the value of each of the following summations.

a) $\sum_{i=1}^{6}(i^2+1)$ **b)** $\sum_{j=-2}^{2}(j^3-1)$ **c)** $\sum_{i=0}^{10}[1+(-1)^i]$

d) $\sum_{k=n}^{2n}(-1)^k$, where n is an odd positive integer

e) $\sum_{i=1}^{6} i(-1)^i$

17. Express each of the following using the summation (or Sigma) notation. In parts (a), (d), and (e), n denotes a positive integer.

a) $\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{n!},\ n \geq 2$

b) $1 + 4 + 9 + 16 + 25 + 36 + 49$

c) $1^3 - 2^3 + 3^3 - 4^3 + 5^3 - 6^3 + 7^3$

d) $\dfrac{1}{n} + \dfrac{2}{n+1} + \dfrac{3}{n+2} + \cdots + \dfrac{n+1}{2n}$

e) $n - \left(\dfrac{n+1}{2!}\right) + \left(\dfrac{n+2}{4!}\right) - \left(\dfrac{n+3}{6!}\right) + \cdots$
$+ (-1)^n \left(\dfrac{2n}{(2n)!}\right)$

18. For the strings of length 10 in Example 24, how many have (a) four 0's, three 1's, and three 2's; (b) at least eight 1's; (c) weight 4?

19. Consider the collection of all strings of length 10 made up from the alphabet 0, 1, 2, and 3. How many of these strings have weight 3? How many have weight 4? How many have even weight?

20. In the three parts of Fig. 8, eight points are equally spaced and marked on the circumference of a given circle.

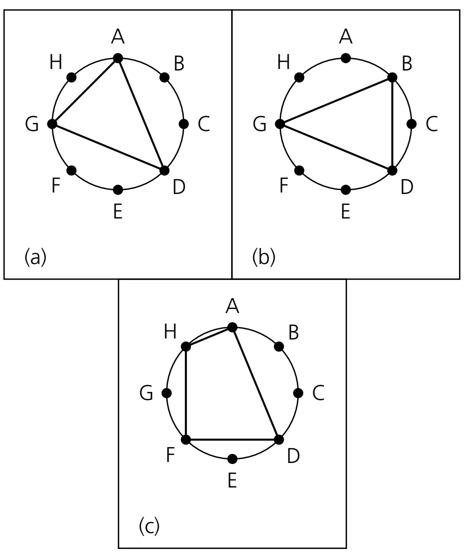

Figure 8

a) For parts (a) and (b) of Fig. 8 we have two different (though congruent) triangles. These two triangles (distinguished by their vertices) result from two selections of size 3 from the vertices A, B, C, D, E, F, G, H. How many different (whether congruent or not) triangles can we inscribe in the circle in this way?

b) How many different quadrilaterals can we inscribe in the circle, using the marked vertices? [One such quadrilateral appears in part (c) of Fig. 8.]

c) How many different polygons of three or more sides can we inscribe in the given circle by using three or more of the marked vertices?

21. How many triangles are determined by the vertices of a regular polygon of n sides? How many if no side of the polygon is to be a side of any triangle?

22. a) In the complete expansion of $(a + b + c + d) \cdot (e + f + g + h)(u + v + w + x + y + z)$ one obtains the sum of terms such as agw, cfx, and dgv. How many such terms appear in this complete expansion?

b) Which of the following terms do *not* appear in the complete expansion from part (a)?

i) afx **ii)** bvx **iii)** chz
iv) cgw **v)** egu **vi)** dfz

23. Determine the coefficient of x^9y^3 in the expansions of (a) $(x + y)^{12}$, (b) $(x + 2y)^{12}$, and (c) $(2x - 3y)^{12}$.

24. Complete the details in the proof of the multinomial theorem.

25. Determine the coefficient of

a) xyz^2 in $(x + y + z)^4$

b) xyz^2 in $(w + x + y + z)^4$

c) xyz^2 in $(2x - y - z)^4$

d) xyz^{-2} in $(x - 2y + 3z^{-1})^4$

e) $w^3x^2yz^2$ in $(2w - x + 3y - 2z)^8$

26. Find the coefficient of $w^2x^2y^2z^2$ in the expansion of (a) $(w + x + y + z + 1)^{10}$, (b) $(2w - x + 3y + z - 2)^{12}$, and (c) $(v + w - 2x + y + 5z + 3)^{12}$.

27. Determine the sum of all the coefficients in the expansions of

a) $(x + y)^3$ **b)** $(x + y)^{10}$ **c)** $(x + y + z)^{10}$

d) $(w + x + y + z)^5$

e) $(2s - 3t + 5u + 6v - 11w + 3x + 2y)^{10}$

28. For any positive integer n determine

a) $\displaystyle\sum_{i=0}^{n} \frac{1}{i!(n-i)!}$ **b)** $\displaystyle\sum_{i=0}^{n} \frac{(-1)^i}{i!(n-i)!}$

29. Show that for all positive integers m and n,

$$n\binom{m+n}{m} = (m+1)\binom{m+n}{m+1}.$$

30. With n a positive integer, evaluate the sum

$$\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \cdots + 2^k\binom{n}{k} + \cdots + 2^n\binom{n}{n}.$$

31. For x a real number and n a positive integer, show that

a) $1 = (1 + x)^n - \dbinom{n}{1}x^1(1 + x)^{n-1}$
$+ \dbinom{n}{2}x^2(1 + x)^{n-2} - \cdots + (-1)^n\dbinom{n}{n}x^n$

b) $1 = (2 + x)^n - \dbinom{n}{1}(x + 1)(2 + x)^{n-1}$
$+ \dbinom{n}{2}(x + 1)^2(2 + x)^{n-2} - \cdots + (-1)^n\dbinom{n}{n}(x + 1)^n$

c) $2^n = (2+x)^n - \binom{n}{1}x^1(2+x)^{n-1} + \binom{n}{2}x^2(2+x)^{n-2} - \cdots + (-1)^n\binom{n}{n}x^n$

32. Determine x if $\sum_{i=0}^{50}\binom{50}{i}8^i = x^{100}$.

33. **a)** If a_0, a_1, a_2, a_3 is a list of four real numbers, what is $\sum_{i=1}^{3}(a_i - a_{i-1})$?

b) Given a list — $a_0, a_1, a_2, \ldots, a_n$ — of $n+1$ real numbers, where n is a positive integer, determine $\sum_{i=1}^{n}(a_i - a_{i-1})$.

c) Determine the value of $\sum_{i=1}^{100}\left(\frac{1}{i+2} - \frac{1}{i+1}\right)$.

34. **a)** Write a computer program (or develop an algorithm) that lists all selections of size 2 from the objects 1, 2, 3, 4, 5, 6.

b) Repeat part (a) for selections of size 3.

4 Combinations with Repetition

When repetitions are allowed, we have seen that for n distinct objects an arrangement of size r of these objects can be obtained in n^r ways, for an integer $r \geq 0$. We now turn to the comparable problem for combinations and once again obtain a related problem whose solution follows from our previous enumeration principles.

EXAMPLE 28

On their way home from track practice, seven high school freshmen stop at a restaurant, where each of them has one of the following: a cheeseburger, a hot dog, a taco, or a fish sandwich. How many different purchases are possible (from the viewpoint of the restaurant)?

Let c, h, t, and f represent cheeseburger, hot dog, taco, and fish sandwich, respectively. Here we are concerned with how many of each item are purchased, not with the order in which they are purchased, so the problem is one of selections, or combinations, with repetition.

In Table 6 we list some possible purchases in column (a) and another means of representing each purchase in column (b).

Table 6

(a)	(b)
1. c, c, h, h, t, t, f	1. x x | x x | x x | x
2. c, c, c, c, h, t, f	2. x x x x | x | x | x
3. c, c, c, c, c, c, f	3. x x x x x x | | | x
4. h, t, t, f, f, f, f	4. | x | x x | x x x x
5. t, t, t, t, t, f, f	5. | | x x x x x | x x
6. t, t, t, t, t, t, t	6. | | x x x x x x x |
7. f, f, f, f, f, f, f	7. | | | x x x x x x x

For a purchase in column (b) of Table 6 we realize that each x to the left of the first bar (|) represents a c, each x between the first and second bars represents an h, the x's between the second and third bars stand for t's, and each x to the right of the third bar stands for an f. The third purchase, for example, has three consecutive bars because no one bought a hot dog or taco; the bar at the start of the fourth purchase indicates that there were no cheeseburgers in that purchase.

Once again a correspondence has been established between two collections of objects, where we know how to count the number in one collection. For the representations in column (b) of Table 6, we are enumerating all arrangements of 10 symbols consisting of seven x's and three |'s, so by our correspondence the number of different purchases for column (a) is

$$\frac{10!}{7!\,3!} = \binom{10}{7}.$$

In this example we note that the seven x's (one for each freshman) correspond to the size of the selection and that the three bars are needed to separate the $3 + 1 = 4$ possible food items that can be chosen.

When we wish to select, *with repetition*, r of n distinct objects, we find (as in Table 6) that we are considering all arrangements of r x's and $n - 1$ |'s and that their number is

$$\frac{(n + r - 1)!}{r!(n - 1)!} = \binom{n + r - 1}{r}.$$

Consequently, the number of combinations of n objects taken r at a time, *with repetition*, is $C(n + r - 1, r)$.

(In Example 28, $n = 4$, $r = 7$, so it is possible for r to exceed n when repetitions are allowed.)

EXAMPLE 29

A donut shop offers 20 kinds of donuts. Assuming that there are at least a dozen of each kind when we enter the shop, we can select a dozen donuts in $C(20 + 12 - 1, 12) = C(31, 12) = 141{,}120{,}525$ ways. (Here $n = 20$, $r = 12$.)

EXAMPLE 30

President Helen has four vice presidents: (1) Betty, (2) Goldie, (3) Mary Lou, and (4) Mona. She wishes to distribute among them \$1000 in Christmas bonus checks, where each check will be written for a multiple of \$100.

a) Allowing the situation in which one or more of the vice presidents get nothing, President Helen is making a selection of size 10 (one for each unit of \$100) from a collection of size 4 (four vice presidents), with repetition. This can be done in $C(4 + 10 - 1, 10) = C(13, 10) = 286$ ways.

b) If there are to be no hard feelings, each vice president should receive at least \$100. With this restriction, President Helen is now faced with making a selection of size 6 (the remaining six units of \$100) from the same collection of size 4, and the choices now number $C(4 + 6 - 1, 6) = C(9, 6) = 84$. [For example, here the selection 2, 3, 3, 4, 4, 4 is interpreted as follows: Betty does not get anything extra — for there is no 1 in the selection. The one 2 in the selection indicates that Goldie gets an additional \$100. Mary Lou receives an additional \$200 (\$100 for each of the two 3's in the selection). Due to the three 4's, Mona's bonus check will total $\$100 + 3(\$100) = \$400$.]

c) If each vice president must get at least \$100 and Mona, as executive vice president, gets at least \$500, then the number of ways President Helen can distribute the bonus checks is

$$\underbrace{C(3+2-1,\,2)}_{\substack{\textbf{Mona gets}\\ \textbf{exactly \$500}}} + \underbrace{C(3+1-1,\,1)}_{\substack{\textbf{Mona gets}\\ \textbf{exactly \$600}}} + \underbrace{C(3+0-1,\,0)}_{\substack{\textbf{Mona gets}\\ \textbf{exactly \$700}}} = 10 = \underbrace{C(4+2-1,\,2)}_{\substack{\textbf{Using the}\\ \textbf{technique in part (b)}}}$$

Having worked examples utilizing combinations with repetition, we now consider two examples involving other counting principles as well.

EXAMPLE 31

In how many ways can we distribute seven bananas and six oranges among four children so that each child receives at least one banana?

After giving each child one banana, consider the number of ways the remaining three bananas can be distributed among these four children. Table 7 shows four of the distributions we are considering here. For example, the second distribution in part (a) of Table 7 — namely, 1, 3, 3 — indicates that we have given the first child (designated by 1) one additional banana and the third child (designated by 3) two additional bananas. The corresponding arrangement in part (b) of Table 7 represents this distribution in terms of three b's and three bars. These six symbols — three of one type (the b's) and three others of a second type (the bars) — can be arranged in $6!/(3!\,3!) = C(6, 3) = C(4+3-1, 3) = 20$ ways. [Here $n = 4$, $r = 3$.] Consequently, there are 20 ways in which we can distribute the three additional bananas among these four children. Table 8 provides the comparable situation for distributing the six oranges. In this case we are arranging nine symbols — six of one type (the o's) and three of a second type (the bars). So now we learn that the number of ways we can distribute the six oranges among these four children is $9!/(6!\,3!) = C(9, 6) = C(4+6-1, 6) = 84$ ways. [Here $n = 4$, $r = 6$.] Therefore, by the rule of product, there are $20 \times 84 = 1680$ ways to distribute the fruit under the stated conditions.

Table 7

(a)	(b)
1) 1, 2, 3	1) $b \mid b \mid b \mid$
2) 1, 3, 3	2) $b \mid \mid b\,b \mid$
3) 3, 4, 4	3) $\mid \mid b \mid b\,b$
4) 4, 4, 4	4) $\mid \mid \mid b\,b\,b$

Table 8

(a)	(b)
1) 1, 2, 2, 3, 3, 4	1) $o \mid o\,o \mid o\,o \mid o$
2) 1, 2, 2, 4, 4, 4	2) $o \mid o\,o \mid \mid o\,o\,o$
3) 2, 2, 2, 3, 3, 3	3) $\mid o\,o\,o \mid o\,o\,o \mid$
4) 4, 4, 4, 4, 4, 4	4) $\mid \mid \mid o\,o\,o\,o\,o\,o$

EXAMPLE 32

A message is made up of 12 different symbols and is to be transmitted through a communication channel. In addition to the 12 symbols, the transmitter will also send a total of 45 (blank) spaces between the symbols, with at least three spaces between each pair of consecutive symbols. In how many ways can the transmitter send such a message?

There are 12! ways to arrange the 12 different symbols, and for each of these arrangements there are 11 positions between the 12 symbols. Because there must be at least three spaces between successive symbols, we use up 33 of the 45 spaces and must now locate the remaining 12 spaces. This is now a selection, with repetition, of size 12 (the spaces) from a collection of size 11 (the locations), and this can be accomplished in $C(11+12-1, 12) = 646{,}646$ ways.

Consequently, by the rule of product the transmitter can send such messages with the required spacing in $(12!)\binom{22}{12} \doteq 3.097 \times 10^{14}$ ways.

In the next example an idea is introduced that appears to have more to do with number theory than with combinations or arrangements. Nonetheless, the solution of this example will turn out to be equivalent to counting combinations with repetitions.

EXAMPLE 33

Determine all integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 7, \qquad \text{where } x_i \geq 0 \quad \text{for all } 1 \leq i \leq 4.$$

One solution of the equation is $x_1 = 3$, $x_2 = 3$, $x_3 = 0$, $x_4 = 1$. (This is different from a solution such as $x_1 = 1$, $x_2 = 0$, $x_3 = 3$, $x_4 = 3$, even though the same four integers are being used.) A possible interpretation for the solution $x_1 = 3$, $x_2 = 3$, $x_3 = 0$, $x_4 = 1$ is that we are distributing seven pennies (identical objects) among four children (distinct containers), and here we have given three pennies to each of the first two children, nothing to the third child, and the last penny to the fourth child. Continuing with this interpretation, we see that each nonnegative integer solution of the equation corresponds to a selection, with repetition, of size 7 (the *identical* pennies) from a collection of size 4 (the *distinct* children), so there are $C(4 + 7 - 1, 7) = 120$ solutions.

At this point it is crucial that we recognize the equivalence of the following:

a) The number of integer solutions of the equation

$$x_1 + x_2 + \cdots + x_n = r, \qquad x_i \geq 0, \qquad 1 \leq i \leq n.$$

b) The number of selections, with repetition, of size r from a collection of size n.

c) The number of ways r identical objects can be distributed among n distinct containers.

In terms of distributions, part (c) is valid only when the r objects being distributed are identical and the n containers are distinct. When both the r objects and the n containers are distinct, we can select any of the n containers for each one of the objects and get n^r distributions by the rule of product.

When the objects are distinct but the containers are identical, we shall solve the problem using the Stirling numbers of the second kind. For the final case, in which both objects and containers are identical, the theory of partitions of integers will provide some necessary results.

EXAMPLE 34

In how many ways can one distribute 10 (identical) white marbles among six distinct containers?

Solving this problem is equivalent to finding the number of nonnegative integer solutions to the equation $x_1 + x_2 + \cdots + x_6 = 10$. That number is the number of selections of size 10, with repetition, from a collection of size 6. Hence the answer is $C(6 + 10 - 1, 10) = 3003$.

We now examine two other examples related to the theme of this section.

EXAMPLE 35

From Example 34 we know that there are 3003 nonnegative integer solutions to the equation $x_1 + x_2 + \cdots + x_6 = 10$. How many such solutions are there to the inequality $x_1 + x_2 + \cdots + x_6 < 10$?

One approach that may seem feasible in dealing with this inequality is to determine the number of such solutions to $x_1 + x_2 + \cdots + x_6 = k$, where k is an integer and $0 \le k \le 9$. Although feasible now, the technique becomes unrealistic if 10 is replaced by a somewhat larger number, say 100. However, we shall establish a combinatorial identity that will help us obtain an alternative solution to the problem by using this approach.

For the present we transform the problem by noting the correspondence between the nonnegative integer solutions of

$$x_1 + x_2 + \cdots + x_6 < 10 \qquad (1)$$

and the integer solutions of

$$x_1 + x_2 + \cdots + x_6 + x_7 = 10, \qquad 0 \le x_i, \qquad 1 \le i \le 6, \qquad 0 < x_7. \qquad (2)$$

The number of solutions of Eq. (2) is the same as the number of nonnegative integer solutions of $y_1 + y_2 + \cdots + y_6 + y_7 = 9$, where $y_i = x_i$ for $1 \le i \le 6$, and $y_7 = x_7 - 1$. This is $C(7 + 9 - 1, 9) = 5005$.

Our next result takes us back to the binomial and multinomial expansions.

EXAMPLE 36

In the binomial expansion for $(x + y)^n$, each term is of the form $\binom{n}{k}x^k y^{n-k}$, so the total number of terms in the expansion is the number of nonnegative integer solutions of $n_1 + n_2 = n$ (n_1 is the exponent for x, n_2 the exponent for y). This number is $C(2 + n - 1, n) = n + 1$.

Perhaps it seems that we have used a rather long-winded argument to get this result. Many of us would probably be willing to believe the result on the basis of our experiences in expanding $(x + y)^n$ for various small values of n.

Although experience is worthwhile in pattern recognition, it is not always enough to find a general principle. Here it would prove of little value if we wanted to know how many terms there are in the expansion of $(w + x + y + z)^{10}$.

Each distinct term here is of the form $\binom{10}{n_1, n_2, n_3, n_4} w^{n_1} x^{n_2} y^{n_3} z^{n_4}$, where $0 \le n_i$ for $1 \le i \le 4$, and $n_1 + n_2 + n_3 + n_4 = 10$. This last equation can be solved in $C(4 + 10 - 1, 10) = 286$ ways, so there are 286 terms in the expansion of $(w + x + y + z)^{10}$.

And now once again the binomial expansion will come into play, as we find ourselves using part (a) of Corollary 1

EXAMPLE 37

a) Let us determine all the different ways in which we can write the number 4 as a sum of positive integers, where the order of the summands is considered relevant. These representations are called the *compositions* of 4 and may be listed as follows:

1) 4
2) $3 + 1$
3) $1 + 3$
4) $2 + 2$
5) $2 + 1 + 1$
6) $1 + 2 + 1$
7) $1 + 1 + 2$
8) $1 + 1 + 1 + 1$

Here we include the sum consisting of only one summand — namely, 4. We find that for the number 4 there are eight compositions in total. (If we do *not* care about the order of the summands, then the representations in (2) and (3) are no longer considered to be different — nor are the representations in (5), (6), and (7). Under these circumstances we find that there are five *partitions* for the number 4 — namely, 4; $3+1$; $2+2$; $2+1+1$; and $1+1+1+1$.)

b) Now suppose that we wish to *count* the number of compositions for the number 7. Here we do *not* want to list all of the possibilities — which include 7; $6+1$; $1+6$; $5+2$; $1+2+4$; $2+4+1$; and $3+1+2+1$. To count all of these compositions, let us consider the number of possible summands.

i) For one summand there is only one composition — namely, 7.

ii) If there are two (positive) summands, we want to count the number of integer solutions for

$$w_1 + w_2 = 7, \qquad \text{where } w_1, w_2 > 0.$$

This is equal to the number of integer solutions for

$$x_1 + x_2 = 5, \qquad \text{where } x_1, x_2 \geq 0.$$

The number of such solutions is $\binom{2+5-1}{5} = \binom{6}{5}$.

iii) Continuing with our next case, we examine the compositions with three (positive) summands. So now we want to count the number of *positive* integer solutions for

$$y_1 + y_2 + y_3 = 7.$$

This is equal to the number of *nonnegative* integer solutions for

$$z_1 + z_2 + z_3 = 4,$$

and that number is $\binom{3+4-1}{4} = \binom{6}{4}$.

We summarize cases (i), (ii), and (iii), and the other four cases in Table 9, where we recall for case (i) that $1 = \binom{6}{6}$.

Table 9

n = The Number of Summands in a Composition of 7	The Number of Compositions of 7 with n Summands
(i) $n = 1$	(i) $\binom{6}{6}$
(ii) $n = 2$	(ii) $\binom{6}{5}$
(iii) $n = 3$	(iii) $\binom{6}{4}$
(iv) $n = 4$	(iv) $\binom{6}{3}$
(v) $n = 5$	(v) $\binom{6}{2}$
(vi) $n = 6$	(vi) $\binom{6}{1}$
(vii) $n = 7$	(vii) $\binom{6}{0}$

Consequently, the results from the right-hand side of our table tell us that the (total) number of compositions of 7 is

$$\binom{6}{6}+\binom{6}{5}+\binom{6}{4}+\binom{6}{3}+\binom{6}{2}+\binom{6}{1}+\binom{6}{0}=\sum_{k=0}^{6}\binom{6}{k}.$$

From part (a) of Corollary 1 this reduces to 2^6.

In general, one finds that for each positive integer m, there are $\sum_{k=0}^{m-1}\binom{m-1}{k}=2^{m-1}$ compositions.

EXAMPLE 38

From Example 37 we know that there are $2^{12-1}=2^{11}=2048$ compositions of 12. If our interest is in those compositions where each summand is even, then we consider, for instance, compositions such as

$$\begin{array}{ll} 2+4+6=2(1+2+3) & 2+8+2=2(1+4+1) \\ 8+2+2=2(4+1+1) & \quad 6+6=2(3+3). \end{array}$$

In each of these four examples, the parenthesized expression is a composition of 6. This observation indicates that the number of compositions of 12, where each summand is even, equals the number of (all) compositions of 6, which is $2^{6-1}=2^5=32$.

Our next two examples provide applications from the area of computer science. Furthermore, the second example will lead to an important summation formula.

EXAMPLE 39

Consider the following program segment, where i, j, and k are integer variables.

```
for i := 1 to 20 do
  for j := 1 to i do
    for k := 1 to j do
      print (i * j + k)
```

How many times is the **print** statement executed in this program segment?

Among the possible choices for i, j, and k (in the order i–first, j–second, k–third) that will lead to execution of the **print** statement, we list (1) 1, 1, 1; (2) 2, 1, 1; (3) 15, 10, 1; and (4) 15, 10, 7. We note that $i=10$, $j=12$, $k=5$ is not one of the selections to be considered, because $j=12>10=i$; this violates the condition set forth in the second **for** loop. Each of the above four selections where the **print** statement is executed satisfies the condition $1\le k\le j\le i\le 20$. In fact, any selection a, b, c ($a\le b\le c$) of size 3, with repetitions allowed, from the list 1, 2, 3, . . . , 20 results in one of the correct selections: here, $k=a$, $j=b$, $i=c$. Consequently the **print** statement is executed

$$\binom{20+3-1}{3}=\binom{22}{3}=1540 \text{ times.}$$

If there had been r (≥ 1) **for** loops instead of three, the **print** statement would have been executed $\binom{20+r-1}{r}$ times.

EXAMPLE 40

Here we use a program segment to derive a summation formula. In this program segment, the variables i, j, n, and *counter* are integer variables. Furthermore, we assume that the value of n has been set prior to this segment.

```
counter := 0
for i := 1 to n do
  for j := 1 to i do
    counter := counter + 1
```

From the results in Example 39, after this segment is executed the value of (the variable) *counter* will be $\binom{n+2-1}{2} = \binom{n+1}{2}$. (This is also the number of times that the statement

(*) `counter := counter + 1`

is executed.)

This result can also be obtained as follows: When $i := 1$, then j varies from 1 to 1 and (*) is executed once; when i is assigned the value 2, then j varies from 1 to 2 and (*) is executed twice; j varies from 1 to 3 when i is assigned the value 3, and (*) is executed three times; in general, for $1 \le k \le n$, when $i := k$, then j varies from 1 to k and (*) is executed k times. In total, the variable *counter* is incremented [and the statement (*) is executed] $1 + 2 + 3 + \cdots + n$ times.

Consequently,

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \binom{n+1}{2} = \frac{n(n+1)}{2}.$$

The derivation of this summation formula, obtained by counting the same result in two different ways, constitutes a combinatorial proof.

Our last example for this section introduces the idea of a run, a notion that arises in statistics — in particular, in the detecting of trends in a statistical process.

EXAMPLE 41 The counter at Patti and Terri's Bar has 15 bar stools. Upon entering the bar Darrell finds the stools occupied as follows:

O O E O O O O E E E O O O E O,

where O indicates an occupied stool and E an empty one. (Here we are not concerned with the occupants of the stools, just whether or not a stool is occupied.) In this case we say that the occupancy of the 15 stools determines seven runs, as shown:

$$\underbrace{\text{OO}}_{\text{Run}}\ \underbrace{\text{E}}_{\text{Run}}\ \underbrace{\text{OOOO}}_{\text{Run}}\ \underbrace{\text{EEE}}_{\text{Run}}\ \underbrace{\text{OOO}}_{\text{Run}}\ \underbrace{\text{E}}_{\text{Run}}\ \underbrace{\text{O}}_{\text{Run}}.$$

In general, a *run* is a consecutive list of identical entries that are preceded and followed by different entries or no entries at all.

A second way in which five E's and 10 O's can be arranged to provide seven runs is

E O O O E E O O E O O O O O E.

We want to find the total number of ways five E's and 10 O's can determine seven runs. If the first run starts with an E, then there must be four runs of E's and three runs of O's. Consequently, the last run must end with an E.

Let x_1 count the number of E's in the first run, x_2 the number of O's in the second run, x_3 the number of E's in the third run, . . . , and x_7 the number of E's in the seventh run. We want to find the number of integer solutions for

$$x_1 + x_3 + x_5 + x_7 = 5, \qquad x_1, x_3, x_5, x_7 > 0 \tag{3}$$

and

$$x_2 + x_4 + x_6 = 10, \qquad x_2, x_4, x_6 > 0. \tag{4}$$

The number of integer solutions for Eq. (3) equals the number of integer solutions for

$$y_1 + y_3 + y_5 + y_7 = 1, \qquad y_1, y_3, y_5, y_7 \geq 0.$$

This number is $\binom{4+1-1}{1} = \binom{4}{1} = 4$. Similarly, for Eq. (4), the number of solutions is $\binom{3+7-1}{7} = \binom{9}{7} = 36$. Consequently, by the rule of product there are $4 \cdot 36 = 144$ arrangements of five E's and 10 O's that determine seven runs, the first run starting with E.

The seven runs may also have the first run starting with an O and the last run ending with an O. So now let w_1 count the number of O's in the first run, w_2 the number of E's in the second run, w_3 the number of O's in the third run, . . . , and w_7 the number of O's in the seventh run. Here we want the number of integer solutions for

$$w_1 + w_3 + w_5 + w_7 = 10, \qquad w_1, w_3, w_5, w_7 > 0$$

and

$$w_2 + w_4 + w_6 = 5, \qquad w_2, w_4, w_6 > 0.$$

Arguing as above, we find that the number of ways to arrange five E's and 10 O's, resulting in seven runs where the first run starts with an O, is $\binom{4+6-1}{6}\binom{3+2-1}{2} = \binom{9}{6}\binom{4}{2} = 504$.

Consequently, by the rule of sum, the five E's and 10 O's can be arranged in $144 + 504 = 648$ ways to produce seven runs.

EXERCISES 4

1. In how many ways can 10 (identical) dimes be distributed among five children if (a) there are no restrictions? (b) each child gets at least one dime? (c) the oldest child gets at least two dimes?

2. In how many ways can 15 (identical) candy bars be distributed among five children so that the youngest gets only one or two of them?

3. Determine how many ways 20 coins can be selected from four large containers filled with pennies, nickels, dimes, and quarters. (Each container is filled with only one type of coin.)

4. A certain ice cream store has 31 flavors of ice cream available. In how many ways can we order a dozen ice cream cones if (a) we do not want the same flavor more than once? (b) a flavor may be ordered as many as 12 times? (c) a flavor may be ordered no more than 11 times?

5. **a)** In how many ways can we select five coins from a collection of 10 consisting of one penny, one nickel, one dime, one quarter, one half-dollar, and five (identical) Susan B. Anthony dollars?

b) In how many ways can we select n objects from a collection of size $2n$ that consists of n distinct and n identical objects?

6. Answer Example 32, where the 12 symbols being transmitted are four A's, four B's, and four C's.

7. Determine the number of integer solutions of

$$x_1 + x_2 + x_3 + x_4 = 32,$$

where

a) $x_i \geq 0, \quad 1 \leq i \leq 4$ **b)** $x_i > 0, \quad 1 \leq i \leq 4$

c) $x_1, x_2 \geq 5, \quad x_3, x_4 \geq 7$

d) $x_i \geq 8, \quad 1 \leq i \leq 4$ **e)** $x_i \geq -2, \quad 1 \leq i \leq 4$

f) $x_1, x_2, x_3 > 0, \quad 0 < x_4 \leq 25$

8. In how many ways can a teacher distribute eight chocolate donuts and seven jelly donuts among three student helpers if each helper wants at least one donut of each kind?

9. Columba has two dozen each of n different colored beads. If she can select 20 beads (with repetitions of colors allowed) in 230,230 ways, what is the value of n?

10. In how many ways can Lisa toss 100 (identical) dice so that at least three of each type of face will be showing?

11. Two n-digit integers (leading zeros allowed) are considered equivalent if one is a rearrangement of the other. (For example, 12033, 20331, and 01332 are considered equivalent five-digit integers.) (a) How many five-digit integers are not equivalent? (b) If the digits 1, 3, and 7 can appear at most once, how many nonequivalent five-digit integers are there?

12. Determine the number of integer solutions for

$$x_1 + x_2 + x_3 + x_4 + x_5 < 40,$$

where

a) $x_i \geq 0, \quad 1 \leq i \leq 5$

b) $x_i \geq -3, \quad 1 \leq i \leq 5$

13. In how many ways can we distribute eight identical white balls into four distinct containers so that (a) no container is left empty? (b) the fourth container has an odd number of balls in it?

14. a) Find the coefficient of v^2w^4xz in the expansion of $(3v + 2w + x + y + z)^8$.

b) How many distinct terms arise in the expansion in part (a)?

15. In how many ways can Beth place 24 different books on four shelves so that there is at least one book on each shelf? (For any of these arrangements consider the books on each shelf to be placed one next to the other, with the first book at the left of the shelf.)

16. For which positive integer n will the equations

(1) $x_1 + x_2 + x_3 + \cdots + x_{19} = n$, and

(2) $y_1 + y_2 + y_3 + \cdots + y_{64} = n$

have the same number of positive integer solutions?

17. How many ways are there to place 12 marbles of the same size in five distinct jars if (a) the marbles are all black? (b) each marble is a different color?

18. a) How many nonnegative integer solutions are there to the pair of equations $x_1 + x_2 + x_3 + \cdots + x_7 = 37$, $x_1 + x_2 + x_3 = 6$?

b) How many solutions in part (a) have $x_1, x_2, x_3 > 0$?

19. How many times is the **print** statement executed for the following program segment? (Here, i, j, k, and m are integer variables.)

```
for i := 1 to 20 do
  for j := 1 to i do
    for k := 1 to j do
      for m := 1 to k do
        print (i * j) + (k * m)
```

20. In the following program segment, i, j, k, and *counter* are integer variables. Determine the value that the variable *counter* will have after the segment is executed.

```
counter := 10
for i := 1 to 15 do
  for j := i to 15 do
    for k := j to 15 do
      counter := counter + 1
```

21. Find the value of *sum* after the given program segment is executed. (Here i, j, k, *increment*, and *sum* are integer variables.)

```
increment := 0
sum := 0
for i := 1 to 10 do
  for j := 1 to i do
    for k := 1 to j do
      begin
        increment := increment + 1
        sum := sum + increment
      end
```

22. Consider the following program segment, where i, j, k, n, and *counter* are integer variables and the value of n (a positive integer) is set prior to this segment.

```
counter := 0
for i := 1 to n do
  for j := 1 to i do
    for k := 1 to j do
      counter := counter + 1
```

We shall determine, in two different ways, the number of times the statement

```
counter := counter + 1
```

is executed. (This is also the value of *counter* after execution of the program segment.) From the result in Example 39, we know that the statement is executed $\binom{n+3-1}{3} = \binom{n+2}{3}$ times. For a fixed value of i, the **for** loops involving j and k result in $\binom{i+1}{2}$ executions of the counter increment statement. Consequently, $\binom{n+2}{3} = \sum_{i=1}^{n} \binom{i+1}{2}$. Use this result to obtain a summation formula for

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2.$$

23. a) Given positive integers m, n with $m \geq n$, show that the number of ways to distribute m identical objects into n distinct containers with no container left empty is

$$C(m-1, m-n) = C(m-1, n-1).$$

b) Show that the number of distributions in part (a) where each container holds at least r objects ($m \geq nr$) is

$$C(m-1+(1-r)n, n-1).$$

24. Write a computer program (or develop an algorithm) to list the integer solutions for

a) $x_1 + x_2 + x_3 = 10, \quad 0 \leq x_i, \quad 1 \leq i \leq 3$

b) $x_1 + x_2 + x_3 + x_4 = 4, \quad -2 \leq x_i, \quad 1 \leq i \leq 4$

25. Consider the 2^{19} compositions of 20. (a) How many have each summand even? (b) How many have each summand a multiple of 4?

26. Let n, m, k be positive integers with $n = mk$. How many compositions of n have each summand a multiple of k?

27. Frannie tosses a coin 12 times and gets five heads and seven tails. In how many ways can these tosses result in (a) two runs of heads and one run of tails; (b) three runs; (c) four runs;

(d) five runs; (e) six runs; and (f) equal numbers of runs of heads and runs of tails?

28. **a)** For $n \geq 4$, consider the strings made up of n bits — that is, a total of n 0's and 1's. In particular, consider those strings where there are (exactly) two occurrences of 01. For example, if $n = 6$ we want to include strings such as 010010 and 100101, but not 101111 or 010101. How many such strings are there?

b) For $n \geq 6$, how many strings of n 0's and 1's contain (exactly) three occurrences of 01?

c) Provide a combinatorial proof for the following: For $n \geq 1$,

$$2^n = \binom{n+1}{1} + \binom{n+1}{3} + \cdots + \begin{cases} \binom{n+1}{n}, & n \text{ odd} \\ \binom{n+1}{n+1}, & n \text{ even.} \end{cases}$$

5 The Catalan Numbers (Optional)

In this section a very prominent sequence of numbers is introduced. This sequence arises in a wide variety of combinatorial situations. We'll begin by examining one specific instance where it is found.

EXAMPLE 42

Let us start at the point (0, 0) in the xy-plane and consider two kinds of moves:

$$\text{R: } (x, y) \rightarrow (x + 1, y) \qquad \text{U: } (x, y) \rightarrow (x, y + 1).$$

We want to know how we can move from (0, 0) to (5, 5) using such moves — one unit to the right or one unit up. So we'll need five R's and five U's. At this point we have a situation like that in Example 14, so we know there are $10!/(5!\,5!) = \binom{10}{5}$ such paths. But now we'll add a twist! In going from (0, 0) to (5, 5) one may touch but *never* rise above the line $y = x$. Consequently, we want to include paths such as those shown in parts (a) and (b) of Fig. 9 but not the path shown in part (c).

The first thing that is evident is that each such arrangement of five R's and five U's must start with an R (and end with a U). Then as we move across this type of arrangement — going from left to right — the number of R's at any point must equal or exceed the number of U's. Note how this happens in parts (a) and (b) of Fig. 9 but not in part (c). Now we can solve the problem at hand if we can count the paths [like the one in part (c)] that go from (0, 0) to (5, 5) but rise above the line $y = x$. Look again at the path in part (c) of Fig. 9. Where does the situation there break down for the first time? After all, we start with the requisite R — then follow it by a U. So far, so good! But then there is a second U and, at this (first) time, the number of U's exceeds the number of R's.

Now let us consider the following transformation:

$$\text{R, U, U, } \mid \text{ U, R, R, R, U, U, R} \leftrightarrow \text{R, U, U, } \mid \text{ R, U, U, U, R, R, U.}$$

What have we done here? For the path on the left-hand side of the transformation, we located the first move (the second U) where the path rose above the line $y = x$. The moves up to and including this move (the second U) remain as is, but the moves that follow are interchanged — each U is replaced by an R and each R by a U. The result is the path on the right-hand side of the transformation — an arrangement of four R's and six U's, as seen in part (d) of Fig. 9. Part (e) of that figure provides another path to be avoided; part (f) shows what happens when this path is transformed by the method described above. Now suppose we start with an arrangement of six U's and four R's, say

$$\text{R, U, R, R, U, U, U, } \mid \text{ U, U, R.}$$

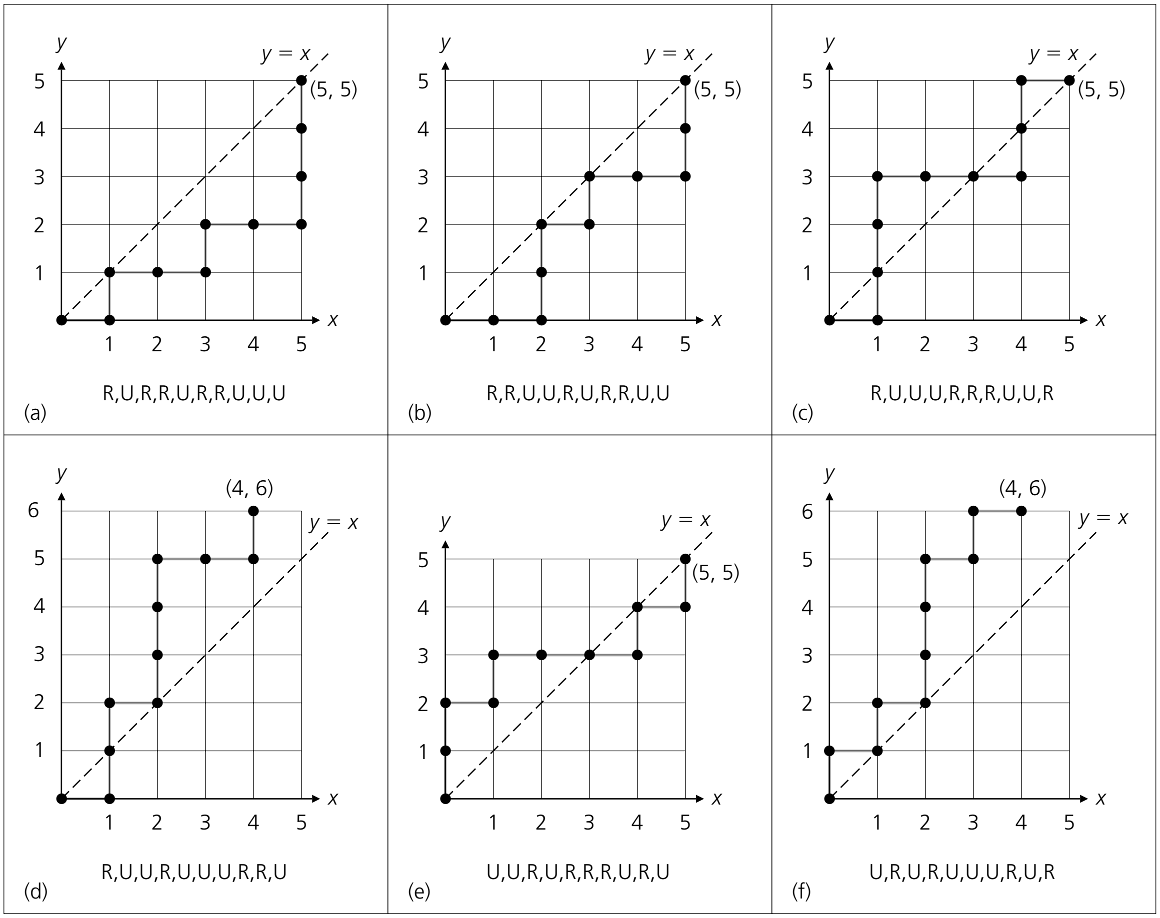

Figure 9

Focus on the first place where the number of U's exceeds the number of R's. Here it is in the seventh position, the location of the fourth U. This arrangement is now transformed as follows: The moves up to and including the fourth U remain as they are; the last three moves are interchanged — each U is replaced by an R, each R by a U. This results in the arrangement

R, U, R, R, U, U, U, ┆ R, R, U.

— one of the *bad* arrangements (of five R's and five U's) we wish to avoid as we go from (0, 0) to (5, 5). The correspondence established by these transformations gives us a way to count the number of bad arrangements. We alternatively count the number of ways to arrange four R's and six U's — this is $10!/(4!\,6!) = \binom{10}{4}$. Consequently, the number of ways to go from (0, 0) to (5, 5) without rising above the line $y = x$ is

$$\binom{10}{5} - \binom{10}{4} = \frac{10!}{5!\,5!} - \frac{10!}{4!\,6!} = \frac{6(10)! - 5(10)!}{6!\,5!}$$
$$= \left(\frac{1}{6}\right)\left(\frac{10!}{5!\,5!}\right) = \frac{1}{(5+1)}\binom{10}{5} = \frac{1}{(5+1)}\binom{2\cdot 5}{5} = 42.$$

The above result generalizes as follows. For any integer $n \geq 0$, the number of paths (made up of n R's and n U's) going from (0, 0) to (n, n), without rising above the line $y = x$, is

$$b_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}, \qquad n \geq 1, \qquad b_0 = 1.$$

The numbers $b_0, b_1, b_2, \ldots$ are called the *Catalan numbers*, after the Belgian mathematician Eugène Charles Catalan (1814–1894), who used them in determining the number of ways to parenthesize the product $x_1x_2x_3x_4 \cdots x_n$. For instance, the five $(= b_3)$ ways to parenthesize $x_1x_2x_3x_4$ are:

$$(((x_1x_2)x_3)x_4) \quad ((x_1(x_2x_3))x_4) \quad ((x_1x_2)(x_3x_4)) \quad (x_1((x_2x_3)x_4)) \quad (x_1(x_2(x_3x_4))).$$

The first seven Catalan numbers are $b_0 = 1$, $b_1 = 1$, $b_2 = 2$, $b_3 = 5$, $b_4 = 14$, $b_5 = 42$, and $b_6 = 132$.

EXAMPLE 43

Here are some other situations where the Catalan numbers arise. Some of these examples are very much like the result in Example 42. A change in vocabulary is often the only difference.

a) In how many ways can one arrange three 1's and three −1's so that all six partial sums (starting with the first summand) are nonnegative? There are five $(= b_3)$ such arrangements:

1, 1, 1, −1, −1, −1	1, 1, −1, −1, 1, −1	1, −1, 1, 1, −1, −1
	1, 1, −1, 1, −1, −1	1, −1, 1, −1, 1, −1

In general, for $n \geq 0$, one can arrange n 1's and n −1's, with all $2n$ partial sums nonnegative, in b_n ways.

b) Given four 1's and four 0's, there are 14 $(= b_4)$ ways to list these eight symbols so that in each list the number of 0's never exceeds the number of 1's (as a list is read from left to right). The following shows these 14 lists:

10101010	11001010	11100010
10101100	11001100	11100100
10110010	11010010	11101000
10110100	11010100	
10111000	11011000	11110000

For $n \geq 0$, there are b_n such lists of n 1's and n 0's.

c) Table 10

$(((ab)c)d)$	$(((abc$	111000
$((a(bc))d)$	$((a(bc$	110100
$((ab)(cd))$	$((ab(c$	110010
$(a((bc)d))$	$(a((bc$	101100
$(a(b(cd)))$	$(a(b(c$	101010

Consider the first column in Table 10. Here we find five ways to parenthesize the product $abcd$. The first of these is $(((ab)c)d)$. Reading left to right, we list the three occurrences of the left parenthesis "(" and the letters a, b, c — maintaining the order in which these six symbols occur. This results in $(((abc$, the first expression in column 2 of Table 10. Likewise, $((a(bc))d)$ in column 1 corresponds to $((a(bc$ in column 2 — and so on, for the other three entries in each of columns 1 and 2. Now one can also go backward, from column 2 to column 1. Take an expression in column 2 and append "d)" to the right end. For instance, $((ab(c$ becomes $((ab(cd)$. Reading this new expression from left to right, we now insert a right parenthesis ")" whenever a product of two results arises. So, for example, $((ab(cd)$ becomes

$$((ab)(cd))$$

For the product of *a* and *b*

For the product of (*ab*) and (*cd*)

The correspondence between the entries in columns 2 and 3 is more immediate. For an entry in column 2 replace each "(" by a "1" and each letter by a "0". Reversing this process, we replace each "1" by a "(", the first 0 by a, the second by b, and the third by c. This takes us from the entries in column 3 to those in column 2.

Now consider the correspondence between columns 1 and 3. (This correspondence arises from the correspondence between columns 1 and 2 and the one between columns 2 and 3.) It shows us that the number of ways to parenthesize the product $abcd$ equals the number of ways to list three 1's and three 0's so that, as such a list is read from left to right, the number of 1's always equals or exceeds the number of 0's. The number of ways here is 5 $(= b_3)$.

In general, one can parenthesize the product $x_1 x_2 x_3 \cdots x_n$ in b_{n-1} ways.

d) Let us arrange the integers 1, 2, 3, 4, 5, 6 in two rows of three so that (1) the integers increase in value as each row is read, from left to right, and (2) in any column the smaller integer is on top. For example, one way to do this is

$$\begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & 6 \end{array}$$

Now consider three 1's and three 0's. Arrange these six symbols in a list so that the 1's are in positions 1, 2, 4 (the top row) and the 0's are in positions 3, 5, 6 (the bottom row). The result is 110100. Reversing the process, start with another list, say 101100 (where the number of 0's never exceeds the number of 1's, as the list is read from left to right). The 1's are in positions 1, 3, 4 and the 0's are in positions 2, 5, 6. This corresponds to the arrangement

$$\begin{array}{ccc} 1 & 3 & 4 \\ 2 & 5 & 6 \end{array}$$

which satisfies conditions (1) and (2), as stated above. From this correspondence we learn that the number of ways to arrange 1, 2, 3, 4, 5, 6, so that conditions (1) and (2) are satisfied, is the number of ways to arrange three 1's and three 0's in a list so that as the six symbols are read, from left to right, the number of 0's never exceeds the number of 1's. Consequently, one can arrange 1, 2, 3, 4, 5, 6 and satisfy conditions (1) and (2) in b_3 $(= 5)$ ways.

EXERCISES 5

1. Verify that for each integer $n \geq 1$,

$$\binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}.$$

2. Determine the value of b_7, b_8, b_9, and b_{10}.

3. a) In how many ways can one travel in the xy-plane from (0, 0) to (3, 3) using the moves R: $(x, y) \to (x + 1, y)$ and U: $(x, y) \to (x, y + 1)$, if the path taken may touch but *never* fall below the line $y = x$? In how many ways from (0, 0) to (4, 4)?

b) Generalize the results in part (a).

c) What can one say about the first and last moves of the paths in parts (a) and (b)?

4. Consider the moves

$$\text{R}: (x, y) \to (x + 1, y) \quad \text{and} \quad \text{U}: (x, y) \to (x, y + 1),$$

as in Example 42. In how many ways can one go

a) from (0, 0) to (6, 6) and not rise above the line $y = x$?

b) from (2, 1) to (7, 6) and not rise above the line $y = x - 1$?

c) from (3, 8) to (10, 15) and not rise above the line $y = x + 5$?

5. Find the other three ways to arrange 1, 2, 3, 4, 5, 6 in two rows of three so that the conditions in part (d) of Example 43 are satisfied.

6. There are b_4 (= 14) ways to arrange 1, 2, 3, . . . , 8 in two rows of four so that (1) the integers increase in value as each row is read, from left to right, and (2) in any column the smaller integer is on top. Find, as in part (d) of Example 43,

a) the arrangements that correspond to each of the following.

i) 10110010 **ii)** 11001010 **iii)** 11101000

b) the lists of four 1's and four 0's that correspond to each of these arrangements of 1, 2, 3, . . . , 8.

i)
1 3 4 5
2 6 7 8

ii)
1 2 3 7
4 5 6 8

iii)
1 2 4 5
3 6 7 8

7. In how many ways can one parenthesize the product $abcdef$?

8. There are 132 ways in which one can parenthesize the product $abcdefg$.

a) Determine, as in part (c) of Example 43, the list of five 1's and five 0's that corresponds to each of the following.

i) $(((ab)c)(d(ef)))$
ii) $(a(b(c(d(ef)))))$
iii) $((((ab)(cd))e)f)$

b) Find, as in Example 43, the way to parenthesize $abcdef$ that corresponds to each given list of five 1's and five 0's.

i) 1110010100
ii) 1100110010
iii) 1011100100

9. Consider drawing n semicircles on and above a horizontal line, with no two semicircles intersecting. In parts (a) and (b) of Fig. 10 we find the two ways this can be done for $n = 2$; the results for $n = 3$ are shown in parts (c)–(g).

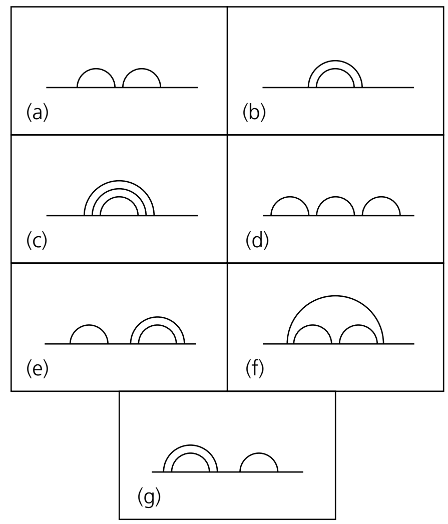

Figure 10

i) How many different drawings are there for four semicircles?

ii) How many for any $n \geq 0$? Explain why.

10. a) In how many ways can one go from (0, 0) to (7, 3) if the only moves permitted are R: $(x, y) \to (x + 1, y)$ and U: $(x, y) \to (x, y + 1)$, and the number of U's may never exceed the number of R's along the path taken?

b) Let m, n be positive integers with $m > n$. Answer the question posed in part (a), upon replacing 7 by m and 3 by n.

11. Twelve patrons, six each with a \$5 bill and the other six each with a \$10 bill, are the first to arrive at a movie theater, where the price of admission is five dollars. In how many ways can these 12 individuals (all loners) line up so that the number with a \$5 bill is never exceeded by the number with a \$10 bill (and, as a result, the ticket seller is always able to make any necessary change from the bills taken in from the first 11 of these 12 patrons)?

6
Summary and Historical Review

In this chapter we introduced the fundamentals for counting combinations, permutations, and arrangements in a large variety of problems. The breakdown of problems into components requiring the same or different formulas for their solutions provided a key insight into the areas of discrete and combinatorial mathematics. This is somewhat similar to the *top-down approach* for developing algorithms in a structured programming language. Here one develops the algorithm for the solution of a difficult problem by first considering major subproblems that need to be solved. These subproblems are then further *refined* — subdivided into more easily workable programming tasks. Each level of refinement improves on the clarity, precision, and thoroughness of the algorithm until it is readily translatable into the code of the programming language.

Table 11 summarizes the major counting formulas we have developed so far. Here we are dealing with a collection of n distinct objects. The formulas count the number of ways to select, or order, with or without repetitions, r of these n objects.

Table 11

Order Is Relevant	Repetitions Are Allowed	Type of Result	Formula
Yes	No	Permutation	$P(n, r) = n!/(n-r)!,$ $0 \leq r \leq n$
Yes	Yes	Arrangement	$n^r, \quad n, r \geq 0$
No	No	Combination	$C(n, r) = n!/[r!(n-r)!] = \binom{n}{r},$ $0 \leq r \leq n$
No	Yes	Combination with repetition	$\binom{n+r-1}{r}, \quad n, r \geq 0$

As we continue to investigate further principles of enumeration, as well as discrete mathematical structures for applications in coding theory, enumeration, optimization, and sorting schemes in computer science, we shall rely on the fundamental ideas introduced in this chapter.

The notion of permutation can be found in the Hebrew work *Sefer Yetzirah* (*The Book of Creation*), a manuscript written by a mystic sometime between 200 and 600. However, even earlier, it is of interest to note that a result of Xenocrates of Chalcedon (396–314 B.C.) may possibly contain “the first attempt on record to solve a difficult problem in permutations and combinations.” For further details consult page 319 of the text by T. L. Heath [4], as well as page 113 of the article by N. L. Biggs [1], a valuable source on the history of enumeration. The first textbook dealing with some of the material we discussed in this chapter was *Ars Conjectandi* by the Swiss mathematician Jakob Bernoulli (1654–1705). The text was published posthumously in 1713 and contained a reprint of the first formal treatise on probability. This treatise had been written in 1657 by Christiaan Huygens (1629–1695), the Dutch physicist, mathematician, and astronomer who discovered the rings of Saturn.

The binomial theorem for $n = 2$ appears in the work of Euclid (300 B.C.), but it was not until the sixteenth century that the term "binomial coefficient" was actually introduced by Michel Stifel (1486–1567). In his *Arithmetica Integra* (1544) he gives the binomial coefficients up to the order of $n = 17$. Blaise Pascal (1623–1662), in his research on probability, published in the 1650s a treatise dealing with the relationships among binomial coefficients, combinations, and polynomials. These results were used by Jakob Bernoulli in proving the general form of the binomial theorem in a manner analogous to that presented in this chapter. Actual use of the symbol $\binom{n}{r}$ did not begin until the nineteenth century, when it was used by Andreas von Ettinghausen (1796–1878).

Countess of Lovelace

Blaise Pascal (1623–1662)

It was not until the twentieth century, however, that the advent of the computer made possible the systematic analysis of processes and algorithms used to generate permutations and combinations.

The first comprehensive textbook dealing with topics in combinations and permutations was written by W. A. Whitworth [10]. Also dealing with the material of this chapter are Chapter 2 of D. I. Cohen [2], Chapter 1 of C. L. Liu [5], Chapter 2 of F. S. Roberts [6], Chapter 4 of K. H. Rosen [7], Chapter 1 of H. J. Ryser [8], and Chapter 5 of A. Tucker [9].

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SUPPLEMENTARY EXERCISES

1. In the manufacture of a certain type of automobile, four kinds of major defects and seven kinds of minor defects can occur. For those situations in which defects do occur, in how many ways can there be twice as many minor defects as there are major ones?

2. A machine has nine different dials, each with five settings labeled 0, 1, 2, 3, and 4.

a) In how many ways can all the dials on the machine be set?

b) If the nine dials are arranged in a line at the top of the machine, how many of the machine settings have no two adjacent dials with the same setting?

3. Twelve points are placed on the circumference of a circle and all the chords connecting these points are drawn. What is the largest number of points of intersection for these chords?

4. A choir director must select six hymns for a Sunday church service. She has three hymn books, each containing 25 hymns (there are 75 different hymns in all). In how many ways can she select the hymns if she wishes to select (a) two hymns from each book? (b) at least one hymn from each book?

5. How many ways are there to place 25 different flags on 10 numbered flagpoles if the order of the flags on a flagpole is (a) not relevant? (b) relevant? (c) relevant and every flagpole flies at least one flag?

6. A penny is tossed 60 times yielding 45 heads and 15 tails. In how many ways could this have happened so that there were no consecutive tails?

7. There are 12 men at a dance. (a) In how many ways can eight of them be selected to form a cleanup crew? (b) How many ways are there to pair off eight women at the dance with eight of these 12 men?

8. In how many ways can the letters in WONDERING be arranged with exactly two consecutive vowels?

9. Dustin has a set of 180 distinct blocks. Each of these blocks is made of either wood or plastic and comes in one of three sizes (small, medium, large), five colors (red, white, blue, yellow, green), and six shapes (triangular, square, rectangular, hexagonal, octagonal, circular). How many of the blocks in this set differ from

a) the *small red wooden square* block in exactly one way? (For example, the *small red plastic square* block is one such block.)

b) the *large blue plastic hexagonal* block in exactly two ways? (For example, the *small red plastic hexagonal* block is one such block.)

10. Mr. and Mrs. Richardson want to name their new daughter so that her initials (first, middle, and last) will be in alphabetical order with no repeated initial. How many such triples of initials can occur under these circumstances?

11. In how many ways can the 11 identical horses on a carousel be painted so that three are brown, three are white, and five are black?

12. In how many ways can a teacher distribute 12 different science books among 16 students if (a) no student gets more than one book? (b) the oldest student gets two books but no other student gets more than one book?

13. Four numbers are selected from the following list of numbers: $-5, -4, -3, -2, -1, 1, 2, 3, 4$. (a) In how many ways can the selections be made so that the product of the four numbers is positive and (i) the numbers are distinct? (ii) each number may be selected as many as four times? (iii) each number may be selected at most three times? (b) Answer part (a) with the product of the four numbers negative.

14. Waterbury Hall, a university residence hall for men, is operated under the supervision of Mr. Kelly. The residence has three floors, each of which is divided into four sections. This coming fall Mr. Kelly will have 12 resident assistants (one for each of the 12 sections). Among these 12 assistants are the four senior assistants — Mr. DiRocco, Mr. Fairbanks, Mr. Hyland, and Mr. Thornhill. (The other eight assistants will be new this fall and are designated as junior assistants.) In how many ways can Mr. Kelly assign his 12 assistants if

a) there are no restrictions?

b) Mr. DiRocco and Mr. Fairbanks must both be assigned to the first floor?

c) Mr. Hyland and Mr. Thornhill must be assigned to different floors?

15. a) How many of the 9000 four-digit integers 1000, 1001, 1002, . . . , 9998, 9999 have four distinct digits that are either increasing (as in 1347 and 6789) or decreasing (as in 6421 and 8653)?

b) How many of the 9000 four-digit integers 1000, 1001, 1002, . . . , 9998, 9999 have four digits that are either nondecreasing (as in 1347, 1226, and 7778) or nonincreasing (as in 6421, 6622, and 9888)?

16. a) Find the coefficient of x^2yz^2 in the expansion of $[(x/2) + y - 3z]^5$.

b) How many distinct terms are there in the complete expansion of

$$\left(\frac{x}{2} + y - 3z\right)^5 ?$$

c) What is the sum of all coefficients in the complete expansion?

17. a) In how many ways can 10 people, denoted A, B, . . . , I, J, be seated about the rectangular table shown in Fig. 11, where Figs. 11(a) and 11(b) are considered the same but are considered different from Fig. 11(c)?

b) In how many of the arrangements of part (a) are A and B seated on longer sides of the table across from each other?

18. a) Determine the number of nonnegative integer solutions to the pair of equations

$$x_1 + x_2 + x_3 = 6, \qquad x_1 + x_2 + \cdots + x_5 = 15,$$
$$x_i \geq 0, \quad 1 \leq i \leq 5.$$

b) Answer part (a) with the pair of equations replaced by the pair of inequalities

$$x_1 + x_2 + x_3 \leq 6, \qquad x_1 + x_2 + \cdots + x_5 \leq 15,$$
$$x_i \geq 0, \quad 1 \leq i \leq 5.$$

19. For any given set in a tennis tournament, opponent A can beat opponent B in seven different ways. (At 6–6 they play a tie breaker.) The first opponent to win three sets wins the tournament. (a) In how many ways can scores be recorded with A winning in five sets? (b) In how many ways can scores be recorded with the tournament requiring at least four sets?

20. Given n distinct objects, determine in how many ways r of these objects can be arranged in a circle, where arrangements are considered the same if one can be obtained from the other by rotation.

21. For every positive integer n, show that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

22. a) In how many ways can the letters in UNUSUAL be arranged?

b) For the arrangements in part (a), how many have all three U's together?

c) How many of the arrangements in part (a) have no consecutive U's?

23. Francesca has 20 different books but the shelf in her dormitory residence will hold only 12 of them.

a) In how many ways can Francesca line up 12 of these books on her bookshelf?

b) How many of the arrangements in part (a) include Francesca's three books on tennis?

24. Determine the value of the integer variable *counter* after execution of the following program segment. (Here i, j, k, l, m, and n are integer variables. The variables r, s, and t are also integer variables; their values — where $r \geq 1$, $s \geq 5$, and $t \geq 7$ — have been set prior to this segment.)

```
counter := 10
for i := 1 to 12 do
  for j := 1 to r do
    counter := counter + 2
for k := 5 to s do
  for l := 3 to k do
    counter := counter + 4
for m := 3 to 12 do
  counter := counter + 6
for n := t downto 7 do
  counter := counter + 8
```

25. a) Find the number of ways to write 17 as a sum of 1's and 2's if order is relevant.

b) Answer part (a) for 18 in place of 17.

c) Generalize the results in parts (a) and (b) for n odd and for n even.

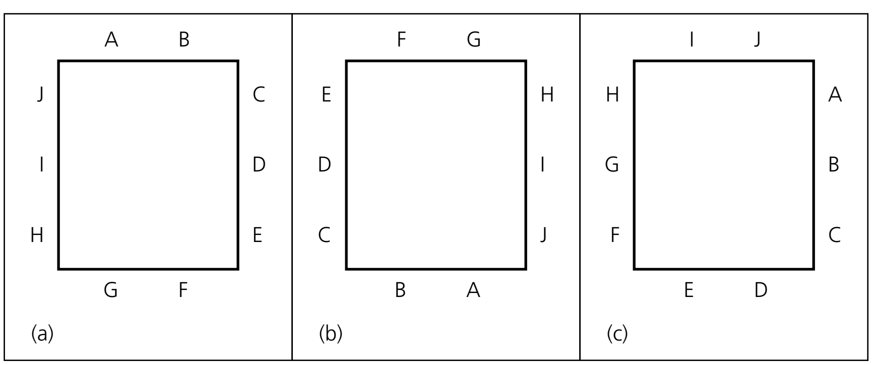


Figure 11

26. a) In how many ways can 17 be written as a sum of 2's and 3's if the order of the summands is (i) not relevant? (ii) relevant?

b) Answer part (a) for 18 in place of 17.

27. a) If n and r are positive integers with $n \geq r$, how many solutions are there to

$$x_1 + x_2 + \cdots + x_r = n,$$

where each x_i is a positive integer, for $1 \leq i \leq r$?

b) In how many ways can a positive integer n be written as a sum of r positive integer summands ($1 \leq r \leq n$) if the order of the summands is relevant?

28. a) In how many ways can one travel in the xy-plane from (1, 2) to (5, 9) if each move is one of the following types:

$$\text{(R)}: (x, y) \to (x+1, y); \quad \text{(U)}: (x, y) \to (x, y+1)?$$

b) Answer part (a) if a third (diagonal) move

$$\text{(D)}: (x, y) \to (x+1, y+1)$$

is also possible.

29. a) In how many ways can a particle move in the xy-plane from the origin to the point (7, 4) if the moves that are allowed are of the form:

$$\text{(R)}: (x, y) \to (x+1, y); \quad \text{(U)}: (x, y) \to (x, y+1)?$$

b) How many of the paths in part (a) do not use the path from (2, 2) to (3, 2) to (4, 2) to (4, 3) shown in Fig. 12?

c) Answer parts (a) and (b) if a third type of move

$$\text{(D)}: (x, y) \to (x+1, y+1)$$

is also allowed.

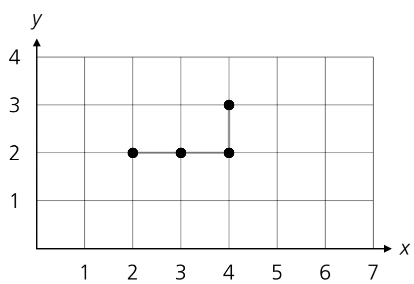


Figure 12

30. Due to their outstanding academic records, Donna and Katalin are the finalists for the outstanding physics student (in their college graduating class). A committee of 14 faculty members will each select one of the candidates to be the winner and place his or her choice (checked off on a ballot) into the ballot box. Suppose that Katalin receives nine votes and Donna receives five. In how many ways can the ballots be selected, one at a time, from the ballot box so that there are always more votes in favor of Katalin? [This is a special case of a general problem called, appropriately, *the ballot problem*. This problem was solved by Joseph Louis François Bertrand (1822–1900).]

31. Consider the 8×5 grid shown in Fig. 13. How many different rectangles (with integer-coordinate corners) does this grid contain? [For example, there is a rectangle (square) with corners (1, 1), (2, 1), (2, 2), (1, 2), a second rectangle with corners (3, 2), (4, 2), (4, 4), (3, 4), and a third with corners (5, 0), (7, 0), (7, 3) (5, 3).]

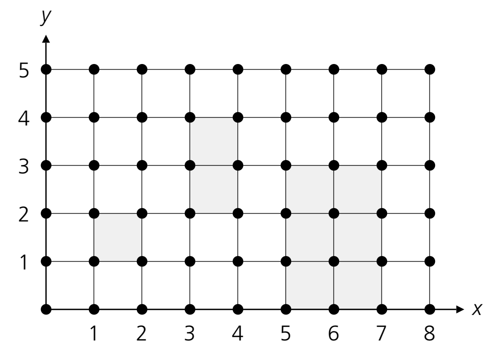


Figure 13

32. As head of quality control, Silvia examined 15 motors, one at a time, and found six defective (D) motors and nine in good (G) working condition. If she listed each finding (of D or G) after examining each individual motor, in how many ways could Silvia's list start with a run of three G's and have six runs in total?

33. In order to graduate on schedule, Hunter must take (and pass) four mathematics electives during his final six quarters. If he may select these electives from a list of 12 (that are offered every quarter) and he does not want to take more than one of these electives in any given quarter, in how many ways can he select and schedule these four electives?

34. In how many ways can a family of four (mother, father, and two children) be seated at a round table, with eight other people, so that the parents are seated next to each other and there is one child on a side of each parent? (Two seatings are considered the same if one can be rotated to look like the other.)

Solutions

Fundamental Principles of Counting

Sections 1 and 2

1. a) 13 **b)** 40 **c)** The rule of sum in part (a); the rule of product in part (b)
3. a) 288 **b)** 24
5. $2 \times 2 \times 1 \times 10 \times 10 \times 2 = 800$ different license plates
7. 2^9 **9. a)** $(14)(12) = 168$ **b)** $(14)(12)(6)(18) = 18{,}144$ **c)** 73,156,608
11. a) $12 + 2 = 14$ **b)** $14 \times 14 = 196$ **c)** 182
13. a) $P(8, 8) = 8!$ **b)** $7!$ $6!$ **15.** $4! = 24$
17. Class A: $(2^7 - 2)(2^{24} - 2) = 2{,}113{,}928{,}964$
Class B: $2^{14}(2^{16} - 2) = 1{,}073{,}709{,}056$
Class C: $2^{12}(2^8 - 2) = 1{,}040{,}384$
19. a) $7! = 5040$ **b)** $(4!)(3!) = 144$ **c)** $(5!)(3!) = 720$ **d)** 288
21. a) $12!/(3!\,2!\,2!\,2!)$ **b)** $2[11!/(3!\,2!\,2!\,2!)]$ **c)** $[7!/(2!\,2!)][6!/(3!\,2!)]$
23. $12!/(4!\,3!\,2!\,3!) = 277{,}200$ **25. a)** $n = 10$ **b)** $n = 5$ **c)** $n = 5$
27. a) $(10!)/(2!\,7!) = 360$ **b)** 360
c) Let x, y, and z be any real numbers and let m, n, and p be any nonnegative integers. The number of paths from (x, y, z) to $(x + m, y + n, z + p)$, as described in part (a), is $(m + n + p)!/(m!\,n!\,p!)$.
29. a) 576 **b)** The rule of product
31. a) $9 \times 9 \times 8 \times 7 \times 6 \times 5 = 136{,}080$ **b)** 9×10^5
(i) (a) 68,880 (b) 450,000
(ii) (a) 28,560 (b) 180,000
(iii) (a) 33,600 (b) 225,000
33. a) 2^{10} **b)** 3^{10} **35. a)** $6!$ **b)** $2(5!) = 240$
37. $\binom{16}{10}9!\,5! = 348{,}713{,}164{,}800$

Section 3

1. $\binom{6}{2} = 6!/(2!\,4!) = 15$. The selections of size 2 are ab, ac, ad, ae, af, bc, bd, be, bf, cd, ce, cf, de, df, and ef.
3. a) $C(10, 4) = 10!/(4!\,6!) = 210$ **b)** $\binom{12}{7} = 12!/(7!\,5!) = 792$
c) $C(14, 12) = 91$ **d)** $\binom{15}{10} = 3003$
5. a) $P(5, 3) = 60$
b) a, f, m a, f, r a, f, t a, m, r a, m, t
a, r, t f, m, r f, m, t f, r, t m, r, t
7. a) $\binom{20}{12} = 125{,}970$ **b)** $\binom{10}{6}\binom{10}{6} = 44{,}100$ **c)** $\sum_{i=1}^{5}\binom{10}{12-2i}\binom{10}{2i}$
d) $\sum_{i=7}^{10}\binom{10}{i}\binom{10}{12-i}$ **e)** $\sum_{i=8}^{10}\binom{10}{i}\binom{10}{12-i}$
9. a) $\binom{8}{2} = 28$ **b)** 70 **c)** $\binom{8}{6} = 28$ **d)** 37
11. a) 120 **b)** 56 **c)** 100
13. $\binom{8}{4}\left(\dfrac{7!}{4!\,2!}\right) = 7350$
15. a) $\binom{15}{2} = 105$ **b)** $\binom{25}{3} = 2300$; $\binom{25}{3}$; $\binom{25}{4} = 12{,}650$
17. a) $\sum_{k=2}^{n}\dfrac{1}{k!}$ **c)** $\sum_{j=1}^{7}(-1)^{j-1}j^3 = \sum_{k=1}^{7}(-1)^{k+1}k^3$ **d)** $\sum_{i=0}^{n}\dfrac{i+1}{n+i}$
19. $\binom{10}{3} + \binom{10}{1}\binom{9}{1} + \binom{10}{1} = 220$ $\binom{10}{4} + \binom{10}{2} + \binom{10}{1}\binom{9}{2} + \binom{10}{1}\binom{9}{1} = 705$
$2^{10}\left(\sum_{i=0}^{5}\binom{10}{2i}\right)$
21. $\binom{n}{3}$ $\binom{n}{3} - n - n(n-4)$, $n \geq 4$
23. a) $\binom{12}{9}$ **b)** $\binom{12}{9}(2^3)$ **c)** $\binom{12}{9}(2^9)(-3)^3$
25. a) $\binom{4}{1,1,2} = 12$ **b)** 12 **c)** $\binom{4}{1,1,2}(2)(-1)(-1)^2 = -24$
d) -216 **e)** $\binom{8}{3,2,1,2}(2^3)(-1)^2(3)(-2)^2 = 161{,}280$
27. a) 2^3 **b)** 2^{10} **c)** 3^{10} **d)** 4^5 **e)** 4^{10}

29. $n\binom{m+n}{m} = n\dfrac{(m+n)!}{m!\,n!} = \dfrac{(m+n)!}{m!(n-1)!} = (m+1)\dfrac{(m+n)!}{(m+1)(m!)(n-1)!}$

$= (m+1)\dfrac{(m+n)!}{(m+1)!(n-1)!} = (m+1)\binom{m+n}{m+1}$

31. Consider the expansions of (a) $[(1+x)-x]^n$; (b) $[(2+x)-(x+1)]^n$; and (c) $[(2+x)-x]^n$.

33. **a)** $a_3 - a_0$ **b)** $a_n - a_0$ **c)** $\frac{1}{102} - \frac{1}{2} = \frac{-25}{51}$

Section 4

1. **a)** $\binom{14}{10}$ **b)** $\binom{9}{5}$ **c)** $\binom{12}{8}$ **3.** $\binom{23}{20}$ **5.** **a)** 2^5 **b)** 2^n

7. **a)** $\binom{35}{32}$ **b)** $\binom{31}{28}$ **c)** $\binom{11}{8}$ **d)** 1 **e)** $\binom{43}{40}$ **f)** $\binom{31}{28} - \binom{6}{3}$

9. $n = 7$ **11.** **a)** $\binom{14}{5}$ **b)** $\binom{11}{5} + 3\binom{10}{4} + 3\binom{9}{3} + \binom{8}{2}$

13. **a)** $\binom{7}{4}$ **b)** $\sum_{i=0}^{3}\binom{9-2i}{7-2i}$ **15.** $\binom{23}{20}(24!)$ **17.** **a)** $\binom{16}{12}$ **b)** 5^{12}

19. $\binom{23}{4}$ **21.** $24{,}310 = \sum_{i=1}^{n} i$ [for $n = \binom{12}{3}$]

23. **a)** Place one of the m identical objects into each of the n distinct containers. This leaves $m-n$ identical objects to be placed into the n distinct containers, resulting in $\binom{n+(m-n)-1}{m-n} = \binom{m-1}{m-n} = \binom{m-1}{n-1}$ distributions.

25. **a)** 2^9 **b)** 2^4

27. **a)** $\binom{2+3-1}{3} = 4$ **b)** 10 **c)** 48 **d)** $\binom{3+4-1}{4}\binom{2+3-1}{3} + \binom{3+2-1}{2}\binom{2+5-1}{5} = 96$
e) 180 **f)** 420

Section 5

1. $\binom{2n}{n} - \binom{2n}{n-1} = \dfrac{(2n)!}{n!\,n!} - \dfrac{(2n)!}{(n-1)!(n+1)!} = \dfrac{(2n)!(n+1)}{(n+1)!\,n!} - \dfrac{(2n)!\,n}{n!(n+1)!} =$
$\dfrac{(2n)![(n+1)-n]}{(n+1)!\,n!} = \dfrac{1}{(n+1)}\dfrac{(2n)!}{n!\,n!} = \left(\dfrac{1}{n+1}\right)\binom{2n}{n}$

3. **a)** 5 $(= b_3)$; 14 $(= b_4)$

b) For $n \geq 0$ there are b_n $\left(= \dfrac{1}{(n+1)}\binom{2n}{n}\right)$ such paths from $(0, 0)$ to (n, n).

c) For $n \geq 0$ the first move is U and the last is R.

5. Using the results in the third column of Table 10 we have:

111000	110010	101010
1 2 3	1 2 5	1 3 5
4 5 6	3 4 6	2 4 6

7. There are $b_5 (= 42)$ ways.

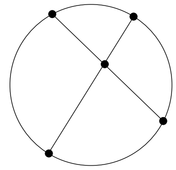
9. (i) When $n = 4$ there are 14 $(= b_4)$ such diagrams.
(ii) For each $n \geq 0$, there are b_n different drawings of n semicircles on and above a horizontal line, with no two semicircles intersecting. Consider, for instance, the diagram in part (f) of Fig. 10. Going from left to right, write 1 the first time you encounter a semicircle and write 0 the second time that semicircle is encountered. Here we get the list 110100. The list 110010 corresponds with the drawing in part (g). This correspondence shows that the number of such drawings for n semicircles is the same as the number of lists of n 1's and n 0's where, as the list is read from left to right, the number of 0's never exceeds the number of 1's.

11. $\left(\dfrac{1}{7}\right)\binom{12}{6}(6!)(6!) = \left(\dfrac{1}{7}\right)(12!) = 68{,}428{,}800$

Supplementary Exercises

1. $\binom{4}{1}\binom{7}{2} + \binom{4}{2}\binom{7}{4} + \binom{4}{3}\binom{7}{6}$

3. Select any four of these twelve points (on the circumference). As seen in the figure, these points determine a pair of chords that intersect. Consequently, the largest number of points of intersection for all possible chords is $\binom{12}{4} = 495$.



5. **a)** 10^{25} **b)** $(10)(11)(12)\cdots(34) = 34!/9!$ **c)** $(25!)\binom{24}{9}$
7. **a)** $C(12, 8)$ **b)** $P(12, 8)$ **9.** **a)** 12 **b)** 49
11. $(1/11)[11!/(5!\,3!\,3!)]$
13. **a)** (i) $\binom{5}{4} + \binom{5}{2}\binom{4}{2} + \binom{4}{4}$ (ii) $\binom{8}{4} + \binom{6}{2}\binom{5}{2} + \binom{7}{4}$ (iii) $\binom{8}{4} + \binom{6}{2}\binom{5}{2} + \binom{7}{4} - 9$
b) (i) $\binom{5}{1}\binom{4}{3} + \binom{5}{3}\binom{4}{1}$ (ii) and (iii) $\binom{5}{1}\binom{6}{3} + \binom{7}{3}\binom{4}{1}$
15. **a)** $2\binom{9}{4} + \binom{9}{3} = 343$ **b)** $[2\binom{12}{4} - 9] + [\binom{12}{3} - 1] = 1200$
17. **a)** $(5)(9!)$ **b)** $(3)(8!)$
19. **a)** $\binom{4}{2}7^5$ **b)** $2[\binom{3}{2}7^4 + \binom{4}{2}7^5]$
21. $0 = (1 + (-1))^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n\binom{n}{n}$, so
$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$
23. **a)** $P(20, 12) = 20!/8!$ **b)** $\binom{17}{9}(12!)$
25. **a)** $\binom{9}{1} + \binom{10}{3} + \cdots + \binom{16}{15} + \binom{17}{17} = \sum_{k=0}^{8}\binom{9+k}{1+2k}$ **b)** $\sum_{k=0}^{9}\binom{9+k}{2k}$
c) $n = 2k+1,\ k \geq 0$: $\sum_{i=0}^{k}\binom{k+1+i}{1+2i}$
$n = 2k,\ k \geq 1$: $\sum_{i=0}^{k}\binom{k+i}{2i}$
27. **a)** $\binom{r+(n-r)-1}{n-r} = \binom{n-1}{n-r} = \binom{n-1}{r-1}$
b) $\sum_{r=1}^{n}\binom{n-1}{r-1} = \binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{n-1} = 2^{n-1}$
29. **a)** $11!/(7!\,4!)$ **b)** $[11!/(7!\,4!)] - [4!/(2!\,2!)][4!/(3!\,1!)]$
c) $[11!/(7!\,4!)] + [10!/(6!\,3!\,1!)] + [9!/(5!\,2!\,2!)] + [8!/(4!\,1!\,3!)] + [7!/(3!\,4!)]$ [in part (a)]
$\{[11!/(7!\,4!)] + [10!/(6!\,3!\,1!)] + [9!/(5!\,2!\,2!)] + [8!/(4!\,1!\,3!)] + [7!/(3!\,4!)]\}$
$- [\{[4!/(2!\,2!)] + [3!/(1!\,1!\,1!)] + [2!/2!]\} \times \{[4!/(3!\,1!)] + [3!/(2!\,1!)]\}]$
[in part (b)]
31. $\binom{9}{2}\binom{6}{2} = 540$ **33.** $\binom{6}{4}(12)(11)(10)(9) = 178{,}200$

Fundamentals of Logic

In this chapter we take a close look at what constitutes a valid argument and a more conventional proof. When a mathematician wishes to provide a proof for a given situation, he or she must use a system of logic. This is also true when a computer scientist develops the algorithms needed for a program or system of programs. The logic of mathematics is applied to decide whether one statement follows from, or is a logical consequence of, one or more other statements.

Some of the rules that govern this process are described in this chapter. We shall use these rules in proofs (provided in the text and required in the exercises). However, at no time can we hope to arrive at a point at which we can apply the rules in an automatic fashion. We should always analyze and seek to understand the situation given. This often calls for attributes we cannot learn in a book, such as insight and creativity. Merely trying to apply formulas or invoke rules will not get us very far either in proving results (such as theorems) or in doing enumeration problems.

1 Basic Connectives and Truth Tables

In the development of any mathematical theory, assertions are made in the form of sentences. Such verbal or written assertions, called *statements* (or *propositions*), are declarative sentences that are either true or false — but *not* both. For example, the following are statements, and we use the lowercase letters of the alphabet (such as p, q, and r) to represent these statements.

p: Combinatorics is a required course for sophomores.

q: Margaret Mitchell wrote *Gone with the Wind.*

r: $2 + 3 = 5$.

On the other hand, we do not regard sentences such as the exclamation

"What a beautiful evening!"

or the command

"Get up and do your exercises."

as statements since they do not have *truth values* (true or false).

The preceding statements represented by the letters p, q, and r are considered to be *primitive* statements, for there is really no way to break them down into anything simpler. New statements can be obtained from existing ones in two ways.

1) Transform a given statement p into the statement $\neg p$, which denotes its *negation* and is read "Not p."

For the statement p above, $\neg p$ is the statement "Combinatorics is not a required course for sophomores." (We do not consider the negation of a primitive statement to be a primitive statement.)

2) Combine two or more statements into a *compound* statement, using the following *logical connectives.*

a) Conjunction: The *conjunction* of the statements p, q is denoted by $p \wedge q$, which is read "p and q." In our example the compound statement $p \wedge q$ is read "Combinatorics is a required course for sophomores, **and** Margaret Mitchell wrote *Gone with the Wind*."

b) Disjunction: The expression $p \vee q$ denotes the *disjunction* of the statements p, q and is read "p or q." Hence "Combinatorics is a required course for sophomores, **or** Margaret Mitchell wrote *Gone with the Wind*" is the verbal translation for $p \vee q$, when p, q are as above. We use the word "or" in the *inclusive* sense here. Consequently, $p \vee q$ is true if one or the other of p, q is true or if *both* of the statements p, q are true. In English we sometimes write "and/or" to point this out. The *exclusive* "or" is denoted by $p \veebar q$. The compound statement $p \veebar q$ is true if one or the other of p, q is true but *not both* of the statements p, q are true. One way to express $p \veebar q$ for the example here is "Combinatorics is a required course for sophomores, or Margaret Mitchell wrote *Gone with the Wind*, but not both."

c) Implication: We say that "p implies q" and write $p \rightarrow q$ to designate the statement, which is the *implication* of q by p. Alternatively, we can also say

(i) "If p, then q." **(ii)** "p is *sufficient* for q."
(iii) "p is a *sufficient condition* for q." **(iv)** "q is *necessary* for p."
(v) "q is a *necessary condition* for p." **(vi)** "p only if q."

A verbal translation of $p \rightarrow q$ for our example is "If combinatorics is a required course for sophomores, then Margaret Mitchell wrote *Gone with the Wind*." The statement p is called the *hypothesis* of the implication; q is called the *conclusion*. When statements are combined in this manner, there need not be any causal relationship between the statements for the implication to be true.

d) Biconditional: Last, the *biconditional* of two statements p, q, is denoted by $p \leftrightarrow q$, which is read "p if and only if q," or "p is necessary and sufficient for q." For our p, q, "Combinatorics is a required course for sophomores if and only if Margaret Mitchell wrote *Gone with the Wind*" conveys the meaning of $p \leftrightarrow q$. We sometimes abbreviate "p if and only if q" as "p iff q."

Throughout our discussion on logic we must realize that a sentence such as

"The number x is an integer."

is *not* a statement because its truth value (true or false) cannot be determined until a numerical value is assigned for x. If x were assigned the value 7, the result would be a true statement. Assigning x a value such as $\frac{1}{2}$, $\sqrt{2}$, or π, however, would make the resulting statement false. (We shall encounter this type of situation again in Sections 4 and 5 of this chapter.)

In the foregoing discussion, we mentioned the circumstances under which the *compound* statements $p \vee q$, $p \veebar q$ are considered true, on the basis of the truth of their components p, q. This idea of the truth or falsity of a compound statement being dependent only on the truth values of its components is worth further investigation. Tables 1 and 2 summarize the truth and falsity of the negation and the different kinds of compound statements on the basis of the truth values of their components. In constructing such *truth tables*, we write "0" for false and "1" for true.

Table 1

p	$\neg p$
0	1
1	0

Table 2

p	q	$p \wedge q$	$p \vee q$	$p \veebar q$	$p \rightarrow q$	$p \leftrightarrow q$
0	0	0	0	0	1	1
0	1	0	1	1	1	0
1	0	0	1	1	0	0
1	1	1	1	0	1	1

The four possible truth assignments for p, q can be listed in any order. For later work, the particular order presented here will prove useful.

We see that the columns of truth values for p and $\neg p$ are the opposite of each other. The statement $p \wedge q$ is true only when both p, q are true, whereas $p \vee q$ is false only when both the component statements p, q are false. As we noted before, $p \veebar q$ is true when exactly one of p, q is true.

For the implication $p \rightarrow q$, the result is true in all cases except where p is true and q is false. We do not want a true statement to lead us into believing something that is false. However, we regard as true a statement such as "If $2 + 3 = 6$, then $2 + 4 = 7$," even though the statements "$2 + 3 = 6$" and "$2 + 4 = 7$" are both false.

Finally, the biconditional $p \leftrightarrow q$ is true when the statements p, q have the same truth value and is false otherwise.

Now that we have been introduced to certain concepts, let us investigate a little further some of these initial ideas about connectives. Our first two examples should prove useful for such an investigation.

EXAMPLE 1

Let s, t, and u denote the following primitive statements:

s: Phyllis goes out for a walk.

t: The moon is out.

u: It is snowing.

The following English sentences provide possible translations for the given (symbolic) compound statements.

a) $(t \wedge \neg u) \rightarrow s$: If the moon is out and it is not snowing, then Phyllis goes out for a walk.

b) $t \to (\neg u \to s)$: If the moon is out, then if it is not snowing Phyllis goes out for a walk. [So $\neg u \to s$ is understood to mean $(\neg u) \to s$ as opposed to $\neg(u \to s)$.]

c) $\neg(s \leftrightarrow (u \vee t))$: It is not the case that Phyllis goes out for a walk if and only if it is snowing or the moon is out.

Now we will work in reverse order and examine the logical (or symbolic) notation for three given English sentences:

d) "Phyllis will go out walking if and only if the moon is out." Here the words "if and only if" indicate that we are dealing with a biconditional. In symbolic form this becomes $s \leftrightarrow t$.

e) "If it is snowing and the moon is not out, then Phyllis will not go out for a walk." This compound statement is an implication where the hypothesis is also a compound statement. One may express this statement in symbolic form as $(u \wedge \neg t) \to \neg s$.

f) "It is snowing but Phyllis will still go out for a walk." Now we come across a new connective — namely, *but*. In our study of logic we shall follow the convention that the connectives *but* and *and* convey the same meaning. Consequently, this sentence may be represented as $u \wedge s$.

Now let us return to the results in Table 2, particularly the sixth column. For if this is one's first encounter with the truth table for the implication $p \to q$, then it may be somewhat difficult to accept the stated entries — especially the results in the first two rows (where p has the truth value 0). The following example should help make these truth value assignments easier to grasp.

EXAMPLE 2

Consider the following scenario. It is almost the week before Christmas and Penny will be attending several parties that week. Ever conscious of her weight, she plans not to weigh herself until the day after Christmas. Considering what those parties may do to her waistline by then, she makes the following resolution for the December 26 outcome: "If I weigh more than 120 pounds, then I shall enroll in an exercise class."

Here we let p and q denote the (primitive) statements

p: I weigh more than 120 pounds.

q: I shall enroll in an exercise class.

Then Penny's statement (implication) is given by $p \to q$.

We shall consider the truth values of this particular example of $p \to q$ for the rows of Table 2. Consider first the easier cases in rows 4 and 3.

- Row 4: p and q both have the truth value 1. On December 26 Penny finds that she weighs more than 120 pounds and promptly enrolls in an exercise class, just as she said she would. Here we consider $p \to q$ to be true and assign it the truth value 1.
- Row 3: p has the truth value 1, q has the truth value 0. Now that December 26 has arrived, Penny finds her weight to be over 120 pounds, but she makes no attempt to enroll in an exercise class. In this case we feel that Penny has broken her resolution — in other words, the implication $p \to q$ is false (and has the truth value 0).

The cases in rows 1 and 2 may not immediately agree with our intuition, but the example should make these results a little easier to accept.

- Row 1: p and q both have the truth value 0. Here Penny finds that on December 26 her weight is 120 pounds or less and she does not enroll in an exercise class. She has not violated her resolution; we take her statement $p \rightarrow q$ to be true and assign it the truth value 1.
- Row 2: p has the truth value 0, q has the truth value 1. This last case finds Penny weighing 120 pounds or less on December 26 but still enrolling in an exercise class. Perhaps her weight is 119 or 120 pounds and she feels this is still too high. Or maybe she wants to join an exercise class because she thinks it will be good for her health. No matter what the reason, she has not gone against her resolution $p \rightarrow q$. Once again, we accept this compound statement as true, assigning it the truth value 1.

Our next example discusses a related notion: the *decision* (or *selection*) structure in computer programming.

EXAMPLE 3

In computer science the **if-then** and **if-then-else** decision structures arise (in various formats) in high-level programming languages such as Java and C++. The hypothesis p is often a relational expression such as $x > 2$. This expression then becomes a (logical) statement that has the truth value 0 or 1, depending on the value of the variable x at that point in the program. The conclusion q is usually an "executable statement." (So q is not one of the logical statements that we have been discussing.) When dealing with "**if** p **then** q," in this context, the computer executes q only on the condition that p is true. For p false, the computer goes to the next instruction in the program sequence. For the decision structure "**if** p **then** q **else** r," q is executed when p is true and r is executed when p is false.

Before continuing, a word of caution: Be careful when using the symbols $\rightarrow$ and $\leftrightarrow$. The implication and the biconditional are not the same, as evidenced by the last two columns of Table 2.

In our everyday language, however, we often find situations where an implication is used when the intention actually calls for a biconditional. For example, consider the following implications that a certain parent might direct to his or her child.

$s \rightarrow t$: If you do your homework, then you will get to watch the baseball game.

$t \rightarrow s$: You will get to watch the baseball game only if you do your homework.

- Case 1: The implication $s \rightarrow t$. When the parent says to the child, "If you do your homework, then you will get to watch the baseball game," he or she is trying a positive approach by emphasizing the enjoyment in watching the baseball game.
- Case 2: The implication $t \rightarrow s$. Here we find the negative approach and the parent who warns the child in saying, "You will get to watch the baseball game only if you do your homework." This parent places the emphasis on the punishment (lack of enjoyment) to be incurred.

In either case, the parent probably wants his or her implication — be it $s \rightarrow t$ or $t \rightarrow s$ — to be understood as the biconditional $s \leftrightarrow t$. For in case 1 the parent wants to hint at the punishment while promising the enjoyment; in case 2, where the punishment has been used (perhaps, to threaten), if the child does in fact do the homework, then that child will definitely be given the opportunity to enjoy watching the baseball game.

In scientific writing one must make every effort to be unambiguous — when an implication is given, it ordinarily cannot, and should not, be interpreted as a biconditional. Definitions are a notable exception, which we shall discuss in Section 5.

Before we continue let us take a step back. When we summarized the material that gave us Tables 1 and 2, we may not have stressed enough that the results were for any statements p, q — *not* just primitive statements p, q. Examples 4 through 6 should help to reinforce this.

EXAMPLE 4

Let us examine the truth table for the compound statement "Margaret Mitchell wrote *Gone with the Wind*, and if $2 + 3 \neq 5$, then combinatorics is a required course for sophomores." In symbolic notation this statement is written as $q \wedge (\neg r \rightarrow p)$, where p, q, and r represent the primitive statements introduced at the start of this section. The last column of Table 3 contains the truth values for this result. We obtained these truth values by using the fact that the conjunction of any two statements is true if and only if both statements are true. This is what we said earlier in Table 2, and now one of our statements — namely, the implication $\neg r \rightarrow p$ — is definitely a compound statement, not a primitive one. Columns 4, 5, and 6 in this table show how we build the truth table up by considering smaller parts of the compound statement and by using the results from Tables 1 and 2.

Table 3

p	q	r	$\neg r$	$\neg r \rightarrow p$	$q \wedge (\neg r \rightarrow p)$
0	0	0	1	0	0
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	0	1	1
1	0	0	1	1	0
1	0	1	0	1	0
1	1	0	1	1	1
1	1	1	0	1	1

EXAMPLE 5

In Table 4 we develop the truth tables for the compound statements $p \vee (q \wedge r)$ (column 5) and $(p \vee q) \wedge r$ (column 7).

Table 4

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$(p \vee q) \wedge r$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	1	0
0	1	1	1	1	1	1
1	0	0	0	1	1	0
1	0	1	0	1	1	1
1	1	0	0	1	1	0
1	1	1	1	1	1	1

Because the truth values in columns 5 and 7 differ (in rows 5 and 7), we must avoid writing a compound statement such as $p \vee q \wedge r$. Without parentheses to indicate which of the connectives $\vee$ and $\wedge$ should be applied first, we have no idea whether we are dealing with $p \vee (q \wedge r)$ or $(p \vee q) \wedge r$.

Our last example for this section illustrates two special types of statements.

EXAMPLE 6

The results in columns 4 and 7 of Table 5 reveal that the statement $p \rightarrow (p \vee q)$ is true and that the statement $p \wedge (\neg p \wedge q)$ is false for all truth value assignments for the component statements p, q.

Table 5

p	q	$p \vee q$	$p \rightarrow (p \vee q)$	$\neg p$	$\neg p \wedge q$	$p \wedge (\neg p \wedge q)$
0	0	0	1	1	0	0
0	1	1	1	1	1	0
1	0	1	1	0	0	0
1	1	1	1	0	0	0

Definition 1

A compound statement is called a *tautology* if it is true for all truth value assignments for its component statements. If a compound statement is false for all such assignments, then it is called a *contradiction.*

Throughout this chapter we shall use the symbol T_0 to denote any tautology and the symbol F_0 to denote any contradiction.

We can use the ideas of tautology and implication to describe what we mean by a valid argument. This will be of primary interest to us in Section 3, and it will help us develop needed skills for proving mathematical theorems. In general, an argument starts with a list of *given* statements called *premises* and a statement called the *conclusion* of the argument. We examine these premises, say $p_1, p_2, p_3, \ldots, p_n$, and try to show that the conclusion q follows logically from these given statements — that is, we try to show that if each of $p_1, p_2, p_3, \ldots, p_n$ is a true statement, then the statement q is also true. To do so one way is to examine the implication

$$(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n)^{\dagger} \rightarrow q,$$

where the hypothesis is the conjunction of the n premises. If any one of $p_1, p_2, p_3, \ldots, p_n$ is false, then no matter what truth value q has, the implication $(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \rightarrow q$ is true. Consequently, if we start with the premises $p_1, p_2, p_3, \ldots, p_n$ — each with truth value 1 — and find that under these circumstances q also has the value 1, then the implication

$$(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \rightarrow q$$

is a *tautology* and we have a *valid argument*.

† At this point we have dealt only with the conjunction of two statements, so we must point out that the conjunction $p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n$ of n statements is true if and only if each p_i, $1 \leq i \leq n$, is true.

EXERCISES 1

1. Determine whether each of the following sentences is a statement.

a) In 2003 George W. Bush was the president of the United States.

b) $x + 3$ is a positive integer.

c) Fifteen is an even number.

d) If Jennifer is late for the party, then her cousin Zachary will be quite angry.

e) What time is it?

f) As of June 30, 2003, Christine Marie Evert had won the French Open a record seven times.

2. Identify the primitive statements in Exercise 1.

3. Let p, q be primitive statements for which the implication $p \to q$ is false. Determine the truth values for each of the following.

a) $p \wedge q$ **b)** $\neg p \vee q$ **c)** $q \to p$ **d)** $\neg q \to \neg p$

4. Let p, q, r, s denote the following statements:

p: I finish writing my computer program before lunch.
q: I shall play tennis in the afternoon.
r: The sun is shining.
s: The humidity is low.

Write the following in symbolic form.

a) If the sun is shining, I shall play tennis this afternoon.

b) Finishing the writing of my computer program before lunch is necessary for my playing tennis this afternoon.

c) Low humidity and sunshine are sufficient for me to play tennis this afternoon.

5. Let p, q, r denote the following statements about a particular triangle ABC.

p: Triangle ABC is isosceles.
q: Triangle ABC is equilateral.
r: Triangle ABC is equiangular.

Translate each of the following into an English sentence.

a) $q \to p$ **b)** $\neg p \to \neg q$

c) $q \leftrightarrow r$ **d)** $p \wedge \neg q$

e) $r \to p$

6. Determine the truth value of each of the following implications.

a) If $3 + 4 = 12$, then $3 + 2 = 6$.

b) If $3 + 3 = 6$, then $3 + 4 = 9$.

c) If Thomas Jefferson was the third president of the United States, then $2 + 3 = 5$.

7. Rewrite each of the following statements as an implication in the **if-then** form.

a) Practicing her serve daily is a sufficient condition for Darci to have a good chance of winning the tennis tournament.

b) Fix my air conditioner or I won't pay the rent.

c) Mary will be allowed on Larry's motorcycle only if she wears her helmet.

8. Construct a truth table for each of the following compound statements, where p, q, r denote primitive statements.

a) $\neg(p \vee \neg q) \to \neg p$ **b)** $p \to (q \to r)$

c) $(p \to q) \to r$ **d)** $(p \to q) \to (q \to p)$

e) $[p \wedge (p \to q)] \to q$ **f)** $(p \wedge q) \to p$

g) $q \leftrightarrow (\neg p \vee \neg q)$

h) $[(p \to q) \wedge (q \to r)] \to (p \to r)$

9. Which of the compound statements in Exercise 8 are tautologies?

10. Verify that $[p \to (q \to r)] \to [(p \to q) \to (p \to r)]$ is a tautology.

11. a) How many rows are needed for the truth table of the compound statement $(p \vee \neg q) \leftrightarrow [(\neg r \wedge s) \to t]$, where p, q, r, s, and t are primitive statements?

b) Let $p_1, p_2, \ldots, p_n$ denote n primitive statements. Let p be a compound statement that contains at least one occurrence each of p_i, for $1 \le i \le n$—and p contains no other primitive statement. How many rows are needed to construct the truth table for p?

12. Determine all truth value assignments, if any, for the primitive statements p, q, r, s, t that make each of the following compound statements false.

a) $[(p \wedge q) \wedge r] \to (s \vee t)$

b) $[p \wedge (q \wedge r)] \to (s \veebar t)$

13. If statement q has the truth value 1, determine all truth value assignments for the primitive statements, p, r, and s for which the truth value of the statement

$$(q \to [(\neg p \vee r) \wedge \neg s]) \wedge [\neg s \to (\neg r \wedge q)]$$

is 1.

14. At the start of a program (written in pseudocode) the integer variable n is assigned the value 7. Determine the value of n after each of the following *successive* statements is encountered during the execution of this program. [Here the value of n following the execution of the statement in part (a) becomes the value of n for the statement in part (b), and so on, through the statement in part (d). For positive integers a, b, $\lfloor a/b \rfloor$ returns the integer part of the quotient—for example, $\lfloor 6/2 \rfloor = 3$, $\lfloor 7/2 \rfloor = 3$, $\lfloor 2/5 \rfloor = 0$, and $\lfloor 8/3 \rfloor = 2$.]

a) `if n > 5 then n := n + 2`

b) **if** (($n + 2 = 8$) **or** ($n - 3 = 6$)) **then**
   $n := 2 * n + 1$

c) **if** (($n - 3 = 16$) **and** ($\lfloor n/6 \rfloor = 1$)) **then**
   $n := n + 3$

d) **if** (($n \neq 21$) **and** ($n - 7 = 15$)) **then**
   $n := n - 4$

15. The integer variables m and n are assigned the values 3 and 8, respectively, during the execution of a program (written in pseudocode). Each of the following *successive* statements is then encountered during program execution. [Here the values of m, n following the execution of the statement in part (a) become the values of m, n for the statement in part (b), and so on, through the statement in part (e).] What are the values of m, n after each of these statements is encountered?

a) **if** $n - m = 5$ **then** $n := n - 2$

b) **if** (($2 * m = n$) **and** ($\lfloor n/4 \rfloor = 1$)) **then**
   $n := 4 * m - 3$

c) **if** (($n < 8$) **or** ($\lfloor m/2 \rfloor = 2$)) **then** $n := 2 * m$
   else $m := 2 * n$

d) **if** (($m < 20$) **and** ($\lfloor n/6 \rfloor = 1$)) **then**
   $m := m - n - 5$

e) **if** (($n = 2 * m$) **or** ($\lfloor n/2 \rfloor = 5$)) **then**
   $m := m + 2$

16. In the following program segment i, j, m, and n are integer variables. The values of m and n are supplied by the user earlier in the execution of the total program.

```
for i := 1 to m do
  for j := 1 to n do
    if i ≠ j then
      print i + j
```

How many times is the **print** statement in the segment executed when (a) $m = 10$, $n = 10$; (b) $m = 20$, $n = 20$; (c) $m = 10$, $n = 20$; (d) $m = 20$, $n = 10$?

17. After baking a pie for the two nieces and two nephews who are visiting her, Aunt Nellie leaves the pie on her kitchen table to cool. Then she drives to the mall to close her boutique for the day. Upon her return she finds that someone has eaten one-quarter of the pie. Since no one was in her house that day — except for the four visitors — Aunt Nellie questions each niece and nephew about who ate the piece of pie. The four "suspects" tell her the following:

Charles:	Kelly ate the piece of pie.
Dawn:	I did not eat the piece of pie.
Kelly:	Tyler ate the pie.
Tyler:	Kelly lied when she said I ate the pie.

If only one of these four statements is true and only one of the four committed this heinous crime, who is the vile culprit that Aunt Nellie will have to punish severely?

2
Logical Equivalence: The Laws of Logic

In all areas of mathematics we need to know when the entities we are studying are equal or essentially the same. For example, in arithmetic and algebra we know that two nonzero real numbers are equal when they have the same magnitude and algebraic sign. Hence, for two nonzero real numbers x, y, we have $x = y$ if $|x| = |y|$ and $xy > 0$, and conversely (that is, if $x = y$, then $|x| = |y|$ and $xy > 0$). When we deal with triangles in geometry, the notion of congruence arises. Here triangle ABC and triangle DEF are congruent if, for instance, they have equal corresponding sides — that is, the length of side $AB =$ the length of side DE, the length of side $BC =$ the length of side EF, and the length of side $CA =$ the length of side FD.

Our study of logic is often referred to as the *algebra of propositions* (as opposed to the algebra of real numbers). In this algebra we shall use the truth tables of the statements, or propositions, to develop an idea of when two such entities are essentially the same. We begin with an example.

EXAMPLE 7

For primitive statements p and q, Table 6 provides the truth tables for the compound statements $\neg p \vee q$ and $p \rightarrow q$. Here we see that the corresponding truth tables for the two statements $\neg p \vee q$ and $p \rightarrow q$ are exactly the same.

Table 6

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

This situation leads us to the following idea.

Definition 2 Two statements s_1, s_2 are said to be *logically equivalent*, and we write $s_1 \Longleftrightarrow s_2$, when the statement s_1 is true (respectively, false) if and only if the statement s_2 is true (respectively, false).

Note that when $s_1 \Longleftrightarrow s_2$ the statements s_1 and s_2 provide the same truth tables because s_1, s_2 have the same truth values for *all* choices of truth values for their primitive components.

As a result of this concept we see that we can express the connective for the implication (of primitive statements) in terms of negation and disjunction — that is, $(p \rightarrow q) \Longleftrightarrow \neg p \vee q$. In the same manner, from the result in Table 7 we have $(p \leftrightarrow q) \Longleftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$, and this helps validate the use of the term *biconditional.* Using the logical equivalence from Table 6, we find that we can also write $(p \leftrightarrow q) \Longleftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$. Consequently, if we so choose, we can eliminate the connectives $\rightarrow$ and $\leftrightarrow$ from compound statements.

Table 7

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	1	1	1	1

Examining Table 8, we find that negation, along with the connectives $\wedge$ and $\vee$, are all we need to replace the *exclusive or* connective, $\veebar$. In fact, we may even eliminate either $\wedge$ or $\vee$. However, for the related applications we want to study later in the text, we shall need both $\wedge$ and $\vee$ as well as negation.

Table 8

p	q	$p \veebar q$	$p \vee q$	$p \wedge q$	$\neg(p \wedge q)$	$(p \vee q) \wedge \neg(p \wedge q)$
0	0	0	0	0	1	0
0	1	1	1	0	1	1
1	0	1	1	0	1	1
1	1	0	1	1	0	0

We now use the idea of logical equivalence to examine some of the important properties that hold for the algebra of propositions.

For all real numbers a, b, we know that $-(a+b) = (-a) + (-b)$. Is there a comparable result for primitive statements p, q?

EXAMPLE 8

In Table 9 we have constructed the truth tables for the statements $\neg(p \wedge q)$, $\neg p \vee \neg q$, $\neg(p \vee q)$, and $\neg p \wedge \neg q$, where p, q are primitive statements. Columns 4 and 7 reveal that $\neg(p \wedge q) \Longleftrightarrow \neg p \vee \neg q$; columns 9 and 10 reveal that $\neg(p \vee q) \Longleftrightarrow \neg p \wedge \neg q$. These results are known as *DeMorgan's Laws.* They are similar to the familiar law for real numbers,

$$-(a+b) = (-a) + (-b),$$

already noted, which shows the negative of a sum to be equal to the sum of the negatives. Here, however, a crucial difference emerges: The negation of the *conjunction* of two primitive statements p, q results in the *disjunction* of their negations $\neg p$, $\neg q$, whereas the negation of the *disjunction* of these same statements p, q is logically equivalent to the *conjunction* of their negations $\neg p$, $\neg q$.

Table 9

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
0	0	0	1	1	1	1	0	1	1
0	1	0	1	1	0	1	1	0	0
1	0	0	1	0	1	1	1	0	0
1	1	1	0	0	0	0	1	0	0

Although p, q were primitive statements in the preceding example we shall soon learn that DeMorgan's Laws hold for any two arbitrary statements.

In the arithmetic of real numbers, the operations of addition and multiplication are both involved in the principle called the Distributive Law of Multiplication over Addition: For all real numbers a, b, c,

$$a \times (b+c) = (a \times b) + (a \times c).$$

The next example shows that there is a similar law for primitive statements. There is also a second related law (for primitive statements) that has no counterpart in the arithmetic of real numbers.

EXAMPLE 9

Table 10 contains the truth tables for the statements $p \wedge (q \vee r)$, $(p \wedge q) \vee (p \wedge r)$, $p \vee (q \wedge r)$, and $(p \vee q) \wedge (p \vee r)$. From the table it follows that for all primitive statements p, q, and r,

$p \wedge (q \vee r) \Longleftrightarrow (p \wedge q) \vee (p \wedge r)$ The Distributive Law of $\wedge$ over $\vee$

$p \vee (q \wedge r) \Longleftrightarrow (p \vee q) \wedge (p \vee r)$ The Distributive Law of $\vee$ over $\wedge$

The second distributive law has no counterpart in the arithmetic of real numbers. That is, it is not true for all real numbers a, b, and c that the following holds: $a + (b \times c) = (a+b) \times (a+c)$. For $a = 2$, $b = 3$, and $c = 5$, for instance, $a + (b \times c) = 17$ but $(a+b) \times (a+c) = 35$.

Table 10

p	q	r	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	0	0	1	1
1	0	0	0	0	1	1
1	0	1	1	1	1	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

Before going any further, we note that, in general, if s_1, s_2 are statements and $s_1 \leftrightarrow s_2$ is a tautology, then s_1, s_2 must have the same corresponding truth values (that is, for each assignment of truth values to the primitive statements in s_1 and s_2, s_1 is true if and only if s_2 is true and s_1 is false if and only if s_2 is false) and $s_1 \Longleftrightarrow s_2$. When s_1 and s_2 are logically equivalent statements (that is, $s_1 \Longleftrightarrow s_2$), then the compound statement $s_1 \leftrightarrow s_2$ is a tautology. Under these circumstances it is also true that $\neg s_1 \Longleftrightarrow \neg s_2$, and $\neg s_1 \leftrightarrow \neg s_2$ is a tautology.

If s_1, s_2, and s_3 are statements where $s_1 \Longleftrightarrow s_2$ and $s_2 \Longleftrightarrow s_3$ then $s_1 \Longleftrightarrow s_3$. When two statements s_1 and s_2 are not logically equivalent, we may write $s_1 \not\Longleftrightarrow s_2$ to designate this situation.

Using the concepts of logical equivalence, tautology, and contradiction, we state the following list of laws for the algebra of propositions.

The Laws of Logic

For any primitive statements p, q, r, any tautology T_0, and any contradiction F_0,

1)	$\neg\neg p \Longleftrightarrow p$	Law of *Double Negation*
2)	$\neg(p \vee q) \Longleftrightarrow \neg p \wedge \neg q$ $\neg(p \wedge q) \Longleftrightarrow \neg p \vee \neg q$	*DeMorgan's* Laws
3)	$p \vee q \Longleftrightarrow q \vee p$ $p \wedge q \Longleftrightarrow q \wedge p$	*Commutative* Laws
4)	$p \vee (q \vee r) \Longleftrightarrow (p \vee q) \vee r$[†] $p \wedge (q \wedge r) \Longleftrightarrow (p \wedge q) \wedge r$	*Associative* Laws
5)	$p \vee (q \wedge r) \Longleftrightarrow (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \Longleftrightarrow (p \wedge q) \vee (p \wedge r)$	*Distributive* Laws
6)	$p \vee p \Longleftrightarrow p$ $p \wedge p \Longleftrightarrow p$	*Idempotent* Laws
7)	$p \vee F_0 \Longleftrightarrow p$ $p \wedge T_0 \Longleftrightarrow p$	*Identity* Laws

[†]We note that because of the Associative Laws, there is no ambiguity in statements of the form $p \vee q \vee r$ or $p \wedge q \wedge r$.

8)	$p \vee \neg p \Longleftrightarrow T_0$ $p \wedge \neg p \Longleftrightarrow F_0$	*Inverse* Laws
9)	$p \vee T_0 \Longleftrightarrow T_0$ $p \wedge F_0 \Longleftrightarrow F_0$	*Domination* Laws
10)	$p \vee (p \wedge q) \Longleftrightarrow p$ $p \wedge (p \vee q) \Longleftrightarrow p$	*Absorption* Laws

We now turn our attention to proving all of these properties. In so doing we realize that we could simply construct the truth tables and compare the results for the corresponding truth values in each case — as we did in Examples 8 and 9. However, before we start writing, let us take one more look at this list of 19 laws, which, aside from the Law of Double Negation, fall naturally into pairs. This pairing idea will help us after we examine the following concept.

Definition 3 Let s be a statement. If s contains no logical connectives other than $\wedge$ and $\vee$, then the *dual* of s, denoted s^d, is the statement obtained from s by replacing each occurrence of $\wedge$ and $\vee$ by $\vee$ and $\wedge$, respectively, and each occurrence of T_0 and F_0 by F_0 and T_0, respectively.

If p is any primitive statement, then p^d is the same as p — that is, the dual of a primitive statement is simply the same primitive statement. And $(\neg p)^d$ is the same as $\neg p$. The statements $p \vee \neg p$ and $p \wedge \neg p$ are duals of each other whenever p is primitive — and so are the statements $p \vee T_0$ and $p \wedge F_0$.

Given the primitive statements p, q, r and the compound statement

$$s: \quad (p \wedge \neg q) \vee (r \wedge T_0),$$

we find that the dual of s is

$$s^d: \quad (p \vee \neg q) \wedge (r \vee F_0).$$

(Note that $\neg q$ is unchanged as we go from s to s^d.)

We now state and use a theorem without proving it. However, we shall justify the result that appears here.

THEOREM 1 *The Principle of Duality.* Let s and t be statements that contain no logical connectives other than $\wedge$ and $\vee$. If $s \Longleftrightarrow t$, then $s^d \Longleftrightarrow t^d$.

As a result, laws 2 through 10 in our list can be established by proving one of the laws in each pair and then invoking this principle.

We also find that it is possible to derive many other logical equivalences. For example, if q, r, s are primitive statements, the results in columns 5 and 7 of Table 11 show us that

$$(r \wedge s) \rightarrow q \Longleftrightarrow \neg(r \wedge s) \vee q$$

or that $[(r \wedge s) \rightarrow q] \leftrightarrow [\neg(r \wedge s) \vee q]$ is a tautology. However, instead of always constructing more (and, unfortunately, larger) truth tables it might be a good idea to recall from Example 7 that for primitive statements p, q, the compound statement

$$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$$

Table 11

q	r	s	$r \wedge s$	$(r \wedge s) \to q$	$\neg(r \wedge s)$	$\neg(r \wedge s) \vee q$
0	0	0	0	1	1	1
0	0	1	0	1	1	1
0	1	0	0	1	1	1
0	1	1	1	0	0	0
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	0	1	1	1
1	1	1	1	1	0	1

is a tautology. If we were to *replace* each occurrence of this primitive statement p by the compound statement $r \wedge s$, then we would obtain the earlier tautology

$$[(r \wedge s) \to q] \leftrightarrow [\neg(r \wedge s) \vee q].$$

What has happened here illustrates the first of the following two *substitution rules*:

1) Suppose that the compound statement P is a tautology. If p is a *primitive* statement that appears in P and we replace *each* occurrence of p by the *same* statement q, then the resulting compound statement P_1 is also a tautology.

2) Let P be a compound statement where p is an arbitrary statement that appears in P, and let q be a statement such that $q \Longleftrightarrow p$. Suppose that in P we replace one or more occurrences of p by q. Then this replacement yields the compound statement P_1. Under these circumstances $P_1 \Longleftrightarrow P$.

These rules are further illustrated in the following two examples.

EXAMPLE 10

a) From the first of DeMorgan's Laws we know that for all primitive statements p, q, the compound statement

$$P: \quad \neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$$

is a tautology. When we replace each occurrence of p by $r \wedge s$, it follows from the first substitution rule that

$$P_1: \quad \neg[(r \wedge s) \vee q] \leftrightarrow [\neg(r \wedge s) \wedge \neg q]$$

is also a tautology. Extending this result one step further, we may replace each occurrence of q by $t \to u$. The same substitution rule now yields the tautology

$$P_2: \quad \neg[(r \wedge s) \vee (t \to u)] \leftrightarrow [\neg(r \wedge s) \wedge \neg(t \to u)],$$

and hence, by the remarks following shortly after Example 9, the logical equivalence

$$\neg[(r \wedge s) \vee (t \to u)] \Longleftrightarrow [\neg(r \wedge s) \wedge \neg(t \to u)].$$

b) For primitive statements p, q, we learn from the last column of Table 12 that the compound statement $[p \wedge (p \to q)] \to q$ is a tautology. Consequently, if r, s, t, u are any statements, then by the first substitution rule we obtain the new tautology

$$[(r \to s) \wedge [(r \to s) \to (\neg t \vee u)]] \to (\neg t \vee u)$$

when we replace each occurrence of p by $r \to s$ and each occurrence of q by $\neg t \vee u$.

Table 12

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

EXAMPLE 11

a) For an application of the second substitution rule, let P denote the compound statement $(p \to q) \to r$. Because $(p \to q) \Longleftrightarrow \neg p \vee q$ (as shown in Example 7 and Table 6), if P_1 denotes the compound statement $(\neg p \vee q) \to r$, then $P_1 \Longleftrightarrow P$. (We also find that $[(p \to q) \to r] \leftrightarrow [(\neg p \vee q) \to r]$ is a tautology.)

b) Now let P represent the compound statement (actually a tautology) $p \to (p \vee q)$. Since $\neg\neg p \Longleftrightarrow p$, the compound statement P_1: $p \to (\neg\neg p \vee q)$ is derived from P by replacing *only the second occurrence* (but *not* the first occurrence) of p by $\neg\neg p$. The second substitution rule still implies that $P_1 \Longleftrightarrow P$. [Note that P_2: $\neg\neg p \to (\neg\neg p \vee q)$, derived by replacing *both* occurrences of p by $\neg\neg p$, is also logically equivalent to P.]

Our next example demonstrates how we can use the idea of logical equivalence together with the laws of logic and the substitution rules.

EXAMPLE 12

Negate and simplify the compound statement $(p \vee q) \to r$.

We organize our explanation as follows:

1) $(p \vee q) \to r \Longleftrightarrow \neg(p \vee q) \vee r$ [by the first substitution rule because $(s \to t) \leftrightarrow (\neg s \vee t)$ is a tautology for primitive statements s, t].

2) Negating the statements in step (1), we have $\neg[(p \vee q) \to r] \Longleftrightarrow \neg[\neg(p \vee q) \vee r]$.

3) From the first of DeMorgan's Laws and the first substitution rule, $\neg[\neg(p \vee q) \vee r] \Longleftrightarrow \neg\neg(p \vee q) \wedge \neg r$.

4) The Law of Double Negation and the second substitution rule now gives us $\neg\neg(p \vee q) \wedge \neg r \Longleftrightarrow (p \vee q) \wedge \neg r$.

From steps (1) through (4) we have $\neg[(p \vee q) \to r] \Longleftrightarrow (p \vee q) \wedge \neg r$.

When we wanted to write the negation of an implication, as in Example 12, we found that the concept of logical equivalence played a key role — in conjunction with the laws of logic and the substitution rules. This idea is important enough to warrant a second look.

EXAMPLE 13

Let p, q denote the primitive statements

p: Joan goes to Lake George. q: Mary pays for Joan's shopping spree.

and consider the implication

$p \to q$: If Joan goes to Lake George, then Mary will pay for Joan's shopping spree.

Here we want to write the negation of $p \rightarrow q$ in a way other than simply $\neg(p \rightarrow q)$. We want to avoid writing the negation as "It is not the case that if Joan goes to Lake George, then Mary will pay for Joan's shopping spree."

To accomplish this we consider the following. Since $p \rightarrow q \Longleftrightarrow \neg p \vee q$, it follows that $\neg(p \rightarrow q) \Longleftrightarrow \neg(\neg p \vee q)$. Then by DeMorgan's Law we have $\neg(\neg p \vee q) \Longleftrightarrow \neg\neg p \wedge \neg q$, and from the Law of Double Negation and the second substitution rule it follows that $\neg\neg p \wedge \neg q \Longleftrightarrow p \wedge \neg q$. Consequently,

$$\neg(p \rightarrow q) \Longleftrightarrow \neg(\neg p \vee q) \Longleftrightarrow \neg\neg p \wedge \neg q \Longleftrightarrow p \wedge \neg q,$$

and we may write the negation of $p \rightarrow q$ in this case as

$\neg(p \rightarrow q)$: Joan goes to Lake George, but Mary does not pay for Joan's shopping spree.

(*Note*: The negation of an if-then statement does *not* begin with the word *if*. It is *not* another *implication*.)

EXAMPLE 14

In Definition 3 the dual s^d of a statement s was defined only for statements involving negation and the basic connectives $\wedge$ and $\vee$. How does one determine the dual of a statement such as s: $p \rightarrow q$, where p, q are primitive?

Because $(p \rightarrow q) \Longleftrightarrow \neg p \vee q$, s^d is logically equivalent to the statement $(\neg p \vee q)^d$, which is $\neg p \wedge q$.

The implication $p \rightarrow q$ and certain statements related to it are now examined in the following example.

EXAMPLE 15

Table 13 gives the truth tables for the statements $p \rightarrow q$, $\neg q \rightarrow \neg p$, $q \rightarrow p$, and $\neg p \rightarrow \neg q$. The third and fourth columns of the table reveal that

$$(p \rightarrow q) \Longleftrightarrow (\neg q \rightarrow \neg p).$$

Table 13

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$	$q \rightarrow p$	$\neg p \rightarrow \neg q$
0	0	1	1	1	1
0	1	1	1	0	0
1	0	0	0	1	1
1	1	1	1	1	1

The statement $\neg q \rightarrow \neg p$ is called the *contrapositive* of the implication $p \rightarrow q$. Columns 5 and 6 of the table show that

$$(q \rightarrow p) \Longleftrightarrow (\neg p \rightarrow \neg q).$$

The statement $q \rightarrow p$ is called the *converse* of $p \rightarrow q$; $\neg p \rightarrow \neg q$ is called the *inverse* of $p \rightarrow q$. We also see from Table 13 that

$$(p \rightarrow q) \not\Leftrightarrow (q \rightarrow p) \quad \text{and} \quad (\neg p \rightarrow \neg q) \not\Leftrightarrow (\neg q \rightarrow \neg p).$$

Consequently, we must keep the implication and its converse straight. The fact that a certain implication $p \rightarrow q$ is true (in particular, as in row 2 of the table) does *not* require that the converse $q \rightarrow p$ also be true. However, it does necessitate the truth of the contrapositive $\neg q \rightarrow \neg p$.

Let us consider a specific example where p, q represent the statements

p: Jeff is concerned about his cholesterol (HDL and LDL) levels.

q: Jeff walks at least two miles three times a week.

Then we obtain

- (The implication: $p \rightarrow q$). If Jeff is concerned about his cholesterol levels, then he will walk at least two miles three times a week.
- (The contrapositive: $\neg q \rightarrow \neg p$). If Jeff does not walk at least two miles three times a week, then he is not concerned about his cholesterol levels.
- (The converse: $q \rightarrow p$). If Jeff walks at least two miles three times a week, then he is concerned about his cholesterol levels.
- (The inverse: $\neg p \rightarrow \neg q$). If Jeff is not concerned about his cholesterol levels, then he will not walk at least two miles three times a week.

If p is true and q is false, then the implication $p \rightarrow q$ and the contrapositive $\neg q \rightarrow \neg p$ are false, while the converse $q \rightarrow p$ and the inverse $\neg p \rightarrow \neg q$ are true. For the case where p is false and q is true, the implication $p \rightarrow q$ and the contrapositive $\neg q \rightarrow \neg p$ are now true, while the converse $q \rightarrow p$ and the inverse $\neg p \rightarrow \neg q$ are false. When p, q are both true or both false, then the implication is true, as are the contrapositive, converse, and inverse.

We turn now to two examples involving the simplification of compound statements. For simplicity, we shall list the major laws of logic being used, but we shall not mention any applications of our two substitution rules.

EXAMPLE 16

For primitive statements p, q, is there any simpler way to express the compound statement $(p \vee q) \wedge \neg(\neg p \wedge q)$—that is, can we find a simpler statement that is logically equivalent to the one given?

Here one finds that

$(p \vee q) \wedge \neg(\neg p \wedge q)$	**Reasons**
$\Longleftrightarrow (p \vee q) \wedge (\neg\neg p \vee \neg q)$	DeMorgan's Law
$\Longleftrightarrow (p \vee q) \wedge (p \vee \neg q)$	Law of Double Negation
$\Longleftrightarrow (p \vee (q \wedge \neg q)$	Distributive Law of $\vee$ over $\wedge$
$\Longleftrightarrow p \vee F_0$	Inverse Law
$\Longleftrightarrow p$	Identity Law

Consequently, we see that

$$(p \vee q) \wedge \neg(\neg p \wedge q) \Longleftrightarrow p,$$

so we can express the given compound statement by the simpler logically equivalent statement p.

EXAMPLE 17

Consider the compound statement

$$\neg[\neg[(p \vee q) \wedge r] \vee \neg q],$$

where p, q, r are primitive statements. This statement contains four occurrences of primitive statements, three negation symbols, and three connectives.

From the laws of logic it follows that

	Reasons
$\neg[\neg[(p \vee q) \wedge r] \vee \neg q]$	
$\Longleftrightarrow \neg\neg[(p \vee q) \wedge r] \wedge \neg\neg q$	DeMorgan's Law
$\Longleftrightarrow [(p \vee q) \wedge r] \wedge q$	Law of Double Negation
$\Longleftrightarrow (p \vee q) \wedge (r \wedge q)$	Associative Law of $\wedge$
$\Longleftrightarrow (p \vee q) \wedge (q \wedge r)$	Commutative Law of $\wedge$
$\Longleftrightarrow [(p \vee q) \wedge q] \wedge r$	Associative Law of $\wedge$
$\Longleftrightarrow q \wedge r$	Absorption Law (as well as the Commutative Laws for $\wedge$ and $\vee$)

Consequently, the original statement

$$\neg[\neg[(p \vee q) \wedge r] \vee \neg q]$$

is logically equivalent to the much simpler statement

$$q \wedge r,$$

where we find only two primitive statements, no negation symbols, and only one connective.

Note further that from Example 7 we have

$$\neg[[(p \vee q) \wedge r] \rightarrow \neg q] \Longleftrightarrow \neg[\neg[(p \vee q) \wedge r] \vee \neg q],$$

so it follows that

$$\neg[[(p \vee q) \wedge r] \rightarrow \neg q] \Longleftrightarrow q \wedge r.$$

We close this section with an application on how the ideas in Examples 16 and 17 can be used in simplifying switching networks.

EXAMPLE 18

A switching network is made up of wires and switches connecting two terminals T_1 and T_2. In such a network, each switch is either open (0), so that no current flows through it, or closed (1), so that current does flow through it.

In Fig. 1(a) we have a network with one switch. Each of parts (b) and (c) contains two (independent) switches.

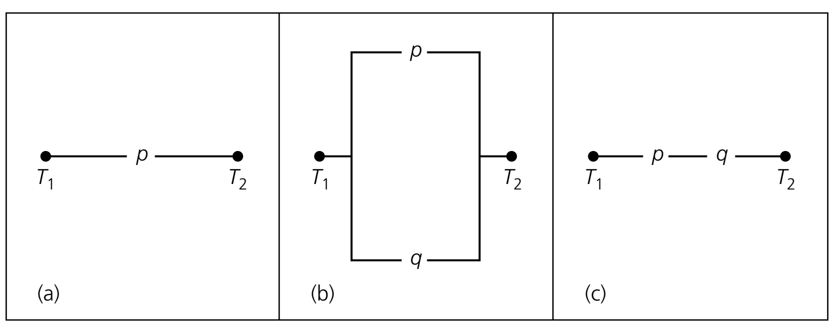


Figure 1

For the network in part (b), current flows from T_1 to T_2 if either of the switches p, q is closed. We call this a *parallel* network and represent it by $p \vee q$. The network in part (c) requires that each of the switches p, q be closed in order for current to flow from T_1 to T_2. Here the switches are in *series*; this network is represented by $p \wedge q$.

The switches in a network need not act independently of each other. Consider the network shown in Fig. 2(a). Here the switches labeled t and $\neg t$ are not independent. We have coupled these two switches so that t is open (closed) if and only if $\neg t$ is simultaneously closed (open). The same is true for the switches at q, $\neg q$. (Also, for example, the three switches labeled p are not independent.)

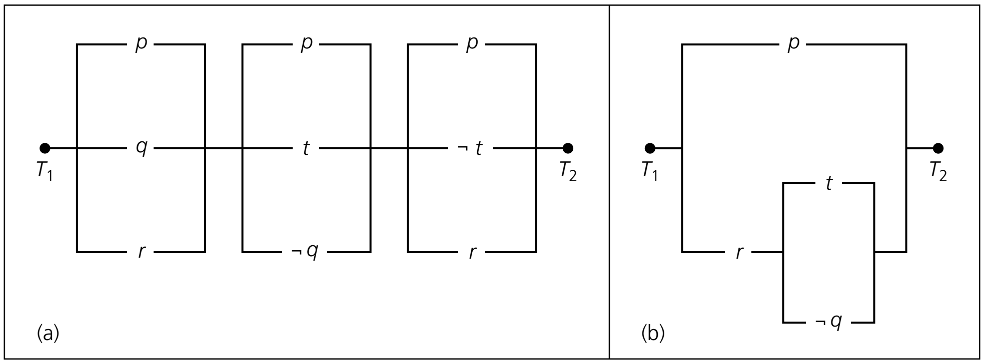


Figure 2

This network is represented by the statement $(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r)$. Using the laws of logic, we may simplify this statement as follows.

	Reasons
$(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r)$	
$\Longleftrightarrow p \vee [(q \vee r) \wedge (t \vee \neg q) \wedge (\neg t \vee r)]$	Distributive Law of $\vee$ over $\wedge$
$\Longleftrightarrow p \vee [(q \vee r) \wedge (\neg t \vee r) \wedge (t \vee \neg q)]$	Commutative Law of $\wedge$
$\Longleftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge (t \vee \neg q)]$	Distributive Law of $\vee$ over $\wedge$
$\Longleftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge (\neg\neg t \vee \neg q)]$	Law of Double Negation
$\Longleftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge \neg(\neg t \wedge q)]$	DeMorgan's Law
$\Longleftrightarrow p \vee [\neg(\neg t \wedge q) \wedge ((\neg t \wedge q) \vee r)]$	Commutative Law of $\wedge$ (twice)
$\Longleftrightarrow p \vee [(\neg(\neg t \wedge q) \wedge (\neg t \wedge q)) \vee (\neg(\neg t \wedge q) \wedge r)]$	Distributive Law of $\wedge$ over $\vee$
$\Longleftrightarrow p \vee [F_0 \vee (\neg(\neg t \wedge q) \wedge r)]$	$\neg s \wedge s \Longleftrightarrow F_0$, for any statement s
$\Longleftrightarrow p \vee [(\neg(\neg t \wedge q)) \wedge r]$	F_0 is the identity for $\vee$
$\Longleftrightarrow p \vee [r \wedge \neg(\neg t \wedge q)]$	Commutative Law of $\wedge$
$\Longleftrightarrow p \vee [r \wedge (t \vee \neg q)]$	DeMorgan's Law and the Law of Double Negation

Hence $(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r) \Longleftrightarrow p \vee [r \wedge (t \vee \neg q)]$, and the network shown in Fig. 2(b) is equivalent to the original network in the sense that current flows from T_1 to T_2 in network (a) exactly when it does so in network (b). But network (b) has only four switches, five fewer than network (a).

EXERCISES 2

1. Let p, q, r denote primitive statements.

a) Use truth tables to verify the following logical equivalences.

i) $p \to (q \land r) \Longleftrightarrow (p \to q) \land (p \to r)$
ii) $[(p \lor q) \to r] \Longleftrightarrow [(p \to r) \land (q \to r)]$
iii) $[p \to (q \lor r)] \Longleftrightarrow [\neg r \to (p \to q)]$

b) Use the substitution rules to show that

$$[p \to (q \lor r)] \Longleftrightarrow [(p \land \neg q) \to r].$$

2. Verify the first Absorption Law by means of a truth table.

3. Use the substitution rules to verify that each of the following is a tautology. (Here p, q, and r are primitive statements.)

a) $[p \lor (q \land r)] \lor \neg[p \lor (q \land r)]$

b) $[(p \lor q) \to r] \leftrightarrow [\neg r \to \neg(p \lor q)]$

4. For primitive statements p, q, r, and s, simplify the compound statement

$$[[[(p \land q) \land r] \lor [(p \land q) \land \neg r]] \lor \neg q] \to s.$$

5. Negate and express each of the following statements in smooth English.

a) Kelsey will get a good education if she puts her studies before her interest in cheerleading.

b) Norma is doing her homework, and Karen is practicing her piano lessons.

c) If Harold passes his C++ course and finishes his data structures project, then he will graduate at the end of the semester.

6. Negate each of the following and simplify the resulting statement.

a) $p \land (q \lor r) \land (\neg p \lor \neg q \lor r)$

b) $(p \land q) \to r$

c) $p \to (\neg q \land r)$

d) $p \lor q \lor (\neg p \land \neg q \land r)$

7. a) If p, q are primitive statements, prove that

$$(\neg p \lor q) \land (p \land (p \land q)) \Longleftrightarrow (p \land q).$$

b) Write the dual of the logical equivalence in part (a).

8. Write the dual for (a) $q \to p$, (b) $p \to (q \land r)$, (c) $p \leftrightarrow q$, and (d) $p \veebar q$, where p, q, and r are primitive statements.

9. Write the converse, inverse, and contrapositive of each of the following implications. For each implication, determine its truth value as well as the truth values of its corresponding converse, inverse, and contrapositive.

a) If $0 + 0 = 0$, then $1 + 1 = 1$.

b) If $-1 < 3$ and $3 + 7 = 10$, then $\sin\left(\frac{3\pi}{2}\right) = -1$.

10. Determine whether each of the following is true or false. Here p, q are arbitrary statements.

a) An equivalent way to express the converse of "p is sufficient for q" is "p is necessary for q."

b) An equivalent way to express the inverse of "p is necessary for q" is "$\neg q$ is sufficient for $\neg p$."

c) An equivalent way to express the contrapositive of "p is necessary for q" is "$\neg q$ is necessary for $\neg p$."

11. Let p, q, and r denote primitive statements. Find a form of the contrapositive of $p \to (q \to r)$ with (a) only one occurrence of the connective $\to$; (b) no occurrences of the connective $\to$.

12. Show that for primitive statements p, q,

$$p \veebar q \Longleftrightarrow [(p \land \neg q) \lor (\neg p \land q)] \Longleftrightarrow \neg(p \leftrightarrow q).$$

13. Verify that $[(p \leftrightarrow q) \land (q \leftrightarrow r) \land (r \leftrightarrow p)] \Longleftrightarrow [(p \to q) \land (q \to r) \land (r \to p)]$, for primitive statements p, q, and r.

14. For primitive statements p, q,

a) verify that $p \to [q \to (p \land q)]$ is a tautology.

b) verify that $(p \lor q) \to [q \to q]$ is a tautology by using the result from part (a) along with the substitution rules and the laws of logic.

c) is $(p \lor q) \to [q \to (p \land q)]$ a tautology?

15. Define the connective "Nand" or "Not . . . and . . ." by $(p \uparrow q) \Longleftrightarrow \neg(p \land q)$, for any statements p, q. Represent the following using only this connective.

a) $\neg p$ **b)** $p \lor q$ **c)** $p \land q$
d) $p \to q$ **e)** $p \leftrightarrow q$

16. The connective "Nor" or "Not . . . or . . ." is defined for any statements p, q by $(p \downarrow q) \Longleftrightarrow \neg(p \lor q)$. Represent the statements in parts (a) through (e) of Exercise 15, using only this connective.

17. For any statements p, q, prove that

a) $\neg(p \downarrow q) \Longleftrightarrow (\neg p \uparrow \neg q)$

b) $\neg(p \uparrow q) \Longleftrightarrow (\neg p \downarrow \neg q)$

18. Give the reasons for each step in the following simplifications of compound statements.

a)

	Reasons
$[(p \lor q) \land (p \lor \neg q)] \lor q$	
$\Longleftrightarrow [p \lor (q \land \neg q)] \lor q$	
$\Longleftrightarrow (p \lor F_0) \lor q$	
$\Longleftrightarrow p \lor q$	

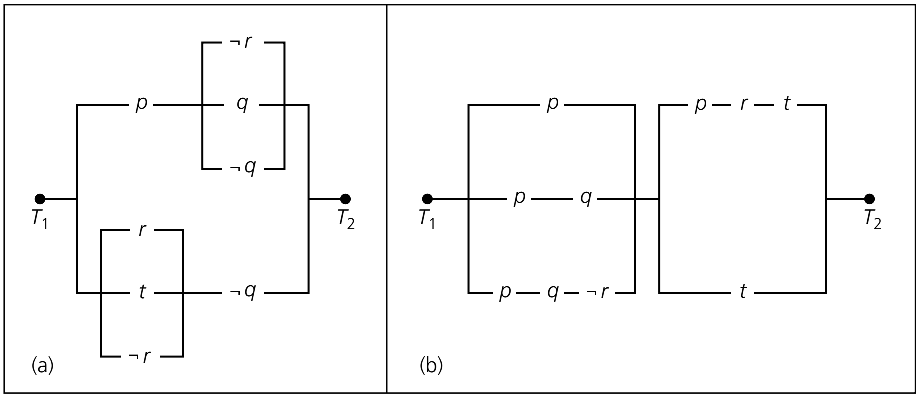


Figure 3

b) $(p \to q) \land [\neg q \land (r \lor \neg q)]$ **Reasons**

$\Longleftrightarrow (p \to q) \land \neg q$
$\Longleftrightarrow (\neg p \lor q) \land \neg q$
$\Longleftrightarrow \neg q \land (\neg p \lor q)$
$\Longleftrightarrow (\neg q \land \neg p) \lor (\neg q \land q)$
$\Longleftrightarrow (\neg q \land \neg p) \lor F_0$
$\Longleftrightarrow \neg q \land \neg p$
$\Longleftrightarrow \neg(q \lor p)$

19. Provide the steps and reasons, as in Exercise 18, to establish the following logical equivalences.

a) $p \lor [p \land (p \lor q)] \Longleftrightarrow p$

b) $p \lor q \lor (\neg p \land \neg q \land r) \Longleftrightarrow p \lor q \lor r$

c) $[(\neg p \lor \neg q) \to (p \land q \land r)] \Longleftrightarrow p \land q$

20. Simplify each of the networks shown in Fig. 3.

3
Logical Implication: Rules of Inference

At the end of Section 1 we mentioned the notion of a valid argument. Now we will begin a formal study of what we shall mean by an argument and when such an argument is valid. This in turn will help us when we investigate how to prove theorems throughout the text.

We start by considering the general form of an argument, one we wish to show is valid. So let us consider the implication

$$(p_1 \land p_2 \land p_3 \land \cdots \land p_n) \to q.$$

Here n is a positive integer, the statements $p_1, p_2, p_3, \ldots, p_n$ are called the *premises* of the argument, and the statement q is the *conclusion* for the argument.

The preceding argument is called *valid* if whenever each of the premises $p_1, p_2, p_3, \ldots, p_n$ is true, then the conclusion q is likewise true. [Note that if any one of $p_1, p_2, p_3, \ldots, p_n$ is false, then the hypothesis $p_1 \land p_2 \land p_3 \land \cdots \land p_n$ is false and the implication $(p_1 \land p_2 \land p_3 \land \cdots \land p_n) \to q$ is automatically true, regardless of the truth value of q.] Consequently, one way to establish the validity of a given argument is to show that the statement $(p_1 \land p_2 \land p_3 \land \cdots \land p_n) \to q$ is a tautology.

The following examples illustrate this particular approach.

EXAMPLE 19

Let p, q, r denote the primitive statements given as

p: Roger studies.
q: Roger plays racketball.
r: Roger passes discrete mathematics.

Now let p_1, p_2, p_3 denote the premises

p_1: If Roger studies, then he will pass discrete mathematics.

p_2: If Roger doesn't play racketball, then he'll study.

p_3: Roger failed discrete mathematics.

We want to determine whether the argument

$$(p_1 \wedge p_2 \wedge p_3) \rightarrow q$$

is valid. To do so, we rewrite p_1, p_2, p_3 as

$$p_1: \quad p \rightarrow r \qquad p_2: \quad \neg q \rightarrow p \qquad p_3: \quad \neg r$$

and examine the truth table for the implication

$$[(p \rightarrow r) \wedge (\neg q \rightarrow p) \wedge \neg r] \rightarrow q$$

given in Table 14. Because the final column in Table 14 contains all 1's, the implication is a tautology. Hence we can say that $(p_1 \wedge p_2 \wedge p_3) \rightarrow q$ is a valid argument.

Table 14

			p_1	p_2	p_3	$(p_1 \wedge p_2 \wedge p_3) \rightarrow q$
p	q	r	$p \rightarrow r$	$\neg q \rightarrow p$	$\neg r$	$[(p \rightarrow r) \wedge (\neg q \rightarrow p) \wedge \neg r] \rightarrow q$
0	0	0	1	0	1	1
0	0	1	1	0	0	1
0	1	0	1	1	1	1
0	1	1	1	1	0	1
1	0	0	0	1	1	1
1	0	1	1	1	0	1
1	1	0	0	1	1	1
1	1	1	1	1	0	1

EXAMPLE 20

Let us now consider the truth table in Table 15. The results in the last column of this table show that for any primitive statements p, r, and s, the implication

$$[p \wedge ((p \wedge r) \rightarrow s)] \rightarrow (r \rightarrow s)$$

Table 15

p_1				p_2	q	$(p_1 \wedge p_2) \rightarrow q$
p	r	s	$p \wedge r$	$(p \wedge r) \rightarrow s$	$r \rightarrow s$	$[(p \wedge ((p \wedge r) \rightarrow s)] \rightarrow (r \rightarrow s)$
0	0	0	0	1	1	1
0	0	1	0	1	1	1
0	1	0	0	1	0	1
0	1	1	0	1	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	0	0	1
1	1	1	1	1	1	1

is a tautology. Consequently, for premises

$$p_1: \quad p \qquad p_2: \quad (p \wedge r) \rightarrow s$$

and conclusion q: $(r \rightarrow s)$, we know that $(p_1 \wedge p_2) \rightarrow q$ is a valid argument, and we may say that the truth of the conclusion q is *deduced* or *inferred* from the truth of the premises p_1, p_2.

The idea presented in the preceding two examples leads to the following.

Definition 4 If p, q are arbitrary statements such that $p \rightarrow q$ is a tautology, then we say that p *logically implies* q and we write $p \Rightarrow q$ to denote this situation.

When p, q are statements and $p \Rightarrow q$, the implication $p \rightarrow q$ is a tautology and we refer to $p \rightarrow q$ as a *logical implication.* Note that we can avoid dealing with the idea of a tautology here by saying that $p \Rightarrow q$ (that is, p logically implies q) if q is true whenever p is true.

In Example 6 we found that for primitive statements p, q, the implication $p \rightarrow (p \vee q)$ is a tautology. In this case, therefore, we can say that p logically implies $p \vee q$ and write $p \Rightarrow (p \vee q)$. Furthermore, because of the first substitution rule, we also find that $p \Rightarrow (p \vee q)$ for any statements p, q — that is, $p \rightarrow (p \vee q)$ is a tautology for any statements p, q, whether or not they are primitive statements.

Let p, q be arbitrary statements.

1) If $p \Longleftrightarrow q$, then the statement $p \leftrightarrow q$ is a tautology, so the statements p, q have the same (corresponding) truth values. Under these conditions the statements $p \rightarrow q$, $q \rightarrow p$ are tautologies, and we have $p \Rightarrow q$ and $q \Rightarrow p$.

2) Conversely, suppose that $p \Rightarrow q$ and $q \Rightarrow p$. The logical implication $p \rightarrow q$ tells us that we never have statement p with the truth value 1 and statement q with the truth value 0. But could we have q with the truth value 1 and p with the truth value 0? If this occurred, we could not have the logical implication $q \rightarrow p$. Therefore, when $p \Rightarrow q$ and $q \Rightarrow p$, the statements p, q have the same (corresponding) truth values and $p \Longleftrightarrow q$.

Finally, the notation $p \not\Rightarrow q$ is used to indicate that $p \rightarrow q$ is *not* a tautology — so the given implication (namely, $p \rightarrow q$) is *not* a logical implication.

EXAMPLE 21

From the results in Example 8 (Table 9) and the first substitution rule, we know that for statements p, q,

$$\neg(p \wedge q) \Longleftrightarrow \neg p \vee \neg q.$$

Consequently,

$$\neg(p \wedge q) \Rightarrow (\neg p \vee \neg q) \quad \text{and} \quad (\neg p \vee \neg q) \Rightarrow \neg(p \wedge q)$$

for all statements p, q. Alternatively, because each of the implications

$$\neg(p \wedge q) \rightarrow (\neg p \vee \neg q) \quad \text{and} \quad (\neg p \vee \neg q) \rightarrow \neg(p \wedge q)$$

is a tautology, we may also write

$$[\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)] \Longleftrightarrow T_0 \quad \text{and} \quad [(\neg p \vee \neg q) \rightarrow \neg(p \wedge q)] \Longleftrightarrow T_0.$$

Returning now to our study of techniques for establishing the validity of an argument, we must take a careful look at the size of Tables 14 and 15. Each table has eight rows. For Table 14 we were able to express the three premises p_1, p_2, and p_3, and the conclusion q, in terms of the three primitive statements p, q, and r. A similar situation arose for the argument we analyzed in Table 15, where we had only two premises. But if we were confronted, for example, with establishing whether

$$[(p \to r) \wedge (r \to s) \wedge (t \vee \neg s) \wedge (\neg t \vee u) \wedge \neg u] \to \neg p$$

is a logical implication (or presents a valid argument), the needed table would require $2^5 = 32$ rows. As the number of premises gets larger and our truth tables grow to 64, 128, 256, or more rows, this first technique for establishing the validity of an argument rapidly loses its appeal.

Furthermore, looking at Table 14 once again, we realize that in order to establish whether

$$[(p \to r) \wedge (\neg q \to p) \wedge \neg r] \to q$$

is a valid argument, we need to consider only those rows of the table where each of the three premises $p \to r$, $\neg q \to p$, and $\neg r$ has the truth value 1. (Remember that if the hypothesis — consisting of the conjunction of all of the premises — is false, then the implication is true regardless of the truth value of the conclusion.) This happens only in the third row, so a good deal of Table 14 is not really necessary. (It is not always the case that only one row has all of the premises true. Note that in Table 15 we would be concerned with the results in rows 5, 6, and 8.)

Consequently, what these observations are telling us is that we can possibly eliminate a great deal of the effort put into constructing the truth tables in Table 14 and Table 15. And since we want to avoid even larger tables, we are persuaded to develop a list of techniques called *rules of inference* that will help us as follows:

1) Using these techniques will enable us to consider only the cases wherein all the premises are true. Hence we consider the conclusion only for those rows of a truth table wherein each premise has the truth value 1 — and we do *not* construct the truth table.

2) The rules of inference are fundamental in the development of a step-by-step validation of how the conclusion q logically follows from the premises $p_1, p_2, p_3, \ldots, p_n$ in an implication of the form

$$(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \to q.$$

Such a development will establish the validity of the given argument, for it will show how the truth of the conclusion can be deduced from the truth of the premises.

Each rule of inference arises from a logical implication. In some cases, the logical implication is stated without proof. (However, several of these proofs will be dealt with in the Section Exercises.)

Many rules of inference arise in the study of logic. We concentrate on those that we need to help us validate the arguments that arise in our study of logic. These rules will also help us later when we turn to methods for proving theorems throughout the remainder of the text. Table 19 summarizes the rules we shall now start to investigate.

EXAMPLE 22

For a first example we consider the rule of inference called *Modus Ponens,* or the *Rule of Detachment.* (*Modus Ponens* comes from Latin and may be translated as "the method of affirming.") In symbolic form this rule is expressed by the logical implication

$$[p \wedge (p \to q)] \to q,$$

which is verified in Table 16, where we find that the fourth row is the only one where both of the premises p and $p \to q$ (and the conclusion q) are true.

Table 16

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

The actual rule will be written in the tabular form

$$\begin{array}{l} \quad p \\ \quad p \to q \\ \hline \therefore q \end{array}$$

where the three dots ($\therefore$) stand for the word "therefore," indicating that q is the conclusion for the premises p and $p \to q$, which appear above the horizontal line.

This rule arises when we argue that *if* (1) p is true, *and* (2) $p \to q$ is true (or $p \Rightarrow q$), *then* the conclusion q must also be true. (After all, if q were false and p were true, then we could not have $p \to q$ true.)

The following valid arguments show us how to apply the Rule of Detachment.

a) 1) Lydia wins a ten-million-dollar lottery. $\quad p$
2) If Lydia wins a ten-million-dollar lottery, then Kay will quit her job. $\quad p \to q$
3) Therefore Kay will quit her job. $\quad \therefore q$

b) 1) If Allison vacations in Paris, then she will have to win a scholarship. $\quad p \to q$
2) Allison is vacationing in Paris. $\quad p$
3) Therefore Allison won a scholarship. $\quad \therefore q$

Before closing the discussion on our first rule of inference let us make one final observation. The two examples in (a) and (b) might suggest that the valid argument $[p \wedge (p \to q)] \to q$ is appropriate only for primitive statements p, q. However, since $[p \wedge (p \to q)] \to q$ is a tautology for primitive statements p, q, it follows from the first substitution rule that (all occurrences of) p or q may be replaced by compound statements—and the resulting implication will also be a tautology. Consequently, if r, s, t, and u are primitive statements, then

$$\begin{array}{l} \quad r \vee s \\ \quad (r \vee s) \to (\neg t \wedge u) \\ \hline \therefore \neg t \wedge u \end{array}$$

is a valid argument, by the Rule of Detachment—just as $[(r \vee s) \wedge [(r \vee s) \to (\neg t \wedge u)]] \to (\neg t \wedge u)$ is a tautology.

A similar situation — in which we can apply the first substitution rule — occurs for each of the rules of inference we shall study. However, we shall not mention this so explicitly with these other rules of inference.

EXAMPLE 23

A second rule of inference is given by the logical implication

$$[(p \to q) \land (q \to r)] \to (p \to r),$$

where p, q, and r are any statements. In tabular form it is written

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore p \to r \end{array}$$

This rule, which is referred to as the *Law of the Syllogism,* arises in many arguments. For example, we may use it as follows:

1) If the integer 35244 is divisible by 396, then the integer 35244 is divisible by 66. $\qquad p \to q$

2) If the integer 35244 is divisible by 66, then the integer 35244 is divisible by 3. $\qquad \underline{q \to r}$

3) Therefore, if the integer 35244 is divisible by 396, then the integer 35244 is divisible by 3. $\qquad \therefore p \to r$

The next example involves a slightly longer argument that uses the rules of inference developed in Examples 22 and 23. In fact, we find here that there may be more than one way to establish the validity of an argument.

EXAMPLE 24

Consider the following argument.

1) Rita is baking a cake.

2) If Rita is baking a cake, then she is not practicing her flute.

3) If Rita is not practicing her flute, then her father will not buy her a car.

4) Therefore Rita's father will not buy her a car.

Concentrating on the forms of the statements in the preceding argument, we may write the argument as

$$\begin{array}{l} p \\ p \to \neg q \\ \underline{\neg q \to \neg r} \\ \therefore \neg r \end{array} \qquad (*)$$

Now we need no longer worry about what the statements actually stand for. Our objective is to use the two rules of inference that we have studied so far in order to deduce the truth of the statement $\neg r$ from the truth of the three premises p, $p \to \neg q$, and $\neg q \to \neg r$.

We establish the validity of the argument as follows:

Steps	**Reasons**
1) $p \to \neg q$	Premise
2) $\neg q \to \neg r$	Premise
3) $p \to \neg r$	This follows from steps (1) and (2) and the Law of the Syllogism
4) p	Premise
5) $\therefore \neg r$	This follows from steps (4) and (3) and the Rule of Detachment

Before continuing with a third rule of inference we shall show that the argument presented at (*) can be validated in a second way. Here our "reasons" will be shortened to the form we shall use for the rest of the section. However, we shall always list whatever is needed to demonstrate how each step in an argument comes about, or follows, from prior steps.

A second way to validate the argument follows.

Steps	**Reasons**
1) p	Premise
2) $p \to \neg q$	Premise
3) $\neg q$	Steps (1) and (2) and the Rule of Detachment
4) $\neg q \to \neg r$	Premise
5) $\therefore \neg r$	Steps (3) and (4) and the Rule of Detachment

EXAMPLE 25

The rule of inference called *Modus Tollens* is given by

$$\begin{array}{l} p \to q \\ \underline{\neg q} \\ \therefore \neg p \end{array}$$

This follows from the logical implication $[(p \to q) \wedge \neg q] \to \neg p$. *Modus Tollens* comes from Latin and can be translated as "method of denying." This is appropriate because we deny the conclusion, q, so as to prove $\neg p$. (Note that we can also obtain this rule from the one for Modus Ponens by using the fact that $p \to q \Longleftrightarrow \neg q \to \neg p$.)

The following exemplifies the use of Modus Tollens is making a valid inference:

1) If Connie is elected president of Phi Delta sorority, then Helen will pledge that sorority. $\quad p \to q$

2) Helen did not pledge Phi Delta sorority. $\quad \neg q$

3) Therefore Connie was not elected president of Phi Delta sorority. $\quad \therefore \neg p$

And now we shall use Modus Tollens to show that the following argument is valid (for primitive statements p, r, s, t, and u).

$$\begin{array}{l} p \to r \\ r \to s \\ t \vee \neg s \\ \neg t \vee u \\ \underline{\neg u} \\ \therefore \neg p \end{array}$$

Both Modus Tollens and the Law of the Syllogism come into play, along with the logical equivalence we developed in Example 7.

Steps	**Reasons**
1) $p \to r,\ r \to s$	Premises
2) $p \to s$	Step (1) and the Law of the Syllogism
3) $t \vee \neg s$	Premise
4) $\neg s \vee t$	Step (3) and the Commutative Law of $\vee$
5) $s \to t$	Step (4) and the fact that $\neg s \vee t \Longleftrightarrow s \to t$
6) $p \to t$	Steps (2) and (5) and the Law of the Syllogism
7) $\neg t \vee u$	Premise
8) $t \to u$	Step (7) and the fact that $\neg t \vee u \Longleftrightarrow t \to u$
9) $p \to u$	Steps (6) and (8) and the Law of the Syllogism
10) $\neg u$	Premise
11) $\therefore \neg p$	Steps (9) and (10) and Modus Tollens

Before continuing with another rule of inference let us summarize what we have just accomplished (and *not* accomplished). The preceding argument shows that

$$[(p \to r) \wedge (r \to s) \wedge (t \vee \neg s) \wedge (\neg t \vee u) \wedge \neg u] \Rightarrow \neg p.$$

We have *not* used the laws of logic, as in Section 2, to express the statement

$$(p \to r) \wedge (r \to s) \wedge (t \vee \neg s) \wedge (\neg t \vee u) \wedge \neg u$$

as a simpler logically equivalent statement. Note that

$$[(p \to r) \wedge (r \to s) \wedge (t \vee \neg s) \wedge (\neg t \vee u) \wedge \neg u] \not\Leftrightarrow \neg p.$$

For when p has the truth value 0 and u has the truth value 1, the truth value of $\neg p$ is 1 while that of $\neg u$ and $(p \to r) \wedge (r \to s) \wedge (t \vee \neg s) \wedge (\neg t \vee u) \wedge \neg u$ is 0.

Let us once more examine a tabular form for each of the two related rules of inference, Modus Ponens and Modus Tollens.

Modus Ponens: $\begin{array}{l} p \to q \\ \underline{p \qquad} \\ \therefore q \end{array}$ Modus Tollens: $\begin{array}{l} p \to q \\ \underline{\neg q \qquad} \\ \therefore \neg p \end{array}$

The reason we wish to do this is that there are other tabular forms that may arise — and these are similar in appearance but present *invalid* arguments — where each of the premises is true but the conclusion is false.

a) Consider the following argument:

1) If Margaret Thatcher is the president of the United States, then she is at least 35 years old. $p \to q$

2) Margaret Thatcher is at least 35 years old. $\underline{q \qquad}$

3) Therefore Margaret Thatcher is the president of the United States. $\therefore p$

Here we find that $[(p \to q) \wedge q] \to p$ is *not* a tautology. For if we consider the truth value assignments p: 0 and q: 1, then each of the premises $p \to q$ and q is true while the conclusion p is false. This *invalid* argument results from the *fallacy* (error in reasoning) where we try to argue by the converse — that is, while $[(p \to q) \wedge p] \Rightarrow q$, it is *not the case* that $[(p \to q) \wedge q] \Rightarrow p$.

b) A second argument where the conclusion doesn't necessarily follow from the premises may be given by:

1) If $2 + 3 = 6$, then $2 + 4 = 6$. $\quad p \to q$
2) $2 + 3 \neq 6$. $\quad \neg p$
3) Therefore $2 + 4 \neq 6$. $\quad \therefore \neg q$

In this case we find that $[(p \to q) \wedge \neg p] \to \neg q$ is *not* a tautology. Once again the truth value assignments p: 0 and q: 1 show us that the premises $p \to q$ and $\neg p$ can both be true while the conclusion $\neg q$ is false. The fallacy behind this invalid argument arises from our attempt to argue by the inverse — for although $[(p \to q) \wedge \neg q] \Rightarrow \neg p$, it does *not* follow that $[(p \to q) \wedge \neg p] \Rightarrow \neg q$.

Before proceeding further we now mention a rather simple but important rule of inference.

EXAMPLE 26

The following rule of inference arises from the observation that if p, q are true statements, then $p \wedge q$ is a true statement.

Now suppose that statements p, q occur in the development of an argument. These statements may be (given) premises or results that are derived from premises and/or from results developed earlier in the argument. Then under these circumstances the two statements p, q can be combined into their conjunction $p \wedge q$, and this new statement can be used in later steps as the argument continues.

We call this rule the *Rule of Conjunction* and write it in tabular form as

$$\begin{array}{l} p \\ \underline{q \quad\quad} \\ \therefore p \wedge q \end{array}$$

As we proceed further with our study of rules of inference, we find another fairly simple but important rule.

EXAMPLE 27

The following rule of inference — one we may feel just illustrates good old common sense — is called the *Rule of Disjunctive Syllogism*. This rule comes about from the logical implication

$$[(p \vee q) \wedge \neg p] \to q,$$

which we can derive from Modus Ponens by observing that $p \vee q \Longleftrightarrow \neg p \to q$.

In tabular form we write

$$\begin{array}{l} p \vee q \\ \underline{\neg p \quad\quad} \\ \therefore q \end{array}$$

This rule of inference arises when there are exactly two possibilities to consider and we are able to eliminate one of them as being true. Then the other possibility has to be true. The following illustrates one such application of this rule.

1) Bart's wallet is in his back pocket or it is on his desk. $\quad p \vee q$
2) Bart's wallet is not in his back pocket. $\quad \neg p$
3) Therefore Bart's wallet is on his desk. $\quad \therefore q$

At this point we have examined five rules of inference. But before we try to validate any more arguments like the one (with 11 steps) in Example 25, we shall look at one more of these rules. This one underlies a method of proof that is sometimes confused with the contrapositive method (or proof) given in Modus Tollens. The confusion arises because both methods involve the negation of a statement. However, we will soon realize that these are two distinct methods. (Toward the end of Section 5 we shall compare and contrast these two methods once again.)

EXAMPLE 28

Let p denote an arbitrary statement, and F_0 a contradiction. The results in column 5 of Table 17 show that the implication $(\neg p \rightarrow F_0) \rightarrow p$ is a tautology, and this provides us with the rule of inference called the *Rule of Contradiction.* In tabular form this rule is written as

$$\begin{array}{l} \neg p \rightarrow F_0 \\ \hline \therefore p \end{array}$$

Table 17

p	$\neg p$	F_0	$\neg p \rightarrow F_0$	$(\neg p \rightarrow F_0) \rightarrow p$
1	0	0	1	1
0	1	0	0	1

This rule tells us that if p is a statement and $\neg p \rightarrow F_0$ is true, then $\neg p$ must be false because F_0 is false. So then we have p true.

The Rule of Contradiction is the basis of a method for establishing the validity of an argument — namely, the method of *Proof by Contradiction*, or *Reductio ad Absurdum*. The idea behind the method of Proof by Contradiction is to establish a statement (namely, the conclusion of an argument) by showing that, if this statement were false, then we would be able to deduce an impossible consequence. The use of this method arises in certain arguments which we shall now describe.

In general, when we want to establish the validity of the argument

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q,$$

we can establish the validity of the logically equivalent argument

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge \neg q) \rightarrow F_0.$$

[This follows from the tautology in column 7 of Table 18 and the first substitution rule — where we replace the primitive statement p by the statement $(p_1 \wedge p_2 \wedge \cdots \wedge p_n)$.]

Table 18

p	q	F_0	$p \wedge \neg q$	$(p \wedge \neg q) \rightarrow F_0$	$p \rightarrow q$	$(p \rightarrow q) \leftrightarrow [(p \wedge \neg q) \rightarrow F_0]$
0	0	0	0	1	1	1
0	1	0	0	1	1	1
1	0	0	1	0	0	1
1	1	0	0	1	1	1

When we apply the method of Proof by Contradiction, we first assume that what we are trying to validate (or prove) is actually false. Then we use this assumption as an additional premise in order to produce a contradiction (or impossible situation) of the form $s \wedge \neg s$, for some statement s. Once we have derived this contradiction we may then conclude that the statement we were given was in fact true — and this validates the argument (or completes the proof).

We shall turn to the method of Proof by Contradiction when it is (or appears to be) easier to use $\neg q$ *in conjunction with* the premises $p_1, p_2, \ldots, p_n$ in order to deduce a contradiction than it is to deduce the conclusion q directly from the premises $p_1, p_2, \ldots, p_n$. The method of Proof by Contradiction will be used in some of the later examples for this section — namely, Examples 32 and 35.

Now that we have examined six rules of inference, we summarize these rules and introduce several others in Table 19 (on the following page).

The next five examples will present valid arguments. In so doing, these examples will show us how to apply the rules listed in Table 19 in conjunction with other results, such as the laws of logic.

EXAMPLE 29

Our first example demonstrates the validity of the argument

$$\begin{array}{l} p \rightarrow r \\ \neg p \rightarrow q \\ q \rightarrow s \\ \hline \therefore \neg r \rightarrow s \end{array}$$

Steps	**Reasons**
1) $p \rightarrow r$	Premise
2) $\neg r \rightarrow \neg p$	Step (1) and $p \rightarrow r \Longleftrightarrow \neg r \rightarrow \neg p$
3) $\neg p \rightarrow q$	Premise
4) $\neg r \rightarrow q$	Steps (2) and (3) and the Law of the Syllogism
5) $q \rightarrow s$	Premise
6) $\therefore \neg r \rightarrow s$	Steps (4) and (5) and the Law of the Syllogism

A second way to validate the given argument proceeds as follows.

Steps	**Reasons**
1) $p \rightarrow r$	Premise
2) $q \rightarrow s$	Premise
3) $\neg p \rightarrow q$	Premise
4) $p \vee q$	Step (3) and $(\neg p \rightarrow q) \Longleftrightarrow (\neg\neg p \vee q) \Longleftrightarrow (p \vee q)$, where the second logical equivalence follows by the Law of Double Negation
5) $r \vee s$	Steps (1), (2), and (4) and the Rule of the Constructive Dilemma
6) $\therefore \neg r \rightarrow s$	Step (5) and $(r \vee s) \Longleftrightarrow (\neg\neg r \vee s) \Longleftrightarrow (\neg r \rightarrow s)$, where the Law of Double Negation is used in the first logical equivalence

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The next example is somewhat more involved.

Table 19

Rule of Inference	Related Logical Implication	Name of Rule
1) $\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$	$[p \wedge (p \rightarrow q)] \rightarrow q$	Rule of Detachment (Modus Ponens)
2) $\begin{array}{l} p \rightarrow q \\ \underline{q \rightarrow r} \\ \therefore p \rightarrow r \end{array}$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Law of the Syllogism
3) $\begin{array}{l} p \rightarrow q \\ \underline{\neg q} \\ \therefore \neg p \end{array}$	$[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$	Modus Tollens
4) $\begin{array}{l} p \\ \underline{q} \\ \therefore p \wedge q \end{array}$		Rule of Conjunction
5) $\begin{array}{l} p \vee q \\ \underline{\neg p} \\ \therefore q \end{array}$	$[(p \vee q) \wedge \neg p] \rightarrow q$	Rule of Disjunctive Syllogism
6) $\begin{array}{l} \underline{\neg p \rightarrow F_0} \\ \therefore p \end{array}$	$(\neg p \rightarrow F_0) \rightarrow p$	Rule of Contradiction
7) $\begin{array}{l} \underline{p \wedge q} \\ \therefore p \end{array}$	$(p \wedge q) \rightarrow p$	Rule of Conjunctive Simplification
8) $\begin{array}{l} \underline{p} \\ \therefore p \vee q \end{array}$	$p \rightarrow p \vee q$	Rule of Disjunctive Amplification
9) $\begin{array}{l} p \wedge q \\ \underline{p \rightarrow (q \rightarrow r)} \\ \therefore r \end{array}$	$[(p \wedge q) \wedge [p \rightarrow (q \rightarrow r)]] \rightarrow r$	Rule of Conditional Proof
10) $\begin{array}{l} p \rightarrow r \\ \underline{q \rightarrow r} \\ \therefore (p \vee q) \rightarrow r \end{array}$	$[(p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow [(p \vee q) \rightarrow r]$	Rule for Proof by Cases
11) $\begin{array}{l} p \rightarrow q \\ r \rightarrow s \\ \underline{p \vee r} \\ \therefore q \vee s \end{array}$	$[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r)] \rightarrow (q \vee s)$	Rule of the Constructive Dilemma
12) $\begin{array}{l} p \rightarrow q \\ r \rightarrow s \\ \underline{\neg q \vee \neg s} \\ \therefore \neg p \vee \neg r \end{array}$	$[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (\neg q \vee \neg s)] \rightarrow (\neg p \vee \neg r)$	Rule of the Destructive Dilemma

EXAMPLE 30

Establish the validity of the argument

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow (r \wedge s) \\ \neg r \vee (\neg t \vee u) \\ \underline{p \wedge t} \\ \therefore u \end{array}$$

Steps	Reasons
1) $p \rightarrow q$	Premise
2) $q \rightarrow (r \wedge s)$	Premise
3) $p \rightarrow (r \wedge s)$	Steps (1) and (2) and the Law of the Syllogism
4) $p \wedge t$	Premise
5) p	Step (4) and the Rule of Conjunctive Simplification
6) $r \wedge s$	Steps (5) and (3) and the Rule of Detachment
7) r	Step (6) and the Rule of Conjunctive Simplification
8) $\neg r \vee (\neg t \vee u)$	Premise
9) $\neg(r \wedge t) \vee u$	Step (8), the Associative Law of $\vee$, and DeMorgan's Laws
10) t	Step (4) and the Rule of Conjunctive Simplification
11) $r \wedge t$	Steps (7) and (10) and the Rule of Conjunction
12) $\therefore u$	Steps (9) and (11), the Law of Double Negation, and the Rule of Disjunctive Syllogism

EXAMPLE 31

This example will provide a way to show that the following argument is valid.

> If the band could not play rock music or the refreshments were not delivered on time, then the New Year's party would have been canceled and Alicia would have been angry. If the party were canceled, then refunds would have had to be made. No refunds were made.
>
> Therefore the band could play rock music.

First we convert the given argument into symbolic form by using the following statement assignments:

p: The band could play rock music.
q: The refreshments were delivered on time.
r: The New Year's party was canceled.
s: Alicia was angry.
t: Refunds had to be made.

The argument above now becomes

$$\begin{array}{l} (\neg p \vee \neg q) \rightarrow (r \wedge s) \\ r \rightarrow t \\ \neg t \\ \hline \therefore p \end{array}$$

We can establish the validity of this argument as follows.

Steps	Reasons
1) $r \rightarrow t$	Premise
2) $\neg t$	Premise
3) $\neg r$	Steps (1) and (2) and Modus Tollens
4) $\neg r \vee \neg s$	Step (3) and the Rule of Disjunctive Amplification
5) $\neg(r \wedge s)$	Step (4) and DeMorgan's Laws
6) $(\neg p \vee \neg q) \rightarrow (r \wedge s)$	Premise
7) $\neg(\neg p \vee \neg q)$	Steps (6) and (5) and Modus Tollens
8) $p \wedge q$	Step (7), DeMorgan's Laws, and the Law of Double Negation
9) $\therefore p$	Step (8) and the Rule of Conjunctive Simplification

EXAMPLE 32

In this instance we shall use the method of Proof by Contradiction. Consider the argument

$$\begin{array}{l} \neg p \leftrightarrow q \\ q \rightarrow r \\ \underline{\neg r} \\ \therefore p \end{array}$$

To establish the validity for this argument, we assume the negation $\neg p$ of the conclusion p as another premise. The objective now is to use these four premises to derive a contradiction F_0. Our derivation follows.

Steps	**Reasons**
1) $\neg p \leftrightarrow q$	Premise
2) $(\neg p \rightarrow q) \wedge (q \rightarrow \neg p)$	Step (1) and $(\neg p \leftrightarrow q) \Longleftrightarrow [(\neg p \rightarrow q) \wedge (q \rightarrow \neg p)]$
3) $\neg p \rightarrow q$	Step (2) and the Rule of Conjunctive Simplification
4) $q \rightarrow r$	Premise
5) $\neg p \rightarrow r$	Steps (3) and (4) and the Law of the Syllogism
6) $\neg p$	Premise (the one assumed)
7) r	Steps (5) and (6) and the Rule of Detachment
8) $\neg r$	Premise
9) $r \wedge \neg r (\Longleftrightarrow F_0)$	Steps (7) and (8) and the Rule of Conjunction
10) $\therefore p$	Steps (6) and (9) and the method of Proof by Contradiction

If we examine further what has happened here, we find that

$$[(\neg p \leftrightarrow q) \wedge (q \rightarrow r) \wedge \neg r \wedge \neg p] \Rightarrow F_0.$$

This requires the truth value of $[(\neg p \leftrightarrow q) \wedge (q \rightarrow r) \wedge \neg r \wedge \neg p]$ to be 0. Because $\neg p \leftrightarrow q$, $q \rightarrow r$, and $\neg r$ are the given premises, each of these statements has the truth value 1. Consequently, for $[(\neg p \leftrightarrow q) \wedge (q \rightarrow r) \wedge \neg r \wedge \neg p]$ to have the truth value 0, the statement $\neg p$ must have the truth value 0. Therefore p has the truth value 1, and the conclusion p of the argument is true.

Before we consider our next example, we need to examine columns 5 and 7 of Table 20. These identical columns tell us that for primitive statements p, q, and r,

$$[p \rightarrow (q \rightarrow r)] \Longleftrightarrow [(p \wedge q) \rightarrow r].$$

Using the first substitution rule, let us replace each occurrence of p by the compound statement $(p_1 \wedge p_2 \wedge \cdots \wedge p_n)$. Then we obtain the new result

$$[(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow (q \rightarrow r)] \Longleftrightarrow [(p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge q) \rightarrow r].$$

Table 20

p	q	r	$p \wedge q$	$(p \wedge q) \rightarrow r$	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$
0	0	0	0	1	1	1
0	0	1	0	1	1	1
0	1	0	0	1	0	1
0	1	1	0	1	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	0	0	0
1	1	1	1	1	1	1

This result tells us that if we wish to establish the validity of the argument (*) we may be able to do so by establishing the validity of the corresponding argument (**).

$$(*)\quad \begin{array}{l} p_1 \\ p_2 \\ \vdots \\ \underline{p_n} \\ \therefore q \rightarrow r \end{array} \qquad\qquad (**)\quad \begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \underline{q} \\ \therefore r \end{array}$$

After all, suppose we want to show that $q \rightarrow r$ has the truth value 1, when each of $p_1, p_2, \ldots, p_n$ does. If the truth value for q is 0, then there is nothing left to do, since the truth value for $q \rightarrow r$ is 1. Hence the real problem is to show that $q \rightarrow r$ has truth value 1, when each of $p_1, p_2, \ldots, p_n$, *and* q does—that is, we need to show that when $p_1, p_2, \ldots, p_n, q$ each have truth value 1, then the truth value of r is 1.

We demonstrate this principle in the next example.

EXAMPLE 33

In order to establish the validity of the argument

$$(*)\qquad \begin{array}{l} u \rightarrow r \\ (r \wedge s) \rightarrow (p \vee t) \\ q \rightarrow (u \wedge s) \\ \underline{\neg t} \\ \therefore q \rightarrow p \end{array}$$

we consider the corresponding argument

$$(**)\qquad \begin{array}{l} u \rightarrow r \\ (r \wedge s) \rightarrow (p \vee t) \\ q \rightarrow (u \wedge s) \\ \neg t \\ \underline{q} \\ \therefore p \end{array}$$

[Note that q is the hypothesis of the conclusion $q \rightarrow p$ for argument (*) and that it becomes another premise for argument (**) where the conclusion is p.]

To validate the argument (**) we proceed as follows.

Steps	**Reasons**
1) q	Premise
2) $q \rightarrow (u \wedge s)$	Premise
3) $u \wedge s$	Steps (1) and (2) and the Rule of Detachment
4) u	Step (3) and the Rule of Conjunctive Simplification
5) $u \rightarrow r$	Premise
6) r	Steps (4) and (5) and the Rule of Detachment
7) s	Step (3) and the Rule of Conjunctive Simplification
8) $r \wedge s$	Steps (6) and (7) and the Rule of Conjunction
9) $(r \wedge s) \rightarrow (p \vee t)$	Premise
10) $p \vee t$	Steps (8) and (9) and the Rule of Detachment
11) $\neg t$	Premise
12) $\therefore p$	Steps (10) and (11) and the Rule of Disjunctive Syllogism

We now know that for argument (**)

$$[(u \rightarrow r) \wedge [(r \wedge s) \rightarrow (p \vee t)] \wedge [q \rightarrow (u \wedge s)] \wedge \neg t \wedge q] \Rightarrow p,$$

and for argument (*) it follows that

$$[(u \rightarrow r) \wedge [(r \wedge s) \rightarrow (p \vee t)] \wedge [q \rightarrow (u \wedge s)] \wedge \neg t] \Rightarrow (q \rightarrow p).$$

Examples 29 through 33 have given us some idea of how to establish the validity of an argument. Following Example 25 we discussed two situations indicating when an argument is invalid — namely, when we try to argue by the converse or the inverse. So now it is time for us to learn a little more about how to determine when an argument is invalid.

Given an argument

$$\begin{array}{r} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

we say that the argument is invalid if it is possible for each of the premises $p_1, p_2, p_3, \ldots, p_n$ to be true (with truth value 1), while the conclusion q is false (with truth value 0).

The next example illustrates an indirect method whereby we may be able to show that an argument we *feel* is invalid (perhaps because we cannot find a way to show that it is valid) actually *is* invalid.

EXAMPLE 34

Consider the primitive statements p, q, r, s, and t and the argument

$$\begin{array}{l} p \\ p \vee q \\ q \rightarrow (r \rightarrow s) \\ t \rightarrow r \\ \hline \therefore \neg s \rightarrow \neg t \end{array}$$

To show that this is an invalid argument, we need *one* assignment of truth values for each of the statements p, q, r, s, and t such that the conclusion $\neg s \rightarrow \neg t$ is false (has the truth value 0) while the four premises are all true (have the truth value 1). The only time the conclusion $\neg s \rightarrow \neg t$ is false is when $\neg s$ is true and $\neg t$ is false. This implies that the truth value for s is 0 and that the truth value for t is 1.

Because p is one of the premises, its truth value must be 1. For the premise $p \vee q$ to have the truth value 1, q may be either true (1) or false (0). So let us consider the premise $t \rightarrow r$ where we know that t is true. If $t \rightarrow r$ is to be true, then r must be true (have the truth value 1). Now with r true (1) and s false (0), it follows that $r \rightarrow s$ is false (0), and that the truth value of the premise $q \rightarrow (r \rightarrow s)$ will be 1 only when q is false (0).

Consequently, under the truth value assignments

$$p\colon\ 1 \qquad q\colon\ 0 \qquad r\colon\ 1 \qquad s\colon\ 0 \qquad t\colon\ 1,$$

the four premises

$$p \qquad p \vee q \qquad q \rightarrow (r \rightarrow s) \qquad t \rightarrow r$$

all have the truth value 1, while the conclusion

$$\neg s \rightarrow \neg t$$

has the truth value 0. In this case we have shown the given argument to be invalid.

The truth value assignments p: 1, q: 0, r: 1, s: 0, and t: 1 of Example 34 provide one case that *disproves* what we thought might have been a valid argument. We should now start to realize that in trying to show that an implication of the form

$$(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \rightarrow q$$

presents a valid argument, we need to consider *all* cases where the premises $p_1, p_2, p_3, \ldots, p_n$ are true. [Each such case is an assignment of truth values for the primitive statements (that make up the premises) where $p_1, p_2, p_3, \ldots, p_n$ are true.] In order to do so — namely, to cover the cases without writing out the truth table — we have been using the rules of inference together with the laws of logic and other logical equivalences. To cover all the necessary cases, we cannot use one specific example (or case) as a means of establishing the validity of the argument (for all possible cases). However, whenever we wish to show that an implication (of the preceding form) is not a tautology, all we need to find is one case for which the implication is false — that is, one case in which all the premises are true but the conclusion is false. This *one* case provides a *counterexample* for the argument and shows it to be invalid.

Let us consider a second example wherein we try the indirect approach of Example 34.

EXAMPLE 35

What can we say about the validity or invalidity of the following argument? Here p, q, r, and s denote primitive statements.)

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow s \\ r \rightarrow \neg s \\ \underline{\neg p \veebar r} \\ \therefore \neg p \end{array}$$

Can the conclusion $\neg p$ be false while the four premises are all true? The conclusion $\neg p$ is false when p has the truth value 1. So for the premise $p \rightarrow q$ to be true, the truth value of q must be 1. From the truth of the premise $q \rightarrow s$, the truth of q forces the truth of s. Consequently, at this point we have statements p, q, and s all with the truth value 1.

Continuing with the premise $r \to \neg s$, we find that because s has the truth value 1, the truth value of r must be 0. Hence r is false. But with $\neg p$ false and the premise $\neg p \veebar r$ true, we also have r true. Therefore we find that $p \Rightarrow (\neg r \wedge r)$.

We have failed in our attempt to find a counterexample to the validity of the given argument. However, this failure has shown us that the given argument is valid — and the validity follows by using the method of Proof by Contradiction.

This introduction to the rules of inference has been far from exhaustive. Several of the books cited among the references listed near the end of this chapter offer additional material for the reader who wishes to pursue this topic further. In Section 5 we shall apply the ideas developed in this section to statements of a more mathematical nature. For we shall want to learn how to develop a proof for a theorem.

EXERCISES 3

1. The following are three valid arguments. Establish the validity of each by means of a truth table. In each case, determine which rows of the table are crucial for assessing the validity of the argument and which rows can be ignored.

a) $[p \wedge (p \to q) \wedge r] \to [(p \vee q) \to r]$

b) $[[(p \wedge q) \to r] \wedge \neg q \wedge (p \to \neg r)] \to (\neg p \vee \neg q)$

c) $[[p \vee (q \vee r)] \wedge \neg q] \to (p \vee r)$

2. Use truth tables to verify that each of the following is a logical implication.

a) $[(p \to q) \wedge (q \to r)] \to (p \to r)$

b) $[(p \to q) \wedge \neg q] \to \neg p$

c) $[(p \vee q) \wedge \neg p] \to q$

d) $[(p \to r) \wedge (q \to r)] \to [(p \vee q) \to r]$

3. Verify that each of the following is a logical implication by showing that it is impossible for the conclusion to have the truth value 0 while the hypothesis has the truth value 1.

a) $(p \wedge q) \to p$

b) $p \to (p \vee q)$

c) $[(p \vee q) \wedge \neg p] \to q$

d) $[(p \to q) \wedge (r \to s) \wedge (p \vee r)] \to (q \vee s)$

e) $[(p \to q) \wedge (r \to s) \wedge (\neg q \vee \neg s)] \to (\neg p \vee \neg r)$

4. For each of the following pairs of statements, use Modus Ponens or Modus Tollens to fill in the blank line so that a valid argument is presented.

a) If Janice has trouble starting her car, then her daughter Angela will check Janice's spark plugs.
Janice had trouble starting her car.
$\therefore$ ______________________________

b) If Brady solved the first problem correctly, then the answer he obtained is 137.
Brady's answer to the first problem is not 137.
$\therefore$ ______________________________

c) If this is a **repeat-until** loop, then the body of this loop is executed at least once.
____
$\therefore$ The body of the loop is executed at least once.

d) If Tim plays basketball in the afternoon, then he will not watch television in the evening.
____
$\therefore$ Tim didn't play basketball in the afternoon.

5. Consider each of the following arguments. If the argument is valid, identify the rule of inference that establishes its validity. If not, indicate whether the error is due to an attempt to argue by the converse or by the inverse.

a) Andrea can program in C++, and she can program in Java.
Therefore Andrea can program in C++.

b) A sufficient condition for Bubbles to win the golf tournament is that her opponent Meg not sink a birdie on the last hole.
Bubbles won the golf tournament.
Therefore Bubbles' opponent Meg did not sink a birdie on the last hole.

c) If Ron's computer program is correct, then he'll be able to complete his computer science assignment in at most two hours.
It takes Ron over two hours to complete his computer science assignment.
Therefore Ron's computer program is not correct.

d) Eileen's car keys are in her purse, or they are on the kitchen table.

Eileen's car keys are not on the kitchen table.
Therefore Eileen's car keys are in her purse.

e) If interest rates fall, then the stock market will rise.
Interest rates are not falling.
Therefore the stock market will not rise.

6. For primitive statements p, q, and r, let P denote the statement

$$[p \wedge (q \wedge r)] \vee \neg[p \vee (q \wedge r)],$$

while P_1 denotes the statement

$$[p \wedge (q \vee r)] \vee \neg[p \vee (q \vee r)].$$

a) Use the rules of inference to show that

$$q \wedge r \Rightarrow q \vee r.$$

b) Is it true that $P \Rightarrow P_1$?

7. Give the reason(s) for each step needed to show that the following argument is valid.

$$[p \wedge (p \rightarrow q) \wedge (s \vee r) \wedge (r \rightarrow \neg q)] \rightarrow (s \vee t)$$

Steps	Reasons
1) p	
2) $p \rightarrow q$	
3) q	
4) $r \rightarrow \neg q$	
5) $q \rightarrow \neg r$	
6) $\neg r$	
7) $s \vee r$	
8) s	
9) $\therefore s \vee t$	

8. Give the reasons for the steps verifying the following argument.

$$\begin{array}{l} (\neg p \vee q) \rightarrow r \\ r \rightarrow (s \vee t) \\ \neg s \wedge \neg u \\ \neg u \rightarrow \neg t \\ \hline \therefore p \end{array}$$

Steps	Reasons
1) $\neg s \wedge \neg u$	
2) $\neg u$	
3) $\neg u \rightarrow \neg t$	
4) $\neg t$	
5) $\neg s$	
6) $\neg s \wedge \neg t$	
7) $r \rightarrow (s \vee t)$	
8) $\neg(s \vee t) \rightarrow \neg r$	
9) $(\neg s \wedge \neg t) \rightarrow \neg r$	
10) $\neg r$	
11) $(\neg p \vee q) \rightarrow r$	
12) $\neg r \rightarrow \neg(\neg p \vee q)$	
13) $\neg r \rightarrow (p \wedge \neg q)$	
14) $p \wedge \neg q$	
15) $\therefore p$	

9. a) Give the reasons for the steps given to validate the argument

$$[(p \rightarrow q) \wedge (\neg r \vee s) \wedge (p \vee r)] \rightarrow (\neg q \rightarrow s).$$

Steps	Reasons
1) $\neg(\neg q \rightarrow s)$	
2) $\neg q \wedge \neg s$	
3) $\neg s$	
4) $\neg r \vee s$	
5) $\neg r$	
6) $p \rightarrow q$	
7) $\neg q$	
8) $\neg p$	
9) $p \vee r$	
10) r	
11) $\neg r \wedge r$	
12) $\therefore \neg q \rightarrow s$	

b) Give a direct proof for the result in part (a).

c) Give a direct proof for the result in Example 32.

10. Establish the validity of the following arguments.

a) $[(p \wedge \neg q) \wedge r] \rightarrow [(p \wedge r) \vee q]$

b) $[p \wedge (p \rightarrow q) \wedge (\neg q \vee r)] \rightarrow r$

c) $$\begin{array}{l} p \rightarrow q \\ \neg q \\ \neg r \\ \hline \therefore \neg(p \vee r) \end{array}$$

d) $$\begin{array}{l} p \rightarrow q \\ r \rightarrow \neg q \\ r \\ \hline \therefore \neg p \end{array}$$

e) $$\begin{array}{l} p \rightarrow (q \rightarrow r) \\ \neg q \rightarrow \neg p \\ p \\ \hline \therefore r \end{array}$$

f) $$\begin{array}{l} p \wedge q \\ p \rightarrow (r \wedge q) \\ r \rightarrow (s \vee t) \\ \neg s \\ \hline \therefore t \end{array}$$

g) $$\begin{array}{l} p \rightarrow (q \rightarrow r) \\ p \vee s \\ t \rightarrow q \\ \neg s \\ \hline \therefore \neg r \rightarrow \neg t \end{array}$$

h) $$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \neg r \\ \hline \therefore q \end{array}$$

11. Show that each of the following arguments is invalid by providing a counterexample — that is, an assignment of truth values for the given primitive statements p, q, r, and s such that all premises are true (have the truth value 1) while the conclusion is false (has the truth value 0).

a) $[(p \wedge \neg q) \wedge [p \rightarrow (q \rightarrow r)]] \rightarrow \neg r$

b) $[[(p \wedge q) \rightarrow r] \wedge (\neg q \vee r)] \rightarrow p$

c) $$\begin{array}{l} p \leftrightarrow q \\ q \rightarrow r \\ r \vee \neg s \\ \neg s \rightarrow q \\ \hline \therefore s \end{array}$$

d) $$\begin{array}{l} p \\ p \rightarrow r \\ p \rightarrow (q \vee \neg r) \\ \neg q \vee \neg s \\ \hline \therefore s \end{array}$$

12. Write each of the following arguments in symbolic form. Then establish the validity of the argument or give a counterexample to show that it is invalid.

a) If Rochelle gets the supervisor's position and works hard, then she'll get a raise. If she gets the raise, then she'll buy a new car. She has not purchased a new car. Therefore either Rochelle did not get the supervisor's position or she did not work hard.

b) If Dominic goes to the racetrack, then Helen will be mad. If Ralph plays cards all night, then Carmela will be mad. If either Helen or Carmela gets mad, then Veronica (their attorney) will be notified. Veronica has not heard from either of these two clients. Consequently, Dominic didn't make it to the racetrack and Ralph didn't play cards all night.

c) If there is a chance of rain or her red headband is missing, then Lois will not mow her lawn. Whenever the temperature is over 80°F, there is no chance for rain. Today the temperature is 85°F and Lois is wearing her red headband. Therefore (sometime today) Lois will mow her lawn.

13. **a)** Given primitive statements p, q, r, show that the implication

$$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$$

is a tautology.

b) The tautology in part (a) provides the rule of inference known as *resolution*, where the conclusion $(q \vee r)$ is called the *resolvent*. This rule was proposed in 1965 by J. A. Robinson and is the basis of many computer programs designed to automate a reasoning system.

In applying resolution each premise (in the hypothesis) and the conclusion are written as *clauses*. A clause is a primitive statement or its negation, or it is the disjunction of terms each of which is a primitive statement or the negation of such a statement. Hence the given rule has the clauses $(p \vee q)$ and $(\neg p \vee r)$ as premises and the clause $(q \vee r)$ as its conclusion (or, resolvent). Should we have the premise $\neg(p \wedge q)$, we replace this by the logically equivalent clause $\neg p \vee \neg q$, by the first of DeMorgan's Laws. The premise $\neg(p \vee q)$ can be replaced by the two clauses $\neg p$, $\neg q$. This is due to the second DeMorgan Law and the Rule of Conjunctive Simplification. For the premise $p \vee (q \wedge r)$, we apply the Distributive Law of $\vee$ over $\wedge$ and the Rule of Conjunctive Simplification to arrive at either of the two clauses $p \vee q$, $p \vee r$. Finally, the premise $p \rightarrow q$ becomes the clause $\neg p \vee q$.

Establish the validity of the following arguments, using resolution (along with the rules of inference and the laws of logic).

(i)
$$\begin{array}{l} p \vee (q \wedge r) \\ \underline{p \rightarrow s} \\ \therefore r \vee s \end{array}$$

(ii)
$$\begin{array}{l} p \\ \underline{p \leftrightarrow q} \\ \therefore q \end{array}$$

(iii)
$$\begin{array}{l} p \vee q \\ p \rightarrow r \\ \underline{r \rightarrow s} \\ \therefore q \vee s \end{array}$$

(iv)
$$\begin{array}{l} \neg p \vee q \vee r \\ \neg q \\ \underline{\neg r} \\ \therefore \neg p \end{array}$$

(v)
$$\begin{array}{l} \neg p \vee s \\ \neg t \vee (s \wedge r) \\ \neg q \vee r \\ \underline{p \vee q \vee t} \\ \therefore r \vee s \end{array}$$

c) Write the following argument in symbolic form, then use resolution (along with the rules of inference and the laws of logic) to establish its validity.

Jonathan does not have his driver's license or his new car is out of gas. Jonathan has his driver's license or he does not like to drive his new car. Jonathan's new car is not out of gas or he does not like to drive his new car. Therefore, Jonathan does not like to drive his new car.

4 The Use of Quantifiers

In Section 1, we mentioned how sentences that involve a variable, such as x, need not be statements. For example, the sentence "The number $x + 2$ is an even integer" is not necessarily true or false unless we know what value is substituted for x. If we restrict our choices to integers, then when x is replaced by -5, -1, or 3, for instance, the resulting statement is false. In fact, it is false whenever x is replaced by an odd integer. When an even integer is substituted for x, however, the resulting statement is true.

We refer to the sentence "The number $x + 2$ is an even integer" as an *open statement*, which we formally define as follows.

Definition 5 A declarative sentence is an *open statement* if

1) it contains one or more variables, and

2) it is not a statement, but

3) it becomes a statement when the variables in it are replaced by certain allowable choices.

When we examine the sentence "The number $x + 2$ is an even integer" in light of this definition, we find it is an open statement that contains the single variable x. With regard to the third element of the definition, in our earlier discussion we restricted the "certain allowable choices" to integers. These allowable choices constitute what is called the *universe* or *universe of discourse* for the open statement. The universe comprises the choices we wish to consider or allow for the variable(s) in the open statement.

In dealing with open statements, we use the following notation:

The open statement "The number $x + 2$ is an even integer" is denoted by $p(x)$ [or $q(x)$, $r(x)$, etc.]. Then $\neg p(x)$ may be read "The number $x + 2$ is *not* an even integer."

We shall use $q(x, y)$ to represent an open statement that contains two variables. For example, consider

$q(x, y)$: The numbers $y + 2$, $x - y$, and $x + 2y$ are even integers.

In the case of $q(x, y)$, there is more than one occurrence of each of the variables x, y. It is understood that when we replace one of the x's by a choice from our universe, we replace the other x by the same choice. Likewise, when a substitution (from the universe) is made for one occurrence of y, that same substitution is made for all other occurrences of the variable y.

With $p(x)$ and $q(x, y)$ as above, and the universe still stipulating the integers as our only allowable choices, we get the following results when we make some replacements for the variables x, y.

$p(5)$: The number $7 (= 5 + 2)$ is an even integer. (FALSE)

$\neg p(7)$: The number 9 is not an even integer. (TRUE)

$q(4, 2)$: The numbers 4, 2, and 8 are even integers. (TRUE)

We also note, for example, that $q(5, 2)$ and $q(4, 7)$ are both false statements, whereas $\neg q(5, 2)$ and $\neg q(4, 7)$ are true.

Consequently, we see that for both $p(x)$ and $q(x, y)$, as already given, some substitutions result in true statements and others in false statements. Therefore we can make the following true statements.

1) For some x, $p(x)$.

2) For some x, y, $q(x, y)$.

Note that in this situation, the statements "For some x, $\neg p(x)$" and "For some x, y, $\neg q(x, y)$" are also true. [Since the statements "For some x, $p(x)$" and "For some x, $\neg p(x)$" are both true, we realize that the second statement is *not* the negation of the first—even though the open statement $\neg p(x)$ is the negation of the open statement $p(x)$. And a similar result is true for the statements involving $q(x, y)$ and $\neg q(x, y)$.]

The phrases "For some x" and "For some x, y" are said to *quantify* the open statements $p(x)$ and $q(x, y)$, respectively. Many postulates, definitions, and theorems in mathematics involve statements that are quantified open statements. These result from the two types of *quantifiers*, which are called the *existential* and the *universal quantifiers*.

Statement (1) uses the *existential quantifier* "For some x," which can also be expressed as "For at least one x" or "There exists an x such that." This quantifier is written in symbolic form as $\exists x$. Hence the statement "For some x, $p(x)$" becomes $\exists x\ p(x)$, in symbolic form.

Statement (2) becomes $\exists x\ \exists y\ q(x, y)$ in symbolic form. The notation $\exists x, y$ can be used to abbreviate $\exists x\ \exists y\ q(x, y)$ to $\exists x, y\ q(x, y)$.

The *universal quantifier* is denoted by $\forall x$ and is read "For all x," "For any x," "For each x," or "For every x." "For all x, y," "For any x, y," "For every x, y," or "For all x and y" is denoted by $\forall x\ \ \forall y$, which can be abbreviated to $\forall x, y$.

Taking $p(x)$ as defined earlier and using the universal quantifier, we can change the open statement $p(x)$ into the (quantified) statement $\forall x\ p(x)$, a false statement.

If we consider the open statement $r(x)$: "$2x$ is an even integer" with the same universe (of all integers), then the (quantified) statement $\forall x\ r(x)$ is a true statement. When we say that $\forall x\ r(x)$ is true, we mean that no matter which integer (from our universe) is substituted for x in $r(x)$, the resulting statement is true. Also note that the statement $\exists x\ r(x)$ is a true statement, whereas $\forall x\ \neg r(x)$ and $\exists x\ \neg r(x)$ are both false.

The variable x in each of open statements $p(x)$ and $r(x)$ is called a *free variable* (of the open statement). As x varies over the universe for an open statement, the truth value of the statement (that results upon the replacement of each occurrence of x) may vary. For instance, in the case of $p(x)$, we found $p(5)$ to be false — while $p(6)$ turns out to be a true statement. The open statement $r(x)$, however, becomes a true statement for every replacement (for x) taken from the universe of all integers. In contrast to the open statement $p(x)$ the statement $\exists x\ p(x)$ has a fixed truth value — namely, true. And in the symbolic representation $\exists x\ p(x)$ the variable x is said to be a *bound* variable — it is bound by the existential quantifier $\exists$. This is also the case for the statements $\forall x\ r(x)$ and $\forall x\ \neg r(x)$, where in each case the variable x is bound by the universal quantifier $\forall$.

For the open statement $q(x, y)$ we have two free variables, each of which is bound by the quantifier $\exists$ in either of the statements $\exists x\ \exists y\ q(x, y)$ or $\exists x, y\ q(x, y)$.

The following example shows how these new ideas about quantifiers can be used in conjunction with the logical connectives.

EXAMPLE 36

Here the universe comprises all real numbers. The open statements $p(x)$, $q(x)$, $r(x)$, and $s(x)$ are given by

$$\begin{array}{llll} p(x): & x \geq 0 & r(x): & x^2 - 3x - 4 = 0 \\ q(x): & x^2 \geq 0 & s(x): & x^2 - 3 > 0. \end{array}$$

Then the following statements are true.

1) $$\exists x\ [p(x) \wedge r(x)]$$

This follows because the real number 4, for example, is a member of the universe and is such that both of the statements $p(4)$ and $r(4)$ are true.

2) $$\forall x\ [p(x) \rightarrow q(x)]$$

If we replace x in $p(x)$ by a negative real number a, then $p(a)$ is false, but $p(a) \rightarrow q(a)$ is true regardless of the truth value of $q(a)$. Replacing x in $p(x)$ by a nonnegative real number b, we find that $p(b)$ and $q(b)$ are both true, as is $p(b) \rightarrow q(b)$. Consequently, $p(x) \rightarrow q(x)$ is true for all replacements x taken from the universe of all real numbers, and the (quantified) statement $\forall x\ [p(x) \rightarrow q(x)]$ is true.

This statement may be translated into any of the following:

a) For every real number x, if $x \geq 0$, then $x^2 \geq 0$.

b) Every nonnegative real number has a nonnegative square.

c) The square of any nonnegative real number is a nonnegative real number.

d) All nonnegative real numbers have nonnegative squares.

Also, the statement $\exists x\ [p(x) \rightarrow q(x)]$ is true.

The next statements we examine are false.

1′) $$\forall\, x\ [q(x) \rightarrow s(x)]$$

We want to show that the statement is false, so we need exhibit only one *counterexample* — that is, *one value of* x for which $q(x) \rightarrow s(x)$ is false — rather than prove something for all x as we did for statement (2). Replacing x by 1, we find that $q(1)$ is true and $s(1)$ is false. Therefore $q(1) \rightarrow s(1)$ is false, and consequently the (quantified) statement $\forall\, x\ [q(x) \rightarrow s(x)]$ is false. [Note that $x = 1$ does not produce the only counterexample: Every real number a between $-\sqrt{3}$ and $\sqrt{3}$ will make $q(a)$ true and $s(a)$ false.]

2′) $$\forall\, x\ [r(x) \vee s(x)]$$

Here there are many values for x, such as 1, $\frac{1}{2}$, $-\frac{3}{2}$, and 0, that produce counterexamples. Upon changing quantifiers, however, we find that the statement $\exists x\ [r(x) \vee s(x)]$ is true.

3′) $$\forall\, x\ [r(x) \rightarrow p(x)]$$

The real number -1 is a solution of the equation $x^2 - 3x - 4 = 0$, so $r(-1)$ is true while $p(-1)$ is false. Therefore the choice of -1 provides the unique counterexample we need to show that this (quantified) statement is false.

Statement (3′) may be translated into either of the following:

a) For every real number x, if $x^2 - 3x - 4 = 0$, then $x \geq 0$.

b) For every real number x, if x is a solution of the equation $x^2 - 3x - 4 = 0$, then $x \geq 0$.

Now we make the following observations. Let $p(x)$ denote any open statement (in the variable x) with a prescribed *nonempty* universe (that is, the universe contains at least one member). Then if $\forall\, x\ p(x)$ is true, so is $\exists x\ p(x)$, or

$$\forall\, x\ p(x) \Rightarrow \exists x\ p(x).$$

When we write $\forall\, x\ p(x) \Rightarrow \exists x\ p(x)$ we are saying that the implication $\forall\, x\ p(x) \rightarrow \exists x\ p(x)$ is a logical implication — that is, $\exists x\ p(x)$ is true whenever $\forall\, x\ p(x)$ is true. Also, we realize that the hypothesis of this implication is the quantified *statement* $\forall\, x\ p(x)$, and the conclusion is $\exists x\ p(x)$, another quantified *statement*. On the other hand, it does not follow that if $\exists x\ p(x)$ is true, then $\forall\, x\ p(x)$ must be true. Hence $\exists x\ p(x)$ does not logically imply $\forall\, x\ p(x)$, in general.

Our next example brings out the fact that the quantification of an open statement may not be as explicit as we might prefer.

EXAMPLE 37

a) Let us consider the universe of all real numbers and examine the sentences:

1) If a number is rational, then it is a real number.

2) If x is rational, then x is real.