PEARSON NEW INTERNATIONAL EDITION

Precalculus

Robert F. Blitzer

Fifth Edition

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FUNDAMENTAL CONCEPTS OF ALGEBRA

What can algebra possibly have to tell me about

- the skyrocketing cost of a college education?
- my workouts?
- the effects of alcohol?
- the meaning of the national debt that exceeds $15 trillion?
- time dilation on a futuristic high-speed journey to a nearby star?
- ethnic diversity in the United States?
- the widening imbalance between numbers of women and men on college campuses?

This chapter reviews fundamental concepts of algebra that are prerequisites for the study of precalculus. Throughout the chapter, you will see how the special language of algebra describes your world.

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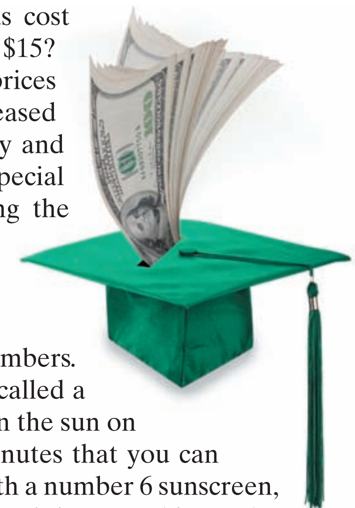
SECTION 1 Algebraic Expressions, Mathematical Models, and Real Numbers

Objectives

1. Evaluate algebraic expressions.
2. Use mathematical models.
3. Find the intersection of two sets.
4. Find the union of two sets.
5. Recognize subsets of the real numbers.
6. Use inequality symbols.
7. Evaluate absolute value.
8. Use absolute value to express distance.
9. Identify properties of the real numbers.
10. Simplify algebraic expressions.



How would your lifestyle change if a gallon of gas cost \$9.15? Or if the price of a staple such as milk was \$15? That's how much those products would cost if their prices had increased at the same rate college tuition has increased since 1980. (*Source:* Center for College Affordability and Productivity) In this section, you will learn how the special language of algebra describes your world, including the skyrocketing cost of a college education.

Algebraic Expressions

Algebra uses letters, such as x and y, to represent numbers. If a letter is used to represent various numbers, it is called a **variable**. For example, imagine that you are basking in the sun on the beach. We can let x represent the number of minutes that you can stay in the sun without burning with no sunscreen. With a number 6 sunscreen, exposure time without burning is six times as long, or 6 times x. This can be written $6 \cdot x$, but it is usually expressed as $6x$. Placing a number and a letter next to one another indicates multiplication.

Notice that $6x$ combines the number 6 and the variable x using the operation of multiplication. A combination of variables and numbers using the operations of addition, subtraction, multiplication, or division, as well as powers or roots, is called an **algebraic expression**. Here are some examples of algebraic expressions:

$$x + 6, \quad x - 6, \quad 6x, \quad \frac{x}{6}, \quad 3x + 5, \quad x^2 - 3, \quad \sqrt{x} + 7.$$

Many algebraic expressions involve *exponents*. For example, the algebraic expression

$$4x^2 + 341x + 3194$$

approximates the average cost of tuition and fees at public U.S. colleges for the school year ending x years after 2000. The expression x^2 means $x \cdot x$ and is read "x to the second power" or "x squared." The exponent, 2, indicates that the base, x, appears as a factor two times.

Exponential Notation

If n is a counting number (1, 2, 3, and so on),

Exponent or Power

$$b^n = \underbrace{b \cdot b \cdot b \cdot \cdots \cdot b}_{\text{b appears as a factor n times}}.$$

Base

b^n is read "the nth power of b" or "b to the nth power." Thus, the nth power of b is defined as the product of n factors of b. The expression b^n is called an **exponential expression**. Furthermore, $b^1 = b$.

For example,

$$8^2 = 8 \cdot 8 = 64, \quad 5^3 = 5 \cdot 5 \cdot 5 = 125, \quad \text{and} \quad 2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16.$$

1 Evaluate algebraic expressions.

Evaluating Algebraic Expressions

Evaluating an algebraic expression means to find the value of the expression for a given value of the variable.

Many algebraic expressions involve more than one operation. Evaluating an algebraic expression without a calculator involves carefully applying the following order of operations agreement:

The Order of Operations Agreement

1. Perform operations within the innermost parentheses and work outward. If the algebraic expression involves a fraction, treat the numerator and the denominator as if they were each enclosed in parentheses.
2. Evaluate all exponential expressions.
3. Perform multiplications and divisions **as they occur**, working **from left to right**.
4. Perform additions and subtractions **as they occur**, working **from left to right**.

EXAMPLE 1 Evaluating an Algebraic Expression

Evaluate $7 + 5(x - 4)^3$ for $x = 6$.

SOLUTION

$$
\begin{aligned}
7 + 5(x - 4)^3 &= 7 + 5(6 - 4)^3 && \text{Replace } x \text{ with } 6.\\
&= 7 + 5(2)^3 && \text{First work inside parentheses: } 6 - 4 = 2.\\
&= 7 + 5(8) && \text{Evaluate the exponential expression: } 2^3 = 2\cdot 2\cdot 2 = 8.\\
&= 7 + 40 && \text{Multiply: } 5(8) = 40.\\
&= 47 && \text{Add.}
\end{aligned}
$$

Check Point 1 Evaluate $8 + 6(x - 3)^2$ for $x = 13$.

2 Use mathematical models.

Formulas and Mathematical Models

An **equation** is formed when an equal sign is placed between two algebraic expressions. One aim of algebra is to provide a compact, symbolic description of the world. These descriptions involve the use of *formulas*. A **formula** is an equation that uses variables to express a relationship between two or more quantities.

Here are two examples of formulas related to heart rate and exercise.

Couch-Potato Exercise

$$H = \frac{1}{5}(220 - a)$$

Heart rate, in beats per minute, | is | $\frac{1}{5}$ of | the difference between 220 and your age.

Working It

$$H = \frac{9}{10}(220 - a)$$

Heart rate, in beats per minute, | is | $\frac{9}{10}$ of | the difference between 220 and your age.

The process of finding formulas to describe real-world phenomena is called **mathematical modeling**. Such formulas, together with the meaning assigned to the variables, are called **mathematical models**. We often say that these formulas model, or describe, the relationships among the variables.

EXAMPLE 2 Modeling the Cost of Attending a Public College

The bar graph in **Figure 1** shows the average cost of tuition and fees for public four-year colleges, adjusted for inflation. The formula

$$T = 4x^2 + 341x + 3194$$

models the average cost of tuition and fees, T, for public U.S. colleges for the school year ending x years after 2000.

a. Use the formula to find the average cost of tuition and fees at public U.S. colleges for the school year ending in 2010.

b. By how much does the formula underestimate or overestimate the actual cost shown in **Figure 1**?

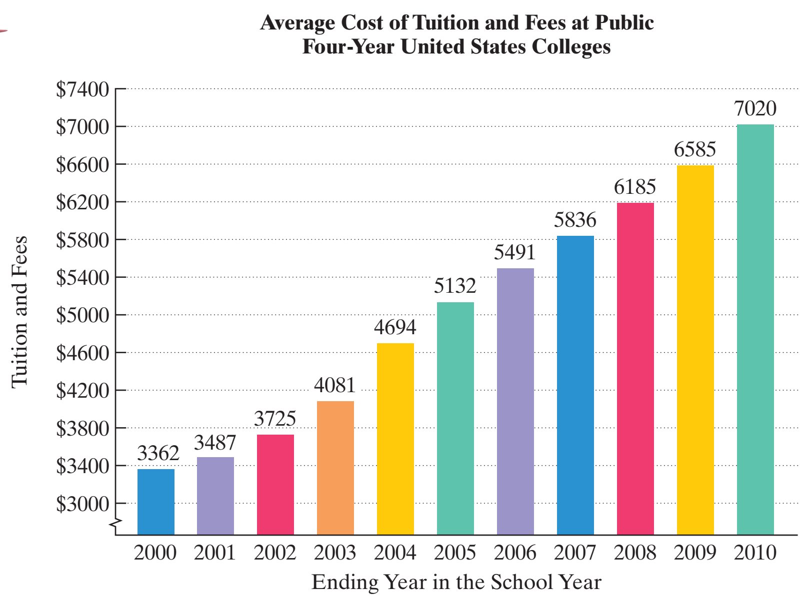


FIGURE 1
Source: The College Board

SOLUTION

a. Because 2010 is 10 years after 2000, we substitute 10 for x in the given formula. Then we use the order of operations to find T, the average cost of tuition and fees for the school year ending in 2010.

$T = 4x^2 + 341x + 3194$	This is the given mathematical model.
$T = 4(10)^2 + 341(10) + 3194$	Replace each occurrence of x with 10.
$T = 4(100) + 341(10) + 3194$	Evaluate the exponential expression: $10^2 = 10 \cdot 10 = 100$.
$T = 400 + 3410 + 3194$	Multiply from left to right: $4(100) = 400$ and $341(10) = 3410$.
$T = 7004$	Add.

The formula indicates that for the school year ending in 2010, the average cost of tuition and fees at public U.S. colleges was $7004.

b. **Figure 1** shows that the average cost of tuition and fees for the school year ending in 2010 was $7020.

The cost obtained from the formula, $7004, underestimates the actual data value by $7020 − $7004, or by $16.

Check Point 2 Assuming trends indicated by the data in **Figure 1** continue, use the formula $T = 4x^2 + 341x + 3194$, described in Example 2, to project the average cost of tuition and fees at public U.S. colleges for the school year ending in 2015.

Sometimes a mathematical model gives an estimate that is not a good approximation or is extended to include values of the variable that do not make sense. In these cases, we say that **model breakdown** has occurred. For example, it is not likely that the formula in Example 2 would give a good estimate of tuition and fees in 2050 because it is too far in the future. Thus, model breakdown would occur.

Sets

Before we describe the set of real numbers, let's be sure you are familiar with some basic ideas about sets. A **set** is a collection of objects whose contents can be clearly determined. The objects in a set are called the **elements** of the set. For example, the set of numbers used for counting can be represented by

$$\{1, 2, 3, 4, 5, \ldots\}.$$

The braces, { }, indicate that we are representing a set. This form of representation, called the **roster method**, uses commas to separate the elements of the set. The symbol consisting of three dots after the 5, called an *ellipsis*, indicates that there is no final element and that the listing goes on forever.

A set can also be written in **set-builder notation**. In this notation, the elements of the set are described but not listed. Here is an example:

$$\{x \mid x \text{ is a counting number less than } 6\}.$$

The set of all x | such that | x is a counting number less than 6.

The same set written using the roster method is

$$\{1, 2, 3, 4, 5\}.$$

GREAT QUESTION!

Can I use symbols other than braces when writing sets using the roster method?

No. Grouping symbols such as parentheses, ( ), and square brackets, [ ], are not used to represent sets in the roster method. Furthermore, only commas are used to separate the elements of a set. Separators such as colons or semicolons are not used.

3 Find the intersection of two sets.

If A and B are sets, we can form a new set consisting of all elements that are in both A and B. This set is called the *intersection* of the two sets.

Definition of the Intersection of Sets

The **intersection** of sets A and B, written $A \cap B$, is the set of elements common to both set A **and** set B. This definition can be expressed in set-builder notation as follows:

$$A \cap B = \{x \mid x \text{ is an element of } A \text{ AND } x \text{ is an element of } B\}.$$

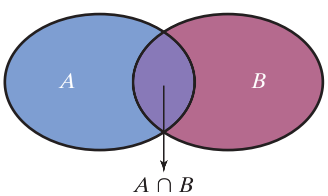


FIGURE 2 Picturing the intersection of two sets

Figure 2 shows a useful way of picturing the intersection of sets A and B. The figure indicates that $A \cap B$ contains those elements that belong to both A and B at the same time.

EXAMPLE 3 Finding the Intersection of Two Sets

Find the intersection: $\{7, 8, 9, 10, 11\} \cap \{6, 8, 10, 12\}$.

SOLUTION

The elements common to $\{7, 8, 9, 10, 11\}$ and $\{6, 8, 10, 12\}$ are 8 and 10. Thus,

$$\{7, 8, 9, 10, 11\} \cap \{6, 8, 10, 12\} = \{8, 10\}.$$

Check Point 3 Find the intersection: $\{3, 4, 5, 6, 7\} \cap \{3, 7, 8, 9\}$.

If a set has no elements, it is called the **empty set**, or the **null set**, and is represented by the symbol $\varnothing$ (the Greek letter phi). Here is an example that shows how the empty set can result when finding the intersection of two sets:

$$\{2, 4, 6\} \cap \{3, 5, 7\} = \varnothing.$$

These sets have no common elements.

Their intersection has no elements and is the empty set.

4 Find the union of two sets.

Another set that we can form from sets A and B consists of elements that are in A or B or in both sets. This set is called the *union* of the two sets.

Definition of the Union of Sets

The **union** of sets A and B, written $A \cup B$, is the set of elements that are members of set A **or** of set B or of both sets. This definition can be expressed in set-builder notation as follows:

$$A \cup B = \{x \mid x \text{ is an element of } A \text{ OR } x \text{ is an element of } B\}.$$

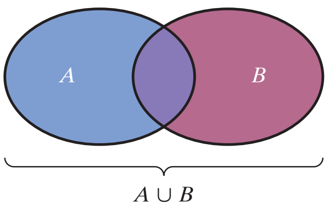


FIGURE 3 Picturing the union of two sets

Figure 3 shows a useful way of picturing the union of sets A and B. The figure indicates that $A \cup B$ is formed by joining the sets together.

We can find the union of set A and set B by listing the elements of set A. Then we include any elements of set B that have not already been listed. Enclose all elements that are listed with braces. This shows that the union of two sets is also a set.

GREAT QUESTION!

How can I use the words *union* and *intersection* to help me distinguish between these two operations?

Union, as in a marriage union, suggests joining things, or uniting them. Intersection, as in the intersection of two crossing streets, brings to mind the area common to both, suggesting things that overlap.

EXAMPLE 4 Finding the Union of Two Sets

Find the union: $\{7, 8, 9, 10, 11\} \cup \{6, 8, 10, 12\}$.

SOLUTION

To find $\{7, 8, 9, 10, 11\} \cup \{6, 8, 10, 12\}$, start by listing all the elements from the first set, namely, 7, 8, 9, 10, and 11. Now list all the elements from the second set that are not in the first set, namely, 6 and 12. The union is the set consisting of all these elements. Thus,

$$\{7, 8, 9, 10, 11\} \cup \{6, 8, 10, 12\} = \{6, 7, 8, 9, 10, 11, 12\}.$$

Although 8 and 10 appear in both sets, do not list 8 and 10 twice.

Check Point 4 Find the union: $\{3, 4, 5, 6, 7\} \cup \{3, 7, 8, 9\}$.

5 Recognize subsets of the real numbers.

The Set of Real Numbers

The sets that make up the real numbers are summarized in **Table 1** at the top of the next page. We refer to these sets as **subsets** of the real numbers, meaning that all elements in each subset are also elements in the set of real numbers.

Notice the use of the symbol $\approx$ in the examples of irrational numbers. The symbol means "is approximately equal to." Thus,

$$\sqrt{2} \approx 1.414214.$$

TECHNOLOGY

A calculator with a square root key gives a decimal approximation for $\sqrt{2}$, not the exact value.

We can verify that this is only an approximation by multiplying 1.414214 by itself. The product is very close to, but not exactly, 2:

$$1.414214 \times 1.414214 = 2.000001237796.$$

Table 1 Important Subsets of the Real Numbers

Name/Symbol	Description	Examples
Natural numbers $\mathbb{N}$	$\{1, 2, 3, 4, 5, \ldots\}$ These are the numbers that we use for counting.	2, 3, 5, 17
Whole numbers $\mathbb{W}$	$\{0, 1, 2, 3, 4, 5, \ldots\}$ The set of whole numbers includes 0 and the natural numbers.	0, 2, 3, 5, 17
Integers $\mathbb{Z}$	$\{\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$ The set of integers includes the negatives of the natural numbers and the whole numbers.	$-17, -5, -3, -2, 0, 2, 3, 5, 17$
Rational numbers $\mathbb{Q}$	$\left\{\frac{a}{b} \,\middle\vert\, a \text{ and } b \text{ are integers and } b \neq 0\right\}$ This means that b is not equal to zero. The set of rational numbers is the set of all numbers that can be expressed as a quotient of two integers, with the denominator not 0. Rational numbers can be expressed as terminating or repeating decimals.	$-17 = \frac{-17}{1}, -5 = \frac{-5}{1}, -3, -2,$ $0, 2, 3, 5, 17,$ $\frac{2}{5} = 0.4,$ $\frac{-2}{3} = -0.6666\ldots = -0.\overline{6}$
Irrational numbers $\mathbb{I}$	The set of irrational numbers is the set of all numbers whose decimal representations are neither terminating nor repeating. Irrational numbers cannot be expressed as a quotient of integers.	$\sqrt{2} \approx 1.414214$ $-\sqrt{3} \approx -1.73205$ $\pi \approx 3.142$ $-\frac{\pi}{2} \approx -1.571$

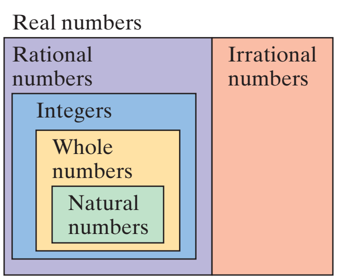


FIGURE 4 Every real number is either rational or irrational.

Not all square roots are irrational. For example, $\sqrt{25} = 5$ because $5^2 = 5 \cdot 5 = 25$. Thus, $\sqrt{25}$ is a natural number, a whole number, an integer, and a rational number $\left(\sqrt{25} = \frac{5}{1}\right)$.

The set of *real numbers* is formed by taking the union of the sets of rational numbers and irrational numbers. Thus, every real number is either rational or irrational, as shown in **Figure 4**.

Real Numbers

The set of **real numbers** is the set of numbers that are either rational or irrational:

$$\{x \mid x \text{ is rational or } x \text{ is irrational}\}.$$

The symbol $\mathbb{R}$ is used to represent the set of real numbers. Thus,

$$\mathbb{R} = \{x \mid x \text{ is rational}\} \cup \{x \mid x \text{ is irrational}\}.$$

EXAMPLE 5 Recognizing Subsets of the Real Numbers

Consider the following set of numbers:

$$\left\{-7, -\frac{3}{4}, 0, 0.\overline{6}, \sqrt{5}, \pi, 7.3, \sqrt{81}\right\}.$$

List the numbers in the set that are

a. natural numbers. **b.** whole numbers. **c.** integers.
d. rational numbers. **e.** irrational numbers. **f.** real numbers.

SOLUTION

a. Natural numbers: The natural numbers are the numbers used for counting. The only natural number in the set $\left\{-7, -\frac{3}{4}, 0, 0.\overline{6}, \sqrt{5}, \pi, 7.3, \sqrt{81}\right\}$ is $\sqrt{81}$ because $\sqrt{81} = 9$. (9 multiplied by itself, or 9^2, is 81.)

b. Whole numbers: The whole numbers consist of the natural numbers and 0. The elements of the set $\left\{-7, -\frac{3}{4}, 0, 0.\overline{6}, \sqrt{5}, \pi, 7.3, \sqrt{81}\right\}$ that are whole numbers are 0 and $\sqrt{81}$.

c. Integers: The integers consist of the natural numbers, 0, and the negatives of the natural numbers. The elements of the set $\left\{-7, -\frac{3}{4}, 0, 0.\overline{6}, \sqrt{5}, \pi, 7.3, \sqrt{81}\right\}$ that are integers are $\sqrt{81}$, 0, and -7.

d. Rational numbers: All numbers in the set $\left\{-7, -\frac{3}{4}, 0, 0.\overline{6}, \sqrt{5}, \pi, 7.3, \sqrt{81}\right\}$ that can be expressed as the quotient of integers are rational numbers. These include $-7\left(-7 = \frac{-7}{1}\right), -\frac{3}{4}, 0\left(0 = \frac{0}{1}\right)$, and $\sqrt{81}\left(\sqrt{81} = \frac{9}{1}\right)$. Furthermore, all numbers in the set that are terminating or repeating decimals are also rational numbers. These include $0.\overline{6}$ and 7.3.

e. Irrational numbers: The irrational numbers in the set $\left\{-7, -\frac{3}{4}, 0, 0.\overline{6}, \sqrt{5}, \pi, 7.3, \sqrt{81}\right\}$ are $\sqrt{5}\left(\sqrt{5} \approx 2.236\right)$ and $\pi (\pi \approx 3.14)$. Both $\sqrt{5}$ and π are only approximately equal to 2.236 and 3.14, respectively. In decimal form, $\sqrt{5}$ and π neither terminate nor have blocks of repeating digits.

f. Real numbers: All the numbers in the given set $\left\{-7, -\frac{3}{4}, 0, 0.\overline{6}, \sqrt{5}, \pi, 7.3, \sqrt{81}\right\}$ are real numbers. ● ● ●

Check Point 5 Consider the following set of numbers:

$$\left\{-9, -1.3, 0, 0.\overline{3}, \frac{\pi}{2}, \sqrt{9}, \sqrt{10}\right\}.$$

List the numbers in the set that are

a. natural numbers. **b.** whole numbers. **c.** integers.
d. rational numbers. **e.** irrational numbers. **f.** real numbers.

The Real Number Line

The **real number line** is a graph used to represent the set of real numbers. An arbitrary point, called the **origin**, is labeled 0. Select a point to the right of 0 and label it 1. The distance from 0 to 1 is called the **unit distance**. Numbers to the right of the origin are **positive** and numbers to the left of the origin are **negative**. The real number line is shown in **Figure 5**.

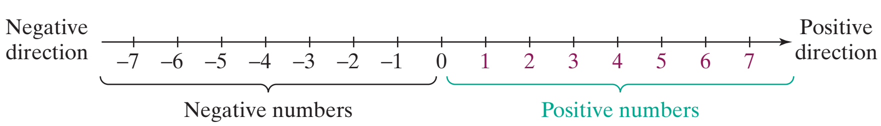


FIGURE 5 The real number line

Real numbers are **graphed** on a number line by placing a dot at the correct location for each number. The integers are easiest to locate. In **Figure 6**, we've graphed six rational numbers and three irrational numbers on a real number line.

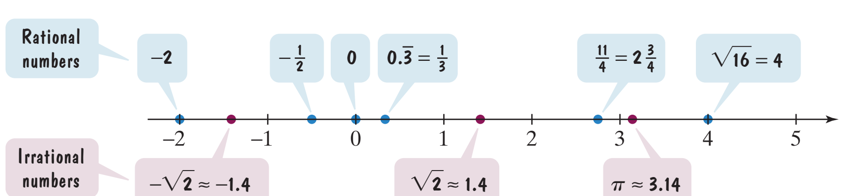


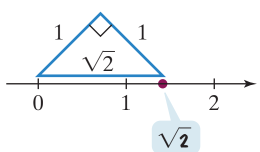
FIGURE 6 Graphing numbers on a real number line

Every real number corresponds to a point on the number line and every point on the number line corresponds to a real number. We say that there is a **one-to-one correspondence** between all the real numbers and all points on a real number line.

GREAT QUESTION!

How did you locate $\sqrt{2}$ as a precise point on the number line in Figure 6?

We used a right triangle with two legs of length 1. The remaining side has a length measuring $\sqrt{2}$.



We'll have lots more to say about right triangles later in the text.

6 Use inequality symbols.

Ordering the Real Numbers

On the real number line, the real numbers increase from left to right. The lesser of two real numbers is the one farther to the left on a number line. The greater of two real numbers is the one farther to the right on a number line.

Look at the number line in **Figure 7**. The integers -4 and -1 are graphed.

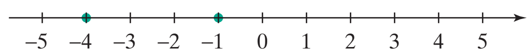


FIGURE 7

Observe that -4 is to the left of -1 on the number line. This means that -4 is less than -1.

$$-4 < -1$$

-4 is less than -1 because -4 is to the **left** of -1 on the number line.

In **Figure 7**, we can also observe that -1 is to the right of -4 on the number line. This means that -1 is greater than -4.

$$-1 > -4$$

-1 is greater than -4 because -1 is to the **right** of -4 on the number line.

The symbols $<$ and $>$ are called **inequality symbols**. These symbols always point to the lesser of the two real numbers when the inequality statement is true.

-4 is less than -1. $\quad -4 < -1 \quad$ The symbol points to -4, the lesser number.

-1 is greater than -4. $\quad -1 > -4 \quad$ The symbol still points to -4, the lesser number.

The symbols $<$ and $>$ may be combined with an equal sign, as shown in the following table:

This inequality is true if either the $<$ part or the $=$ part is true.

This inequality is true if either the $>$ part or the $=$ part is true.

Symbols	Meaning	Examples	Explanation
$a \le b$	a is less than or equal to b.	$2 \le 9$ $9 \le 9$	Because $2 < 9$ Because $9 = 9$
$b \ge a$	b is greater than or equal to a.	$9 \ge 2$ $2 \ge 2$	Because $9 > 2$ Because $2 = 2$

7 Evaluate absolute value.

Absolute Value

The **absolute value** of a real number a, denoted by $|a|$, is the distance from 0 to a on the number line. This distance is always taken to be nonnegative. For example, the real number line in **Figure 8** shows that

$$|-3| = 3 \quad \text{and} \quad |5| = 5.$$

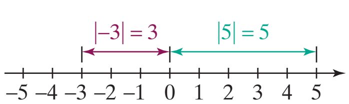


FIGURE 8 Absolute value as the distance from 0

The absolute value of -3 is 3 because -3 is 3 units from 0 on the number line. The absolute value of 5 is 5 because 5 is 5 units from 0 on the number line. The absolute value of a positive real number or 0 is the number itself. The absolute value of a negative real number, such as -3, is the number without the negative sign.

We can define the absolute value of the real number x without referring to a number line. The algebraic definition of the absolute value of x is given as follows:

Definition of Absolute Value

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

If x is nonnegative (that is, $x \geq 0$), the absolute value of x is the number itself. For example,

$$|5| = 5 \qquad |\pi| = \pi \qquad \left|\frac{1}{3}\right| = \frac{1}{3} \qquad |0| = 0.$$

Zero is the only number whose absolute value is 0.

If x is a negative number (that is, $x < 0$), the absolute value of x is the opposite of x. This makes the absolute value positive. For example,

$$|-3| = -(-3) = 3 \qquad |-\pi| = -(-\pi) = \pi \qquad \left|-\frac{1}{3}\right| = -\left(-\frac{1}{3}\right) = \frac{1}{3}.$$

This middle step is usually omitted.

EXAMPLE 6 Evaluating Absolute Value

Rewrite each expression without absolute value bars:

a. $|\sqrt{3} - 1|$ **b.** $|2 - \pi|$ **c.** $\dfrac{|x|}{x}$ if $x < 0$.

SOLUTION

a. Because $\sqrt{3} \approx 1.7$, the number inside the absolute value bars, $\sqrt{3} - 1$, is positive. The absolute value of a positive number is the number itself. Thus,

$$|\sqrt{3} - 1| = \sqrt{3} - 1.$$

b. Because $\pi \approx 3.14$, the number inside the absolute value bars, $2 - \pi$, is negative. The absolute value of x when $x < 0$ is $-x$. Thus,

$$|2 - \pi| = -(2 - \pi) = \pi - 2.$$

c. If $x < 0$, then $|x| = -x$. Thus,

$$\frac{|x|}{x} = \frac{-x}{x} = -1.$$

Check Point 6 Rewrite each expression without absolute value bars:

a. $|1 - \sqrt{2}|$ **b.** $|\pi - 3|$ **c.** $\dfrac{|x|}{x}$ if $x > 0$.

DISCOVERY

Verify the triangle inequality if $a = 4$ and $b = 5$. Verify the triangle inequality if $a = 4$ and $b = -5$.

When does equality occur in the triangle inequality and when does inequality occur? Verify your observation with additional number pairs.

Listed below are several basic properties of absolute value. Each of these properties can be derived from the definition of absolute value.

Properties of Absolute Value

For all real numbers a and b,

1. $|a| \geq 0$ **2.** $|-a| = |a|$ **3.** $a \leq |a|$

4. $|ab| = |a||b|$ **5.** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}, \quad b \neq 0$

6. $|a + b| \leq |a| + |b|$ (called the triangle inequality)

8 Use absolute value to express distance.

Distance between Points on a Real Number Line

Absolute value is used to find the distance between two points on a real number line. If a and b are any real numbers, the **distance between *a* and *b*** is the absolute value of their difference. For example, the distance between 4 and 10 is 6. Using absolute value, we find this distance in one of two ways:

$$|10 - 4| = |6| = 6 \qquad \text{or} \qquad |4 - 10| = |-6| = 6.$$

The distance between 4 and 10 on the real number line is 6.

Notice that we obtain the same distance regardless of the order in which we subtract.

Distance between Two Points on the Real Number Line

If a and b are any two points on a real number line, then the distance between a and b is given by

$$|a - b| \quad \text{or} \quad |b - a|.$$

EXAMPLE 7 Distance between Two Points on a Number Line

Find the distance between -5 and 3 on the real number line.

SOLUTION

Because the distance between a and b is given by $|a - b|$, the distance between -5 and 3 is

$$|-5 - 3| = |-8| = 8.$$

$a = -5$ $b = 3$

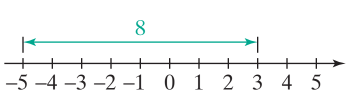


FIGURE 9 The distance between -5 and 3 is 8.

Figure 9 verifies that there are 8 units between -5 and 3 on the real number line. We obtain the same distance if we reverse the order of the subtraction:

$$|3 - (-5)| = |8| = 8.$$

Check Point 7 Find the distance between -4 and 5 on the real number line.

9 Identify properties of the real numbers.

Properties of Real Numbers and Algebraic Expressions

When you use your calculator to add two real numbers, you can enter them in any order. The fact that two real numbers can be added in any order is called the **commutative property of addition**. You probably use this property, as well as other properties of real numbers listed in **Table 2**, without giving it much thought. The properties of the real numbers are especially useful when working with algebraic expressions. For each property listed in **Table 2**, a, b, and c represent real numbers, variables, or algebraic expressions.

Blitzer Bonus

The Associative Property and the English Language

In the English language, phrases can take on different meanings depending on the way the words are associated with commas.

Here are three examples.

- Woman, without her man, is nothing.
  Woman, without her, man is nothing.
- What's the latest dope?
  What's the latest, dope?
- Population of Amsterdam broken down by age and sex
  Population of Amsterdam, broken down by age and sex

Table 2 Properties of the Real Numbers

Name	Meaning	Examples
Commutative Property of Addition	Changing order when adding does not affect the sum. $a + b = b + a$	• $13 + 7 = 7 + 13$ • $13x + 7 = 7 + 13x$
Commutative Property of Multiplication	Changing order when multiplying does not affect the product. $ab = ba$	• $\sqrt{2}\cdot\sqrt{5} = \sqrt{5}\cdot\sqrt{2}$ • $x\cdot 6 = 6x$
Associative Property of Addition	Changing grouping when adding does not affect the sum. $(a + b) + c = a + (b + c)$	• $3 + (8 + x) = (3 + 8) + x = 11 + x$
Associative Property of Multiplication	Changing grouping when multiplying does not affect the product. $(ab)c = a(bc)$	• $-2(3x) = (-2\cdot 3)x = -6x$
Distributive Property of Multiplication over Addition	Multiplication distributes over addition. $a\cdot(b + c) = a\cdot b + a\cdot c$	• $7(4 + \sqrt{3}) = 7\cdot 4 + 7\cdot\sqrt{3} = 28 + 7\sqrt{3}$ • $5(3x + 7) = 5\cdot 3x + 5\cdot 7 = 15x + 35$
Identity Property of Addition	Zero can be deleted from a sum. $a + 0 = a$ $0 + a = a$	• $\sqrt{3} + 0 = \sqrt{3}$ • $0 + 6x = 6x$
Identity Property of Multiplication	One can be deleted from a product. $a\cdot 1 = a$ $1\cdot a = a$	• $1\cdot\pi = \pi$ • $13x\cdot 1 = 13x$
Inverse Property of Addition	The sum of a real number and its additive inverse gives 0, the additive identity. $a + (-a) = 0$ $(-a) + a = 0$	• $\sqrt{5} + (-\sqrt{5}) = 0$ • $-\pi + \pi = 0$ • $6x + (-6x) = 0$ • $(-4y) + 4y = 0$
Inverse Property of Multiplication	The product of a nonzero real number and its multiplicative inverse gives 1, the multiplicative identity. $a\cdot\frac{1}{a} = 1,\quad a \neq 0$ $\frac{1}{a}\cdot a = 1,\quad a \neq 0$	• $7\cdot\frac{1}{7} = 1$ • $\left(\frac{1}{x-3}\right)(x - 3) = 1,\quad x \neq 3$

The properties of the real numbers in **Table 2** apply to the operations of addition and multiplication. Subtraction and division are defined in terms of addition and multiplication.

Definitions of Subtraction and Division

Let a and b represent real numbers.

Subtraction: $a - b = a + (-b)$

We call $-b$ the **additive inverse** or **opposite** of b.

Division: $a \div b = a \cdot \frac{1}{b}$, where $b \neq 0$

We call $\frac{1}{b}$ the **multiplicative inverse** or **reciprocal** of b. The quotient of a and b, $a \div b$, can be written in the form $\frac{a}{b}$, where a is the **numerator** and b the **denominator** of the fraction.

Because subtraction is defined in terms of adding an inverse, the distributive property can be applied to subtraction:

$$a(b - c) = ab - ac$$

$$(b - c)a = ba - ca.$$

For example,

$$4(2x - 5) = 4 \cdot 2x - 4 \cdot 5 = 8x - 20.$$

10 Simplify algebraic expressions.

Simplifying Algebraic Expressions

The **terms** of an algebraic expression are those parts that are separated by addition. For example, consider the algebraic expression

$$7x - 9y + z - 3,$$

which can be expressed as

$$7x + (-9y) + z + (-3).$$

This expression contains four terms, namely, $7x$, $-9y$, z, and -3.

The numerical part of a term is called its **coefficient**. In the term $7x$, the 7 is the coefficient. If a term containing one or more variables is written without a coefficient, the coefficient is understood to be 1. Thus, z means $1z$. If a term is a constant, its coefficient is that constant. Thus, the coefficient of the constant term -3 is -3.

$$7x + (-9y) + z + (-3)$$

Coefficient is 7.

Coefficient is −9.

Coefficient is 1; z means $1z$.

Coefficient is −3.

The parts of each term that are multiplied are called the **factors** of the term. The factors of the term $7x$ are 7 and x.

Like terms are terms that have exactly the same variable factors. For example, $3x$ and $7x$ are like terms. The distributive property in the form

$$ba + ca = (b + c)a$$

enables us to add or subtract like terms. For example,

$$3x + 7x = (3 + 7)x = 10x$$

$$7y^2 - y^2 = 7y^2 - 1y^2 = (7 - 1)y^2 = 6y^2.$$

This process is called **combining like terms**.

GREAT QUESTION!

What is the bottom line for combining like terms?

To combine like terms mentally, add or subtract the coefficients of the terms. Use this result as the coefficient of the terms' variable factor(s).

An algebraic expression is **simplified** when parentheses have been removed and like terms have been combined.

EXAMPLE 8 Simplifying an Algebraic Expression

Simplify: $6(2x^2 + 4x) + 10(4x^2 + 3x)$.

SOLUTION

$$\begin{aligned} & 6(2x^2 + 4x) + 10(4x^2 + 3x) \\ &= 6 \cdot 2x^2 + 6 \cdot 4x + 10 \cdot 4x^2 + 10 \cdot 3x && \text{Use the distributive property to remove the parentheses.} \\ &= 12x^2 + 24x + 40x^2 + 30x && \text{Multiply.} \\ &= (12x^2 + 40x^2) + (24x + 30x) && \text{Group like terms.} \\ &= 52x^2 + 54x && \text{Combine like terms.} \end{aligned}$$

$52x^2$ and $54x$ are not like terms. They contain different variable factors, x^2 and x, and cannot be combined.

Check Point 8 Simplify: $7(4x^2 + 3x) + 2(5x^2 + x)$.

Properties of Negatives

The distributive property can be extended to cover more than two terms within parentheses. For example,

This sign represents subtraction. This sign tells us that the number is negative.

$$\begin{aligned} -3(4x - 2y + 6) &= -3 \cdot 4x - (-3) \cdot 2y - 3 \cdot 6 \\ &= -12x - (-6y) - 18 \\ &= -12x + 6y - 18. \end{aligned}$$

The voice balloons illustrate that negative signs can appear side by side. They can represent the operation of subtraction or the fact that a real number is negative. Here is a list of properties of negatives and how they are applied to algebraic expressions:

Properties of Negatives

Let a and b represent real numbers, variables, or algebraic expressions.

Property	Examples
1. $(-1)a = -a$	$(-1)4xy = -4xy$
2. $-(-a) = a$	$-(-6y) = 6y$
3. $(-a)b = -ab$	$(-7)4xy = -7 \cdot 4xy = -28xy$
4. $a(-b) = -ab$	$5x(-3y) = -5x \cdot 3y = -15xy$
5. $-(a + b) = -a - b$	$-(7x + 6y) = -7x - 6y$
6. $-(a - b) = -a + b$ $= b - a$	$-(3x - 7y) = -3x + 7y$ $= 7y - 3x$

It is not uncommon to see algebraic expressions with parentheses preceded by a negative sign or subtraction. Properties 5 and 6 in the box, $-(a + b) = -a - b$ and $-(a - b) = -a + b$, are related to this situation. An expression of the form $-(a + b)$ can be simplified as follows:

$$-(a + b) = -1(a + b) = (-1)a + (-1)b = -a + (-b) = -a - b.$$

Do you see a fast way to obtain the simplified expression on the right in the preceding equation? **If a negative sign or a subtraction symbol appears outside parentheses, drop the parentheses and change the sign of every term within the parentheses.** For example,

$$-(3x^2 - 7x - 4) = -3x^2 + 7x + 4.$$

EXAMPLE 9 Simplifying an Algebraic Expression

Simplify: $8x + 2[5 - (x - 3)]$.

SOLUTION

$8x + 2[5 - (x - 3)]$	
$= 8x + 2[5 - x + 3]$	Drop parentheses and change the sign of each term in parentheses: $-(x - 3) = -x + 3$.
$= 8x + 2[8 - x]$	Simplify inside brackets: $5 + 3 = 8$.
$= 8x + 16 - 2x$	Apply the distributive property: $2[8 - x] = 2 \cdot 8 - 2x = 16 - 2x$.
$= (8x - 2x) + 16$	Group like terms.
$= (8 - 2)x + 16$	Apply the distributive property.
$= 6x + 16$	Simplify.

Check Point 9 Simplify: $6 + 4[7 - (x - 2)]$.

Blitzer Bonus || Using Algebra to Measure Blood-Alcohol Concentration

The amount of alcohol in a person's blood is known as blood-alcohol concentration (BAC), measured in grams of alcohol per deciliter of blood. A BAC of 0.08, meaning 0.08%, indicates that a person has 8 parts alcohol per 10,000 parts blood. In every state in the United States, it is illegal to drive with a BAC of 0.08 or higher.

How Do I Measure My Blood-Alcohol Concentration?

Here's a formula that models BAC for a person who weighs w pounds and who has n drinks* per hour.

$$\text{BAC} = \frac{600n}{w(0.6n + 169)}$$

Number of drinks consumed in an hour

Blood-alcohol concentration

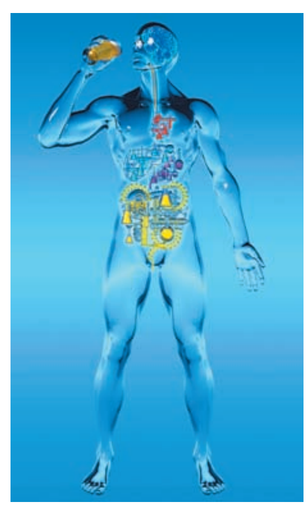
Body weight, in pounds

*A drink can be a 12-ounce can of beer, a 5-ounce glass of wine, or a 1.5-ounce shot of liquor. Each contains approximately 14 grams, or $\frac{1}{2}$ ounce, of alcohol.

Blood-alcohol concentration can be used to quantify the meaning of "tipsy."

BAC	Effects on Behavior
0.05	Feeling of well-being; mild release of inhibitions; absence of observable effects
0.08	Feeling of relaxation; mild sedation; exaggeration of emotions and behavior; slight impairment of motor skills; increase in reaction time
0.12	Muscle control and speech impaired; difficulty performing motor skills; uncoordinated behavior
0.15	Euphoria; major impairment of physical and mental functions; irresponsible behavior; some difficulty standing, walking, and talking
0.35	Surgical anesthesia; lethal dosage for a small percentage of people
0.40	Lethal dosage for 50% of people; severe circulatory and respiratory depression; alcohol poisoning/overdose

Source: National Clearinghouse for Alcohol and Drug Information



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Keeping in mind the meaning of "tipsy," we can use our model to compare blood-alcohol concentrations of a 120-pound person and a 200-pound person for various numbers of drinks. We determined each BAC using a calculator, rounding to three decimal places.

Blood-Alcohol Concentrations of a 120-Pound Person

$$\text{BAC} = \frac{600n}{120(0.6n + 169)}$$

n (number of drinks per hour)	1	2	3	4	5	6	7	8	9	10
BAC (blood-alcohol concentration)	0.029	0.059	0.088	0.117	0.145	0.174	0.202	0.230	0.258	0.286

$$\underbrace{\qquad 3 \text{ through } 10 \qquad}_{\text{Illegal to drive}}$$

Illegal to drive

Blood-Alcohol Concentrations of a 200-Pound Person

$$\text{BAC} = \frac{600n}{200(0.6n + 169)}$$

n (number of drinks per hour)	1	2	3	4	5	6	7	8	9	10
BAC (blood-alcohol concentration)	0.018	0.035	0.053	0.070	0.087	0.104	0.121	0.138	0.155	0.171

Illegal to drive

Like all mathematical models, the formula for BAC gives approximate rather than exact values. There are other variables that influence blood-alcohol concentration that are not contained in the model. These include the rate at which an individual's body processes alcohol, how quickly one drinks, sex, age, physical condition, and the amount of food eaten prior to drinking.

CONCEPT AND VOCABULARY CHECK

Fill in each blank so that the resulting statement is true.

1. A combination of numbers, variables, and operation symbols is called an algebraic ________.

2. If n is a counting number, b^n, read ____________, indicates that there are n factors of b. The number b is called the _____ and the number n is called the ________.

3. An equation that expresses a relationship between two or more variables, such as $H = \frac{9}{10}(220 - a)$, is called a/an _______. The process of finding such equations to describe real-world phenomena is called mathematical ________. Such equations, together with the meaning assigned to the variables, are called mathematical _______.

4. The set of elements common to both set A and set B is called the _________ of sets A and B, and is symbolized by _____.

5. The set of elements that are members of set A or set B or of both sets is called the ______ of sets A or B and is symbolized by ______.

6. The set $\{1, 2, 3, 4, 5, \ldots\}$ is called the set of _______ numbers.

7. The set $\{0, 1, 2, 3, 4, 5, \ldots\}$ is called the set of ______ numbers.

8. The set $\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$ is called the set of _______.

9. The set of numbers in the form $\frac{a}{b}$, where a and b belong to the set in Exercise 8 and $b \neq 0$, is called the set of _______ numbers.

10. The set of numbers whose decimal representations are neither terminating nor repeating is called the set of ________ numbers.

11. Every real number is either a/an _______ number or a/an ________ number.

12. The notation $|x|$ is read the ___________ of x. If $x \geq 0$, then $|x| =$ ___. If $x < 0$, then $|x| =$ ___.

13. The commutative properties state that $a + b =$ ____ and $ab =$ ___.

14. The associative properties state that $(a + b) + c =$ _______ and ____ $= a(bc)$.

15. The distributive property states that $a(b + c) =$ _____.

16. $a + (-a) =$ __ : The sum of a real number and its additive _______ is __, the additive _______.

17. $a \cdot \frac{1}{a} = 1, a \neq 0$: The product of a nonzero real number and its multiplicative _______ is __, the multiplicative _______.

18. An algebraic expression is ________ when parentheses have been removed and like terms have been combined.

19. $-(-a) =$ __.

EXERCISE SET 1

Practice Exercises

In Exercises 1–16, evaluate each algebraic expression for the given value or values of the variable(s).

1. $7 + 5x$, for $x = 10$

2. $8 + 6x$, for $x = 5$

3. $6x - y$, for $x = 3$ and $y = 8$

4. $8x - y$, for $x = 3$ and $y = 4$

5. $x^2 + 3x$, for $x = 8$

6. $x^2 + 5x$, for $x = 6$

7. $x^2 - 6x + 3$, for $x = 7$

8. $x^2 - 7x + 4$, for $x = 8$

9. $4 + 5(x - 7)^3$, for $x = 9$

10. $6 + 5(x - 6)^3$, for $x = 8$

11. $x^2 - 3(x - y)$, for $x = 8$ and $y = 2$

12. $x^2 - 4(x - y)$, for $x = 8$ and $y = 3$

13. $\frac{5(x + 2)}{2x - 14}$, for $x = 10$

14. $\frac{7(x - 3)}{2x - 16}$, for $x = 9$

15. $\frac{2x + 3y}{x + 1}$, for $x = -2$ and $y = 4$

16. $\frac{2x + y}{xy - 2x}$, for $x = -2$ and $y = 4$

The formula

$$C = \frac{5}{9}(F - 32)$$

expresses the relationship between Fahrenheit temperature, F, and Celsius temperature, C. In Exercises 17–18, use the formula to convert the given Fahrenheit temperature to its equivalent temperature on the Celsius scale.

17. 50°F

18. 86°F

A football was kicked vertically upward from a height of 4 feet with an initial speed of 60 feet per second. The formula

$$h = 4 + 60t - 16t^2$$

describes the ball's height above the ground, h, in feet, t seconds after it was kicked. Use this formula to solve Exercises 19–20.

19. What was the ball's height 2 seconds after it was kicked?

20. What was the ball's height 3 seconds after it was kicked?

In Exercises 21–28, find the intersection of the sets.

21. $\{1, 2, 3, 4\} \cap \{2, 4, 5\}$

22. $\{1, 3, 7\} \cap \{2, 3, 8\}$

23. $\{s, e, t\} \cap \{t, e, s\}$

24. $\{r, e, a, l\} \cap \{l, e, a, r\}$

25. $\{1, 3, 5, 7\} \cap \{2, 4, 6, 8, 10\}$

26. $\{0, 1, 3, 5\} \cap \{-5, -3, -1\}$

27. $\{a, b, c, d\} \cap \varnothing$

28. $\{w, y, z\} \cap \varnothing$

In Exercises 29–34, find the union of the sets.

29. $\{1, 2, 3, 4\} \cup \{2, 4, 5\}$

30. $\{1, 3, 7, 8\} \cup \{2, 3, 8\}$

31. $\{1, 3, 5, 7\} \cup \{2, 4, 6, 8, 10\}$

32. $\{0, 1, 3, 5\} \cup \{2, 4, 6\}$

33. $\{a, e, i, o, u\} \cup \varnothing$

34. $\{e, m, p, t, y\} \cup \varnothing$

In Exercises 35–38, list all numbers from the given set that are **a.** *natural numbers,* **b.** *whole numbers,* **c.** *integers,* **d.** *rational numbers,* **e.** *irrational numbers,* **f.** *real numbers.*

35. $\{-9, -\frac{4}{5}, 0, 0.25, \sqrt{3}, 9.2, \sqrt{100}\}$

36. $\{-7, -0.\overline{6}, 0, \sqrt{49}, \sqrt{50}\}$

37. $\{-11, -\frac{5}{6}, 0, 0.75, \sqrt{5}, \pi, \sqrt{64}\}$

38. $\{-5, -0.\overline{3}, 0, \sqrt{2}, \sqrt{4}\}$

39. Give an example of a whole number that is not a natural number.

40. Give an example of a rational number that is not an integer.

41. Give an example of a number that is an integer, a whole number, and a natural number.

42. Give an example of a number that is a rational number, an integer, and a real number.

Determine whether each statement in Exercises 43–50 is true or false.

43. $-13 \le -2$

44. $-6 > 2$

45. $4 \ge -7$

46. $-13 < -5$

47. $-\pi \ge -\pi$

48. $-3 > -13$

49. $0 \ge -6$

50. $0 \ge -13$

In Exercises 51–60, rewrite each expression without absolute value bars.

51. $|300|$

52. $|-203|$

53. $|12 - \pi|$

54. $|7 - \pi|$

55. $|\sqrt{2} - 5|$

56. $|\sqrt{5} - 13|$

57. $\frac{-3}{|-3|}$

58. $\frac{-7}{|-7|}$

59. $||-3| - |-7||$

60. $||-5| - |-13||$

In Exercises 61–66, evaluate each algebraic expression for $x = 2$ and $y = -5$.

61. $|x + y|$

62. $|x - y|$

63. $|x| + |y|$

64. $|x| - |y|$

65. $\frac{y}{|y|}$

66. $\frac{|x|}{x} + \frac{|y|}{y}$

In Exercises 67–74, express the distance between the given numbers using absolute value. Then find the distance by evaluating the absolute value expression.

67. 2 and 17

68. 4 and 15

69. -2 and 5

70. -6 and 8

71. -19 and -4

72. -26 and -3

73. -3.6 and -1.4

74. -5.4 and -1.2

In Exercises 75–84, state the name of the property illustrated.

75. $6 + (-4) = (-4) + 6$

76. $11 \cdot (7 + 4) = 11 \cdot 7 + 11 \cdot 4$

77. $6 + (2 + 7) = (6 + 2) + 7$

78. $6 \cdot (2 \cdot 3) = 6 \cdot (3 \cdot 2)$

79. $(2 + 3) + (4 + 5) = (4 + 5) + (2 + 3)$

80. $7 \cdot (11 \cdot 8) = (11 \cdot 8) \cdot 7$

81. $2(-8 + 6) = -16 + 12$

82. $-8(3 + 11) = -24 + (-88)$

83. $\frac{1}{(x+3)}(x+3) = 1, x \neq -3$

84. $(x+4) + [-(x+4)] = 0$

In Exercises 85–96, simplify each algebraic expression.

85. $5(3x+4) - 4$

86. $2(5x+4) - 3$

87. $5(3x-2) + 12x$

88. $2(5x-1) + 14x$

89. $7(3y-5) + 2(4y+3)$

90. $4(2y-6) + 3(5y+10)$

91. $5(3y-2) - (7y+2)$

92. $4(5y-3) - (6y+3)$

93. $7 - 4[3 - (4y-5)]$

94. $6 - 5[8 - (2y-4)]$

95. $18x^2 + 4 - [6(x^2-2) + 5]$

96. $14x^2 + 5 - [7(x^2-2) + 4]$

In Exercises 97–102, write each algebraic expression without parentheses.

97. $-(-14x)$

98. $-(-17y)$

99. $-(2x - 3y - 6)$

100. $-(5x - 13y - 1)$

101. $\frac{1}{3}(3x) + [(4y) + (-4y)]$

102. $\frac{1}{2}(2y) + [(-7x) + 7x]$

Practice Plus

In Exercises 103–110, insert either $<$, $>$, or $=$ in the shaded area to make a true statement.

103. $|-6| \;\square\; |-3|$

104. $|-20| \;\square\; |-50|$

105. $\left|\frac{3}{5}\right| \;\square\; |-0.6|$

106. $\left|\frac{5}{2}\right| \;\square\; |-2.5|$

107. $\frac{30}{40} - \frac{3}{4} \;\square\; \frac{14}{15} \cdot \frac{15}{14}$

108. $\frac{17}{18} \cdot \frac{18}{17} \;\square\; \frac{50}{60} - \frac{5}{6}$

109. $\frac{8}{13} \div \frac{8}{13} \;\square\; |-1|$

110. $|-2| \;\square\; \frac{4}{17} \div \frac{4}{17}$

In Exercises 111–120, use the order of operations to simplify each expression.

111. $8^2 - 16 \div 2^2 \cdot 4 - 3$

112. $10^2 - 100 \div 5^2 \cdot 2 - 3$

113. $\frac{5 \cdot 2 - 3^2}{[3^2 - (-2)]^2}$

114. $\frac{10 \div 2 + 3 \cdot 4}{(12 - 3 \cdot 2)^2}$

115. $8 - 3[-2(2-5) - 4(8-6)]$

116. $8 - 3[-2(5-7) - 5(4-2)]$

117. $\frac{2(-2) - 4(-3)}{5 - 8}$

118. $\frac{6(-4) - 5(-3)}{9 - 10}$

119. $\frac{(5-6)^2 - 2|3-7|}{89 - 3 \cdot 5^2}$

120. $\frac{12 \div 3 \cdot 5|2^2 + 3^2|}{7 + 3 - 6^2}$

In Exercises 121–128, write each English phrase as an algebraic expression. Then simplify the expression. Let x represent the number.

121. A number decreased by the sum of the number and four

122. A number decreased by the difference between eight and the number

123. Six times the product of negative five and a number

124. Ten times the product of negative four and a number

125. The difference between the product of five and a number and twice the number

126. The difference between the product of six and a number and negative two times the number

127. The difference between eight times a number and six more than three times the number

128. Eight decreased by three times the sum of a number and six

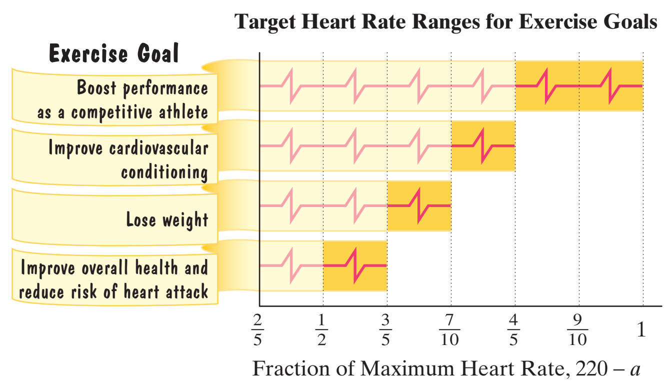
Application Exercises

The maximum heart rate, in beats per minute, that you should achieve during exercise is 220 minus your age:

$$220 - a.$$

This algebraic expression gives maximum heart rate in terms of age, a.

The following bar graph shows the target heart rate ranges for four types of exercise goals. The lower and upper limits of these ranges are fractions of the maximum heart rate, $220 - a$. Exercises 129–130 are based on the information in the graph.



129. If your exercise goal is to improve cardiovascular conditioning, the graph shows the following range for target heart rate, H, in beats per minute:

Lower limit of range: $H = \frac{7}{10}(220 - a)$

Upper limit of range: $H = \frac{4}{5}(220 - a)$

a. What is the lower limit of the heart range, in beats per minute, for a 20-year-old with this exercise goal?

b. What is the upper limit of the heart range, in beats per minute, for a 20-year-old with this exercise goal?

130. If your exercise goal is to improve overall health, the graph shows the following range for target heart rate, H, in beats per minute:

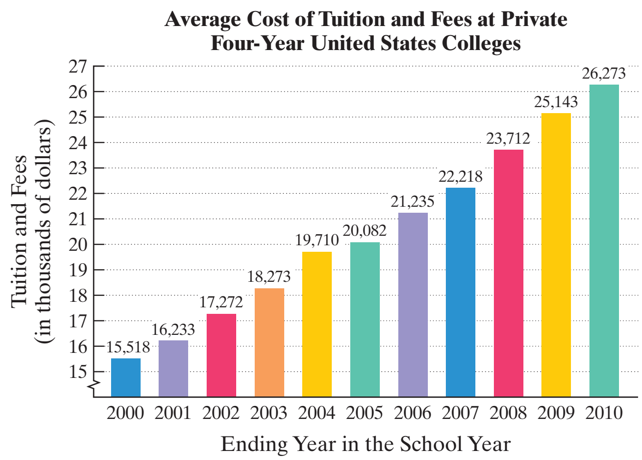
Lower limit of range: $H = \frac{1}{2}(220 - a)$

Upper limit of range: $H = \frac{3}{5}(220 - a)$

a. What is the lower limit of the heart range, in beats per minute, for a 30-year-old with this exercise goal?

b. What is the upper limit of the heart range, in beats per minute, for a 30-year-old with this exercise goal?

The bar graph shows the average cost of tuition and fees at private four-year colleges in the United States.



Source: The College Board

The formula

$$T = 26x^2 + 819x + 15{,}527$$

models the average cost of tuition and fees, T, at private U.S. colleges for the school year ending x years after 2000. Use this information to solve Exercises 131–132.

131. a. Use the formula to find the average cost of tuition and fees at private U.S. colleges for the school year ending in 2010.

b. By how much does the formula underestimate or overestimate the actual cost shown by the graph for the school year ending in 2010?

c. Use the formula to project the average cost of tuition and fees at private U.S. colleges for the school year ending in 2013.

132. a. Use the formula to find the average cost of tuition and fees at private U.S. colleges for the school year ending in 2009.

b. By how much does the formula underestimate or overestimate the actual cost shown by the graph for the school year ending in 2009?

c. Use the formula to project the average cost of tuition and fees at private U.S. colleges for the school year ending in 2012.

133. You had \$10,000 to invest. You put x dollars in a safe, government-insured certificate of deposit paying 5% per year. You invested the remainder of the money in noninsured corporate bonds paying 12% per year. Your total interest earned at the end of the year is given by the algebraic expression

$$0.05x + 0.12(10{,}000 - x).$$

a. Simplify the algebraic expression.

b. Use each form of the algebraic expression to determine your total interest earned at the end of the year if you invested \$6000 in the safe, government-insured certificate of deposit.

134. It takes you 50 minutes to get to campus. You spend t minutes walking to the bus stop and the rest of the time riding the bus. Your walking rate is 0.06 mile per minute and the bus travels at a rate of 0.5 mile per minute. The total distance walking and traveling by bus is given by the algebraic expression

$$0.06t + 0.5(50 - t).$$

a. Simplify the algebraic expression.

b. Use each form of the algebraic expression to determine the total distance that you travel if you spend 20 minutes walking to the bus stop.

135. Read the Blitzer Bonus. Use the formula

$$\text{BAC} = \frac{600n}{w(0.6n + 169)}$$

and replace w with your body weight. Using this formula and a calculator, compute your BAC for integers from $n = 1$ to $n = 10$. Round to three decimal places. According to this model, how many drinks can you consume in an hour without exceeding the legal measure of drunk driving?

Writing in Mathematics

Writing about mathematics will help you learn mathematics. For all writing exercises in this text, use complete sentences to respond to the question. Some writing exercises can be answered in a sentence; others require a paragraph or two. You can decide how much you need to write as long as your writing clearly and directly answers the question in the exercise. Standard references such as a dictionary and a thesaurus should be helpful.

136. What is an algebraic expression? Give an example with your explanation.

137. If n is a natural number, what does b^n mean? Give an example with your explanation.

138. What does it mean when we say that a formula models real-world phenomena?

139. What is the intersection of sets A and B?

140. What is the union of sets A and B?

141. How do the whole numbers differ from the natural numbers?

142. Can a real number be both rational and irrational? Explain your answer.

143. If you are given two real numbers, explain how to determine which is the lesser.

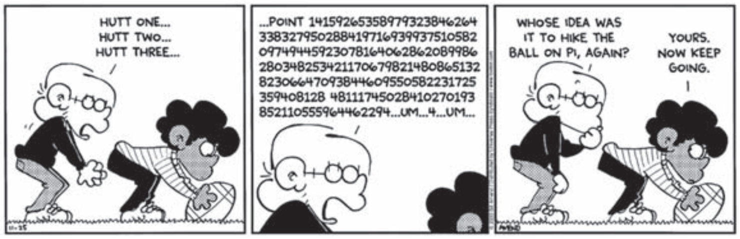
Critical Thinking Exercises

Make Sense? *In Exercises 144–147, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

144. My mathematical model describes the data for tuition and fees at public four-year colleges for the past ten years extremely well, so it will serve as an accurate prediction for the cost of public colleges in 2050.

145. A model that describes the average cost of tuition and fees at private U.S. colleges for the school year ending x years after 2000 cannot be used to estimate the cost of private education for the school year ending in 2000.

146. The humor in this cartoon is based on the fact that the football will never be hiked.



Foxtrot © 2003, 2009 by Bill Amend/Used by permission of Universal Uclick.

147. Just as the commutative properties change groupings, the associative properties change order.

In Exercises 148–155, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

148. Every rational number is an integer.

149. Some whole numbers are not integers.

150. Some rational numbers are not positive.

151. Irrational numbers cannot be negative.

152. The term x has no coefficient.

153. $5 + 3(x - 4) = 8(x - 4) = 8x - 32$

154. $-x - x = -x + (-x) = 0$

155. $x - 0.02(x + 200) = 0.98x - 4$

In Exercises 156–158, insert either < or > in the shaded area between the numbers to make the statement true.

156. $\sqrt{2}\ \boxed{}\ 1.5$ **157.** $-\pi\ \boxed{}\ -3.5$

158. $-\dfrac{3.14}{2}\ \boxed{}\ -\dfrac{\pi}{2}$

Preview Exercises

Exercises 159–161 will help you prepare for the material covered in the next section.

159. In parts (a) and (b), complete each statement.

a. $b^4 \cdot b^3 = (b \cdot b \cdot b \cdot b)(b \cdot b \cdot b) = b^?$

b. $b^5 \cdot b^5 = (b \cdot b \cdot b \cdot b \cdot b)(b \cdot b \cdot b \cdot b \cdot b) = b^?$

c. Generalizing from parts (a) and (b), what should be done with the exponents when multiplying exponential expressions with the same base?

160. In parts (a) and (b), complete each statement.

a. $\dfrac{b^7}{b^3} = \dfrac{\cancel{b} \cdot \cancel{b} \cdot \cancel{b} \cdot b \cdot b \cdot b \cdot b}{\cancel{b} \cdot \cancel{b} \cdot \cancel{b}} = b^?$

b. $\dfrac{b^8}{b^2} = \dfrac{\cancel{b} \cdot \cancel{b} \cdot b \cdot b \cdot b \cdot b \cdot b \cdot b}{\cancel{b} \cdot \cancel{b}} = b^?$

c. Generalizing from parts (a) and (b), what should be done with the exponents when dividing exponential expressions with the same base?

161. If 6.2 is multiplied by 10^3, what does this multiplication do to the decimal point in 6.2?

SECTION 2 Exponents and Scientific Notation

Objectives

1. Use properties of exponents.
2. Simplify exponential expressions.
3. Use scientific notation.

Bigger than the biggest thing ever and then some. Much bigger than that in fact, really amazingly immense, a totally stunning size, real 'wow, that's big', time … Gigantic multiplied by colossal multiplied by staggeringly huge is the sort of concept we're trying to get across here.

Douglas Adams, *The Restaurant at the End of the Universe*

Although Adams's description may not quite apply to this \$15.2 trillion national debt, exponents can be used to explore the meaning of this "staggeringly huge" number. In this section, you will learn to use exponents to provide a way of putting large and small numbers in perspective.

1 Use properties of exponents.

Properties of Exponents

The major properties of exponents are summarized in the box that follows on the next page.

GREAT QUESTION!

Cut to the chase. What do I do with negative exponents?

When a negative integer appears as an exponent, switch the position of the base (from numerator to denominator or from denominator to numerator) and make the exponent positive.

GREAT QUESTION!

What's the difference between $\frac{4^3}{4^5}$ and $\frac{4^5}{4^3}$?

These quotients represent different numbers:

$$\frac{4^3}{4^5} = 4^{3-5} = 4^{-2} = \frac{1}{4^2} = \frac{1}{16}$$

$$\frac{4^5}{4^3} = 4^{5-3} = 4^2 = 16.$$

Properties of Exponents

Property | **Examples**

The Negative-Exponent Rule

If b is any real number other than 0 and n is a natural number, then

$$b^{-n} = \frac{1}{b^n}.$$

- $5^{-3} = \frac{1}{5^3} = \frac{1}{125}$
- $\frac{1}{4^{-2}} = \frac{1}{\frac{1}{4^2}} = 4^2 = 16$

The Zero-Exponent Rule

If b is any real number other than 0,

$$b^0 = 1.$$

- $7^0 = 1$
- $(-5)^0 = 1$
- $-5^0 = -1$

Only 5 is raised to the zero power.

The Product Rule

If b is a real number or algebraic expression, and m and n are integers,

$$b^m \cdot b^n = b^{m+n}.$$

- $2^2 \cdot 2^3 = 2^{2+3} = 2^5 = 32$
- $x^{-3} \cdot x^7 = x^{-3+7} = x^4$

When multiplying exponential expressions with the same base, add the exponents. Use this sum as the exponent of the common base.

The Power Rule

If b is a real number or algebraic expression, and m and n are integers,

$$(b^m)^n = b^{mn}.$$

- $(2^2)^3 = 2^{2\cdot3} = 2^6 = 64$
- $(x^{-3})^4 = x^{-3\cdot4} = x^{-12} = \frac{1}{x^{12}}$

When an exponential expression is raised to a power, multiply the exponents. Place the product of the exponents on the base and remove the parentheses.

The Quotient Rule

If b is a nonzero real number or algebraic expression, and m and n are integers,

$$\frac{b^m}{b^n} = b^{m-n}.$$

- $\frac{2^8}{2^4} = 2^{8-4} = 2^4 = 16$
- $\frac{x^3}{x^7} = x^{3-7} = x^{-4} = \frac{1}{x^4}$

When dividing exponential expressions with the same nonzero base, subtract the exponent in the denominator from the exponent in the numerator. Use this difference as the exponent of the common base.

Products Raised to Powers

If a and b are real numbers or algebraic expressions, and n is an integer,

$$(ab)^n = a^n b^n.$$

- $(-2y)^4 = (-2)^4 y^4 = 16y^4$
- $(-2xy)^3 = (-2)^3 x^3 y^3 = -8x^3y^3$

When a product is raised to a power, raise each factor to that power.

Quotients Raised to Powers

If a and b are real numbers, $b \neq 0$, or algebraic expressions, and n is an integer,

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

- $\left(\frac{2}{5}\right)^4 = \frac{2^4}{5^4} = \frac{16}{625}$
- $\left(-\frac{3}{x}\right)^3 = \frac{(-3)^3}{x^3} = -\frac{27}{x^3}$

When a quotient is raised to a power, raise the numerator to that power and divide by the denominator to that power.

2 Simplify exponential expressions.

Simplifying Exponential Expressions

Properties of exponents are used to simplify exponential expressions. An exponential expression is **simplified** when

- No parentheses appear.
- No powers are raised to powers.
- Each base occurs only once.
- No negative or zero exponents appear.

Simplifying Exponential Expressions

	Example
1. If necessary, remove parentheses by using $(ab)^n = a^n b^n$ or $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.	$(xy)^3 = x^3y^3$
2. If necessary, simplify powers to powers by using $(b^m)^n = b^{mn}$.	$(x^4)^3 = x^{4 \cdot 3} = x^{12}$
3. If necessary, be sure that each base appears only once by using $b^m \cdot b^n = b^{m+n}$ or $\frac{b^m}{b^n} = b^{m-n}$.	$x^4 \cdot x^3 = x^{4+3} = x^7$
4. If necessary, rewrite exponential expressions with zero powers as 1 ($b^0 = 1$). Furthermore, write the answer with positive exponents by using $b^{-n} = \frac{1}{b^n}$ or $\frac{1}{b^{-n}} = b^n$.	$\frac{x^5}{x^8} = x^{5-8} = x^{-3} = \frac{1}{x^3}$

The following example shows how to simplify exponential expressions. Throughout the example, assume that no variable in a denominator is equal to zero.

EXAMPLE 1 Simplifying Exponential Expressions

Simplify:

a. $(-3x^4y^5)^3$

b. $(-7xy^4)(-2x^5y^6)$

c. $\frac{-35x^2y^4}{5x^6y^{-8}}$

d. $\left(\frac{4x^2}{y}\right)^{-3}$.

SOLUTION

a. $(-3x^4y^5)^3 = (-3)^3(x^4)^3(y^5)^3$ Raise each factor inside the parentheses to the third power.

$= (-3)^3x^{4\cdot 3}y^{5\cdot 3}$ Multiply the exponents when raising powers to powers.

$= -27x^{12}y^{15}$ $(-3)^3 = (-3)(-3)(-3) = -27$

b. $(-7xy^4)(-2x^5y^6) = (-7)(-2)xx^5y^4y^6$ Group factors with the same base.

$= 14x^{1+5}y^{4+6}$ When multiplying expressions with the same base, add the exponents.

$= 14x^6y^{10}$ Simplify.

c. $\dfrac{-35x^2y^4}{5x^6y^{-8}} = \left(\dfrac{-35}{5}\right)\left(\dfrac{x^2}{x^6}\right)\left(\dfrac{y^4}{y^{-8}}\right)$ Group factors with the same base.

$= -7x^{2-6}y^{4-(-8)}$ When dividing expressions with the same base, subtract the exponents.

$= -7x^{-4}y^{12}$ Simplify. Notice that $4 - (-8) = 4 + 8 = 12$.

$= \dfrac{-7y^{12}}{x^4}$ Write as a fraction and move the base with the negative exponent, x^{-4}, to the other side of the fraction bar and make the negative exponent positive.

d. $\left(\dfrac{4x^2}{y}\right)^{-3} = \dfrac{(4x^2)^{-3}}{y^{-3}}$ Raise the numerator and the denominator to the -3 power.

$= \dfrac{4^{-3}(x^2)^{-3}}{y^{-3}}$ Raise each factor inside the parentheses to the -3 power.

$= \dfrac{4^{-3}x^{-6}}{y^{-3}}$ Multiply the exponents when raising a power to a power: $(x^2)^{-3} = x^{2(-3)} = x^{-6}$.

$= \dfrac{y^3}{4^3x^6}$ Move each base with a negative exponent to the other side of the fraction bar and make each negative exponent positive.

$= \dfrac{y^3}{64x^6}$ $4^3 = 4\cdot 4\cdot 4 = 64$

Check Point 1 Simplify:

a. $(2x^3y^6)^4$ **b.** $(-6x^2y^5)(3xy^3)$

c. $\dfrac{100x^{12}y^2}{20x^{16}y^{-4}}$ **d.** $\left(\dfrac{5x}{y^4}\right)^{-2}$.

GREAT QUESTION!

Simplifying exponential expressions seems to involve lots of steps. Are there common errors I can avoid along the way?

Yes. Here's a list. The first column has the correct simplification. The second column contains common errors you should try to avoid.

Correct	Incorrect	Description of Error
$b^3 \cdot b^4 = b^7$	$b^3 \cdot b^4 = b^{12}$	The exponents should be added, not multiplied.
$3^2 \cdot 3^4 = 3^6$	$3^2 \cdot 3^4 = 9^6$	The common base should be retained, not multiplied.
$\frac{5^{16}}{5^4} = 5^{12}$	$\frac{5^{16}}{5^4} = 5^4$	The exponents should be subtracted, not divided.
$(4a)^3 = 64a^3$	$(4a)^3 = 4a^3$	Both factors should be cubed.
$b^{-n} = \frac{1}{b^n}$	$b^{-n} = -\frac{1}{b^n}$	Only the exponent should change sign.
$(a+b)^{-1} = \frac{1}{a+b}$	$(a+b)^{-1} = \frac{1}{a} + \frac{1}{b}$	The exponent applies to the entire expression $a + b$.

3 Use scientific notation.

Scientific Notation

As of December 2011, the national debt of the United States was about \$15.2 trillion. This is the amount of money the government has had to borrow over the years, mostly by selling bonds, because it has spent more than it has collected in taxes. A stack of \$1 bills equaling the national debt would measure more than 950,000 miles. That's more than two round trips from Earth to the moon. Because a trillion is 10^{12} (see **Table 3**), the national debt can be expressed as

$$15.2 \times 10^{12}.$$

Table 3 Names of Large Numbers

10^2	hundred
10^3	thousand
10^6	million
10^9	billion
10^{12}	trillion
10^{15}	quadrillion
10^{18}	quintillion
10^{21}	sextillion
10^{24}	septillion
10^{27}	octillion
10^{30}	nonillion
10^{100}	googol
10^{googol}	googolplex

Because $15.2 = 1.52 \times 10$, the national debt can be expressed as

$$\begin{aligned} 15.2 \times 10^{12} &= (1.52 \times 10) \times 10^{12} = 1.52 \times (10 \times 10^{12}) \\ &= 1.52 \times 10^{1+12} = 1.52 \times 10^{13} \end{aligned}$$

The number 1.52×10^{13} is written in a form called *scientific notation*.

Scientific Notation

A number is written in **scientific notation** when it is expressed in the form

$$a \times 10^n,$$

where the absolute value of a is greater than or equal to 1 and less than 10 ($1 \le |a| < 10$), and n is an integer.

It is customary to use the multiplication symbol, $\times$, rather than a dot, when writing a number in scientific notation.

Converting from Scientific to Decimal Notation

Here are two examples of numbers in scientific notation:

$$6.4 \times 10^5 \quad \text{means} \quad 640{,}000.$$

$$2.17 \times 10^{-3} \quad \text{means} \quad 0.00217.$$

Do you see that the number with the positive exponent is relatively large and the number with the negative exponent is relatively small?

We can use n, the exponent on the 10 in $a \times 10^n$, to change a number in scientific notation to decimal notation. If n is **positive**, move the decimal point in a to the **right** n places. If n is **negative**, move the decimal point in a to the **left** $|n|$ places.

EXAMPLE 2 Converting from Scientific to Decimal Notation

Write each number in decimal notation:

a. 6.2×10^7 **b.** -6.2×10^7 **c.** 2.019×10^{-3} **d.** -2.019×10^{-3}.

SOLUTION

In each case, we use the exponent on the 10 to determine how far to move the decimal point and in which direction. In parts (a) and (b), the exponent is positive, so we move the decimal point to the right. In parts (c) and (d), the exponent is negative, so we move the decimal point to the left.

a. $6.2 \times 10^7 = 62{,}000{,}000$

$n = 7$

Move the decimal point 7 places to the right.

b. $-6.2 \times 10^7 = -62{,}000{,}000$

$n = 7$

Move the decimal point 7 places to the right.

c. $2.019 \times 10^{-3} = 0.002019$

$n = -3$

Move the decimal point $|-3|$ places, or 3 places, to the left.

d. $-2.019 \times 10^{-3} = -0.002019$

$n = -3$

Move the decimal point $|-3|$ places, or 3 places, to the left.

Check Point 2 Write each number in decimal notation:

a. -2.6×10^9 **b.** 3.017×10^{-6}.

Converting from Decimal to Scientific Notation

To convert from decimal notation to scientific notation, we reverse the procedure of Example 2.

Converting from Decimal to Scientific Notation

Write the number in the form $a \times 10^n$.

- Determine a, the numerical factor. Move the decimal point in the given number to obtain a number whose absolute value is between 1 and 10, including 1.
- Determine n, the exponent on 10^n. The absolute value of n is the number of places the decimal point was moved. The exponent n is positive if the decimal point was moved to the left, negative if the decimal point was moved to the right, and 0 if the decimal point was not moved.

EXAMPLE 3 Converting from Decimal Notation to Scientific Notation

Write each number in scientific notation:

a. 34,970,000,000,000 **b.** −34,970,000,000,000

c. 0.0000000000802 **d.** −0.0000000000802.

SOLUTION

a. $34{,}970{,}000{,}000{,}000 = 3.497 \times 10^{13}$

Move the decimal point to get a number whose absolute value is between 1 and 10.

The decimal point was moved 13 places to the left, so $n = 13$.

b. $-34{,}970{,}000{,}000{,}000 = -3.497 \times 10^{13}$

TECHNOLOGY

You can use your calculator's EE (enter exponent) or EXP key to convert from decimal to scientific notation. Here is how it's done for 0.0000000000802.

Many Scientific Calculators

Keystrokes

.0000000000802 EE =

Display

8.02 − 11

Many Graphing Calculators

Use the mode setting for scientific notation.

Keystrokes

.0000000000802 ENTER

Display

8.02E − 11

c. $0.0000000000802 = 8.02 \times 10^{-11}$

Move the decimal point to get a number whose absolute value is between 1 and 10.

The decimal point was moved 11 places to the right, so $n = -11$.

d. $-0.0000000000802 = -8.02 \times 10^{-11}$

GREAT QUESTION!

In scientific notation, which numbers have positive exponents and which have negative exponents?

If the absolute value of a number is greater than 10, it will have a positive exponent in scientific notation. If the absolute value of a number is less than 1, it will have a negative exponent in scientific notation.

Check Point 3 Write each number in scientific notation:

a. 5,210,000,000 **b.** −0.00000006893.

EXAMPLE 4 Expressing the U.S. Population in Scientific Notation

As of December 2011, the population of the United States was approximately 312 million. Express the population in scientific notation.

SOLUTION

Because a million is 10^6, the 2011 population can be expressed as

$$312 \times 10^6.$$

This factor is not between 1 and 10, so the number is not in scientific notation.

The voice balloon indicates that we need to convert 312 to scientific notation.

$$312 \times 10^6 = (3.12 \times 10^2) \times 10^6 = 3.12 \times 10^{2+6} = 3.12 \times 10^8$$

$312 = 3.12 \times 10^2$

In scientific notation, the population is 3.12×10^8.

Check Point 4 Express 410×10^7 in scientific notation.

Computations with Scientific Notation

Properties of exponents are used to perform computations with numbers that are expressed in scientific notation.

TECHNOLOGY

$(6.1 \times 10^5)(4 \times 10^{-9})$

On a Calculator:

Many Scientific Calculators

6.1 EE 5 × 4 EE 9 +/− =

Display

2.44 − 03

Many Graphing Calculators

6.1 EE 5 × 4 EE (−) 9 ENTER

Display (in scientific notation mode)

2.44E − 3

EXAMPLE 5 Computations with Scientific Notation

Perform the indicated computations, writing the answers in scientific notation:

a. $(6.1 \times 10^5)(4 \times 10^{-9})$ **b.** $\dfrac{1.8 \times 10^4}{3 \times 10^{-2}}$.

SOLUTION

a. $(6.1 \times 10^5)(4 \times 10^{-9})$

$= (6.1 \times 4) \times (10^5 \times 10^{-9})$ Regroup factors.

$= 24.4 \times 10^{5+(-9)}$ Add the exponents on 10 and multiply the other parts.

$= 24.4 \times 10^{-4}$ Simplify.

$= (2.44 \times 10^1) \times 10^{-4}$ Convert 24.4 to scientific notation: $24.4 = 2.44 \times 10^1$.

$= 2.44 \times 10^{-3}$ $10^1 \times 10^{-4} = 10^{1+(-4)} = 10^{-3}$

b.
$$\begin{aligned}
\frac{1.8 \times 10^4}{3 \times 10^{-2}} &= \left(\frac{1.8}{3}\right) \times \left(\frac{10^4}{10^{-2}}\right) && \text{Regroup factors.} \\
&= 0.6 \times 10^{4-(-2)} && \text{Subtract the exponents on 10 and divide the other parts.} \\
&= 0.6 \times 10^6 && \text{Simplify: } 4 - (-2) = 4 + 2 = 6. \\
&= (6 \times 10^{-1}) \times 10^6 && \text{Convert 0.6 to scientific notation: } 0.6 = 6 \times 10^{-1}. \\
&= 6 \times 10^5 && 10^{-1} \times 10^6 = 10^{-1+6} = 10^5
\end{aligned}$$

Check Point 5 Perform the indicated computations, writing the answers in scientific notation:

a. $(7.1 \times 10^5)(5 \times 10^{-7})$ **b.** $\frac{1.2 \times 10^6}{3 \times 10^{-3}}$.

Applications: Putting Numbers in Perspective

Due to tax cuts and spending increases, the United States began accumulating large deficits in the 1980s. To finance the deficit, the government had borrowed \$15.2 trillion as of December 2011. The graph in **Figure 10** shows the national debt increasing over time.

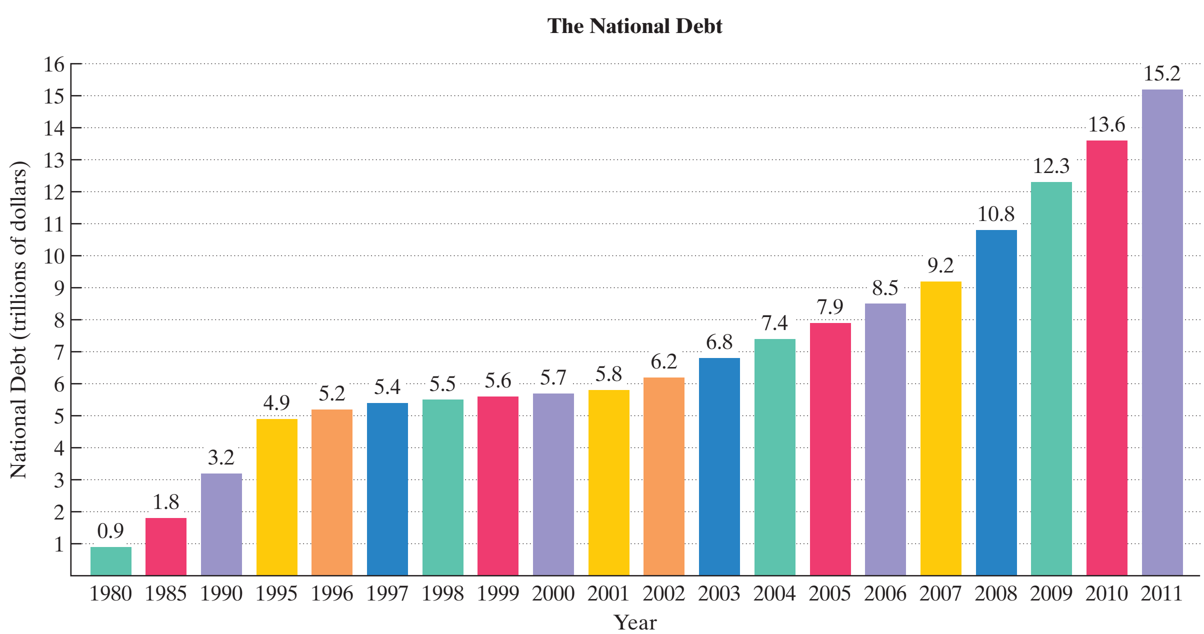


FIGURE 10
Source: Office of Management and Budget

Example 6 shows how we can use scientific notation to comprehend the meaning of a number such as 15.2 trillion.

EXAMPLE 6 The National Debt

As of December 2011, the national debt was \$15.2 trillion, or 15.2×10^{12} dollars. At that time, the U.S. population was approximately 312,000,000 (312 million), or 3.12×10^8. If the national debt was evenly divided among every individual in the United States, how much would each citizen have to pay?

SOLUTION

The amount each citizen must pay is the total debt, 15.2×10^{12} dollars, divided by the number of citizens, 3.12×10^8.

$$\frac{15.2 \times 10^{12}}{3.12 \times 10^8} = \left(\frac{15.2}{3.12}\right) \times \left(\frac{10^{12}}{10^8}\right)$$
$$\approx 4.87 \times 10^{12-8}$$
$$= 4.87 \times 10^4$$
$$= 48{,}700$$

Every U.S. citizen would have to pay approximately \$48,700 to the federal government to pay off the national debt.

Check Point 6 As of December 2011, the United States had spent \$2.6 trillion for the wars in Iraq and Afghanistan. (Source: costsofwar.org) At that time, the U.S. population was approximately 312 million (3.12×10^8). If the cost of these wars was evenly divided among every individual in the United States, how much, to the nearest hundred dollars, would each citizen have to pay?

An Application: Black Holes in Space

The concept of a black hole, a region in space where matter appears to vanish, intrigues scientists and nonscientists alike. Scientists theorize that when massive stars run out of nuclear fuel, they begin to collapse under the force of their own gravity. As the star collapses, its density increases. In turn, the force of gravity increases so tremendously that even light cannot escape from the star. Consequently, it appears black.

A mathematical model, called the Schwarzchild formula, describes the critical value to which the radius of a massive body must be reduced for it to become a black hole. This model forms the basis of our next example.

EXAMPLE 7 An Application of Scientific Notation

Use the Schwarzchild formula

$$R_s = \frac{2GM}{c^2}$$

where

R_s = Radius of the star, in meters, that would cause it to become a black hole
M = Mass of the star, in kilograms
G = A constant, called the gravitational constant
$= 6.7 \times 10^{-11} \dfrac{\text{m}^3}{\text{kg} \cdot \text{s}^2}$
c = Speed of light
$= 3 \times 10^8$ meters per second

to determine to what length the radius of the sun must be reduced for it to become a black hole. The sun's mass is approximately 2×10^{30} kilograms.

SOLUTION

$$R_s = \frac{2GM}{c^2}$$ Use the given model.

$$= \frac{2 \times 6.7 \times 10^{-11} \times 2 \times 10^{30}}{(3 \times 10^8)^2}$$ Substitute the given values: $G = 6.7 \times 10^{-11}$, $M = 2 \times 10^{30}$, and $c = 3 \times 10^8$.

$$= \frac{(2 \times 6.7 \times 2) \times (10^{-11} \times 10^{30})}{(3 \times 10^8)^2}$$ Rearrange factors in the numerator.

$$= \frac{26.8 \times 10^{-11+30}}{3^2 \times (10^8)^2}$$ Add exponents in the numerator. Raise each factor in the denominator to the power.

$$= \frac{26.8 \times 10^{19}}{9 \times 10^{16}}$$ Multiply powers to powers: $(10^8)^2 = 10^{8 \cdot 2} = 10^{16}$.

$$= \frac{26.8}{9} \times 10^{19-16}$$ When dividing expressions with the same base, subtract the exponents.

$$\approx 2.978 \times 10^3$$ Simplify.

$$= 2978$$

Although the sun is not massive enough to become a black hole (its radius is approximately 700,000 kilometers), the Schwarzchild model theoretically indicates that if the sun's radius were reduced to approximately 2978 meters, that is, about $\frac{1}{235{,}000}$ its present size, it would become a black hole. ● ● ●

Check Point 7 The speed of blood, S, in centimeters per second, located r centimeters from the central axis of an artery is modeled by

$$S = (1.76 \times 10^5)[(1.44 \times 10^{-2}) - r^2].$$

Find the speed of blood at the central axis of this artery.

Blitzer Bonus || Seven Ways to Spend \$1 Trillion

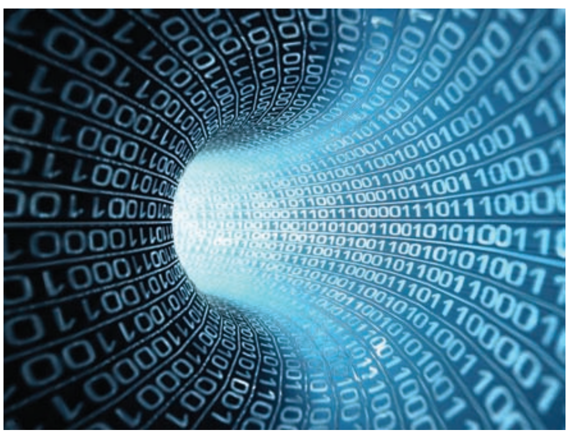


Image © photobank.kiev.ua, 2009

Confronting a national debt of \$15.2 trillion starts with grasping just how colossal \$1 trillion (1×10^{12}) actually is. To help you wrap your head around this mind-boggling number, and to put the national debt in further perspective, consider what \$1 trillion will buy:

- 40,816,326 new cars based on an average sticker price of \$24,500 each
- 5,574,136 homes based on the national median price of \$179,400 for existing single-family homes
- one year's salary for 14.7 million teachers based on the average teacher salary of \$68,000 in California
- the annual salaries of all 535 members of Congress for the next 10,742 years based on current salaries of \$174,000 per year
- the salary of basketball superstar LeBron James for 50,000 years based on an annual salary of \$20 million
- annual base pay for 59.5 million U.S. privates (that's 100 times the total number of active-duty soldiers in the Army) based on basic pay of \$16,794 per year
- salaries to hire all 2.8 million residents of the state of Kansas in full-time minimum-wage jobs for the next 23 years based on the federal minimum wage of \$7.25 per hour

Source: Kiplinger.com

CONCEPT AND VOCABULARY CHECK

Fill in each blank so that the resulting statement is true.

1. The product rule for exponents states that $b^m \cdot b^n =$ ____. When multiplying exponential expressions with the same base, ______ the exponents.
2. The quotient rule for exponents states that $\frac{b^m}{b^n} =$ ______, $b \neq 0$. When dividing exponential expressions with the same nonzero base, ________ the exponents.
3. If $b \neq 0$, then $b^0 =$ ____.
4. The negative-exponent rule states that $b^{-n} =$ ____, $b \neq 0$.
5. True or false: $5^{-2} = -5^2$ ______
6. Negative exponents in denominators can be evaluated using $\frac{1}{b^{-n}} =$ ___, $b \neq 0$.
7. True or false: $\frac{1}{8^{-2}} = 8^2$ ______
8. A positive number is written in scientific notation when it is expressed in the form $a \times 10^n$, where a is ______________________________ and n is a/an ________.
9. True or false: 4×10^3 is written in scientific notation. ____
10. True or false: 40×10^2 is written in scientific notation. ______

EXERCISE SET 2

Practice Exercises

Evaluate each exponential expression in Exercises 1–22.

1. $5^2 \cdot 2$

2. $6^2 \cdot 2$

3. $(-2)^6$

4. $(-2)^4$

5. -2^6

6. -2^4

7. $(-3)^0$

8. $(-9)^0$

9. -3^0

10. -9^0

11. 4^{-3}

12. 2^{-6}

13. $2^2 \cdot 2^3$

14. $3^3 \cdot 3^2$

15. $(2^2)^3$

16. $(3^3)^2$

17. $\frac{2^8}{2^4}$

18. $\frac{3^8}{3^4}$

19. $3^{-3} \cdot 3$

20. $2^{-3} \cdot 2$

21. $\frac{2^3}{2^7}$

22. $\frac{3^4}{3^7}$

Simplify each exponential expression in Exercises 23–64.

23. $x^{-2}y$

24. xy^{-3}

25. x^0y^5

26. x^7y^0

27. $x^3 \cdot x^7$

28. $x^{11} \cdot x^5$

29. $x^{-5} \cdot x^{10}$

30. $x^{-6} \cdot x^{12}$

31. $(x^3)^7$

32. $(x^{11})^5$

33. $(x^{-5})^3$

34. $(x^{-6})^4$

35. $\frac{x^{14}}{x^7}$

36. $\frac{x^{30}}{x^{10}}$

37. $\frac{x^{14}}{x^{-7}}$

38. $\frac{x^{30}}{x^{-10}}$

39. $(8x^3)^2$

40. $(6x^4)^2$

41. $\left(-\frac{4}{x}\right)^3$

42. $\left(-\frac{6}{y}\right)^3$

43. $(-3x^2y^5)^2$

44. $(-3x^4y^6)^3$

45. $(3x^4)(2x^7)$

46. $(11x^5)(9x^{12})$

47. $(-9x^3y)(-2x^6y^4)$

48. $(-5x^4y)(-6x^7y^{11})$

49. $\frac{8x^{20}}{2x^4}$

50. $\frac{20x^{24}}{10x^6}$

51. $\frac{25a^{13}b^4}{-5a^2b^3}$

52. $\frac{35a^{14}b^6}{-7a^7b^3}$

53. $\frac{14b^7}{7b^{14}}$

54. $\frac{20b^{10}}{10b^{20}}$

55. $(4x^3)^{-2}$

56. $(10x^2)^{-3}$

57. $\frac{24x^3y^5}{32x^7y^{-9}}$

58. $\frac{10x^4y^9}{30x^{12}y^{-3}}$

59. $\left(\frac{5x^3}{y}\right)^{-2}$

60. $\left(\frac{3x^4}{y}\right)^{-3}$

61. $\left(\frac{-15a^4b^2}{5a^{10}b^{-3}}\right)^3$

62. $\left(\frac{-30a^{14}b^8}{10a^{17}b^{-2}}\right)^3$

63. $\left(\frac{3a^{-5}b^2}{12a^3b^{-4}}\right)^0$

64. $\left(\frac{4a^{-5}b^3}{12a^3b^{-5}}\right)^0$

In Exercises 65–76, write each number in decimal notation without the use of exponents.

65. 3.8×10^2

66. 9.2×10^2

67. 6×10^{-4}

68. 7×10^{-5}

69. -7.16×10^6

70. -8.17×10^6

71. 7.9×10^{-1}

72. 6.8×10^{-1}

73. -4.15×10^{-3}

74. -3.14×10^{-3}

75. -6.00001×10^{10}

76. -7.00001×10^{10}

In Exercises 77–86, write each number in scientific notation.

77. 32,000

78. 64,000

79. 638,000,000,000,000,000

80. 579,000,000,000,000,000

81. −5716

82. −3829

83. 0.0027

84. 0.0083

85. −0.00000000504

86. −0.00000000405

In Exercises 87–106, perform the indicated computations. Write the answers in scientific notation. If necessary, round the decimal factor in your scientific notation answer to two decimal places.

87. $(3 \times 10^4)(2.1 \times 10^3)$

88. $(2 \times 10^4)(4.1 \times 10^3)$

89. $(1.6 \times 10^{15})(4 \times 10^{-11})$

90. $(1.4 \times 10^{15})(3 \times 10^{-11})$

91. $(6.1 \times 10^{-8})(2 \times 10^{-4})$

92. $(5.1 \times 10^{-8})(3 \times 10^{-4})$

93. $(4.3 \times 10^8)(6.2 \times 10^4)$

94. $(8.2 \times 10^8)(4.6 \times 10^4)$

95. $\frac{8.4 \times 10^8}{4 \times 10^5}$

96. $\frac{6.9 \times 10^8}{3 \times 10^5}$

97. $\frac{3.6 \times 10^4}{9 \times 10^{-2}}$

98. $\frac{1.2 \times 10^4}{2 \times 10^{-2}}$

99. $\frac{4.8 \times 10^{-2}}{2.4 \times 10^6}$

100. $\frac{7.5 \times 10^{-2}}{2.5 \times 10^6}$

101. $\frac{2.4 \times 10^{-2}}{4.8 \times 10^{-6}}$

102. $\frac{1.5 \times 10^{-2}}{3 \times 10^{-6}}$

103. $\frac{480{,}000{,}000{,}000}{0.00012}$

104. $\frac{282{,}000{,}000{,}000}{0.00141}$

105. $\frac{0.00072 \times 0.003}{0.00024}$

106. $\frac{66{,}000 \times 0.001}{0.003 \times 0.002}$

Practice Plus

In Exercises 107–114, simplify each exponential expression. Assume that variables represent nonzero real numbers.

107. $\frac{(x^{-2}y)^{-3}}{(x^2y^{-1})^3}$

108. $\frac{(xy^{-2})^{-2}}{(x^{-2}y)^{-3}}$

109. $(2x^{-3}yz^{-6})(2x)^{-5}$

110. $(3x^{-4}yz^{-7})(3x)^{-3}$

111. $\left(\frac{x^3y^4z^5}{x^{-3}y^{-4}z^{-5}}\right)^{-2}$

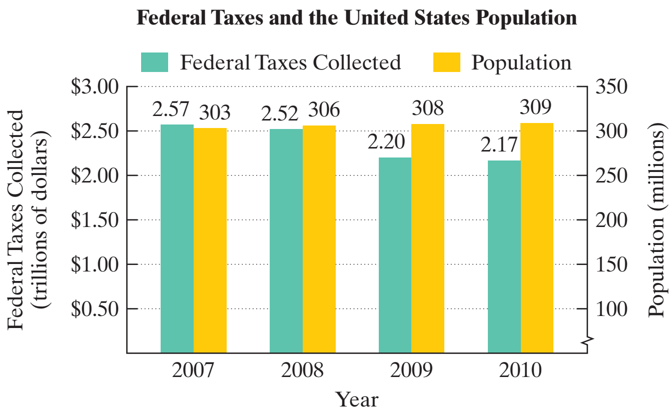
112. $\left(\frac{x^4y^5z^6}{x^{-4}y^{-5}z^{-6}}\right)^{-4}$

113. $\dfrac{(2^{-1}x^{-2}y^{-1})^{-2}(2x^{-4}y^{3})^{-2}(16x^{-3}y^{3})^{0}}{(2x^{-3}y^{-5})^{2}}$

114. $\dfrac{(2^{-1}x^{-3}y^{-1})^{-2}(2x^{-6}y^{4})^{-2}(9x^{3}y^{-3})^{0}}{(2x^{-4}y^{-6})^{2}}$

Application Exercises

The bar graph shows the total amount Americans paid in federal taxes, in trillions of dollars, and the U.S. population, in millions, from 2007 through 2010. Exercises 115–116 are based on the numbers displayed by the graph.



Sources: Internal Revenue Service and U.S. Census Bureau

115. a. In 2010, the United States government collected \$2.17 trillion in taxes. Express this number in scientific notation.

b. In 2010, the population of the United States was approximately 309 million. Express this number in scientific notation.

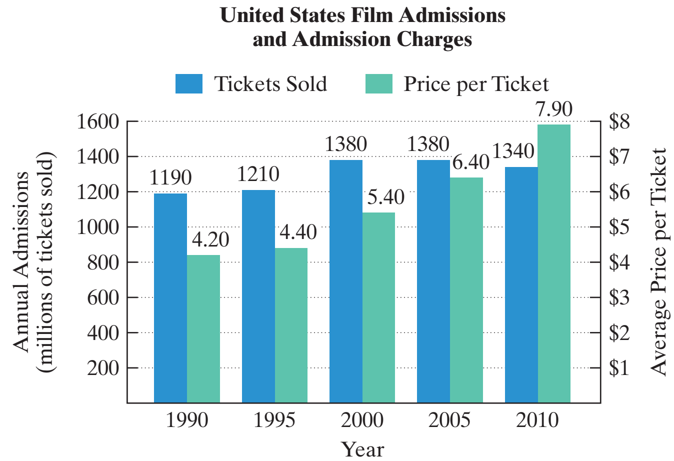
c. Use your scientific notation answers from parts (a) and (b) to answer this question: If the total 2010 tax collections were evenly divided among all Americans, how much would each citizen pay? Express the answer in decimal notation, rounded to the nearest dollar.

116. a. In 2009, the United States government collected \$2.20 trillion in taxes. Express this number in scientific notation.

b. In 2009, the population of the United States was approximately 308 million. Express this number in scientific notation.

c. Use your scientific notation answers from parts (a) and (b) to answer this question: If the total 2009 tax collections were evenly divided among all Americans, how much would each citizen pay? Express the answer in decimal notation, rounded to the nearest dollar.

In the dramatic arts, ours is the era of the movies. As individuals and as a nation, we've grown up with them. Our images of love, war, family, country—even of things that terrify us—owe much to what we've seen on screen. The bar graph at the top of the next column quantifies our love for movies by showing the number of tickets sold, in millions, and the average price per ticket for five selected years. Exercises 117–118 are based on the numbers displayed by the graph.



Source: Motion Picture Association of America

117. Use scientific notation to compute the amount of money that the motion picture industry made from box-office receipts in 2010. Express the answer in scientific notation.

118. Use scientific notation to compute the amount of money that the motion picture industry made from box office receipts in 2005. Express the answer in scientific notation.

119. The mass of one oxygen molecule is 5.3×10^{-23} gram. Find the mass of 20,000 molecules of oxygen. Express the answer in scientific notation.

120. The mass of one hydrogen atom is 1.67×10^{-24} gram. Find the mass of 80,000 hydrogen atoms. Express the answer in scientific notation.

121. There are approximately 3.2×10^{7} seconds in a year. According to the United States Department of Agriculture, Americans consume 127 chickens per second. How many chickens are eaten per year in the United States? Express the answer in scientific notation.

122. Convert 365 days (one year) to hours, to minutes, and, finally, to seconds, to determine how many seconds there are in a year. Express the answer in scientific notation.

Writing in Mathematics

123. Describe what it means to raise a number to a power. In your description, include a discussion of the difference between -5^2 and $(-5)^2$.

124. Explain the product rule for exponents. Use $2^3 \cdot 2^5$ in your explanation.

125. Explain the power rule for exponents. Use $(3^2)^4$ in your explanation.

126. Explain the quotient rule for exponents. Use $\dfrac{5^8}{5^2}$ in your explanation.

127. Why is $(-3x^2)(2x^{-5})$ not simplified? What must be done to simplify the expression?

128. How do you know if a number is written in scientific notation?

129. Explain how to convert from scientific to decimal notation and give an example.

130. Explain how to convert from decimal to scientific notation and give an example.

131. Refer to the Blitzer Bonus. Use scientific notation to verify any three of the bulleted items on ways to spend \$1 trillion.

Critical Thinking Exercises

Make Sense? *In Exercises 132–135, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

132. There are many exponential expressions that are equal to $36x^{12}$, such as $(6x^6)^2$, $(6x^3)(6x^9)$, $36(x^3)^9$, and $6^2(x^2)^6$.

133. If 5^{-2} is raised to the third power, the result is a number between 0 and 1.

134. The population of Colorado is approximately 4.6×10^{12}.

135. I just finished reading a book that contained approximately 1.04×10^5 words.

In Exercises 136–143, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

136. $4^{-2} < 4^{-3}$

137. $5^{-2} > 2^{-5}$

138. $(-2)^4 = 2^{-4}$

139. $5^2 \cdot 5^{-2} > 2^5 \cdot 2^{-5}$

140. $534.7 = 5.347 \times 10^3$

141. $\dfrac{8 \times 10^{30}}{4 \times 10^{-5}} = 2 \times 10^{25}$

142. $(7 \times 10^5) + (2 \times 10^{-3}) = 9 \times 10^2$

143. $(4 \times 10^3) + (3 \times 10^2) = 4.3 \times 10^3$

144. The mad Dr. Frankenstein has gathered enough bits and pieces (so to speak) for $2^{-1} + 2^{-2}$ of his creature-to-be. Write a fraction that represents the amount of his creature that must still be obtained.

145. If $b^A = MN$, $b^C = M$, and $b^D = N$, what is the relationship among A, C, and D?

146. Our hearts beat approximately 70 times per minute. Express in scientific notation how many times the heart beats over a lifetime of 80 years. Round the decimal factor in your scientific notation answer to two decimal places.

Group Exercise

147. Putting Numbers into Perspective. A large number can be put into perspective by comparing it with another number. For example, we put the \$15.2 trillion national debt (Example 12) and the \$2.17 trillion the government collected in taxes (Exercise 115) by comparing these numbers to the number of U.S. citizens.

For this project, each group member should consult an almanac, a newspaper, or the Internet to find a number greater than one million. Explain to other members of the group the context in which the large number is used. Express the number in scientific notation. Then put the number into perspective by comparing it with another number.

Preview Exercises

Exercises 148–150 will help you prepare for the material covered in the next section.

148. a. Find $\sqrt{16} \cdot \sqrt{4}$.

b. Find $\sqrt{16 \cdot 4}$.

c. Based on your answers to parts (a) and (b), what can you conclude?

149. a. Use a calculator to approximate $\sqrt{300}$ to two decimal places.

b. Use a calculator to approximate $10\sqrt{3}$ to two decimal places.

c. Based on your answers to parts (a) and (b), what can you conclude?

150. a. Simplify: $21x + 10x$.

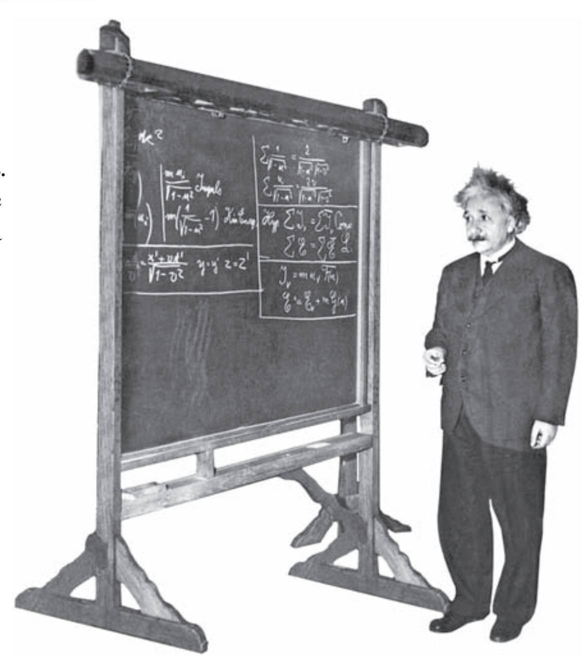
b. Simplify: $21\sqrt{2} + 10\sqrt{2}$.

SECTION 3 Radicals and Rational Exponents

Objectives

1. Evaluate square roots.
2. Simplify expressions of the form $\sqrt{a^2}$.
3. Use the product rule to simplify square roots.
4. Use the quotient rule to simplify square roots.
5. Add and subtract square roots.
6. Rationalize denominators.
7. Evaluate and perform operations with higher roots.
8. Understand and use rational exponents.

This photograph shows mathematical models used by Albert Einstein at a lecture on relativity. Notice the radicals that appear in many of the formulas. Among these models, there is one describing how an astronaut in a moving spaceship ages more slowly than friends who remain on Earth. No description of your world can be complete without roots and radicals. In this section, in addition to reviewing the basics of radical expressions and the use of rational exponents to indicate radicals, you will see how radicals model time dilation for a futuristic high-speed trip to a nearby star.



1 Evaluate square roots.

Square Roots

From our earlier work with exponents, we are aware that the square of both 5 and -5 is 25:

$$5^2 = 25 \quad \text{and} \quad (-5)^2 = 25.$$

The reverse operation of squaring a number is finding the *square root* of the number. For example,

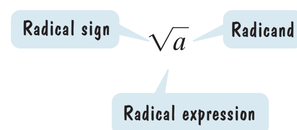
- One square root of 25 is 5 because $5^2 = 25$.
- Another square root of 25 is -5 because $(-5)^2 = 25$.

In general, **if $b^2 = a$, then b is a square root of a.**

The symbol $\sqrt{}$ is used to denote the *nonnegative* or *principal square root* of a number. For example,

- $\sqrt{25} = 5$ because $5^2 = 25$ and 5 is positive.
- $\sqrt{100} = 10$ because $10^2 = 100$ and 10 is positive.

The symbol $\sqrt{}$ that we use to denote the principal square root is called a **radical sign**. The number under the radical sign is called the **radicand**. Together we refer to the radical sign and its radicand as a **radical expression**.



Definition of the Principal Square Root

If a is a nonnegative real number, the nonnegative number b such that $b^2 = a$, denoted by $b = \sqrt{a}$, is the **principal square root** of a.

The symbol $-\sqrt{}$ is used to denote the negative square root of a number. For example,

- $-\sqrt{25} = -5$ because $(-5)^2 = 25$ and -5 is negative.
- $-\sqrt{100} = -10$ because $(-10)^2 = 100$ and -10 is negative.

EXAMPLE 1 Evaluating Square Roots

Evaluate:

a. $\sqrt{64}$ **b.** $-\sqrt{49}$ **c.** $\sqrt{\frac{1}{4}}$ **d.** $\sqrt{9+16}$ **e.** $\sqrt{9} + \sqrt{16}$.

SOLUTION

a. $\sqrt{64} = 8$ The principal square root of 64 is 8. Check: $8^2 = 64$.

b. $-\sqrt{49} = -7$ The negative square root of 49 is -7. Check: $(-7)^2 = 49$.

c. $\sqrt{\frac{1}{4}} = \frac{1}{2}$ The principal square root of $\frac{1}{4}$ is $\frac{1}{2}$. Check: $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

d. $\sqrt{9+16} = \sqrt{25}$ First simplify the expression under the radical sign.

$= 5$ Then take the principal square root of 25, which is 5.

e. $\sqrt{9} + \sqrt{16} = 3 + 4$ $\sqrt{9} = 3$ because $3^2 = 9$. $\sqrt{16} = 4$ because $4^2 = 16$.

$= 7$

GREAT QUESTION!

Is $\sqrt{a+b}$ equal to $\sqrt{a} + \sqrt{b}$?

No. In Example 1, parts (d) and (e), observe that $\sqrt{9+16}$ is not equal to $\sqrt{9} + \sqrt{16}$. In general,

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$$

and

$$\sqrt{a-b} \neq \sqrt{a} - \sqrt{b}.$$

Check Point 1 Evaluate:

a. $\sqrt{81}$ b. $-\sqrt{9}$ c. $\sqrt{\frac{1}{25}}$

d. $\sqrt{36 + 64}$ e. $\sqrt{36} + \sqrt{64}$.

A number that is the square of a rational number is called a **perfect square**. All the radicands in Example 1 and Check Point 1 are perfect squares.

- 64 is a perfect square because $64 = 8^2$. Thus, $\sqrt{64} = 8$.
- $\frac{1}{4}$ is a perfect square because $\frac{1}{4} = \left(\frac{1}{2}\right)^2$. Thus, $\sqrt{\frac{1}{4}} = \frac{1}{2}$.

Let's see what happens to the radical expression $\sqrt{x}$ if x is a negative number. Is the square root of a negative number a real number? For example, consider $\sqrt{-25}$. Is there a real number whose square is -25? No. Thus, $\sqrt{-25}$ is not a real number. In general, **a square root of a negative number is not a real number**.

If a number a is nonnegative ($a \geq 0$), then $(\sqrt{a})^2 = a$. For example,

$$(\sqrt{2})^2 = 2, \quad (\sqrt{3})^2 = 3, \quad (\sqrt{4})^2 = 4, \quad \text{and} \quad (\sqrt{5})^2 = 5.$$

2 Simplify expressions of the form $\sqrt{a^2}$.

Simplifying Expressions of the Form $\sqrt{a^2}$

You may think that $\sqrt{a^2} = a$. However, this is not necessarily true. Consider the following examples:

$$\sqrt{4^2} = \sqrt{16} = 4$$
$$\sqrt{(-4)^2} = \sqrt{16} = 4.$$

The result is not -4, but rather the absolute value of -4, or 4.

Here is a rule for simplifying expressions of the form $\sqrt{a^2}$:

Simplifying $\sqrt{a^2}$

For any real number a,

$$\sqrt{a^2} = |a|.$$

In words, the principal square root of a^2 is the absolute value of a.

For example, $\sqrt{6^2} = |6| = 6$ and $\sqrt{(-6)^2} = |-6| = 6$.

3 Use the product rule to simplify square roots.

The Product Rule for Square Roots

A rule for multiplying square roots can be generalized by comparing $\sqrt{25} \cdot \sqrt{4}$ and $\sqrt{25 \cdot 4}$. Notice that

$$\sqrt{25} \cdot \sqrt{4} = 5 \cdot 2 = 10 \quad \text{and} \quad \sqrt{25 \cdot 4} = \sqrt{100} = 10.$$

Because we obtain 10 in both situations, the original radical expressions must be equal. That is,

$$\sqrt{25} \cdot \sqrt{4} = \sqrt{25 \cdot 4}.$$

This result is a special case of the **product rule for square roots** that can be generalized as follows:

The Product Rule for Square Roots

If a and b represent nonnegative real numbers, then

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \quad \text{and} \quad \sqrt{a} \cdot \sqrt{b} = \sqrt{ab}.$$

The square root of a product is the product of the square roots.

The product of two square roots is the square root of the product of the radicands.

A square root is **simplified** when its radicand has no factors other than 1 that are perfect squares. For example, $\sqrt{500}$ is not simplified because it can be expressed as $\sqrt{100 \cdot 5}$ and 100 is a perfect square. Example 2 shows how the product rule is used to remove from the square root any perfect squares that occur as factors.

EXAMPLE 2 Using the Product Rule to Simplify Square Roots

Simplify:

a. $\sqrt{500}$ **b.** $\sqrt{6x} \cdot \sqrt{3x}$.

SOLUTION

a.
$$\sqrt{500} = \sqrt{100 \cdot 5}$$
Factor 500. 100 is the greatest perfect square factor.
$$= \sqrt{100}\sqrt{5}$$
Use the product rule: $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.
$$= 10\sqrt{5}$$
Write $\sqrt{100}$ as 10. We read $10\sqrt{5}$ as "ten times the square root of 5."

b. We can simplify $\sqrt{6x} \cdot \sqrt{3x}$ using the product rule only if $6x$ and $3x$ represent nonnegative real numbers. Thus, $x \geq 0$.

$$\sqrt{6x} \cdot \sqrt{3x} = \sqrt{6x \cdot 3x}$$
Use the product rule: $\sqrt{a}\sqrt{b} = \sqrt{ab}$.
$$= \sqrt{18x^2}$$
Multiply in the radicand.
$$= \sqrt{9x^2 \cdot 2}$$
Factor 18. 9 is the greatest perfect square factor.
$$= \sqrt{9x^2}\sqrt{2}$$
Use the product rule: $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.
$$= \sqrt{9}\sqrt{x^2}\sqrt{2}$$
Use the product rule to write $\sqrt{9x^2}$ as the product of two square roots.
$$= 3x\sqrt{2}$$
$\sqrt{x^2} = |x| = x$ because $x \geq 0$.

GREAT QUESTION!

When simplifying square roots, what happens if I use a perfect square factor that isn't the greatest perfect square factor possible?

You'll need to simplify even further. For example, consider the following factorization:

$$\sqrt{500} = \sqrt{25 \cdot 20} = \sqrt{25}\sqrt{20} = 5\sqrt{20}.$$

25 is a perfect square factor of 500, but not the greatest perfect square factor.

Because 20 contains a perfect square factor, 4, the simplification is not complete.

$$5\sqrt{20} = 5\sqrt{4 \cdot 5} = 5\sqrt{4}\sqrt{5} = 5 \cdot 2\sqrt{5} = 10\sqrt{5}$$

Although the result checks with our simplification using $\sqrt{500} = \sqrt{100 \cdot 5}$, **more work is required when the greatest perfect square factor is not used**.

Check Point 2 Simplify:

a. $\sqrt{75}$ **b.** $\sqrt{5x} \cdot \sqrt{10x}$.

4 Use the quotient rule to simplify square roots.

The Quotient Rule for Square Roots

Another property for square roots involves division.

The Quotient Rule for Square Roots

If a and b represent nonnegative real numbers and $b \neq 0$, then

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad \text{and} \quad \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}.$$

The square root of a quotient is the quotient of the square roots.

The quotient of two square roots is the square root of the quotient of the radicands.

EXAMPLE 3 Using the Quotient Rule to Simplify Square Roots

Simplify:

a. $\sqrt{\dfrac{100}{9}}$ **b.** $\dfrac{\sqrt{48x^3}}{\sqrt{6x}}$.

SOLUTION

a. $\sqrt{\dfrac{100}{9}} = \dfrac{\sqrt{100}}{\sqrt{9}} = \dfrac{10}{3}$

b. We can simplify the quotient of $\sqrt{48x^3}$ and $\sqrt{6x}$ using the quotient rule only if $48x^3$ and $6x$ represent nonnegative real numbers and $6x \neq 0$. Thus, $x > 0$.

$$\frac{\sqrt{48x^3}}{\sqrt{6x}} = \sqrt{\frac{48x^3}{6x}} = \sqrt{8x^2} = \sqrt{4x^2}\sqrt{2} = \sqrt{4}\sqrt{x^2}\sqrt{2} = 2x\sqrt{2}$$

$\sqrt{x^2} = |x| = x$ because $x > 0$.

Check Point 3 Simplify:

a. $\sqrt{\dfrac{25}{16}}$ **b.** $\dfrac{\sqrt{150x^3}}{\sqrt{2x}}$.

5 Add and subtract square roots.

Adding and Subtracting Square Roots

Two or more square roots can be combined using the distributive property provided that they have the same radicand. Such radicals are called **like radicals**. For example,

$$7\sqrt{11} + 6\sqrt{11} = (7 + 6)\sqrt{11} = 13\sqrt{11}.$$

7 square roots of 11 plus 6 square roots of 11 result in 13 square roots of 11.

GREAT QUESTION!

Should like radicals remind me of like terms?

Yes. Adding or subtracting like radicals is similar to adding or subtracting like terms:

$$7x + 6x = 13x$$

and

$$7\sqrt{11} + 6\sqrt{11} = 13\sqrt{11}.$$

EXAMPLE 4 Adding and Subtracting Like Radicals

Add or subtract as indicated:

a. $7\sqrt{2} + 5\sqrt{2}$ **b.** $\sqrt{5x} - 7\sqrt{5x}$.

SOLUTION

a. $7\sqrt{2} + 5\sqrt{2} = (7 + 5)\sqrt{2}$ Apply the distributive property.

$= 12\sqrt{2}$ Simplify.

b. $\sqrt{5x} - 7\sqrt{5x} = 1\sqrt{5x} - 7\sqrt{5x}$ Write $\sqrt{5x}$ as $1\sqrt{5x}$.

$= (1 - 7)\sqrt{5x}$ Apply the distributive property.

$= -6\sqrt{5x}$ Simplify.

Check Point 4 Add or subtract as indicated:

a. $8\sqrt{13} + 9\sqrt{13}$ **b.** $\sqrt{17x} - 20\sqrt{17x}$.

In some cases, radicals can be combined once they have been simplified. For example, to add $\sqrt{2}$ and $\sqrt{8}$, we can write $\sqrt{8}$ as $\sqrt{4\cdot 2}$ because 4 is a perfect square factor of 8.

$$\sqrt{2} + \sqrt{8} = \sqrt{2} + \sqrt{4\cdot 2} = 1\sqrt{2} + 2\sqrt{2} = (1 + 2)\sqrt{2} = 3\sqrt{2}$$

EXAMPLE 5 Combining Radicals That First Require Simplification

Add or subtract as indicated:

a. $7\sqrt{3} + \sqrt{12}$ **b.** $4\sqrt{50x} - 6\sqrt{32x}$.

SOLUTION

a. $7\sqrt{3} + \sqrt{12}$

$= 7\sqrt{3} + \sqrt{4 \cdot 3}$ Split 12 into two factors such that one is a perfect square.

$= 7\sqrt{3} + 2\sqrt{3}$ $\sqrt{4 \cdot 3} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$

$= (7 + 2)\sqrt{3}$ Apply the distributive property. You will find that this step is usually done mentally.

$= 9\sqrt{3}$ Simplify.

b. $4\sqrt{50x} - 6\sqrt{32x}$

$= 4\sqrt{25 \cdot 2x} - 6\sqrt{16 \cdot 2x}$ 25 is the greatest perfect square factor of $50x$ and 16 is the greatest perfect square factor of $32x$.

$= 4 \cdot 5\sqrt{2x} - 6 \cdot 4\sqrt{2x}$ $\sqrt{25 \cdot 2x} = \sqrt{25}\sqrt{2x} = 5\sqrt{2x}$ and $\sqrt{16 \cdot 2x} = \sqrt{16}\sqrt{2x} = 4\sqrt{2x}$.

$= 20\sqrt{2x} - 24\sqrt{2x}$ Multiply: $4 \cdot 5 = 20$ and $6 \cdot 4 = 24$.

$= (20 - 24)\sqrt{2x}$ Apply the distributive property.

$= -4\sqrt{2x}$ Simplify.

Check Point 5 Add or subtract as indicated:

a. $5\sqrt{27} + \sqrt{12}$ **b.** $6\sqrt{18x} - 4\sqrt{8x}$.

6 Rationalize denominators.

Rationalizing Denominators

The calculator screen in **Figure 11** shows approximate values for $\frac{1}{\sqrt{3}}$ and $\frac{\sqrt{3}}{3}$. The two approximations are the same. This is not a coincidence:

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \boxed{\frac{\sqrt{3}}{\sqrt{3}}} = \frac{\sqrt{3}}{\sqrt{9}} = \frac{\sqrt{3}}{3}.$$

Any number divided by itself is 1. Multiplication by 1 does not change the value of $\frac{1}{\sqrt{3}}$.

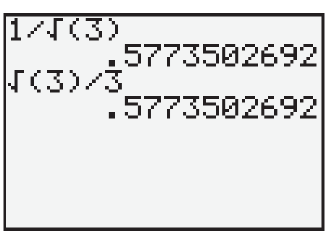


FIGURE 11 The calculator screen shows approximate values for $\frac{1}{\sqrt{3}}$ and $\frac{\sqrt{3}}{3}$.

This process involves rewriting a radical expression as an equivalent expression in which the denominator no longer contains any radicals. The process is called **rationalizing the denominator**. If the denominator consists of the square root of a natural number that is not a perfect square, **multiply the numerator and the denominator by the smallest number that produces the square root of a perfect square in the denominator**.

EXAMPLE 6 Rationalizing Denominators

Rationalize the denominator:

a. $\frac{15}{\sqrt{6}}$ **b.** $\frac{12}{\sqrt{8}}$.

GREAT QUESTION!

What exactly does rationalizing a denominator do to an irrational number in the denominator?

Rationalizing a numerical denominator makes that denominator a rational number.

SOLUTION

a. If we multiply the numerator and the denominator of $\frac{15}{\sqrt{6}}$ by $\sqrt{6}$, the denominator becomes $\sqrt{6}\cdot\sqrt{6} = \sqrt{36} = 6$. Therefore, we multiply by 1, choosing $\frac{\sqrt{6}}{\sqrt{6}}$ for 1.

$$\frac{15}{\sqrt{6}} = \underbrace{\frac{15}{\sqrt{6}}\cdot\frac{\sqrt{6}}{\sqrt{6}}}_{\text{Multiply by 1.}} = \frac{15\sqrt{6}}{\sqrt{36}} = \frac{15\sqrt{6}}{6} = \underbrace{\frac{5\sqrt{6}}{2}}_{\text{Simplify: } \frac{15}{6} = \frac{15\div 3}{6\div 3} = \frac{5}{2}.}$$

b. The *smallest* number that will produce the square root of a perfect square in the denominator of $\frac{12}{\sqrt{8}}$ is $\sqrt{2}$, because $\sqrt{8}\cdot\sqrt{2} = \sqrt{16} = 4$. We multiply by 1, choosing $\frac{\sqrt{2}}{\sqrt{2}}$ for 1.

$$\frac{12}{\sqrt{8}} = \frac{12}{\sqrt{8}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{12\sqrt{2}}{\sqrt{16}} = \frac{12\sqrt{2}}{4} = 3\sqrt{2}$$

Check Point 6 Rationalize the denominator:

a. $\frac{5}{\sqrt{3}}$ **b.** $\frac{6}{\sqrt{12}}$.

Radical expressions that involve the sum and difference of the same two terms are called **conjugates**. Thus,

$$\sqrt{a} + \sqrt{b} \quad \text{and} \quad \sqrt{a} - \sqrt{b}$$

are conjugates. Conjugates are used to rationalize denominators because the product of such a pair contains no radicals:

Multiply each term of $\sqrt{a}-\sqrt{b}$ by each term of $\sqrt{a}+\sqrt{b}$.

$$\begin{aligned}
&(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})\\
&= \underbrace{\sqrt{a}(\sqrt{a}-\sqrt{b})}_{\text{Distribute } \sqrt{a} \text{ over } \sqrt{a}-\sqrt{b}.} + \underbrace{\sqrt{b}(\sqrt{a}-\sqrt{b})}_{\text{Distribute } \sqrt{b} \text{ over } \sqrt{a}-\sqrt{b}.}\\
&= \sqrt{a}\cdot\sqrt{a} - \sqrt{a}\cdot\sqrt{b} + \sqrt{b}\cdot\sqrt{a} - \sqrt{b}\cdot\sqrt{b}\\
&= (\sqrt{a})^2 \underbrace{- \sqrt{ab} + \sqrt{ab}}_{-\sqrt{ab}+\sqrt{ab}=0} - (\sqrt{b})^2\\
&= (\sqrt{a})^2 - (\sqrt{b})^2\\
&= a - b.
\end{aligned}$$

Multiplying Conjugates

$$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b$$

How can we rationalize a denominator if the denominator contains two terms with one or more square roots? **Multiply the numerator and the denominator by the conjugate of the denominator.** Here are three examples of such expressions:

- $\dfrac{7}{5+\sqrt{3}}$ The conjugate of the denominator is $5-\sqrt{3}$.
- $\dfrac{8}{3\sqrt{2}-4}$ The conjugate of the denominator is $3\sqrt{2}+4$.
- $\dfrac{h}{\sqrt{x+h}-\sqrt{x}}$ The conjugate of the denominator is $\sqrt{x+h}+\sqrt{x}$.

The product of the denominator and its conjugate is found using the formula

$$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b.$$

The simplified product will not contain a radical.

EXAMPLE 7 Rationalizing a Denominator Containing Two Terms

Rationalize the denominator: $\dfrac{7}{5+\sqrt{3}}$.

SOLUTION

The conjugate of the denominator is $5-\sqrt{3}$. If we multiply the numerator and denominator by $5-\sqrt{3}$, the simplified denominator will not contain a radical. Therefore, we multiply by 1, choosing $\dfrac{5-\sqrt{3}}{5-\sqrt{3}}$ for 1.

$$\frac{7}{5+\sqrt{3}} = \underbrace{\frac{7}{5+\sqrt{3}} \cdot \frac{5-\sqrt{3}}{5-\sqrt{3}}}_{\text{Multiply by 1.}} = \underbrace{\frac{7(5-\sqrt{3})}{5^2-(\sqrt{3})^2}}_{(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = (\sqrt{a})^2-(\sqrt{b})^2} = \frac{7(5-\sqrt{3})}{25-3}$$

$$= \frac{7(5-\sqrt{3})}{22} \text{ or } \frac{35-7\sqrt{3}}{22}$$

In either form of the answer, there is no radical in the denominator.

Check Point 7 Rationalize the denominator: $\dfrac{8}{4+\sqrt{5}}$.

7 Evaluate and perform operations with higher roots.

Other Kinds of Roots

We define the **principal *n*th root** of a real number a, symbolized by $\sqrt[n]{a}$, as follows:

Definition of the Principal *n*th Root of a Real Number

$$\sqrt[n]{a} = b \text{ means that } b^n = a.$$

If n, the **index**, is even, then a is nonnegative ($a \geq 0$) and b is also nonnegative ($b \geq 0$). If n is odd, a and b can be any real numbers.

For example,

$$\sqrt[3]{64} = 4 \text{ because } 4^3 = 64 \quad \text{and} \quad \sqrt[5]{-32} = -2 \text{ because } (-2)^5 = -32.$$

The same vocabulary that we learned for square roots applies to nth roots. The symbol $\sqrt[n]{}$ is called a **radical** and the expression under the radical is called the **radicand**.

GREAT QUESTION!

Should I know the higher roots of certain numbers by heart?

Some higher roots occur so frequently that you might want to memorize them.

Cube Roots	
$\sqrt[3]{1} = 1$	$\sqrt[3]{125} = 5$
$\sqrt[3]{8} = 2$	$\sqrt[3]{216} = 6$
$\sqrt[3]{27} = 3$	$\sqrt[3]{1000} = 10$
$\sqrt[3]{64} = 4$	

Fourth Roots
$\sqrt[4]{1} = 1$
$\sqrt[4]{16} = 2$
$\sqrt[4]{81} = 3$
$\sqrt[4]{256} = 4$
$\sqrt[4]{625} = 5$

Fifth Roots
$\sqrt[5]{1} = 1$
$\sqrt[5]{32} = 2$
$\sqrt[5]{243} = 3$

A number that is the nth power of a rational number is called a **perfect *n*th power**. For example, 8 is a perfect third power, or perfect cube, because $8 = 2^3$. Thus, $\sqrt[3]{8} = \sqrt[3]{2^3} = 2$. In general, one of the following rules can be used to find the nth root of a perfect nth power:

Finding *n*th Roots of Perfect *n*th Powers

If n is odd, $\sqrt[n]{a^n} = a$.
If n is even, $\sqrt[n]{a^n} = |a|$.

For example,

$$\sqrt[3]{(-2)^3} = -2 \qquad \text{and} \qquad \sqrt[4]{(-2)^4} = |-2| = 2.$$

Absolute value is not needed with odd roots, but is necessary with even roots.

The Product and Quotient Rules for Other Roots

The product and quotient rules apply to cube roots, fourth roots, and all higher roots.

The Product and Quotient Rules for *n*th Roots

For all real numbers a and b, where the indicated roots represent real numbers,

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \qquad \text{and} \qquad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

The nth root of a product is the product of the nth roots.

The product of two nth roots is the nth root of the product of the radicands.

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}},\ b \neq 0 \qquad \text{and} \qquad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}},\ b \neq 0.$$

The nth root of a quotient is the quotient of the nth roots.

The quotient of two nth roots is the nth root of the quotient of the radicands.

EXAMPLE 8 Simplifying, Multiplying, and Dividing Higher Roots

Simplify: **a.** $\sqrt[3]{24}$ **b.** $\sqrt[4]{8} \cdot \sqrt[4]{4}$ **c.** $\sqrt[4]{\frac{81}{16}}$.

SOLUTION

a. $\sqrt[3]{24} = \sqrt[3]{8 \cdot 3}$ Find the greatest perfect cube that is a factor of 24. $2^3 = 8$, so 8 is a perfect cube and is the greatest perfect cube factor of 24.

$= \sqrt[3]{8} \cdot \sqrt[3]{3}$ $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

$= 2\sqrt[3]{3}$ $\sqrt[3]{8} = 2$

b. $\sqrt[4]{8} \cdot \sqrt[4]{4} = \sqrt[4]{8 \cdot 4}$ $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

$= \sqrt[4]{32}$ Find the greatest perfect fourth power that is a factor of 32.

$= \sqrt[4]{16 \cdot 2}$ $2^4 = 16$, so 16 is a perfect fourth power and is the greatest perfect fourth power that is a factor of 32.

$= \sqrt[4]{16} \cdot \sqrt[4]{2}$ $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

$= 2\sqrt[4]{2}$ $\sqrt[4]{16} = 2$

c. $\sqrt[4]{\dfrac{81}{16}} = \dfrac{\sqrt[4]{81}}{\sqrt[4]{16}}$ $\quad \sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

$= \dfrac{3}{2}$ $\quad \sqrt[4]{81} = 3$ because $3^4 = 81$ and $\sqrt[4]{16} = 2$ because $2^4 = 16$.

Check Point 8 Simplify:

a. $\sqrt[3]{40}$ **b.** $\sqrt[5]{8} \cdot \sqrt[5]{8}$ **c.** $\sqrt[3]{\dfrac{125}{27}}$.

We have seen that adding and subtracting square roots often involves simplifying terms. The same idea applies to adding and subtracting higher roots.

EXAMPLE 9 Combining Cube Roots

Subtract: $5\sqrt[3]{16} - 11\sqrt[3]{2}$.

SOLUTION

$5\sqrt[3]{16} - 11\sqrt[3]{2}$

$= 5\sqrt[3]{8 \cdot 2} - 11\sqrt[3]{2}$ — Factor 16. 8 is the greatest perfect cube factor: $2^3 = 8$ and $\sqrt[3]{8} = 2$.

$= 5 \cdot 2\sqrt[3]{2} - 11\sqrt[3]{2}$ — $\sqrt[3]{8 \cdot 2} = \sqrt[3]{8}\sqrt[3]{2} = 2\sqrt[3]{2}$

$= 10\sqrt[3]{2} - 11\sqrt[3]{2}$ — Multiply: $5 \cdot 2 = 10$.

$= (10 - 11)\sqrt[3]{2}$ — Apply the distributive property.

$= -1\sqrt[3]{2}$ or $-\sqrt[3]{2}$ — Simplify.

Check Point 9 Subtract: $3\sqrt[3]{81} - 4\sqrt[3]{3}$.

8 Understand and use rational exponents.

Rational Exponents

We define rational exponents so that their properties are the same as the properties for integer exponents. For example, we know that exponents are multiplied when an exponential expression is raised to a power. For this to be true,

$$\left(7^{\frac{1}{2}}\right)^2 = 7^{\frac{1}{2} \cdot 2} = 7^1 = 7.$$

We also know that

$$(\sqrt{7})^2 = \sqrt{7} \cdot \sqrt{7} = \sqrt{49} = 7.$$

Can you see that the square of both $7^{\frac{1}{2}}$ and $\sqrt{7}$ is 7? It is reasonable to conclude that

$$7^{\frac{1}{2}} \quad \text{means} \quad \sqrt{7}.$$

We can generalize the fact that $7^{\frac{1}{2}}$ means $\sqrt{7}$ with the following definition:

The Definition of $a^{\frac{1}{n}}$

If $\sqrt[n]{a}$ represents a real number, where $n \geq 2$ is an integer, then

$$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

The denominator of the rational exponent is the radical's index.

Furthermore,

$$a^{-\frac{1}{n}} = \frac{1}{a^{\frac{1}{n}}} = \frac{1}{\sqrt[n]{a}}, \quad a \neq 0.$$

EXAMPLE 10 Using the Definition of $a^{\frac{1}{n}}$

Simplify:

a. $64^{\frac{1}{2}}$ **b.** $125^{\frac{1}{3}}$ **c.** $-16^{\frac{1}{4}}$ **d.** $(-27)^{\frac{1}{3}}$ **e.** $64^{-\frac{1}{3}}$.

SOLUTION

a. $64^{\frac{1}{2}} = \sqrt{64} = 8$

b. $125^{\frac{1}{3}} = \sqrt[3]{125} = 5$

The denominator is the index.

c. $-16^{\frac{1}{4}} = -(\sqrt[4]{16}) = -2$

The base is 16 and the negative sign is not affected by the exponent.

d. $(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = -3$

Parentheses show that the base is −27 and that the negative sign is affected by the exponent.

e. $64^{-\frac{1}{3}} = \dfrac{1}{64^{\frac{1}{3}}} = \dfrac{1}{\sqrt[3]{64}} = \dfrac{1}{4}$

Check Point 10 Simplify:

a. $25^{\frac{1}{2}}$ **b.** $8^{\frac{1}{3}}$ **c.** $-81^{\frac{1}{4}}$ **d.** $(-8)^{\frac{1}{3}}$ **e.** $27^{-\frac{1}{3}}$.

In Example 10 and Check Point 10, each rational exponent had a numerator of 1. If the numerator is some other integer, we still want to multiply exponents when raising a power to a power. For this reason,

$$a^{\frac{2}{3}} = (a^{\frac{1}{3}})^2 \quad \text{and} \quad a^{\frac{2}{3}} = (a^2)^{\frac{1}{3}}.$$

This means $(\sqrt[3]{a})^2$.

This means $\sqrt[3]{a^2}$.

Thus,

$$a^{\frac{2}{3}} = \left(\sqrt[3]{a}\right)^2 = \sqrt[3]{a^2}.$$

Do you see that the denominator, 3, of the rational exponent is the same as the index of the radical? The numerator, 2, of the rational exponent serves as an exponent in each of the two radical forms. We generalize these ideas with the following definition:

The Definition of $a^{\frac{m}{n}}$

If $\sqrt[n]{a}$ represents a real number and $\dfrac{m}{n}$ is a positive rational number, $n \geq 2$, then

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m.$$

Also,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

Furthermore, if $a^{-\frac{m}{n}}$ is a nonzero real number, then

$$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}.$$

The first form of the definition of $a^{\frac{m}{n}}$, shown again below, involves taking the root first. This form is often preferable because smaller numbers are involved. Notice that the rational exponent consists of two parts, indicated by the following voice balloons:

The numerator is the exponent.

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m.$$

The denominator is the radical's index.

EXAMPLE 11 Using the Definition of $a^{\frac{m}{n}}$

Simplify:

a. $27^{\frac{2}{3}}$ **b.** $9^{\frac{3}{2}}$ **c.** $81^{-\frac{3}{4}}$.

SOLUTION

a. $27^{\frac{2}{3}} = (\sqrt[3]{27})^2 = 3^2 = 9$

b. $9^{\frac{3}{2}} = (\sqrt{9})^3 = 3^3 = 27$

c. $81^{-\frac{3}{4}} = \dfrac{1}{81^{\frac{3}{4}}} = \dfrac{1}{(\sqrt[4]{81})^3} = \dfrac{1}{3^3} = \dfrac{1}{27}$

TECHNOLOGY

Here are the calculator keystroke sequences for $81^{-\frac{3}{4}}$:

Many Scientific Calculators

81 $\boxed{y^x}$ $\boxed{(}$ 3 $\boxed{+/-}$ $\boxed{\div}$ 4 $\boxed{)}$ $\boxed{=}$

Many Graphing Calculators

81 $\boxed{\wedge}$ $\boxed{(}$ $\boxed{(-)}$ 3 $\boxed{\div}$ 4 $\boxed{)}$ $\boxed{\text{ENTER}}$.

Check Point 11 Simplify:

a. $27^{\frac{4}{3}}$ **b.** $4^{\frac{3}{2}}$ **c.** $32^{-\frac{2}{5}}$.

Properties of exponents can be applied to expressions containing rational exponents.

EXAMPLE 12 Simplifying Expressions with Rational Exponents

Simplify using properties of exponents:

a. $\left(5x^{\frac{1}{2}}\right)\left(7x^{\frac{3}{4}}\right)$ **b.** $\dfrac{32x^{\frac{5}{3}}}{16x^{\frac{3}{4}}}$.

SOLUTION

a. $\left(5x^{\frac{1}{2}}\right)\left(7x^{\frac{3}{4}}\right) = 5 \cdot 7x^{\frac{1}{2}} \cdot x^{\frac{3}{4}}$ Group numerical factors and group variable factors with the same base.

$= 35x^{\frac{1}{2}+\frac{3}{4}}$ When multiplying expressions with the same base, add the exponents.

$= 35x^{\frac{5}{4}}$ $\frac{1}{2} + \frac{3}{4} = \frac{2}{4} + \frac{3}{4} = \frac{5}{4}$

b. $\dfrac{32x^{\frac{5}{3}}}{16x^{\frac{3}{4}}} = \left(\dfrac{32}{16}\right)\left(\dfrac{x^{\frac{5}{3}}}{x^{\frac{3}{4}}}\right)$ Group numerical factors and group variable factors with the same base.

$= 2x^{\frac{5}{3}-\frac{3}{4}}$ When dividing expressions with the same base, subtract the exponents.

$= 2x^{\frac{11}{12}}$ $\frac{5}{3} - \frac{3}{4} = \frac{20}{12} - \frac{9}{12} = \frac{11}{12}$

Check Point 12 Simplify using properties of exponents:

a. $\left(2x^{\frac{4}{3}}\right)\left(5x^{\frac{8}{3}}\right)$ **b.** $\dfrac{20x^4}{5x^{\frac{3}{2}}}$.

Rational exponents are sometimes useful for simplifying radicals by reducing the index.

EXAMPLE 13 Reducing the Index of a Radical

Simplify: $\sqrt[9]{x^3}$.

SOLUTION

$$\sqrt[9]{x^3} = x^{\frac{3}{9}} = x^{\frac{1}{3}} = \sqrt[3]{x}$$

Check Point 13 Simplify: $\sqrt[6]{x^3}$.

Blitzer Bonus || A Radical Idea: Time Is Relative

The Persistence of Memory (1931), Salvador Dali:. © 2011 MOMA/ARS.

What does travel in space have to do with radicals? Imagine that in the future we will be able to travel at velocities approaching the speed of light (approximately 186,000 miles per second). According to Einstein's theory of special relativity, time would pass more quickly on Earth than it would in the moving spaceship. The special-relativity equation

$$R_a = R_f\sqrt{1 - \left(\frac{v}{c}\right)^2}$$

gives the aging rate of an astronaut, R_a, relative to the aging rate of a friend, R_f, on Earth. In this formula, v is the astronaut's speed and c is the speed of light. As the astronaut's speed approaches the speed of light, we can substitute c for v.

$R_a = R_f\sqrt{1 - \left(\frac{v}{c}\right)^2}$ Einstein's equation gives the aging rate of an astronaut, R_a, relative to the aging rate of a friend, R_f, on Earth.

$R_a = R_f\sqrt{1 - \left(\frac{c}{c}\right)^2}$ The velocity, v, is approaching the speed of light, c, so let $v = c$.

$= R_f\sqrt{1 - 1}$ $\left(\frac{c}{c}\right)^2 = 1^2 = 1 \cdot 1 = 1$

$= R_f\sqrt{0}$ Simplify the radicand: $1 - 1 = 0$.

$= R_f \cdot 0$ $\sqrt{0} = 0$

$= 0$ Multiply: $R_f \cdot 0 = 0$.

Close to the speed of light, the astronaut's aging rate, R_a, relative to a friend, R_f, on Earth is nearly 0. What does this mean? As we age here on Earth, the space traveler would barely get older. The space traveler would return to an unknown futuristic world in which friends and loved ones would be long gone.

CONCEPT AND VOCABULARY CHECK

Fill in each blank so that the resulting statement is true.

1. The symbol $\sqrt{\ }$ is used to denote the nonnegative, or ________, square root of a number.
2. $\sqrt{64} = 8$ because ___ $= 64$.
3. $\sqrt{a^2} =$ ___
4. The product rule for square roots states that if a and b are nonnegative, then $\sqrt{ab} =$ ______.
5. The quotient rule for square roots states that if a and b are nonnegative and $b \neq 0$, then $\sqrt{\frac{a}{b}} = \frac{}{\_\_\_\_}$.
6. $8\sqrt{3} + 10\sqrt{3} =$ ______
7. $\sqrt{3} + \sqrt{75} = \sqrt{3} + \sqrt{25 \cdot 3} = \sqrt{3} +$ __$\sqrt{3} =$ _____
8. The conjugate of $7 + \sqrt{3}$ is _______.
9. We rationalize the denominator of $\frac{5}{\sqrt{10} - \sqrt{2}}$ by multiplying the numerator and denominator by _________.
10. In the expression $\sqrt[3]{64}$, the number 3 is called the ______ and the number 64 is called the ________.
11. $\sqrt[5]{-32} = -2$ because ______ $= -32$.
12. If n is odd, $\sqrt[n]{a^n} =$ ____.
    If n is even, $\sqrt[n]{a^n} =$ ____.
13. $a^{\frac{1}{n}} =$ _____
14. $16^{\frac{3}{4}} = (\sqrt[4]{16})^3 = (\_\_)^3 =$ ___

EXERCISE SET 3

Practice Exercises

Evaluate each expression in Exercises 1–12, or indicate that the root is not a real number.

1. $\sqrt{36}$
2. $\sqrt{25}$
3. $-\sqrt{36}$
4. $-\sqrt{25}$
5. $\sqrt{-36}$
6. $\sqrt{-25}$
7. $\sqrt{25 - 16}$
8. $\sqrt{144 + 25}$
9. $\sqrt{25} - \sqrt{16}$
10. $\sqrt{144} + \sqrt{25}$
11. $\sqrt{(-13)^2}$
12. $\sqrt{(-17)^2}$

Use the product rule to simplify the expressions in Exercises 13–22. In Exercises 17–22, assume that variables represent nonnegative real numbers.

13. $\sqrt{50}$
14. $\sqrt{27}$
15. $\sqrt{45x^2}$
16. $\sqrt{125x^2}$
17. $\sqrt{2x} \cdot \sqrt{6x}$
18. $\sqrt{10x} \cdot \sqrt{8x}$
19. $\sqrt{x^3}$
20. $\sqrt{y^3}$
21. $\sqrt{2x^2} \cdot \sqrt{6x}$
22. $\sqrt{6x} \cdot \sqrt{3x^2}$

Use the quotient rule to simplify the expressions in Exercises 23–32. Assume that $x > 0$.

23. $\sqrt{\frac{1}{81}}$
24. $\sqrt{\frac{1}{49}}$
25. $\sqrt{\frac{49}{16}}$
26. $\sqrt{\frac{121}{9}}$
27. $\frac{\sqrt{48x^3}}{\sqrt{3x}}$
28. $\frac{\sqrt{72x^3}}{\sqrt{8x}}$
29. $\frac{\sqrt{150x^4}}{\sqrt{3x}}$
30. $\frac{\sqrt{24x^4}}{\sqrt{3x}}$
31. $\frac{\sqrt{200x^3}}{\sqrt{10x^{-1}}}$
32. $\frac{\sqrt{500x^3}}{\sqrt{10x^{-1}}}$

In Exercises 33–44, add or subtract terms whenever possible.

33. $7\sqrt{3} + 6\sqrt{3}$
34. $8\sqrt{5} + 11\sqrt{5}$
35. $6\sqrt{17x} - 8\sqrt{17x}$
36. $4\sqrt{13x} - 6\sqrt{13x}$
37. $\sqrt{8} + 3\sqrt{2}$
38. $\sqrt{20} + 6\sqrt{5}$
39. $\sqrt{50x} - \sqrt{8x}$
40. $\sqrt{63x} - \sqrt{28x}$
41. $3\sqrt{18} + 5\sqrt{50}$
42. $4\sqrt{12} - 2\sqrt{75}$
43. $3\sqrt{8} - \sqrt{32} + 3\sqrt{72} - \sqrt{75}$
44. $3\sqrt{54} - 2\sqrt{24} - \sqrt{96} + 4\sqrt{63}$

In Exercises 45–54, rationalize the denominator.

45. $\frac{1}{\sqrt{7}}$
46. $\frac{2}{\sqrt{10}}$
47. $\frac{\sqrt{2}}{\sqrt{5}}$
48. $\frac{\sqrt{7}}{\sqrt{3}}$
49. $\frac{13}{3 + \sqrt{11}}$
50. $\frac{3}{3 + \sqrt{7}}$
51. $\frac{7}{\sqrt{5} - 2}$
52. $\frac{5}{\sqrt{3} - 1}$
53. $\frac{6}{\sqrt{5} + \sqrt{3}}$
54. $\frac{11}{\sqrt{7} - \sqrt{3}}$

Evaluate each expression in Exercises 55–66, or indicate that the root is not a real number.

55. $\sqrt[3]{125}$
56. $\sqrt[3]{8}$
57. $\sqrt[3]{-8}$
58. $\sqrt[3]{-125}$
59. $\sqrt[4]{-16}$
60. $\sqrt[4]{-81}$
61. $\sqrt[4]{(-3)^4}$
62. $\sqrt[4]{(-2)^4}$
63. $\sqrt[5]{(-3)^5}$
64. $\sqrt[5]{(-2)^5}$
65. $\sqrt[5]{-\frac{1}{32}}$
66. $\sqrt[6]{\frac{1}{64}}$

Simplify the radical expressions in Exercises 67–74 if possible.

67. $\sqrt[3]{32}$
68. $\sqrt[3]{150}$
69. $\sqrt[3]{x^4}$
70. $\sqrt[3]{x^5}$
71. $\sqrt[3]{9} \cdot \sqrt[3]{6}$
72. $\sqrt[3]{12} \cdot \sqrt[3]{4}$
73. $\frac{\sqrt[5]{64x^6}}{\sqrt[5]{2x}}$
74. $\frac{\sqrt[4]{162x^5}}{\sqrt[4]{2x}}$

In Exercises 75–82, add or subtract terms whenever possible.

75. $4\sqrt[5]{2} + 3\sqrt[5]{2}$

76. $6\sqrt[5]{3} + 2\sqrt[5]{3}$

77. $5\sqrt[3]{16} + \sqrt[3]{54}$

78. $3\sqrt[3]{24} + \sqrt[3]{81}$

79. $\sqrt[3]{54xy^3} - y\sqrt[3]{128x}$

80. $\sqrt[3]{24xy^3} - y\sqrt[3]{81x}$

81. $\sqrt{2} + \sqrt[3]{8}$

82. $\sqrt{3} + \sqrt[3]{15}$

In Exercises 83–90, evaluate each expression without using a calculator.

83. $36^{\frac{1}{2}}$

84. $121^{\frac{1}{2}}$

85. $8^{\frac{1}{3}}$

86. $27^{\frac{1}{3}}$

87. $125^{\frac{2}{3}}$

88. $8^{\frac{2}{3}}$

89. $32^{-\frac{4}{5}}$

90. $16^{-\frac{5}{2}}$

In Exercises 91–100, simplify using properties of exponents.

91. $\left(7x^{\frac{1}{3}}\right)\left(2x^{\frac{1}{4}}\right)$

92. $\left(3x^{\frac{2}{3}}\right)\left(4x^{\frac{3}{4}}\right)$

93. $\dfrac{20x^{\frac{1}{2}}}{5x^{\frac{1}{4}}}$

94. $\dfrac{72x^{\frac{3}{4}}}{9x^{\frac{1}{3}}}$

95. $\left(x^{\frac{2}{3}}\right)^3$

96. $\left(x^{\frac{4}{5}}\right)^5$

97. $(25x^4y^6)^{\frac{1}{2}}$

98. $(125x^9y^6)^{\frac{1}{3}}$

99. $\dfrac{\left(3y^{\frac{1}{4}}\right)^3}{y^{\frac{1}{12}}}$

100. $\dfrac{\left(2y^{\frac{1}{5}}\right)^4}{y^{\frac{3}{10}}}$

In Exercises 101–108, simplify by reducing the index of the radical.

101. $\sqrt[4]{5^2}$

102. $\sqrt[4]{7^2}$

103. $\sqrt[3]{x^6}$

104. $\sqrt[4]{x^{12}}$

105. $\sqrt[6]{x^4}$

106. $\sqrt[9]{x^6}$

107. $\sqrt[9]{x^6y^3}$

108. $\sqrt[12]{x^4y^8}$

Practice Plus

In Exercises 109–110, evaluate each expression.

109. $\sqrt[3]{\sqrt[4]{16} + \sqrt{625}}$

110. $\sqrt[3]{\sqrt{\sqrt{169} + \sqrt{9}} + \sqrt{\sqrt[3]{1000} + \sqrt[3]{216}}}$

In Exercises 111–114, simplify each expression. Assume that all variables represent positive numbers.

111. $(49x^{-2}y^4)^{-\frac{1}{2}}\left(xy^{\frac{1}{2}}\right)$

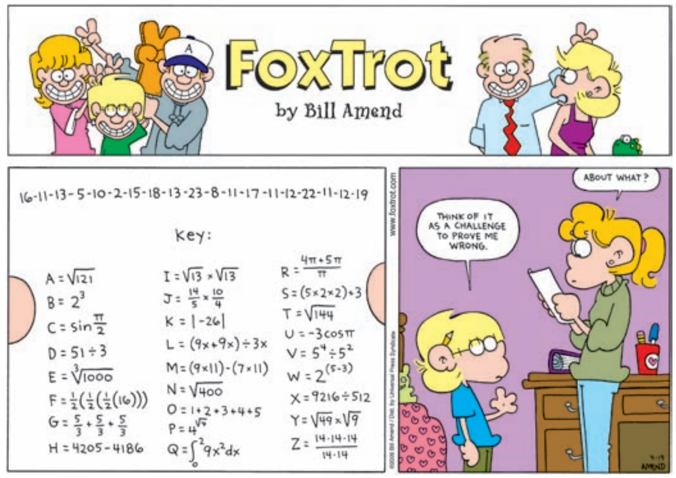
112. $(8x^{-6}y^3)^{\frac{1}{3}}\left(x^{\frac{5}{6}}y^{-\frac{1}{3}}\right)^6$

113. $\left(\dfrac{x^{-\frac{5}{4}}y^{\frac{1}{3}}}{x^{-\frac{3}{4}}}\right)^{-6}$

114. $\left(\dfrac{x^{\frac{1}{2}}y^{-\frac{7}{4}}}{y^{-\frac{5}{4}}}\right)^{-4}$

Application Exercises

115. The popular comic strip *FoxTrot* follows the off-the-wall lives of the Fox family. Youngest son Jason is forever obsessed by his love of math. In the math-themed strip shown at the top of the next column, Jason shares his opinion in a coded message about the mathematical abilities of his sister Paige.



Solve problems A through Z in the left panel. Then decode Jason Fox's message involving his opinion about the mathematical abilities of his sister Paige shown on the first line.

Hints: Here is the solution for problem C and partial solutions for problems Q and U.

These are from trigonometry.

$$C = \sin\frac{\pi}{2} = \sin 90° = 1$$

$$Q = \int_0^2 9x^2\,dx = 3x^3\Big|_0^2 = 3\cdot 2^3 - 3\cdot 0^3 = \underline{\qquad}$$

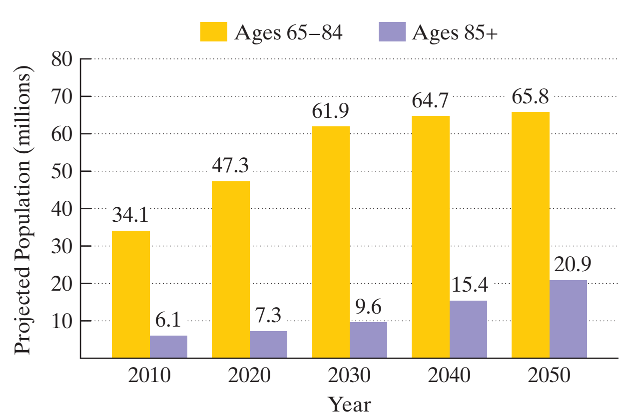
This is from calculus.

$$U = -3\cos\pi = -3\cos 180° = -3(-1) = \underline{\qquad}$$

Note: The comic strip *FoxTrot* is now printed in more than one thousand newspapers. What made cartoonist Bill Amend, a college physics major, put math in the comic? "I always try to use math in the strip to make the joke accessible to anyone," he said. "But if you understand math, hopefully you'll like it that much more!" We highly recommend the math humor in Amend's *FoxTrot* collection *Math, Science, and Unix Underpants* (Andrews McMeel Publishing, 2009).

116. America is getting older. The graph shows the projected elderly U.S. population for ages 65–84 and for ages 85 and older.

Projected Elderly United States Population



Source: U.S. Census Bureau

The formula $E = 5\sqrt{x} + 34.1$ models the projected number of elderly Americans ages 65–84, E, in millions, x years after 2010.

a. Use the formula to find the projected increase in the number of Americans ages 65–84, in millions, from 2020 to 2050. Express this difference in simplified radical form.

b. Use a calculator and write your answer in part (a) to the nearest tenth. Does this rounded decimal overestimate or underestimate the difference in the projected data shown by the bar graph on the previous page? By how much?

117. The early Greeks believed that the most pleasing of all rectangles were **golden rectangles**, whose ratio of width to height is

$$\frac{w}{h} = \frac{2}{\sqrt{5} - 1}.$$

The Parthenon at Athens fits into a golden rectangle once the triangular pediment is reconstructed.

Rationalize the denominator of the golden ratio. Then use a calculator and find the ratio of width to height, correct to the nearest hundredth, in golden rectangles.

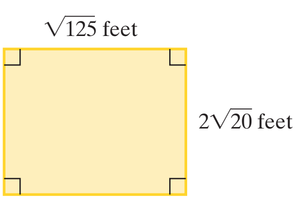
118. Use Einstein's special-relativity equation

$$R_a = R_f\sqrt{1 - \left(\frac{v}{c}\right)^2},$$

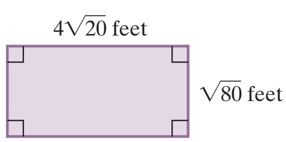
described in the Blitzer Bonus, to solve this exercise. You are moving at 90% of the speed of light. Substitute $0.9c$ for v, your velocity, in the equation. What is your aging rate, correct to two decimal places, relative to a friend on Earth? If you are gone for 44 weeks, approximately how many weeks have passed for your friend?

The perimeter, P, of a rectangle with length l and width w is given by the formula $P = 2l + 2w$. The area, A, is given by the formula $A = lw$. In Exercises 119–120, use these formulas to find the perimeter and area of each rectangle. Express answers in simplified radical form. Remember that perimeter is measured in linear units, such as feet or meters, and area is measured in square units, such as square feet, ft^2, or square meters, m^2.

119.



120.



Writing in Mathematics

121. Explain how to simplify $\sqrt{10} \cdot \sqrt{5}$.

122. Explain how to add $\sqrt{3} + \sqrt{12}$.

123. Describe what it means to rationalize a denominator. Use both $\frac{1}{\sqrt{5}}$ and $\frac{1}{5 + \sqrt{5}}$ in your explanation.

124. What difference is there in simplifying $\sqrt[3]{(-5)^3}$ and $\sqrt[4]{(-5)^4}$?

125. What does $a^{\frac{m}{n}}$ mean?

126. Describe the kinds of numbers that have rational fifth roots.

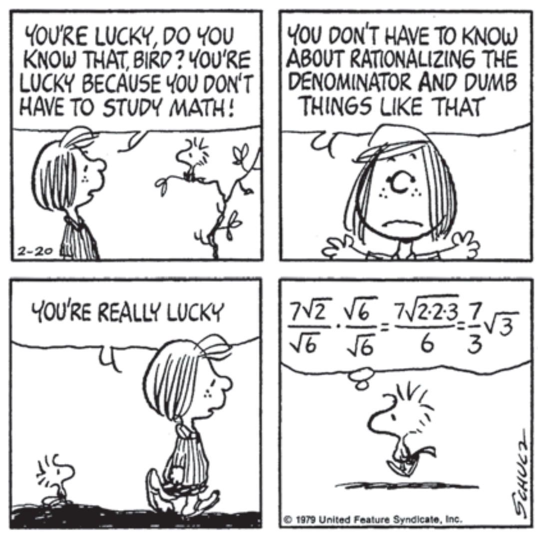
127. Why must a and b represent nonnegative numbers when we write $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$? Is it necessary to use this restriction in the case of $\sqrt[3]{a} \cdot \sqrt[3]{b} = \sqrt[3]{ab}$? Explain.

128. Read the Blitzer Bonus. The future is now: You have the opportunity to explore the cosmos in a starship traveling near the speed of light. The experience will enable you to understand the mysteries of the universe in deeply personal ways, transporting you to unimagined levels of knowing and being. The downside: You return from your two-year journey to a futuristic world in which friends and loved ones are long gone. Do you explore space or stay here on Earth? What are the reasons for your choice?

Critical Thinking Exercises

Make Sense? *In Exercises 129–132, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

129. The joke in this Peanuts cartoon would be more effective if Woodstock had rationalized the denominator correctly in the last frame.



Peanuts © 1978 Peanuts Worldwide LLC. Used by permission of Universal Uclick. All rights reserved.

130. Using my calculator, I determined that $6^7 = 279{,}936$, so 6 must be a seventh root of 279,936.

131. I simplified the terms of $2\sqrt{20} + 4\sqrt{75}$, and then I was able to add the like radicals.

132. When I use the definition for $a^{\frac{m}{n}}$, I usually prefer to first raise a to the m power because smaller numbers are involved.

In Exercises 133–136, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

133. $7^{\frac{1}{2}} \cdot 7^{\frac{1}{2}} = 49$

134. $8^{-\frac{1}{3}} = -2$

135. The cube root of -8 is not a real number.

136. $\frac{\sqrt{20}}{8} = \frac{\sqrt{10}}{4}$

In Exercises 137–138, fill in each box to make the statement true.

137. $(5 + \sqrt{\square})(5 - \sqrt{\square}) = 22$

138. $\sqrt{\square x^{\square}} = 5x^7$

139. Find the exact value of $\sqrt{13 + \sqrt{2} + \frac{7}{3 + \sqrt{2}}}$ without the use of a calculator.

140. Place the correct symbol, $>$ or $<$, in the shaded area between the given numbers. *Do not use a calculator*. Then check your result with a calculator.

a. $3^{\frac{1}{2}} \square 3^{\frac{1}{3}}$ **b.** $\sqrt{7} + \sqrt{18} \square \sqrt{7 + 18}$

141. a. A mathematics professor recently purchased a birthday cake for her son with the inscription

$$\text{Happy}\left(2^{\frac{5}{2}} \cdot 2^{\frac{3}{4}} \div 2^{\frac{1}{4}}\right)\text{th Birthday.}$$

How old is the son?

b. The birthday boy, excited by the inscription on the cake, tried to wolf down the whole thing. Professor Mom, concerned about the possible metamorphosis of her son into a blimp, exclaimed, "Hold on! It is your birthday, so why not take $\frac{8^{-\frac{4}{3}} + 2^{-2}}{16^{-\frac{3}{4}} + 2^{-1}}$ of the cake? I'll eat half of what's left over." How much of the cake did the professor eat?

Preview Exercises

Exercises 142–144 will help you prepare for the material covered in the next section.

142. Multiply: $(2x^3y^2)(5x^4y^7)$.

143. Use the distributive property to multiply:

$$2x^4(8x^4 + 3x).$$

144. Simplify and express the answer in descending powers of x:

$$2x(x^2 + 4x + 5) + 3(x^2 + 4x + 5).$$

SECTION 4 Polynomials

Objectives

1. Understand the vocabulary of polynomials.
2. Add and subtract polynomials.
3. Multiply polynomials.
4. Use FOIL in polynomial multiplication.
5. Use special products in polynomial multiplication.
6. Perform operations with polynomials in several variables.

Can that be Axl, your author's yellow lab, sharing a special moment with a baby chick? And if it is (it is), what possible relevance can this have to polynomials? An answer is promised before you reach the Exercise Set. For now, we open the section by defining and describing polynomials.

Old Dog…New Chicks

How We Define Polynomials

More education results in a higher income. The mathematical models

$$M = 0.6x^3 + 285x^2 - 2256x + 15{,}112$$

$$\text{and} \quad W = -1.2x^3 + 367x^2 - 4900x + 26{,}561$$

describe the median, or middlemost, annual income for men, M, and women, W, who have completed x years of education. We'll be working with these models and the data upon which they are based in the Exercise Set.

The algebraic expressions that appear on the right sides of the models are examples of *polynomials*. A **polynomial** is a single term or the sum of two or more terms containing variables with whole-number exponents. The polynomials above each contain four terms. Equations containing polynomials are used in such diverse areas as science, business, medicine, psychology, and sociology. In this section, we review basic ideas about polynomials and their operations.

1 Understand the vocabulary of polynomials.

How We Describe Polynomials

Consider the polynomial

$$7x^3 - 9x^2 + 13x - 6.$$

We can express $7x^3 - 9x^2 + 13x - 6$ as

$$7x^3 + (-9x^2) + 13x + (-6).$$

The polynomial contains four terms. It is customary to write the terms in the order of descending powers of the variable. This is the **standard form** of a polynomial.

Some polynomials contain only one variable. Each term of such a polynomial in x is of the form ax^n. If $a \neq 0$, the degree of ax^n is n. For example, the degree of the term $7x^3$ is 3.

GREAT QUESTION!

Why doesn't the constant 0 have a degree?

We can express 0 in many ways, including $0x$, $0x^2$, and $0x^3$. It is impossible to assign a single exponent on the variable. This is why 0 has no defined degree.

The Degree of ax^n

If $a \neq 0$, the degree of ax^n is n. The degree of a nonzero constant is 0. The constant 0 has no defined degree.

Here is an example of a polynomial and the degree of each of its four terms:

$$\underbrace{6x^4}_{\text{degree 4}} - \underbrace{3x^3}_{\text{degree 3}} + \underbrace{2x}_{\text{degree 1}} - \underbrace{5}_{\text{degree of nonzero constant: 0}}.$$

Notice that the exponent on x for the term $2x$, meaning $2x^1$, is understood to be 1. For this reason, the degree of $2x$ is 1. You can think of -5 as $-5x^0$; thus, its degree is 0.

A polynomial is simplified when it contains no grouping symbols and no like terms. A simplified polynomial that has exactly one term is called a **monomial**. A **binomial** is a simplified polynomial that has two terms. A **trinomial** is a simplified polynomial with three terms. Simplified polynomials with four or more terms have no special names.

The **degree of a polynomial** is the greatest degree of all the terms of the polynomial. For example, $4x^2 + 3x$ is a binomial of degree 2 because the degree of the first term is 2, and the degree of the other term is less than 2. Also, $7x^5 - 2x^2 + 4$ is a trinomial of degree 5 because the degree of the first term is 5, and the degrees of the other terms are less than 5.

Up to now, we have used x to represent the variable in a polynomial. However, any letter can be used. For example,

- $7x^5 - 3x^3 + 8$ is a polynomial (in x) of degree 5. Because there are three terms, the polynomial is a trinomial.
- $6y^3 + 4y^2 - y + 3$ is a polynomial (in y) of degree 3. Because there are four terms, the polynomial has no special name.
- $z^7 + \sqrt{2}$ is a polynomial (in z) of degree 7. Because there are two terms, the polynomial is a binomial.

We can tie together the threads of our discussion with the formal definition of a polynomial in one variable. In this definition, the coefficients of the terms are represented by a_n (read "a sub n"), a_{n-1} (read "a sub n minus 1"), a_{n-2}, and so on. The small letters to the lower right of each a are called **subscripts** and are *not exponents*. Subscripts are used to distinguish one constant from another when a large and undetermined number of such constants are needed.

Definition of a Polynomial in x

A **polynomial in x** is an algebraic expression of the form

$$a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0,$$

where $a_n, a_{n-1}, a_{n-2}, \ldots, a_1$, and a_0 are real numbers, $a_n \neq 0$, and n is a nonnegative integer. The polynomial is of **degree n**, a_n is the **leading coefficient**, and a_0 is the **constant term**.

2 Add and subtract polynomials.

Adding and Subtracting Polynomials

Polynomials are added and subtracted by combining like terms. For example, we can combine the monomials $-9x^3$ and $13x^3$ using addition as follows:

$$-9x^3 + 13x^3 = (-9 + 13)x^3 = 4x^3.$$

These like terms both contain x to the third power.

Add coefficients and keep the same variable factor, x^3.

EXAMPLE 1 Adding and Subtracting Polynomials

Perform the indicated operations and simplify:

a. $(-9x^3 + 7x^2 - 5x + 3) + (13x^3 + 2x^2 - 8x - 6)$

b. $(7x^3 - 8x^2 + 9x - 6) - (2x^3 - 6x^2 - 3x + 9)$.

SOLUTION

a. $(-9x^3 + 7x^2 - 5x + 3) + (13x^3 + 2x^2 - 8x - 6)$

$= (-9x^3 + 13x^3) + (7x^2 + 2x^2) + (-5x - 8x) + (3 - 6)$ Group like terms.

$= 4x^3 + 9x^2 + (-13x) + (-3)$ Combine like terms.

$= 4x^3 + 9x^2 - 13x - 3$ Simplify.

b. $(7x^3 - 8x^2 + 9x - 6) - (2x^3 - 6x^2 - 3x + 9)$ Rewrite subtraction as addition of the additive inverse.

Change the sign of each coefficient.

$= (7x^3 - 8x^2 + 9x - 6) + (-2x^3 + 6x^2 + 3x - 9)$

$= (7x^3 - 2x^3) + (-8x^2 + 6x^2) + (9x + 3x) + (-6 - 9)$ Group like terms.

$= 5x^3 + (-2x^2) + 12x + (-15)$ Combine like terms.

$= 5x^3 - 2x^2 + 12x - 15$ Simplify. ●●●

GREAT QUESTION!

Can I use a vertical format to add and subtract polynomials?

Yes. Arrange like terms in columns and combine vertically:

$$\begin{array}{r} 7x^3 - 8x^2 + 9x - 6 \\ \underline{-2x^3 + 6x^2 + 3x - 9} \\ 5x^3 - 2x^2 + 12x - 15 \end{array}$$

The like terms can be combined by adding their coefficients and keeping the same variable factor.

Check Point 1 Perform the indicated operations and simplify:

a. $(-17x^3 + 4x^2 - 11x - 5) + (16x^3 - 3x^2 + 3x - 15)$

b. $(13x^3 - 9x^2 - 7x + 1) - (-7x^3 + 2x^2 - 5x + 9)$.

3 Multiply polynomials.

Multiplying Polynomials

The product of two monomials is obtained by using properties of exponents. For example,

$$(-8x^6)(5x^3) = -8 \cdot 5x^{6+3} = -40x^9.$$

Multiply coefficients and add exponents.

Furthermore, we can use the distributive property to multiply a monomial and a polynomial that is not a monomial. For example,

$$3x^4(2x^3 - 7x + 3) = 3x^4 \cdot 2x^3 - 3x^4 \cdot 7x + 3x^4 \cdot 3 = 6x^7 - 21x^5 + 9x^4.$$

Monomial

Trinomial

How do we multiply two polynomials if neither is a monomial? For example, consider

$$(2x + 3)(x^2 + 4x + 5).$$

Binomial

Trinomial

GREAT QUESTION!

Because monomials with the same base and different exponents can be multiplied, can they also be added?

No. Don't confuse adding and multiplying monomials.

Addition:

$$5x^4 + 6x^4 = 11x^4$$

Multiplication:

$$\begin{aligned}(5x^4)(6x^4) &= (5 \cdot 6)(x^4 \cdot x^4) \\ &= 30x^{4+4} \\ &= 30x^8\end{aligned}$$

Only like terms can be added or subtracted, but unlike terms may be multiplied.

Addition:

$5x^4 + 3x^2$ cannot be simplified.

Multiplication:

$$\begin{aligned}(5x^4)(3x^2) &= (5 \cdot 3)(x^4 \cdot x^2) \\ &= 15x^{4+2} \\ &= 15x^6\end{aligned}$$

One way to perform $(2x + 3)(x^2 + 4x + 5)$ is to distribute $2x$ throughout the trinomial

$$2x(x^2 + 4x + 5)$$

and 3 throughout the trinomial

$$3(x^2 + 4x + 5).$$

Then combine the like terms that result.

Multiplying Polynomials When Neither Is a Monomial

Multiply each term of one polynomial by each term of the other polynomial. Then combine like terms.

EXAMPLE 2 Multiplying a Binomial and a Trinomial

Multiply: $(2x + 3)(x^2 + 4x + 5)$.

SOLUTION

$$(2x + 3)(x^2 + 4x + 5)$$

$$= 2x(x^2 + 4x + 5) + 3(x^2 + 4x + 5)$$ Multiply the trinomial by each term of the binomial.

$$= 2x \cdot x^2 + 2x \cdot 4x + 2x \cdot 5 + 3x^2 + 3 \cdot 4x + 3 \cdot 5$$ Use the distributive property.

$$= 2x^3 + 8x^2 + 10x + 3x^2 + 12x + 15$$ Multiply monomials: Multiply coefficients and add exponents.

$$= 2x^3 + 11x^2 + 22x + 15$$ Combine like terms: $8x^2 + 3x^2 = 11x^2$ and $10x + 12x = 22x$.

Another method for performing the multiplication is to use a vertical format similar to that used for multiplying whole numbers.

$$\begin{array}{rrrr}
 & x^2 & + 4x & + 5 \\
 & & 2x & + 3 \\
\hline
 & 3x^2 & + 12x & + 15 \\
2x^3 + & 8x^2 & + 10x & \\
\hline
2x^3 + & 11x^2 & + 22x & + 15
\end{array}$$

$3(x^2 + 4x + 5)$

$2x(x^2 + 4x + 5)$

Write like terms in the same column.

Combine like terms.

Check Point 2 Multiply: $(5x - 2)(3x^2 - 5x + 4)$.

4 Use FOIL in polynomial multiplication.

The Product of Two Binomials: FOIL

Frequently, we need to find the product of two binomials. One way to perform this multiplication is to distribute each term in the first binomial through the second binomial. For example, we can find the product of the binomials $3x + 2$ and $4x + 5$ as follows:

$$(3x + 2)(4x + 5) = 3x(4x + 5) + 2(4x + 5)$$

$$= 3x(4x) + 3x(5) + 2(4x) + 2(5)$$

$$= 12x^2 + 15x + 8x + 10.$$

Distribute $3x$ over $4x + 5$.

Distribute 2 over $4x + 5$.

We'll combine these like terms later. For now, our interest is in how to obtain *each* of these four terms.

We can also find the product of $3x + 2$ and $4x + 5$ using a method called FOIL, which is based on our preceding work. Any two binomials can be quickly multiplied by using the FOIL method, in which **F** represents the product of the **first** terms in each binomial, **O** represents the product of the **outside** terms, **I** represents the product of the **inside** terms, and **L** represents the product of the **last**, or second, terms in each binomial. For example, we can use the FOIL method to find the product of the binomials $3x + 2$ and $4x + 5$ as follows:

$$\underset{\text{inside}\quad\text{outside}}{\overset{\text{first}\quad\text{last}}{(3x+2)(4x+5)}} = \underbrace{\overset{\text{F}}{12x^2}}_{\substack{\text{Product}\\\text{of}\\\text{First}\\\text{terms}}} + \underbrace{\overset{\text{O}}{15x}}_{\substack{\text{Product}\\\text{of}\\\text{Outside}\\\text{terms}}} + \underbrace{\overset{\text{I}}{8x}}_{\substack{\text{Product}\\\text{of}\\\text{Inside}\\\text{terms}}} + \underbrace{\overset{\text{L}}{10}}_{\substack{\text{Product}\\\text{of}\\\text{Last}\\\text{terms}}}$$

$$= 12x^2 + 23x + 10 \qquad \text{Combine like terms.}$$

In general, here's how to use the FOIL method to find the product of $ax + b$ and $cx + d$:

Using the FOIL Method to Multiply Binomials

$$\underset{\text{inside}\quad\text{outside}}{\overset{\text{first}\quad\text{last}}{(ax+b)(cx+d)}} = \underbrace{\overset{\text{F}}{ax \cdot cx}}_{\substack{\text{Product}\\\text{of}\\\text{First}\\\text{terms}}} + \underbrace{\overset{\text{O}}{ax \cdot d}}_{\substack{\text{Product}\\\text{of}\\\text{Outside}\\\text{terms}}} + \underbrace{\overset{\text{I}}{b \cdot cx}}_{\substack{\text{Product}\\\text{of}\\\text{Inside}\\\text{terms}}} + \underbrace{\overset{\text{L}}{b \cdot d}}_{\substack{\text{Product}\\\text{of}\\\text{Last}\\\text{terms}}}$$

EXAMPLE 3 Using the FOIL Method

Multiply: $(3x + 4)(5x - 3)$.

SOLUTION

$$\underset{\text{inside}\quad\text{outside}}{\overset{\text{first}\quad\text{last}}{(3x+4)(5x-3)}} = \overset{\text{F}}{3x \cdot 5x} + \overset{\text{O}}{3x(-3)} + \overset{\text{I}}{4 \cdot 5x} + \overset{\text{L}}{4(-3)}$$

$$= 15x^2 - 9x + 20x - 12$$

$$= 15x^2 + 11x - 12 \qquad \text{Combine like terms.}$$

Check Point 3 Multiply: $(7x - 5)(4x - 3)$.

5 Use special products in polynomial multiplication.

Special Products

There are several products that occur so frequently that it's convenient to memorize the form, or pattern, of these formulas.

GREAT QUESTION!

Do I have to memorize the special products shown in the table on the right?

Not necessarily. Although it's convenient to memorize these forms, the FOIL method can be used on all five examples in the box. To cube $x + 4$, you can first square $x + 4$ using FOIL and then multiply this result by $x + 4$. We suggest memorizing these special forms because they let you multiply far more rapidly than using the FOIL method.

Special Products

Let A and B represent real numbers, variables, or algebraic expressions.

Special Product	Example
Sum and Difference of Two Terms	
$(A + B)(A - B) = A^2 - B^2$	$(2x + 3)(2x - 3) = (2x)^2 - 3^2$ $= 4x^2 - 9$
Squaring a Binomial	
$(A + B)^2 = A^2 + 2AB + B^2$	$(y + 5)^2 = y^2 + 2 \cdot y \cdot 5 + 5^2$ $= y^2 + 10y + 25$
$(A - B)^2 = A^2 - 2AB + B^2$	$(3x - 4)^2$ $= (3x)^2 - 2 \cdot 3x \cdot 4 + 4^2$ $= 9x^2 - 24x + 16$
Cubing a Binomial	
$(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$	$(x + 4)^3$ $= x^3 + 3x^2(4) + 3x(4)^2 + 4^3$ $= x^3 + 12x^2 + 48x + 64$
$(A - B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$	$(x - 2)^3$ $= x^3 - 3x^2(2) + 3x(2)^2 - 2^3$ $= x^3 - 6x^2 + 12x - 8$

6 Perform operations with polynomials in several variables.

Polynomials in Several Variables

A **polynomial in two variables**, x and y, contains the sum of one or more monomials in the form ax^ny^m. The constant, a, is the **coefficient**. The exponents, n and m, represent whole numbers. The **degree** of the monomial ax^ny^m is $n + m$.

Here is an example of a polynomial in two variables:

The coefficients are 7, −17, 1, −6, and 9.

$$7x^2y^3 - 17x^4y^2 + xy - 6y^2 + 9.$$

Degree of monomial: $2 + 3 = 5$

Degree of monomial: $4 + 2 = 6$

Degree of monomial $(1x^1y^1)$: $1 + 1 = 2$

Degree of monomial $(-6x^0y^2)$: $0 + 2 = 2$

Degree of monomial $(9x^0y^0)$: $0 + 0 = 0$

The **degree of a polynomial in two variables** is the highest degree of all its terms. For the preceding polynomial, the degree is 6.

Polynomials containing two or more variables can be added, subtracted, and multiplied just like polynomials that contain only one variable. For example, we can add the monomials $-7xy^2$ and $13xy^2$ as follows:

$$-7xy^2 + 13xy^2 = (-7 + 13)xy^2 = 6xy^2.$$

These like terms both contain the variable factors x and y^2.

Add coefficients and keep the same variable factors, xy^2.

EXAMPLE 4 Multiplying Polynomials in Two Variables

Multiply: **a.** $(x + 4y)(3x - 5y)$ **b.** $(5x + 3y)^2$.

SOLUTION

We will perform the multiplication in part (a) using the FOIL method. We will multiply in part (b) using the formula for the square of a binomial sum, $(A + B)^2$.

a. $(x + 4y)(3x - 5y)$ Multiply these binomials using the FOIL method.

F O I L

$$= (x)(3x) + (x)(-5y) + (4y)(3x) + (4y)(-5y)$$
$$= 3x^2 - 5xy + 12xy - 20y^2$$
$$= 3x^2 + 7xy - 20y^2 \quad \text{Combine like terms.}$$

$$(A + B)^2 = A^2 + 2 \cdot A \cdot B + B^2$$

b. $$(5x + 3y)^2 = (5x)^2 + 2(5x)(3y) + (3y)^2$$
$$= 25x^2 + 30xy + 9y^2$$

Check Point 4 Multiply:

a. $(7x - 6y)(3x - y)$ **b.** $(2x + 4y)^2$.

Special products can sometimes be used to find the products of certain trinomials, as illustrated in Example 5.

EXAMPLE 5 Using the Special Products

Multiply: **a.** $(7x + 5 + 4y)(7x + 5 - 4y)$ **b.** $(3x + y + 1)^2$.

SOLUTION

a. By grouping the first two terms within each of the parentheses, we can find the product using the form for the sum and difference of two terms.

$$(A + B) \cdot (A - B) = A^2 - B^2$$

$$[(7x + 5) + 4y] \cdot [(7x + 5) - 4y] = (7x + 5)^2 - (4y)^2$$
$$= (7x)^2 + 2 \cdot 7x \cdot 5 + 5^2 - (4y)^2$$
$$= 49x^2 + 70x + 25 - 16y^2$$

b. We can group the terms of $(3x + y + 1)^2$ so that the formula for the square of a binomial can be applied.

$$(A + B)^2 = A^2 + 2 \cdot A \cdot B + B^2$$

$$[(3x + y) + 1]^2 = (3x + y)^2 + 2 \cdot (3x + y) \cdot 1 + 1^2$$
$$= 9x^2 + 6xy + y^2 + 6x + 2y + 1$$

Check Point 5 Multiply:

a. $(3x + 2 + 5y)(3x + 2 - 5y)$ **b.** $(2x + y + 3)^2$.

Blitzer Bonus || Labrador Retrievers and Polynomial Multiplication

The color of a Labrador retriever is determined by its pair of genes. A single gene is inherited at random from each parent. The black-fur gene, B, is dominant. The yellow-fur gene, Y, is recessive. This means that labs with at least one black-fur gene (BB or BY) have black coats. Only labs with two yellow-fur genes (YY) have yellow coats.

Axl, your author's yellow lab, inherited his genetic makeup from two black BY parents.

Second BY parent, a black lab with a recessive yellow-fur gene

First BY parent, a black lab with a recessive yellow-fur gene

	B	Y
B	BB	BY
Y	BY	YY

The table shows the four possible combinations of color genes that BY parents can pass to their offspring.

Because YY is one of four possible outcomes, the probability that a yellow lab like Axl will be the offspring of these black parents is $\frac{1}{4}$.

The probabilities suggested by the table can be modeled by the expression $\left(\frac{1}{2}B + \frac{1}{2}Y\right)^2$.

$$\left(\frac{1}{2}B + \frac{1}{2}Y\right)^2 = \left(\frac{1}{2}B\right)^2 + 2\left(\frac{1}{2}B\right)\left(\frac{1}{2}Y\right) + \left(\frac{1}{2}Y\right)^2$$

$$= \frac{1}{4}BB + \frac{1}{2}BY + \frac{1}{4}YY$$

The probability of a black lab with two dominant black genes is $\frac{1}{4}$.

The probability of a black lab with a recessive yellow gene is $\frac{1}{2}$.

The probability of a yellow lab with two recessive yellow genes is $\frac{1}{4}$.

CONCEPT AND VOCABULARY CHECK

Fill in each blank so that the resulting statement is true.

1. A polynomial is a single term or the sum of two or more terms containing variables with exponents that are _______ numbers.
2. It is customary to write the terms of a polynomial in the order of descending powers of the variable. This is called the ________ form of a polynomial.
3. A simplified polynomial that has exactly one term is called a/an _________.
4. A simplified polynomial that has two terms is called a/an _________.
5. A simplified polynomial that has three terms is called a/an _________.
6. If $a \neq 0$, the degree of ax^n is ____.
7. Polynomials are added by combining _____ terms.
8. To multiply $7x^3(4x^5 - 8x^2 + 6)$, use the __________ property to multiply each term of the trinomial _____________ by the monomial ______.
9. To multiply $(5x + 3)(x^2 + 8x + 7)$, begin by multiplying each term of $x^2 + 8x + 7$ by _____. Then multiply each term of $x^2 + 8x + 7$ by _____. Then combine _____ terms.
10. When using the FOIL method to find $(x + 7)(3x + 5)$, the product of the first terms is _____, the product of the outside terms is _____, the product of the inside terms is _____, and the product of the last terms is _____.
11. $(A + B)(A - B) =$ _________. The product of the sum and difference of the same two terms is the square of the first term ______ the square of the second term.
12. $(A + B)^2 =$ ______________. The square of a binomial sum is the first term ________ plus 2 times the _______________ plus the last term ________.
13. $(A - B)^2 =$ ______________. The square of a binomial difference is the first term squared _______ 2 times the _______________ ______ the last term squared.

    plus or minus?
14. If $a \neq 0$, the degree of ax^ny^m is ______.

EXERCISE SET 4

Practice Exercises

In Exercises 1–4, is the algebraic expression a polynomial? If it is, write the polynomial in standard form.

1. $2x + 3x^2 - 5$

2. $2x + 3x^{-1} - 5$

3. $\dfrac{2x + 3}{x}$

4. $x^2 - x^3 + x^4 - 5$

In Exercises 5–8, find the degree of the polynomial.

5. $3x^2 - 5x + 4$

6. $-4x^3 + 7x^2 - 11$

7. $x^2 - 4x^3 + 9x - 12x^4 + 63$

8. $x^2 - 8x^3 + 15x^4 + 91$

In Exercises 9–14, perform the indicated operations. Write the resulting polynomial in standard form and indicate its degree.

9. $(-6x^3 + 5x^2 - 8x + 9) + (17x^3 + 2x^2 - 4x - 13)$

10. $(-7x^3 + 6x^2 - 11x + 13) + (19x^3 - 11x^2 + 7x - 17)$

11. $(17x^3 - 5x^2 + 4x - 3) - (5x^3 - 9x^2 - 8x + 11)$

12. $(18x^4 - 2x^3 - 7x + 8) - (9x^4 - 6x^3 - 5x + 7)$

13. $(5x^2 - 7x - 8) + (2x^2 - 3x + 7) - (x^2 - 4x - 3)$

14. $(8x^2 + 7x - 5) - (3x^2 - 4x) - (-6x^3 - 5x^2 + 3)$

In Exercises 15–82, find each product.

15. $(x + 1)(x^2 - x + 1)$

16. $(x + 5)(x^2 - 5x + 25)$

17. $(2x - 3)(x^2 - 3x + 5)$

18. $(2x - 1)(x^2 - 4x + 3)$

19. $(x + 7)(x + 3)$

20. $(x + 8)(x + 5)$

21. $(x - 5)(x + 3)$

22. $(x - 1)(x + 2)$

23. $(3x + 5)(2x + 1)$

24. $(7x + 4)(3x + 1)$

25. $(2x - 3)(5x + 3)$

26. $(2x - 5)(7x + 2)$

27. $(5x^2 - 4)(3x^2 - 7)$

28. $(7x^2 - 2)(3x^2 - 5)$

29. $(8x^3 + 3)(x^2 - 5)$

30. $(7x^3 + 5)(x^2 - 2)$

31. $(x + 3)(x - 3)$

32. $(x + 5)(x - 5)$

33. $(3x + 2)(3x - 2)$

34. $(2x + 5)(2x - 5)$

35. $(5 - 7x)(5 + 7x)$

36. $(4 - 3x)(4 + 3x)$

37. $(4x^2 + 5x)(4x^2 - 5x)$

38. $(3x^2 + 4x)(3x^2 - 4x)$

39. $(1 - y^5)(1 + y^5)$

40. $(2 - y^5)(2 + y^5)$

41. $(x + 2)^2$

42. $(x + 5)^2$

43. $(2x + 3)^2$

44. $(3x + 2)^2$

45. $(x - 3)^2$

46. $(x - 4)^2$

47. $(4x^2 - 1)^2$

48. $(5x^2 - 3)^2$

49. $(7 - 2x)^2$

50. $(9 - 5x)^2$

51. $(x + 1)^3$

52. $(x + 2)^3$

53. $(2x + 3)^3$

54. $(3x + 4)^3$

55. $(x - 3)^3$

56. $(x - 1)^3$

57. $(3x - 4)^3$

58. $(2x - 3)^3$

59. $(x + 5y)(7x + 3y)$

60. $(x + 9y)(6x + 7y)$

61. $(x - 3y)(2x + 7y)$

62. $(3x - y)(2x + 5y)$

63. $(3xy - 1)(5xy + 2)$

64. $(7x^2y + 1)(2x^2y - 3)$

65. $(7x + 5y)^2$

66. $(9x + 7y)^2$

67. $(x^2y^2 - 3)^2$

68. $(x^2y^2 - 5)^2$

69. $(x - y)(x^2 + xy + y^2)$

70. $(x + y)(x^2 - xy + y^2)$

71. $(3x + 5y)(3x - 5y)$

72. $(7x + 3y)(7x - 3y)$

73. $(x + y + 3)(x + y - 3)$

74. $(x + y + 5)(x + y - 5)$

75. $(3x + 7 - 5y)(3x + 7 + 5y)$

76. $(5x + 7y - 2)(5x + 7y + 2)$

77. $[5y - (2x + 3)][5y + (2x + 3)]$

78. $[8y + (7 - 3x)][8y - (7 - 3x)]$

79. $(x + y + 1)^2$

80. $(x + y + 2)^2$

81. $(2x + y + 1)^2$

82. $(5x + 1 + 6y)^2$

Practice Plus

In Exercises 83–90, perform the indicated operation or operations.

83. $(3x + 4y)^2 - (3x - 4y)^2$

84. $(5x + 2y)^2 - (5x - 2y)^2$

85. $(5x - 7)(3x - 2) - (4x - 5)(6x - 1)$

86. $(3x + 5)(2x - 9) - (7x - 2)(x - 1)$

87. $(2x + 5)(2x - 5)(4x^2 + 25)$

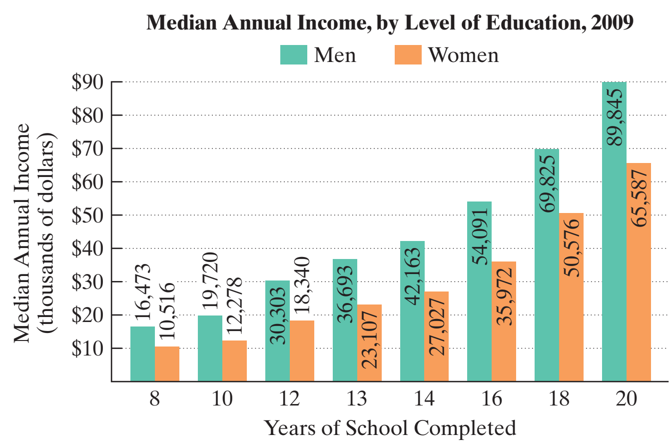
88. $(3x + 4)(3x - 4)(9x^2 + 16)$

89. $\dfrac{(2x - 7)^5}{(2x - 7)^3}$

90. $\dfrac{(5x - 3)^6}{(5x - 3)^4}$

Application Exercises

As you complete more years of education, you can count on a greater income. The bar graph shows the median, or middlemost, annual income for Americans, by level of education, in 2009.



Source: Bureau of the Census

Here are polynomial models that describe the median annual income for men, M, and for women, W, who have completed x years of education:

$$M = 312x^2 - 2615x + 16{,}615$$
$$W = 316x^2 - 4224x + 23{,}730$$
$$M = 0.6x^3 + 285x^2 - 2256x + 15{,}112$$
$$W = -1.2x^3 + 367x^2 - 4900x + 26{,}561$$

Exercises 91–92 are based on these models and the data displayed by the graph above.

91. a. Use the equation defined by a polynomial of degree 2 to find the median annual income for a man with 16 years of education. Does this underestimate or overestimate the median income shown by the bar graph? By how much?

b. Use the equations defined by polynomials of degree 3 to find a mathematical model for $M - W$.

c. According to the model in part (b), what is the difference, rounded to the nearest dollar, in the median annual income between men and women with 14 years of education?

d. According to the data displayed by the graph, what is the actual difference in the median annual income between men and women with 14 years of education? Did the result of part (c) underestimate or overestimate this difference? By how much?

92. a. Use the equation defined by a polynomial of degree 2 to find the median annual income for a woman with 18 years of education. Does this underestimate or overestimate the median income shown by the bar graph? By how much?

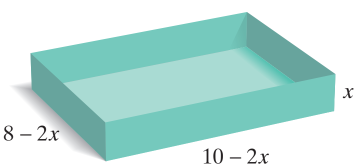
b. Use the equations defined by polynomials of degree 3 to find a mathematical model for $M - W$.

c. According to the model in part (b), what is the difference, rounded to the nearest dollar, in the median annual income between men and women with 16 years of education?

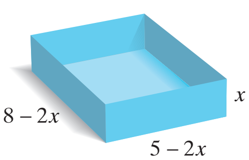
d. According to the data displayed by the graph, what is the actual difference in the median annual income between men and women with 16 years of education? Did the result of part (c) underestimate or overestimate this difference? By how much?

The volume, V, of a rectangular solid with length l, width w, and height h is given by the formula $V = lwh$. In Exercises 93–94, use this formula to write a polynomial in standard form that models, or represents, the volume of the open box.

93.

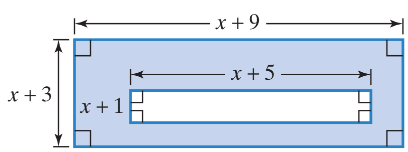


94.

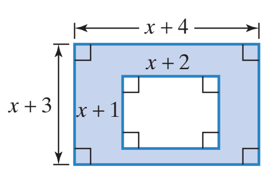


In Exercises 95–96, write a polynomial in standard form that models, or represents, the area of the shaded region.

95.



96.



Writing in Mathematics

97. What is a polynomial in x?

98. Explain how to subtract polynomials.

99. Explain how to multiply two binomials using the FOIL method. Give an example with your explanation.

100. Explain how to find the product of the sum and difference of two terms. Give an example with your explanation.

101. Explain how to square a binomial difference. Give an example with your explanation.

102. Explain how to find the degree of a polynomial in two variables.

Critical Thinking Exercises

Make Sense? *In Exercises 103–106, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

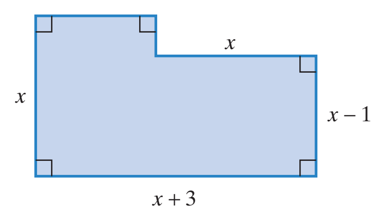
103. Knowing the difference between factors and terms is important: In $(3x^2y)^2$, I can distribute the exponent 2 on each factor, but in $(3x^2 + y)^2$, I cannot do the same thing on each term.

104. I used the FOIL method to find the product of $x + 5$ and $x^2 + 2x + 1$.

105. Many English words have prefixes with meanings similar to those used to describe polynomials, such as *monologue*, *binocular*, and *tricuspid*.

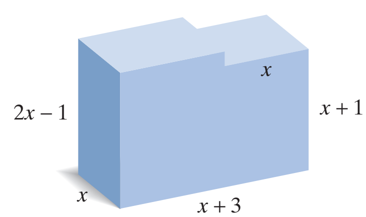
106. Special-product formulas have patterns that make their multiplications quicker than using the FOIL method.

107. Express the area of the plane figure shown as a polynomial in standard form.

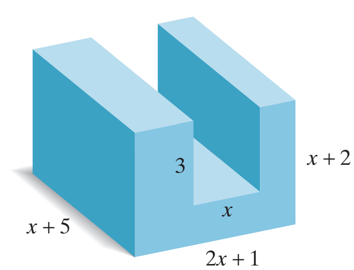


In Exercises 108–109, represent the volume of each figure as a polynomial in standard form.

108.



109.



110. Simplify: $(y^n + 2)(y^n - 2) - (y^n - 3)^2$.

Preview Exercises

Exercises 111–113 will help you prepare for the material covered in the next section. In each exercise, replace the boxed question mark with an integer that results in the given product. Some trial and error may be necessary.

111. $(x + 3)(x + \boxed{?}) = x^2 + 7x + 12$

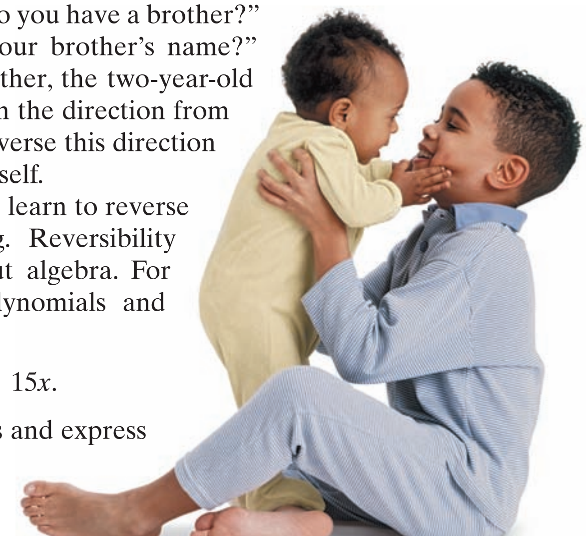
112. $(x - \boxed{?})(x - 12) = x^2 - 14x + 24$

113. $(4x + 1)(2x - \boxed{?}) = 8x^2 - 10x - 3$

SECTION 5 Factoring Polynomials

Objectives

1. Factor out the greatest common factor of a polynomial.
2. Factor by grouping.
3. Factor trinomials.
4. Factor the difference of squares.
5. Factor perfect square trinomials.
6. Factor the sum or difference of two cubes.
7. Use a general strategy for factoring polynomials.
8. Factor algebraic expressions containing fractional and negative exponents.



A two-year-old boy is asked, "Do you have a brother?" He answers, "Yes." "What is your brother's name?" "Tom." Asked if Tom has a brother, the two-year-old replies, "No." The child can go in the direction from self to brother, but he cannot reverse this direction and move from brother back to self.

As our intellects develop, we learn to reverse the direction of our thinking. Reversibility of thought is found throughout algebra. For example, we can multiply polynomials and show that

$$5x(2x + 3) = 10x^2 + 15x.$$

We can also reverse this process and express the resulting polynomial as

$$10x^2 + 15x = 5x(2x + 3).$$

Factoring a polynomial expressed as the sum of monomials means finding an equivalent expression that is a product.

Factoring $10x^2 + 15x$

Sum of monomials

Equivalent expression that is a product

$$10x^2 + 15x = 5x(2x + 3)$$

The factors of $10x^2 + 15x$ are $5x$ and $2x + 3$.

In this section, we will be **factoring over the set of integers**, meaning that the coefficients in the factors are integers. Polynomials that cannot be factored using integer coefficients are called **irreducible over the integers**, or **prime**.

The goal in factoring a polynomial is to use one or more factoring techniques until each of the polynomial's factors, except possibly for a monomial factor, is prime or irreducible. In this situation, the polynomial is said to be **factored completely**.

We will now discuss basic techniques for factoring polynomials.

1 Factor out the greatest common factor of a polynomial.

Common Factors

In any factoring problem, the first step is to look for the *greatest common factor*. The **greatest common factor**, abbreviated GCF, is an expression of the highest degree that divides each term of the polynomial. The distributive property in the reverse direction

$$ab + ac = a(b + c)$$

can be used to factor out the greatest common factor.

EXAMPLE 1 Factoring Out the Greatest Common Factor

Factor:

a. $18x^3 + 27x^2$ **b.** $x^2(x + 3) + 5(x + 3)$.

GREAT QUESTION!

Is there a rule that can help me determine the greatest common factor?

Yes. The variable part of the greatest common factor always contains the *smallest* power of a variable or algebraic expression that appears in all terms of the polynomial.

SOLUTION

a. First, determine the greatest common factor.

9 is the greatest integer that divides 18 and 27.

$$18x^3 + 27x^2$$

x^2 is the greatest expression that divides x^3 and x^2.

The GCF of the two terms of the polynomial is $9x^2$.

$$18x^3 + 27x^2$$
$$= 9x^2(2x) + 9x^2(3) \quad \text{Express each term as the product of the GCF and its other factor.}$$
$$= 9x^2(2x + 3) \quad \text{Factor out the GCF.}$$

b. In this situation, the greatest common factor is the common binomial factor $(x + 3)$. We factor out this common factor as follows:

$$x^2(x + 3) + 5(x + 3) = (x + 3)(x^2 + 5). \quad \text{Factor out the common binomial factor.}$$

●●●

Check Point 1 Factor:

a. $10x^3 - 4x^2$ **b.** $2x(x - 7) + 3(x - 7)$.

2 Factor by grouping.

Factoring by Grouping

Some polynomials have only a greatest common factor of 1. However, by a suitable grouping of the terms, it still may be possible to factor. This process, called **factoring by grouping**, is illustrated in Example 2.

EXAMPLE 2 Factoring by Grouping

Factor: $x^3 + 4x^2 + 3x + 12$.

SOLUTION

There is no factor other than 1 common to all terms. However, we can group terms that have a common factor:

$$\boxed{x^3 + 4x^2} + \boxed{3x + 12}.$$

Common factor is x^2. Common factor is 3.

We now factor the given polynomial as follows:

$$x^3 + 4x^2 + 3x + 12$$
$$= (x^3 + 4x^2) + (3x + 12) \quad \text{Group terms with common factors.}$$
$$= x^2(x + 4) + 3(x + 4) \quad \text{Factor out the greatest common factor from the grouped terms. The remaining two terms have } x + 4 \text{ as a common binomial factor.}$$
$$= (x + 4)(x^2 + 3). \quad \text{Factor out the GCF, } x + 4.$$

Thus, $x^3 + 4x^2 + 3x + 12 = (x + 4)(x^2 + 3)$. Check the factorization by multiplying the right side of the equation using the FOIL method. Because the factorization is correct, you should obtain the original polynomial. ●●●

DISCOVERY

In Example 2, group the terms as follows:

$$(x^3 + 3x) + (4x^2 + 12).$$

Factor out the greatest common factor from each group and complete the factoring process. Describe what happens. What can you conclude?

Check Point 2 Factor: $x^3 + 5x^2 - 2x - 10$.

3 Factor trinomials.

Factoring Trinomials

To factor a trinomial of the form $ax^2 + bx + c$, a little trial and error may be necessary.

A Strategy for Factoring $ax^2 + bx + c$

Assume, for the moment, that there is no greatest common factor.

1. Find two **F**irst terms whose product is ax^2:

$$(\square x + \quad)(\square x + \quad) = ax^2 + bx + c.$$

2. Find two **L**ast terms whose product is c:

$$(\square x + \square)(\square x + \square) = ax^2 + bx + c.$$

3. By trial and error, perform steps 1 and 2 until the sum of the **O**utside product and **I**nside product is bx:

$$(\square x + \square)(\square x + \square) = ax^2 + bx + c.$$

I

O

Sum of O + I

If no such combination exists, the polynomial is prime.

GREAT QUESTION!

Should I feel discouraged if it takes me a while to get the correct factorization?

The *error* part of the factoring strategy plays an important role in the process. If you do not get the correct factorization the first time, this is not a bad thing. This error is often helpful in leading you to the correct factorization.

EXAMPLE 3 Factoring a Trinomial Whose Leading Coefficient Is 1

Factor: $x^2 + 6x + 8$.

SOLUTION

Step 1 Find two First terms whose product is x^2.

$$x^2 + 6x + 8 = (x \qquad)(x \qquad)$$

Step 2 Find two Last terms whose product is 8.

Factors of 8	8, 1	4, 2	−8, −1	−4, −2

Step 3 Try various combinations of these factors. The correct factorization of $x^2 + 6x + 8$ is the one in which the sum of the **O**utside and **I**nside products is equal to $6x$. Here is a list of the possible factorizations:

Possible Factorizations of $x^2 + 6x + 8$	Sum of Outside and Inside Products (Should Equal $6x$)
$(x + 8)(x + 1)$	$x + 8x = 9x$
$(x + 4)(x + 2)$	$2x + 4x = 6x$
$(x - 8)(x - 1)$	$-x - 8x = -9x$
$(x - 4)(x - 2)$	$-2x - 4x = -6x$

This is the required middle term.

Thus, $x^2 + 6x + 8 = (x + 4)(x + 2)$ or $(x + 2)(x + 4)$.

In factoring a trinomial of the form $x^2 + bx + c$, you can speed things up by listing the factors of c and then finding their sums. We are interested in a sum of b. For example, in factoring $x^2 + 6x + 8$, we are interested in the factors of 8 whose sum is 6.

Factors of 8	8, 1	4, 2	−8, −1	−4, −2
Sum of Factors	9	6	−9	−6

This is the desired sum.

Thus, $x^2 + 6x + 8 = (x + 4)(x + 2)$.

Check Point 3 Factor: $x^2 + 13x + 40$.

EXAMPLE 4 Factoring a Trinomial Whose Leading Coefficient Is 1

Factor: $x^2 + 3x - 18$.

SOLUTION

Step 1 Find two First terms whose product is x^2.

$$x^2 + 3x - 18 = (x \quad)(x \quad)$$

To find the second term of each factor, we must find two integers whose product is -18 and whose sum is 3.

Step 2 Find two Last terms whose product is -18.

Factors of -18	$18, -1$	$-18, 1$	$9, -2$	$-9, 2$	$6, -3$	$-6, 3$

Step 3 Try various combinations of these factors. We are looking for the pair of factors whose sum is 3.

Factors of -18	$18, -1$	$-18, 1$	$9, -2$	$-9, 2$	$6, -3$	$-6, 3$
Sum of Factors	17	-17	7	-7	3	-3

This is the desired sum.

Thus, $x^2 + 3x - 18 = (x + 6)(x - 3)$ or $(x - 3)(x + 6)$.

GREAT QUESTION!

Is there a way to eliminate some of the combinations of factors for a trinomial whose leading coefficient is 1?

Yes. To factor $x^2 + bx + c$ when c is positive, find two numbers with the same sign as the middle term.

$$x^2 + 6x + 8 = (x + 4)(x + 2) \qquad x^2 - 5x + 6 = (x - 3)(x - 2)$$

Same signs

Same signs

To factor $x^2 + bx + c$ when c is negative, find two numbers with opposite signs whose sum is the coefficient of the middle term.

$$x^2 + 3x - 18 = (x + 6)(x - 3) \qquad x^2 - 2x - 99 = (x + 9)(x - 11)$$

Negative

Opposite signs

Negative

Opposite signs

Check Point 4 Factor: $x^2 - 5x - 14$.

EXAMPLE 5 Factoring a Trinomial Whose Leading Coefficient Is Not 1

Factor: $8x^2 - 10x - 3$.

GREAT QUESTION!

When factoring trinomials, must I list every possible factorization before getting the correct one?

With practice, you will find that it is not necessary to list every possible factorization of the trinomial. As you practice factoring, you will be able to narrow down the list of possible factors to just a few. When it comes to factoring, practice makes perfect.

SOLUTION

Step 1 Find two First terms whose product is $8x^2$.

$$8x^2 - 10x - 3 \stackrel{?}{=} (8x \quad)(x \quad)$$
$$8x^2 - 10x - 3 \stackrel{?}{=} (4x \quad)(2x \quad)$$

Step 2 Find two Last terms whose product is -3. The possible factorizations are $1(-3)$ and $-1(3)$.

Step 3 Try various combinations of these factors. The correct factorization of $8x^2 - 10x - 3$ is the one in which the sum of the **O**utside and **I**nside products is equal to $-10x$. Here is a list of the possible factorizations:

These four factorizations use $(8x \quad)(x \quad)$ with $1(-3)$ and $-1(3)$ as factorizations of -3.

These four factorizations use $(4x \quad)(2x \quad)$ with $1(-3)$ and $-1(3)$ as factorizations of -3.

Possible Factorizations of $8x^2 - 10x - 3$	Sum of Outside and Inside Products (Should Equal $-10x$)
$(8x + 1)(x - 3)$	$-24x + x = -23x$
$(8x - 3)(x + 1)$	$8x - 3x = 5x$
$(8x - 1)(x + 3)$	$24x - x = 23x$
$(8x + 3)(x - 1)$	$-8x + 3x = -5x$
$(4x + 1)(2x - 3)$	$-12x + 2x = -10x$
$(4x - 3)(2x + 1)$	$4x - 6x = -2x$
$(4x - 1)(2x + 3)$	$12x - 2x = 10x$
$(4x + 3)(2x - 1)$	$-4x + 6x = 2x$

This is the required middle term.

Thus, $8x^2 - 10x - 3 = (4x + 1)(2x - 3)$ or $(2x - 3)(4x + 1)$.

Use FOIL multiplication to check either of these factorizations. • • •

Check Point 5 Factor: $6x^2 + 19x - 7$.

EXAMPLE 6 Factoring a Trinomial in Two Variables

Factor: $2x^2 - 7xy + 3y^2$.

SOLUTION

Step 1 Find two First terms whose product is $2x^2$.

$$2x^2 - 7xy + 3y^2 = (2x \quad)(x \quad)$$

Step 2 Find two Last terms whose product is $3y^2$. The possible factorizations are $(y)(3y)$ and $(-y)(-3y)$.

Step 3 Try various combinations of these factors. The correct factorization of $2x^2 - 7xy + 3y^2$ is the one in which the sum of the **O**utside and **I**nside products is equal to $-7xy$. Here is a list of possible factorizations:

Possible Factorizations of $2x^2 - 7xy + 3y^2$	Sum of Outside and Inside Products (Should Equal $-7xy$)
$(2x + 3y)(x + y)$	$2xy + 3xy = 5xy$
$(2x + y)(x + 3y)$	$6xy + xy = 7xy$
$(2x - 3y)(x - y)$	$-2xy - 3xy = -5xy$
$(2x - y)(x - 3y)$	$-6xy - xy = -7xy$

This is the required middle term.

GREAT QUESTION!

I zone out reading your long lists of possible factorizations. Are there any rules for shortening these lists?

Here are some suggestions for reducing the list of possible factorizations for $ax^2 + bx + c$:

1. If b is relatively small, avoid the larger factors of a.
2. If c is positive, the signs in both binomial factors must match the sign of b.
3. If the trinomial has no common factor, no binomial factor can have a common factor.
4. Reversing the signs in the binomial factors changes the sign of bx, the middle term.

Thus,

$$2x^2 - 7xy + 3y^2 = (2x - y)(x - 3y) \quad \text{or} \quad (x - 3y)(2x - y).$$

Use FOIL multiplication to check either of these factorizations.

Check Point 6 Factor: $3x^2 - 13xy + 4y^2$.

4 Factor the difference of squares.

Factoring the Difference of Two Squares

A method for factoring the difference of two squares is obtained by reversing the special product for the sum and difference of two terms.

The Difference of Two Squares

If A and B are real numbers, variables, or algebraic expressions, then

$$A^2 - B^2 = (A + B)(A - B).$$

In words: The difference of the squares of two terms factors as the product of a sum and a difference of those terms.

EXAMPLE 7 Factoring the Difference of Two Squares

Factor: **a.** $x^2 - 4$ **b.** $81x^2 - 49$.

SOLUTION

We must express each term as the square of some monomial. Then we use the formula for factoring $A^2 - B^2$.

a. $x^2 - 4 = x^2 - 2^2 = (x + 2)(x - 2)$

$A^2 - B^2 = (A + B)(A - B)$

b. $81x^2 - 49 = (9x)^2 - 7^2 = (9x + 7)(9x - 7)$

Check Point 7 Factor:

a. $x^2 - 81$ **b.** $36x^2 - 25$.

We have seen that a polynomial is factored completely when it is written as the product of prime polynomials. To be sure that you have factored completely, check to see whether any factors with more than one term in the factored polynomial can be factored further. If so, continue factoring.

GREAT QUESTION!

Why isn't factoring $x^4 - 81$ as $(x^2 + 9)(x^2 - 9)$ a complete factorization?

The second factor, $x^2 - 9$, is itself a difference of two squares and can be factored.

EXAMPLE 8 A Repeated Factorization

Factor completely: $x^4 - 81$.

SOLUTION

$x^4 - 81 = (x^2)^2 - 9^2$ Express as the difference of two squares.

$= (x^2 + 9)(x^2 - 9)$ The factors are the sum and the difference of the expressions being squared.

$= (x^2 + 9)(x^2 - 3^2)$ The factor $x^2 - 9$ is the difference of two squares and can be factored.

$= (x^2 + 9)(x + 3)(x - 3)$ The factors of $x^2 - 9$ are the sum and the difference of the expressions being squared.

Check Point 8 Factor completely: $81x^4 - 16$.

5 Factor perfect square trinomials.

Factoring Perfect Square Trinomials

Our next factoring technique is obtained by reversing the special products for squaring binomials. The trinomials that are factored using this technique are called **perfect square trinomials**.

Factoring Perfect Square Trinomials

Let A and B be real numbers, variables, or algebraic expressions.

1. $A^2 + 2AB + B^2 = (A + B)^2$ Same sign
2. $A^2 - 2AB + B^2 = (A - B)^2$ Same sign

The two items in the box show that perfect square trinomials, $A^2 + 2AB + B^2$ and $A^2 - 2AB + B^2$, come in two forms: one in which the coefficient of the middle term is positive and one in which the coefficient of the middle term is negative. Here's how to recognize a perfect square trinomial:

1. The first and last terms are squares of monomials or integers.
2. The middle term is twice the product of the expressions being squared in the first and last terms.

EXAMPLE 9 Factoring Perfect Square Trinomials

Factor: **a.** $x^2 + 6x + 9$ **b.** $25x^2 - 60x + 36$.

SOLUTION

a. $x^2 + 6x + 9 = x^2 + 2 \cdot x \cdot 3 + 3^2 = (x + 3)^2$

$A^2 + 2AB + B^2 = (A + B)^2$

The middle term has a positive sign.

b. We suspect that $25x^2 - 60x + 36$ is a perfect square trinomial because $25x^2 = (5x)^2$ and $36 = 6^2$. The middle term can be expressed as twice the product of $5x$ and 6.

$$25x^2 - 60x + 36 = (5x)^2 - 2 \cdot 5x \cdot 6 + 6^2 = (5x - 6)^2$$

$A^2 - 2AB + B^2 = (A - B)^2$

Check Point 9 Factor:

a. $x^2 + 14x + 49$ **b.** $16x^2 - 56x + 49$.

6 Factor the sum or difference of two cubes.

Factoring the Sum or Difference of Two Cubes

We can use the following formulas to factor the sum or the difference of two cubes:

Factoring the Sum or Difference of Two Cubes

1. Factoring the Sum of Two Cubes

$$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$

Same signs   Opposite signs   Always positive

2. Factoring the Difference of Two Cubes

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

Same signs   Opposite signs   Always positive

GREAT QUESTION!

A Cube of SOAP

The formulas for factoring $A^3 + B^3$ and $A^3 - B^3$ are difficult to remember and easy to confuse. Can you help me out?

When factoring sums or differences of cubes, observe the sign patterns shown by the voice balloons in the box. The word *SOAP* is a way to remember these patterns:

S O A P.

Same signs   Opposite signs   Always Positive

EXAMPLE 10 Factoring Sums and Differences of Two Cubes

Factor: **a.** $x^3 + 8$ **b.** $64x^3 - 125$.

SOLUTION

a. To factor $x^3 + 8$, we must express each term as the cube of some monomial. Then we use the formula for factoring $A^3 + B^3$.

$$x^3 + 8 = x^3 + 2^3 = (x + 2)(x^2 - x \cdot 2 + 2^2) = (x + 2)(x^2 - 2x + 4)$$

$$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$

b. To factor $64x^3 - 125$, we must express each term as the cube of some monomial. Then we use the formula for factoring $A^3 - B^3$.

$$64x^3 - 125 = (4x)^3 - 5^3 = (4x - 5)[(4x)^2 + (4x)(5) + 5^2]$$

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

$$= (4x - 5)(16x^2 + 20x + 25)$$

Check Point 10 Factor:

a. $x^3 + 1$ **b.** $125x^3 - 8$.

7 Use a general strategy for factoring polynomials.

A Strategy for Factoring Polynomials

It is important to practice factoring a wide variety of polynomials so that you can quickly select the appropriate technique. The polynomial is factored completely when all its polynomial factors, except possibly for monomial factors, are prime. Because of the commutative property, the order of the factors does not matter.

A Strategy for Factoring a Polynomial

1. If there is a common factor, factor out the GCF.

2. Determine the number of terms in the polynomial and try factoring as follows:

a. If there are two terms, can the binomial be factored by using one of the following special forms?

Difference of two squares: $A^2 - B^2 = (A + B)(A - B)$

Sum of two cubes: $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$

Difference of two cubes: $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$

b. If there are three terms, is the trinomial a perfect square trinomial? If so, factor by using one of the following special forms:

$$A^2 + 2AB + B^2 = (A + B)^2$$
$$A^2 - 2AB + B^2 = (A - B)^2.$$

If the trinomial is not a perfect square trinomial, try factoring by trial and error.

c. If there are four or more terms, try factoring by grouping.

3. Check to see if any factors with more than one term in the factored polynomial can be factored further. If so, factor completely.

EXAMPLE 11 Factoring a Polynomial

Factor: $2x^3 + 8x^2 + 8x$.

SOLUTION

Step 1 If there is a common factor, factor out the GCF. Because $2x$ is common to all terms, we factor it out.

$$2x^3 + 8x^2 + 8x = 2x(x^2 + 4x + 4) \quad \text{Factor out the GCF.}$$

Step 2 Determine the number of terms and factor accordingly. The factor $x^2 + 4x + 4$ has three terms and is a perfect square trinomial. We factor using $A^2 + 2AB + B^2 = (A + B)^2$.

$$\begin{aligned} 2x^3 + 8x^2 + 8x &= 2x(x^2 + 4x + 4) \\ &= 2x(\underbrace{x^2}_{A^2} + \underbrace{2 \cdot x \cdot 2}_{2AB} + \underbrace{2^2}_{B^2}) \\ &= 2x(x + 2)^2 \qquad A^2 + 2AB + B^2 = (A + B)^2 \end{aligned}$$

Step 3 Check to see if factors can be factored further. In this problem, they cannot. Thus,

$$2x^3 + 8x^2 + 8x = 2x(x + 2)^2.$$

Check Point 11 Factor: $3x^3 - 30x^2 + 75x$.

EXAMPLE 12 Factoring a Polynomial

Factor: $x^2 - 25a^2 + 8x + 16$.

SOLUTION

Step 1 If there is a common factor, factor out the GCF. Other than 1 or −1, there is no common factor.

Step 2 Determine the number of terms and factor accordingly. There are four terms. We try factoring by grouping. It can be shown that grouping into two groups of two terms does not result in a common binomial factor. Let's try grouping as a difference of squares.

$$x^2 - 25a^2 + 8x + 16$$

$$= (x^2 + 8x + 16) - 25a^2$$ Rearrange terms and group as a perfect square trinomial minus $25a^2$ to obtain a difference of squares.

$$= (x + 4)^2 - (5a)^2$$ Factor the perfect square trinomial.

$$= (x + 4 + 5a)(x + 4 - 5a)$$ Factor the difference of squares. The factors are the sum and difference of the expressions being squared.

Step 3 Check to see if factors can be factored further. In this case, they cannot, so we have factored completely. ● ● ●

Check Point 12 Factor: $x^2 - 36a^2 + 20x + 100$.

8 Factor algebraic expressions containing fractional and negative exponents.

Factoring Algebraic Expressions Containing Fractional and Negative Exponents

Although expressions containing fractional and negative exponents are not polynomials, they can be simplified using factoring techniques.

EXAMPLE 13 Factoring Involving Fractional and Negative Exponents

Factor and simplify: $x(x + 1)^{-\frac{3}{4}} + (x + 1)^{\frac{1}{4}}$.

SOLUTION

The greatest common factor of $x(x + 1)^{-\frac{3}{4}} + (x + 1)^{\frac{1}{4}}$ is $x + 1$ with the *smaller exponent* in the two terms. Thus, the greatest common factor is $(x + 1)^{-\frac{3}{4}}$.

$$x(x + 1)^{-\frac{3}{4}} + (x + 1)^{\frac{1}{4}}$$

$$= (x + 1)^{-\frac{3}{4}}x + (x + 1)^{-\frac{3}{4}}(x + 1)^{\frac{4}{4}}$$ Express each term as the product of the greatest common factor and its other factor. Note that $(x + 1)^{-\frac{3}{4}}(x + 1)^{\frac{4}{4}} = (x + 1)^{-\frac{3}{4}+\frac{4}{4}} = (x + 1)^{\frac{1}{4}}$.

$$= (x + 1)^{-\frac{3}{4}}x + (x + 1)^{-\frac{3}{4}}(x + 1)$$ Simplify: $(x + 1)^{\frac{4}{4}} = (x + 1)$.

$$= (x + 1)^{-\frac{3}{4}}[x + (x + 1)]$$ Factor out the greatest common factor.

$$= \frac{2x + 1}{(x + 1)^{\frac{3}{4}}}$$ $b^{-n} = \frac{1}{b^n}$ ● ● ●

Check Point 13 Factor and simplify: $x(x - 1)^{-\frac{1}{2}} + (x - 1)^{\frac{1}{2}}$

CONCEPT AND VOCABULARY CHECK

Here is a list of the factoring techniques that we have discussed.

a. Factoring out the GCF
b. Factoring by grouping
c. Factoring trinomials by trial and error
d. Factoring the difference of two squares

$$A^2 - B^2 = (A + B)(A - B)$$

e. Factoring perfect square trinomials

$$A^2 + 2AB + B^2 = (A + B)^2$$
$$A^2 - 2AB + B^2 = (A - B)^2$$

f. Factoring the sum of two cubes

$$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$

g. Factoring the difference of two cubes

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

Fill in each blank by writing the letter of the technique (a through g) for factoring the polynomial.

1. $16x^2 - 25$ ______
2. $27x^3 - 1$ ______
3. $x^2 + 7x + xy + 7y$ ______
4. $4x^2 + 8x + 3$ ______
5. $9x^2 + 24x + 16$ ______
6. $5x^2 + 10x$ ______
7. $x^3 + 1000$ ______
8. The algebraic expression $(x + 1)^{\frac{1}{2}} - \frac{1}{3}(x + 1)^{\frac{3}{2}}$ can be factored using ________ as the greatest common factor.

EXERCISE SET 5

Practice Exercises

In Exercises 1–10, factor out the greatest common factor.

1. $18x + 27$
2. $16x - 24$
3. $3x^2 + 6x$
4. $4x^2 - 8x$
5. $9x^4 - 18x^3 + 27x^2$
6. $6x^4 - 18x^3 + 12x^2$
7. $x(x + 5) + 3(x + 5)$
8. $x(2x + 1) + 4(2x + 1)$
9. $x^2(x - 3) + 12(x - 3)$
10. $x^2(2x + 5) + 17(2x + 5)$

In Exercises 11–16, factor by grouping.

11. $x^3 - 2x^2 + 5x - 10$
12. $x^3 - 3x^2 + 4x - 12$
13. $x^3 - x^2 + 2x - 2$
14. $x^3 + 6x^2 - 2x - 12$
15. $3x^3 - 2x^2 - 6x + 4$
16. $x^3 - x^2 - 5x + 5$

In Exercises 17–38, factor each trinomial, or state that the trinomial is prime.

17. $x^2 + 5x + 6$
18. $x^2 + 8x + 15$
19. $x^2 - 2x - 15$
20. $x^2 - 4x - 5$
21. $x^2 - 8x + 15$
22. $x^2 - 14x + 45$
23. $3x^2 - x - 2$
24. $2x^2 + 5x - 3$
25. $3x^2 - 25x - 28$
26. $3x^2 - 2x - 5$
27. $6x^2 - 11x + 4$
28. $6x^2 - 17x + 12$
29. $4x^2 + 16x + 15$
30. $8x^2 + 33x + 4$
31. $9x^2 - 9x + 2$
32. $9x^2 + 5x - 4$
33. $20x^2 + 27x - 8$
34. $15x^2 - 19x + 6$
35. $2x^2 + 3xy + y^2$
36. $3x^2 + 4xy + y^2$
37. $6x^2 - 5xy - 6y^2$
38. $6x^2 - 7xy - 5y^2$

In Exercises 39–48, factor the difference of two squares.

39. $x^2 - 100$
40. $x^2 - 144$
41. $36x^2 - 49$
42. $64x^2 - 81$
43. $9x^2 - 25y^2$
44. $36x^2 - 49y^2$
45. $x^4 - 16$
46. $x^4 - 1$
47. $16x^4 - 81$
48. $81x^4 - 1$

In Exercises 49–56, factor each perfect square trinomial.

49. $x^2 + 2x + 1$
50. $x^2 + 4x + 4$
51. $x^2 - 14x + 49$
52. $x^2 - 10x + 25$
53. $4x^2 + 4x + 1$
54. $25x^2 + 10x + 1$
55. $9x^2 - 6x + 1$
56. $64x^2 - 16x + 1$

In Exercises 57–64, factor using the formula for the sum or difference of two cubes.

57. $x^3 + 27$
58. $x^3 + 64$
59. $x^3 - 64$
60. $x^3 - 27$
61. $8x^3 - 1$
62. $27x^3 - 1$
63. $64x^3 + 27$
64. $8x^3 + 125$

In Exercises 65–92, factor completely, or state that the polynomial is prime.

65. $3x^3 - 3x$
66. $5x^3 - 45x$
67. $4x^2 - 4x - 24$
68. $6x^2 - 18x - 60$
69. $2x^4 - 162$
70. $7x^4 - 7$
71. $x^3 + 2x^2 - 9x - 18$
72. $x^3 + 3x^2 - 25x - 75$
73. $2x^2 - 2x - 112$
74. $6x^2 - 6x - 12$
75. $x^3 - 4x$
76. $9x^3 - 9x$
77. $x^2 + 64$
78. $x^2 + 36$
79. $x^3 + 2x^2 - 4x - 8$
80. $x^3 + 2x^2 - x - 2$
81. $y^5 - 81y$
82. $y^5 - 16y$
83. $20y^4 - 45y^2$
84. $48y^4 - 3y^2$
85. $x^2 - 12x + 36 - 49y^2$
86. $x^2 - 10x + 25 - 36y^2$
87. $9b^2x - 16y - 16x + 9b^2y$
88. $16a^2x - 25y - 25x + 16a^2y$

89. $x^2y - 16y + 32 - 2x^2$ **90.** $12x^2y - 27y - 4x^2 + 9$

91. $2x^3 - 8a^2x + 24x^2 + 72x$

92. $2x^3 - 98a^2x + 28x^2 + 98x$

In Exercises 93–102, factor and simplify each algebraic expression.

93. $x^{\frac{3}{2}} - x^{\frac{1}{2}}$ **94.** $x^{\frac{3}{4}} - x^{\frac{1}{4}}$

95. $4x^{-\frac{2}{3}} + 8x^{\frac{1}{3}}$ **96.** $12x^{-\frac{3}{4}} + 6x^{\frac{1}{4}}$

97. $(x + 3)^{\frac{1}{2}} - (x + 3)^{\frac{3}{2}}$ **98.** $(x^2 + 4)^{\frac{3}{2}} + (x^2 + 4)^{\frac{7}{2}}$

99. $(x + 5)^{-\frac{1}{2}} - (x + 5)^{-\frac{3}{2}}$

100. $(x^2 + 3)^{-\frac{2}{3}} + (x^2 + 3)^{-\frac{5}{3}}$

101. $(4x - 1)^{\frac{1}{2}} - \frac{1}{3}(4x - 1)^{\frac{3}{2}}$

102. $-8(4x + 3)^{-2} + 10(5x + 1)(4x + 3)^{-1}$

Practice Plus

In Exercises 103–114, factor completely.

103. $10x^2(x + 1) - 7x(x + 1) - 6(x + 1)$

104. $12x^2(x - 1) - 4x(x - 1) - 5(x - 1)$

105. $6x^4 + 35x^2 - 6$ **106.** $7x^4 + 34x^2 - 5$

107. $y^7 + y$ **108.** $(y + 1)^3 + 1$

109. $x^4 - 5x^2y^2 + 4y^4$ **110.** $x^4 - 10x^2y^2 + 9y^4$

111. $(x - y)^4 - 4(x - y)^2$ **112.** $(x + y)^4 - 100(x + y)^2$

113. $2x^2 - 7xy^2 + 3y^4$ **114.** $3x^2 + 5xy^2 + 2y^4$

Application Exercises

115. Your computer store is having an incredible sale. The price on one model is reduced by 40%. Then the sale price is reduced by another 40%. If x is the computer's original price, the sale price can be modeled by

$$(x - 0.4x) - 0.4(x - 0.4x).$$

a. Factor out $(x - 0.4x)$ from each term. Then simplify the resulting expression.

b. Use the simplified expression from part (a) to answer these questions. With a 40% reduction followed by a 40% reduction, is the computer selling at 20% of its original price? If not, at what percentage of the original price is it selling?

116. Your local electronics store is having an end-of-the-year sale. The price on a plasma television had been reduced by 30%. Now the sale price is reduced by another 30%. If x is the television's original price, the sale price can be modeled by

$$(x - 0.3x) - 0.3(x - 0.3x).$$

a. Factor out $(x - 0.3x)$ from each term. Then simplify the resulting expression.

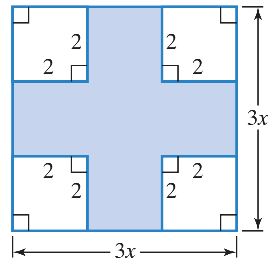
b. Use the simplified expression from part (a) to answer these questions. With a 30% reduction followed by a 30% reduction, is the television selling at 40% of its original price? If not, at what percentage of the original price is it selling?

In Exercises 117–120,

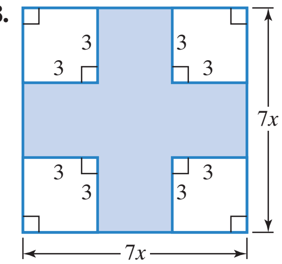
a. *Write an expression for the area of the shaded region.*

b. *Write the expression in factored form.*

117.



118.



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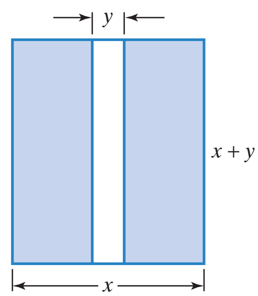

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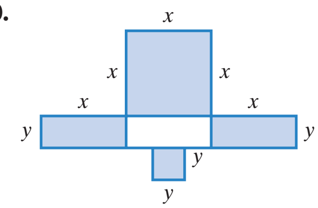
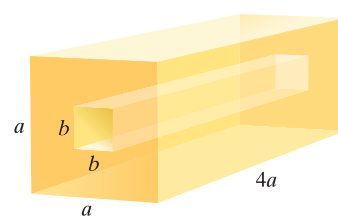
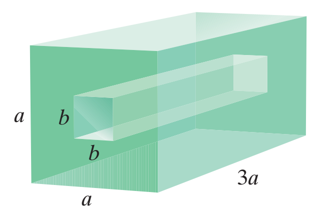


In Exercises 121–122, find the formula for the volume of the region outside the smaller rectangular solid and inside the larger rectangular solid. Then express the volume in factored form.

121.



122.



Writing in Mathematics

123. Using an example, explain how to factor out the greatest common factor of a polynomial.

124. Suppose that a polynomial contains four terms. Explain how to use factoring by grouping to factor the polynomial.

125. Explain how to factor $3x^2 + 10x + 8$.

126. Explain how to factor the difference of two squares. Provide an example with your explanation.

127. What is a perfect square trinomial and how is it factored?

128. Explain how to factor $x^3 + 1$.

129. What does it mean to factor completely?

Critical Thinking Exercises

Make Sense? *In Exercises 130–133, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

130. Although $20x^3$ appears in both $20x^3 + 8x^2$ and $20x^3 + 10x$, I'll need to factor $20x^3$ in different ways to obtain each polynomial's factorization.

131. You grouped the polynomial's terms using different groupings than I did, yet we both obtained the same factorization.

132. I factored $4x^2 - 100$ completely and obtained $(2x + 10)(2x - 10)$.

133. First factoring out the greatest common factor makes it easier for me to determine how to factor the remaining factor, assuming that it is not prime.

In Exercises 134–137, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

134. $x^4 - 16$ is factored completely as $(x^2 + 4)(x^2 - 4)$.

135. The trinomial $x^2 - 4x - 4$ is a prime polynomial.

136. $x^2 + 36 = (x + 6)^2$

137. $x^3 - 64 = (x + 4)(x^2 + 4x - 16)$

In Exercises 138–141, factor completely.

138. $x^{2n} + 6x^n + 8$

139. $-x^2 - 4x + 5$

140. $x^4 - y^4 - 2x^3y + 2xy^3$

141. $(x - 5)^{-\frac{1}{2}}(x + 5)^{-\frac{1}{2}} - (x + 5)^{\frac{1}{2}}(x - 5)^{-\frac{3}{2}}$

In Exercises 142–143, find all integers b so that the trinomial can be factored.

142. $x^2 + bx + 15$

143. $x^2 + 4x + b$

Preview Exercises

Exercises 144–146 will help you prepare for the material covered in the next section.

144. Factor the numerator and the denominator. Then simplify by dividing out the common factor in the numerator and the denominator.

$$\frac{x^2 + 6x + 5}{x^2 - 25}$$

In Exercises 145–146, perform the indicated operation. Where possible, reduce the answer to its lowest terms.

145. $\frac{5}{4} \cdot \frac{8}{15}$

146. $\frac{1}{2} + \frac{2}{3}$

Mid-Chapter Check Point

WHAT YOU KNOW: We defined the real numbers $[\{x | x \text{ is rational}\} \cup \{x | x \text{ is irrational}\}]$ and graphed them as points on a number line. We reviewed the basic rules of algebra, using these properties to simplify algebraic expressions. We expanded our knowledge of exponents to include exponents other than natural numbers:

$$b^0 = 1; \quad b^{-n} = \frac{1}{b^n}; \quad \frac{1}{b^{-n}} = b^n; \quad b^{\frac{1}{n}} = \sqrt[n]{b};$$

$$b^{\frac{m}{n}} = (\sqrt[n]{b})^m = \sqrt[n]{b^m}; \quad b^{-\frac{m}{n}} = \frac{1}{b^{\frac{m}{n}}}.$$

We used properties of exponents to simplify exponential expressions and properties of radicals to simplify radical expressions. We performed operations with polynomials. We used a number of fast methods for finding products of polynomials, including the FOIL method for multiplying binomials, a special-product formula for the product of the sum and difference of two terms $[(A + B)(A - B) = A^2 - B^2]$, and special-product formulas for squaring binomials $[(A + B)^2 = A^2 + 2AB + B^2; (A - B)^2 = A^2 - 2AB + B^2]$. We reversed the direction of these formulas and reviewed how to factor polynomials. We used a general strategy, summarized in the box, for factoring a wide variety of polynomials.

In Exercises 1–27, simplify the given expression or perform the indicated operation (and simplify, if possible), whichever is appropriate.

1. $(3x + 5)(4x - 7)$

2. $(3x + 5) - (4x - 7)$

3. $\sqrt{6} + 9\sqrt{6}$

4. $3\sqrt{12} - \sqrt{27}$

5. $7x + 3[9 - (2x - 6)]$

6. $(8x - 3)^2$

7. $\left(x^{\frac{1}{3}}y^{-\frac{1}{2}}\right)^6$

8. $\left(\frac{2}{7}\right)^0 - 32^{-\frac{2}{5}}$

9. $(2x - 5) - (x^2 - 3x + 1)$

10. $(2x - 5)(x^2 - 3x + 1)$

11. $x^3 + x^3 - x^3 \cdot x^3$

12. $(9a - 10b)(2a + b)$

13. $\{a, c, d, e\} \cup \{c, d, f, h\}$

14. $\{a, c, d, e\} \cap \{c, d, f, h\}$

15. $(3x^2y^3 - xy + 4y^2) - (-2x^2y^3 - 3xy + 5y^2)$

16. $\frac{24x^2y^{13}}{-2x^5y^{-2}}$

17. $\left(\frac{1}{3}x^{-5}y^4\right)(18x^{-2}y^{-1})$

18. $\sqrt[12]{x^4}$

19. $[4y - (3x + 2)][4y + (3x + 2)]$

20. $(x - 2y - 1)^2$

21. $\frac{24 \times 10^3}{2 \times 10^6}$ (Express the answer in scientific notation.)

22. $\frac{\sqrt[3]{32}}{\sqrt[3]{2}}$

23. $(x^3 + 2)(x^3 - 2)$

24. $(x^2 + 2)^2$

25. $\sqrt{50} \cdot \sqrt{6}$

26. $\frac{11}{7 - \sqrt{3}}$

27. $\frac{11}{\sqrt{3}}$

In Exercises 28–34, factor completely, or state that the polynomial is prime.

28. $7x^2 - 22x + 3$

29. $x^2 - 2x + 4$

30. $x^3 + 5x^2 + 3x + 15$

31. $3x^2 - 4xy - 7y^2$

32. $64y - y^4$

33. $50x^3 + 20x^2 + 2x$

34. $x^2 - 6x + 9 - 49y^2$

In Exercises 35–36, factor and simplify each algebraic expression.

35. $x^{-\frac{3}{2}} - 2x^{-\frac{1}{2}} + x^{\frac{1}{2}}$

36. $(x^2 + 1)^{\frac{1}{2}} - 10(x^2 + 1)^{-\frac{1}{2}}$

37. List all the rational numbers in this set:

$$\left\{-11, -\frac{3}{7}, 0, 0.45, \sqrt{23}, \sqrt{25}\right\}.$$

In Exercises 38–39, rewrite each expression without absolute value bars.

38. $|2 - \sqrt{13}|$

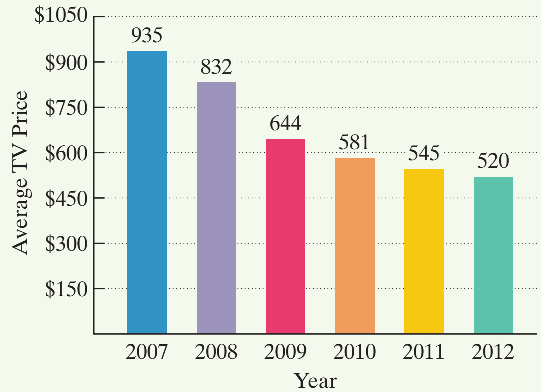
39. $x^2|x| \quad \text{if} \quad x < 0$

40. If the population of the United States is approximately 3.0×10^8 and each person produces about 4.6 pounds of garbage per day, express the total number of pounds of garbage produced in the United States in one day in scientific notation.

41. A human brain contains 3×10^{10} neurons and a gorilla brain contains 7.5×10^9 neurons. How many times as many neurons are in the brain of a human as in the brain of a gorilla?

42. TVs keep getting fancier and bigger, but prices do not. The bar graph at the top of the next column shows the average price of a TV in the United States from 2007 through 2012.

Average Price of a TV



Source: Consumer Electronics Association

Here are two mathematical models for the data shown by the graph. In each formula, P represents the average price of a TV x years after 2007.

Model 1 $\quad P = -86x + 890$

Model 2 $\quad P = 18x^2 - 175x + 950$

a. Which model better describes the data for 2007?

b. Does the polynomial model of degree 2 underestimate or overestimate the average TV price for 2012? By how much?

SECTION 6 Rational Expressions

Objectives

1. Specify numbers that must be excluded from the domain of a rational expression.
2. Simplify rational expressions.
3. Multiply rational expressions.
4. Divide rational expressions.
5. Add and subtract rational expressions.
6. Simplify complex rational expressions.
7. Simplify fractional expressions that occur in calculus.
8. Rationalize numerators.

How do we describe the costs of reducing environmental pollution? We often use algebraic expressions involving quotients of polynomials. For example, the algebraic expression

$$\frac{250x}{100 - x}$$

describes the cost, in millions of dollars, to remove x percent of the pollutants that are discharged into a river. Removing a modest percentage of pollutants, say 40%, is far less costly than removing a substantially greater percentage, such as 95%. We see this by evaluating the algebraic expression for $x = 40$ and $x = 95$.

Evaluating $\dfrac{250x}{100 - x}$ for

$x = 40$: Cost is $\dfrac{250(40)}{100 - 40} \approx 167.$

$x = 95$: Cost is $\dfrac{250(95)}{100 - 95} = 4750.$

The cost increases from approximately \$167 million to a possibly prohibitive \$4750 million, or \$4.75 billion. Costs spiral upward as the percentage of removed pollutants increases.

Many algebraic expressions that describe costs of environmental projects are examples of *rational expressions*. First we will define rational expressions. Then we will review how to perform operations with such expressions.

1 Specify numbers that must be excluded from the domain of a rational expression.

Rational Expressions

A **rational expression** is the quotient of two polynomials. Some examples are

$$\frac{x-2}{4},\quad \frac{4}{x-2},\quad \frac{x}{x^2-1},\quad \text{and}\quad \frac{x^2+1}{x^2+2x-3}.$$

The set of real numbers for which an algebraic expression is defined is the **domain** of the expression. Because rational expressions indicate division and division by zero is undefined, **we must exclude numbers from a rational expression's domain that make the denominator zero**.

EXAMPLE 1 Excluding Numbers from the Domain

Find all the numbers that must be excluded from the domain of each rational expression:

a. $\frac{4}{x-2}$ **b.** $\frac{x}{x^2-1}$ **c.** $\frac{9x}{x^2+3x-18}$.

SOLUTION

To determine the numbers that must be excluded from each domain, examine the denominators.

a. $\frac{4}{x-2}$ This denominator would equal zero if $x = 2$.

b. $\frac{x}{x^2-1} = \frac{x}{(x+1)(x-1)}$ This factor would equal zero if $x = -1$. This factor would equal zero if $x = 1$.

c. $\frac{9x}{x^2+3x-18} = \frac{9x}{(x+6)(x-3)}$ This factor would equal zero if $x = -6$. This factor would equal zero if $x = 3$.

For the rational expression in part (a), we must exclude 2 from the domain. For the rational expression in part (b), we must exclude both -1 and 1 from the domain. For the rational expression in part (c), we must exclude both -6 and 3 from the domain. These excluded numbers are often written to the right of a rational expression:

$$\frac{4}{x-2},\ x \neq 2 \qquad \frac{x}{x^2-1},\ x \neq -1, x \neq 1 \qquad \frac{9x}{x^2+3x-18},\ x \neq -6, x \neq 3.$$

Check Point 1 Find all the numbers that must be excluded from the domain of each rational expression:

a. $\frac{7}{x+5}$ **b.** $\frac{x}{x^2-36}$ **c.** $\frac{7x}{x^2-5x-14}$.

2 Simplify rational expressions.

Simplifying Rational Expressions

A rational expression is **simplified** if its numerator and denominator have no common factors other than 1 or -1. The following procedure can be used to simplify rational expressions:

Simplifying Rational Expressions

1. Factor the numerator and the denominator completely.
2. Divide both the numerator and the denominator by any common factors.

EXAMPLE 2 Simplifying Rational Expressions

Simplify:

a. $\dfrac{x^3 + x^2}{x + 1}$ **b.** $\dfrac{x^2 + 6x + 5}{x^2 - 25}$.

SOLUTION

a. $\dfrac{x^3 + x^2}{x + 1} = \dfrac{x^2(x + 1)}{x + 1}$ Factor the numerator. Because the denominator is $x + 1$, $x \neq -1$.

$= \dfrac{x^2\cancel{(x + 1)}^{1}}{\cancel{x + 1}_{1}}$ Divide out the common factor, $x + 1$.

$= x^2, x \neq -1$ Denominators of 1 need not be written because $\frac{a}{1} = a$.

b. $\dfrac{x^2 + 6x + 5}{x^2 - 25} = \dfrac{(x + 5)(x + 1)}{(x + 5)(x - 5)}$ Factor the numerator and denominator. Because the denominator is $(x + 5)(x - 5)$, $x \neq -5$ and $x \neq 5$.

$= \dfrac{\cancel{(x + 5)}^{1}(x + 1)}{\cancel{(x + 5)}_{1}(x - 5)}$ Divide out the common factor, $x + 5$.

$= \dfrac{x + 1}{x - 5}, \quad x \neq -5, \quad x \neq 5$

Check Point 2 Simplify:

a. $\dfrac{x^3 + 3x^2}{x + 3}$ **b.** $\dfrac{x^2 - 1}{x^2 + 2x + 1}$.

3 Multiply rational expressions.

Multiplying Rational Expressions

The product of two rational expressions is the product of their numerators divided by the product of their denominators. Here is a step-by-step procedure for multiplying rational expressions:

Multiplying Rational Expressions

1. Factor all numerators and denominators completely.
2. Divide numerators and denominators by common factors.
3. Multiply the remaining factors in the numerators and multiply the remaining factors in the denominators.

EXAMPLE 3 Multiplying Rational Expressions

Multiply: $\dfrac{x - 7}{x - 1} \cdot \dfrac{x^2 - 1}{3x - 21}$.

SOLUTION

$$\frac{x-7}{x-1}\cdot\frac{x^2-1}{3x-21}$$ This is the given multiplication problem.

$$=\frac{x-7}{x-1}\cdot\frac{(x+1)(x-1)}{3(x-7)}$$ Factor as many numerators and denominators as possible. Because the denominators have factors of x − 1 and x − 7, x ≠ 1 and x ≠ 7.

$$=\frac{\overset{1}{\cancel{x-7}}}{\underset{1}{\cancel{x-1}}}\cdot\frac{(x+1)\overset{1}{\cancel{(x-1)}}}{3\underset{1}{\cancel{(x-7)}}}$$ Divide numerators and denominators by common factors.

$$=\frac{x+1}{3},\ x\neq 1,\ x\neq 7$$ Multiply the remaining factors in the numerators and denominators.

These excluded numbers from the domain must also be excluded from the simplified expression's domain.

Check Point 3 Multiply:

$$\frac{x+3}{x^2-4}\cdot\frac{x^2-x-6}{x^2+6x+9}.$$

4 Divide rational expressions.

Dividing Rational Expressions

The quotient of two rational expressions is the product of the first expression and the multiplicative inverse, or reciprocal, of the second expression. The reciprocal is found by interchanging the numerator and the denominator. Thus, **we find the quotient of two rational expressions by inverting the divisor and multiplying**.

EXAMPLE 4 Dividing Rational Expressions

Divide: $\frac{x^2-2x-8}{x^2-9}\div\frac{x-4}{x+3}$.

SOLUTION

$$\frac{x^2-2x-8}{x^2-9}\div\frac{x-4}{x+3}$$ This is the given division problem.

$$=\frac{x^2-2x-8}{x^2-9}\cdot\frac{x+3}{x-4}$$ Invert the divisor and multiply.

$$=\frac{(x-4)(x+2)}{(x+3)(x-3)}\cdot\frac{x+3}{x-4}$$ Factor as many numerators and denominators as possible. For nonzero denominators, x ≠ −3, x ≠ 3, and x ≠ 4.

$$=\frac{\overset{1}{\cancel{(x-4)}}(x+2)}{\underset{1}{\cancel{(x+3)}}(x-3)}\cdot\frac{\overset{1}{\cancel{(x+3)}}}{\underset{1}{\cancel{(x-4)}}}$$ Divide numerators and denominators by common factors.

$$=\frac{x+2}{x-3},\ x\neq -3,\ x\neq 3,\ x\neq 4$$ Multiply the remaining factors in the numerators and in the denominators.

Check Point 4 Divide:

$$\frac{x^2 - 2x + 1}{x^3 + x} \div \frac{x^2 + x - 2}{3x^2 + 3}.$$

5 Add and subtract rational expressions.

Adding and Subtracting Rational Expressions with the Same Denominator

We add or subtract rational expressions with the same denominator by (1) adding or subtracting the numerators, (2) placing this result over the common denominator, and (3) simplifying, if possible.

EXAMPLE 5 Subtracting Rational Expressions with the Same Denominator

Subtract: $\frac{5x + 1}{x^2 - 9} - \frac{4x - 2}{x^2 - 9}$.

SOLUTION

Don't forget the parentheses.

$$\frac{5x + 1}{x^2 - 9} - \frac{4x - 2}{x^2 - 9} = \frac{5x + 1 - (4x - 2)}{x^2 - 9}$$

Subtract numerators and include parentheses to indicate that both terms are subtracted. Place this difference over the common denominator.

$$= \frac{5x + 1 - 4x + 2}{x^2 - 9}$$

Remove parentheses and then change the sign of each term in parentheses.

$$= \frac{x + 3}{x^2 - 9}$$

Combine like terms.

$$= \frac{\cancel{x + 3}^{1}}{\cancel{(x + 3)}_{1}(x - 3)}$$

Factor and simplify ($x \neq -3$ and $x \neq 3$).

$$= \frac{1}{x - 3}, x \neq -3, x \neq 3$$

GREAT QUESTION!

When subtracting a numerator containing more than one term, do I have to insert parentheses like you did in Example 5?

Yes. When a numerator is being subtracted, we must **subtract every term in that expression**. Parentheses indicate that every term inside is being subtracted.

Check Point 5 Subtract: $\frac{x}{x + 1} - \frac{3x + 2}{x + 1}$.

Adding and Subtracting Rational Expressions with Different Denominators

Rational expressions that have no common factors in their denominators can be added or subtracted using one of the following properties:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}, b \neq 0, d \neq 0.$$

The denominator, bd, is the product of the factors in the two denominators. Because we are considering rational expressions that have no common factors in their denominators, the product bd gives the least common denominator.

EXAMPLE 6 Subtracting Rational Expressions Having No Common Factors in Their Denominators

Subtract: $\dfrac{x+2}{2x-3} - \dfrac{4}{x+3}$.

SOLUTION

We need to find the least common denominator. This is the product of the distinct factors in each denominator, namely $(2x-3)(x+3)$. We can therefore use the subtraction property given previously as follows:

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

$$\frac{x+2}{2x-3} - \frac{4}{x+3} = \frac{(x+2)(x+3) - (2x-3)4}{(2x-3)(x+3)}$$

Observe that $a = x + 2$, $b = 2x - 3$, $c = 4$, and $d = x + 3$.

$$= \frac{x^2 + 5x + 6 - (8x - 12)}{(2x-3)(x+3)}$$

Multiply in the numerator.

$$= \frac{x^2 + 5x + 6 - 8x + 12}{(2x-3)(x+3)}$$

Remove parentheses and then change the sign of each term in parentheses.

$$= \frac{x^2 - 3x + 18}{(2x-3)(x+3)}, x \neq \frac{3}{2}, x \neq -3$$

Combine like terms in the numerator.

Check Point 6 Add: $\dfrac{3}{x+1} + \dfrac{5}{x-1}$.

The **least common denominator**, or LCD, of several rational expressions is a polynomial consisting of the product of all prime factors in the denominators, with each factor raised to the greatest power of its occurrence in any denominator. When adding and subtracting rational expressions that have different denominators with one or more common factors in the denominators, it is efficient to find the least common denominator first.

Finding the Least Common Denominator

1. Factor each denominator completely.
2. List the factors of the first denominator.
3. Add to the list in step 2 any factors of the second denominator that do not appear in the list.
4. Form the product of the factors from the list in step 3. This product is the least common denominator.

EXAMPLE 7 Finding the Least Common Denominator

Find the least common denominator of

$$\frac{7}{5x^2 + 15x} \quad \text{and} \quad \frac{9}{x^2 + 6x + 9}.$$

SOLUTION

Step 1 Factor each denominator completely.

$$5x^2 + 15x = 5x(x + 3)$$
$$x^2 + 6x + 9 = (x + 3)^2 \quad \text{or} \quad (x + 3)(x + 3)$$

Factors are 5, x, and $x + 3$. $\quad \dfrac{7}{5x^2 + 15x} \qquad \dfrac{9}{x^2 + 6x + 9}$ Factors are $x + 3$ and $x + 3$.

Step 2 List the factors of the first denominator.

$$5, x, x + 3$$

Step 3 Add any unlisted factors from the second denominator. One factor of $x^2 + 6x + 9$ is already in our list. That factor is $x + 3$. However, the other factor of $x + 3$ is not listed in step 2. We add a second factor of $x + 3$ to the list. We have

$$5, x, x + 3, x + 3.$$

Step 4 The least common denominator is the product of all factors in the final list. Thus,

$$5x(x + 3)(x + 3) \quad \text{or} \quad 5x(x + 3)^2$$

is the least common denominator.

Check Point 7 Find the least common denominator of

$$\frac{3}{x^2 - 6x + 9} \quad \text{and} \quad \frac{7}{x^2 - 9}.$$

Finding the least common denominator for two (or more) rational expressions is the first step needed to add or subtract the expressions.

Adding and Subtracting Rational Expressions That Have Different Denominators

1. Find the LCD of the rational expressions.
2. Rewrite each rational expression as an equivalent expression whose denominator is the LCD. To do so, multiply the numerator and the denominator of each rational expression by any factor(s) needed to convert the denominator into the LCD.
3. Add or subtract numerators, placing the resulting expression over the LCD.
4. If possible, simplify the resulting rational expression.

EXAMPLE 8 Adding Rational Expressions with Different Denominators

Add: $\dfrac{x + 3}{x^2 + x - 2} + \dfrac{2}{x^2 - 1}$.

SOLUTION

Step 1 Find the least common denominator. Start by factoring the denominators.

$$x^2 + x - 2 = (x + 2)(x - 1)$$
$$x^2 - 1 = (x + 1)(x - 1)$$

The factors of the first denominator are $x + 2$ and $x - 1$. The only factor from the second denominator that is not listed is $x + 1$. Thus, the least common denominator is

$$(x + 2)(x - 1)(x + 1).$$

Step 2 Write equivalent expressions with the LCD as denominators. We must rewrite each rational expression with a denominator of $(x + 2)(x - 1)(x + 1)$. We do so by multiplying both the numerator and the denominator of each rational expression by any factor(s) needed to convert the expression's denominator into the LCD.

$$\frac{x+3}{(x+2)(x-1)} \cdot \frac{x+1}{x+1} = \frac{(x+3)(x+1)}{(x+2)(x-1)(x+1)} \qquad \frac{2}{(x+1)(x-1)} \cdot \frac{x+2}{x+2} = \frac{2(x+2)}{(x+2)(x-1)(x+1)}$$

Multiply the numerator and denominator by $x + 1$ to get $(x + 2)(x - 1)(x + 1)$, the LCD.

Multiply the numerator and denominator by $x + 2$ to get $(x + 2)(x - 1)(x + 1)$, the LCD.

Because $\frac{x+1}{x+1} = 1$ and $\frac{x+2}{x+2} = 1$, we are not changing the value of either rational expression, only its appearance.

Now we are ready to perform the indicated addition.

$$\frac{x+3}{x^2+x-2} + \frac{2}{x^2-1}$$ This is the given problem.

$$= \frac{x+3}{(x+2)(x-1)} + \frac{2}{(x+1)(x-1)}$$ Factor the denominators. The LCD is $(x + 2)(x - 1)(x + 1)$.

$$= \frac{(x+3)(x+1)}{(x+2)(x-1)(x+1)} + \frac{2(x+2)}{(x+2)(x-1)(x+1)}$$ Rewrite equivalent expressions with the LCD.

Step 3 Add numerators, putting this sum over the LCD.

$$= \frac{(x+3)(x+1) + 2(x+2)}{(x+2)(x-1)(x+1)}$$

$$= \frac{x^2 + 4x + 3 + 2x + 4}{(x+2)(x-1)(x+1)}$$ Perform the multiplications in the numerator.

$$= \frac{x^2 + 6x + 7}{(x+2)(x-1)(x+1)}, x \neq -2, x \neq 1, x \neq -1$$ Combine like terms in the numerator: $4x + 2x = 6x$ and $3 + 4 = 7$.

Step 4 If necessary, simplify. Because the numerator is prime, no further simplification is possible.

Check Point 8 Subtract: $\frac{x}{x^2 - 10x + 25} - \frac{x-4}{2x-10}$.

6 Simplify complex rational expressions.

Complex Rational Expressions

Complex rational expressions, also called **complex fractions**, have numerators or denominators containing one or more rational expressions. Here are two examples of such expressions:

$$\frac{1 + \frac{1}{x}}{1 - \frac{1}{x}}$$ Separate rational expressions occur in the numerator and the denominator.

$$\frac{\frac{1}{x+h} - \frac{1}{x}}{h}.$$ Separate rational expressions occur in the numerator.

One method for simplifying a complex rational expression is to combine its numerator into a single expression and combine its denominator into a single expression. Then perform the division by inverting the denominator and multiplying.

EXAMPLE 9 Simplifying a Complex Rational Expression

Simplify: $\dfrac{1+\dfrac{1}{x}}{1-\dfrac{1}{x}}$.

SOLUTION

Step 1 Add to get a single rational expression in the numerator.

$$1+\frac{1}{x}=\frac{1}{1}+\frac{1}{x}=\frac{1\cdot x}{1\cdot x}+\frac{1}{x}=\frac{x}{x}+\frac{1}{x}=\frac{x+1}{x}$$

The LCD is $1\cdot x$, or x.

Step 2 Subtract to get a single rational expression in the denominator.

$$1-\frac{1}{x}=\frac{1}{1}-\frac{1}{x}=\frac{1\cdot x}{1\cdot x}-\frac{1}{x}=\frac{x}{x}-\frac{1}{x}=\frac{x-1}{x}$$

The LCD is $1\cdot x$, or x.

Step 3 Perform the division indicated by the main fraction bar: Invert and multiply. If possible, simplify.

$$\frac{1+\frac{1}{x}}{1-\frac{1}{x}}=\frac{\frac{x+1}{x}}{\frac{x-1}{x}}=\frac{x+1}{x}\cdot\frac{x}{x-1}=\frac{x+1}{\cancel{x}_1}\cdot\frac{\overset{1}{\cancel{x}}}{x-1}=\frac{x+1}{x-1}$$

Invert and multiply.

Check Point 9 Simplify: $\dfrac{\dfrac{1}{x}-\dfrac{3}{2}}{\dfrac{1}{x}+\dfrac{3}{4}}$.

A second method for simplifying a complex rational expression is to find the least common denominator of all the rational expressions in its numerator and denominator. Then multiply each term in its numerator and denominator by this least common denominator. Because we are multiplying by a form of 1, we will obtain an equivalent expression that does not contain fractions in its numerator or denominator. Here we use this method to simplify the complex rational expression in Example 9.

$$\frac{1+\frac{1}{x}}{1-\frac{1}{x}}=\frac{\left(1+\frac{1}{x}\right)}{\left(1-\frac{1}{x}\right)}\cdot\frac{x}{x}$$

The least common denominator of all the rational expressions is x. Multiply the numerator and denominator by x. Because $\frac{x}{x}=1$, we are not changing the complex fraction ($x\neq 0$).

$$=\frac{1\cdot x+\frac{1}{x}\cdot x}{1\cdot x-\frac{1}{x}\cdot x}$$

Use the distributive property. Be sure to distribute x to every term.

$$=\frac{x+1}{x-1},\ x\neq 0,\ x\neq 1$$

Multiply. The complex rational expression is now simplified.

EXAMPLE 10 Simplifying a Complex Rational Expression

Simplify: $\dfrac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h}$.

SOLUTION

We will use the method of multiplying each of the three terms, $\dfrac{1}{x+h}$, $\dfrac{1}{x}$, and h, by the least common denominator. The least common denominator is $x(x + h)$.

$$\frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h}$$

$$= \frac{\left(\dfrac{1}{x+h} - \dfrac{1}{x}\right)x(x+h)}{hx(x+h)}$$

Multiply the numerator and denominator by $x(x + h)$, $h \neq 0$, $x \neq 0$, $x \neq -h$.

$$= \frac{\dfrac{1}{x+h} \cdot x(x+h) - \dfrac{1}{x} \cdot x(x+h)}{hx(x+h)}$$

Use the distributive property in the numerator.

$$= \frac{x - (x+h)}{hx(x+h)}$$

Simplify: $\dfrac{1}{\cancel{x+h}} \cdot x\cancel{(x+h)} = x$ and $\dfrac{1}{\cancel{x}} \cdot \cancel{x}(x+h) = x + h$.

$$= \frac{x - x - h}{hx(x+h)}$$

Subtract in the numerator. Remove parentheses and change the sign of each term in parentheses.

$$= \frac{-\overset{1}{\cancel{h}}}{\underset{1}{\cancel{h}}x(x+h)}$$

Simplify: $x - x - h = -h$.

$$= -\frac{1}{x(x+h)}, h \neq 0, x \neq 0, x \neq -h$$

Divide the numerator and denominator by h.

Check Point 10 Simplify: $\dfrac{\dfrac{1}{x+7} - \dfrac{1}{x}}{7}$.

7 Simplify fractional expressions that occur in calculus.

Fractional Expressions in Calculus

Fractional expressions containing radicals occur frequently in calculus. Because of the radicals, these expressions are not rational expressions. However, they can often be simplified using the procedure for simplifying complex rational expressions.

EXAMPLE 11 Simplifying a Fractional Expression Containing Radicals

Simplify: $\dfrac{\sqrt{9 - x^2} + \dfrac{x^2}{\sqrt{9 - x^2}}}{9 - x^2}$.

SOLUTION

$$\frac{\sqrt{9-x^2}+\dfrac{x^2}{\sqrt{9-x^2}}}{9-x^2}$$

The least common denominator is $\sqrt{9-x^2}$.

$$=\frac{\sqrt{9-x^2}+\dfrac{x^2}{\sqrt{9-x^2}}}{9-x^2}\cdot\frac{\sqrt{9-x^2}}{\sqrt{9-x^2}}$$

Multiply the numerator and the denominator by $\sqrt{9-x^2}$.

$$=\frac{\sqrt{9-x^2}\sqrt{9-x^2}+\dfrac{x^2}{\sqrt{9-x^2}}\sqrt{9-x^2}}{(9-x^2)\sqrt{9-x^2}}$$

Use the distributive property in the numerator.

$$=\frac{(9-x^2)+x^2}{(9-x^2)^{\frac{3}{2}}}$$

In the denominator:
$(9-x^2)^1(9-x^2)^{\frac{1}{2}}=(9-x^2)^{1+\frac{1}{2}}$
$=(9-x^2)^{\frac{3}{2}}$.

$$=\frac{9}{\sqrt{(9-x^2)^3}}$$

Because the original expression was in radical form, write the denominator in radical form.

Check Point 11 Simplify: $\dfrac{\sqrt{x}+\dfrac{1}{\sqrt{x}}}{x}$.

7 Rationalize numerators.

Another fractional expression that you will encounter in calculus is

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}.$$

Can you see that this expression is not defined if $h=0$? However, in calculus, you will ask the following question:

What happens to the expression as h takes on values that get closer and closer to 0, such as $h=0.1$, $h=0.01$, $h=0.001$, $h=0.0001$, and so on?

The question is answered by first **rationalizing the numerator**. This process involves rewriting the fractional expression as an equivalent expression in which the numerator no longer contains any radicals. **To rationalize a numerator, multiply by 1 to eliminate the radicals in the *numerator*. Multiply the numerator and the denominator by the conjugate of the numerator.**

EXAMPLE 12 Rationalizing a Numerator

Rationalize the numerator:

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}.$$

SOLUTION

The conjugate of the numerator is $\sqrt{x+h}+\sqrt{x}$. If we multiply the numerator and denominator by $\sqrt{x+h}+\sqrt{x}$, the simplified numerator will not contain a radical. Therefore, we multiply by 1, choosing $\dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$ for 1.

$$\frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$ Multiply by 1.

$$= \frac{\left(\sqrt{x+h}\right)^2 - (\sqrt{x})^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$ $(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2$

$$= \frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$ $\left(\sqrt{x+h}\right)^2 = x+h$ and $(\sqrt{x})^2 = x$.

$$= \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$ Simplify: $x+h-x=h$.

$$= \frac{1}{\sqrt{x+h}+\sqrt{x}}, \quad h \neq 0$$ Divide both the numerator and denominator by h.

What happens to $\frac{\sqrt{x+h}-\sqrt{x}}{h}$ as h gets closer and closer to 0? In Example 12, we showed that

$$\frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{1}{\sqrt{x+h}+\sqrt{x}}.$$

As h gets closer to 0, the expression on the right gets closer to $\frac{1}{\sqrt{x+0}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}}$, or $\frac{1}{2\sqrt{x}}$. Thus, the fractional expression $\frac{\sqrt{x+h}-\sqrt{x}}{h}$ approaches $\frac{1}{2\sqrt{x}}$ as h gets closer to 0.

Blitzer Bonus
Calculus Preview

In calculus, you will summarize the discussion on the right using the special notation

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{1}{2\sqrt{x}}.$$

This is read "the limit of $\frac{\sqrt{x+h}-\sqrt{x}}{h}$ as h approaches 0 equals $\frac{1}{2\sqrt{x}}$."

Check Point 12 Rationalize the numerator: $\frac{\sqrt{x+3}-\sqrt{x}}{3}$.

CONCEPT AND VOCABULARY CHECK

Fill in each blank so that the resulting statement is true.

1. A rational expression is the quotient of two ________.
2. The set of real numbers for which a rational expression is defined is the ________ of the expression. We must exclude all numbers from this set that make the denominator of the rational expression __.
3. We simplify a rational expression by ________ the numerator and the denominator completely. Then we divide the numerator and the denominator by any ____________.
4. $\frac{x}{5}\cdot\frac{x}{3} =$ ___
5. $\frac{x}{5} \div \frac{x}{3} =$ ___, $x \neq 0$
6. $\frac{x^2}{3} - \frac{x-4}{3} =$ ________
7. Consider the following subtraction problem:

$$\frac{x-1}{x^2+x-6} - \frac{x-2}{x^2+4x+3}.$$

The factors of the first denominator are ____________. The factors of the second denominator are ____________. The LCD is ____________.

8. An equivalent expression for $\frac{3x+2}{x-5}$ with a denominator of $(3x+4)(x-5)$ can be obtained by multiplying the numerator and denominator by ______.
9. A rational expression whose numerator or denominator or both contain rational expressions is called a/an ________ rational expression or a/an ________ fraction.
10. $$\frac{\frac{1}{x+3}-\frac{1}{x}}{3} = \frac{x(x+3)}{x(x+3)} \cdot \frac{\left(\frac{1}{x+3}-\frac{1}{x}\right)}{3} = \frac{\_ - (\_\_\_\_)}{3x(x+3)}$$
$$= \frac{\_\_}{3x(x+3)}$$
$$= \_\_\_\_\_\_$$
11. We can simplify

$$\frac{\sqrt{x}+\frac{1}{\sqrt{x}}}{x}$$

by multiplying the numerator and the denominator by __________.

12. We can rationalize the numerator of $\frac{\sqrt{x+7}-\sqrt{x}}{7}$ by multiplying the numerator and the denominator by __________.

EXERCISE SET 6

Practice Exercises

In Exercises 1–6, find all numbers that must be excluded from the domain of each rational expression.

1. $\frac{7}{x-3}$

2. $\frac{13}{x+9}$

3. $\frac{x+5}{x^2-25}$

4. $\frac{x+7}{x^2-49}$

5. $\frac{x-1}{x^2+11x+10}$

6. $\frac{x-3}{x^2+4x-45}$

In Exercises 7–14, simplify each rational expression. Find all numbers that must be excluded from the domain of the simplified rational expression.

7. $\frac{3x-9}{x^2-6x+9}$

8. $\frac{4x-8}{x^2-4x+4}$

9. $\frac{x^2-12x+36}{4x-24}$

10. $\frac{x^2-8x+16}{3x-12}$

11. $\frac{y^2+7y-18}{y^2-3y+2}$

12. $\frac{y^2-4y-5}{y^2+5y+4}$

13. $\frac{x^2+12x+36}{x^2-36}$

14. $\frac{x^2-14x+49}{x^2-49}$

In Exercises 15–32, multiply or divide as indicated.

15. $\frac{x-2}{3x+9}\cdot\frac{2x+6}{2x-4}$

16. $\frac{6x+9}{3x-15}\cdot\frac{x-5}{4x+6}$

17. $\frac{x^2-9}{x^2}\cdot\frac{x^2-3x}{x^2+x-12}$

18. $\frac{x^2-4}{x^2-4x+4}\cdot\frac{2x-4}{x+2}$

19. $\frac{x^2-5x+6}{x^2-2x-3}\cdot\frac{x^2-1}{x^2-4}$

20. $\frac{x^2+5x+6}{x^2+x-6}\cdot\frac{x^2-9}{x^2-x-6}$

21. $\frac{x^3-8}{x^2-4}\cdot\frac{x+2}{3x}$

22. $\frac{x^2+6x+9}{x^3+27}\cdot\frac{1}{x+3}$

23. $\frac{x+1}{3}\div\frac{3x+3}{7}$

24. $\frac{x+5}{7}\div\frac{4x+20}{9}$

25. $\frac{x^2-4}{x}\div\frac{x+2}{x-2}$

26. $\frac{x^2-4}{x-2}\div\frac{x+2}{4x-8}$

27. $\frac{4x^2+10}{x-3}\div\frac{6x^2+15}{x^2-9}$

28. $\frac{x^2+x}{x^2-4}\div\frac{x^2-1}{x^2+5x+6}$

29. $\frac{x^2-25}{2x-2}\div\frac{x^2+10x+25}{x^2+4x-5}$

30. $\frac{x^2-4}{x^2+3x-10}\div\frac{x^2+5x+6}{x^2+8x+15}$

31. $\frac{x^2+x-12}{x^2+x-30}\cdot\frac{x^2+5x+6}{x^2-2x-3}\div\frac{x+3}{x^2+7x+6}$

32. $\frac{x^3-25x}{4x^2}\cdot\frac{2x^2-2}{x^2-6x+5}\div\frac{x^2+5x}{7x+7}$

In Exercises 33–58, add or subtract as indicated.

33. $\frac{4x+1}{6x+5}+\frac{8x+9}{6x+5}$

34. $\frac{3x+2}{3x+4}+\frac{3x+6}{3x+4}$

35. $\frac{x^2-2x}{x^2+3x}+\frac{x^2+x}{x^2+3x}$

36. $\frac{x^2-4x}{x^2-x-6}+\frac{4x-4}{x^2-x-6}$

37. $\frac{4x-10}{x-2}-\frac{x-4}{x-2}$

38. $\frac{2x+3}{3x-6}-\frac{3-x}{3x-6}$

39. $\frac{x^2+3x}{x^2+x-12}-\frac{x^2-12}{x^2+x-12}$

40. $\frac{x^2-4x}{x^2-x-6}-\frac{x-6}{x^2-x-6}$

41. $\frac{3}{x+4}+\frac{6}{x+5}$

42. $\frac{8}{x-2}+\frac{2}{x-3}$

43. $\frac{3}{x+1}-\frac{3}{x}$

44. $\frac{4}{x}-\frac{3}{x+3}$

45. $\frac{2x}{x+2}+\frac{x+2}{x-2}$

46. $\frac{3x}{x-3}-\frac{x+4}{x+2}$

47. $\frac{x+5}{x-5}+\frac{x-5}{x+5}$

48. $\frac{x+3}{x-3}+\frac{x-3}{x+3}$

49. $\frac{3}{2x+4}+\frac{2}{3x+6}$

50. $\frac{5}{2x+8}+\frac{7}{3x+12}$

51. $\frac{4}{x^2+6x+9}+\frac{4}{x+3}$

52. $\frac{3}{5x+2}+\frac{5x}{25x^2-4}$

53. $\frac{3x}{x^2+3x-10}-\frac{2x}{x^2+x-6}$

54. $\frac{x}{x^2-2x-24}-\frac{x}{x^2-7x+6}$

55. $\frac{x+3}{x^2-1}-\frac{x+2}{x-1}$

56. $\frac{x+5}{x^2-4}-\frac{x+1}{x-2}$

57. $\frac{4x^2+x-6}{x^2+3x+2}-\frac{3x}{x+1}+\frac{5}{x+2}$

58. $\frac{6x^2+17x-40}{x^2+x-20}+\frac{3}{x-4}-\frac{5x}{x+5}$

In Exercises 59–72, simplify each complex rational expression.

59. $\dfrac{\frac{x}{3}-1}{x-3}$

60. $\dfrac{\frac{x}{4}-1}{x-4}$

61. $\dfrac{1+\frac{1}{x}}{3-\frac{1}{x}}$

62. $\dfrac{8+\frac{1}{x}}{4-\frac{1}{x}}$

63. $\dfrac{\frac{1}{x}+\frac{1}{y}}{x+y}$

64. $\dfrac{1-\frac{1}{x}}{xy}$

65. $\dfrac{x-\frac{x}{x+3}}{x+2}$

66. $\dfrac{x-3}{x-\frac{3}{x-2}}$

67. $\dfrac{\frac{3}{x-2}-\frac{4}{x+2}}{\frac{7}{x^2-4}}$

68. $\dfrac{\frac{x}{x-2}+1}{\frac{3}{x^2-4}+1}$

69. $\dfrac{\dfrac{1}{x+1}}{\dfrac{1}{x^2-2x-3}+\dfrac{1}{x-3}}$

70. $\dfrac{\dfrac{6}{x^2+2x-15}-\dfrac{1}{x-3}}{\dfrac{1}{x+5}+1}$

71. $\dfrac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}$

72. $\dfrac{\dfrac{x+h}{x+h+1}-\dfrac{x}{x+1}}{h}$

Exercises 73–78 contain fractional expressions that occur frequently in calculus. Simplify each expression.

73. $\dfrac{\sqrt{x}-\dfrac{1}{3\sqrt{x}}}{\sqrt{x}}$

74. $\dfrac{\sqrt{x}-\dfrac{1}{4\sqrt{x}}}{\sqrt{x}}$

75. $\dfrac{\dfrac{x^2}{\sqrt{x^2+2}}-\sqrt{x^2+2}}{x^2}$

76. $\dfrac{\sqrt{5-x^2}+\dfrac{x^2}{\sqrt{5-x^2}}}{5-x^2}$

77. $\dfrac{\dfrac{1}{\sqrt{x+h}}-\dfrac{1}{\sqrt{x}}}{h}$

78. $\dfrac{\dfrac{1}{\sqrt{x+3}}-\dfrac{1}{\sqrt{x}}}{3}$

In Exercises 79–82, rationalize the numerator.

79. $\dfrac{\sqrt{x+5}-\sqrt{x}}{5}$

80. $\dfrac{\sqrt{x+7}-\sqrt{x}}{7}$

81. $\dfrac{\sqrt{x}+\sqrt{y}}{x^2-y^2}$

82. $\dfrac{\sqrt{x}-\sqrt{y}}{x^2-y^2}$

Practice Plus

In Exercises 83–90, perform the indicated operations. Simplify the result, if possible.

83. $\left(\dfrac{2x+3}{x+1}\cdot\dfrac{x^2+4x-5}{2x^2+x-3}\right)-\dfrac{2}{x+2}$

84. $\dfrac{1}{x^2-2x-8}\div\left(\dfrac{1}{x-4}-\dfrac{1}{x+2}\right)$

85. $\left(2-\dfrac{6}{x+1}\right)\left(1+\dfrac{3}{x-2}\right)$

86. $\left(4-\dfrac{3}{x+2}\right)\left(1+\dfrac{5}{x-1}\right)$

87. $\dfrac{y^{-1}-(y+5)^{-1}}{5}$

88. $\dfrac{y^{-1}-(y+2)^{-1}}{2}$

89. $\left(\dfrac{1}{a^3-b^3}\cdot\dfrac{ac+ad-bc-bd}{1}\right)-\dfrac{c-d}{a^2+ab+b^2}$

90. $\dfrac{ab}{a^2+ab+b^2}+\left(\dfrac{ac-ad-bc+bd}{ac-ad+bc-bd}\div\dfrac{a^3-b^3}{a^3+b^3}\right)$

Application Exercises

91. The rational expression

$$\frac{130x}{100-x}$$

describes the cost, in millions of dollars, to inoculate x percent of the population against a particular strain of flu.

a. Evaluate the expression for $x = 40$, $x = 80$, and $x = 90$. Describe the meaning of each evaluation in terms of percentage inoculated and cost.

b. For what value of x is the expression undefined?

c. What happens to the cost as x approaches 100%? How can you interpret this observation?

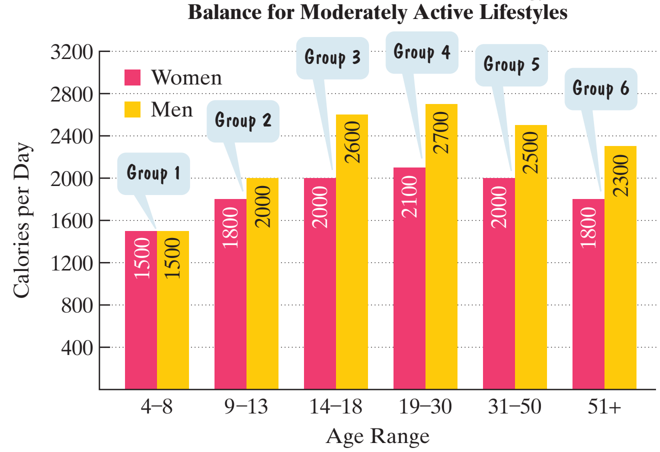
92. The average rate on a round-trip commute having a one-way distance d is given by the complex rational expression

$$\frac{2d}{\dfrac{d}{r_1}+\dfrac{d}{r_2}},$$

in which r_1 and r_2 are the average rates on the outgoing and return trips, respectively. Simplify the expression. Then find your average rate if you drive to campus averaging 40 miles per hour and return home on the same route averaging 30 miles per hour. Explain why the answer is not 35 miles per hour.

93. The bar graph shows the estimated number of calories per day needed to maintain energy balance for various gender and age groups for moderately active lifestyles. (Moderately active means a lifestyle that includes physical activity equivalent to walking 1.5 to 3 miles per day at 3 to 4 miles per hour, in addition to the light physical activity associated with typical day-to-day life.)

Calories Needed to Maintain Energy Balance for Moderately Active Lifestyles



Source: U.S.D.A.

a. The mathematical model

$$W = -66x^2 + 526x + 1030$$

describes the number of calories needed per day, W, by women in age group x with moderately active lifestyles. According to the model, how many calories per day are needed by women between the ages of 19 and 30, inclusive, with this lifestyle? Does this underestimate or overestimate the number shown by the graph? By how much?

b. The mathematical model

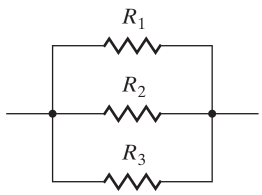
$$M = -120x^2 + 998x + 590$$

describes the number of calories needed per day, M, by men in age group x with moderately active lifestyles. According to the model, how many calories per day are needed by men between the ages of 19 and 30, inclusive, with this lifestyle? Does this underestimate or overestimate the number shown by the graph? By how much?

c. Write a simplified rational expression that describes the ratio of the number of calories needed per day by women in age group x to the number of calories needed per day by men in age group x for people with moderately active lifestyles.

94. If three resistors with resistances $R_1, R_2,$ and R_3 are connected in parallel, their combined resistance is given by the expression

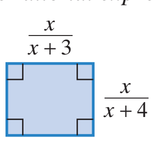
$$\frac{1}{\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}}.$$



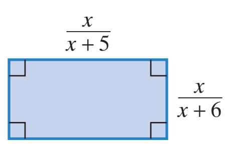
Simplify the complex rational expression. Then find the combined resistance when R_1 is 4 ohms, R_2 is 8 ohms, and R_3 is 12 ohms.

In Exercises 95–96, express the perimeter of each rectangle as a single rational expression.

95.



96.



Writing in Mathematics

97. What is a rational expression?

98. Explain how to determine which numbers must be excluded from the domain of a rational expression.

99. Explain how to simplify a rational expression.

100. Explain how to multiply rational expressions.

101. Explain how to divide rational expressions.

102. Explain how to add or subtract rational expressions with the same denominators.

103. Explain how to add rational expressions having no common factors in their denominators. Use $\frac{3}{x+5}+\frac{7}{x+2}$ in your explanation.

104. Explain how to find the least common denominator for denominators of x^2-100 and $x^2-20x+100$.

105. Describe two ways to simplify $\dfrac{\frac{3}{x}+\frac{2}{x^2}}{\frac{1}{x^2}+\frac{2}{x}}$.

Explain the error in Exercises 106–108. Then rewrite the right side of the equation to correct the error that now exists.

106. $\frac{1}{a}+\frac{1}{b}=\frac{1}{a+b}$

107. $\frac{1}{x}+7=\frac{1}{x+7}$

108. $\frac{a}{x}+\frac{a}{b}=\frac{a}{x+b}$

Critical Thinking Exercises

Make Sense? *In Exercises 109–112, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

109. I evaluated $\frac{3x-3}{4x(x-1)}$ for $x=1$ and obtained 0.

110. The rational expressions

$$\frac{7}{14x} \quad \text{and} \quad \frac{7}{14+x}$$

can both be simplified by dividing each numerator and each denominator by 7.

111. When performing the division

$$\frac{7x}{x+3} \div \frac{(x+3)^2}{x-5},$$

I began by dividing the numerator and the denominator by the common factor, $x+3$.

112. I subtracted $\frac{3x-5}{x-1}$ from $\frac{x-3}{x-1}$ and obtained a constant.

In Exercises 113–116, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

113. $\frac{x^2-25}{x-5}=x-5$

114. The expression $\frac{-3y-6}{y+2}$ simplifies to the consecutive integer that follows -4.

115. $\frac{2x-1}{x-7}+\frac{3x-1}{x-7}-\frac{5x-2}{x-7}=0$

116. $6+\frac{1}{x}=\frac{7}{x}$

In Exercises 117–119, perform the indicated operations.

117. $\frac{1}{x^n-1}-\frac{1}{x^n+1}-\frac{1}{x^{2n}-1}$

118. $\left(1-\frac{1}{x}\right)\left(1-\frac{1}{x+1}\right)\left(1-\frac{1}{x+2}\right)\left(1-\frac{1}{x+3}\right)$

119. $(x-y)^{-1}+(x-y)^{-2}$

120. In one short sentence, five words or less, explain what

$$\frac{\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}}{\frac{1}{x^4}+\frac{1}{x^5}+\frac{1}{x^6}}$$

does to each number x.

Preview Exercises

Exercises 121–123 will help you prepare for the material covered in the next section.

121. If 6 is substituted for x in the equation

$$2(x-3)-17=13-3(x+2),$$

is the resulting statement true or false?

122. Multiply and simplify: $12\left(\frac{x+2}{4}-\frac{x-1}{3}\right)$.

123. Evaluate

$$\frac{-b-\sqrt{b^2-4ac}}{2a}$$

for $a=2$, $b=9$, and $c=-5$.

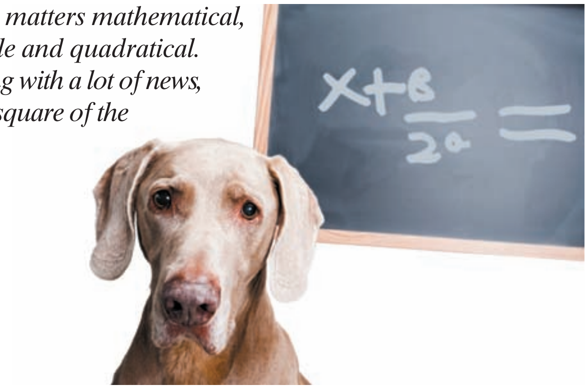
SECTION 7 Equations

Objectives

1. Solve linear equations in one variable.
2. Solve linear equations containing fractions.
3. Solve rational equations with variables in the denominators.
4. Solve a formula for a variable.
5. Solve equations involving absolute value.
6. Solve quadratic equations by factoring.
7. Solve quadratic equations by the square root property.
8. Solve quadratic equations by completing the square.
9. Solve quadratic equations using the quadratic formula.
10. Use the discriminant to determine the number and type of solutions of quadratic equations.
11. Determine the most efficient method to use when solving a quadratic equation.
12. Solve radical equations.

I'm very well acquainted, too, with matters mathematical,
I understand equations, both simple and quadratical.
About binomial theorem I'm teeming with a lot of news,
With many cheerful facts about the square of the hypotenuse.

—Gilbert and Sullivan,
The Pirates of Penzance



Equations quadratical? Cheerful news about the square of the hypotenuse? You've come to the right place. In this section, we will review how to solve a variety of equations, including linear equations, quadratic equations, and radical equations. (Yes, it's *quadratic* and not *quadratical*, despite the latter's rhyme with mathematical.) In the next section, we will look at applications of quadratic equations, introducing (cheerfully, of course) the Pythagorean Theorem and the square of the hypotenuse.

Solving Linear Equations in One Variable

We begin with a general definition of a linear equation in one variable.

Definition of a Linear Equation

A **linear equation in one variable x** is an equation that can be written in the form

$$ax + b = 0,$$

where a and b are real numbers, and $a \neq 0$.

An example of a linear equation in one variable is

$$4x + 12 = 0.$$

Solving an equation in x involves determining all values of x that result in a true statement when substituted into the equation. Such values are **solutions**, or **roots**, of the equation. For example, substitute -3 for x in $4x + 12 = 0$. We obtain

$$4(-3) + 12 = 0, \quad \text{or} \quad -12 + 12 = 0.$$

This simplifies to the true statement $0 = 0$. Thus, -3 is a solution of the equation $4x + 12 = 0$. We also say that -3 **satisfies** the equation $4x + 12 = 0$, because when we substitute -3 for x, a true statement results. The set of all such solutions is called the equation's **solution set**. For example, the solution set of the equation $4x + 12 = 0$ is $\{-3\}$ because -3 is the equation's only solution.

Two or more equations that have the same solution set are called **equivalent equations**. For example, the equations

$$4x + 12 = 0 \quad \text{and} \quad 4x = -12 \quad \text{and} \quad x = -3$$

are equivalent equations because the solution set for each is $\{-3\}$. To solve a linear equation in x, we transform the equation into an equivalent equation one or more times. Our final equivalent equation should be of the form

$$x = \text{a number}.$$

The solution set of this equation is the set consisting of the number.

To generate equivalent equations, we will use the following principles:

Generating Equivalent Equations

An equation can be transformed into an equivalent equation by one or more of the following operations:

Operation	Example
1. Simplify an expression by removing grouping symbols and combining like terms.	• $3(x - 6) = 6x - x$ $3x - 18 = 5x$
2. Add (or subtract) the same real number or variable expression on *both* sides of the equation.	• $3x - 18 = 5x$ $3x - 18 - 3x = 5x - 3x$ Subtract 3x from both sides of the equation. $-18 = 2x$
3. Multiply (or divide) by the same *nonzero* quantity on *both* sides of the equation.	• $-18 = 2x$ $\frac{-18}{2} = \frac{2x}{2}$ Divide both sides of the equation by 2. $-9 = x$
4. Interchange the two sides of the equation.	• $-9 = x$ $x = -9$

If you look closely at the equations in the box, you will notice that we have solved the equation $3(x - 6) = 6x - x$. The final equation, $x = -9$, with x isolated on the left side, shows that $\{-9\}$ is the solution set. The idea in solving a linear equation is to get the variable by itself on one side of the equal sign and a number by itself on the other side.

Here is a step-by-step procedure for solving a linear equation in one variable. Not all of these steps are necessary to solve every equation.

1 Solve linear equations in one variable.

Solving a Linear Equation

1. Simplify the algebraic expression on each side by removing grouping symbols and combining like terms.
2. Collect all the variable terms on one side and all the numbers, or constant terms, on the other side.
3. Isolate the variable and solve.
4. Check the proposed solution in the original equation.

EXAMPLE 1 Solving a Linear Equation

Solve and check: $2(x - 3) - 17 = 13 - 3(x + 2)$.

SOLUTION

Step 1 Simplify the algebraic expression on each side.

Do not begin with 13 − 3. Multiplication (the distributive property) is applied before subtraction.

$2(x - 3) - 17 = 13 - 3(x + 2)$ This is the given equation.

$2x - 6 - 17 = 13 - 3x - 6$ Use the distributive property.

$2x - 23 = -3x + 7$ Combine like terms.

Step 2 Collect variable terms on one side and constant terms on the other side. We will collect variable terms of $2x - 23 = -3x + 7$ on the left by adding $3x$ to both sides. We will collect the numbers on the right by adding 23 to both sides.

$$2x - 23 + 3x = -3x + 7 + 3x \quad \text{Add } 3x \text{ to both sides.}$$
$$5x - 23 = 7 \quad \text{Simplify: } 2x + 3x = 5x.$$
$$5x - 23 + 23 = 7 + 23 \quad \text{Add 23 to both sides.}$$
$$5x = 30 \quad \text{Simplify.}$$

Step 3 Isolate the variable and solve. We isolate the variable, x, by dividing both sides of $5x = 30$ by 5.

$$\frac{5x}{5} = \frac{30}{5} \quad \text{Divide both sides by 5.}$$
$$x = 6 \quad \text{Simplify.}$$

Step 4 Check the proposed solution in the original equation. Substitute 6 for x in the original equation.

$$2(x - 3) - 17 = 13 - 3(x + 2) \quad \text{This is the original equation.}$$
$$2(6 - 3) - 17 \stackrel{?}{=} 13 - 3(6 + 2) \quad \text{Substitute 6 for } x.$$
$$2(3) - 17 \stackrel{?}{=} 13 - 3(8) \quad \text{Simplify inside parentheses.}$$
$$6 - 17 \stackrel{?}{=} 13 - 24 \quad \text{Multiply.}$$
$$-11 = -11 \quad \text{Subtract.}$$

The true statement $-11 = -11$ verifies that the solution set is $\{6\}$. • • •

DISCOVERY

Solve the equation in Example 1 by collecting terms with the variable on the right and numerical terms on the left. What do you observe?

Check Point 1 Solve and check: $4(2x + 1) = 29 + 3(2x - 5)$.

2 Solve linear equations containing fractions.

Linear Equations with Fractions

Equations are easier to solve when they do not contain fractions. How do we remove fractions from an equation? We begin by multiplying both sides of the equation by the least common denominator of any fractions in the equation. The least common denominator is the smallest number that all denominators will divide into. Multiplying every term on both sides of the equation by the least common denominator will eliminate the fractions in the equation. Example 2 shows how we "clear an equation of fractions."

EXAMPLE 2 Solving a Linear Equation Involving Fractions

Solve and check: $\frac{x + 2}{4} - \frac{x - 1}{3} = 2.$

SOLUTION

The fractional terms have denominators of 4 and 3. The smallest number that is divisible by 4 and 3 is 12. We begin by multiplying both sides of the equation by 12, the least common denominator.

$$\frac{x + 2}{4} - \frac{x - 1}{3} = 2 \quad \text{This is the given equation.}$$
$$12\left(\frac{x + 2}{4} - \frac{x - 1}{3}\right) = 12 \cdot 2 \quad \text{Multiply both sides by 12.}$$

$$12\left(\frac{x+2}{4}\right) - 12\left(\frac{x-1}{3}\right) = 24$$ Use the distributive property and multiply each term on the left by 12.

$$\overset{3}{\cancel{12}}\left(\frac{x+2}{\underset{1}{\cancel{4}}}\right) - \overset{4}{\cancel{12}}\left(\frac{x-1}{\underset{1}{\cancel{3}}}\right) = 24$$ Divide out common factors in each multiplication on the left.

$$3(x+2) - 4(x-1) = 24$$ The fractions are now cleared.

$$3x + 6 - 4x + 4 = 24$$ Use the distributive property.

$$-x + 10 = 24$$ Combine like terms: $3x - 4x = -x$ and $6 + 4 = 10$.

$$-x + 10 - 10 = 24 - 10$$ Subtract 10 from both sides.

$$-x = 14$$ Simplify.

We're not finished. A negative sign should not precede the variable.

Isolate x by multiplying or dividing both sides of this equation by -1.

$$\frac{-x}{-1} = \frac{14}{-1}$$ Divide both sides by -1.

$$x = -14$$ Simplify.

Check the proposed solution. Substitute -14 for x in the original equation. You should obtain $2 = 2$. This true statement verifies that the solution set is $\{-14\}$. ● ● ●

Check Point 2 Solve and check: $\dfrac{x-3}{4} = \dfrac{5}{14} - \dfrac{x+5}{7}$.

3 Solve rational equations with variables in the denominators.

Rational Equations

A **rational equation** is an equation containing one or more rational expressions. In Example 2, we solved a rational equation with constants in the denominators. This rational equation was a linear equation. Now, let's consider a rational equation such as

$$\frac{3}{x+6} + \frac{1}{x-2} = \frac{4}{x^2 + 4x - 12}.$$

Can you see how this rational equation differs from the rational equation that we solved earlier? The variable appears in the denominators. Although this rational equation is not a linear equation, the solution procedure still involves multiplying each side by the least common denominator. However, we must avoid any values of the variable that make a denominator zero.

EXAMPLE 3 Solving a Rational Equation

Solve: $\dfrac{3}{x+6} + \dfrac{1}{x-2} = \dfrac{4}{x^2 + 4x - 12}$.

SOLUTION

To identify values of x that make denominators zero, let's factor $x^2 + 4x - 12$, the denominator on the right. This factorization is also necessary in identifying the least common denominator.

$$\frac{3}{x+6} + \frac{1}{x-2} = \frac{4}{(x+6)(x-2)}$$

This denominator is zero if $x = -6$.

This denominator is zero if $x = 2$.

This denominator is zero if $x = -6$ or $x = 2$.

We see that x cannot equal -6 or 2. The least common denominator is $(x + 6)(x - 2)$.

$$\frac{3}{x+6} + \frac{1}{x-2} = \frac{4}{(x+6)(x-2)}, \quad x \neq -6, x \neq 2$$

This is the given equation with a denominator factored.

$$(x+6)(x-2)\left(\frac{3}{x+6} + \frac{1}{x-2}\right) = (x+6)(x-2) \cdot \frac{4}{(x+6)(x-2)}$$

Multiply both sides by $(x + 6)(x - 2)$, the LCD.

$$\cancel{(x+6)}(x-2) \cdot \frac{3}{\cancel{x+6}} + (x+6)\cancel{(x-2)} \cdot \frac{1}{\cancel{x-2}} = \cancel{(x+6)}\cancel{(x-2)} \cdot \frac{4}{\cancel{(x+6)}\cancel{(x-2)}}$$

Use the distributive property and divide out common factors.

$$3(x-2) + 1(x+6) = 4$$

Simplify. This equation is cleared of fractions.

$$3x - 6 + x + 6 = 4$$

Use the distributive property.

$$4x = 4$$

Combine like terms.

$$\frac{4x}{4} = \frac{4}{4}$$

Divide both sides by 4.

$$x = 1$$

Simplify. This is not part of the restriction that $x \neq -6$ and $x \neq 2$.

Check the proposed solution. Substitute 1 for x in the original equation. You should obtain $-\frac{4}{7} = -\frac{4}{7}$. This true statement verifies that the solution set is $\{1\}$. ● ● ●

Check Point 3 Solve: $\dfrac{6}{x+3} - \dfrac{5}{x-2} = \dfrac{-20}{x^2 + x - 6}$.

EXAMPLE 4 Solving a Rational Equation

Solve: $\dfrac{1}{x+1} = \dfrac{2}{x^2 - 1} - \dfrac{1}{x-1}$.

SOLUTION

We begin by factoring $x^2 - 1$.

$$\frac{1}{x+1} = \frac{2}{(x+1)(x-1)} - \frac{1}{x-1}$$

This denominator is zero if $x = -1$.

This denominator is zero if $x = -1$ or $x = 1$.

This denominator is zero if $x = 1$.

We see that x cannot equal -1 or 1. The least common denominator is $(x + 1)(x - 1)$.

$$\frac{1}{x+1} = \frac{2}{(x+1)(x-1)} - \frac{1}{x-1}, \quad x \neq -1, x \neq 1$$

This is the given equation with a denominator factored.

$$(x+1)(x-1) \cdot \frac{1}{x+1} = (x+1)(x-1)\left(\frac{2}{(x+1)(x-1)} - \frac{1}{x-1}\right)$$

Multiply both sides by $(x + 1)(x - 1)$, the LCD.

$$\cancel{(x+1)}(x-1) \cdot \frac{1}{\cancel{x+1}} = \cancel{(x+1)}\cancel{(x-1)} \cdot \frac{2}{\cancel{(x+1)}\cancel{(x-1)}} - (x+1)\cancel{(x-1)} \cdot \frac{1}{\cancel{(x-1)}}$$

Use the distributive property and divide out common factors.

$$1(x-1) = 2-(x+1)$$ Simplify. This equation is cleared of fractions.

$$x-1 = 2-x-1$$ Simplify.

$$x-1 = -x+1$$ Combine numerical terms.

$$x+x-1 = -x+x+1$$ Add x to both sides.

$$2x-1 = 1$$ Simplify.

$$2x-1+1 = 1+1$$ Add 1 to both sides.

$$2x = 2$$ Simplify.

$$\frac{2x}{2} = \frac{2}{2}$$ Divide both sides by 2.

$$x = 1$$ Simplify.

GREAT QUESTION!

When do I get rid of proposed solutions in rational equations?

Reject any proposed solution that causes any denominator in an equation to equal 0.

The proposed solution, 1, is *not* a solution because of the restriction that $x \neq 1$. There is *no solution to this equation*. The solution set for this equation contains no elements. The solution set is $\varnothing$, the empty set. ● ● ●

Check Point 4 Solve: $\frac{1}{x+2} = \frac{4}{x^2-4} - \frac{1}{x-2}$.

4 Solve a formula for a variable.

Solving a Formula for One of Its Variables

Solving a formula for a variable means rewriting the formula so that the variable is isolated on one side of the equation. It does not mean obtaining a numerical value for that variable.

To solve a formula for one of its variables, treat that variable as if it were the only variable in the equation. Think of the other variables as if they were numbers.

EXAMPLE 5 Solving a Formula for a Variable

If you wear glasses, did you know that each lens has a measurement called its focal length, f? When an object is in focus, its distance from the lens, p, and the distance from the lens to your retina, q, satisfy the formula

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}.$$

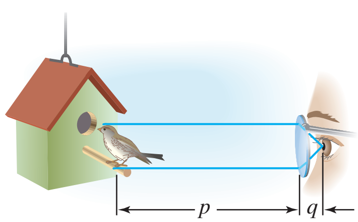


FIGURE 12

(See **Figure 12**.) Solve this formula for p.

SOLUTION

Our goal is to isolate the variable p. We begin by multiplying both sides by the least common denominator, pqf, to clear the equation of fractions.

We need to isolate p.

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$$ This is the given formula.

$$pqf\left(\frac{1}{p} + \frac{1}{q}\right) = pqf\left(\frac{1}{f}\right)$$ Multiply both sides by pqf, the LCD.

$$\cancel{p}qf\left(\frac{1}{\cancel{p}}\right) + p\cancel{q}f\left(\frac{1}{\cancel{q}}\right) = pq\cancel{f}\left(\frac{1}{\cancel{f}}\right)$$ Use the distributive property on the left side and divide out common factors.

$$qf + pf = pq$$ Simplify. The formula is cleared of fractions.

We need to isolate p.

GREAT QUESTION!

Can I solve $qf + pf = pq$ for p by dividing both sides by q and writing

$$\frac{qf + pf}{q} = p?$$

No. When a formula is solved for a specified variable, that variable must be isolated on one side. The variable p occurs on both sides of

$$\frac{qf + pf}{q} = p.$$

To collect terms with p on one side of $qf + pf = pq$, subtract pf from both sides. Then factor p from the two resulting terms on the right to convert two occurrences of p into one.

$$qf + pf = pq \quad \text{This is the equation cleared of fractions.}$$

$$qf + pf - pf = pq - pf \quad \text{Subtract } pf \text{ from both sides.}$$

$$qf = pq - pf \quad \text{Simplify.}$$

$$qf = p(q - f) \quad \text{Factor out } p\text{, the specified variable.}$$

$$\frac{qf}{q - f} = \frac{p(q - f)}{q - f} \quad \text{Divide both sides by } q - f \text{ and solve for } p.$$

$$\frac{qf}{q - f} = p \quad \text{Simplify.}$$

Check Point 5 Solve for q: $\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$.

5 Solve equations involving absolute value.

Equations Involving Absolute Value

We have seen that the absolute value of x, denoted $|x|$, describes the distance of x from zero on a number line. Now consider an **absolute value equation**, such as

$$|x| = 2.$$

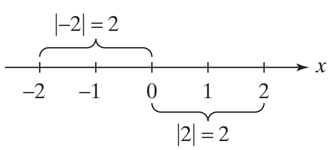


FIGURE 13

This means that we must determine real numbers whose distance from the origin on a number line is 2. **Figure 13** shows that there are two numbers such that $|x| = 2$, namely, 2 and -2. We write $x = 2$ or $x = -2$. This observation can be generalized as follows:

Rewriting an Absolute Value Equation without Absolute Value Bars

If c is a positive real number and u represents any algebraic expression, then $|u| = c$ is equivalent to $u = c$ or $u = -c$.

EXAMPLE 6 Solving an Equation Involving Absolute Value

Solve: $5|1 - 4x| - 15 = 0$.

SOLUTION

$$5|1 - 4x| - 15 = 0 \quad \text{This is the given equation.}$$

We need to isolate $|1 - 4x|$, the absolute value expression.

$$5|1 - 4x| = 15 \quad \text{Add 15 to both sides.}$$

$$|1 - 4x| = 3 \quad \text{Divide both sides by 5.}$$

$$1 - 4x = 3 \quad \text{or} \quad 1 - 4x = -3 \quad \text{Rewrite } |u| = c \text{ as } u = c \text{ or } u = -c.$$

$$-4x = 2 \qquad -4x = -4 \quad \text{Subtract 1 from both sides of each equation.}$$

$$x = -\tfrac{1}{2} \qquad x = 1 \quad \text{Divide both sides of each equation by } -4.$$

Take a moment to check $-\frac{1}{2}$ and 1, the proposed solutions, in the original equation, $5|1 - 4x| - 15 = 0$. In each case, you should obtain the true statement $0 = 0$. The solution set is $\left\{-\frac{1}{2}, 1\right\}$.

Check Point 6 Solve: $4|1 - 2x| - 20 = 0$.

The absolute value of a number is never negative. Thus, if u is an algebraic expression and c is a negative number, then $|u| = c$ has no solution. For example, the equation $|3x - 6| = -2$ has no solution because $|3x - 6|$ cannot be negative. The solution set is $\varnothing$, the empty set.

The absolute value of 0 is 0. Thus, if u is an algebraic expression and $|u| = 0$, the solution is found by solving $u = 0$. For example, the solution of $|x - 2| = 0$ is obtained by solving $x - 2 = 0$. The solution is 2 and the solution set is $\{2\}$.

6 Solve quadratic equations by factoring.

Quadratic Equations and Factoring

Linear equations are first-degree polynomial equations of the form $ax + b = 0$. *Quadratic equations* are second-degree polynomial equations and contain an additional term involving the square of the variable.

Definition of a Quadratic Equation

A **quadratic equation** in x is an equation that can be written in the **general form**

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers, with $a \neq 0$. A quadratic equation in x is also called a **second-degree polynomial equation** in x.

Here are examples of quadratic equations in general form:

$$4x^2 - 2x \quad = 0 \qquad\qquad 2x^2 + 7x - 4 = 0.$$

$a = 4$ $b = -2$ $c = 0$ $\qquad$ $a = 2$ $b = 7$ $c = -4$

Some quadratic equations, including the two shown above, can be solved by factoring and using the **zero-product principle**.

The Zero-Product Principle

If the product of two algebraic expressions is zero, then at least one of the factors is equal to zero.

$$\text{If } AB = 0, \text{ then } A = 0 \text{ or } B = 0.$$

The zero-product principle can be applied only when a quadratic equation is in general form, with zero on one side of the equation.

Solving a Quadratic Equation by Factoring

1. If necessary, rewrite the equation in the general form $ax^2 + bx + c = 0$, moving all terms to one side, thereby obtaining zero on the other side.
2. Factor completely.
3. Apply the zero-product principle, setting each factor containing a variable equal to zero.
4. Solve the equations in step 3.
5. Check the solutions in the original equation.

EXAMPLE 7 Solving Quadratic Equations by Factoring

Solve by factoring:

a. $4x^2 - 2x = 0$ $\qquad$ **b.** $2x^2 + 7x = 4$.

SOLUTION

a. $4x^2 - 2x = 0$ — The given equation is in general form, with zero on one side.

$2x(2x - 1) = 0$ — Factor.

$2x = 0$ or $2x - 1 = 0$ — Use the zero-product principle and set each factor equal to zero.

$x = 0$ $\quad$ $2x = 1$ — Solve the resulting equations.

$x = \frac{1}{2}$

Check the proposed solutions, 0 and $\frac{1}{2}$, in the original equation.

Check 0:

$$4x^2 - 2x = 0$$
$$4 \cdot 0^2 - 2 \cdot 0 \stackrel{?}{=} 0$$
$$0 - 0 \stackrel{?}{=} 0$$
$$0 = 0, \quad \text{true}$$

Check $\frac{1}{2}$:

$$4x^2 - 2x = 0$$
$$4\left(\tfrac{1}{2}\right)^2 - 2\left(\tfrac{1}{2}\right) \stackrel{?}{=} 0$$
$$4\left(\tfrac{1}{4}\right) - 2\left(\tfrac{1}{2}\right) \stackrel{?}{=} 0$$
$$1 - 1 \stackrel{?}{=} 0$$
$$0 = 0, \quad \text{true}$$

The solution set is $\left\{0, \frac{1}{2}\right\}$.

b. $2x^2 + 7x = 4$ — This is the given equation.

$2x^2 + 7x - 4 = 4 - 4$ — Subtract 4 from both sides and write the quadratic equation in general form.

$2x^2 + 7x - 4 = 0$ — Simplify.

$(2x - 1)(x + 4) = 0$ — Factor.

$2x - 1 = 0$ or $x + 4 = 0$ — Use the zero-product principle and set each factor equal to zero.

$2x = 1$ $\quad$ $x = -4$ — Solve the resulting equations.

$x = \frac{1}{2}$

Check the proposed solutions, $\frac{1}{2}$ and -4, in the original equation.

Check $\frac{1}{2}$:

$$2x^2 + 7x = 4$$
$$2\left(\tfrac{1}{2}\right)^2 + 7\left(\tfrac{1}{2}\right) \stackrel{?}{=} 4$$
$$\tfrac{1}{2} + \tfrac{7}{2} \stackrel{?}{=} 4$$
$$4 = 4, \quad \text{true}$$

Check -4:

$$2x^2 + 7x = 4$$
$$2(-4)^2 + 7(-4) \stackrel{?}{=} 4$$
$$32 + (-28) \stackrel{?}{=} 4$$
$$4 = 4, \quad \text{true}$$

The solution set is $\left\{-4, \frac{1}{2}\right\}$.

Check Point 7 Solve by factoring:

a. $3x^2 - 9x = 0$ $\qquad$ **b.** $2x^2 + x = 1$.

7 Solve quadratic equations by the square root property.

Quadratic Equations and the Square Root Property

Quadratic equations of the form $u^2 = d$, where u is an algebraic expression and d is a positive real number, can be solved by the *square root property*. First, isolate the squared expression u^2 on one side of the equation and the number d on the other side. Then take the square root of both sides. Remember, there are two numbers whose square is d. One number is $\sqrt{d}$ and one is $-\sqrt{d}$.

We can use factoring to verify that $u^2 = d$ has these two solutions.

$$u^2 = d \qquad \text{This is the given equation.}$$

$$u^2 - d = 0 \qquad \text{Move all terms to one side and obtain zero on the other side.}$$

$$(u + \sqrt{d})(u - \sqrt{d}) = 0 \qquad \text{Factor.}$$

$$u + \sqrt{d} = 0 \quad \text{or} \quad u - \sqrt{d} = 0 \qquad \text{Set each factor equal to zero.}$$

$$u = -\sqrt{d} \qquad\qquad u = \sqrt{d} \qquad \text{Solve the resulting equations.}$$

Because the solutions differ only in sign, we can write them in abbreviated notation as $u = \pm\sqrt{d}$. We read this as "u equals positive or negative square root of d" or "u equals plus or minus square root of d."

Now that we have verified these solutions, we can solve $u^2 = d$ directly by taking square roots. This process is called the **square root property**.

The Square Root Property

If u is an algebraic expression and d is a positive real number, then $u^2 = d$ has exactly two solutions:

$$\text{If } u^2 = d, \text{ then } u = \sqrt{d} \text{ or } u = -\sqrt{d}.$$

Equivalently,

$$\text{If } u^2 = d, \text{ then } u = \pm\sqrt{d}.$$

EXAMPLE 8 Solving Quadratic Equations by the Square Root Property

Solve by the square root property:

a. $3x^2 - 15 = 0$ **b.** $(x - 2)^2 = 6$.

SOLUTION

To apply the square root property, we need a squared expression by itself on one side of the equation.

$3x^2 - 15 = 0$ We want x^2 by itself.

$(x - 2)^2 = 6$ The squared expression is by itself.

a.

$$3x^2 - 15 = 0 \qquad \text{This is the original equation.}$$

$$3x^2 = 15 \qquad \text{Add 15 to both sides.}$$

$$x^2 = 5 \qquad \text{Divide both sides by 3.}$$

$$x = \sqrt{5} \text{ or } x = -\sqrt{5} \qquad \text{Apply the square root property. Equivalently, } x = \pm\sqrt{5}.$$

By checking both proposed solutions in the original equation, we can confirm that the solution set is $\{-\sqrt{5}, \sqrt{5}\}$ or $\{\pm\sqrt{5}\}$.

b.

$$(x - 2)^2 = 6 \qquad \text{This is the original equation.}$$

$$x - 2 = \pm\sqrt{6} \qquad \text{Apply the square root property.}$$

$$x = 2 \pm \sqrt{6} \qquad \text{Add 2 to both sides.}$$

By checking both values in the original equation, we can confirm that the solution set is $\{2 + \sqrt{6}, 2 - \sqrt{6}\}$ or $\{2 \pm \sqrt{6}\}$.

Check Point 8 Solve by the square root property:

a. $3x^2 - 21 = 0$ **b.** $(x + 5)^2 = 11$.

8 Solve quadratic equations by completing the square.

Quadratic Equations and Completing the Square

How do we solve an equation in the form $ax^2 + bx + c = 0$ if the trinomial $ax^2 + bx + c$ cannot be factored? We cannot use the zero-product principle in such a case. However, we can convert the equation into an equivalent equation that can be solved using the square root property. This is accomplished by **completing the square**.

Completing the Square

If $x^2 + bx$ is a binomial, then by adding $\left(\frac{b}{2}\right)^2$, which is the square of half the coefficient of x, a perfect square trinomial will result. That is,

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2.$$

We can solve any quadratic equation by completing the square. If the coefficient of the x^2-term is one, we add the square of half the coefficient of x to both sides of the equation. **When you add a constant term to one side of the equation to complete the square, be certain to add the same constant to the other side of the equation.** These ideas are illustrated in Example 9.

EXAMPLE 9 Solving a Quadratic Equation by Completing the Square

Solve by completing the square: $x^2 - 6x + 4 = 0$.

SOLUTION

We begin by subtracting 4 from both sides. This is done to isolate the binomial $x^2 - 6x$ so that we can complete the square.

$x^2 - 6x + 4 = 0$ This is the original equation.

$x^2 - 6x = -4$ Subtract 4 from both sides.

Next, we work with $x^2 - 6x = -4$ and complete the square. Find half the coefficient of the x-term and square it. The coefficient of the x-term is -6. Half of -6 is -3 and $(-3)^2 = 9$. Thus, we add 9 to both sides of the equation.

$x^2 - 6x + 9 = -4 + 9$ Add 9 to both sides of $x^2 - 6x = -4$ to complete the square.

$(x - 3)^2 = 5$ Factor and simplify.

$x - 3 = \sqrt{5}$ or $x - 3 = -\sqrt{5}$ Apply the square root property.

$x = 3 + \sqrt{5}$ $\quad$ $x = 3 - \sqrt{5}$ Add 3 to both sides in each equation.

The solutions are $3 \pm \sqrt{5}$ and the solution set is $\{3 + \sqrt{5}, 3 - \sqrt{5}\}$, or $\{3 \pm \sqrt{5}\}$.

Check Point 9 Solve by completing the square: $x^2 + 4x - 1 = 0$.

9 Solve quadratic equations using the quadratic formula.

Quadratic Equations and the Quadratic Formula

We can use the method of completing the square to derive a formula that can be used to solve all quadratic equations. The derivation given here also shows a particular quadratic equation, $3x^2 - 2x - 4 = 0$, to specifically illustrate each of the steps.

Notice that if the coefficient of the x^2-term in a quadratic equation is not one, you must divide each side of the equation by this coefficient before completing the square.

Deriving the Quadratic Formula

General Form of a Quadratic Equation	Comment	A Specific Example
$ax^2 + bx + c = 0, a > 0$	This is the given equation.	$3x^2 - 2x - 4 = 0$
$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$	Divide both sides by a so that the coefficient of x^2 is 1.	$x^2 - \frac{2}{3}x - \frac{4}{3} = 0$
$x^2 + \frac{b}{a}x = -\frac{c}{a}$	Isolate the binomial by adding $-\frac{c}{a}$ on both sides of the equation.	$x^2 - \frac{2}{3}x = \frac{4}{3}$
$x^2 + \underbrace{\frac{b}{a}x + \left(\frac{b}{2a}\right)^2}_{(\text{half})^2} = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$ $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$	Complete the square. Add the square of half the coefficient of x to both sides.	$x^2 - \underbrace{\frac{2}{3}x + \left(-\frac{1}{3}\right)^2}_{(\text{half})^2} = \frac{4}{3} + \left(-\frac{1}{3}\right)^2$ $x^2 - \frac{2}{3}x + \frac{1}{9} = \frac{4}{3} + \frac{1}{9}$
$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a}\cdot\frac{4a}{4a} + \frac{b^2}{4a^2}$	Factor on the left side and obtain a common denominator on the right side.	$\left(x - \frac{1}{3}\right)^2 = \frac{4}{3}\cdot\frac{3}{3} + \frac{1}{9}$
$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$ $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$	Add fractions on the right side.	$\left(x - \frac{1}{3}\right)^2 = \frac{12 + 1}{9}$ $\left(x - \frac{1}{3}\right)^2 = \frac{13}{9}$
$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$	Apply the square root property.	$x - \frac{1}{3} = \pm\sqrt{\frac{13}{9}}$
$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$	Take the square root of the quotient, simplifying the denominator.	$x - \frac{1}{3} = \pm\frac{\sqrt{13}}{3}$
$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$	Solve for x by subtracting $\frac{b}{2a}$ from both sides.	$x = \frac{1}{3} \pm \frac{\sqrt{13}}{3}$
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Combine fractions on the right side.	$x = \frac{1 \pm \sqrt{13}}{3}$

The formula shown at the bottom of the left column is called the *quadratic formula*. A similar proof shows that the same formula can be used to solve quadratic equations if a, the coefficient of the x^2-term, is negative.

The Quadratic Formula

The solutions of a quadratic equation in general form $ax^2 + bx + c = 0$, with $a \neq 0$, are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

x equals negative b plus or minus the square root of $b^2 - 4ac$, all divided by $2a$.

To use the quadratic formula, write the quadratic equation in general form if necessary. Then determine the numerical values for a (the coefficient of the x^2-term), b (the coefficient of the x-term), and c (the constant term). Substitute the values of a, b, and c into the quadratic formula and evaluate the expression. The $\pm$ sign indicates that there are two solutions of the equation.

EXAMPLE 10 Solving a Quadratic Equation Using the Quadratic Formula

Solve using the quadratic formula: $2x^2 - 6x + 1 = 0$.

SOLUTION

The given equation is in general form. Begin by identifying the values for a, b, and c.

$$\underbrace{2}_{a=2}x^2 \underbrace{-6}_{b=-6}x + \underbrace{1}_{c=1} = 0$$

Substituting these values into the quadratic formula and simplifying gives the equation's solutions.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Use the quadratic formula.

$$= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)(1)}}{2 \cdot 2}$$ Substitute the values for a, b, and c: $a = 2$, $b = -6$, and $c = 1$.

$$= \frac{6 \pm \sqrt{36 - 8}}{4}$$ $-(-6) = 6$, $(-6)^2 = (-6)(-6) = 36$, and $4(2)(1) = 8$.

$$= \frac{6 \pm \sqrt{28}}{4}$$ Complete the subtraction under the radical.

$$= \frac{6 \pm 2\sqrt{7}}{4}$$ $\sqrt{28} = \sqrt{4 \cdot 7} = \sqrt{4}\sqrt{7} = 2\sqrt{7}$

$$= \frac{2(3 \pm \sqrt{7})}{4}$$ Factor out 2 from the numerator.

$$= \frac{3 \pm \sqrt{7}}{2}$$ Divide the numerator and denominator by 2.

The solution set is $\left\{\frac{3 + \sqrt{7}}{2}, \frac{3 - \sqrt{7}}{2}\right\}$ or $\left\{\frac{3 \pm \sqrt{7}}{2}\right\}$. • • •

GREAT QUESTION!

Should I check irrational solutions by substitution in the given quadratic equation?

No. Checking irrational solutions can be time-consuming. The solutions given by the quadratic formula are always correct, unless you have made a careless error. Checking for computational errors or errors in simplification is sufficient.

Check Point 10 Solve using the quadratic formula:

$$2x^2 + 2x - 1 = 0.$$

10 Use the discriminant to determine the number and type of solutions of quadratic equations.

Quadratic Equations and the Discriminant

The quantity $b^2 - 4ac$, which appears under the radical sign in the quadratic formula, is called the **discriminant**. **Table 4** shows how the discriminant of the quadratic equation $ax^2 + bx + c = 0$ determines the number and type of solutions.

Table 4 The Discriminant and the Kinds of Solutions to $ax^2 + bx + c = 0$

Discriminant $b^2 - 4ac$	Kinds of Solutions to $ax^2 + bx + c = 0$
$b^2 - 4ac > 0$	**Two unequal real solutions:** If a, b, and c are rational numbers and the discriminant is a perfect square, the solutions are rational. If the discriminant is not a perfect square, the solutions are irrational.
$b^2 - 4ac = 0$	**One solution (a repeated solution) that is a real number:** If a, b, and c are rational numbers, the repeated solution is also a rational number.
$b^2 - 4ac < 0$	**No real solutions**

GREAT QUESTION!

Is the square root sign part of the discriminant?

No. The discriminant is $b^2 - 4ac$. It is not $\sqrt{b^2 - 4ac}$, so do not give the discriminant as a radical.

EXAMPLE 11 Using the Discriminant

Compute the discriminant of $4x^2 - 8x + 1 = 0$. What does the discriminant indicate about the number and type of solutions?

SOLUTION

Begin by identifying the values for a, b, and c.

$$4x^2 - 8x + 1 = 0$$

$a = 4$ $b = -8$ $c = 1$

Substitute and compute the discriminant:

$$b^2 - 4ac = (-8)^2 - 4 \cdot 4 \cdot 1 = 64 - 16 = 48.$$

The discriminant is 48. Because the discriminant is positive, the equation $4x^2 - 8x + 1 = 0$ has two unequal real solutions. • • •

Check Point 11 Compute the discriminant of $3x^2 - 2x + 5 = 0$. What does the discriminant indicate about the number and type of solutions?

11 Determine the most efficient method to use when solving a quadratic equation.

Determining Which Method to Use

All quadratic equations can be solved by the quadratic formula. However, if an equation is in the form $u^2 = d$, such as $x^2 = 5$ or $(2x + 3)^2 = 8$, it is faster to use the square root property, taking the square root of both sides. If the equation is not in the form $u^2 = d$, write the quadratic equation in general form ($ax^2 + bx + c = 0$). Try to solve the equation by factoring. If $ax^2 + bx + c$ cannot be factored, then solve the quadratic equation by the quadratic formula.

Because we used the method of completing the square to derive the quadratic formula, we no longer need it for solving quadratic equations. However, we will use completing the square later in the text to help graph circles and other kinds of equations.

Table 5 on the next page summarizes our observations about which technique to use when solving a quadratic equation.

Table 5 Determining the Most Efficient Technique to Use When Solving a Quadratic Equation

Description and Form of the Quadratic Equation	Most Efficient Solution Method	Example
$ax^2 + bx + c = 0$ and $ax^2 + bx + c$ can be factored easily.	Factor and use the zero-product principle.	$3x^2 + 5x - 2 = 0$ $(3x - 1)(x + 2) = 0$ $3x - 1 = 0$ or $x + 2 = 0$ $x = \frac{1}{3}$ $\quad x = -2$
$ax^2 + bx = 0$ The quadratic equation has no constant term. $(c = 0)$	Factor and use the zero-product principle.	$6x^2 + 9x = 0$ $3x(2x + 3) = 0$ $3x = 0$ or $2x + 3 = 0$ $x = 0$ $\quad 2x = -3$ $x = -\frac{3}{2}$
$ax^2 + c = 0$ The quadratic equation has no x-term. $(b = 0)$	Solve for x^2 and apply the square root property.	$7x^2 - 4 = 0$ $7x^2 = 4$ $x^2 = \frac{4}{7}$ $x = \pm\sqrt{\frac{4}{7}}$ $= \pm\frac{2}{\sqrt{7}} = \pm\frac{2}{\sqrt{7}}\cdot\frac{\sqrt{7}}{\sqrt{7}} = \pm\frac{2\sqrt{7}}{7}$
$u^2 = d$; u is a first-degree polynomial.	Use the square root property.	$(x + 4)^2 = 5$ $x + 4 = \pm\sqrt{5}$ $x = -4 \pm \sqrt{5}$
$ax^2 + bx + c = 0$ and $ax^2 + bx + c$ cannot be factored or the factoring is too difficult.	Use the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.	$x^2 - 2x - 6 = 0$ $a = 1$ $\quad b = -2$ $\quad c = -6$ $x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-6)}}{2(1)}$ $= \frac{2 \pm \sqrt{4 + 24}}{2(1)}$ $= \frac{2 \pm \sqrt{28}}{2} = \frac{2 \pm \sqrt{4}\sqrt{7}}{2}$ $= \frac{2 \pm 2\sqrt{7}}{2} = \frac{2(1 \pm \sqrt{7})}{2}$ $= 1 \pm \sqrt{7}$

12 Solve radical equations.

Radical Equations

A **radical equation** is an equation in which the variable occurs in a square root, cube root, or any higher root. An example of a radical equation is

$$\sqrt{x} = 9.$$

We solve the equation by squaring both sides:

Squaring both sides eliminates the square root.

$$(\sqrt{x})^2 = 9^2$$
$$x = 81.$$

The proposed solution, 81, can be checked in the original equation, $\sqrt{x} = 9$. Because $\sqrt{81} = 9$, the solution is 81 and the solution set is $\{81\}$.

In general, we solve radical equations with square roots by squaring both sides of the equation. We solve radical equations with nth roots by raising both sides of the equation to the nth power. Unfortunately, if n is even, all the solutions of the equation raised to the even power may not be solutions of the original equation. Consider, for example, the equation

$$x = 4.$$

If we square both sides, we obtain

$$x^2 = 16.$$

Solving this equation using the square root property, we obtain

$$x = \pm\sqrt{16} = \pm 4.$$

The new equation $x^2 = 16$ has two solutions, -4 and 4. By contrast, only 4 is a solution of the original equation, $x = 4$. For this reason, **when raising both sides of an equation to an even power, always check proposed solutions in the original equation**.

Here is a general method for solving radical equations with nth roots:

Solving Radical Equations Containing *n*th Roots

1. If necessary, arrange terms so that one radical is isolated on one side of the equation.
2. Raise both sides of the equation to the nth power to eliminate the isolated nth root.
3. Solve the resulting equation. If this equation still contains radicals, repeat steps 1 and 2.
4. Check all proposed solutions in the original equation.

Extra solutions may be introduced when you raise both sides of a radical equation to an even power. Such solutions, which are not solutions of the given equation, are called **extraneous solutions** or **extraneous roots**.

EXAMPLE 12 Solving a Radical Equation

Solve: $\sqrt{2x-1} + 2 = x$.

GREAT QUESTION!

Can I square the right side of $\sqrt{2x-1} = x - 2$ by first squaring x and then squaring 2?

No. Be sure to square *both sides* of an equation. Do *not* square each term.

Correct:

$$(\sqrt{2x-1})^2 = (x-2)^2$$

Incorrect!

$$\cancel{(\sqrt{2x-1})^2 = x^2 - 2^2}$$

SOLUTION

Step 1 Isolate a radical on one side. We isolate the radical, $\sqrt{2x-1}$, by subtracting 2 from both sides.

$$\sqrt{2x-1} + 2 = x \qquad \text{This is the given equation.}$$
$$\sqrt{2x-1} = x - 2 \qquad \text{Subtract 2 from both sides.}$$

Step 2 Raise both sides to the *n*th power. Because n, the index, is 2, we square both sides.

$$(\sqrt{2x-1})^2 = (x-2)^2$$
$$2x - 1 = x^2 - 4x + 4 \qquad \text{Simplify. Use the formula } (A-B)^2 = A^2 - 2AB + B^2 \text{ on the right side.}$$

Step 3 Solve the resulting equation. Because of the x^2-term in $2x - 1 = x^2 - 4x + 4$, the resulting equation is a quadratic equation. We can obtain 0 on the left side by subtracting $2x$ and adding 1 on both sides.

$$2x - 1 = x^2 - 4x + 4 \quad \text{The resulting equation is quadratic.}$$
$$0 = x^2 - 6x + 5 \quad \text{Write in general form, subtracting 2x and adding 1 on both sides.}$$
$$0 = (x - 1)(x - 5) \quad \text{Factor.}$$
$$x - 1 = 0 \quad \text{or} \quad x - 5 = 0 \quad \text{Set each factor equal to 0.}$$
$$x = 1 \qquad\qquad x = 5 \quad \text{Solve the resulting equations.}$$

Step 4 Check the proposed solutions in the original equation.

Check 1:

$$\sqrt{2x - 1} + 2 = x$$
$$\sqrt{2 \cdot 1 - 1} + 2 \stackrel{?}{=} 1$$
$$\sqrt{1} + 2 \stackrel{?}{=} 1$$
$$1 + 2 \stackrel{?}{=} 1$$
$$3 = 1, \quad \text{false}$$

Check 5:

$$\sqrt{2x - 1} + 2 = x$$
$$\sqrt{2 \cdot 5 - 1} + 2 \stackrel{?}{=} 5$$
$$\sqrt{9} + 2 \stackrel{?}{=} 5$$
$$3 + 2 \stackrel{?}{=} 5$$
$$5 = 5, \quad \text{true}$$

Thus, 1 is an extraneous solution. The only solution is 5, and the solution set is $\{5\}$.

Check Point 12 Solve: $\sqrt{x + 3} + 3 = x$.

CONCEPT AND VOCABULARY CHECK

Fill in each blank so that the resulting statement is true.

1. An equation in the form $ax + b = 0$, $a \neq 0$, such as $3x + 17 = 0$, is called a/an ________ equation in one variable.
2. Two or more equations that have the same solution set are called __________ equations.
3. The first step in solving $7 + 3(x - 2) = 2x + 10$ is to ________________________.
4. The fractions in the equation
$$\frac{x}{4} = 2 + \frac{x - 3}{3}$$
can be eliminated by multiplying both sides by the ____________________ of $\frac{x}{4}$ and $\frac{x-3}{3}$, which is _____.
5. We reject any proposed solution of a rational equation that causes a denominator to equal _____.
6. The restrictions on the variable in the rational equation
$$\frac{1}{x - 2} - \frac{2}{x + 4} = \frac{2x - 1}{x^2 + 2x - 8}$$
are ______ and ________.
7. $$\frac{5}{x + 4} + \frac{3}{x + 3} = \frac{12x + 9}{(x + 4)(x + 3)}$$
$$(x + 4)(x + 3)\left(\frac{5}{x + 4} + \frac{3}{x + 3}\right) = (x + 4)(x + 3)\left(\frac{12x + 9}{(x + 4)(x + 3)}\right)$$
The resulting equation cleared of fractions is ________________________.
8. Solving a formula for a variable means rewriting the formula so that the variable is ________________.
9. The first step in solving $IR + Ir = E$ for I is to obtain a single occurrence of I by _________ I from the two terms on the left.
10. If $c > 0$, $|u| = c$ is equivalent to $u =$ __ or $u =$ ___.
11. $|3x - 1| = 7$ is equivalent to __________ or __________.
12. An equation that can be written in the general form $ax^2 + bx + c = 0$, $a \neq 0$, is called a/an ____________ equation.
13. The zero-product principle states that if $AB = 0$, then ______________.
14. The square root property states that if $u^2 = d$, then $u =$ ________.
15. If $x^2 = 7$, then $x =$ __________.
16. To solve $x^2 + 6x = 7$ by completing the square, add _______ to both sides of the equation.
17. The solutions of a quadratic equation in the general form $ax^2 + bx + c = 0$, $a \neq 0$, are given by the quadratic formula $x =$ ______________.
18. In order to solve $2x^2 + 9x - 5 = 0$ by the quadratic formula, we use $a =$ _______, $b =$ _______, and $c =$ ________.
19. In order to solve $x^2 = 4x + 1$ by the quadratic formula, we use $a =$ _______, $b =$ ________, and $c =$ ________.

20. $x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)}$ simplifies to $x =$ ______.

21. The discriminant of $ax^2 + bx + c = 0$ is defined by ______.

22. If the discriminant of $ax^2 + bx + c = 0$ is negative, the quadratic equation has ______ real solutions.

23. If the discriminant of $ax^2 + bx + c = 0$ is positive, the quadratic equation has ______ real solutions.

24. The most efficient technique for solving $(2x + 7)^2 = 25$ is by using ______.

25. The most efficient technique for solving $x^2 + 5x - 10 = 0$ is by using ______.

26. The most efficient technique for solving $x^2 + 8x + 15 = 0$ is by using ______.

27. An equation in which the variable occurs in a square root, cube root, or any higher root is called a/an ______ equation.

28. Solutions of a squared equation that are not solutions of the original equation are called ______ solutions.

29. Consider the equation

$$\sqrt{2x + 1} = x - 7.$$

Squaring the left side and simplifying results in ______. Squaring the right side and simplifying results in ______.

EXERCISE SET 7

Practice Exercises

In Exercises 1–16, solve each linear equation.

1. $7x - 5 = 72$

2. $6x - 3 = 63$

3. $11x - (6x - 5) = 40$

4. $5x - (2x - 10) = 35$

5. $2x - 7 = 6 + x$

6. $3x + 5 = 2x + 13$

7. $7x + 4 = x + 16$

8. $13x + 14 = 12x - 5$

9. $3(x - 2) + 7 = 2(x + 5)$

10. $2(x - 1) + 3 = x - 3(x + 1)$

11. $\dfrac{x + 3}{6} = \dfrac{3}{8} + \dfrac{x - 5}{4}$

12. $\dfrac{x + 1}{4} = \dfrac{1}{6} + \dfrac{2 - x}{3}$

13. $\dfrac{x}{4} = 2 + \dfrac{x - 3}{3}$

14. $5 + \dfrac{x - 2}{3} = \dfrac{x + 3}{8}$

15. $\dfrac{x + 1}{3} = 5 - \dfrac{x + 2}{7}$

16. $\dfrac{3x}{5} - \dfrac{x - 3}{2} = \dfrac{x + 2}{3}$

Exercises 17–26 contain rational equations with variables in denominators. For each equation, **a.** *Write the value or values of the variable that make a denominator zero. These are the restrictions on the variable.* **b.** *Keeping the restrictions in mind, solve the equation.*

17. $\dfrac{1}{x - 1} + 5 = \dfrac{11}{x - 1}$

18. $\dfrac{3}{x + 4} - 7 = \dfrac{-4}{x + 4}$

19. $\dfrac{8x}{x + 1} = 4 - \dfrac{8}{x + 1}$

20. $\dfrac{2}{x - 2} = \dfrac{x}{x - 2} - 2$

21. $\dfrac{3}{2x - 2} + \dfrac{1}{2} = \dfrac{2}{x - 1}$

22. $\dfrac{3}{x + 3} = \dfrac{5}{2x + 6} + \dfrac{1}{x - 2}$

23. $\dfrac{2}{x + 1} - \dfrac{1}{x - 1} = \dfrac{2x}{x^2 - 1}$

24. $\dfrac{4}{x + 5} + \dfrac{2}{x - 5} = \dfrac{32}{x^2 - 25}$

25. $\dfrac{1}{x - 4} - \dfrac{5}{x + 2} = \dfrac{6}{x^2 - 2x - 8}$

26. $\dfrac{1}{x - 3} - \dfrac{2}{x + 1} = \dfrac{8}{x^2 - 2x - 3}$

In Exercises 27–42, solve each formula for the specified variable. Do you recognize the formula? If so, what does it describe?

27. $I = Prt$ for P

28. $C = 2\pi r$ for r

29. $T = D + pm$ for p

30. $P = C + MC$ for M

31. $A = \frac{1}{2}h(a + b)$ for a

32. $A = \frac{1}{2}h(a + b)$ for b

33. $S = P + Prt$ for r

34. $S = P + Prt$ for t

35. $B = \dfrac{F}{S - V}$ for S

36. $S = \dfrac{C}{1 - r}$ for r

37. $IR + Ir = E$ for I

38. $A = 2lw + 2lh + 2wh$ for h

39. $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{f}$ for f

40. $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$ for R_1

41. $f = \dfrac{f_1 f_2}{f_1 + f_2}$ for f_1

42. $f = \dfrac{f_1 f_2}{f_1 + f_2}$ for f_2

In Exercises 43–54, solve each absolute value equation or indicate the equation has no solution.

43. $|x - 2| = 7$

44. $|x + 1| = 5$

45. $|2x - 1| = 5$

46. $|2x - 3| = 11$

47. $2|3x - 2| = 14$

48. $3|2x - 1| = 21$

49. $2\left|4 - \dfrac{5}{2}x\right| + 6 = 18$

50. $4\left|1 - \dfrac{3}{4}x\right| + 7 = 10$

51. $|x + 1| + 5 = 3$

52. $|x + 1| + 6 = 2$

53. $|2x - 1| + 3 = 3$

54. $|3x - 2| + 4 = 4$

In Exercises 55–60, solve each quadratic equation by factoring.

55. $x^2 - 3x - 10 = 0$

56. $x^2 - 13x + 36 = 0$

57. $x^2 = 8x - 15$

58. $x^2 = -11x - 10$

59. $5x^2 = 20x$

60. $3x^2 = 12x$

In Exercises 61–66, solve each quadratic equation by the square root property.

61. $3x^2 = 27$

62. $5x^2 = 45$

63. $5x^2 + 1 = 51$

64. $3x^2 - 1 = 47$

65. $3(x - 4)^2 = 15$

66. $3(x + 4)^2 = 21$

In Exercises 67–74, solve each quadratic equation by completing the square.

67. $x^2 + 6x = 7$

68. $x^2 + 6x = -8$

69. $x^2 - 2x = 2$

70. $x^2 + 4x = 12$

71. $x^2 - 6x - 11 = 0$

72. $x^2 - 2x - 5 = 0$

73. $x^2 + 4x + 1 = 0$

74. $x^2 + 6x - 5 = 0$

In Exercises 75–82, solve each quadratic equation using the quadratic formula.

75. $x^2 + 8x + 15 = 0$
76. $x^2 + 8x + 12 = 0$
77. $x^2 + 5x + 3 = 0$
78. $x^2 + 5x + 2 = 0$
79. $3x^2 - 3x - 4 = 0$
80. $5x^2 + x - 2 = 0$
81. $4x^2 = 2x + 7$
82. $3x^2 = 6x - 1$

Compute the discriminant of each equation in Exercises 83–90. What does the discriminant indicate about the number and type of solutions?

83. $x^2 - 4x - 5 = 0$
84. $4x^2 - 2x + 3 = 0$
85. $2x^2 - 11x + 3 = 0$
86. $2x^2 + 11x - 6 = 0$
87. $x^2 = 2x - 1$
88. $3x^2 = 2x - 1$
89. $x^2 - 3x - 7 = 0$
90. $3x^2 + 4x - 2 = 0$

In Exercises 91–114, solve each quadratic equation by the method of your choice.

91. $2x^2 - x = 1$
92. $3x^2 - 4x = 4$
93. $5x^2 + 2 = 11x$
94. $5x^2 = 6 - 13x$
95. $3x^2 = 60$
96. $2x^2 = 250$
97. $x^2 - 2x = 1$
98. $2x^2 + 3x = 1$
99. $(2x + 3)(x + 4) = 1$
100. $(2x - 5)(x + 1) = 2$
101. $(3x - 4)^2 = 16$
102. $(2x + 7)^2 = 25$
103. $3x^2 - 12x + 12 = 0$
104. $9 - 6x + x^2 = 0$
105. $4x^2 - 16 = 0$
106. $3x^2 - 27 = 0$
107. $x^2 = 4x - 2$
108. $x^2 = 6x - 7$
109. $2x^2 - 7x = 0$
110. $2x^2 + 5x = 3$
111. $\frac{1}{x} + \frac{1}{x+2} = \frac{1}{3}$
112. $\frac{1}{x} + \frac{1}{x+3} = \frac{1}{4}$
113. $\frac{2x}{x-3} + \frac{6}{x+3} = -\frac{28}{x^2-9}$
114. $\frac{3}{x-3} + \frac{5}{x-4} = \frac{x^2-20}{x^2-7x+12}$

In Exercises 115–124, solve each radical equation. Check all proposed solutions.

115. $\sqrt{3x + 18} = x$
116. $\sqrt{20 - 8x} = x$
117. $\sqrt{x + 3} = x - 3$
118. $\sqrt{x + 10} = x - 2$
119. $\sqrt{2x + 13} = x + 7$
120. $\sqrt{6x + 1} = x - 1$
121. $x - \sqrt{2x + 5} = 5$
122. $x - \sqrt{x + 11} = 1$
123. $\sqrt{2x + 19} - 8 = x$
124. $\sqrt{2x + 15} - 6 = x$

Practice Plus

In Exercises 125–134, solve each equation.

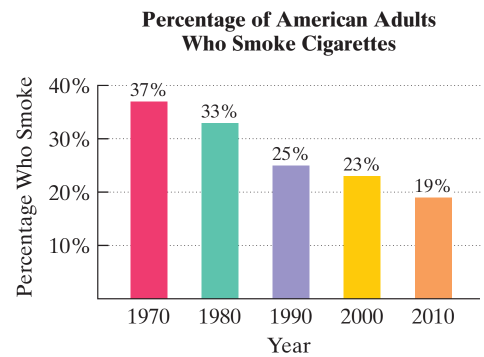
125. $25 - [2 + 5x - 3(x + 2)] = -3(2x - 5) - [5(x - 1) - 3x + 3]$
126. $45 - [4 - 2x - 4(x + 7)] = -4(1 + 3x) - [4 - 3(x + 2) - 2(2x - 5)]$
127. $7 - 7x = (3x + 2)(x - 1)$
128. $10x - 1 = (2x + 1)^2$
129. $|x^2 + 2x - 36| = 12$
130. $|x^2 + 6x + 1| = 8$
131. $\frac{1}{x^2 - 3x + 2} = \frac{1}{x+2} + \frac{5}{x^2-4}$
132. $\frac{x-1}{x-2} + \frac{x}{x-3} = \frac{1}{x^2 - 5x + 6}$
133. $\sqrt{x + 8} - \sqrt{x - 4} = 2$
134. $\sqrt{x + 5} - \sqrt{x - 3} = 2$

In Exercises 135–136, list all numbers that must be excluded from the domain of each rational expression.

135. $\frac{3}{2x^2 + 4x - 9}$
136. $\frac{7}{2x^2 - 8x + 5}$

Application Exercises

In the years after warning labels were put on cigarette packs, the number of smokers dropped from approximately two in five adults to one in five. The bar graph shows the percentage of American adults who smoked cigarettes for selected years from 1970 through 2010.



Source: Centers for Disease Control and Prevention

The mathematical model

$$p + \frac{x}{2} = 37$$

describes the percentage of Americans who smoked cigarettes, p, x years after 1970. Use this model to solve Exercises 137–138.

137. a. Does the mathematical model underestimate or overestimate the percentage of American adults who smoked cigarettes in 2010? By how much?

b. Use the mathematical model to project the year when only 7% of American adults will smoke cigarettes.

138. a. Does the mathematical model underestimate or overestimate the percentage of American adults who smoked cigarettes in 2000? By how much?

b. Use the mathematical model to project the year when only 2% of American adults will smoke cigarettes.

139. A company wants to increase the 10% peroxide content of its product by adding pure peroxide (100% peroxide). If x liters of pure peroxide are added to 500 liters of its 10% solution, the concentration, C, of the new mixture is given by

$$C = \frac{x + 0.1(500)}{x + 500}.$$

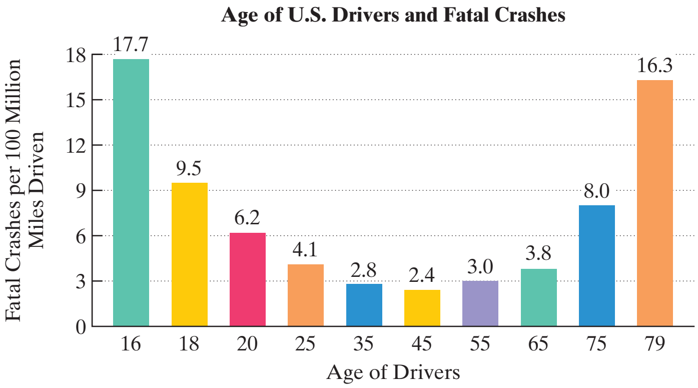
How many liters of pure peroxide should be added to produce a new product that is 28% peroxide?

140. Suppose that x liters of pure acid are added to 200 liters of a 35% acid solution.

a. Write a formula that gives the concentration, C, of the new mixture. (*Hint:* See Exercise 139.)

b. How many liters of pure acid should be added to produce a new mixture that is 74% acid?

A driver's age has something to do with his or her chance of getting into a fatal car crash. The bar graph shows the number of fatal vehicle crashes per 100 million miles driven for drivers of various age groups. For example, 25-year-old drivers are involved in 4.1 fatal crashes per 100 million miles driven. Thus, when a group of 25-year-old Americans have driven a total of 100 million miles, approximately 4 have been in accidents in which someone died.



Source: Insurance Institute for Highway Safety

The number of fatal vehicle crashes per 100 million miles, y, for drivers of age x can be modeled by the formula

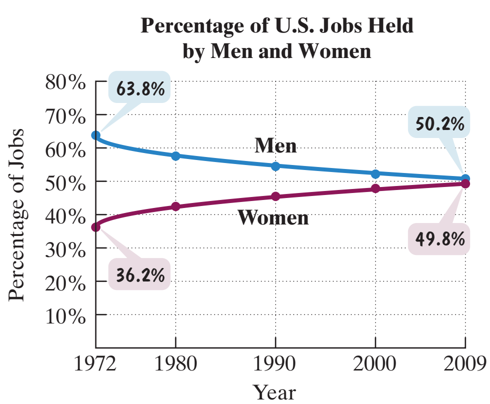
$$y = 0.013x^2 - 1.19x + 28.24.$$

Use the formula above to solve Exercises141–142.

141. What age groups are expected to be involved in 3 fatal crashes per 100 million miles driven? How well does the formula model the trend in the actual data shown in the bar graph?

142. What age groups are expected to be involved in 10 fatal crashes per 100 million miles driven? How well does the formula model the trend in the actual data shown by the bar graph?

By the end of 2010, women made up more than half of the labor force in the United States for the first time in the country's history. The graphs show the percentage of jobs in the U.S. labor force held by men and by women from 1972 through 2009. Exercises 143–144 are based on the data displayed by the graphs.



Source: Bureau of Labor Statistics

143. The formula

$$p = 2.2\sqrt{t} + 36.2$$

models the percentage of jobs in the U.S. labor force, p, held by women t years after 1972.

a. Use the appropriate graph to estimate the percentage of jobs in the U.S. labor force held by women in 2000. Give your estimation to the nearest percent.

b. Use the mathematical model to determine the percentage of jobs in the U.S. labor force held by women in 2000. Round to the nearest tenth of a percent.

c. According to the formula, when will 52% of jobs in the U.S. labor force be held by women? Round to the nearest year.

144. The formula

$$p = -2.2\sqrt{t} + 63.8$$

models the percentage of jobs in the U.S. labor force, p, held by men t years after 1972.

a. Use the appropriate graph to estimate the percentage of jobs in the U.S. labor force held by men in 2000. Give your estimation to the nearest percent.

b. Use the mathematical model to determine the percentage of jobs in the U.S. labor force held by men in 2000. Round to the nearest tenth of a percent.

c. According to the formula, when will 48% of jobs in the U.S. labor force be held by men? Round to the nearest year.

Writing in Mathematics

145. What is a linear equation in one variable? Give an example of this type of equation.

146. Explain how to determine the restrictions on the variable for the equation

$$\frac{3}{x+5} + \frac{4}{x-2} = \frac{7}{x^2 + 3x - 6}.$$

147. What does it mean to solve a formula for a variable?

148. Explain how to solve an equation involving absolute value.

149. Why does the procedure that you explained in Exercise 148 not apply to the equation $|x - 2| = -3$? What is the solution set for this equation?

150. What is a quadratic equation?

151. Explain how to solve $x^2 + 6x + 8 = 0$ using factoring and the zero-product principle.

152. Explain how to solve $x^2 + 6x + 8 = 0$ by completing the square.

153. Explain how to solve $x^2 + 6x + 8 = 0$ using the quadratic formula.

154. How is the quadratic formula derived?

155. What is the discriminant and what information does it provide about a quadratic equation?

156. If you are given a quadratic equation, how do you determine which method to use to solve it?

157. In solving $\sqrt{2x - 1} + 2 = x$, why is it a good idea to isolate the radical term? What if we don't do this and simply square each side? Describe what happens.

158. What is an extraneous solution to a radical equation?

Critical Thinking Exercises

Make Sense? *In Exercises 159–162, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

159. The model $P = -0.18n + 2.1$ describes the number of pay phones, P, in millions, n years after 2000, so I have to solve a linear equation to determine the number of pay phones in 2010.

160. Although I can solve $3x + \frac{1}{5} = \frac{1}{4}$ by first subtracting $\frac{1}{5}$ from both sides, I find it easier to begin by multiplying both sides by 20, the least common denominator.

161. Because I want to solve $25x^2 - 169 = 0$ fairly quickly, I'll use the quadratic formula.

162. When checking a radical equation's proposed solution, I can substitute into the original equation or any equation that is part of the solution process.

In Exercises 163–166, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

163. The equation $(2x - 3)^2 = 25$ is equivalent to $2x - 3 = 5$.

164. Every quadratic equation has two distinct numbers in its solution set.

165. The equations $3y - 1 = 11$ and $3y - 7 = 5$ are equivalent.

166. The equation $ax^2 + c = 0, a \neq 0$, cannot be solved by the quadratic formula.

167. Find b such that $\frac{7x + 4}{b} + 13 = x$ will have a solution set given by $\{-6\}$.

168. Write a quadratic equation in general form whose solution set is $\{-3, 5\}$.

169. Solve for C: $V = C - \frac{C - S}{L}N$.

170. Solve for t: $s = -16t^2 + v_0t$.

Preview Exercises

Exercises 171–173 will help you prepare for the material covered in the next section.

171. Jane's salary exceeds Jim's by \$150 per week. If x represents Jim's weekly salary, write an algebraic expression that models Jane's weekly salary.

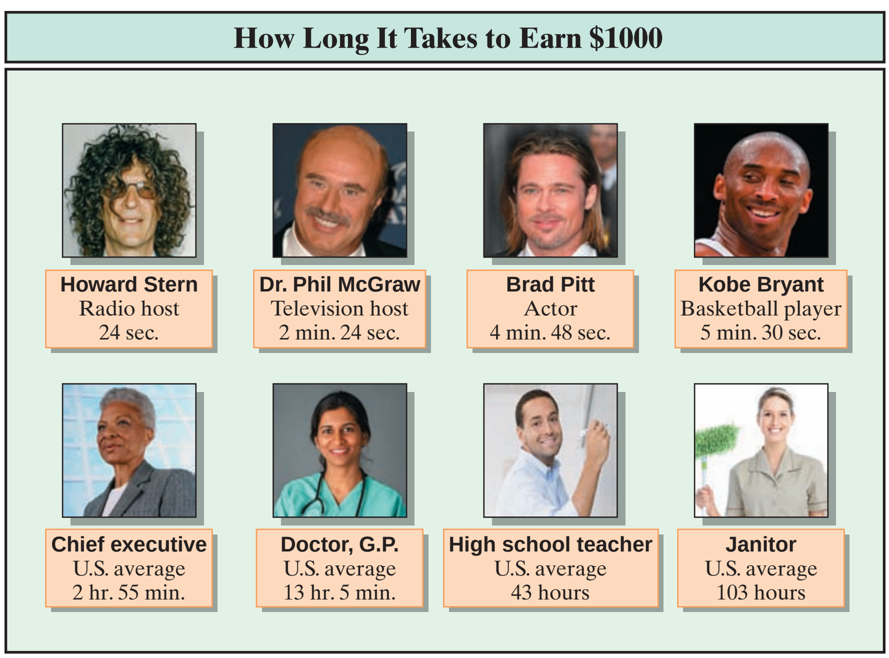
172. A telephone texting plan has a monthly fee of \$20 with a charge of \$0.05 per text. Write an algebraic expression that models the plan's monthly cost for x text messages.

173. If the width of a rectangle is represented by x and the length is represented by $x + 200$, write a simplified algebraic expression that models the rectangle's perimeter.

SECTION 8 Modeling with Equations

Objective

1. Use equations to solve problems.



Source: *Time*

In this section, you'll see examples and exercises focused on how much money Americans earn. These situations illustrate a step-by-step strategy for solving problems. As you become familiar with this strategy, you will learn to solve a wide variety of problems.

1 Use equations to solve problems.

Problem Solving with Equations

We have seen that a model is a mathematical representation of a real-world situation. In this section, we will be solving problems that are presented in English. This means that we must obtain models by translating from the ordinary language of English into the language of algebraic equations. To translate, however, we must understand the English prose and be familiar with the forms of algebraic language. Following are some general steps we will use in solving word problems:

GREAT QUESTION!

Why are word problems important?

There is great value in reasoning through the steps for solving a word problem. This value comes from the problem-solving skills that you will attain and is often more important than the specific problem or its solution.

Strategy for Solving Word Problems

Step 1 Read the problem carefully several times until you can state in your own words what is given and what the problem is looking for. Let x (or any variable) represent one of the unknown quantities in the problem.

Step 2 If necessary, write expressions for any other unknown quantities in the problem in terms of x.

Step 3 Write an equation in x that models the verbal conditions of the problem.

Step 4 Solve the equation and answer the problem's question.

Step 5 Check the solution *in the original wording* of the problem, not in the equation obtained from the words.

EXAMPLE 1 Starting Salaries for College Graduates with Undergraduate Degrees

The bar graph in **Figure 14** shows the ten most popular college majors with median, or middlemost, starting salaries for recent college graduates.

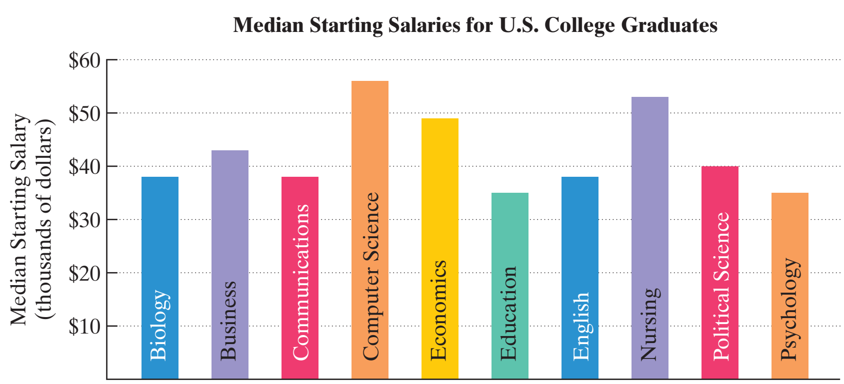


FIGURE 14
Source: PayScale (2010 data)

The median starting salary of a business major exceeds that of a psychology major by \$8 thousand. The median starting salary of an English major exceeds that of a psychology major by \$3 thousand. Combined, their median starting salaries are \$116 thousand. Determine the median starting salaries of psychology majors, business majors, and English majors with bachelor's degrees.

SOLUTION

Step 1 Let x represent one of the unknown quantities. We know something about the median starting salaries of business majors and English majors: Business majors earn \$8 thousand more than psychology majors and English majors earn \$3 thousand more than psychology majors. We will let

x = the median starting salary, in thousands of dollars, of psychology majors.

Step 2 Represent other unknown quantities in terms of *x*. Because business majors earn \$8 thousand more than psychology majors, let

Salary exceeds a psychology major, x, by \$8 thousand.

$x + 8 =$ the median starting salary, in thousands of dollars, of business majors.

Because English majors earn \$3 thousand more than psychology majors, let

Salary exceeds a psychology major, x, by \$3 thousand.

$x + 3 =$ the median starting salary, in thousands of dollars, of English majors.

Step 3 Write an equation in *x* that models the conditions. Combined, the median starting salaries for psychology, business, and English majors are \$116 thousand.

The median starting salary for psychology majors	plus	the median starting salary for business majors	plus	the median starting salary for English majors	is	\$116 thousand.
x	$+$	$(x + 8)$	$+$	$(x + 3)$	$=$	116

Step 4 Solve the equation and answer the question.

$$x + (x + 8) + (x + 3) = 116 \quad \text{This is the equation that models the problem's conditions.}$$
$$3x + 11 = 116 \quad \text{Remove parentheses, regroup, and combine like terms.}$$
$$3x = 105 \quad \text{Subtract 11 from both sides.}$$
$$x = 35 \quad \text{Divide both sides by 3.}$$

Thus,

$$\text{starting salary of psychology majors} = x = 35$$
$$\text{starting salary of business majors} = x + 8 = 35 + 8 = 43$$
$$\text{starting salary of English majors} = x + 3 = 35 + 3 = 38.$$

The median starting salary of psychology majors is \$35 thousand, the median starting salary of business majors is \$43 thousand, and the median starting salary of English majors is \$38 thousand.

Step 5 Check the proposed solution in the original wording of the problem. The problem states that combined, the median starting salaries are \$116 thousand. Using the median salaries we determined in step 4, the sum is

$$\$35 \text{ thousand} + \$43 \text{ thousand} + \$38 \text{ thousand},$$

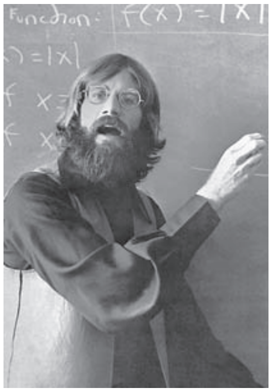
or \$116 thousand, which verifies the problem's conditions. • • •

GREAT QUESTION!

Example 1 involves using the word *exceeds* to represent two of the unknown quantities. Can you help me to write algebraic expressions for quantities described using *exceeds*?

Modeling with the word *exceeds* can be a bit tricky. It's helpful to identify the smaller quantity. Then add to this quantity to represent the larger quantity. For example, suppose that Tim's height exceeds Tom's height by a inches. Tom is the shorter person. If Tom's height is represented by x, then Tim's height is represented by $x + a$.

Check Point 1 Three of the bars in **Figure 14** represent median starting salaries of education, computer science, and economics majors. The median starting salary of a computer science major exceeds that of an education major by \$21 thousand. The median starting salary of an economics major exceeds that of an education major by \$14 thousand. Combined, their median starting salaries are \$140 thousand. Determine the median starting salaries of education majors, computer science majors, and economics majors with bachelor's degrees.



Your author teaching math in 1969

EXAMPLE 2 Modeling Attitudes of College Freshmen

Researchers have surveyed college freshmen every year since 1969. **Figure 15** shows that attitudes about some life goals have changed dramatically over the years. In particular, the freshman class of 2009 was more interested in making money than the freshmen of 1969 had been. In 1969, 42% of first-year college students considered "being well-off financially" essential or very important. For the period from 1969 through 2009, this percentage increased by approximately 0.9 each year. If this trend continues, by which year will all college freshmen consider "being well-off financially" essential or very important?

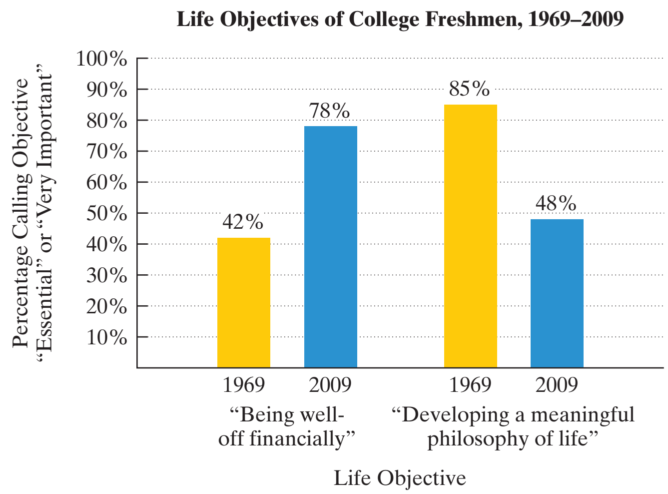


FIGURE 15
Source: Higher Education Research Institute

SOLUTION

Step 1 Let x represent one of the unknown quantities. We are interested in the year when all college freshmen, or 100% of the freshmen, will consider this life objective essential or very important. Let

x = the number of years after 1969 when all freshmen will consider "being well-off financially" essential or very important.

Step 2 Represent other unknown quantities in terms of x. There are no other unknown quantities to find, so we can skip this step.

Step 3 Write an equation in x that models the conditions.

The 1969 percentage	increased by	0.9 each year for x years	equals	100% of the freshmen.
42	$+$	$0.9x$	$=$	100

Step 4 Solve the equation and answer the question.

$$42 + 0.9x = 100$$ This is the equation that models the problem's conditions.

$$42 - 42 + 0.9x = 100 - 42$$ Subtract 42 from both sides.

$$0.9x = 58$$ Simplify.

$$\frac{0.9x}{0.9} = \frac{58}{0.9}$$ Divide both sides by 0.9.

$$x = 64.\overline{4} \approx 64$$ Simplify and round to the nearest whole number.

Using current trends, by approximately 64 years after 1969, or in 2033, all freshmen will consider "being well-off financially" essential or very important.

Step 5 Check the proposed solution in the original wording of the problem. The problem states that all freshmen (100%, represented by 100 using the model) will consider the objective essential or very important. Does this approximately occur if we increase the 1969 percentage, 42%, by 0.9 each year for 64 years, our proposed solution?

$$42 + 0.9(64) = 42 + 57.6 = 99.6 \approx 100$$

This verifies that using trends shown in **Figure 15**, all first-year college students will consider the objective essential or very important approximately 64 years after 1969. ●●●

Check Point 2 **Figure 15** shows that the freshman class of 2009 was less interested in developing a philosophy of life than the freshmen of 1969 had been. In 1969, 85% of the freshmen considered this objective essential or very important. Since then, this percentage has decreased by approximately 0.9 each year. If this trend continues, by which year will only 25% of college freshmen consider "developing a meaningful philosophy of life" essential or very important?

EXAMPLE 3 A Price Reduction on a Digital Camera

Your local computer store is having a terrific sale on digital cameras. After a 40% price reduction, you purchase a digital camera for $276. What was the camera's price before the reduction?

SOLUTION

Step 1 Let *x* represent one of the unknown quantities. We will let

x = the price of the digital camera prior to the reduction.

Step 2 Represent other unknown quantities in terms of *x*. There are no other unknown quantities to find, so we can skip this step.

Step 3 Write an equation in *x* that models the conditions. The camera's original price minus the 40% reduction is the reduced price, $276.

$$\overbrace{x}^{\text{Original price}} \quad \overbrace{-}^{\text{minus}} \quad \overbrace{0.4x}^{\text{the reduction (40\% of the original price)}} \quad \overbrace{=}^{\text{is}} \quad \overbrace{276}^{\text{the reduced price, \$276.}}$$

GREAT QUESTION!

Why is the 40% reduction written as 0.4*x* in Example 3?

- 40% is written 0.40 or 0.4.
- "Of" represents multiplication, so 40% of the original price is $0.4x$.

Notice that the original price, x, reduced by 40% is $x - 0.4x$ and *not* $x - 0.4$.

Step 4 Solve the equation and answer the question.

$x - 0.4x = 276$ This is the equation that models the problem's conditions.

$0.6x = 276$ Combine like terms: $x - 0.4x = 1x - 0.4x = 0.6x$.

$\dfrac{0.6x}{0.6} = \dfrac{276}{0.6}$ Divide both sides by 0.6.

$x = 460$ Simplify: $\begin{array}{r} 460. \\ 0.6\overline{)276.0} \end{array}$

The digital camera's price before the reduction was $460.

Step 5 Check the proposed solution in the original wording of the problem. The price before the reduction, $460, minus the 40% reduction should equal the reduced price given in the original wording, $276:

$$460 - 40\% \text{ of } 460 = 460 - 0.4(460) = 460 - 184 = 276.$$

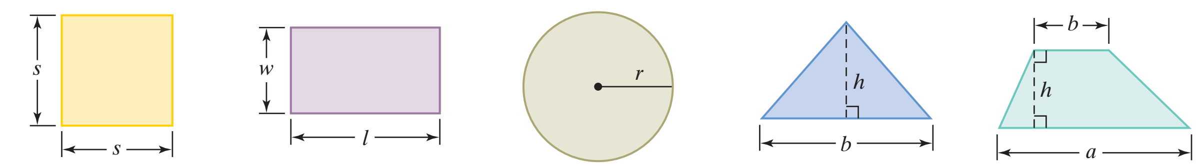
This verifies that the digital camera's price before the reduction was $460. ●●●

Check Point 3 After a 30% price reduction, you purchase a new computer for \$840. What was the computer's price before the reduction?

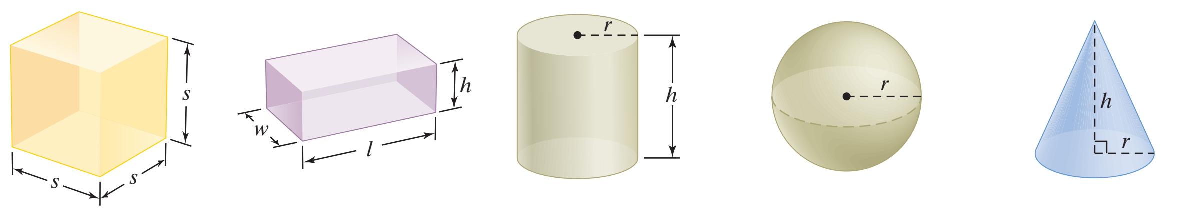
Solving geometry problems usually requires a knowledge of basic geometric ideas and formulas. Formulas for area, perimeter, and volume are given in **Table 6**.

Table 6 Common Formulas for Area, Perimeter, and Volume

Square	Rectangle	Circle	Triangle	Trapezoid
$A = s^2$ $P = 4s$	$A = lw$ $P = 2l + 2w$	$A = \pi r^2$ $C = 2\pi r$	$A = \frac{1}{2}bh$	$A = \frac{1}{2}h(a + b)$



Cube	Rectangular Solid	Circular Cylinder	Sphere	Cone
$V = s^3$	$V = lwh$	$V = \pi r^2 h$	$V = \frac{4}{3}\pi r^3$	$V = \frac{1}{3}\pi r^2 h$



We will be using the formula for the perimeter of a rectangle, $P = 2l + 2w$, in our next example. The formula states that a rectangle's perimeter is the sum of twice its length and twice its width.

EXAMPLE 4 Finding the Dimensions of an American Football Field

The length of an American football field is 200 feet more than the width. If the perimeter of the field is 1040 feet, what are its dimensions?

SOLUTION

Step 1 Let x represent one of the unknown quantities. We know something about the length; the length is 200 feet more than the width. We will let

$$x = \text{the width.}$$

Step 2 Represent other unknown quantities in terms of x. Because the length is 200 feet more than the width, we add 200 to the width to represent the length. Thus,

$$x + 200 = \text{the length.}$$

Figure 16 illustrates an American football field and its dimensions.

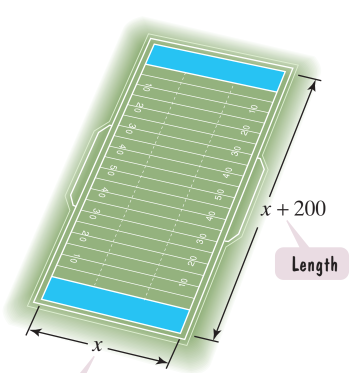


FIGURE 16
An American football field

Step 3 Write an equation in x that models the conditions. Because the perimeter of the field is 1040 feet,

Twice the length	plus	twice the width	is	the perimeter.
$2(x + 200)$	$+$	$2x$	$=$	$1040.$

GREAT QUESTION!

Should I draw pictures like Figure 16 when solving geometry problems?

When solving word problems, particularly problems involving geometric figures, drawing a picture of the situation is often helpful. Label x on your drawing and, where appropriate, label other parts of the drawing in terms of x.

Step 4 Solve the equation and answer the question.

$$\begin{aligned} 2(x+200)+2x &= 1040 && \text{This is the equation that models the problem's conditions.}\\ 2x+400+2x &= 1040 && \text{Apply the distributive property.}\\ 4x+400 &= 1040 && \text{Combine like terms: } 2x+2x=4x.\\ 4x &= 640 && \text{Subtract 400 from both sides.}\\ x &= 160 && \text{Divide both sides by 4.} \end{aligned}$$

Thus,

$$\begin{aligned} \text{width} &= x = 160.\\ \text{length} &= x+200 = 160+200 = 360. \end{aligned}$$

The dimensions of an American football field are 160 feet by 360 feet. (The 360-foot length is usually described as 120 yards.)

Step 5 Check the proposed solution in the original wording of the problem. The perimeter of the football field using the dimensions that we found is

$$2(360\text{ feet}) + 2(160\text{ feet}) = 720\text{ feet} + 320\text{ feet} = 1040\text{ feet}.$$

Because the problem's wording tells us that the perimeter is 1040 feet, our dimensions are correct.

Check Point 4 The length of a rectangular basketball court is 44 feet more than the width. If the perimeter of the basketball court is 288 feet, what are its dimensions?

We will use the formula for the area of a rectangle, $A = lw$, in our next example. The formula states that a rectangle's area is the product of its length and its width.

EXAMPLE 5 Solving a Problem Involving Landscape Design

A rectangular garden measures 80 feet by 60 feet. A large path of uniform width is to be added along both shorter sides and one longer side of the garden. The landscape designer doing the work wants to double the garden's area with the addition of this path. How wide should the path be?

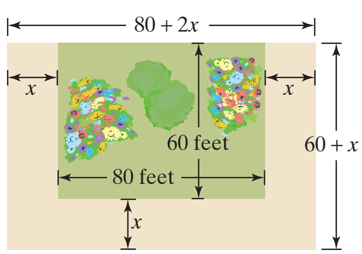


FIGURE 17 The garden's area is to be doubled by adding the path.

SOLUTION

Step 1 Let x represent one of the unknown quantities. We will let

$$x = \text{the width of the path.}$$

The situation is illustrated in **Figure 17**. The figure shows the original 80-by-60 foot rectangular garden and the path of width x added along both shorter sides and one longer side.

Step 2 Represent other unknown quantities in terms of x. Because the path is added along both shorter sides and one longer side, **Figure 17** shows that

$$\begin{aligned} 80+2x &= \text{the length of the new, expanded rectangle}\\ 60+x &= \text{the width of the new, expanded rectangle.} \end{aligned}$$

Step 3 Write an equation in x that models the conditions. The area of the rectangle must be doubled by the addition of the path.

$$\overbrace{(80+2x)(60+x)}^{\text{The area, or length times width, of the new, expanded rectangle}} \quad \overbrace{=}^{\text{must be}} \quad \overbrace{2}^{\text{twice that of}} \cdot \overbrace{80 \cdot 60}^{\text{the area of the garden.}}$$

Step 4 Solve the equation and answer the question.

$(80 + 2x)(60 + x) = 2 \cdot 80 \cdot 60$	This is the equation that models the problem's conditions.
$4800 + 200x + 2x^2 = 9600$	Multiply. Use FOIL on the left side.
$2x^2 + 200x - 4800 = 0$	Subtract 9600 from both sides and write the quadratic equation in general form.
$2(x^2 + 100x - 2400) = 0$	Factor out 2, the GCF.
$2(x - 20)(x + 120) = 0$	Factor the trinomial.
$x - 20 = 0$ or $x + 120 = 0$	Set each variable factor equal to 0.
$x = 20$ $\quad$ $x = -120$	Solve for x.

The path cannot have a negative width. Because -120 is geometrically impossible, we use $x = 20$. The width of the path should be 20 feet.

Step 5 Check the proposed solution in the original wording of the problem. Has the landscape architect doubled the garden's area with the 20-foot-wide path? The area of the garden is 80 feet times 60 feet, or 4800 square feet. Because $80 + 2x$ and $60 + x$ represent the length and width of the expanded rectangle,

$$80 + 2x = 80 + 2 \cdot 20 = 120 \text{ feet is the expanded rectangle's length.}$$
$$60 + x = 60 + 20 = 80 \text{ feet is the expanded rectangle's width.}$$

The area of the expanded rectangle is 120 feet times 80 feet, or 9600 square feet. This is double the area of the garden, 4800 square feet, as specified by the problem's conditions. • • •

Check Point 5 A rectangular garden measures 16 feet by 12 feet. A path of uniform width is to be added so as to surround the entire garden. The landscape artist doing the work wants the garden and path to cover an area of 320 square feet. How wide should the path be?

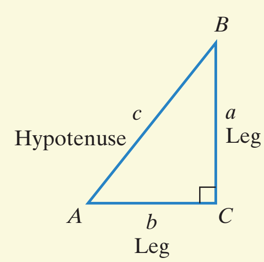
The solution to our next problem relies on knowing the **Pythagorean Theorem**. The theorem relates the lengths of the three sides of a **right triangle**, a triangle with one angle measuring 90°. The side opposite the 90° angle is called the **hypotenuse**. The other sides are called **legs**. The legs form the two sides of the right angle.

The Pythagorean Theorem

The sum of the squares of the lengths of the legs of a right triangle equals the square of the length of the hypotenuse.

If the legs have lengths a and b, and the hypotenuse has length c, then

$$a^2 + b^2 = c^2.$$



EXAMPLE 6 Using the Pythagorean Theorem

a. A wheelchair ramp with a length of 122 inches has a horizontal distance of 120 inches. What is the ramp's vertical distance?

b. Construction laws are very specific when it comes to access ramps for the disabled. Every vertical rise of 1 inch requires a horizontal run of 12 inches. Does this ramp satisfy the requirement?

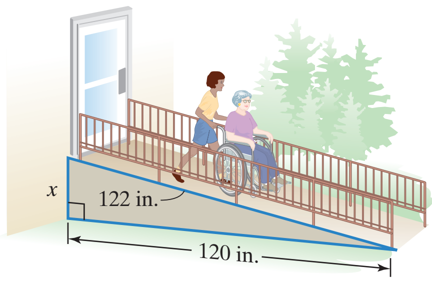


FIGURE 18

SOLUTION

a. Step 1 Let x represent one of the unknown quantities. We will let

$$x = \text{the ramp's vertical distance.}$$

The situation is illustrated in **Figure 18**.

Step 2 Represent other unknown quantities in terms of x. There are no other unknown quantities, so we can skip this step.

Step 3 Write an equation in x that models the conditions. Figure 18 shows the right triangle that is formed by the ramp, the wall, and the ground. We can find x, the ramp's vertical distance, using the Pythagorean Theorem.

$$\underbrace{x^2}_{(\text{leg})^2} \underbrace{+}_{\text{plus}} \underbrace{120^2}_{(\text{leg})^2} \underbrace{=}_{\text{equals}} \underbrace{122^2}_{(\text{hypotenuse})^2}$$

Step 4 Solve the equation and answer the question. The quadratic equation $x^2 + 120^2 = 122^2$ can be solved most efficiently by the square root property.

$x^2 + 120^2 = 122^2$	This is the equation resulting from the Pythagorean Theorem.
$x^2 + 14{,}400 = 14{,}884$	Square 120 and 122.
$x^2 = 484$	Isolate x^2 by subtracting 14,400 from both sides.
$x = \sqrt{484}$ or $x = -\sqrt{484}$	Apply the square root property.
$x = 22$ $\quad$ $x = -22$	Simplify.

Because x represents the ramp's vertical distance, this measurement must be positive. We reject -22. Thus, the ramp's vertical distance is 22 inches.

Step 5 Check the proposed solution in the original wording of the problem. The problem's wording implies that the ramp, the wall, and the ground form a right triangle. This can be checked using the **converse of the Pythagorean Theorem:** If a triangle has sides of lengths a, b, and c, where c is the length of the longest side, and if $a^2 + b^2 = c^2$, then the triangle is a right triangle. Let's check that a vertical distance of 22 inches forms a right triangle with the ramp's length of 122 inches and its horizontal distance of 120 inches. Is $22^2 + 120^2 = 122^2$? Simplifying the arithmetic, we obtain the true statement $14{,}884 = 14{,}884$. Thus, a vertical distance of 22 inches forms a right triangle.

b. Every vertical rise of 1 inch requires a horizontal run of 12 inches. Because the ramp has a vertical distance of 22 inches, it requires a horizontal distance of 22(12) inches, or 264 inches. The horizontal distance is only 120 inches, so this ramp does not satisfy construction laws for access ramps for the disabled. ●●●

GREAT QUESTION!

If the sum of the squares of the lengths of two sides of a triangle equals the square of the length of the third side, must the triangle be a right triangle?

Yes. The Pythagorean Theorem is an *if . . . then* statement: If a triangle is a right triangle, then $a^2 + b^2 = c^2$. The **converse** of *if p then q* is *if q then p*. Although the converse of a true statement may not be true, the converse of the Pythagorean Theorem is also a true statement: If $a^2 + b^2 = c^2$, then a triangle is a right triangle.

Check Point 6 A radio tower is supported by two wires that are each 130 yards long and attached to the ground 50 yards from the base of the tower. How tall is the tower?

In our final example, the conditions are modeled by a rational equation.

EXAMPLE 7 Dividing the Cost of a Yacht

A group of friends agrees to share the cost of a \$50,000 yacht equally. Before the purchase is made, one more person joins the group and enters the agreement. As a result, each person's share is reduced by \$2500. How many people were in the original group?

SOLUTION

Step 1 Let x represent one of the unknown quantities. We will let

$$x = \text{the number of people in the original group.}$$

Step 2 Represent other unknown quantities in terms of *x*. Because one more person joined the original group, let

$$x + 1 = \text{the number of people in the final group.}$$

Step 3 Write an equation in *x* that models the conditions. As a result of one more person joining the original group, each person's share is reduced by \$2500.

$$\underbrace{\overbrace{\frac{50{,}000}{x}}^{\text{Original cost per person}}}_{\text{This is the yacht's cost, \$50,000, divided by the number of people, } x.} \overbrace{-}^{\text{minus}} \overbrace{2500}^{\$2500} \overbrace{=}^{\text{equals}} \underbrace{\overbrace{\frac{50{,}000}{x+1}}^{\text{the final cost per person.}}}_{\text{This is the yacht's cost, \$50,000, divided by the number of people, } x+1.}$$

Step 4 Solve the equation and answer the question.

$$\frac{50{,}000}{x} - 2500 = \frac{50{,}000}{x+1}$$

This is the equation that models the problem's conditions.

$$x(x+1)\left(\frac{50{,}000}{x} - 2500\right) = x(x+1)\cdot\frac{50{,}000}{x+1}$$

Multiply both sides by $x(x + 1)$, the LCD.

$$\cancel{x}(x+1)\cdot\frac{50{,}000}{\cancel{x}} - x(x+1)2500 = x\cancel{(x+1)}\cdot\frac{50{,}000}{\cancel{(x+1)}}$$

Use the distributive property and divide out common factors.

$$50{,}000(x+1) - 2500x(x+1) = 50{,}000x$$

Simplify.

$$50{,}000x + 50{,}000 - 2500x^2 - 2500x = 50{,}000x$$

Use the distributive property.

$$-2500x^2 + 47{,}500x + 50{,}000 = 50{,}000x$$

Combine like terms: $50{,}000x - 2500x = 47{,}500x$.

$$-2500x^2 - 2500x + 50{,}000 = 0$$

Write the quadratic equation in general form, subtracting $50{,}000x$ from both sides.

$$-2500(x^2 + x - 20) = 0$$

Factor out -2500.

$$-2500(x+5)(x-4) = 0$$

Factor completely.

$$x + 5 = 0 \quad \text{or} \quad x - 4 = 0$$

Set each variable factor equal to zero.

$$x = -5 \qquad\qquad x = 4$$

Solve the resulting equations.

Because x represents the number of people in the original group, x cannot be negative. Thus, there were four people in the original group.

Step 5 Check the proposed solution in the original wording of the problem.

$$\text{original cost per person} = \frac{\$50{,}000}{4} = \$12{,}500$$

$$\text{final cost per person} = \frac{\$50{,}000}{5} = \$10{,}000$$

We see that each person's share is reduced by \$12,500 − \$10,000, or \$2500, as specified by the problem's conditions.

Check Point 7 A group of people share equally in a $5,000,000 lottery. Before the money is divided, three more winning ticket holders are declared. As a result, each person's share is reduced by $375,000. How many people were in the original group of winners?

CONCEPT AND VOCABULARY CHECK

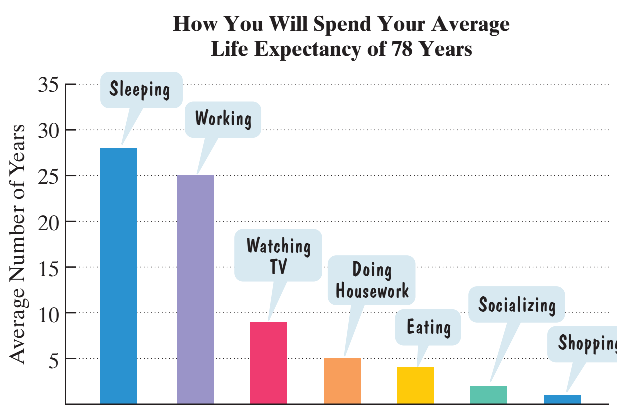
Fill in each blank so that the resulting statement is true.

1. According to the U.S. Office of Management and Budget, the 2011 budget for defense exceeded the budget for education by $658.6 billion. If x represents the budget for education, in billions of dollars, the budget for defense can be represented by ________.
2. In 2000, 31% of U.S. adults viewed a college education as essential for success. For the period from 2000 through 2010, this percentage increased by approximately 2.4 each year. The percentage of U.S. adults who viewed a college education as essential for success x years after 2000 can be represented by ________.
3. I purchased a computer after a 15% price reduction. If x represents the computer's original price, the reduced price can be represented by ________.
4. The length of a rectangle is 5 feet more than the width. If x represents the width, in feet, the length is represented by ______. The perimeter of the rectangle is represented by ________. The area of the rectangle is represented by ________.
5. A triangle with one angle measuring 90° is called a/an ______ triangle. The side opposite the 90° angle is called the ________. The other sides are called ________.
6. The Pythagorean Theorem states that in any ________ triangle, the sum of the squares of the lengths of the ________ equals ________.
7. If x people equally share the cost of a $10,000 boat, the cost per person is represented by ________. If two more people share the cost, the cost per person is represented by ________.

EXERCISE SET 8

Practice and Application Exercises

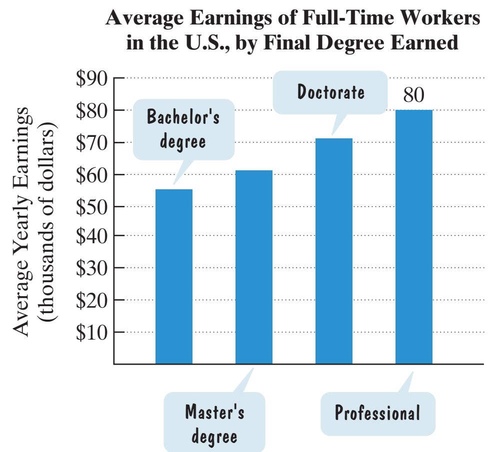
How will you spend your average life expectancy of 78 years? The bar graph shows the average number of years you will devote to each of your most time-consuming activities. Exercises 1–2 are based on the data displayed by the graph.



Source: U.S. Bureau of Labor Statistics

1. According to the American Bureau of Labor Statistics, you will devote 37 years to sleeping and watching TV. The number of years sleeping will exceed the number of years watching TV by 19. Over your lifetime, how many years will you spend on each of these activities?
2. According to the American Bureau of Labor Statistics, you will devote 32 years to sleeping and eating. The number of years sleeping will exceed the number of years eating by 24. Over your lifetime, how many years will you spend on each of these activities?

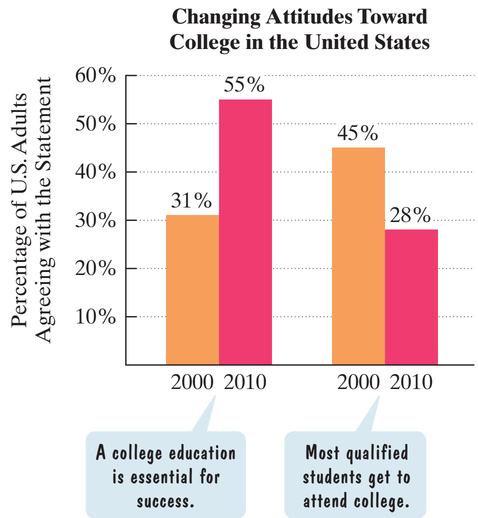
The bar graph shows average yearly earnings in the United States for people with a college education, by final degree earned. Exercises 3–4 are based on the data displayed by the graph.



Source: U.S. Census Bureau

3. The average yearly salary of an American whose final degree is a master's is $49 thousand less than twice that of an American whose final degree is a bachelor's. Combined, two people with each of these educational attainments earn $116 thousand. Find the average yearly salary of Americans with each of these final degrees.
4. The average yearly salary of an American whose final degree is a doctorate is $39 thousand less than twice that of an American whose final degree is a bachelor's. Combined, two people with each of these educational attainments earn $126 thousand. Find the average yearly salary of Americans with each of these final degrees.

Even as Americans increasingly view a college education as essential for success, many believe that a college education is becoming less available to qualified students. Exercises 5–6 are based on the data displayed by the graph.



Source: *Public Agenda*

5. In 2000, 31% of U.S. adults viewed a college education as essential for success. For the period 2000 through 2010, the percentage viewing a college education as essential for success increased on average by approximately 2.4 each year. If this trend continues, by which year will 67% of all American adults view a college education as essential for success?

6. The data displayed by the graph indicate that in 2000, 45% of U.S. adults believed most qualified students get to attend college. For the period from 2000 through 2010, the percentage who believed that a college education is available to most qualified students decreased by approximately 1.7 each year. If this trend continues, by which year will only 11% of all American adults believe that most qualified students get to attend college?

7. A new car worth \$24,000 is depreciating in value by \$3000 per year.

a. Write a formula that models the car's value, y, in dollars, after x years.

b. Use the formula from part (a) to determine after how many years the car's value will be \$9000.

8. A new car worth \$45,000 is depreciating in value by \$5000 per year.

a. Write a formula that models the car's value, y, in dollars, after x years.

b. Use the formula from part (a) to determine after how many years the car's value will be \$10,000.

9. In 2010, there were 13,300 students at college A, with a projected enrollment increase of 1000 students per year. In the same year, there were 26,800 students at college B, with a projected enrollment decline of 500 students per year. According to these projections, when will the colleges have the same enrollment? What will be the enrollment in each college at that time?

10. In 2000, the population of Greece was 10,600,000, with projections of a population decrease of 28,000 people per year. In the same year, the population of Belgium was 10,200,000, with projections of a population decrease of 12,000 people per year. (*Source:* United Nations) According to these projections, when will the two countries have the same population? What will be the population at that time?

11. After a 20% reduction, you purchase a television for \$336. What was the television's price before the reduction?

12. After a 30% reduction, you purchase a dictionary for \$30.80. What was the dictionary's price before the reduction?

13. Including 8% sales tax, an inn charges \$162 per night. Find the inn's nightly cost before the tax is added.

14. Including 5% sales tax, an inn charges \$252 per night. Find the inn's nightly cost before the tax is added.

Exercises 15–16 involve markup, the amount added to the dealer's cost of an item to arrive at the selling price of that item.

15. The selling price of a refrigerator is \$584. If the markup is 25% of the dealer's cost, what is the dealer's cost of the refrigerator?

16. The selling price of a scientific calculator is \$15. If the markup is 25% of the dealer's cost, what is the dealer's cost of the calculator?

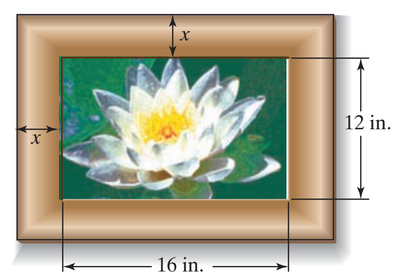
17. A rectangular soccer field is twice as long as it is wide. If the perimeter of the soccer field is 300 yards, what are its dimensions?

18. A rectangular swimming pool is three times as long as it is wide. If the perimeter of the pool is 320 feet, what are its dimensions?

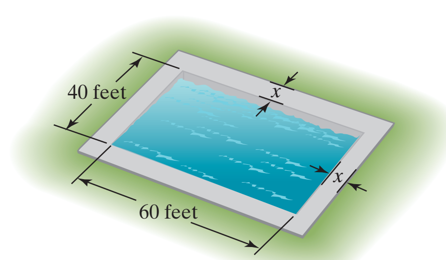
19. The length of the rectangular tennis court at Wimbledon is 6 feet longer than twice the width. If the court's perimeter is 228 feet, what are the court's dimensions?

20. The length of a rectangular pool is 6 meters less than twice the width. If the pool's perimeter is 126 meters, what are its dimensions?

21. The rectangular painting in the figure shown measures 12 inches by 16 inches and includes a frame of uniform width around the four edges. The perimeter of the rectangle formed by the painting and its frame is 72 inches. Determine the width of the frame.



22. The rectangular swimming pool in the figure shown measures 40 feet by 60 feet and includes a path of uniform width around the four edges. The perimeter of the rectangle formed by the pool and the surrounding path is 248 feet. Determine the width of the path.



23. The length of a rectangular sign is 3 feet longer than the width. If the sign's area is 54 square feet, find its length and width.

24. A rectangular parking lot has a length that is 3 yards greater than the width. The area of the parking lot is 180 square yards. Find the length and the width.

25. Each side of a square is lengthened by 3 inches. The area of this new, larger square is 64 square inches. Find the length of a side of the original square.

26. Each side of a square is lengthened by 2 inches. The area of this new, larger square is 36 square inches. Find the length of a side of the original square.

27. A pool measuring 10 meters by 20 meters is surrounded by a path of uniform width. If the area of the pool and the path combined is 600 square meters, what is the width of the path?

28. A vacant rectangular lot is being turned into a community vegetable garden measuring 15 meters by 12 meters. A path of uniform width is to surround the garden. If the area of the lot is 378 square meters, find the width of the path surrounding the garden.

29. As part of a landscaping project, you put in a flower bed measuring 20 feet by 30 feet. To finish off the project, you are putting in a uniform border of pine bark around the outside of the rectangular garden. You have enough pine bark to cover 336 square feet. How wide should the border be?

30. As part of a landscaping project, you put in a flower bed measuring 10 feet by 12 feet. You plan to surround the bed with a uniform border of low-growing plants that require 1 square foot each when mature. If you have 168 of these plants, how wide a strip around the flower bed should you prepare for the border?

31. A 20-foot ladder is 15 feet from a house. How far up the house, to the nearest tenth of a foot, does the ladder reach?

32. The base of a 30-foot ladder is 10 feet from a building. If the ladder reaches the flat roof, how tall, to the nearest tenth of a foot, is the building?

33. A tree is supported by a wire anchored in the ground 5 feet from its base. The wire is 1 foot longer than the height that it reaches on the tree. Find the length of the wire.

34. A tree is supported by a wire anchored in the ground 15 feet from its base. The wire is 4 feet longer than the height that it reaches on the tree. Find the length of the wire.

35. A rectangular piece of land whose length its twice its width has a diagonal distance of 64 yards. How many yards, to the nearest tenth of a yard, does a person save by walking diagonally across the land instead of walking its length and its width?

36. A rectangular piece of land whose length is three times its width has a diagonal distance of 92 yards. How many yards, to the nearest tenth of a yard, does a person save by walking diagonally across the land instead of walking its length and its width?

37. A group of people share equally in a \$20,000,000 lottery. Before the money is divided, two more winning ticket holders are declared. As a result, each person's share is reduced by \$500,000. How many people were in the original group of winners?

38. A group of friends agrees to share the cost of a \$480,000 vacation condominium equally. Before the purchase is made, four more people join the group and enter the agreement. As a result, each person's share is reduced by \$32,000. How many people were in the original group?

In Exercises 39–42, use the formula

$$\text{Time traveled} = \frac{\text{Distance traveled}}{\text{Average velocity}}.$$

39. A car can travel 300 miles in the same amount of time it takes a bus to travel 180 miles. If the average velocity of the bus is 20 miles per hour slower than the average velocity of the car, find the average velocity for each.

40. A passenger train can travel 240 miles in the same amount of time it takes a freight train to travel 160 miles. If the average velocity of the freight train is 20 miles per hour slower than the average velocity of the passenger train, find the average velocity of each.

41. You ride your bike to campus a distance of 5 miles and return home on the same route. Going to campus, you ride mostly downhill and average 9 miles per hour faster than on your return trip home. If the round trip takes one hour and ten minutes—that is $\frac{7}{6}$ hours—what is your average velocity on the return trip?

42. An engine pulls a train 140 miles. Then a second engine, whose average velocity is 5 miles per hour faster than the first engine, takes over and pulls the train 200 miles. The total time required for both engines is 9 hours. Find the average velocity of each engine.

43. An automobile repair shop charged a customer \$448, listing \$63 for parts and the remainder for labor. If the cost of labor is \$35 per hour, how many hours of labor did it take to repair the car?

44. A repair bill on a sailboat came to \$1603, including \$532 for parts and the remainder for labor. If the cost of labor is \$63 per hour, how many hours of labor did it take to repair the sailboat?

45. An HMO pamphlet contains the following recommended weight for women: "Give yourself 100 pounds for the first 5 feet plus 5 pounds for every inch over 5 feet tall." Using this description, what height corresponds to a recommended weight of 135 pounds?

46. A job pays an annual salary of \$33,150, which includes a holiday bonus of \$750. If paychecks are issued twice a month, what is the gross amount for each paycheck?

47. You have 35 hits in 140 times at bat. Your batting average is $\frac{35}{140}$, or 0.25. How many consecutive hits must you get to increase your batting average to 0.30?

48. You have 30 hits in 120 times at bat. Your batting average is $\frac{30}{120}$, or 0.25. How many consecutive hits must you get to increase your batting average to 0.28?

Writing in Mathematics

49. In your own words, describe a step-by-step approach for solving algebraic word problems.

50. Write an original word problem that can be solved using an equation. Then solve the problem.

51. In your own words, state the Pythagorean Theorem.