GLOBAL EDITION

Elementary and Middle School Mathematics

Teaching Developmentally

ELEVENTH EDITION

John A. Van de Walle

Karen S. Karp

Jennifer M. Bay-Williams

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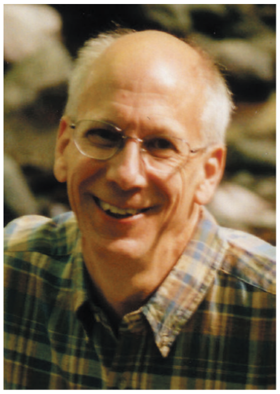
About the Authors



John A. Van de Walle

The late **Dr. John A. Van de Walle** was a professor emeritus at Virginia Commonwealth University. He was a leader in mathematics education who regularly gave professional development workshops focused on mathematics instruction that engage students in mathematical reasoning and problem solving for K–8 teachers in the United States and Canada. He visited and taught in many classrooms and worked with teachers to implement student-centered mathematics lessons. He co-authored the *Scott Foresman-Addison Wesley Mathematics K–6* series and contributed to the original Pearson School mathematics program *enVisionMATH*. Additionally, John was very active in the National Council of Teachers of Mathematics (NCTM), writing book chapters and journal articles, serving on the board of directors, chairing the educational materials committee, and speaking at national and regional meetings.

Karen S. Karp

Dr. Karen S. Karp is a professor at Johns Hopkins University (Maryland). Previously, she was a professor of mathematics education at the University of Louisville for more than twenty years. Prior to entering the field of teacher education, she was an elementary school teacher in New York. She is the coauthor of *Strengths-Based Teaching and Learning in Mathematics: 5 Teaching Turnarounds for Grades K-6*; the three book series, *The Math Pact: Achieving Instructional Cohesion within and across Grades*; and the *What Works Clearinghouse Practice Guide on Assisting Students Struggling with Mathematics: Intervention in the Elementary Grades*. She is a former member of the board of directors for the National Council of Teachers of Mathematics (NCTM) and a former president of the Association of Mathematics Teacher Educators. In 2020, she received the NCTM Lifetime Achievement Award and in 2021 became a member of the United States National Committee on Mathematics Instruction. She continues to work in classrooms to support teachers in ways to instruct students with special needs in math.

Jennifer M. Bay-Williams

Dr. Jennifer Bay-Williams is a professor at the University of Louisville. She is a leader in mathematics education, regularly working to support mathematics instruction. Beyond her work on this book and the related *Teaching Student Centered Mathematics* book series, Jennifer has written other books that focus on effective mathematics teaching. Most recently she has written a book series called *Figuring out Fluency in Mathematics K-12* and *Math Fact Fluency: 60+ Games and Assessment Tools to Help Learning and Retention*. For teacher leaders, she has authored *Everything You Need for Mathematics Coaching*, and in the area of financial literacy, she authored a three-book series titled *On the Money* (a financial literacy book series). Over the years, Jennifer has taught elementary, middle, and high school in the United States and in Peru. She continues work with teachers around the world and in her local school districts to support their work. Jennifer has been actively involved in NCTM, NCSM, AMTE, and TODOS: Mathematics for All. Currently, she is serving as an associate editor for *Mathematics Teacher: Learning and Teaching in preK-12.*

About the Contributor

Jonathan Wray

Dr. Jonathan Wray is the technology contributor to *Elementary and Middle School Mathematics, Teaching Developmentally* (sixth to eleventh editions). He is Coordinator for Secondary Mathematics in the Howard County Public School System. He has served as the president of the Association of Maryland Mathematics Teacher Educators (AMMTE), the Maryland Council of Teachers of Mathematics (MCTM), and the manager of the Elementary Mathematics Specialists and Teacher Leaders (ems&tl) Project. Jon also served as an elected member of the National Council of Teachers of Mathematics (NCTM) Board of Directors (2012–2015). He has been recognized for his expertise in infusing technology in mathematics teaching and was named an outstanding technology leader in education by the Maryland Society for Educational Technology (MSET). Jon is the recipient of the 2020 Ross Taylor/Glen Gilbert National Mathematics Leadership Award given by the National Council of Supervisors of Mathematics.

Preface

All students can learn mathematics with understanding. It is through the teacher's actions that every student can have this experience. We believe that teachers must create a classroom environment in which students are given opportunities to solve problems and work together, using their ideas and strategies to solve them. Effective mathematics instruction involves posing tasks that engage students in the mathematics they are expected to learn. Then, by allowing students to interact with and productively struggle with *their own mathematical ideas* and *their own strategies*, they will learn to see the connections among mathematical topics and the real world. Students value mathematics and feel empowered to use it.

Creating a classroom in which students design solution pathways, engage in productive struggle, and connect one mathematical idea to another is complex. Questions arise, such as, "How do I get students to wrestle with problems if they just want me to show them how to do it? What kinds of tasks lend themselves to this type of engagement? Where can I learn the mathematics content I need to be able to teach in this way?" With these and other questions firmly in mind, the goals of this resource are to:

1. Illustrate what it means to teach mathematics by using a problem-based approach.
2. Focus attention on student thinking, including the ways students might reason about numbers and possible challenges and misconceptions they might have.
3. Serve as a go-to reference for teaching pre-K–8 mathematics content, with useful learning progressions and research-based instructional strategies to guide instructional decision-making.
4. Present a robust collection of high-quality tasks and problem-based activities that can engage students in learning significant mathematical concepts and skills while developing positive mathematics identities.

New to This Edition

The following are the highlights of the most significant changes in the eleventh edition.

Learning Progressions

This book has always focused on teaching mathematics developmentally. In this edition, every chapter has learning progressions. In many cases, these are visuals at the start of the chapter to serve as an advanced organizer of how we approach the content in the book. In other cases, specific ideas within a chapter have a specific learning progression. These progressions can be useful for planning units, understanding prior knowledge and future directions, and can inform assessment development.

Focus on Reasoning Strategy Instruction

As described in Chapters 2 and 3, developing procedural fluency is important and requires that students learn reasoning strategies. We have increased attention to what real fluency looks like in these early chapters, and then increased attention to teaching reasoning strategies to students across the number chapters, beginning with basic fact reasoning strategies in Chapter 9, whole-number reasoning strategies in Chapters 11 and 12, and reasoning strategies for fractions in Chapter 15.

Less Talk, More Action

In every revision, we receive wonderful suggestions on what we can add and rarely a suggestion on what to cut. We have continued the tradition of having this book serve as a reference for teaching all mathematics concepts in K–8. But it is a lot, so in this edition, we worked hard to pare down on narrative, in particular in Part II. The overall pages may not be too different, because we have added more activities and tasks, new figures, and reflection questions. In the end, we believe these changes make this edition more accessible to our readers, whether they are just getting started, looking for ideas for their 20th year of teaching, or preparing a professional learning workshop for their colleagues.

Technology Turnover

Over four years' time, in particular time that included significant online learning, the technology used to support mathematics instruction has changed, so many new apps and more graphing tools are available. Throughout Part II of the book, the suggestions in the activities and the Technology Notes have been significantly revised.

Writing to Learn

Back by popular demand! Each chapter closes with discussion questions focused on the big ideas of the chapter. These prompts can be used for discussions, assignments, or break-out rooms.

Children's Literature

Children's literature that is perfect for integrating into mathematics lessons is peppered throughout the book. For example, in Chapter 6 you will find a collection of children's books that can form the beginning of a culturally diverse mathematics literature library. Additionally, every chapter in Part II includes an updated selection of children's literature. This includes two new extensive listings of children's books at different reading levels on a variety of measurement (103 choices) and geometry topics (62 books).

New Activities

This edition has more than 40 new activities and many new and revised blackline masters.

Major Changes to Chapters

Every chapter in the eleventh edition has been revised to reflect the most current research, standards, and exemplars. This is evident in the approximately *360* new references in the eleventh edition! This represents our ongoing commitment to synthesize and present the most current evidence of effective mathematics teaching. First, we set out to make each chapter more accessible to all teachers. That means we cut some discussion, reorganized a number of chapters, and revised many figures. As we noted in the acknowledgements, we are grateful to reviewers and colleagues for their ideas in making these improvements. Here we share changes *not* already described earlier.

Chapter 1: Developing Confident and Competent Mathematics Learners

The introduction to teaching developmentally has dramatically shifted to emphasize the purpose and focus of teaching mathematics well. Added to this chapter is a strengths-based approach (readers identify their strengths too!), the addition of ideas from the *Catalyzing Change: Initiating Critical Conversations* series, and myths in teaching mathematics.

Chapter 2: Exploring What It Means to Know and Do Mathematics

This chapter is now sequenced like the title; the chapter begins with a focus on mathematical proficiency, now addressing each of the five strands. Then readers have the opportunity to explore three tasks. (The probability task has been moved to the probability chapter.) And this chapter is 1000 words shorter, providing the reader more time to really do the math!

Chapter 3: Teaching Problem-Based Mathematics

Although the teaching practices were an important part of Chapter 3 in the 10th edition, they now provide the organizational structure for this completely revamped chapter, based on the *Mathematics Teaching Framework* (Huinker & Bill, 2017). And there is a new section describing learning progressions, because that was a priority to include in all Part II chapters. Much of the discussions have been updated, and a new table lists all of the problem-solving strategies.

Chapter 4: Planning in the Problem-Based Classroom

This much leaner chapter (3000 fewer words!) has revised, concise sections on families and differentiation. The family section includes a family newsletter, bookmarks to support homework, and ideas for developing a strong relationship with families. The section on differentiation zooms in on tasks and how to make them accessible and challenging to a range of learners. The elements within the before-during-after lesson model were revised.

What grew, in this chapter, was the collection of instructional routines, which now includes Three Reads, Number Strings, Splat!, WODB, and Estimation 180.

Chapter 5: Creating Assessments *for* Learning

Newly revised to reflect the most effective assessment approaches, Chapter 5 has a focus on peer and self-assessments, including a new self-assessment tool to capture students' dispositions toward math. A mathematical practices rubric is also shared as are many new figures and updated resources.

Chapter 6: Teaching Mathematics Equitably to All Students

Chapter 6 is completely redesigned, with a new beginning that connects the Mathematics Teaching Framework (Huinker & Bill, 2017) to equitable mathematics teaching, attending to three domains. Included in this new discussion is attention to developing mathematics identity, developing agency, avoiding implicit bias, and recognizing structural barriers. Each section on specific groups of learners (e.g., students with special needs in math, multilingual learners, etc.) was revised significantly, based on major work that has emerged in the past four years (more than 50 new references). Also, there is a new table providing a list of wonderful culturally diverse children's literature.

Chapter 7: Developing Early Number Concepts and Number Sense

This important first chapter in Part II begins with a look at mathematics content for the youngest learners by including a new learning progression for early numeracy. Added too are discussions of both perceptual and conceptual subitizing and more explicit conversations about linking counting on and counting back to addition and subtraction. There are also more virtual manipulatives and other web-based materials to respond to shifts to online learning options.

Chapter 8: Developing Meaning for the Operations

Every student can develop an understanding of each operation when addition, subtraction, multiplication, and division are learned through a concrete, semi-concrete, abstract (CSA) approach. This chapter has been reorganized to emphasize the importance of CSA. New features in this chapter include a list of attributes of effective contexts, a table of interpretations for remainders, new activities, and more children's literature.

Chapter 9: Developing Basic Fact Fluency

This chapter has a lot of changes. There are two new learning progressions that focus on fact sets and strategies, along with expanded attention to assessment, addressing the problem with timed tests and providing more guidance on effective assessment tools. Enjoy the many new games and activities that are there not for the fun of them (though they are fun), but to provide better ways for students to learn their facts meaningfully. There are also several new figures and tables.

Chapter 10: Developing Whole-Number Place-Value Concepts

There is a major reorganization of the chapter into progressions, including two from Roger Howe on the five stages of place-value notation and the historical evolution of notation. There is also a structuring of the content to reflect the CSA representations. One representation has been enhanced through the addition of the bottom-up hundred chart! Notice the addition of new children's literature for possible integration.

Chapter 11: Developing Strategies for Addition and Subtraction Computation

In this chapter, we added a progression on general computational strategy development that aligns with other computation discussed throughout the book. Importantly, we changed the language from invented strategies to reasoning strategies to reflect current literature in the field. Logically, we also increased attention to teaching reasoning strategies. Many figures were added or reconstructed, and the organization is based again on CSA.

Chapter 12: Developing Strategies for Multiplication and Division Computation

Chapter 12 is reorganized around CSA and focused on strategy use. As such, there are two new tables for reasoning strategy progressions, one for multiplication and one for division. There is also a new table that shows the four steps in the process of using an open array with partial products. The computational estimation section includes a revised table and new activities.

Chapter 13: Algebraic Thinking

Beyond the title change, this chapter has much new content as well as a redesign. Overall, this increases the focus on algebraic thinking in elementary school, building the strong foundation for algebraic thinking and mathematical modeling across K–8. There is an increased focus on generalizations, in particular an expanded section on the properties of the operations.

Chapter 14: Developing Fraction Concepts

New additions to this chapter include an expanded section on equal-sharing that connects to the related learning progression, an increased use of fraction strips and number lines throughout, and more attention to the unit. The "Fractions as Numbers" section was also adapted to be more concise.

Chapter 15: Developing Fraction Operations

Just as Chapters 11 and 12 increased attention to reasoning strategies, so has this chapter on operations with fractions. Furthermore, there is an expanded focus on estimating sums/differences (based on research findings that this improves student success). This chapter also has many new figures to support understanding.

Chapter 16: Developing Decimal and Percent Concepts and Decimal Computation

It is sometimes challenging to come up with realistic contexts for tenths, hundredths, and thousandths, so we've added a table that shares real-world options. Similar to the other chapters on computation, we are focusing on reasoning strategies for addition and subtraction of decimals through a new table. There is also a new progression for percent problem types.

Chapter 17: Ratios and Proportional Reasoning

This chapter had several major changes. First, we increased attention to social justice and real-life contexts throughout the chapter (6 new activities in this chapter alone). Additionally, the proportional reasoning representations and strategies were completely revised. The chapter was reorganized to focus on comparing ratios early in the chapter and to emphasize making connections to other mathematics at the end of the chapter.

Chapter 18: Developing Measurement Concepts

Because it spans K–8, measurement is a large chapter filled with new figures and activities. To begin, you will find a new progression for the measurement process and, later in the chapter, a new progression for learning about money. As in several other chapters, we have added new online manipulatives as options for instruction.

Chapter 19: Developing Geometric Thinking and Geometric Concepts

Similar to Chapter 18, this chapter covers grades K–8 and includes a new table with the van Hieles' Levels of Geometric Thought. The van Hiele progression is one of the most important progressions in mathematics, and we've added research about the importance of teachers studying these levels. Not surprisingly, the chapter is reorganized to reflect the van Hiele levels. A new table shares sites for virtual geometry tools so they can be integrated with online learning if needed. In response to a reviewer's suggestion, there is increased attention to the Pythagorean theorem.

Chapter 20: Developing Concepts of Data and Statistics

This leaner chapter has numerous changes throughout the chapter to reflect the new *GAISE II* recommendations. In addition, there is an increased focus on culturally responsive mathematics instruction and many new and updated figures.

Chapter 21: Exploring Concepts of Probability

One of the significant changes in this chapter was connecting qualitative and quantitative probability descriptions to support student understanding. The probability ladder is new and provides a strong visual to support this connection and explore the probability continuum. There is also more attention to selecting models (tree diagrams or area models) to determine probabilities.

Chapter 22: Exploring Exponents, Negative Numbers, and Real Numbers

This chapter has an expanded, clearer discussion of the challenges of using the mnemonic PEMDAS. Adding and subtracting negative numbers has been revised in various ways, including in a hot-air balloon context. Like other chapters focusing on the operations, this chapter also turns more attention to fluency and flexibility with rational numbers.

An Introduction to *Teaching Developmentally*

If you look at the table of contents, you will see that the chapters are separated into two distinct sections. Part I (Chapters 1–6) addresses important ideas about mathematics teaching that apply across concepts. Among the ideas in Part I, we focus mathematics learning on students' strengths, knowing what it means to do mathematics, determining effective ways to get your students *actively engaged* through intentional planning and implementing effective teaching and assessing strategies. Throughout Part I and especially in Chapter 6, you learn how to ensure that each and every student develops mathematical proficiency and a positive mathematics identity.

Part II (Chapters 7–22) offers specific guidance for teaching every major mathematics topic in the pre-K–8 curriculum. Each chapter begins with big ideas for that content, along with learning outcomes. Each section provides guidance on how to teach that content through and illustrative examples through many figures and problem-based activities. Each of these chapters concludes with a list of challenges (and ways to support students with those challenges). The support found in each chapter can guide your planning and assessing as you work to ensure that every student develops competence and confidence in mathematics.

Hundreds of tasks and activities are embedded in the text. Take out a pencil and paper, or use technology, and try the problems, thinking about how you might solve them *and* how students at the intended grades might solve them. This is one way to engage actively in *your learning* about how *students learn* mathematics. In so doing, this book will increase your own understanding of mathematics, the students you teach, and how to teach them effectively.

Pedagogical Features

By flipping through the book, you will notice many section headings, a large number of figures, and various special features. All are designed to support your teaching, now and into the future. Here are a few features to enhance your reading and support your teaching.

Learning Outcomes

To help readers know what they should expect to learn, each chapter begins with learning outcomes. Numbered self-checks are interspersed along the way to align with each learning outcome.

Developing Fraction Concepts

CHAPTER 14

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

14.1 Describe and give examples for fraction constructs and fraction representations.

14.2 Describe three types of fraction representations, and what situations and manipulatives fit within each.

14.3 Explain foundational concepts of fractional parts, including equal sharing, iterating, and partitioning.

14.4 Illustrate the concept of equivalence across fraction models.

14.5 Describe strategies for comparing fractions and ways to teach this topic conceptually.

Big Ideas

Much of the research and literature on effective teaching of mathematics suggests that teachers plan their instruction around big ideas rather than on isolated skills or concepts. At the beginning of each chapter in Part II, you will find a list of the big mathematical ideas associated with the chapter. Teachers find these lists helpful to envision quickly the mathematics they are to teach.

BIG IDEAS

- Fractions can and should be represented across different interpretations (e.g., part-whole and division) and different representations: area (e.g., $\frac{1}{3}$ of a garden), length (e.g., $\frac{3}{4}$ of an inch), and set (e.g., $\frac{1}{2}$ of the marbles).
- The number line is a critical representation in developing fraction concepts.
- Fractions are equal shares of a whole or a unit. Therefore, equal sharing activities (e.g., 2 sandwiches shared with 4 friends) build on whole-number knowledge to introduce fractional quantities.
- Partitioning and iterating are strategies students can use to understand the meaning of fractions. Partitioning can be thought of as splitting the whole equally (e.g., splitting a whole into fourths), and iterating can be thought of as making a copy of each piece and counting them all (e.g., one-fourth, two-fourths, etc.).

Activities

The numerous activities found in every chapter of Part II have always been rated by readers as one of the most valuable parts of the book. Some activity ideas are described directly in the text and in the illustrations. Others are presented in the numbered activity boxes. Every activity is a problem-based task (as described in Chapter 3) and is designed to engage students in *doing* mathematics.

Activity 21.9 Fair or Unfair Coin Games

CCSS-M: 7.SP.C.6; 7.SP.C.7a

For each of these games, ask students first to predict whether they believe the game is fair and then play the game, considering whether the game is fair or unfair (and why). You can play these games within one lesson or on different days.

Coin Game 1: Three students form a group and are given two like coins (e.g., two pennies). For each flip of both coins, one player gets a point, based on the following rules:

Player A: Two heads; Player B: Two tails; Player C: One of each

The game is over after 20 tosses. The player who has the most points wins. Have students play the game two or three times.

Coin Game 2: Partners compete in this game, flipping two like coins (e.g., two pennies). For each flip of both coins, one player gets a point, based on the following rules:

Player A: Same face (e.g., both heads); Player B: One of each

After playing the game, ask students to create data displays of their group's data. Pool the class data and examine their results. Two-coin experiments lend themselves to many representations (e.g., tables, lists, various versions of tree diagrams), and these representations help students understand theoretical probability (English & Watson, 2016). When the full class has played the game several times, conduct a discussion on the fairness of each game. Challenge students to make an argument based on the data and game rules as to whether the game is fair. For multilingual learners, discuss the meaning of *fair* prior to beginning the game and review the term when asking students to create an argument. As another challenge, have students design their own fair game.

Adaptations to Challenge and Support Students

Many Part II Activities have one or more icons (challenging students, supporting multilingual learners, supporting students with special needs in math). When you see these icons, it means that suggestions are embedded in the activity to differentiate the task so that it is accessible and meaningful to every student.

Activity 9.2 How Many Feet in the Bed?

CCSS-M: 1.OA.A.1; 1.OA.C.6; 2.OA.B.2

Read *How Many Feet in the Bed?* by Diane Johnston Hamm (1991). On the second time reading the book, ask students how many more feet are in the bed each time a new person gets in. Ask students to record the equation (e.g., 6 + 2) and tell how many feet in total. Two less can be considered as family members get out of the bed. Find opportunities to make the connection between counting on and adding using a number line. For multilingual learners, be sure that they know what the phrases "two more" and "two less" mean (and clarify the meaning of *foot,* which is also used for measuring). Acting out this situation with students in the classroom can be a great illustration for both multilingual learners and students with special needs in math.

Formative Assessment Notes

Assessment is an integral process within instruction. Similarly, it makes sense to think about what to be listening for (assessing) as you read about different areas of content development. Throughout the Part II chapters, there are formative assessment notes with brief descriptions of ways to assess a topic in that section. Reading these assessment notes as you read the text can help you understand how best to assist students to grow in their mathematics understanding.

FORMATIVE ASSESSMENT Notes. Any of the preceding problems can be used as a formative assessment with an observation look-for tool. On the look-for tool would be such concepts as: (1) determines a reasonable estimate; (2) models the problem with manipulative or picture; (3) solves using an efficient method; (4) writes an accurate equation. Students may be able to illustrate but then not find fraction equivalences to solve the problem; or they may represent and solve but struggle to write an equation. A look-for tool provides guidance for what future instruction is needed. ■

Technology Notes

Like assessment, technology is infused into instruction. Thus, also in Part II, we include technology notes (updated significantly in this edition!). Descriptions include open-source (free) software, apps, and other Web-based resources.

***TECHNOLOGY* Note.** Dreambox (www.dreambox.com) Teacher Tools offers a bar (length) model for exploring addition and subtraction. Students solve problems visually, trading out equivalent fractions and recording their answers. Other sites offer various virtual manipulatives by which a student can build a problem and solve it. Those sites include Toy Theater Teacher Tools (toytheatre.com), Math Learning Center free apps (mathlearningcenter.org), and Brainingcamp (brainingcamp.com). On each site, you or the student can select the manipulative you will use to represent the problems. ■

Standards for Mathematical Practice Margin Notes

The Standards for Mathematical Practice continue to be an important component of many state standards. They describe what it means to do math! There are eight of them, as described in Chapter 1. In Part II, we use margin notes to showcase where an instructional suggestion connects to a particular mathematical practice.

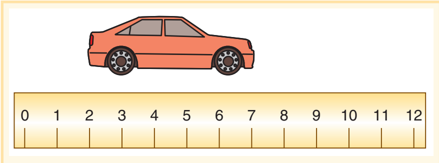
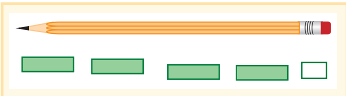
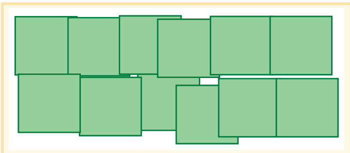
Standards for Mathematical Practice

MP8. Look for and express regularity in repeated reasoning

Tables of Common Challenges

Found in every chapter of Part II are tables of common challenges that students might experience related to each topic. Each challenge has been identified in the research literature. Notice that suggestions for every case list how to support students as they navigate through these potentially difficult areas.

TABLE 18.2 COMMON CHALLENGES IN MEASUREMENT AND HOW TO HELP

Common Challenge	What It Looks Like	How to Help
1. Focus on the ending point of a ruler rather than the length.	Students read the number on the ruler that aligns with the object's right edge. 	• Move items along a ruler and ask students if they believe the item has changed size. • Iterate units to show how long the item is and then show how those units match the units on the ruler. • Give students experiences with the Broken Ruler.
2. Place multiple individual units with gaps between the units or with some of the units overlapping; or students use unequal units.	Students have gaps between units or use unequal units when measuring length:  Students overlap units when measuring area: 	• Ask students to estimate the measure before measuring the object and explain why it is a reasonable estimate. • If students use unequal units, ask what the number refers to (e.g., the number of white rods or brown rods). • If students overlap length units, have them use linking cubes to measure length. • If students overlap area units, use a unit with some thickness, making it difficult to overlap the pieces (e.g., square tiles or pattern blocks).

End-of-Chapter Resources

At the end of each chapter are *Resources*, which include "Writing to Learn" assessments, "Literature Connections" (found in all Part II chapters), and "Recommended Readings."

Writing to Learn. These exercises provide teacher candidates or teachers with a variety of opportunities to articulate, in their own words, the concepts they are learning. Prompts and questions have been designed to emphasize analysis and comprehension.

Literature Connections. Here you will find examples of children's literature for launching into the mathematics concepts in the chapter just read. For each title suggested, there is a brief description of how the mathematics concepts in the chapter can be connected to the story.

Recommended Readings. In this section, you will find an annotated list of articles and books to augment the information found in the chapter. These recommendations include NCTM articles and books and other professional resources designed for the classroom teacher.

RESOURCES FOR CHAPTER 14

WRITING TO LEARN

1. Describe the benefits of using equal sharing tasks.
2. Give examples of representations and contexts that fall into each of the three categories of fraction models (area, length, and set).
3. What does partitioning mean? Explain and illustrate.
4. What does iterating mean? Explain and illustrate.
5. What are two ways to build the conceptual relationship between $\frac{11}{4}$ and $2\frac{3}{4}$?
6. Explain why the number line is so important to developing fraction concepts.
7. Describe two ways to compare $\frac{5}{12}$ and $\frac{5}{8}$ mentally.

LITERATURE CONNECTIONS

Apple Fractions

Pallotta (2002)

Interesting facts about apples are connected to fractions as fair shares. Words for fractions are used and connected to fraction symbols, making it a good connection for fractions in grades 1–3.

Halfsy: A Math Adventure

Deitelbaum (2020)

Halfsy is on a journey from 0 to 1 (a length context for fractions), and on the way notices half of other things: the moon (area), birds (quantity).

The Doorbell Rang

Hutchins (1986)

The story is about sharing cookies, a changing situation as more and more children arrive. A great connection to sharing tasks.

The Lion's Share

McElligott (2009)

The story involves halving a rectangular cake over and over again, and provides a visual opportunity to explore unit fractions.

The Man Who Counted: A Collection of Mathematical Adventures

Tahan (1993)

This book contains a story, "Beasts of Burden," about a wise mathematician, Beremiz, and the narrator, who are traveling together on one camel. They are asked to figure out how to give each brother his share of 35 camels: *one-half to one*, *one-third to one*, *and one-ninth to one* (a set context). See Bresser (1995) for three days of activities with this book.

RECOMMENDED READINGS

Articles

Clarke, D. M., Roche, A., & Mitchell, A. (2008). Ten practical tips for making fractions come alive and make sense. *Mathematics Teaching in the Middle School*, *13*(7), 373–380.

Ten excellent tips for teaching fractions are discussed and favorite activities are shared. An excellent overview of teaching fractions.

Monson, D., Cramer, K., & Ahrendt, S. (2020). Using models to build fraction understanding. *Mathematics Teacher: Learning & Teaching PreK-12*, *113*(2), 117–123.

This article provides excellent tasks and visuals to develop a strong understanding of fractions, including fractions on the number line.

Freeman, D. W., & Jorgensen, T. A. (2015). Moving beyond brownies and pizzas. *Teaching Children Mathematics*, *21*(7), 412–420.

This article describes students' thinking as they compare fractions. In the more4U pages, they offer excellent sets of tasks with a range of contexts, each set focusing on a different reasoning strategy for comparing fractions.

Books

Empson, S. B., & Levi, L. (2011). *Extending children's mathematics: Fractions and decimals.* Portsmouth, NH: Heinemann.

With many tasks and examples of student work, this book is a must-have for supporting students' emerging concepts of fractions and decimals.

Instructor Resources

Instructor's Resource Manual

The Instructor's Resource Manual for the eleventh edition includes a wealth of resources designed to help instructors teach the course, including chapter notes, activity suggestions, and suggested assessment questions.

PowerPoint® Slides

Ideal for instructors to use for lecture presentations or student handouts, the PowerPoint presentation slides provide ready-to-use graphics and text images tied to the individual chapters and content development of the text.

Note: All instructor resources are available for download at www.pearson.com/en-gb.html. Use one of the following methods:

- From the main page, use the search function to look up either the lead author (Van de Walle) or the title (*Elementary and Middle School Mathematics: Teaching Developmentally*). Select the desired search result and then click the Resources tab to view and download all available resources.
- From the main page, use the search function to look up the ISBN (provided earlier) of the specific instructor resource you would like to download. When the product page loads, click the Downloadable Resources tab.

Acknowledgments

With each new edition, more amazing people become part of this book. We always begin by acknowledging those experts who supported the very first edition: Warren Crown, John Dossey, Bob Gilbert, and Steven Willoughby, who gave time and great care in offering suggestions to John Van de Walle. As new editions emerged, so many voices have been heard, from reviewers, to students, to colleagues. We are so grateful!

In preparing this eleventh edition, we received incredibly thoughtful reviews that led us to really revamp Part 1 of the book as well as thoroughly improve the focus within Part II chapters. Those reviewers are Megan Burton, Auburn University; Kelly Byrd, University of South Alabama; Karen Colum, Minnesota State University, Mankato; Jose N. Contreras, Ball State University; Shea Culpepper, University of Houston; Ella-Mae P. Daniel, Florida State University; Melissa Geiselhof, Arizona State University; Susan Gregson, University of Cincinnati; Maria Gross, Azusa Pacific University; Betti C. Kreye, Virginia Tech; Jemma Kwon, California State University, Sacramento; Gregory T. Miyata, California State University, Los Angeles; LeAnn Neel-Romine, Ball State University; Andrew Polly, University of North Carolina at Charlotte; Iris M. Riggs, California State University, San Bernardino; Dorothy L. Shapland, Metropolitan State University of Denver; Sarah Smitherman Pratt, University of North Texas; and Ian Whitacre, Florida State University. Thank you so very much!

Additionally, we are very grateful for the ideas we receive via email. You can email one of us directly or use the book email: teachingdevelopmentally@gmail.com.

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From Karen Karp: I express thanks to my husband, Bob Ronau, who as a mathematics educator graciously helped me think about numerous decisions while offering insights and encouragement. He also used his technology skills to prepare many of the figures and activity pages. In addition, I thank my children, Matthew, Tammy, Joshua, Matt, Christine, Jeffrey, and Pamela for their kind support and inspiration. I also am indebted to my wonderful grandchildren, Jessica, Zane, Madeline, Jack, Emma, and Owen, who consistently deepen my understanding about how learners think and grow mathematically.

From Jennifer Bay-Williams: I am forever thankful and indebted to the many classroom teachers, math teacher leaders, and teacher educators with whom I have worked. It is through our collaborations, teaching opportunities, and study that I learn and grow. My spouse, Mitch Williams, continues to do all he can to allow me to do what I need to do, and is an amazing sounding board. I also want to acknowledge how much I learn from and with my children, MacKenna (18 years) and Nicolas (16 years). This book was revised during their Covid school year, so I benefitted greatly from our nearly 24/7 time together.

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PART I Teaching Mathematics Developmentally: Big Ideas and Research-Based Practices

The fundamental core of effective teaching of mathematics combines an understanding of how students learn, how to promote that learning by teaching through problem solving, and how to plan for and assess that learning daily. That is the focus of these first six chapters, providing discussion, examples, and activities that develop the core ideas of learning, teaching, planning, and assessment for each and every student.

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Developing Confident and Competent Mathematics Learners

CHAPTER 1

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

1.1 Summarize the factors that influence the teaching of mathematics.

1.2 Describe the importance of content standards, process standards, and standards of mathematical practice.

1.3 Explore the qualities needed to learn and grow as a professional teacher of mathematics.

Did you know that the most important factor in a student's learning of mathematics is their teacher? You can ensure that every student becomes confident and competent in mathematics. You likely have many questions about how to make this vision a reality: What direction is there for what mathematics you will teach? What guidance is there for how you should teach? What can you achieve in your teaching to help students feel like they are good at math? How will you know whether students are learning? These questions and many more are addressed in this book. These three things—your knowledge of mathematics, how to teach mathematics, and how to focus on each learner—are the most important tools you can acquire to be a successful teacher of mathematics. We are here to help you hone those skills!

In this chapter, we begin this journey by addressing the big picture of teaching mathematics in K–8. We begin with some big ideas that are important about teaching mathematics developmentally, briefly describe the most significant influences on mathematics learning, and, finally, offer suggestions to you as a *teacher* of mathematics.

Learning How to Teach Mathematics Developmentally

As part of your desire to build successful learners of mathematics, you might recognize the challenge that mathematics is sometimes seen as the subject people love to hate. At social events—even at parent–teacher conferences—other adults may hear that you are a teacher of mathematics and share comments such as, "I could never do math," or, "I can't calculate the tip at a restaurant—I just hope they have suggestions for tips at the bottom of my receipt."

Instead of ignoring these disclosures, consider what positive action you can take. Would people confide that they don't read and hadn't read a book in years? Not likely. Families' and teachers' attitudes toward mathematics may enhance or detract from students' ability to think they can do math. Teaching mathematics developmentally can transform this positioning of mathematics into a positive experience for all students. Here we share some global ways to address misunderstandings and negative feelings about mathematics.

Math Is Important

Learning mathematics is essential to being productive at work and in our communities. It also facilitates curiosity in science and technology and enhances personal activities—not to mention that it's fun! While all students have many reasons to learn mathematics, the National Council of Teachers of Mathematics (NCTM) *Catalyzing Change* series (2018-2020) identifies several key purposes for early childhood and elementary and middle school students' learning mathematics. They are:

- Develop deep mathematical understanding as confident and capable learners.
- Understand and critique the world through mathematics.
- Experience the wonder, joy, and beauty of mathematics (p. 11 and p. 7).

These purposes of learning mathematics express what underlies the most effective ways to teach mathematics K–8. In alignment, the goal of this book is to help you learn these effective instructional methods that are often unique to mathematics. We base this book on the collective wisdom of an organization of mathematics teacher educators and mathematicians from across the United States who developed a set of standards that established what knowledge, skills, and dispositions are important in cultivating a well-prepared beginning teacher of mathematics, the *Standards for Preparing Teachers of Mathematics* (AMTE, 2020). As you dig into the information in the chapters ahead, your vision of what is possible for all your students, and your confidence to explore and teach mathematics, will grow in ways endorsed by experts in the field.

Ultimately, your students need to think of themselves as mathematicians in the same way as they think of themselves as readers. As students interact with our increasingly mathematical and technological world, they need to construct, modify, communicate, or integrate new mathematics-related information in many forms. Solving novel problems and approaching new situations with a mathematical perspective should come as naturally as using reading to comprehend facts, insights, or the latest news. Particularly because this century is a quantitative one (Hacker, 2016), we must prepare students to interpret the language and power of numeracy. Hacker states that "decimals and ratios are now as crucial as nouns and verbs" (p. 2). So, for your future students, consider how important mathematics is to interpreting and successfully surviving in our complex economy and changing environment and find ways to guide them toward these goals. Learning mathematics opens up a world of important ideas.

Mathematics Is about Effort, Not Genetics and Other Myths

Video Example 1.1
In this video, late author John Van de Walle provides his definition of mathematics. https://youtu.be/0gW9g8Ofi8A

Mathematics ability is not inherited—anyone can learn mathematics. Some of the previously mentioned quotations hint at a false assumption prevalent in the United States: that mathematics ability is hereditary. This just isn't true. What leads a person to be successful is their perseverance and support, both in and out of the classroom. In fact, engaging students in productive struggle is one of most effective ways to support their understanding (Baker, Jessup, Jacobs, Empson & Case, 2020). This speaks to the need to do a better job in schools to ensure that every child has access to a strong mathematics program!

The second myth that stands in the way of learning is that learning mathematics is about speed and memorizing facts and procedures. Have you observed a mathematician at work? They grapple with problems for hours, even years. Yes, they know facts and procedures to support their thinking, but the heart of mathematics is mathematical thinking, not being fast. Further, mathematics is not a bunch of isolated skills; it is a collective, interconnected whole.

We must help students see these connections to make sense of mathematics. There are also ways we were taught that didn't help us make such connections, such as talking about borrowing and carrying numbers, or that multiplication "makes numbers bigger," which is not so when you consider fractions, decimals—or zero, for that matter (Karp, Bush & Dougherty, 2014).

A third myth that interferes with learning mathematics is that people believe there is just one way to solve a problem. That is almost never the case. Even for a problem like $29 + 33$, there are numerous ways to solve it. Many adults use the standard algorithm, because that is the way they were taught. Algorithms focus on performing an operation on one place value at a time, moving from ones to tens in this example, and the standard algorithm is not necessarily the best way. (Can you think of an alternative that is more efficient?) Maybe even doing this problem in your head?

We begin thinking about teaching developmentally by tackling the first myth. You must believe that all children can learn mathematics, and that happens when you focus on their strengths!

Strengths-Based Approach

Part of becoming an effective teacher of mathematics is recognizing mathematical strengths—in students *and* in yourself. When you exude confidence and a can-do, problem-solving approach, you play an important role in developing the same qualities in your students. We know that when there is a focus on what a student doesn't do well—such as "you didn't pass that fractions test" or "I think you need more practice with your multiplication facts"—it affects a learner's confidence negatively. As Skinner, Louie, and Baldinger suggest, some students "are conditioned to see themselves as the dust of the mathematical universe and see others as the stars" (2019, p. 339). Rather than focus on deficits, effective mathematics teaching shifts attention to strengths (Kobett & Karp, 2020). All students need to know they are brilliant and capable of learning mathematical concepts and procedures. When you acknowledge their strengths, you can watch them flourish before your eyes.

This is true for teaching as well. Consider how you might react if you were told that "that math lesson you taught today was just not that effective." When you connect to your own humanity and explore your own mathematics identity (Aguirre, Mayfield-Ingram, & Martin 2013), the bond between you and your students grows in depth and breadth. For example, you can use what you know from your own experiences to add your empathy and compassion to new learners' initial experiences with mathematical topics. Here are two examples:

Rather than say, "Oh, this problem is easy," one of your strengths might be remembering how you heard those words as a learner and it made you feel uneasy. Instead, say, "Let's try this challenge together and explore some new ideas."

From your own past experiences, you know you have the strength of using multiple approaches to try to solve a problem. When you see a student who is stumped, you say, "Here's something I found helpful when I was stuck on a problem." Then you walk students down the path of possible solution strategies, including using manipulative materials, making drawings, creating tables of numbers, or even acting out a problem to find a successful solution strategy.

It's time to identify your strengths and acknowledge that you have the knowledge and resources to build and celebrate your students' superpowers (Kobett & Karp, 2020, p. 17.) We will help you in that process, starting right now.

Teaching Developmentally

With your strengths and your students' strengths in mind, you are ready to focus on the way in which mathematics is taught. The aforementioned second myth matters. Mathematics is connected, and that means new knowledge builds on students' prior knowledge. For example, if you are working on sums like $29 + 34$, you can explore what strategies students use to add $9 + 4$. A student might use a Make 10 Strategy, rethinking the problem as $10 + 3$. You explicitly connect their understanding by asking questions like, "Can you use the Make 10 Strategy for $29 + 34$? Try it!" One part of teaching developmentally is to be sure you are purposeful in connecting the topic you are teaching now to related topics students have already learned.

So, let's find ways of countering negative statements about mathematics, especially if they are declared in the presence of students. Point out that, actually, it is a myth that only some people can be successful at learning mathematics. We all can! Only in that way can the chain of passing apprehension from adult to child be broken. There is much joy in solving mathematical problems, and it is essential that you demonstrate an excitement for learning and nurture a passion for mathematics in your students.

What Influences the Mathematics We Teach?

Learning Outcome 1.1 **Summarize the factors that influence the teaching of mathematics.**

How you are teaching mathematics should not look anything like the teaching of mathematics 50 years ago or even 20 years ago. Many factors influence these changes. Here we briefly discuss four major reasons.

The World Is Changing

In *The World Is Flat* (2007), Thomas Friedman discusses how globalization has created the need for people to have skills that are long-lasting and will survive the ever-changing landscape of available jobs. Friedman emphasizes that in a world that is digitized and surrounded by algorithms, math lovers will always have career opportunities and choices. Yet, there is a skills gap of qualified people; reports reveal that the United States will need to fill 3.5 million science, technology, engineering, and mathematics (STEM) jobs by 2025 (Radu, 2018).

Every teacher of mathematics has the job of preparing students with career skills while developing a love of math in students. The mathematician Stefan Banach stated, "Mathematics is the most beautiful and most powerful creation of the human spirit" (as cited in Kaluźa, 1996, n.p.). So, as you can see, there is an array of compelling reasons why children will benefit from the study of mathematics, using the approaches you will learn in this book. Help your students acquire the mental tools to make sense of mathematics—in some cases for mathematical applications that might not yet be known! This knowledge serves as a lens for interpreting the world.

Our changing world influences what should be taught in pre-K–8 mathematics classrooms, because there is a relationship between early mathematics performance and success in middle school (Bailey, Siegler, & Geary, 2014) and high school mathematics (Watts, Duncan, Siegler, & Davis-Kean, 2014), with third-grade mathematics scores pinpointed as a strong predictor of high school AP pass rates (Clark, Yu, Yi & Shi, 2018). As we prepare pre-K–8 students for jobs that possibly do not currently exist, we can predict that there will be few jobs in which just knowing simple computation is enough to be successful. We can also predict that many jobs and much of life will require interpreting complex data, designing algorithms to make predictions, and using multiple strategies to approach new problems.

As you prepare to help students learn mathematics for the future, you will need some perspective on the forces that effect change in the mathematics classroom. This chapter addresses the leadership that you, the teacher, will develop as you shape mathematics experiences for your students. Your beliefs about what it means to know and do mathematics and about how students make sense of mathematics will affect how you approach instruction and the understandings and skills your students take from the classroom. The enthusiasm you demonstrate about mathematical ideas will transform your students' interest in this amazing and beautiful discipline.

International Comparisons

One factor shaping what mathematics education looks like is the public or political pressure for change, due largely to student performance results in national and international studies. These large-scale comparisons of student performance continue to make headlines, provoke public opinion, and pressure legislatures to call for tougher standards backed by testing. These assessments provide strong evidence that mathematics teaching *must* change if U.S. students are to be competitive in the global market and understand the complex issues they will confront as responsible citizens of the world (Green, 2014).

National Assessment of Education Progress (NAEP). Since the 1960s, the United States has used the NAEP test to gather data on how fourth-, eighth-, and twelfth-grade students are doing in mathematics in what is known as the Nation's Report Card (https://nces.ed.gov/nationsreportcard). These data provide a tool for policy makers and educators to measure students' overall improvement over time, using four achievement levels: below basic, basic, proficient, and advanced, with proficient and advanced representing the desired grade-level achievement. In the most recent NAEP mathematics assessment in 2019, 41 percent of all U.S. students in grade 4 and 34 percent of students in grade 8 performed at the desirable levels of proficient and advanced (National Center for Education Statistics, 2019). Despite encouraging gains over the past 30 years, due to shifts in instructional practices (particularly at the elementary level) (Kloosterman, Rutledge, & Kenney, 2009a), students' performance in 2019 still shows room for improvement. We have work to do!

Trends in International Mathematics and Science Study (TIMSS). In 2019, 64 nations, at fourth grade and 46 at eighth grade, participated in TIMSS (TIMSS, 2019), the largest international comparative study of students' mathematics and science achievement—given seven times since it started in 1995. Data were gathered from a randomly selected group of 580,000 students, with approximately 20,000 from the United States. American students performed above the international average but were outperformed at fourth grade by 14 countries, headed by Singapore and including other countries such as Chinese Taipei, Japan, the Russian Federation, Northern Ireland, and Norway, and at eighth grade by 15 countries, also headed by Singapore and including the Republic of Korea, Hong Kong, Ireland, Israel, and Hungary. These data provide valuable benchmarks that allow the United States to reflect on our teaching practices and our overall competitiveness in preparing students for a global economy. If you've heard people talk about how mathematics is taught in Singapore—these rankings are why. We learn a common theme from these examples: These high-performing nations focus on teaching that emphasizes conceptual understanding and procedural fluency, both of which are critically important to the long-term growth of problem-solving skills (OECD, 2016; Rittle-Johnson, Schneider, & Star, 2015). In fact, teaching in these high-performing countries more closely resembles the long-standing recommendations of the National Council of Teachers of Mathematics, the major professional organization for mathematics teachers, discussed in the following section.

Research on Learning and Teaching

Over the years, there have been significant reforms in mathematics education that reflect research on how students learn mathematics. Just as we would not expect doctors to be using exactly the same techniques and medicines that were prevalent when you were a child, teachers' methods are evolving and transforming via a powerful collection of expert knowledge about how the mind functions and how to design effective instruction (Wiggins, 2013). The field of mathematics education actually knows a lot about how students learn (e.g., Cai, 2017; English & Kirshner, 2015; NRC, 2001). Earlier, we shared some of those ideas, and each chapter reflects the most current research in the field (hence all the citations!)

National Council of Teachers of Mathematics (NCTM). One transformative factor in the teaching of mathematics is the leadership of NCTM. The NCTM, with more than 60,000 members, is the world's largest organization of mathematics educators and, as such, holds an influential role in both supporting teachers and emphasizing what's best for learners. NCTM's guidance in the creation and dissemination of standards for curriculum, assessment, and teaching led the way for other disciplines to create standards and for the eventual development of the mathematics standards used in your state today. For example, *Principles to Actions* (NCTM, 2014b) identifies eight effective teaching practices. This book, along with the *Catalyzing Change* series (2019–2020), provides strong support to all teachers. The premiere research journal is housed within NCTM, and the teacher journals, including the *Mathematics Teacher: Learning and Teaching PK-12*, is full of high-quality teaching ideas. Visit the NCTM website at www.nctm.org.

How Do We Use the Standards to Teach?

Learning Outcome 1.2 **Describe the importance of content standards, process standards, and standards of mathematical practice.**

What you teach is based on standards, but where did the standards come from? What we know about mathematics is the result of many thinkers of all races, ethnicities, and gender identities, beginning well before modern times. The heritage of these powerful ideas is diverse, and the legacy of many giants comes before what your state or school chose as the standards for your classroom. For that reason, it is helpful to think about how we came to teach what we teach and why.

In 1989, NCTM published the very first standards document of any discipline: "Curriculum and Evaluation Standards for School Mathematics." In 2010, the National Governors Association (NGA) Center for Best Practices and the Council of Chief State School Officers (CCSSO) presented the *Common Core State Standards* (http://www.corestandards.org/math), which incorporated ideas from many documents from the NCTM, including *Principles and Standards for School Mathematics* (2000) and *Curriculum Focal Points* (2006), as well as international curriculum documents. A large majority of U.S. states adopted these standards, and other states used these standards for reference while creating their own, similar version. (Check your state's website.) At this time, approximately 42 states, Washington, D.C., four territories, and the Department of Defense Schools have adopted standards similar to the CCSS-M. The movement that NCTM started in 1989 transformed the country from having little to no coherent vision on what school mathematics should be taught and when to a widely shared understanding of what students should know and be able to do across the grades.

Mathematics Content Standards

Standards articulate the *critical*, *essential*, or *emphasized areas* of mathematics content for each grade to provide a coherent curriculum built around big mathematical ideas. These larger groups of related standards are commonly called *domains*. These domains range from Counting and Cardinality, to Number and Operations in Base Ten, to Measurement and Data. Topics are sequenced based on what is known from research and practice about learning progressions, which are sometimes referred to as *learning trajectories* (Clements & Sarama, 2021; Clements, Sarama, Baroody, & Joswick, 2020).

Deconstructing and Understanding Standards. One important task you need to think about is learning about the standards you will teach. The actual statements are not often as easily communicated as, "Third graders will learn perimeter." Instead, you may find something like, "Represent a fraction $\frac{1}{b}$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size $\frac{1}{b}$ and that the endpoint of the part based at 0 locates the number $\frac{1}{b}$ on the number line." You can reread that multiple times, but the best way to learn more about the standards incorporates what we know about brain-targeted teaching (Hardiman, 2012). First, consider drawing what each standard means as a way to delve into the topic (Strand & Bailey, 2020).

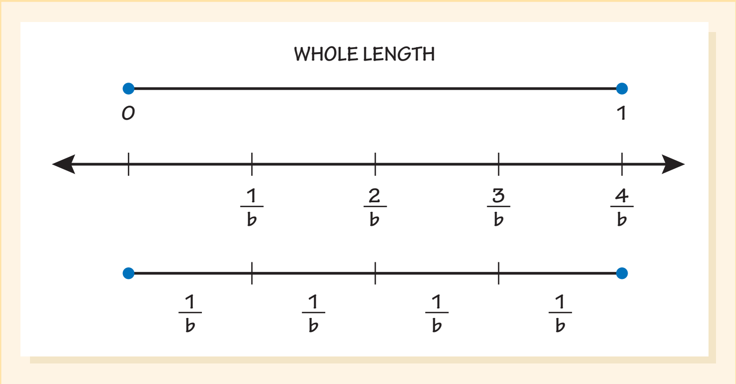


FIGURE 1.1 Decomposing a fraction standard in a drawing

Work with a colleague and divide and conquer as you unpack the ideas. See Figure 1.1 for a way to think about the preceding standard. As you draw, you will focus on the meaning and how you can link a group of standards through their connections and relationships.

Vertical Alignment of Standards. In addition to making sense of the standard at your grade, you need to know what students were taught the previous year (so you can assess whether they currently have that prior knowledge and then build on it), and you need to know where they are headed next year. Learning progressions help teachers know the sequence of what came before a particular concept as well as what to expect next as students reach key points along a pathway to desired learning targets (Corcoran, Mosher, & Rogat, 2009). Although these paths are not identical for all students, they can inform the order of instructional experiences that supports movement toward understanding and application of mathematics concepts. In other words, these progressions help teachers to teach mathematics developmentally! Bookmark the website for Achieve the Core (https://achievethecore.org/page/254/progressions-documents-for-the-common-core-state-standards-for-mathematics), where many helpful resources, including the Progression Documents, can be found. We also share numerous learning progressions in Part II!

Remember that standards set what students need to learn at a given grade but not how to teach that content. Instead, districts and schools make the final decisions about which curriculum to use or if they want to develop their own curriculum. Standards are not a curriculum. In fact, teachers are the essential piece in selecting and presenting the content standards into a coherent and engaging package.

Mathematical Practices

Beyond *what* mathematics is to be learned, standards describe the skills students need to have to do mathematics. For example, students need to be able to justify or explain their strategy. The vision for such actions was first described by NCTM as Process Standards (see, for example, *Principles and Standards for School* NCTM, 2000). The five process standards are: problem solving, reasoning and proof, communication, connections, and representations. Mathematical Practices were also captured in *Adding It Up* (National Research Council, 2001) (see Chapter 2 in this book).

The NCTM Process Standards (https://www.nctm.org/Standards-and-Positions/Principles-and-Standards/Process/) and the Strands of Mathematical Proficiency (https://www.nap.edu/read/9822/chapter/6) were blended into the *Standards for Mathematical Practice*. These are described in Table 1.1 and in Appendix A. Teachers must attend to these practices in the same way they attend to content, identifying particular practices to develop within lessons and units. These practices are blended with content. You will see that in Part II, we place Mathematical Practice notes in the margin to illustrate how these practices can emerge when you are engaging students in doing mathematics.

TABLE 1.1 THE STANDARDS FOR MATHEMATICAL PRACTICE

Mathematical Practice	K–8 Students Should Be Able To:
Make sense of problems and persevere in solving them.	• Explain what the problem is asking. • Describe possible approaches to a solution. • Consider similar problems to gain insights. • Use concrete objects or drawings to think about and solve problems. • Monitor and evaluate their progress and change strategies if needed. • Check their answers by using a different method. • Try again with another approach if one attempt is not successful or when they feel stuck.
Reason abstractly and quantitatively.	• Explain the relationship between quantities in problem situations. • Represent situations using symbols (e.g., writing expressions or equations). • Create representations that fit the word problem. • Use flexibly the different properties of operations and objects.
Construct viable arguments and critique the reasoning of others.	• Understand and use assumptions, definitions, and previous results to explain or justify solutions. • Make conjectures by building a logical set of statements. • Analyze situations and use examples and counterexamples. • Explain their thinking and justify conclusions in ways that are understandable to teachers and peers. • Compare two possible arguments for strengths and weaknesses to enhance the final argument.
Model with mathematics.	• Apply mathematics to solve problems in everyday life. • Make assumptions and approximations to simplify a problem. • Identify important quantities and use tools or representations to connect their relationships. • Reflect on the reasonableness of their answer based on the context of the problem.
Use appropriate tools strategically.	• Consider a variety of tools, choose the most appropriate tool, and use the tool correctly (e.g., manipulative, ruler, technology) to support their problem solving. • Use estimation to detect possible errors and establish a reasonable range of answers. • Use technology to help visualize, explore, and compare information.
Attend to precision.	• Communicate precisely, using clear definitions and appropriate mathematical language. • State accurately the meanings of symbols. • Specify appropriate units of measure and labels of axes. • Use a level of precision suitable for the problem context.
Look for and make use of structure.	• Identify and explain mathematical patterns or structures. • Shift viewpoints and see things as single objects or as composed of multiple objects or see expressions in many equivalent forms. • Explain why and when properties of operations are true in a particular context.
Look for and express regularity in repeated reasoning.	• Notice if patterns in calculations are repeated and use that information to solve other problems. • Use and justify the use of general methods or shortcuts by identifying generalizations. • Self-assess as they work to see whether a strategy makes sense, checking for reasonableness prior to finalizing their answer.

Source: Based on Council of Chief State School Officers. (2010). Common Core State Standards.

How to Teach the Standards Effectively

NCTM also developed a publication that explores the specific learning conditions, school structures, and teaching practices that will be important for a high-quality education for all students. The book *Principles to Actions* (NCTM, 2014b) uses classroom stories and samples of student work to illustrate the careful, reflective work required of effective teachers of mathematics through six guiding principles (see Table 1.2 and Appendix B). A series of presentations

TABLE 1.2 THE SIX GUIDING PRINCIPLES FROM *PRINCIPLES TO ACTIONS*

Guiding Principle	Suggestions for Classroom Actions That Align with the Principles
Teaching and learning	• Select focused mathematics goals. • Use meaningful instructional tasks that develop reasoning, sense making, and problem-solving strategies. • Present and encourage a variety of mathematical representations that connect the same ideas or concepts. • Facilitate student discussions and conversations about important mathematical ideas. • Ask essential questions that are planned to be a catalyst for deeper levels of thinking. • Use a strong foundation of conceptual understanding as a foundation for procedural fluency. • Encourage productive struggle—it is a way to deepen understanding and move toward student application of their learning. • Generate ways for students to provide evidence of their thinking through discussions, illustrations, and written responses.
Access and equity	• Establish high expectations for all students. • Provide supports targeted to student needs (equity not equality). • Provide instructional opportunities for students to demonstrate their competence in different ways—creating tasks with easy entry points for students who struggle and extension options for those who finish quickly. • Identify obstacles to students' success and find ways to bridge or eliminate those barriers. • Develop all students' confidence that they can do mathematics. • Enhance the learning of all by celebrating students' diversity.
Curriculum	• Build connections across mathematics topics to capitalize on broad themes and big ideas. • Look for both horizontal and vertical alignment to build coherence. • Avoid thinking of a curriculum as a look-for tool or disconnected set of daily lessons.
Tools and technology	• Include an array of technological tools and manipulatives to support the exploration of mathematical concepts, structures, and relationships. • Think beyond computation when considering the integration of technology. • Explore connections to how technology use for problem solving links to career readiness.
Assessment	• Incorporate a continuous assessment plan to follow how students are performing and how instruction can be modified and thereby improved. • Move beyond test results that just look at overall increases and decreases to pinpoint specific student needs. • Consider the use of multiple assessments to capture a variety of student performance. • Encourage students to self-assess sometimes by evaluating the work of others to enhance their own performance. • Teach students how to check their work.
Professionalism	• Develop a long-term plan for building your expertise. • Build collaborations that will enhance the work of the group of collaborators as you enhance the performance of the students in the school. • Take advantage of all coaching, mentoring, and professional development opportunities and be a life-long learner. • Structure in time to reflect on and analyze your instructional practices.

(webcasts), led by the authors of the publication, explore several of the guiding principles and are available on the *Principles to Actions* portion of NCTM's website (www.nctm.org).

Pause & Reflect

Take a moment to think about which of these six guiding principles seem the biggest change since you were in school. Why do you think these changes to teaching were necessary?

An Invitation to Learn and Grow

Learning Outcome 1.3 **Explore the qualities needed to learn and grow as a professional teacher of mathematics.**

Think back to when you were a student in pre-K–8 classrooms. What are your remembrances of learning mathematics? Here are some thoughts from in-service and pre-service teachers of whom we asked the same question. Which description resonates with your thinking?

> I was really good at math in lower elementary grades, but because I never understood why math works, it made it very difficult to embrace the concepts as I moved into higher grades. I started believing I wasn't good at math so I didn't get too upset when my grades reflected that. *Kathryn*
>
> As a student, I always felt lost during mathematics instruction. It was as if everyone around me had a magic key or code that I missed out on getting. *Tracy*
>
> I consider myself to be really good at math and I enjoy mathematics-related activities, but I often wonder if I would have been GREAT at math and had a completely different career if I cared about math as much as I do now. Sometimes I feel robbed. *April*
>
> Math went from engaging, interactive instruction that I excelled at and loved, to lecture-style instruction that I struggled with. I could not seek outside help, even though I tried, because the teacher's way was so different from the way of the people trying to help me. I went from getting all As to getting low Bs and Cs without knowing how the change happened. *Janelle*
>
> Math class was full of elimination games where students were pitted against each other to see who could answer a math fact the fastest. Because I have a good memory I did well, but I hated every moment. It was such a nerve-wracking experience and for the longest time that is what I thought math was. *Lawrence*
>
> Math was never a problem because it was logical; everything made sense. *Tova*

As you can see, these memories run the gamut with an array of emotions and experiences. The question now becomes: What do you hope your former students will say as they think back to your mathematics instruction? The challenge is for each and every student to learn mathematics with understanding and enthusiasm. Would you relish hearing your students, 15 years after leaving your classroom, state that you encouraged them to be mathematically minded, curious about solving new problems, self-motivated, able to think critically about both correct and incorrect strategies, and that you nurtured them to be risk takers willing to persevere on challenging tasks? That is our expectation!

The mathematics education described in this book may not be the same as the mathematics content and the mathematics teaching you experienced in your grade K–8 experience. As a practicing or prospective teacher preparing to teach mathematics using a problem-solving approach, this book may require you to confront some of your personal beliefs—beliefs about what it means to *do mathematics*, how one goes about *learning mathematics*, how to *teach mathematics*, and what it means to *assess mathematics*. Success in mathematics isn't merely about speed or the notion that there is one right answer. Thinking and talking about mathematics as a means to sense making is a strategy that will serve us well in becoming a society where all individuals are confident in their ability to do math.

Becoming a Teacher of Mathematics

As we said at the start of this chapter, you bring many strengths to the teaching of mathematics, including your willingness to try new things, a fresh perspective on how technology can be integrated into instruction, and your stance as a lifelong learner. Here we describe characteristics, habits of thought, skills, and dispositions you will continue to cultivate to reach success as an effective teacher of mathematics and, we hope, will embrace as you read this book. This book is about increasing the quality of instruction. We are not about fixing children; we are instead engaged in fixing structures.

Knowledgeable. You will need to have a profound, flexible, and adaptive knowledge of mathematics content (Ma, 1999). This statement is not intended to intimidate if you feel that mathematics is not your strong suit, but it is meant to help you prepare for a serious semester of learning about mathematics and how to teach it. You cannot teach what you do not know. Teachers are the driving factors in students' school performance (Colvin & Edwards, 2018). An absence of high-quality opportunities for students to gain mathematics knowledge can result in students having economic challenges and little potential for social mobility (OECD, 2016). These findings add to the gravity of your responsibility, because a student's learning for the year in mathematics will likely come from you. If you are not sure of a fractional concept or other aspect of mathematics content knowledge, now is the time to make changes in your depth of content understanding to prepare best for your role as an instructional leader. You don't want to work on the brink of your knowledge base—instead you need to soak up the knowledge so you will feel more confident and can speak with added passion and enthusiasm. This book and your instructor will help you in that process.

Persistent. You need the ability to stave off frustration and demonstrate persistence. Dweck (2007) describes the brain as similar to a muscle—one that can be strengthened with a good workout! People are not just wired for learning mathematics; they must perform hard work and persevere to understand new ideas. As you move through this book and work the problems yourself, you will learn methods and strategies that will help you anticipate the experiences and strategies to get students past any stumbling blocks. It is likely that what works for you as a learner might work for some of your students. As you conduct these mental workouts, if you ponder, struggle, talk about your thinking, and reflect on how these new ideas fit or don't fit with your prior knowledge, you will enhance your teaching repertoire. Remember as you model these characteristics for your students, they too will value perseverance more than speed. In fact, Einstein did not describe himself as intelligent. Instead, he stated that he was just someone who continued to work on problems longer than others did. Creating opportunities for your students to engage in productive struggle is part of the learning process (Stigler & Hiebert, 2009; Warshauer, 2015) and will be discussed in greater depth in Chapter 2.

Positive Disposition. Prepare yourself by developing a positive attitude toward the subject of mathematics. Research shows that teachers with positive attitudes teach math in more successful ways that result in their students liking math more (Karp, 1991), performing at higher levels (Palardy & Rumberger, 2008), and developing confidence and skill that are personally meaningful and support the community in which they live. If you have ever thought, "I don't like math," that mindset will be evident in your instruction (Ramirez, Hooper, Kersting, Ferguson, & Yeager, 2018). The good news is that research shows that changing attitudes toward mathematics is relatively easy (Tobias, 1995) and that attitude changes are long-lasting (Dweck, 2007). In addition, math methods courses are found to be effective in increasing positive attitudes, more so than student teaching experiences (Jong & Hodges, 2015). Also, pre-service teachers who studied key concepts in mathematics methods classes were more effective in planning lessons on those big ideas—even years after taking the course (Morris & Hiebert, 2017). Not only can you acquire a more positive attitude toward mathematics, as a professional educator it is essential that you do.

To explore your students' attitudes toward mathematics, consider using an interview protocol. Prepare interview questions in three areas: (1) students' attitudes toward mathematics, (2) the typical math environment, and (3) what learning environment preferences students have. Write two or three questions in each area. Each question can be followed with a "why?" or "why not?" Here you can explore how the classroom environment may affect their attitudes and learning preferences.

Ready for Change. Demonstrate a readiness for change, even for change so radical that it may cause you disequilibrium. You may find that what is familiar will become unfamiliar and, conversely, what is unfamiliar will become familiar. For example, you may have always referred to "reducing fractions" as the process of changing $\frac{2}{4}$ to $\frac{1}{2}$, but this language is misleading, because the fractions are not getting smaller. Such terminology can lead children to mistaken connections when they ask, "Did the reduced fraction go on a diet?" A careful look will point out that *reducing* is not the term to use; rather, you are writing an equivalent fraction that is *simplified* or in *lowest terms*. Even though you may have used the language *reducing* for years, you need to become familiar with more precise language, such as *simplifying* fractions.

On the other hand, what is unfamiliar will become more comfortable. It may initially feel uncomfortable for you to ask students, "Did anyone solve it differently?" especially if you are worried that you might not understand their approach. Yet this question is essential to effective mathematics teaching. As you bravely use this strategy, it will become comfortable (and you will learn many new strategies!).

Another potentially new shift in practice is toward a mixed emphasis on teaching concepts and procedures. What happens in a procedure-focused classroom when a student doesn't understand division of fractions? A teacher with only procedural knowledge may say, "Just change the division sign to multiplication, flip over the second fraction, and multiply." We know the use of a memorized approach doesn't work well if we want students to understand fully the process of dividing fractions, so let's consider an example using $3\frac{1}{2} \div \frac{1}{2} =$ ______. Start by relating this division problem to students' prior knowledge of a whole-number division problem such as $25 \div 5 =$ ______, using a corresponding situation: "How many servings of 5 pizza slices are there in 25 slices?" Then ask students to put these same words to use in the original fraction division problem, such as, "You plan to serve each guest $\frac{1}{2}$ a pizza. If you have $3\frac{1}{2}$ pizzas, how many guests can you serve?" Yes, there are seven halves in $3\frac{1}{2}$; therefore, seven guests can be served. Are you surprised that you can do this division of fractions problem in your head once you learn this connection?

To respond to students' challenges, uncertainties, and frustrations, you may need to unlearn and relearn mathematical concepts, developing comprehensive conceptual understanding and a variety of representations along the way. Unearthing these ideas through looking at student work samples, even including errors, and discussing them is an important approach (McLaren, van Gog, Ganoe, Karabinos, & Yaron, 2016). Supporting your mathematics content knowledge on solid, well-supported terrain is important in making a lasting difference in your students' learning of mathematics—so be ready for change. What you already understand will provide you with many Aha moments as you read this book and connect new information to your current mathematics knowledge.

A Team Player. Your school should work as a unit where all teachers are supporting children not just for the one grade they teach but in a coherent manner across the grades. When this idea of a team is implemented, teachers agree to use the same mathematical language, symbols, models, and notation to give students a familiar thread that ties the concepts and procedures together year after year. This established understanding is called a Mathematics Whole School Agreement (Karp, Dougherty, & Bush, 2021), and your eager collaboration is essential in making this approach work well. These collaborative professional development efforts promote your school's capacity in terms of students' mathematics achievement and the ability to support your ongoing professional learning (Donaldson, 2018).

Self-Aware and Reflective. Maya Angelou said, "Do the best you can until you know better. Then when you know better, do better." No matter whether you are a pre-service teacher or an experienced teacher, there is always more to learn about the content and methodology of teaching mathematics. The ability to examine oneself for strengths or areas that need improvement or to reflect on successes and challenges is critical for growth and development. The best teachers are always trying to improve their practice through reading the latest article, attending the webinar or podcast with the most up-to-date information, or signing up for the next series of professional development opportunities. These teachers don't say, "Oh, that's what I am already doing"; instead, they identify and celebrate each new insight and idea they gain. Highly effective teachers never finish learning or exhaust the number of new mental connections they make, and, as a result, they never burn out. Sherman (1982) suggests, "You can't go back and make a new start, but you can start right now and make a brand new ending" (p. 45).

Think back to the quotations from the teachers at the beginning of this section. Self-reflect upon your strengths and target areas for continued growth. Also reflect upon what memories you will create for your students.

As you begin this adventure, let's be reminded of what John Van de Walle said with every new edition of the book: "Enjoy the journey!"

RESOURCES FOR CHAPTER 1

WRITING TO LEARN

At the end of each chapter of this book, you will find a series of questions under this same heading. The questions are designed to help you reflect on important ideas of the chapter. Writing (or talking with a peer) is an excellent way to explore new ideas and incorporate them into your own knowledge base.

1. What are the eight Standards for Mathematical Practice? How do they relate to the Mathematics Content Standards?
2. What are three purposes for learning mathematics as suggested by the NCTM *Catalyzing Change* document? How do these purposes align with your own mathematics instruction when you were a student?

RECOMMENDED READINGS

Articles

Buckheister, K., Jackson, C., & Taylor, C. (2015). An inside track: Fostering mathematical practices. *Teaching Children Mathematics*, *22*(1), 28–35.

The authors share a game used with early elementary students, and through teacher–student dialogue, they describe how the several mathematical practices can be developed.

Karp, K., Bush, S., & Dougherty, B. (2014). 13 rules that expire. *Teaching Children Mathematics*, *21*(1) 18–25.

This article helps students move away from overgeneralizations and rules that cause confusion while considering developing terminology and notation that enhances student understanding.

Books

Kobett, B., & Karp, K. (2020). *Strengths-based teaching and learning in mathematics: 5 teaching turnarounds for grades K–6*. Thousand Oaks, CA: Corwin: A Sage Publication; and Reston, VA: NCTM.

If you want to focus on students' mathematics strengths rather than identify their deficits, this book is the perfect read.

National Council of Teachers of Mathematics. (2020). *Catalyzing change in early childhood and elementary mathematics: Initiating critical conversations*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2020). *Catalyzing change in middle mathematics: Initiating critical conversations*. Reston, VA: NCTM.

The Catalyzing Change series explores important conversations about the policies, practices, and issues that are factors in influencing mathematics education.

CHAPTER

2

Exploring What It Means to Know and Do Mathematics

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

2.1 Describe essential components of mathematical proficiency.

2.2 Illustrate what it means to do mathematics.

2.3 Connect learning theories to mathematical proficiency and to mathematics teaching.

To help students learn mathematics, you must know *what* is important to learn, as well as *how* students learn. That is the focus of this chapter! Let's look at a relatively poorly understood topic, fraction division. If a student learned this topic, what will they know and what should they be able to do? The answer is more than being able to implement a procedure successfully (e.g., the invert-and-multiply algorithm). To know and understand division of fractions, a student can answer these questions:

- What does $3 \div \frac{1}{4}$ mean conceptually?
- What situation might this expression represent?
- Will the result be greater than or less than 3 and why?
- What strategies can we employ to solve this problem?
- What illustration or manipulative might illustrate this problem?
- How does this expression relate to subtraction? To multiplication?

Wondering the answers to some of these questions? See Chapter 15! The point here is that there is more *to* learn than procedures. This chapter focuses first on what it means to do mathematics, then engages you in doing mathematics, and finally explains how students learn mathematics.

What Is Mathematical Proficiency?

Learning Outcome 2.1 **Describe essential components of mathematical proficiency.**

Doing mathematics means generating strategies for solving a problem, applying an efficient strategy, and checking to see whether the answer makes sense. As you read in Chapter 1, standards documents describe these actions as Mathematical Practices (see Table 1.1 and Appendix A).

The Mathematical Practices describe specific actions of *mathematical proficiency*, as defined by the National Research Council in *Adding It Up* (2001) (see Figure 2.1). A person who is mathematically proficient doesn't just know content; they know how to navigate the process of solving problems. It is our responsibility as teachers of mathematics to help students develop each of these five strands of mathematical proficiency!

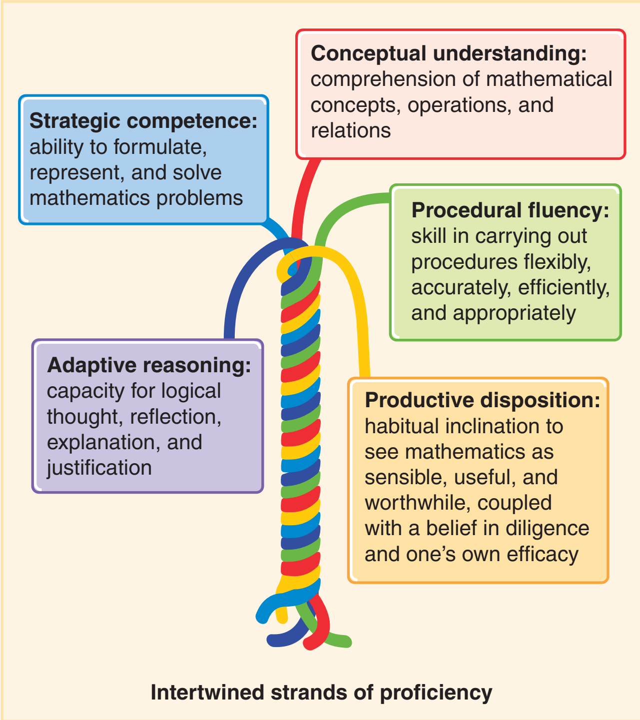


FIGURE 2.1 Five strands of mathematical proficiency

Source: National Research Council. (2001). *Adding It Up: Helping Children Learn Mathematics*, p. 5. Reprinted with permission from the National Academy of Sciences, courtesy of the National Academies Press, Washington, DC.

Conceptual Understanding

Conceptual understanding means having *connected* knowledge: "mental connections among mathematical facts, procedures, and ideas" (Hiebert & Grouws, 2007, p. 380). Consider what a person might know about a fraction such as $\frac{6}{8}$. Here is a partial list of what they might be able to do:

- Read the fraction.
- Identify the 6 and 8 as the numerator and denominator, respectively.
- Recognize that $\frac{6}{8}$ is equivalent to $\frac{3}{4}$.
- Notice that $\frac{6}{8}$ is more than $\frac{1}{2}$ (recognize relative size).
- Draw a region that is shaded in a way to show $\frac{6}{8}$.
- Locate $\frac{6}{8}$ on a number line.
- Illustrate $\frac{6}{8}$ of a set of 48 pennies or counters.
- State that there are an infinitely large number of equivalencies to $\frac{6}{8}$.
- Recognize $\frac{6}{8}$ as a rationale number.
- Describe $\frac{6}{8}$ as a ratio (girls to boys, for example).
- Represent $\frac{6}{8}$ as a decimal fraction.

At what point does a person know enough to claim they understand fractions? A number of items on this list refer to procedural knowledge (e.g., being able to find an equivalent fraction), and others refer to conceptual knowledge (e.g., recognizing that $\frac{6}{8}$ is greater than $\frac{1}{2}$ by analyzing the relationship between the numerator and denominator). A student may know that $\frac{6}{8}$ can be simplified to $\frac{3}{4}$ but not recognize that $\frac{3}{4}$ and $\frac{6}{8}$ are equivalent (having procedural knowledge without conceptual knowledge). A student may be able to find one fraction between $\frac{1}{2}$ and $\frac{6}{8}$ but not be able to find others, meaning they have some procedural knowledge but insufficient connections to recognize that they could also change denominators. The point is that *understanding* a mathematics topic is hard to define. It is best explained as a measure of the quality and quantity of connections a person can make between a new idea and their existing ideas. In other words, conceptual understanding is a flexible web of connections and relationships within and between ideas, interpretations, and images of mathematical concepts—a relational understanding.

Relational Understanding. Understanding exists along a continuum from an *instrumental understanding*—doing something without understanding (see Figure 2.2) to a *relational understanding*—knowing what to do and why (Skemp, 1978). This continuum continues to be a useful way to think about the depth of a student's conceptual understanding. In the $\frac{6}{8}$ example, a student who only knows a procedure for simplifying $\frac{6}{8}$ to $\frac{3}{4}$ has an understanding near the instrumental end of the continuum, whereas a student who can draw diagrams, give examples, and find numerous equivalencies has an understanding moving toward the relational end of the continuum.

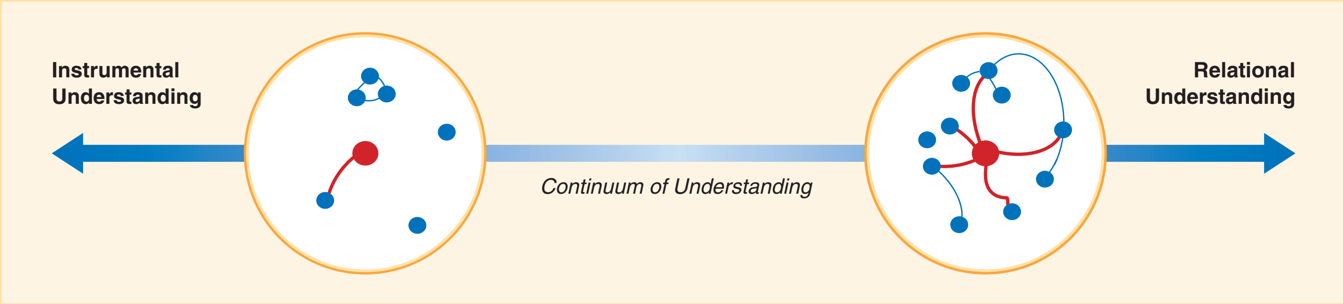


FIGURE 2.2 Continuum of understanding from instrumental to relational

Relational understanding is the goal for students across all mathematics content in the K–8 curriculum. Figure 2.3 provides a visual of skills and concepts related to understanding ratios. Note how much is involved in having a relational understanding.

Procedural Fluency

Procedural fluency is built on the foundation of conceptual understanding (NCTM, 2014b). Procedural fluency is sometimes confused with being able to do standard algorithms correctly and quickly, but it requires a much more comprehensive knowledge than that. *Procedural* fluency includes three components: (1) efficiency, (2) flexibility, and (3) accuracy (see Figure 2.4). Both efficiency and flexibility require appropriate strategy selection. Additionally, connected with procedural fluency is knowing ways to check for reasonableness, including estimating at the beginning of the problem and looking back to see whether the answer is close to that estimate (addressed in Part II of this book).

To illustrate fluency, we will explore two topics: addition with whole numbers and subtraction with decimals.

Adding Two-Digit Numbers. Solve the problem in a way that makes sense to you.

$$37 + 28 = ?$$

Figure 2.5 illustrates four strategies that lead to a correct answer (accurate). Which of these strategies is efficient? A fluent student does not automatically stack the numbers and apply the standard algorithm (though they know how to do this); the fluent student looks at the problem and considers which strategy will be *efficient*, given the numbers in the problem. An efficient strategy is one that may take fewer steps than the alternatives. In this case, a student might move 2 from the 37 to the 28 to adapt the equation to $35 + 30$. This is called a

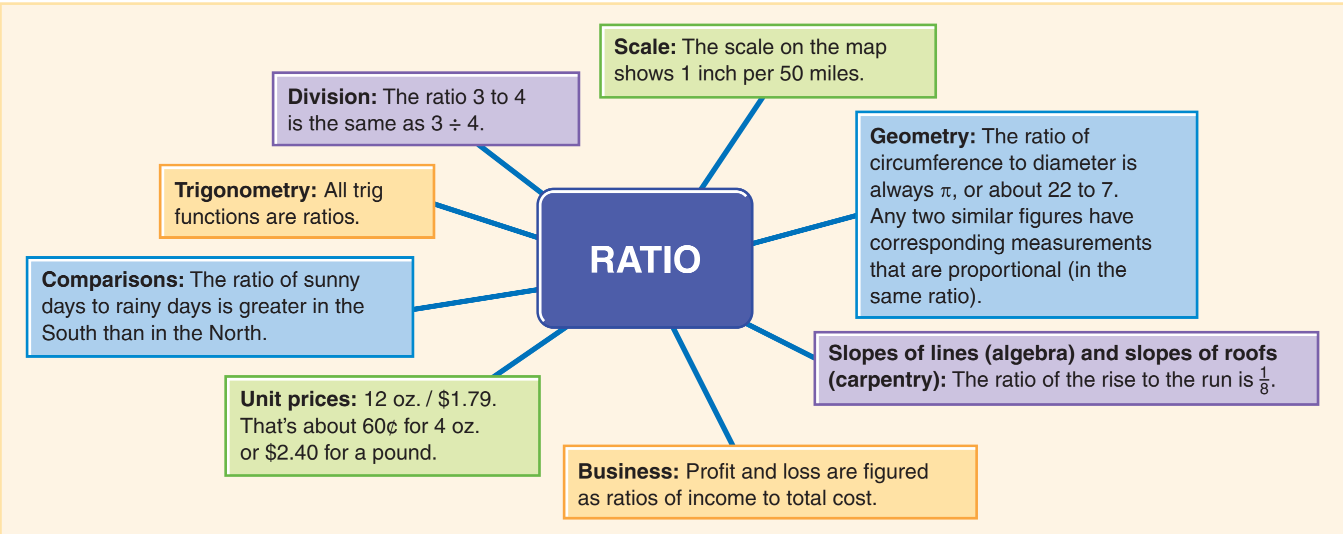


FIGURE 2.3 Connected ideas that contribute to a relational understanding of ratios

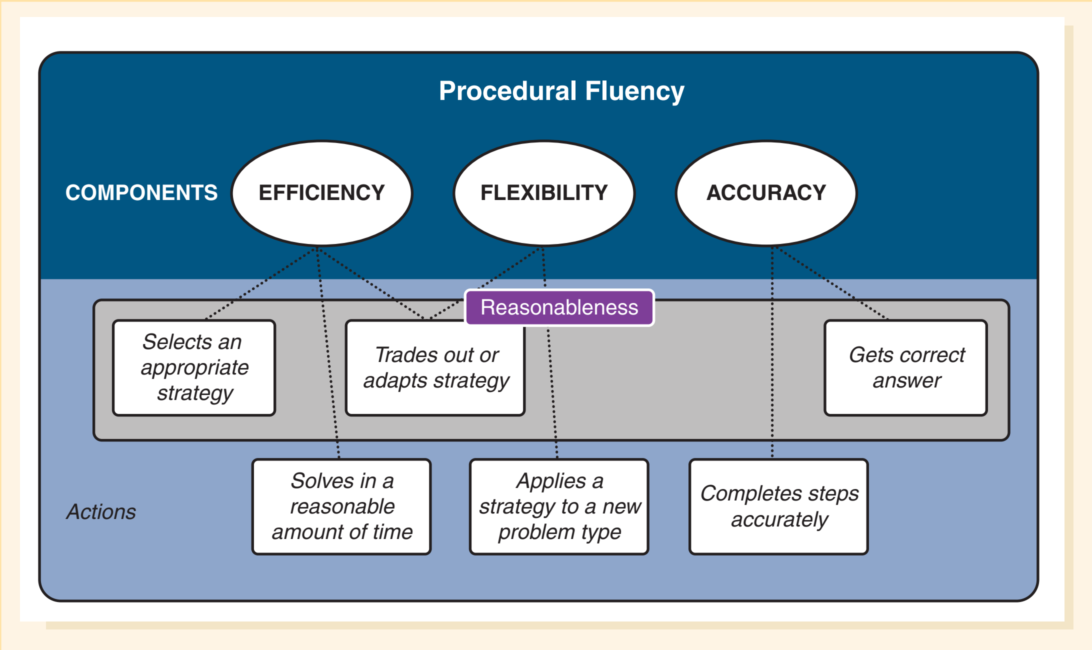


FIGURE 2.4 Procedural fluency components and actions

Source: Bay-Williams, J. M., & San Giovanni, J. J. (2021). *Figuring out fluency in mathematics teaching and learning, K-8.* Corwin. Used with permission.

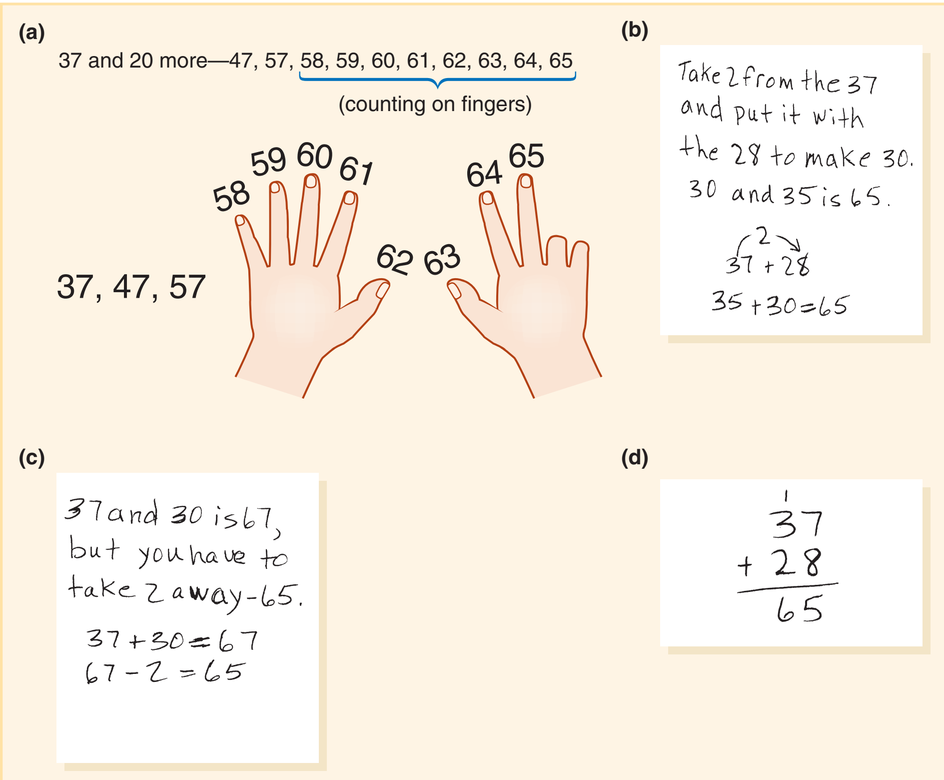


FIGURE 2.5 Different strategies for solving 37 + 28

Make Tens strategy (see Figure 2.5[b]). Or a student might add 2 to 28 to round to an easier number, add $37 + 30$ and then subtract 2 from the answer to counteract the addition. This is called a *Compensation strategy* (see Figure 2.5[c]). Either of these strategies may be done mentally. Which strategy or strategies are efficient? (Which strategy did you use?) Using Make Tens or Compensation uses fewer steps than the standard algorithm and can be done mentally, making them more efficient for this example.

Given different problems, a fluent student opts for different strategies, showing *flexibility*. Practice your fluency by selecting one of these strategies (Make Tens, Compensation, standard algorithm, or another strategy) to solve these four problems.

1. $57 + 98 =$ **2.** $49 + 48 =$ **3.** $62 + 24 =$ **4.** $37 + 46 =$

Make Tens or Compensation strategies are a good fit for the first two problems. The third problem involves no regrouping, so simply adding tens and adding ones is efficient. The final problem is not as good of a fit for Make Tens or Compensation, so a student might use a Partials Sums strategy, adding tens (70) and ones (13) and then combining (83). Or, they may use the standard algorithm (adding the digits in the ones first and then in the tens). How efficient and flexible were you in solving these four two-digit addition problems?

Subtracting Decimals. Solve it using an efficient method:

$$11.24 - 10.9$$

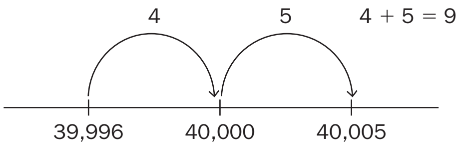
Too many people only know one way to solve this problem, which is to stack the numbers and regroup starting with the hundredths place. But there are more efficient options that require understanding what subtraction *means*. One meaning is "take away." For this equation, a take-away situation might be having \$11.24 and spending \$10.90. Subtraction also means to compare or find the difference. A compare situation might be asking the difference between times for running the 100-meter race for two runners.

Either interpretation (take-away or compare) can be used to find the difference. Here are three fourth-grade explanations:

Elisa: I started at 10.9, counted up 0.1 to 11, plus 0.24 more, so the answer is 0.34.
Monique: I added one tenth to both numbers. That makes 11.34 – 11. Then it was easy: 0.34.
Aaron: I subtracted 11 to get 0.24, then added the 0.1 back on to get 0.34.

Do any of these strategies match how you thought about it? Can you see which ones used take-away thinking and which ones used compare thinking? How do these strategies compare to the standard algorithm in terms of efficiency?

Think about the following problem and what you just read about subtraction: $40{,}005 - 39{,}996 =$ ________. Hopefully, you see how efficient a counting-up strategy is here.



In this example, the standard algorithm for subtraction involves regrouping across zeros, a sometimes tedious, inefficient, and prone-to-error method.

Procedural fluency supports conceptual understanding (and vice versa), and both contribute to student development of procedural flexibility (Schneider, Rittle-Johnson, & Star, 2011). Sadly, procedural fluency is often mistaken for learning standard algorithms and being able to do them quickly and accurately. An overemphasis on teaching standard algorithms—especially too early—can actually interfere with the development of fluency.

Aiden's house is 4 blocks closer to school than Martin's house. Martin's house is 10 blocks from school. How many blocks is Aiden's house from school?

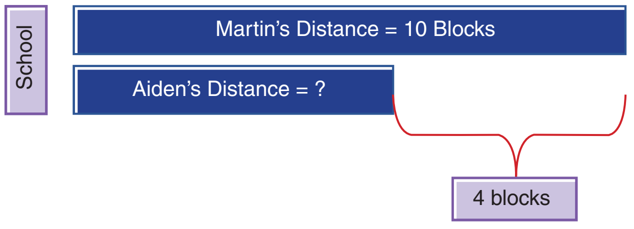


FIGURE 2.6 A story problem translated into a bar diagram

Strategic Competence

This strand of proficiency is what is often called problem solving. *Strategic competence* includes reading a story problem or encountering a mathematical situation and figuring out what the problem is as well as determining a method for solving it. As part of strategic competence, students are able to create useful representations to help them reason about the problem. These depictions might include a number line, as illustrated earlier, or a bar diagram to make sense of a story problem, as illustrated in Figure 2.6.

Strategic competence also means a student can solve both routine and nonroutine problems. A routine problem is one that the student knows how to solve based on previous experiences, for example, the two-digit addition problems described earlier. A student with strategic competence can choose the best strategies for each problem. Nonroutine problems are problems the learner does not immediately know how to solve. Examples are provided in the next section (e.g., Two Machines, One Job). Students with strategic competence consider the situation, possible representations, and different approaches as they work to solve nonroutine problems. Like with procedural fluency, flexibility is the key to strategic competence. Some strategies fit one situation but not another. Strategic competence is knowing what features in a problem make it a good fit for a particular strategy. You can support students' strategic competence and procedural fluency by asking these questions:

- Why does this strategy work?
- Is this an efficient strategy?
- When does this strategy work?

Adaptive Reasoning

As described in Figure 2.1, *adaptive reasoning* is the capacity to think logically about the relationships among concepts and situations. It includes careful consideration at the beginning of solving a problem of a variety of solution approaches, the readiness to switch out for a new strategy in the middle of working a problem if one isn't fruitful, and being able to justify the solution at the end of solving a problem. When a teacher confirms that a student's answer is correct or not, they can undermine the student's development of adaptive reasoning. As described in *Adding It Up* (NRC, 2001), "students who disagree about a mathematical answer need not rely on checking with the teacher… they need only check that their reasoning is valid" (p. 129). Teachers support adaptive reasoning by asking students to justify their strategies and/or their answers. Doing so supports each student's understanding of mathematics and develops productive dispositions and self-regulation.

Productive Disposition

A productive disposition is a favorable attitude about being able to do mathematics. A student with a productive disposition tries to make sense of mathematics and believes that putting effort into learning mathematics is what is needed to be successful. This outlook is called a growth mindset (Boaler, 2015; Dweck, 2016). A person with a productive disposition sees themselves as an effective learner and doer of mathematics (NRC, 2001). They have the confidence to solve complex problems. In essence, they have a positive mathematics identity. A student's identity is a deeply held belief about themselves (Aguirre, Mayfield-Ingram, & Martin, 2013).

Although we addressed this myth in Chapter 1, it bears repeating: Far too many people think that mathematical ability is inherited and thereby limited to only a few people (Grady, 2016). This is simply not true, and such a fixed mindset leads to math anxiety (Sparks, 2020). Sadly, numerous studies show that anxiety specific to math (i.e., not to other areas such as reading) begins as early as first grade, and this anxiety negatively affects performance on achievement tests, on student learning of math skills and concepts, and on problem solving (Ramirez, Shaw, & Maloney, 2018). The truth is that there are not math people and non-math people. There are people who persevere and solve challenging tasks and those who quit. Telling yourself, your colleagues, your parents, and your students that effort matters more so than ability can help to turn their math anxiety into a productive disposition. Because students (and other stakeholders) may have a fixed mindset or experience math anxiety, this fifth strand of mathematical proficiency must receive more attention!

A productive disposition emerges when the other strands of mathematical proficiency are developed, and conversely, it supports the development of these other practices.

For which of the following questions might a student with a productive disposition typically say "yes"?

- When you read a problem you don't know how to solve, do you think, "Great, something challenging. I can solve this"?
- Do you consider several possible approaches before diving in to solve a problem?
- As you work, do you draw a picture or use a manipulative?
- Do you recognize a wrong path and try something else?
- When you finish a problem, do you wonder whether it is right? If there are other answers?

A "yes" response is evidence of mathematical proficiency and connects to at least one of the mathematical practices. When teachers are intentional and explicit about developing the mathematical practices, students' participation and confidence improve—they develop a productive disposition (Wilburne, Wildmann, Morret, & Stipanovic, 2014).

An Invitation to Do Mathematics

Learning Outcome 2.2 **Illustrate what it means to do mathematics.**

The purpose of this section is to provide *you* with opportunities to engage in doing mathematics so you know better how to engage your students. Three problems (tasks) are posed. For each, stop and solve. Then read the "Explore" section. Continue to engage with the task. As you do, reflect on your mathematical proficiency and your use of Mathematical Practices (or processes).

Task 1. One Up, One Down

This task has two parts, addition and multiplication. Either task can be used with students, or you can do one and then the other.

Task 1a One Up, One Down: Addition

When you add $7 + 7$, you get 14. When you make the first number 1 more and the second number 1 less, you get the same answer:

$$\begin{array}{ccccc} \uparrow & & \downarrow & & \\ 7 & + & 7 & = & 14 \\ 8 & + & 6 & = & 14 \end{array}$$

For what other numbers, does this one-up, one-down pattern work?

What else do you notice about this pattern?

Why is this pattern true? Explain and illustrate.

Task 1b One Up, One Down: Multiplication

When you multiply 7×7, you get 49. When you make the first number 1 more and the second number 1 less, you get an answer that is one less:

$$\begin{array}{ccccc} \uparrow & & \downarrow & & \\ 7 & \times & 7 & = & 49 \\ 8 & \times & 6 & = & 48 \end{array}$$

For what other numbers does this one-up, one-down pattern work?

What else do you notice about this pattern?

Why is this pattern true? Explain and illustrate.

Explore. If you explored both of these operations, you may have noticed that there are many more patterns or generalizations in the addition situation than in the multiplication situation. Consider:

- What manipulative or picture might illustrate the patterns?
- How is the pattern altered if the sums/products begin as two consecutive numbers (e.g., 8×7)? If they differed by 2 or by 3?
- What if you instead go two up, two down (e.g., $7 + 7$ to $9 + 5$ OR 7×7 to 9×5)?
- In what ways is the addition situation similar to and different from the multiplication situation?

Strategies. Let's look at the multiplication pattern, using illustrations. Draw a rectangle (or array) with dimensions of the original problem, 7×7 (see Figure 2.7[a]), and then draw the new rectangle (an 8-by-6) and compare the two rectangles. A second option is to represent the situation as equal groups, using stacks of chips (see Figure 2.7[b]). How might you move the counters to represent the one up, one down problem of 8×6? See how the stacks for each expression compare.

Have you made some mathematical connections and conjectures in exploring this problem? In doing so, you have hopefully felt a sense of accomplishment and excitement. Such a feeling strengthens a person's mathematics identity and productive disposition, which is why we must engage students in doing mathematical explorations like this one!

Task 2. Solving the Mystery

This task engages students in looking for a pattern. For middle school students, they can also explore this task by using variables.

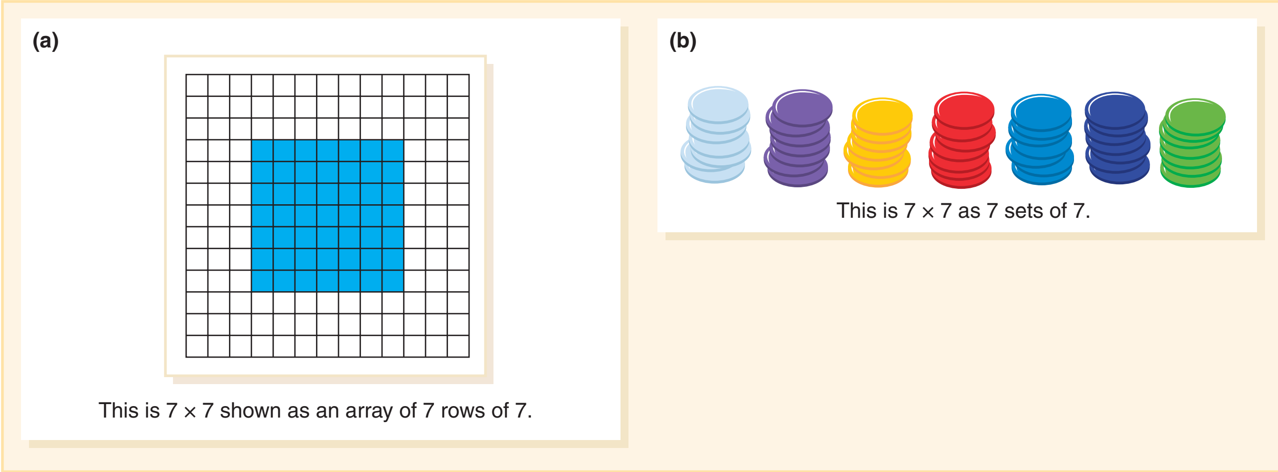


FIGURE 2.7 Two ways to illustrate One Up, One Down: Multiplication

Task 2 Solve the Mystery

Follow these five steps:

1. Write down any whole number.
2. Add to it the number that comes after it in counting.
3. Add 9.
4. Divide by 2.
5. Subtract your original number.

Did you get 5? Wow! How did we know? Can you solve the mystery of why this works?

Explore. Here are ways to explore this mathematical mystery:

- Try different starting numbers and look for a pattern.
- Try to explore with a variable for your number (like *n*).
- Work backward to see what insights you might gain.
- Explain why this series of steps always leads to the answer of 5.

Extend. This four-step mystery could be simplified or made more complex (or students can write their own). The result can be the start number, or a set number (like 5). Try one of these:

- Create a three-step mystery that ends with the starting number. A simple example: Add 10, subtract 7, subtract 3. Be clever!
- Create a mystery puzzle that results in the answer of 0.
- Create a mystery puzzle that uses all four operations.

Task 3. Two Machines, One Job

This task may seem like it is a complicated ratios or proportion problem, but it is really about reasoning.

Task 3 Two Machines, One Job

Sophia's Paper Shredding Shop started with one paper-shredding machine. Business was good, so Sophia bought a new shredding machine. The old machine could shred a truckload of paper in 4 hours. The new machine could shred the same truckload in only 2 hours. How long will it take to shred one truckload of paper if Sophia runs both shredders at the same time?

Explore. This task may take a while to explore. As you practice perseverance, here are some things to try:

- Estimate how much time you think it should take the two machines. Estimating can sometimes lead to a new insight.
- Use a manipulative (chips, plastic cubes, pennies) to represent the size of the truckload.
- Draw a picture. For example, could you draw a rectangle or a line segment to represent the truckload? The time that was used or needed?

Strategies. Hopefully, you have solved this problem in some way that makes sense to you. Understanding other people's strategies can deepen your own understanding. And, teachers are always in a position where they must try to figure out how their students are thinking about a problem.

Figure 2.8 illustrates one method for solving the problem (based on Schifter & Fosnot, 1993). The whole paper strip represents the whole truckload. The new machine completes $\frac{1}{2}$ of the truckload in one hour (see purple shading). The old machine completes $\frac{1}{4}$ of the paper in one hour (green). Together, the machines shred $\frac{3}{4}$ of the truckload in one hour. So, the remaining one-fourth will take 20 minutes.

Another solution is to reason abstractly rather than to use visuals.

4 hours: 3 Truckloads (2 truckloads from new machine + 1 truckload from old machine)
If three truckloads take 4 hours, divide by 3 to figure out how much time for 1 truckload:
$\frac{4}{3}$ hours = $1\frac{1}{3}$ hours = 1 hour and 20 minutes

Reflecting on the Three Tasks

Now that you have solved these tasks, to what extent do you feel you demonstrated mathematical proficiency? In other words, did you notice your knowledge of concepts and procedures working together to solve the problems? Did you try a strategy and trade it out for another strategy along the way, demonstrating strategic competence? Did you adapt strategies and justify why you selected them and/or why they work? And, importantly, did you enter the problem solving with a productive disposition, thinking that with enough effort you could figure it out? More specifically, to what extent did you engage in the mathematical practices?

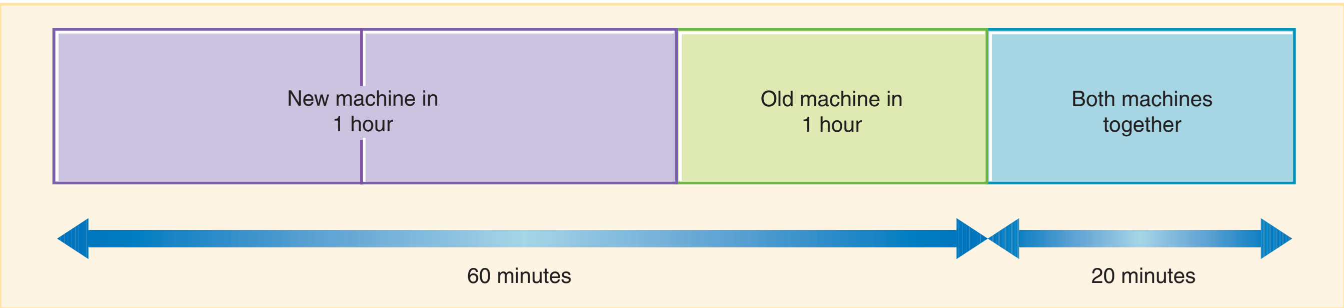


FIGURE 2.8 One illustration of Two Machines, One Job task

For example, to what extent did you make sense of the problem and persevere (MP 1)? Use a manipulative strategically (MP 5) or draw a mathematical representation (MP 4)? Look for structure or a pattern (MP 7 or 8)? Reason about the quantities in the problems (MP 2)? These practices (actions) are what it means to *do mathematics*!

If your answers are yes, you demonstrated mathematical proficiency. If your answers are no, reflect on why this might be the case. You may have learned mathematics by replicating procedures shown to you by your teachers, with the expectation that you should show all your steps. As a result, you may have felt uncomfortable and maybe even incapable of solving these problems. When we learn mathematics in a way that we are just asked to practice skills, rather than to figure out problems, it is more difficult to develop a positive, productive mathematics disposition. As a teacher, developing a productive disposition for yourself and for your students must be a priority! Showing students how to do an algorithm does not prepare them to be mathematically proficient. Imagine how much more engaging and meaningful math class can be when the focus is on thinking! And, in developing students' mathematical proficiency, you are also developing positive mathematics identity!

How Do Students Learn Mathematics?

Learning Outcome 2.3 **Connect learning theories to mathematical proficiency and to mathematics teaching.**

Given that the goal for every student is mathematical proficiency, you may be wondering how to ensure that they learn such things as strategic competence and reasoning. The answers to how you will carry this out are grounded in learning theory.

In this section, we briefly describe two compatible major learning theories and provide mathematics-specific teaching ideas that stem from them (Norton & D'Ambrosio, 2008).

Constructivism

Constructivism is rooted in Jean Piaget's work, developed in the 1930s and translated to English in the 1950s. At the heart of constructivism is the notion that learners are not blank slates but rather creators or constructors of their own learning (Fosnot, 2005). Integrated *networks*, or *cognitive schemas*, are formed by constructing knowledge, and they are used to build new knowledge. Through *reflective thought*—the effort to connect existing ideas to new information—people modify their existing schemas to incorporate new ideas (Fosnot, 2005). All people construct or give meaning to things they perceive or think about. As you read these words, you are giving meaning to them. Whether listening passively to a lecture or actively engaging in synthesizing findings in a project, your brain is applying prior knowledge from your existing schemas to make sense of the new information.

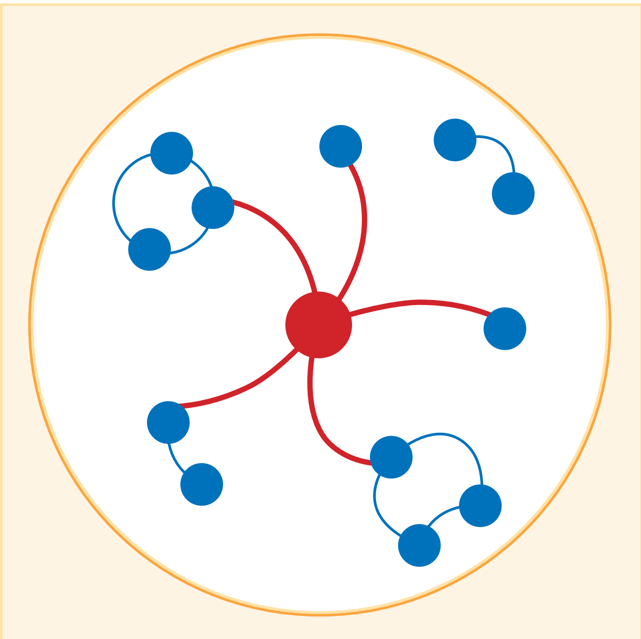


FIGURE 2.9 Learning connects prior knowledge (blue dots) to new knowledge (red dot)

To connect to the metaphor of building construction, the *tools* we use to build understanding are our existing ideas and knowledge. The *materials* we use are things we see, hear, or touch, as well as our own thoughts and ideas. In Figure 2.9, blue and red dots are used as symbols for ideas. The blue dots represent existing ideas and prior knowledge. The lines joining the ideas represent our logical connections or relationships that have developed between and among ideas. The red dot is an emerging idea, one that is being constructed. When existing ideas are used to build new knowledge, those connections remain; when new knowledge is learned in isolation, connections are not formed.

Sociocultural Theory

About 100 years ago, Lev Vygotsky, a Russian psychologist, began developing what is now called sociocultural theory. Fifty years later, it began to gain attention as a way to explain learning. Like constructivism, this theory assumes active meaning-seeking on the part of the learner. Key to this theory is the zone of proximal development (ZPD) (Vygotsky, 1978). Simply put, the ZPD refers to a range of knowledge that is out of reach for a person to learn on his or her own, but is accessible if the learner has support from more knowledgeable others. If a task is too easy, an individual can do it on their own, but they won't learn from it. If it is out of reach, they won't learn from it. An experience must be in the zone for learning to occur. That means for learning to occur, three things must be present: (1) guidance from someone with advanced knowledge, (2) social interaction, and (3) tasks that position the learner in their ZPD (Cherry, 2020). Social interaction is essential for learning! Additionally, the learner is influenced by both the classroom community and the broader social and historical culture of the members of the classroom (Forman, 2003).

Implications for Teaching Mathematics

Learning theories are not teaching strategies—theory *informs* teaching. This section outlines teaching strategies that are informed by constructivist and sociocultural perspectives. You will see these strategies revisited in more detail in Chapters 3 and 4, where a problem-based model for instruction is discussed, and in Part II of this book, where you learn how to apply these ideas to specific mathematics topics.

Use Manipulatives and Other Tools. *Manipulatives* are physical objects that students and teachers can use to illustrate and discover mathematical concepts, whether made specifically for mathematics (e.g., connecting cubes) or for other purposes (e.g., buttons). A *tool* is any object, picture, or drawing that can be used to explore a concept. Choices for manipulatives (including virtual manipulatives) abound—from common objects such as lima beans to commercially produced materials such as pattern blocks. Manipulatives help students explore mathematical relationships concretely.

A tool, however, does not illustrate a concept. The tool is used to visualize a mathematical concept; only your mind can impose the mathematical relationship on the object (Suh, 2007a). Consider each of the concepts and the corresponding manipulative model in Figure 2.10. Try to separate the tool from the mathematical concept.

a. Beyond being part of a counting sequence, 6 represents a quantity. Quantities are relational. Adding 1 counter to a set of 6 counters means there is now *one more*: 7 is *one more than* 6.
b. The concept of "measure of length" is a comparison. The length measure of an object is a comparison relationship of the length of the object to the length of the unit.
c. The concept of "rectangle" includes both spatial and length relationships. The opposite sides are of equal length and parallel, and the adjacent sides meet at right angles.
d. The concept of "hundred" is not in the larger square but in the relationship of that square to the strip ("ten") and to the little square ("one").
e. "Chance" is a relationship between the frequency of an event and all possible outcomes. Spinners can represent relative frequencies with shading (e.g., 25% chance of yellow).
f. Location is based on direction and magnitude. Arrows on the number line can show direction (pointing) and magnitude (length).

Although tools can be used to support learning, they can be ineffective. One such misstep is when a teacher shows the students exactly what to do with the manipulative, and has students imitate that process. It is just as possible to move blocks mindlessly as it is to go through the steps of an algorithm mindlessly. On the other extreme, giving out manipulatives with no purpose for their use can lead to nonproductive and unsystematic investigation (Stein & Bovalino, 2001). The goal is to set up tasks, using the tools as ways to explore the concept so that students notice important mathematical relationships that can be discussed, connecting the concrete representations to abstract concepts.

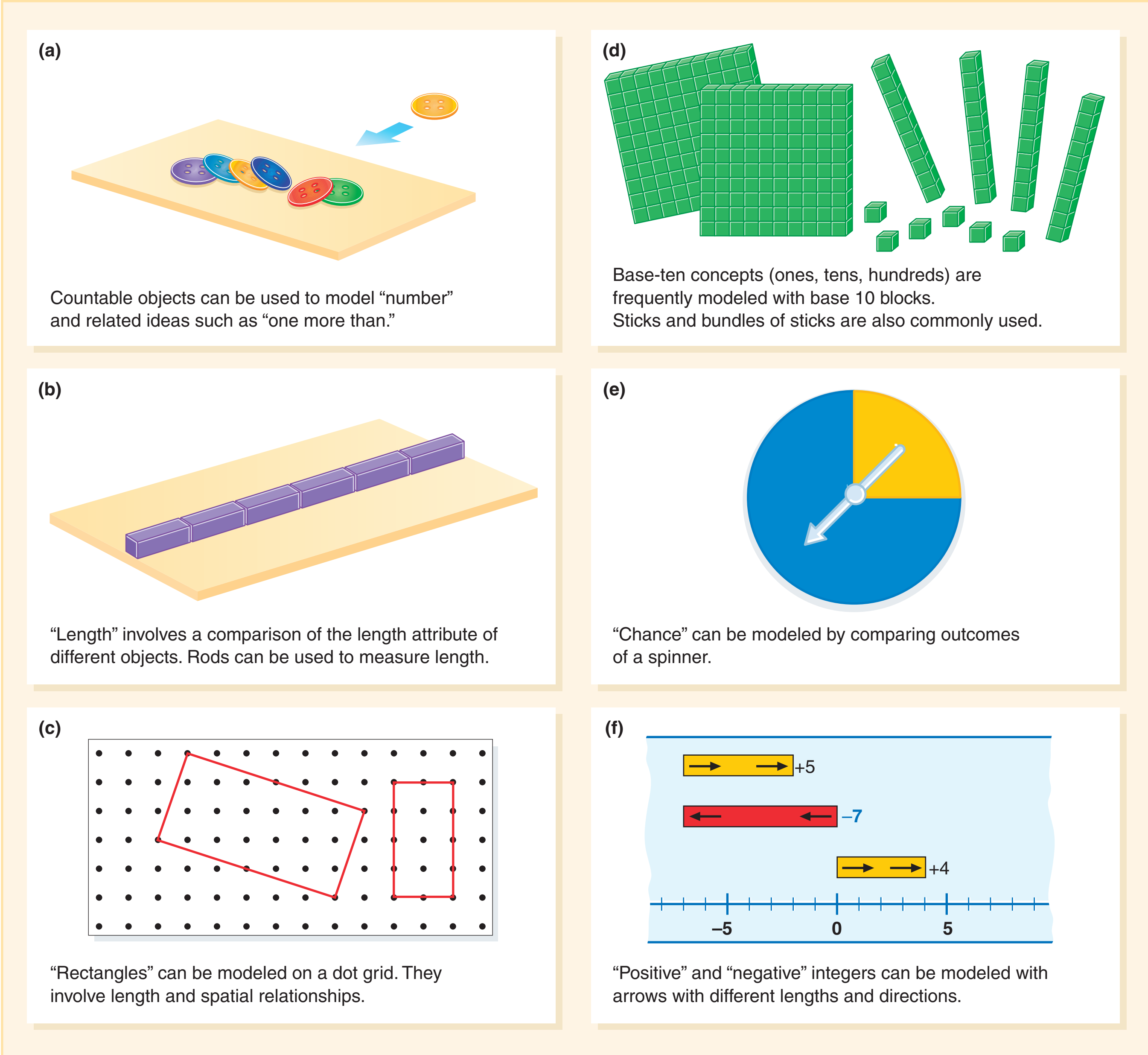


FIGURE 2.10 Examples of tools to illustrate mathematics concepts

Make Connections Explicit. Purposefully having students connect ideas is not only grounded in learning theory, it is well-established through research to improve students’ conceptual understanding (Hiebert & Grouws, 2007). The teacher’s role in making mathematical relationships explicit is to be sure that students are making the connections that are implied in a task. For example, asking students to relate today’s topic to one they investigated last week, or by asking “How is Laila’s strategy like Marco’s strategy?” are ways to be help students be explicit about mathematical relationships.

Use Multiple Representations. Students need to connect ideas within a concept. That includes making connections across different representations. Figure 2.11 illustrates a web of representations showing ways to demonstrate understanding. Students who have difficulty translating a concept from one representation to another also have difficulty solving

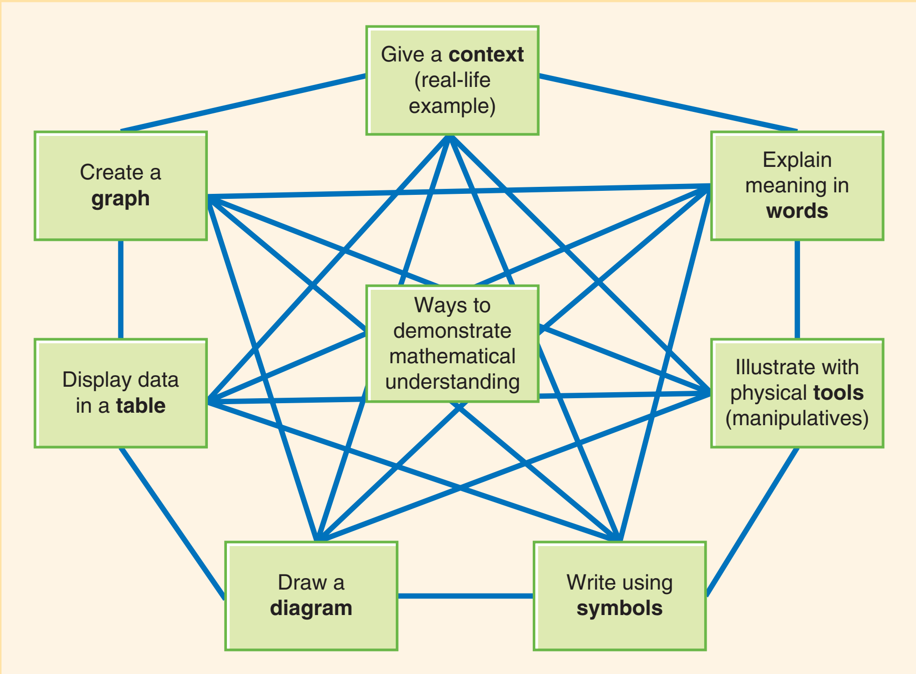


FIGURE 2.11 Web of representations to demonstrate understanding

problems and understanding computations (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). Therefore, strengthening students' ability to move between and among these multiple representations improves their understanding.

Earlier, the topic of ratios was explored to highlight conceptual (relational) knowledge (Figure 2.3). In Figure 2.12, the same topic is used to illustrate a web of representations. Using these visuals and contexts helps learners develop relational understanding.

Encourage Multiple Strategies. Encourage students to use strategies that make sense to them, building on their prior knowledge. Even learning a basic fact like 7×8 can have better results if a teacher promotes multiple strategies, for example asking, "Besides skip counting, how can you figure out 7×8 if you don't know the answer?" Student explanations include:

Jair: I know 5×8 equals 40, and 2×8 equals 16, so 7 groups of 8 is 56.

Lina: I used the square of 7×7, which equals 49, and added one more 7 more to get 56.

Cody: I know 4×7 equals 28, so I doubled 28 to get eight groups of 7, which was 56.

Encouraging different strategies develops strategic competence, fosters a productive disposition, and develops procedural fluency.

Provide Opportunities to Communicate and Reflect. For a new idea to be interconnected in a rich web of interrelated ideas, children must reflect on the relevant ideas they possess (blue dots) and bring them to bear on the development of the new idea (red dot). In discussions with peers, students adapt and expand on their existing networks of concepts as they look at the representations and listen to the thinking of others. Pressing for students to explain further and justify their thinking increases student mathematical reasoning (i.e., use of the Mathematical Practices) (Sussman, Hammerman, Higgins, & Hochberg, 2019).

Engage Students in Productive Struggle. Have you ever just wanted to think through something yourself without being interrupted or told how to do it? Too often, teachers step in to help too soon or give too much guidance. In reality, these moves do not help the student learn. Engaging in productive struggle helps students learn mathematics (Hiebert & Grouws, 2007).

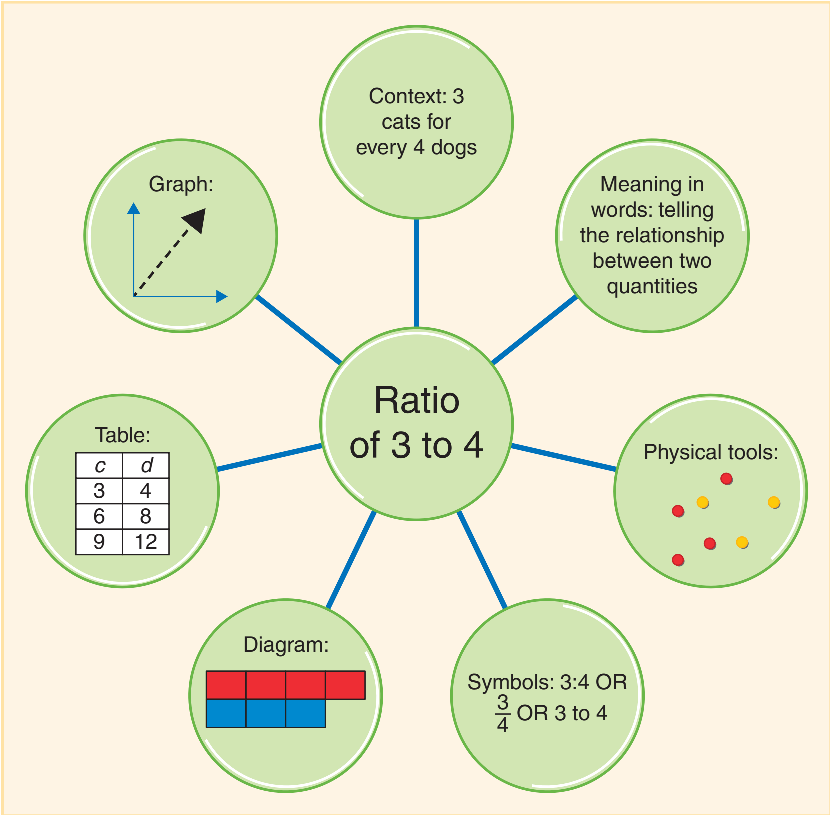


FIGURE 2.12 Multiple representations for ratio of 3 to 4

Productive struggle includes such actions as drawing a representation or seeking out a manipulative, asking a question of a peer or teacher, and persevering through a task by trying a second or third approach if needed (NCTM, 2014b; SanGiovanni, Katt, & Dykema, 2020). Productive struggle is *not* waiting to be shown how to do something, feeling despair because a problem doesn't initially make sense, or becoming frustrated with an inaccessible task (Baker, Jessup, Jacobs, Empson, & Case, 2020). When students are in productive struggle, they are in their ZPD. They are building conceptual connections. That is why they learn better than if they were just shown how to do something.

Treat Errors as Opportunities for Learning. When students make errors, it can mean a misapplication of their prior knowledge in a new situation. Students rarely give random responses, so their errors are an insight into limited understandings or misconceptions they might have. For example, students comparing decimals may incorrectly apply rules of whole numbers, thinking that longer means larger (Martinie, 2014). Often, one student's misconception is shared by others in the class, and discussing an error publicly can help other students understand the correct approach (Hoffman, Breyfogle, & Dressler, 2009; Rathouz, 2011a). This level of transparency about using errors to learn also communicates that doing mathematics involves making mistakes; it is just part of the discipline.

Scaffold Content. The practice of *scaffolding* is based on the idea that a task otherwise outside of a student's ZPD can become accessible if it is carefully structured. When students do not have skills they need to be successful with a grade-level topic, you can plan several days of instruction that start with what students know and sequence tasks each day to explore a culminating task that does meet the grade-level standard (Coleman, 2020). For example, if you are teaching multiplication of a whole number times a fraction (e.g., $5 \times 3\frac{1}{2}$), scaffolding might begin with prior knowledge: (1) exploring and explaining the meaning of whole-number multiplication (5×3), perhaps connecting to story situations, and (2) constructing representations to show 5 groups of 3. Then scaffolding can build meaning, starting with concrete and semi-concrete representations and moving toward abstract understanding (CSA). That would mean exploring stories and using manipulatives and drawings to represent $5 \times 3\frac{1}{2}$. For concepts completely new to students, the learning may require more structure or assistance, including the use of tools (e.g., manipulatives, sketches) or more support from peers or knowledgeable others. As students become more comfortable with the content, the scaffolds are reduced and the student becomes more independent. Scaffolding can also be used to help students develop strategic competence by initially offering students simpler problems or removing the question prompt so that students focus on making sense of the situation first (Barlow, Duncan, Lischka, Hartland, & Willingham, 2017).

Honor Diversity. Finally, and importantly, these theories emphasize that each learner is unique, with a different collection of prior knowledge and cultural experiences. Because new knowledge is built on existing knowledge and experience, effective teaching incorporates and uses what the students bring to the classroom, honoring those earlier experiences. Thus, lesson contexts should be selected based on students' interests, knowledge, background, strengths, and experiences. (See also the discussion of culturally responsive mathematics instruction in Chapter 6.)

Connecting the Dots

It seems appropriate to close this chapter by once again connecting some dots, especially because the ideas represented here are the foundation for the approach to each topic in the content chapters found in Part II. This chapter began with discussing what is important to know about mathematics—the five strands of mathematical proficiency. Second, you explored what *doing* mathematics feels and looks like as you solved and reflected on three tasks. Each of these tasks offered opportunities to make connections between mathematics concepts—connecting the blue dots to new ideas represented as red dots. Finally, you read how learning theories—the importance of having opportunities to connect the dots—connects to mathematics learning. The best learning opportunities, according to constructivism and sociocultural theories, are those that engage learners in productive struggle, making connections, using representations, communicating ideas, and reflecting on their use of strategies, to name a few. Your teaching must focus on purposefully creating these types of opportunities for students to develop their own networks of blue dots and logically connect new red dots to this existing web of ideas. As you carefully plan and design instruction that challenges students to think critically and creatively, you nurture positive mathematics identities and support the development of mathematical proficiency!

RESOURCES FOR CHAPTER 2

WRITING TO LEARN

1. An elevator talk is an explanation that can be offered between an elevator door closing and reopening, usually to a complete stranger. In other words, simple, meaningful, and to the point. What might be your elevator talk for answering, "What are (a) mathematical proficiency and (b) Mathematical Practices or processes and/or (c) doing mathematics?"
2. How can you support students' relational understanding (and how can you avoid their development of only an instrumental understanding)?
3. Describe how to implement the One Up, One Down task with students in a way that reflects constructivist and/or sociocultural learning theory. Link specifically to aspects of the theory.
4. Select a major concept students learn in K–8. For this concept, select at least three of the Implications for Teaching Mathematics and describe how those implications support students in constructing meaning of that concept.

RECOMMENDED READINGS

Articles

Carter, S. (2008). Disequilibrium & questioning in the primary classroom: Establishing routines that help students learn. *Teaching Children Mathematics*, *15*(3), 134–137.

This is a wonderful teacher's story of how she infused the constructivist notion of disequilibrium and the related idea of productive struggle to support learning in her first-grade class.

Whitacre, I., Schoen, R. C., Champagne, Z., & Goddard, A. (2016). Relational thinking: What's the difference? *Teaching Children Mathematics*, *23*(5), 303–309.

These authors illustrate the importance of relational thinking by sharing the results of students' work on subtraction problems. They offer helpful strategies for developing students' procedural flexibility.

Baker, K., Jessup, N. A., Jacobs, V. R., Empson, S. B., & Case, J. (2020). Productive struggle in action. *Mathematics Teacher: Learning & Teaching PK-12*, *113*(5), 361–367.

Through sharing Ryan's experience in solving a problem, these authors show what engaging students in productive struggle looks like. They also offer an excellent list of takeaways.

Books

Kelemanik, G., Lucenta, A., & Creighton, S. J. (2016). *Routines for reasoning: Fostering the mathematical practices in all student*s. Portsmouth, NH: Heinemann.

These authors propose an organizational structure and excellent classroom teaching routines to ensure that your students become mathematically proficient.

Mohr, D., Walcott, C., & Kloosterman, P. (2019). *Mathematical thinking: From assessment items to challenging tasks*. Reston, VA: NCTM.

Using the National Assessment of Educational Progress (NAEP) items for inspiration, the authors share 36 excellent tasks along with full lesson plans. Suggestions for facilitating discussion and differentiating are also included. An excellent resource to get students doing mathematics!

CHAPTER 3

Teaching Problem-Based Mathematics

LEARNING OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

3.1 Describe specific practices for effective mathematics teaching.

3.2 Explain the roles of problem solving and learning progressions in setting goals for student learning.

3.3 Describe features of worthwhile tasks and methods for adapting tasks to make them better.

3.4 Illustrate the relationship between procedural fluency and conceptual understanding.

3.5 Describe and relate the Mathematics Teaching Practices related to facilitating mathematical discourse.

Imagine yourself in a mathematics classroom. What are students working on? What are they talking about? If that classroom embodies the ideas in Chapter 2 of doing mathematics, then you will see students working on a task carefully selected by the teacher that allows them to add to their prior knowledge, connect mathematical ideas, and learn important conceptual and procedural knowledge related to the topic. Effective teaching of mathematics is described in *Principles to Actions* (NCTM, 2014b) through eight Effective Mathematics Teaching Practices. These Teaching Practices support the development of the eight Mathematical Practices for students. These are different sets of practices (possibly confusing because there are eight on both lists and both lists use the word *practice*): The Mathematical Practices describe student actions; the Teaching Practices describe teacher actions. This chapter is about the Teaching Practices, providing lots of tasks, examples, and activities to illuminate what effective mathematics teaching looks like.

Effective Mathematics Teaching

Learning Outcome 3.1 **Describe specific practices for effective mathematics teaching.**

Classrooms where students are learning through doing mathematics do not happen by accident—they happen because the teacher implements specific Teaching Practices and establishes expectations that encourage risk taking, reasoning, the generation and sharing of ideas, and so forth. Teaching this way is a paradigm shift from traditional mathematics teaching where students repeat what the teacher demonstrates. At first glance, it may seem that the teacher's role is less demanding because the students are doing the mathematics, but the teacher's role is actually more demanding in such classrooms. Why work harder? So that each and every student develops competence and confidence in their ability to do mathematics—and that outcome is worth your effort.

Success for Every Student

The NCTM Effective Mathematics Teaching Practices were designed to ensure that every child has access to high-quality mathematics learning experiences: "[E]ffective teaching is the nonnegotiable core that ensures that all students learn mathematics at high levels" (NCTM, 2014b, p. 4). Inclusive teaching, teaching that challenges and supports every child, engages students in *doing* mathematics (not watching the teacher do the mathematics). Teaching through engaging students in problem solving provides opportunities for every student to become mathematically proficient, because this approach:

- *Develops Mathematical Practices and processes*. By definition, teaching through problem solving positions students to be the doers, and as they are doing the mathematics, they are developing the eight Mathematical Practices (see Table 1.1 and Appendix A).
- *Focuses students' attention on ideas and sense making*. When solving problems, students are reflecting on the concepts inherent in the problems and making connections (e.g., between concepts and procedures).
- *Builds on students' strengths*. Students can choose and apply strategies that they understand. For example, students may solve 42 – 26 by applying various mental strategies, using a manipulative or a hundred chart, or by applying an algorithm.
- *Invites creativity*. Students enjoy the creative process of problem solving, searching for patterns, and showing how they figured something out.
- *Allows for extensions and elaborations*. Extensions and what-if questions can motivate and challenge all students as well as provide enrichment for advanced learners or quick finishers. Students can even generate their own mathematical questions to explore.
- *Develops student confidence and identities*. As students learn through problem solving, they begin to identify themselves as doers of mathematics.
- *Engages students so that there are fewer discipline problems*. Many discipline issues are the result of boredom, not understanding an explanation, or finding little relevance. Students like to be challenged, have choice, and work with peers, thus giving them less reason to disengage.
- *Provides formative assessment data*. As students use manipulatives, draw pictures, and talk, they provide the teacher with a steady stream of valuable information about their mathematical thinking that can be used to support continued learning.

When students have confidence, show perseverance, and enjoy mathematics, it makes sense that they will achieve at a higher level and want to continue learning about mathematics, opening many doors to them in the future.

The Mathematics Teaching Practices

Before exploring effective teaching, let's reflect on a traditional mathematics lesson and how it falls short. Stereotypical mathematics lessons include the teacher showing students how to follow the steps of a procedure and then students practicing that procedure on a worksheet with 20 or so straight-forward exercises. With this image in mind, how does that teaching relate to the *Mathematics Teaching Framework* in Figure 3.1? This framework organizes the eight Mathematics Teaching Practices described in *Principles to Actions* (Huinker, 2018; NCTM, 2014b). Let's look at the three boxes at the top of the framework and see how that traditional lesson matches up. The lesson goals focus learning on memorizing a procedure. Not a very robust learning goal for students. The tasks do not promote reasoning or problem solving, and the procedures were not connected to concepts. So far, the lesson is falling short of effective mathematics teaching.

When the top three boxes (Teaching Practices) are not well developed, the rest of the Mathematics Teaching Practices (in the "facilitate meaningful mathematical discourse" box) are virtually impossible to enact. There is little opportunity to pose questions beyond, "What answer did you get?" and therefore, it is difficult to see evidence of student thinking. Meanwhile, the students were shown *how* to do the procedures, so no productive struggle will happen, except when students forget what the teacher said, which is not the meaning of productive struggle (and requires the teacher to repeat constantly). Furthermore, there were likely no representations to connect. This whole scenario leads us to a critical fact about effective mathematics teaching: These first three Teaching Practices must be effectively implemented to open up the chance for effective discourse and meaningful and useful learning.

Too often, mathematics teaching still follows this I-do, We-do, You-do pattern of the teacher showing one way to perform a skill and students practicing that skill first together and then independently, using the same procedure. Unfortunately, this approach to mathematics teaching has not been successful. Here are a few shortcomings of a teach-by-telling approach:

- Students are not thinking (the teacher is doing the thinking for them) (Liljedahl, 2021).
- It communicates that there is only one way to solve the problem, misrepresenting the field of mathematics and disempowering students who naturally may want to make sense of mathematics.
- It positions the student as a passive learner, dependent on the teacher to present ideas, rather than as an independent thinker with the capability and responsibility for solving the problem.
- It assumes that all students have the necessary prior knowledge to understand the teacher's explanations—which is rarely the case.
- It decreases the likelihood of a student attempting to solve a nonroutine problem without instructions on how to solve it. Yet, doing mathematics means you can solve nonroutine problems.

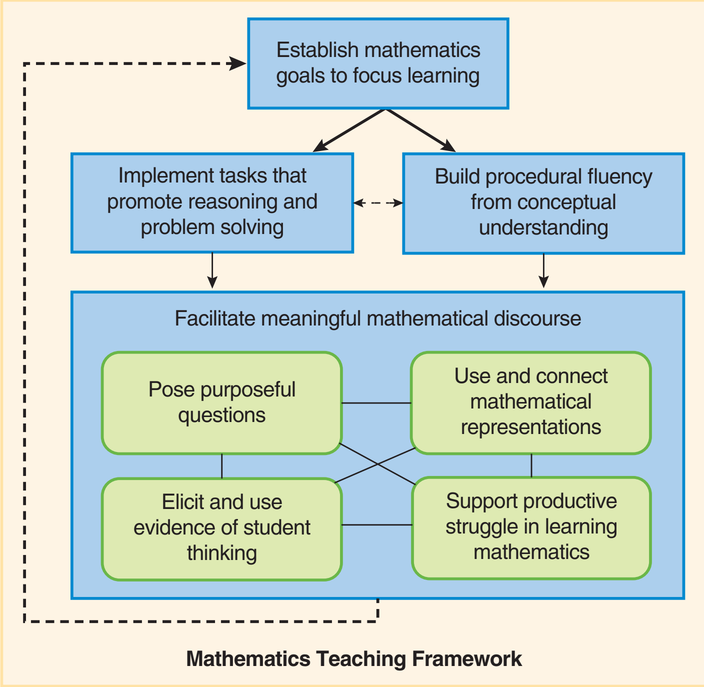


FIGURE 3.1 Mathematics Teaching Framework (Huinker & Bill, 2017).

From D. Huinker & V. Bill. (2017). *Taking Action: Implementing Effective Mathematics Teaching Practices in Kindergarten–Grade 5*. Taking Action Series. NCTM.

We can do better for our students through effective mathematics teaching. That begins with attending to the eight Teaching Practices in Figure 3.1. This chapter begins with the top three and then focuses on the components related to facilitating meaningful discourse.

Establishing Goals to Focus Learning

Learning Outcome 3.2 Explain the roles of problem solving and learning progressions in setting goals for student learning.

A focus on goals for learning a mathematics lesson begins with thinking of goals that are useful in today's society. Skills needed in the 21st-century workplace are less about being able to compute and more about being able to design solutions to authentic problems, from distribution of food to conservation to the spread of disease. Important life skills include critical thinking, communication, collaboration, creativity, and the ability to use technology effectively (Partnership for 21st Century Skills, n.d.). These life skills must be goals within mathematics teaching. We call this *problem-based* teaching.

Problem-Based Mathematics Teaching

Solving problems has always been a part of mathematics learning, but the way in which they are incorporated has varied based on our goals for students. Here we briefly review three approaches to problem solving (Schroeder & Lester, 1989). The distinctions among these ways are important because attending to only one of these approaches will not lead to your students becoming mathematically proficient.

Teaching for Problem Solving. Teaching *for* problem solving starts with learning the abstract concept and then moving to solving problems as a way to apply the learned skills (explain-practice-apply). For example, students learn the algorithm for adding fractions first and then solve story problems that involve adding fractions. This has been the traditional approach (practice skills first, and then solve story problems). The major shortcoming of this approach is that students learn very early in school that the stories they encounter will be solved using the skill they just learned. Therefore, there is no need to read and make sense of the story; they can just lift the numbers out and use the skill of that day's lesson. This habit results in students having difficulties with story problems, multistep problems, and high-level tasks. In other words, the pattern of explain-practice-apply works against preparing students to *do mathematics*. Solving application problems after a skill is learned *is* important. The key is to be sure that the application problem is complex enough that understanding the situation and using strategies are necessary to solving it.

Teaching Problem Solving. Students need guidance on how to problem solve. This includes (1) the process of problem solving and (2) general ways to solve problems (e.g., draw a picture). The mathematician George Pólya wrote a classic book, *How to Solve It* (1945), which outlined four steps for problem solving. These steps continue to be widely used today. For example, they are reflected in the first mathematical practice, "Making sense of problems and persevere in solving them." The four steps are summarized here:

1. **Understand the problem.** First, you must figure out what the problem is about and identify what question or problem is being posed.
2. **Devise a plan.** How might you solve the problem? Consider your options and make a choice: Will you want to write an equation? Will you want to model the problem with a manipulative?
3. **Carry out the plan.** Implement the plan you selected from step 2.
4. **Look back.** Does your answer make sense? If not, loop back to step 2 and select a different strategy to solve the problem, or loop back to step 3 to fix something within your strategy.

The beauty of Pólya's framework is its generalizability; it can and should be applied to many types of problems, from simple computational exercises to authentic and worthwhile multistep problems. Explicitly teaching these four steps to students can improve their ability to think mathematically.

Problem-solving strategies are identifiable methods for approaching a task. Students select or design a strategy as they devise a plan (Pólya's step 2). Table 3.1 provides a list of general problem-solving strategies in grades K–8, though not all of them are used at every grade level. These strategies become the tools by which students can enter into the process of solving unfamiliar and novel tasks. These strategies are not distinct, but interrelated. For example, creating a list is a way of looking for patterns. When students employ one of these strategies, it should be identified, highlighted, and discussed. Over time, these

TABLE 3.1 PROBLEM-SOLVING STRATEGIES IN K–8

Strategy	Brief Explanation	What It Might Look Like
Model the problem	The problem is interpreted in a concrete, visual way	• Uses manipulatives • Acts out the problem • Draws a picture or sketch (on paper or by using dynamic software)
Make an organized list	The problem is explored by recording possible solutions or examples in a logical way.	• Makes an organized list • Creates a table or chart
Predict and check for reasonableness	Possible realistic answers are tried out to see if they work (this is sometimes called the Guess & Check strategy).	• Tries out an answer (e.g., using a calculator or testing it in the equation) • Creates a list of guesses and adjustments (tries a value; if it's too low, tries a higher value, etc.)
Make a simpler problem	Student simplifies the quantities in a problem to make a situation easier to understand and analyze.	• Rereads the problem with no numbers in it or uses easy numbers in it • Writes and solves a similar problem that fits the situation, but with smaller values • Uses visuals to explore the simpler problem
Look for a pattern	Student looks across a series of values or problems to notice similarities to solve the problem.	• Solves simpler problems to see what happens • Looks across problems to see what is different and what is the same
Prove a conjecture	Student formulates a possible solution based on mathematical properties, for example, and justifies (proves) that it works.	• Makes a claim (e.g., the answer will be one more) • Uses manipulatives, visuals, or numbers to prove they are correct
Write equations	Student interprets the story numerically and/or symbolically	• Writes an expression or series of expressions • Writes an equation

problem-solving strategies will become habits of mind and help students become mathematically proficient.

Mathematical problem solving is grounded in curiosity. It is important not to proceduralize it. In other words, do not take the problem solving out of a problem-solving activity by telling students to "make a table to solve this problem." Instead, pose a problem that lends itself to different strategies and ask students to approach the problem in a way that makes the most sense to them. You can post the list of strategies from Table 3.1 (or one like it) to help students select a method.

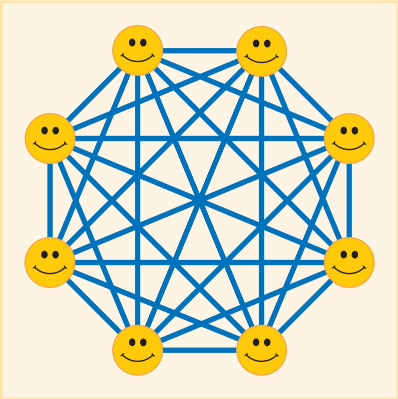
The **Elbow Hello** problem (replacing the classic handshake problem) is an example of a nonroutine task that lends itself to many strategies:

Task 1

Eight friends met for a skating party. Each friend gave one elbow hello to everyone else at the party. How many elbow hellos occurred?

Here are some problem-solving strategies for counting elbow bumps:

Model the Problem (could be a drawn picture or students actually acting it out):



Make an Organized List (also incorporates Make a Simpler Problem):

Number of friends	2	3	4	5	6	7	8
Number of elbow bumps	1	3	6	10	15	21	28

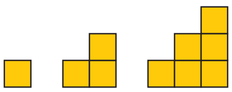
Write an Equation: $7 + 6 + 5 + 4 + 3 + 2 + 1 + 0 = 28$. Student explains: The first friend elbow-bumps 7 friends, the next friend has only 6 friends to greet (she already elbow-bumped the first friend), and so on down to 0 elbow-bumps for the last friend.

During the sharing of strategies, you can help students see other options and make connections among the strategies. Additionally, you can highlight a particular strategy so that more students can use that strategy on a future task. Discussing the many ways to approach this task supports the development of the Mathematical Practices (Albrecht, 2016).

Similar problems can be posed to support the development and reinforcement of these problem-solving strategies; for example:

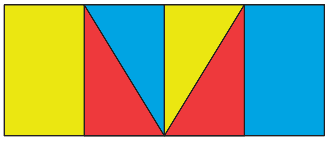
Task 2

1. If six softball teams play each other once in a round-robin tournament, how many games will be needed?
2. How many blocks are needed for the 10th staircase?

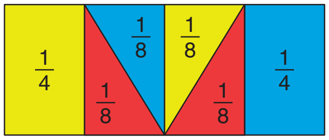


Teaching through Problem Solving. This approach means that students learn mathematics content through doing problems. They explore real contexts, problems, situations, and models, and from those explorations they learn concepts and skills. Teaching *through* problem solving is essentially turning the teaching *for* problem solving approach upside down—with the problem or task presented at the *beginning* of a lesson or unit and related knowledge or skills emerging from exploring the problem(s). As such, the students are involved in *thinking* in order to make sense of the mathematics (Liljedahl, 2021). Consider the following task, given to students who have not learned the algorithm for adding fractions with different denominators:

What fraction of this flag is blue?



By partitioning, students might label each section with their fraction of the whole, labeling the blue parts as illustrated here:



As students work, they recognize that they need equal-sized pieces to combine them. They change $\frac{1}{4}$ to $\frac{2}{8}$ (either by partitioning or knowing the fraction equivalency) and then add the blue pieces to get $\frac{3}{8}$. *After* students have solved the task, the teacher convenes the class to highlight important mathematical ideas—in this case, that you can most efficiently combine same-sized pieces. Notice, with this approach, that mathematical ideas are the outcomes of the problem-solving experience rather than of problem solving explained beforehand. Children are *learning* mathematics by *doing* mathematics; and by doing mathematics, they are learning mathematics (Cai, 2010).

Teaching through problem solving has a positive impact on students' performance. A recent national survey of fourth graders (National Center for Education Statistics, 2019) reports that students with higher mathematics scores (1) have teachers who place a strong emphasis on *using alternate strategies* to solve problems, (2) enjoy solving complex problems, (3) are more confident. Students in problem-based classrooms learn to ask: Why? What would happen if? What is another way? How does this way compare to that way? Will this always work? Generating this habit of consistent inquiry develops a disposition of openness, curiosity, and wonder (Clifford & Marinucci, 2008). This curiosity comes naturally to young students. Our goal as teachers is to nurture this disposition.

Learning Progressions

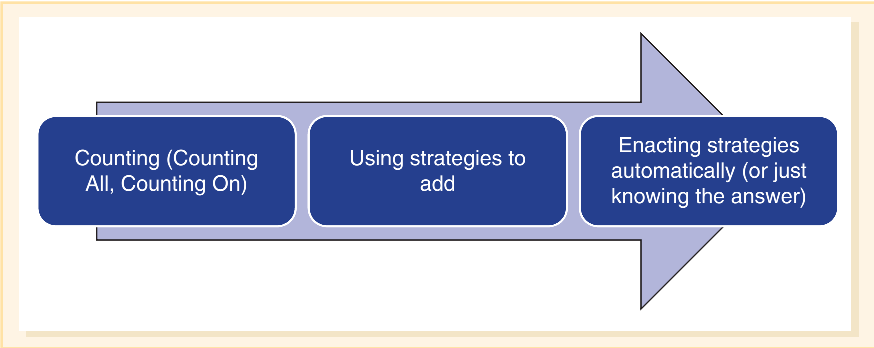
Learning goals (and related objectives) must fit within learning progressions. Mathematics is developmental. Teaching developmentally means that each lesson is part of a learning progression or trajectory (both terms are used interchangeably). *Learning progressions* describe the development of student thinking and strategies over time in terms of sophistication of both conceptual understanding and procedural fluency (Clements & Sarama 2021; Daro, Mosher, & Corcoran, 2011; Siemon et al., 2017; Sztajn, Confrey, Wilson, & Edgington, 2012). Recall that the Mathematics Teaching Framework (Figure 3.1) begins with the Teaching Practice related to establishing goals. That Teaching Practice states:

> Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.

Lesson goals and objectives are the focus of Chapter 4 (planning). Part II of this book includes learning progressions as they relate to each chapter's content. Regardless of what you are teaching, as you establish goals, ask yourself:

- How does this topic relate to students' prior mathematics knowledge and experiences?
- How does this idea connect to future mathematics learning?

After selecting a topic within a learning progression, you establish learning goals and then day-by-day objectives for students. Each day, the lesson goals must make sense to students. As an example, let's look at addition facts. These are the single-digit sums. The learning progression to reach automaticity with these facts looks like this (Baroody, 2006):



Through formative assessments, you notice that students are able to count on and ready to learn strategies. A goal might be to have students use a Make Tens Strategy. Lesson objectives are specific and measureable. Examples for this goal are:

- Students will be able to rethink a single-digit addition problem (e.g., $9 + 6$) as a $10 + \_\_\_\_$ sum, using a Make Tens strategy.
- Students will be able to explain why $9 + 6 = 10 + 5$.

Once students learn this strategy, they keep practicing it until they automatically use the strategy and determine an answer to $9 + 6$. Effective teaching means knowing where your daily goals and objectives fit within the broader landscape.

Just restating the standards is not likely to help your students know what they are doing or why they are doing it and, on many days, you are working on an aspect of a standard, not the whole standard. Instead, think of student-friendly ways to help them understand the focus of that day's lesson. Additionally, on some days, the lesson's purpose reveals itself *after* students explore a task and discover an important mathematical idea. On such days, telling students what they will learn can spoil the surprise. Instead, consider helping students connect the outcome of the task to a big idea in mathematics and conclude by sharing that this was the purpose of the day's lesson. You can pose questions such as: What did we learn from solving this task? What big ideas can we take away from our work today? Ask yourself:

- What mathematics is being learned?
- Why is this mathematics important to learn?
- How can I introduce the big ideas (without taking away an aha in the lesson itself)?

With a focus on problem solving and learning progressions, the next important step is selecting worthwhile tasks.

Tasks That Promote Problem Solving

Learning Outcome 3.3 **Describe features of worthwhile tasks and methods for adapting tasks to make them better.**

To create classroom experiences in which students engage in problem solving, you must select tasks that promote problem solving. Such a worthwhile task may take on many forms. It might be clearly defined or open-ended, it may involve problem solving or problem posing, it may include words or be purely symbols, it may take only a few minutes to solve or weeks to investigate, it may be real-life or abstract. Also, a task can promote problem solving initially but then become routine as a student's knowledge and experience grows.

Worthwhile Tasks

For a task to lend itself to problem solving—in other words, to be problematic—it must pose a question for which (1) there are no prescribed rules or methods to solve and (2) there is not a perception that there is one correct solution method (Hiebert et al., 1997).

Here is a task for you to try:

Task 3

$$10 + \blacksquare = 4 + (3 + \blacktriangle)$$

Find a number to replace the square and a number to replace the triangle so that the equation is true.

Find more pairs of numbers that will make the equation true.

What pattern do you notice about the numbers for any correct solution?

Does this task promote problem solving? It does not have a prescribed approach, and there are numerous ways to approach the problem, so it meets the first criterion of being problematic. Worthwhile tasks have other features, namely that they are cognitively demanding and open, and/or include relevant contexts. Each is discussed here.

High Cognitive Demand. Tasks that promote problem solving are cognitively demanding, meaning they involve high-level thinking. Low cognitive demand–tasks (also called routine problems or lower-level tasks) involve stating facts or following known procedures or computations. High-level cognitive demand tasks, on the other hand, involve (1) procedures with connections or (2) doing mathematics (Smith & Stein, 1998). Such tasks engage students in understanding, analyzing, applying, and evaluating. When trying to determine whether a task is high-level cognitive demand, look for these characteristics (adapted from NCTM, 2014b):

- Has a solution pathway that is not immediately obvious
- Makes connections between procedures and mathematical concepts and ideas
- Focuses attention on generalizations and underlying structures
- Includes the potential use of multiple representations
- Involves non-algorithmic (nonroutine) thinking
- Engages students in exploration and reflection (e.g., Is this approach getting me somewhere?)

Open Tasks. As implied by the list you just read, worthwhile tasks can be solved in a variety of ways, even if it is a computation problem like $597 + 332$. Open-ended tasks are tasks that have more than one answer, like Make the Equation True. Open-ended may also mean that there are different ways to represent your answer (e.g., various pictures, demonstrations with manipulatives, acting out a problem, using a table or a graph, etc.). However, many worthwhile problems have one answer, one end. Yet, how to reach that result is open. In other words, the middle of the task is open, hence the term *open middle tasks* (https://www.openmiddle.com) (Kaplinsky, n.d.). Whether open-middle or open-ended, worthwhile tasks provide students with opportunities to reason and make decisions. Open tasks support diverse learners because students are encouraged to employ their prior learning and experiences, and having a choice of strategies can lower the anxiety of students, particularly that of multilingual learners (Murrey, 2008). Compare the K-1 tasks in Figure 3.2.

TASK 1: [The teacher places a bowl of objects (e.g., toy cars) on the table.] He asks, "Do we have enough [toy cars] for everyone in the class?"

TASK 2: [The teacher gives each student a page with pictures of cars copied in rows as seen below]. He asks, "Do we have enough cars for everyone in the class?"

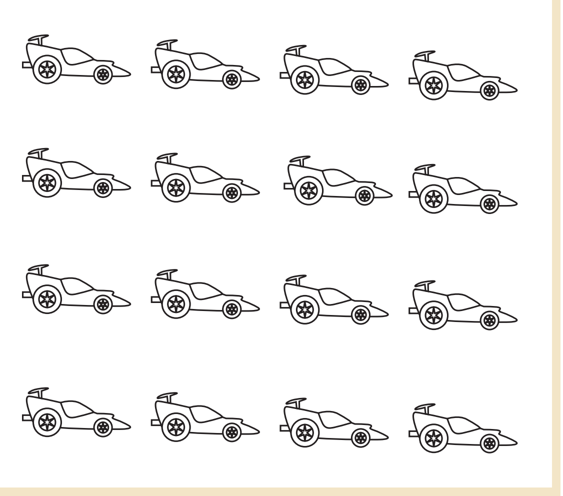


FIGURE 3.2 Open task examples

At first glance, the first task might seem more engaging because it includes actual manipulatives, but in this case, having toy cars (or a cube representing a car) available might lead to a low-level strategy: passing the cars out to see whether there are enough for each child. The second task is more challenging. Students might count the cars. As they count, you can observe their thinking: Do they start at the top and count across the rows? Or do they haphazardly count and miss a car or double-count? Do they count by ones? By twos? By fours? Instead of counting cars, students might count their friends in the class first. Or, students might assign a car to each friend, writing a name on each car. Counting the cars is just one aspect of the task; students must also decide how that number compares to the number of students in the class. Does a student just know that the number of cars is greater or fewer than the number of students? Do they represent each child in the class with a counter and match a counter with a pictured car? Do they look for the two numbers on a hundred chart or number line to compare? Because the second task is more challenging, it offers many more options for how students engage in the task and how they can show a solution, making it an open and worthwhile task for the class.

Figure 3.3(a) provides an open task on probability (grades 7–8).

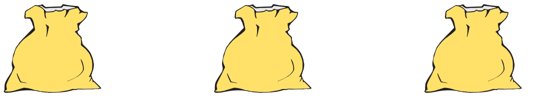
Pause & Reflect

Before studying the solutions in Figure 3.3(b), read the problem, select a strategy, and solve it. ●

Figure 3.3(b) illustrates a range of students' solutions. Student (b) used percentages as a way to compare; student (d) found simplified fractions to compare quantities; student (g) used part–part ratios to reason about the quantities. Solutions (a) and (d) reveal student misconceptions. During a classroom discussion, strategies are shared so that students can make connections among the strategies and to important mathematical ideas (i.e., ratio, fractions, percentages, and probability) and clear up misconceptions (Smith, Bill, & Hughes, 2008).

(a)

Ms. Rhee's mathematics class was studying statistics. She brought in three bags containing red and blue marbles. The three bags were labeled as shown below:



75 red 25 blue	40 red 20 blue	100 red 25 blue
Bag X Total = 100 marbles	Bag Y Total = 60 marbles	Bag Z Total = 125 marbles

Ms. Rhee shook each bag. She asked the class, "If you close your eyes, reach into a bag, and remove 1 marble, which bag would give you the best chance of picking a blue marble?"

Which bag would you choose?

Explain why this bag gives you the best chance of picking a blue marble. You may use the diagram above in your explanation.

(b)

Bag x is $\frac{1}{3}$ blue
Bag y is $\frac{1}{2}$ blue
Bag z is $\frac{1}{4}$ blue
$\frac{1}{2}$ is alot so it must be bag y

a.

I Found the % of blue marbles in each bag.
X $\frac{25}{100} = 25\%$
Y $\frac{20}{60} = 33\frac{1}{3}\%$
Z $\frac{25}{125} = 20\%$

b.

X $\frac{75}{25} = \frac{3}{1} = 3$
y $\frac{40}{20} = \frac{2}{1} = 2$
Z $\frac{100}{25} = \frac{4}{1} = 4$
Since the marbles in bag Z total 125 I think your chances would be higher than the others.

c.

Because bag Y is $\frac{1}{3}$ full of blue marbles and bag X is only $\frac{1}{4}$ full of blue marbles and bag Z is only $\frac{1}{5}$ full of blue marbles.

d.

Bag X is $\frac{1}{4}$ blue and Bag y is $\frac{1}{3}$ blue
better chance Bag y
Bag y has 1 blue to 2 reds and
Bag Z has 1 blue to 4 red
better chance bag y

e.

The X bag has 75 red and 25 blue
there are 50 extra marbles that are red
The Z bag has 100 red and 25 blue
there are 75 extra red than blue
Now Bag X has 40 red a 20 blue
there a 20 extra red than blue.

f.

Notice in the first bag there are 75 red & 25 blue that is a 1:3 chance
Notice the second bag there are 40 red 20 blue that is a 1:2 chance
Notice the third: bag there are 100 red 25 blue that is a 1:4 of a chance.
This shows that in bag y you would be likely to pick a blue marble.

g.

Bag X as 75 red and 25 blues
and bag Z as 100 red and 25 blues
in bags X and Z the blues are the same, so then you would haft look at the red to see which is the last between them, and bag X as 75 red and 75 is lease than 100, so I chose Bag X.

h.

FIGURE 3.3 The best chance of blue: An open task for middle school

Source: Smith, M.S., Bill, V., & Hughes, E. K. (2008). Thinking through a lesson: Successfully implementing high-level tasks. *Mathematics Teaching in the Middle School, 14*(3), 132–138. Reprinted with permission.

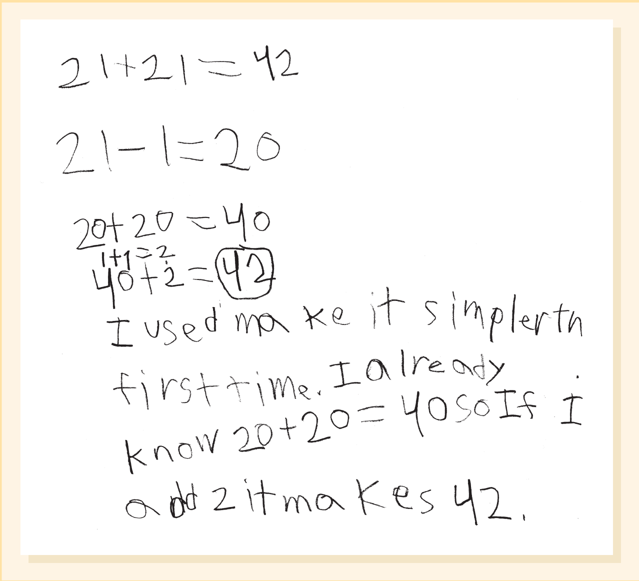
Use Relevant Contexts. A positive feature of a task is when the problem gets students excited about doing mathematics. Compare the following tasks on multiplying 2 two-digit numbers. Which one do you think would be more relevant to fourth-grade students?

Classroom A: "Today we are going to use grid paper to show the sub-products when we multiply 2 two-digit numbers."

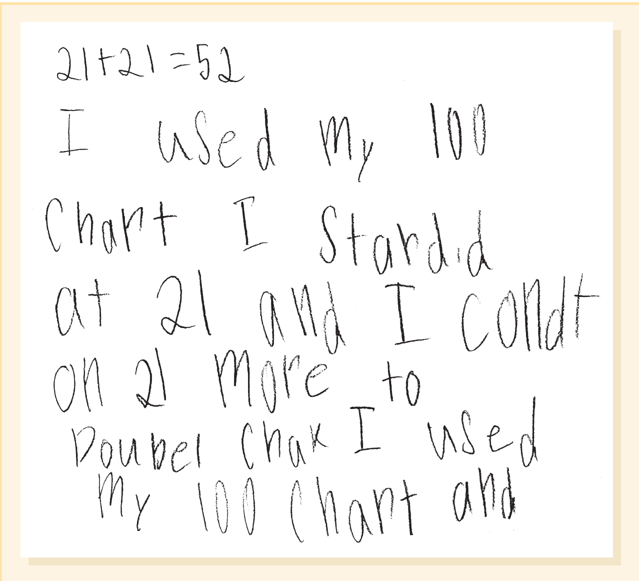
Classroom B: "The school is planning a fall festival and the class is selling water. The principal said we have 14 cases of bottled water in the storage closet. I looked in the storage closet and could see that a case had 7 rows, with 5 bottles in each row. How can we use the information about one case to figure out how many bottles of water we have? If we sell each bottle for $2.00, how much money might we make?"

Contexts must reflect the cultures and interests of the students in your classroom (the focus of Chapter 6). Here we briefly share two ways to incorporate relevant contexts—using literature and connecting to other disciplines.

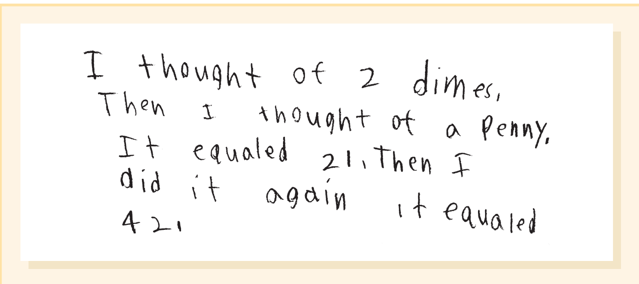
1. Use Literature. Children's literature and young adolescent literature are rich sources of problems. Picture books, poems, media, and chapter books can be used to create cognitively demanding, open tasks. As you read this book, look for "Literature Connections" at the end of every chapter in Part II. Here is one example: Two of Everything (Hong, 1993), a Chinese folktale, involves finding a pot that doubles whatever is put into it. (Imagine where the story goes when Mrs. Haktak falls in the pot!) Like many stories, read-alouds can be found on YouTube (e.g., https://youtu.be/JML_7tsqlmU). Students can explore the following open task: How many students would be in our class if our whole class fell in the Magic Pot? Figure 3.4 illustrates different second-grade solutions.



Robbie adds tens and ones to solve.



Kylee uses a hundred chart and counts on.



Benjamin uses the context of money to combine.

FIGURE 3.4 Three solutions to doubling a class of 21 children

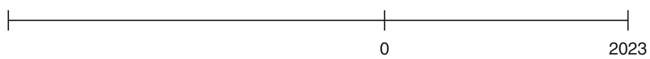
Many books are written to focus on numeracy, but it is also important to see mathematical reasoning opportunities in favorite children's books commonly used in reading aloud (Linder & Bennett, 2020). Exploring the mathematics present in stories that are not explicitly mathematics-oriented helps students see their world mathematically. Another great way to use literature is to explore nonfiction (picture books, chapter books, newspapers, magazines, and the web), which truly is real-world problem solving. There are books of lists (e.g., *The Curious Book of Lists* by Turner & Selmes, 2019) and world-record books, for example, which provide many great contexts for exploring the globe and comparing world data to your class (see Bay-Williams & Martinie, 2009 and Petersen, 2004 for lessons). Epic! (https://www.getepic.com) is a free service for teachers that contains many collections of online books, videos, audiobooks, and so on. It can be searched or filtered by grade, subject, topic, Lexile, Accelerated Reading level, Developmental Reading level, Grade Level reading level, and so on. (Families can also subscribe.)

2. Connect to Other Disciplines. Interdisciplinary lessons help students see connections among the courses and topics they are studying, which may feel completely separate to them. The Mathematical Practices are closely connected to the Science and Engineering Practices, illuminating how closely related these two fields are (de Araujo, Hanuscin, & Otten, 2020). In kindergarten, students can link their study of natural systems in science to mathematics by sorting leaves based on color, smooth or jagged edges, feel of the leaf, and shape. Students learn about rules for sorting and can use Venn diagrams to keep track of their sorts. They can observe and analyze what is common and different in leaves from different trees. Older students can find the perimeter and area of various types of leaves and learn about why these perimeters and areas differ.

The social studies curriculum is rich with opportunities to do mathematics. Timelines of historic events are excellent opportunities for students to work on the relative sizes of numbers and to make better sense of history. Dates lend to creating number lines. For example, the dates for the seven wonders of the ancient world can be used to explore negative numbers.

Where on the number line do these wonders of the ancient world go? Can you find the other wonders and place them on the number line?

Hanging Gardens of Babylon: About 600 B.C.
Statue of Zeus: 435 B.C.
Great Pyramids: About 2500 B.C.
Lighthouse of Alexandria: About 250 B.C.



Evaluating and Adapting Tasks

Throughout this book, in student textbooks, on the Internet, at workshops and webinars you attend, and in articles you read, you will find suggestions for tasks that someone believes

Worthwhile Task Evaluation Guide	
Task Potential	Try it and ask . . . ❏ What mathematical goals does the task address (and are they aligned with what you are seeking)? ❏ What key concepts and/or misconceptions might this task elicit? ❏ What is problematic about the task?
Problem-Solving Strategies	Will the task elicit more than one problem-solving strategy? Which strategies are possible? ❏ Model the Problem ❏ Make an Organized List ❏ Predict and Check for Reasonableness ❏ Make a Simpler Problem ❏ Look for a Pattern ❏ Prove a Conjecture ❏ Write an Equation
Features	To what extent does the task have these key features? ❏ High Cognitive Demand ❏ Open ❏ Relevant Context(s)

FIGURE 3.5 A tool to evaluate a task

are effective for teaching a particular mathematics concept. Yet a large quantity of what is readily available falls short when measured against the standards of being cognitively demanding, open, and relevant to students. Beware of the low-level cognitive-demand tasks cloaked in attractive artwork—they may look fun, but if the mathematics is not problematic, the tasks are not worthwhile. Make sure it is the mathematics itself that is clever and engaging.

An intentional selection process can help you pinpoint high-quality tasks, such as applying the Task Evaluation Guide in Figure 3.5, which includes the ideas discussed in this chapter. All boxes do not need to be checked. A task may include a number of problem-solving strategies but fall short in terms of relevance for students. In this case, you may decide to change the context. Or the task is complete with the features, but it does not match your mathematical goals for the lesson; hence, you don't use it.

Let's explore how this guide can help with selecting a worthwhile task. Imagine you are teaching fourth grade and are seeking a worthwhile task for this goal:

4.G.A.2: Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles. (NGA Center & CCSSO, 2010, p. 32)

You type "classify triangles" into a web search. Hundreds of links and worksheet images appear, like the one in Figure 3.6.

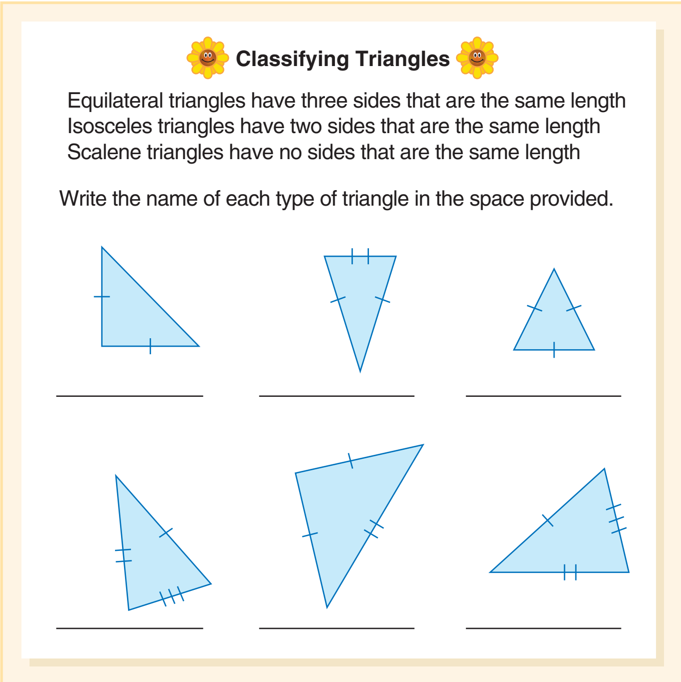


FIGURE 3.6 Example of a categorizing triangles worksheet

Pause & Reflect

How does this task measure up on the Task Evaluation Guide? How might you adapt the task so that it rates higher on some of these measures of a worthwhile task? ●

The worksheet in Figure 3.6 appears to match the learning goals, but it is not cognitively demanding, not open, or has no context. Here are some ways you might adapt it:

1. Remove the tick marks telling which sides are the same length; distribute just the set of triangles, letter them, and ask students to write similarities and differences between pairs of triangles. This opens the task and requires a higher level of cognitive demand.
2. Only use the list of terms with definitions at the top and ask students to identify examples of each in the room or in a picture book.
3. Cut out the triangles; ask students to create piles of what they consider the same triangles and give names to their groups. In a later discussion, different possible ways to sort triangles can be discussed, and appropriate terminology can be reinforced.

Extensions to make this task stronger could be added, such as asking students, "Can you build two triangles of different sizes that are both isosceles?" "Can you create a triangle with three obtuse angles? Why or why not?" "If a triangle is classified as a right triangle, then which classifications for sides are possible or impossible?" Each of these adaptations takes very little time and greatly increases the quality of the task.

Certainly, computation tasks (i.e., worksheets) are often devoid of the features of worthwhile tasks, but they must have these features for students to develop procedural fluency. Here are some suggestions for adapting low-quality computation tasks (adapted from Boaler & Sengupta-Irving, 2016):

1. Allow multiple *methods*: Explicitly ask students to use multiple methods, strategies, and representations to solve.
2. Make it an exploration: Change the task so there is more to it than a single computation. For example, rather than ask kindergartners to add $3 + 5$, ask them to find numbers they can add to equal the number 8.
3. Add a visual requirement: Visualization enhances understanding. Students can use two different manipulatives to justify their solution, or show how an equation fits a drawing of a story situation.
4. Reason and convince: Require students to create convincing mathematical arguments and expect the same from their peers. Ask them to be skeptics and ask clarifying questions of each other. (Modeling the use of logical arguments and follow-up questioning is important to be sure students focus on supporting each other's thinking.)
5. Postpone teaching a standard method: Begin with students' intuition about how to solve a problem type before learning about conventional methods. Delay the teaching of algorithms until other strategies are learned.

Just as students become adept at problem-solving strategies, with time and commitment you will become adept at evaluating and adapting tasks to support student learning better.

Procedural Fluency from Conceptual Understanding

Learning Outcome 3.4 **Illustrate the relationship between procedural fluency and conceptual understanding.**

In Chapter 2, you read that mathematical proficiency includes conceptual understanding *and* procedural proficiency. It is not surprising that effective teaching involves making connections between concepts and procedures. We illustrate this by exploring five tasks.

Tasks That Build Procedural Fluency from Conceptual Understanding

Try out each of the tasks in this section. Ask yourself, "What procedures am I exploring? What concepts am I exploring? How are connections made between concepts and procedures?"

Topic 1: Partitioning **Grades: K–1**
Task: Bowls on Two Tables
Six bowls of cereal are placed on two tables. Draw a picture to show a way that six bowls might be placed on two tables. Write the equation that matches your picture. Find more ways and write the equations. Finally, tell how many ways you think there are to place six bowls on two tables.

In kindergarten or grade 1, contexts are necessary, because they are concrete ways for students to think about the mathematical relationships. Mathematically, students may determine one or two ways to decompose 6 or may find all the ways (open). Students can share how they thought about the problem and what patterns they noticed as they found new combinations (high cognitive demand). For example, a student might note that as one table gets a bowl, the other table loses a bowl, so one bowl moves over. They may notice that there are then seven possible ways. You can extend student thinking, asking, "How do you know you have them all?" The task can also be adapted to other totals and other contexts, such as, "How many ways can you put 10 toys into 2 baskets?" or "If there are 10 bowls, would there be 11 possible ways?" Students begin to notice that numbers can be taken apart and put back together in different ways (conceptual understanding). They also begin to learn about addition by recording number sentences such as $6 = 5 + 1$ (procedural knowledge), discussing the meaning of the symbols, and showing how the numbers represent the situation (connecting concepts and procedures).

Topic 2: Adding Two-Digit Whole Numbers **Grades: 1–2**
Select a strategy to solve $48 + 25$. Illustrate or explain your strategy.

Even though there is no story or situation to resolve, this task can be worthwhile. (Check the list of five suggestions in the previous section.) In this case, students were asked to choose a method. The teacher recorded selected student thinking on the board:

Sam	Erin	Jeremiah	Leah
$40 + 20 = 60$ $8 + 5 = 13$ $60 + 13 = 73$	$50 + 25 = 75$ $75 + 2 - 73$	$50 + 23 = 73$	$48 + 20 = 68$ $68 + 5 = 73$

As students share their solution pathways, they are deepening their conceptual understanding of place value and properties, they are learning procedures, and they are developing a foundation for learning the standard algorithm. All operations for whole numbers, decimals, fractions, and integers can be explored in a similar manner.

Topic 3: Area of a Rectangle **Grades: 3–4**
Task: Tiling Book Covers
Find the area of the cover of your math book by covering it with color tiles. Repeat for the areas of books of various sizes. What patterns do you notice in covering the book? Is this pattern or rule for covering any rectangle? Select 3 other books to measure. For each, predict and measure the number of tiles required to cover it, and reflect on the patterns you notice.

As students begin to cover surfaces, they may run out of tiles, or they may just get tired of placing tiles. They notice that each row has the same number of tiles, so they just need to know how many rows and columns will cover the book and they can skip count or multiply to find the total number of tiles. This investigation develops the concept of area, strengthens understanding of multiplication as equal rows, and leads to the procedure for area of a rectangle.

Topic 4: Division of Fractions **Grades: 5–7**
Task: Making Scarves
Anthony is knitting scarves for gifts for his three sisters. Each scarf is one yard long, and he can knit $\frac{1}{4}$ of a scarf each day. How long will it take him to make the scarves?

When students explore this task without the "division of fractions" label, they can approach the problem in multiple ways. Leah, Kelly, Jaden, and MacKenna solved the task by applying what they knew, including skip counting by fourths (Leah), measuring equivalencies (Kelly), and figuring ratios of yards to days (Jaden) and rates of days per yards (MacKenna). To extend their thinking about scarf-making, they were asked, "What would happen if Anthony decided to make $\frac{3}{4}$ of a scarf in one day?" A review of their second task shows that they used both their strategies and their answers from the first task in interesting ways. After more experiences with scarf-making at other rates, students can begin to connect to the concept of division involving fractions. Series of related tasks can help students connect concepts and procedures. The scarf length and the amount completed per day can vary to help students look for patterns across the problem set.

Topic 5: Ratios and Proportions **Grades: 6–8**
Task: First to the Water
Jack and Jill were at the same location at the bottom of a hill, hoping to fetch a pail of water. They both begin walking up the hill, Jack walking 5 yards every 25 seconds and Jill walking 3 yards every 10 seconds. Assuming a constant walking rate, who will get to the water first?

Students can engage with this task in a variety of ways. They can represent the problem visually with jumps on a number line or symbolically by using a rate approach (determining the number of yards per second for each person). By specializing and considering several examples, students will begin to generalize a procedure for comparing ratios, which is the essence of proportional reasoning.

What about Drill and Practice?

The phrase "drill and practice" slips off the tongue so rapidly that the two words, *drill* and *practice*, appear to be synonyms—and, for the most part, they have been. In the interest of developing a new perspective on drill and practice, consider definitions that differentiate between these terms as different types of activities.

> *Drill* refers to repetitive exercises designed to replicate a procedure or algorithm.
>
> *Practice* refers to varied tasks or experiences focused on a particular concept or procedure.

Drill. Finding pages for drill is easy; there is an endless supply of such worksheets on the Internet. But where has this drill gotten us? Drill has created generations of people who don't remember what they learned, do not like mathematics, and purposefully do not pursue professions that involve mathematics. Drill may improve procedural knowledge but not conceptual understanding, but when the number of problems is reduced and time is then spent discussing problems, both procedural and conceptual knowledge are supported (Franke, Kazemi, & Battey, 2007).

Practice. High-quality practice is not as easy to find. Such practice gives students opportunities to solidify concepts, explore alternative and flexible strategies for solving problems, and make connections between concepts and procedures. Using worthwhile tasks as practice sends a clear message that mathematics is about figuring things out. Routines, games, and activities, like those found throughout this book, can engage students in meaningful, enjoyable practice.

Facilitating Mathematical Discourse

Learning Outcome 3.5 Describe and relate the Mathematics Teaching Practices related to facilitating mathematical discourse.

With the first three Mathematics Teaching Practices implemented effectively, the focus of teaching turns to classroom discourse. To help students become productive mathematical thinkers, teachers must ask key questions, be able to respond with targeted feedback to students, probe student thinking, prompt students to reflect on their thinking, be comfortable with uncertainty, and know the difference between productive and unproductive struggle (Kazemi & Hintz, 2014; NCTM, 2014b). Teachers themselves must display productive dispositions, showing students they are willing to explore, experiment, and make conjectures; recognize multiple solution pathways; make connections among strategies; and monitor and reflect on their work. The goal of productive discourse is to keep the cognitive demand high while students are learning and formalizing mathematical concepts. Note that the purpose is not for students to tell their answers and get validation from the teacher. Orchestrating classroom discourse is so important and complex that it incorporates five of the eight Teaching Practices (see Figure 3.1).

A lot of research connects mathematical discourse to student learning (e.g., Kazemi & Hintz, 2014; Staples & King, 2017; Wood, Williams, & McNeal, 2006). Suggestions from these studies include:

- Encourage student–student dialogue rather than student–teacher conversations that exclude the rest of the class. When students have differing solutions, have them work these ideas out as a class. “George, I noticed that you got a different answer than Tomeka did. What do you think about her explanation?”
- Encourage students to ask questions. “Pete, did you understand how they did that? Do you want to ask Antonio a question?”
- Ask follow-up questions whether the answer is right or wrong. Your role is to understand student thinking, not to lead students to the correct answer. So follow up with probes to learn more about their answers. Sometimes you will find that what you assumed they were thinking is not accurate. And if you only follow up on wrong answers, students quickly figure this out and get nervous when you ask them to explain their thinking.
- Call on students in such a way that, over time, all students are able to participate. Use times when students are working in small groups to identify interesting solutions that you will highlight during the sharing portion of the lesson. Be intentional about the order in which the solutions are shared; for example, select two that you would like to compare presented back to back.
- Demonstrate to students that it is okay to be confused and that asking clarifying questions is expected. Tell them that this confusion, or disequilibrium, just means they are engaged in doing real mathematics and is an indication they are learning.