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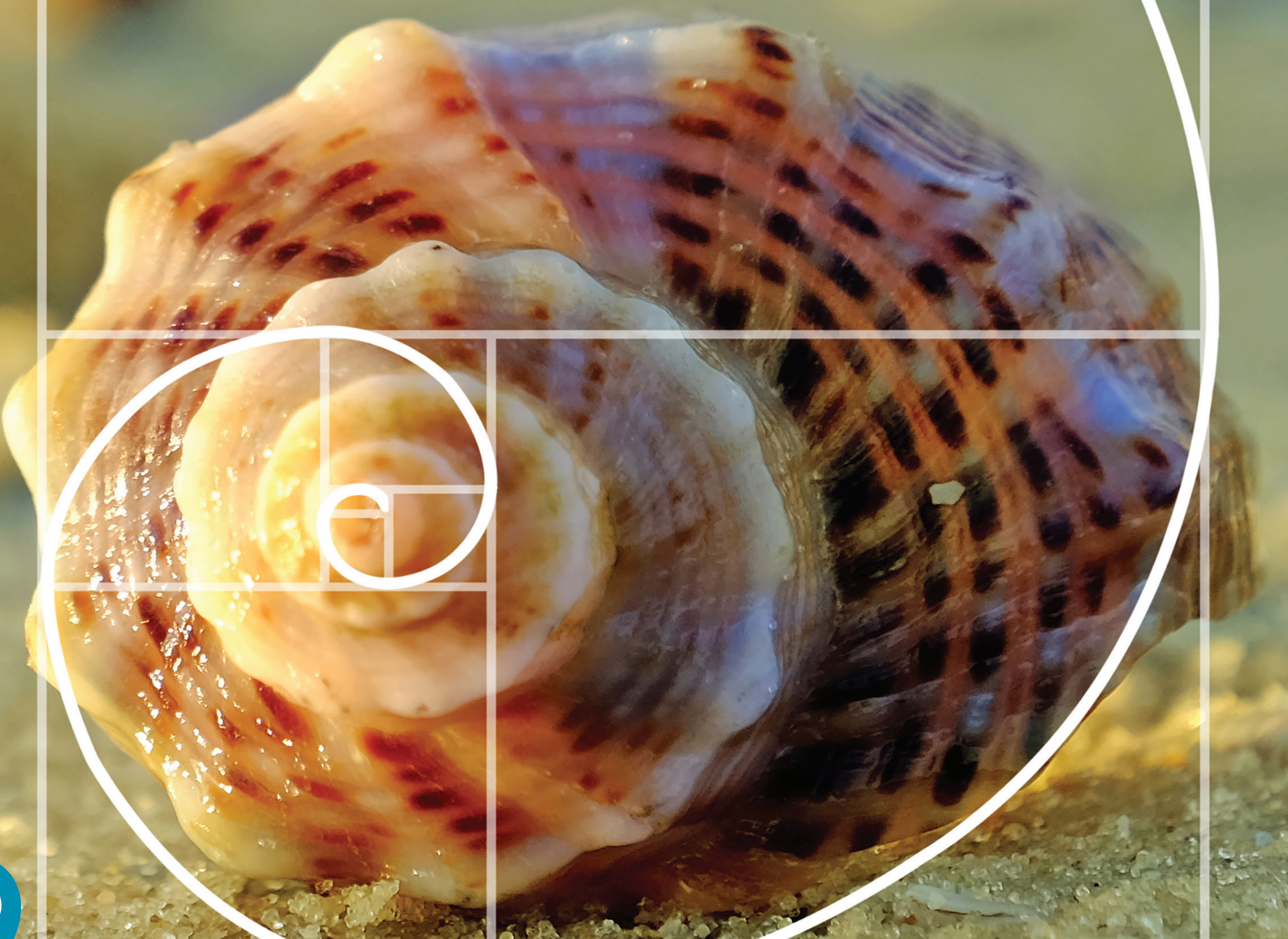
INTRODUCTORY MATHEMATICAL ANALYSIS

FOR BUSINESS, ECONOMICS, AND THE LIFE AND SOCIAL SCIENCES



INTRODUCTORY MATHEMATICAL ANALYSIS

FOR BUSINESS, ECONOMICS, AND THE LIFE AND SOCIAL SCIENCES

FOURTEENTH EDITION
GLOBAL EDITION

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Preface

The fourteenth edition of *Introductory Mathematical Analysis for Business, Economics, and the Life and Social Sciences (IMA)* continues to provide a mathematical foundation for students in a variety of fields and majors, as suggested by the title. As begun in the thirteenth edition, the book has three parts: College Algebra, Chapters 0–4; Finite Mathematics, Chapters 5–9; and Calculus, Chapters 10–17.

Schools that have two academic terms per year tend to give Business students a term devoted to Finite Mathematics and a term devoted to Calculus. For these schools we recommend Chapters 0 through 9 for the first course, starting wherever the preparation of the students allows, and Chapters 10 through 17 for the second, including as much as the students' background allows and their needs dictate.

For schools with three quarter or three semester courses per year there are a number of possible uses for this book. If their program allows three quarters of Mathematics, well-prepared Business students can start a first course on Finite Mathematics with Chapter 1 and proceed through topics of interest up to and including Chapter 9. In this scenario, a second course on Differential Calculus could start with Chapter 10 on Limits and Continuity, followed by the three "differentiation chapters", 11 through 13 inclusive. Here, Section 12.6 on Newton's Method can be omitted without loss of continuity, while some instructors may prefer to review Chapter 4 on Exponential and Logarithmic Functions prior to studying them as differentiable functions. Finally, a third course could comprise Chapters 14 through 17 on Integral Calculus with an introduction to Multivariable Calculus. Note that Chapter 16 is certainly not needed for Chapter 17 and Section 15.8 on Improper Integrals can be safely omitted if Chapter 16 is not covered.

Approach

Introductory Mathematical Analysis for Business, Economics, and the Life and Social Sciences (IMA) takes a unique approach to problem solving. As has been the case in earlier editions of this book, we establish an emphasis on algebraic calculations that sets this text apart from other introductory, applied mathematics books. The process of calculating with variables builds skill in mathematical modeling and paves the way for students to use calculus. The reader will not find a "definition-theorem-proof" treatment, but there is a sustained effort to impart a genuine mathematical treatment of applied problems. In particular, our guiding philosophy leads us to include informal proofs and general calculations that shed light on how the corresponding calculations are done in applied problems. Emphasis on developing algebraic skills is extended to the exercises, of which many, even those of the drill type, are given with general rather than numerical coefficients.

We have refined the organization of our book over many editions to present the content in very manageable portions for optimal teaching and learning. Inevitably, that process tends to put "weight" on a book, and the present edition makes a very concerted effort to pare the book back somewhat, both with respect to design features—making for a cleaner approach—and content—recognizing changing pedagogical needs.

Changes for the Fourteenth Edition

We continue to make the elementary notions in the early chapters pave the way for their use in more advanced topics. For example, while discussing factoring, a topic many students find somewhat arcane, we point out that the principle "$ab = 0$ implies $a = 0$ or $b = 0$", together with factoring, enables the splitting of some complicated equations into several simpler equations. We point out that percentages are just rescaled numbers via the "equation" $p\% = \frac{p}{100}$ so that, in calculus, "relative rate of change" and "percentage rate of change" are related by the "equation" $r = r \cdot 100\%$. We think that at this time, when negative interest rates are often discussed, even if seldom implemented, it is wise to be absolutely precise about simple notions that are often taken for granted. In fact, in the

Finance, Chapter 5, we explicitly discuss negative interest rates and ask, somewhat rhetorically, why banks do not use continuous compounding (given that for a long time now continuous compounding has been able to simplify calculations *in practice* as well as in theory).

Whenever possible, we have tried to incorporate the extra ideas that were in the "Explore and Extend" chapter-closers into the body of the text. For example, the functions tax rate $t(i)$ and tax paid $T(i)$ of income i, are seen for what they are: everyday examples of case-defined functions. We think that in the process of learning about polynomials it is helpful to include Horner's Method for their evaluation, since with even a simple calculator at hand this makes the calculation much faster. While doing linear programming, it sometimes helps to think of lines and planes, etcetera, in terms of intercepts alone, so we include an exercise to show that if a line has (nonzero) intercepts x_0 and y_0 then its equation is given by

$$\frac{x}{x_0} + \frac{y}{y_0} = 1$$

and, moreover, (for positive x_0 and y_0) we ask for a geometric interpretation of the equivalent equation $y_0x + x_0y = x_0y_0$.

But, turning to our "paring" of the previous *IMA*, let us begin with Linear Programming. This is surely one of the most important topics in the book for Business students. We now feel that, while students should know about the possibility of *Multiple Optimum Solutions* and *Degeneracy and Unbounded Solutions*, they do not have enough time to devote an entire, albeit short, section to each of these. The remaining sections of Chapter 7 are already demanding and we now content ourselves with providing simple alerts to these possibilities that are easily seen geometrically. (The deleted sections were always tagged as "omittable".)

We think further that, in Integral Calculus, it is far more important for Applied Mathematics students to be adept at using tables to evaluate integrals than to know about *Integration by Parts* and *Partial Fractions*. In fact, these topics, of endless joy to some as recreational problems, do not seem to fit well into the general scheme of serious problem solving. It is a fact of life that an elementary function (in the technical sense) can easily fail to have an elementary antiderivative, and it seems to us that *Parts* does not go far enough to rescue this difficulty to warrant the considerable time it takes to master the technique. Since *Partial Fractions* ultimately lead to elementary antiderivatives for all *rational* functions, they *are* part of serious problem solving and a better case can be made for their inclusion in an applied textbook. However, it is vainglorious to do so without the inverse tangent function at hand and, by longstanding tacit agreement, applied calculus books do not venture into trigonometry.

After deleting the sections mentioned above, we reorganized the remaining material of the "integration chapters", 14 and 15, to rebalance them. The first concludes with the Fundamental Theorem of Calculus while the second is more properly "applied". We think that the formerly daunting Chapter 17 has benefited from deletion of *Implicit Partial Differentiation*, the *Chain Rule* for partial differentiation, and *Lines of Regression*. Since Multivariable Calculus is extremely important for Applied Mathematics, we hope that this more manageable chapter will encourage instructors to include it in their syllabi.

Examples and Exercises

Most instructors and students will agree that the key to an effective textbook is in the quality and quantity of the examples and exercise sets. To that end, more than 850 examples are worked out in detail. Some of these examples include a *strategy* box designed to guide students through the general steps of the solution before the specific solution is obtained. (See, for example, Section 14.3 Example 4.) In addition, an abundant number of diagrams (almost 500) and exercises (more than 5000) are included. Of the exercises, approximately 20 percent have been either updated or written completely anew. In each exercise set, grouped problems are usually given in increasing order of difficulty. In most exercise sets the problems progress from the basic mechanical drill-type to more interesting thought-provoking problems. The exercises labeled with a coloured exercise number correlate to a "Now Work Problem N" statement and example in the section.

Based on the feedback we have received from users of this text, the diversity of the applications provided in both the exercise sets and examples is truly an asset of this book. Many real applied problems with accurate data are included. Students do not need to look hard to see how the mathematics they are learning is applied to everyday or work-related situations. A great deal of effort has been put into producing a proper balance between drill-type exercises and problems requiring the integration and application of the concepts learned.

Pedagogy and Hallmark Features

- **Applications:** An abundance and variety of applications for the intended audience appear throughout the book so that students see frequently how the mathematics they are learning can be used. These applications cover such diverse areas as business, economics, biology, medicine, sociology, psychology, ecology, statistics, earth science, and archaeology. Many of these applications are drawn from literature and are documented by references, sometimes from the Web. In some, the background and context are given in order to stimulate interest. However, the text is self-contained, in the sense that it assumes no prior exposure to the concepts on which the applications are based. (See, for example, Chapter 15, Section 7, Example 2.)
- **Now Work Problem N:** Throughout the text we have retained the popular *Now Work Problem N* feature. The idea is that after a worked example, students are directed to an end-of-section problem (labeled with a colored exercise number) that reinforces the ideas of the worked example. This gives students an opportunity to practice what they have just learned. Because the majority of these keyed exercises are odd-numbered, students can immediately check their answer in the back of the book to assess their level of understanding.
- **Cautions:** Cautionary warnings are presented in very much the same way an instructor would warn students in class of commonly made errors. These appear in the margin, along with other explanatory notes and emphases.
- **Definitions, key concepts, and important rules and formulas:** These are clearly stated and displayed as a way to make the navigation of the book that much easier for the student. (See, for example, the Definition of Derivative in Section 11.1.)
- **Review material:** Each chapter has a review section that contains a list of important terms and symbols, a chapter summary, and numerous review problems. In addition, key examples are referenced along with each group of important terms and symbols.
- **Inequalities and slack variables:** In Section 1.2, when inequalities are introduced we point out that $a \leq b$ is equivalent to "there exists a non-negative number, s, such that $a + s = b$". The idea is not deep but the pedagogical point is that *slack variables*, key to implementing the simplex algorithm in Chapter 7, should be familiar and not distract from the rather technical material in linear programming.
- **Absolute value:** It is common to note that $|a - b|$ provides the distance from a to b. In Example 4e of Section 1.4 we point out that "x is less than σ units from μ" translates as $|x - \mu| < \sigma$. In Section 1.4 this is but an exercise with the notation, as it should be, but the point here is that later (in Chapter 9) μ will be the mean and σ the standard deviation of a random variable. Again we have separated, in advance, a simple idea from a more advanced one. Of course, Problem 12 of Problems 1.4, which asks the student to set up $|f(x) - L| < \epsilon$, has a similar agenda to Chapter 10 on limits.
- **Early treatment of summation notation:** This topic is necessary for study of the definite integral in Chapter 14, but it is *useful* long before that. Since it is a notation that is new to most students at this level, but no more than a notation, we get it out of the way in Chapter 1. By using it when convenient, *before coverage of the definite integral*, it is not a distraction from that challenging concept.

- **Section 1.6 on sequences:** This section provides several pedagogical advantages. The very definition is stated in a fashion that paves the way for the more important and more basic definition of function in Chapter 2. In summing the terms of a sequence we are able to practice the use of summation notation introduced in the preceding section. The most obvious benefit though is that "sequences" allows us a better organization in the annuities section of Chapter 5. Both the present and the future values of an annuity are obtained by summing (finite) geometric sequences. Later in the text, sequences arise in the definition of the number e in Chapter 4, in Markov chains in Chapter 9, and in Newton's method in Chapter 12, so that a helpful unifying reference is obtained.
- **Sum of an infinite sequence:** In the course of summing the terms of a finite sequence, it is natural to raise the possibility of summing the terms of an infinite sequence. This is a nonthreatening environment in which to provide a first foray into the world of limits. We simply explain how certain infinite geometric sequences have well-defined sums and phrase the results in a way that creates a toehold for the introduction of limits in Chapter 10. These particular infinite sums enable us to introduce the idea of a perpetuity, first informally in the sequence section, and then again in more detail in a separate section in Chapter 5.
- **Section 2.8, Functions of Several Variables:** The introduction to functions of several variables appears in Chapter 2 because it is a topic that should appear long before Calculus. Once we have done some calculus there are particular ways to use calculus in the study of functions of several variables, but these aspects should not be confused with the basics that we use throughout the book. For example, "a-sub-n-angle-r" and "s-sub-n-angle-r" studied in the Mathematics of Finance, Chapter 5, are perfectly good functions of two variables, and Linear Programming seeks to optimize linear functions of several variables subject to linear constraints.
- **Leontief's input-output analysis in Section 6.7:** In this section we have separated various aspects of the total problem. We begin by describing what we call the Leontief matrix A as an encoding of the input and output relationships between sectors of an economy. Since this matrix can often be assumed to be constant for a substantial period of time, we begin by assuming that A is a given. The simpler problem is then to determine the production, X, which is required to meet an external demand, D, for an economy whose Leontief matrix is A. We provide a careful account of this as the solution of $(I-A)X = D$. Since A can be assumed to be fixed while various demands, D, are investigated, there is *some* justification to compute $(I-A)^{-1}$ so that we have $X = (I-A)^{-1}D$. However, use of a matrix inverse should not be considered an essential part of the solution. Finally, we explain how the Leontief matrix can be found from a table of data that might be available to a planner.
- **Birthday probability in Section 8.4:** This is a treatment of the classic problem of determining the probability that at least 2 of n people have their birthday on the same day. While this problem is given as an example in many texts, the recursive formula that we give for calculating the probability as a function of n is not a common feature. It is reasonable to include it in this book because recursively defined sequences appear explicitly in Section 1.6.
- **Markov Chains:** We noticed that considerable simplification of the problem of finding steady state vectors is obtained by writing state vectors as columns rather than rows. This does necessitate that a transition matrix $\mathbf{T} = [t_{ij}]$ have t_{ij} ="probability that next state is i given that current state is j" but avoids several artificial transpositions.
- **Sign Charts for a function in Chapter 10:** The sign charts that we introduced in the 12th edition now make their appearance in Chapter 10. Our point is that these charts can be made for any real-valued function of a real variable and their help in graphing a function begins prior to the introduction of derivatives. Of course we continue to exploit their use in Chapter 13 "Curve Sketching" where, for each function f, we advocate making a sign chart for each of f, f', and f'', interpreted for f itself. When this is possible, the graph of the function becomes almost self-evident. We freely acknowledge that this is a blackboard technique used by many instructors, but it appears too rarely in textbooks.

Supplements

- **MyLab Math Online Course (access code required)** Built around Pearson's best-selling content, MyLab™ Math, is an online homework, tutorial, and assessment program designed to work with this text to engage students and improve results. MyLab Math can be successfully implemented in any classroom environment—lab-based, hybrid, fully online, or traditional. By addressing instructor and student needs, MyLab Math improves student learning. Used by more than 37 million students worldwide, MyLab Math delivers consistent, measurable gains in student learning outcomes, retention and subsequent course success. Visit www.mymathlab.com/results to learn more.

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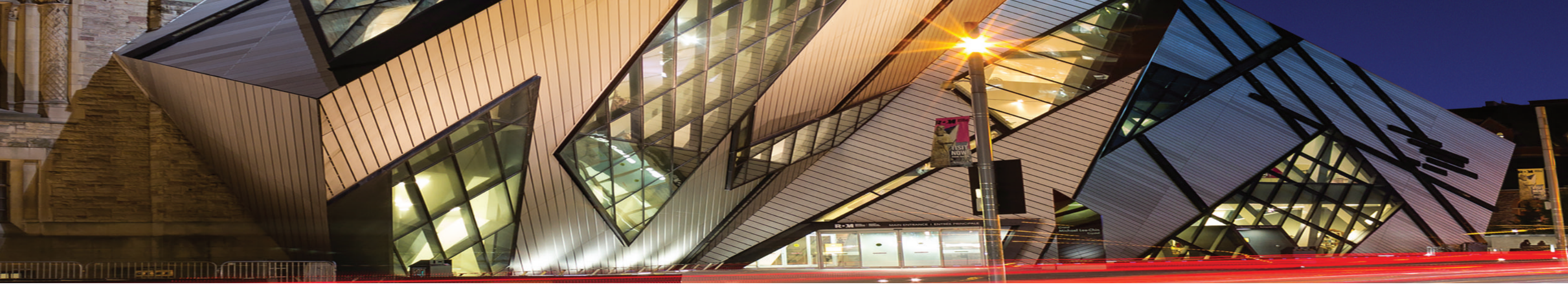


0 Review of Algebra

Lesley Griffith worked for a yacht supply company in Antibes, France. Often, she needed to examine receipts in which only the total paid was reported and then determine the amount of the total which was French “value-added tax”. It is known as TVA for “Taxe à la Value Ajouté”. The French TVA rate was 19.6% (but in January of 2014 it increased to 20%). A lot of Lesley’s business came from Italian suppliers and purchasers, so she also had to deal with the similar problem of receipts containing Italian sales tax at 18% (now 22%).

A problem of this kind demands a formula, so that the user can just plug in a tax rate like 19.6% or 22% to suit a particular place and time, but many people are able to work through a particular case of the problem, using specified numbers, without knowing the formula. Thus, if Lesley had a 200-Euro French receipt, she might have reasoned as follows: If the item cost 100 Euros before tax, then the receipt total would be for 119.6 Euros with tax of 19.6, so *tax in a receipt total of 200 is to 200 as 19.6 is to 119.6*. Stated mathematically,

$$\frac{\text{tax in } 200}{200} = \frac{19.6}{119.6} \approx 0.164 = 16.4\%$$

If her reasoning is correct then the amount of TVA in a 200-Euro receipt is about 16.4% of 200 Euros, which is 32.8 Euros. In fact, many people will now guess that

$$\text{tax in } R = R\left(\frac{p}{100+p}\right)$$

gives the tax in a receipt R, when the tax rate is $p\%$. Thus, if Lesley felt confident about her deduction, she could have multiplied her Italian receipts by $\frac{18}{118}$ to determine the tax they contained.

Of course, most people do not remember formulas for very long and are uncomfortable basing a monetary calculation on an assumption such as the one we italicized above. There are lots of relationships that are more complicated than simple proportionality! The purpose of this chapter is to review the algebra necessary for you to construct your own formulas, *with confidence,* as needed. In particular, we will derive Lesley’s formula from principles with which everybody is familiar. This usage of algebra will appear throughout the book, in the course of making *general calculations with variable quantities*.

In this chapter we will review real numbers and algebraic expressions and the basic operations on them. The chapter is designed to provide a brief review of some terms and methods of symbolic calculation. Probably, you have seen most of this material before. However, because these topics are important in handling the mathematics that comes later, an immediate second exposure to them may be beneficial. Devote whatever time is necessary to the sections in which you need review.

Objective

To become familiar with sets, in particular sets of real numbers, and the real-number line.

0.1 Sets of Real Numbers

A **set** is a collection of objects. For example, we can speak of the set of even numbers between 5 and 11, namely, 6, 8, and 10. An object in a set is called an **element** of that set. If this sounds a little circular, don't worry. The words *set* and *element* are like *line* and *point* in geometry. We cannot define them in more primitive terms. It is only with practice in using them that we come to understand their meaning. The situation is also rather like the way in which a child learns a first language. Without knowing *any* words, a child infers the meaning of a few very simple words by watching and listening to a parent and ultimately uses these very few words to build a working vocabulary. None of us needs to understand the mechanics of this process in order to learn how to speak. In the same way, it is possible to learn practical mathematics without becoming embroiled in the issue of undefined primitive terms.

One way to specify a set is by listing its elements, in any order, inside braces. For example, the previous set is $\{6, 8, 10\}$, which we could denote by a letter such as A, allowing us to write $A = \{6, 8, 10\}$. Note that $\{8, 10, 6\}$ also denotes the same set, as does $\{6, 8, 10, 10\}$. *A set is determined by its elements*, and neither rearrangements nor repetitions in a listing affect the set. A set A is said to be a subset of a set B if and only if every element of A is also an element of B. For example, if $A = \{6, 8, 10\}$ and $B = \{6, 8, 10, 12\}$, then A is a subset of B but B is not a subset of A. There is exactly one set which contains *no* elements. It is called *the empty set* and is denoted by $\emptyset$.

Certain sets of numbers have special names. The numbers 1, 2, 3, and so on form the set of **positive integers**:

$$\textit{set of positive integers} = \{1, 2, 3, \ldots\}$$

The three dots are an informal way of saying that the listing of elements is unending and the reader is expected to generate as many elements as needed from the pattern.

The positive integers together with 0 and the **negative integers** $-1, -2, -3, \ldots,$ form the set of **integers**:

$$\textit{set of integers} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

The reason for $q \neq 0$ is that we cannot divide by zero.

The set of **rational numbers** consists of numbers, such as $\frac{1}{2}$ and $\frac{5}{3}$, that can be written as a quotient of two integers. That is, a rational number is a number that can be written as $\frac{p}{q}$, where p and q are integers and $q \neq 0$. (The symbol "$\neq$" is read "is not equal to.") For example, the numbers $\frac{19}{20}$, $\frac{-2}{7}$, and $\frac{-6}{-2}$ are rational. We remark that $\frac{2}{4}$, $\frac{1}{2}$, $\frac{3}{6}$, $\frac{-4}{-8}$, 0.5, and 50% all represent the same rational number. The integer 2 is rational, since $2 = \frac{2}{1}$. In fact, every integer is rational.

Every integer is a rational number.

All rational numbers can be represented by decimal numbers that *terminate,* such as $\frac{3}{4} = 0.75$ and $\frac{3}{2} = 1.5$, or by *nonterminating, repeating decimal numbers* (composed of a group of digits that repeats without end), such as $\frac{2}{3} = 0.666\ldots$, $\frac{-4}{11} = -0.3636\ldots$, and $\frac{2}{15} = 0.1333\ldots$. Numbers represented by *nonterminating, nonrepeating* decimals are called **irrational numbers**. An irrational number cannot be written as an integer divided by an integer. The numbers π (pi) and $\sqrt{2}$ are examples of irrational numbers. Together, the rational numbers and the irrational numbers form the set of **real numbers**.

Every rational number is a real number.

The set of real numbers consists of all decimal numbers.

Real numbers can be represented by points on a line. First we choose a point on the line to represent zero. This point is called the *origin*. (See Figure 0.1.) Then a standard measure of distance, called a *unit distance,* is chosen and is successively marked off both to the right and to the left of the origin. With each point on the line we associate a directed distance, which depends on the position of the point with respect to the origin.

Some Points and Their Coordinates

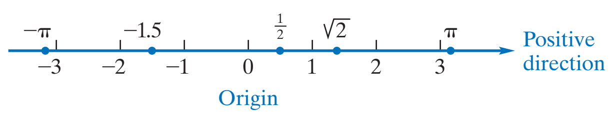


FIGURE 0.1 The real-number line.

Positions to the right of the origin are considered positive (+) and positions to the left are negative (−). For example, with the point $\frac{1}{2}$ unit to the right of the origin there corresponds the number $\frac{1}{2}$, which is called the **coordinate** of that point. Similarly, the coordinate of the point 1.5 units to the left of the origin is −1.5. In Figure 0.1, the coordinates of some points are marked. The arrowhead indicates that the direction to the right along the line is considered the positive direction.

To each point on the line there corresponds a unique real number, and to each real number there corresponds a unique point on the line. There is a *one-to-one correspondence* between points on the line and real numbers. We call such a line, with coordinates marked, a **real-number line**. We feel free to treat real numbers as points on a real-number line and vice versa.

EXAMPLE 1 Identifying Kinds of Real Numbers

Is it true that 0.151515 . . . is an irrational number?

Solution: The dots in 0.151515 . . . are understood to convey repetition of the digit string “15”. Irrational numbers were defined to be real numbers that are represented by a *nonterminating, nonrepeating* decimal, so 0.151515 . . . is not irrational. It is therefore a rational number. It is not immediately clear how to represent 0.151515 . . . as a quotient of integers. In Chapter 1 we will learn how to show that $0.151515\ldots = \frac{5}{33}$. You can check that this is *plausible* by entering $5 \div 33$ on a calculator, but you should also think about why the calculator exercise does not *prove* that $0.151515\ldots = \frac{5}{33}$.

Now Work Problem 7 ◁

PROBLEMS 0.1

In Problems 1–12, determine the truth of each statement. If the statement is false, give a reason why that is so.

1. $\sqrt{-13}$ is an integer.

2. $\frac{-2}{7}$ is rational.

3. −3 is a positive integer.

4. 0 is not rational.

5. $\sqrt{3}$ is rational.

6. $\frac{-1}{0}$ is a rational number.

7. $\sqrt{25}$ is not a positive integer.

8. $\sqrt{2}$ is a real number.

9. $\frac{0}{0}$ is rational.

10. π is a positive integer.

11. 0 is to the right of $-\sqrt{2}$ on the real-number line.

12. Every integer is positive or negative.

13. Every terminating decimal number can be regarded as a repeating decimal number.

14. $\sqrt{-1}$ is a real number.

Objective

To name, illustrate, and relate properties of the real numbers and their operations.

0.2 Some Properties of Real Numbers

We now state a few important properties of the real numbers. Let a, b, and c be real numbers.

1. The Transitive Property of Equality

If $a = b$ and $b = c$, then $a = c$.

Thus, two numbers that are both equal to a third number are equal to each other. For example, if $x = y$ and $y = 7$, then $x = 7$.

2. The Closure Properties of Addition and Multiplication

For all real numbers a and b, there are unique real numbers $a + b$ and ab.

This means that any two numbers can be added and multiplied, and the result in each case is a real number.

3. The Commutative Properties of Addition and Multiplication

$$a + b = b + a \quad \text{and} \quad ab = ba$$

This means that two numbers can be added or multiplied in any order. For example, $3 + 4 = 4 + 3$ and $(7)(-4) = (-4)(7)$.

4. The Associative Properties of Addition and Multiplication

$$a + (b + c) = (a + b) + c \quad \text{and} \quad a(bc) = (ab)c$$

This means that, for both addition and multiplication, numbers can be grouped in any order. For example, $2 + (3 + 4) = (2 + 3) + 4$; in both cases, the sum is 9. Similarly, $2x + (x + y) = (2x + x) + y$, and observe that the right side more obviously simplifies to $3x + y$ than does the left side. Also, $(6 \cdot \frac{1}{3}) \cdot 5 = 6(\frac{1}{3} \cdot 5)$, and here the left side obviously reduces to 10, so the right side does too.

5. The Identity Properties

There are unique real numbers denoted 0 and 1 such that, for each real number a,

$$0 + a = a \quad \text{and} \quad 1a = a$$

6. The Inverse Properties

For each real number a, there is a unique real number denoted $-a$ such that

$$a + (-a) = 0$$

The number $-a$ is called the **negative** of a.

For example, since $6 + (-6) = 0$, the negative of 6 is -6. The negative of a number is not necessarily a negative number. For example, the negative of -6 is 6, since $(-6) + (6) = 0$. That is, the negative of -6 is 6, so we can write $-(-6) = 6$.

For each real number a, *except* 0, there is a unique real number denoted a^{-1} such that

$$a \cdot a^{-1} = 1$$

The number a^{-1} is called the **reciprocal** of a.

Zero does not have a reciprocal because there is no number that when multiplied by 0 gives 1. This is a consequence of $0 \cdot a = 0$ in 7. The Distributive Properties.

Thus, all numbers *except* 0 have a reciprocal. Recall that a^{-1} can be written $\frac{1}{a}$. For example, the reciprocal of 3 is $\frac{1}{3}$, since $3(\frac{1}{3}) = 1$. Hence, $\frac{1}{3}$ is the reciprocal of 3. The reciprocal of $\frac{1}{3}$ is 3, since $(\frac{1}{3})(3) = 1$. *The reciprocal of 0 is not defined*.

7. The Distributive Properties

$$a(b + c) = ab + ac \quad \text{and} \quad (b + c)a = ba + ca$$

$$0 \cdot a = 0 = a \cdot 0$$

For example, although $2(3+4) = 2(7) = 14$, we can also write

$$2(3+4) = 2(3) + 2(4) = 6 + 8 = 14$$

Similarly,

$$(2+3)(4) = 2(4) + 3(4) = 8 + 12 = 20$$

and

$$x(z+4) = x(z) + x(4) = xz + 4x$$

The distributive property can be extended to the form

$$a(b+c+d) = ab + ac + ad$$

In fact, it can be extended to sums involving any number of terms.

Subtraction is defined in terms of addition:

$$a - b \quad \text{means} \quad a + (-b)$$

where $-b$ is the negative of b. Thus, $6-8$ means $6+(-8)$.

In a similar way, we define **division** in terms of multiplication. If $b \neq 0$, then

$$a \div b \quad \text{means} \quad a(b^{-1})$$

Usually, we write either $\frac{a}{b}$ or a/b for $a \div b$. Since $b^{-1} = \frac{1}{b}$,

$$\frac{a}{b} = a(b^{-1}) = a\left(\frac{1}{b}\right)$$

$\frac{a}{b}$ means a times the reciprocal of b.

Thus, $\frac{3}{5}$ means 3 times $\frac{1}{5}$, where $\frac{1}{5}$ is the reciprocal of 5. Sometimes we refer to $\frac{a}{b}$ as the *ratio* of a to b. We remark that since 0 does not have a reciprocal, **division by 0 is not defined**.

The following examples show some manipulations involving the preceding properties.

EXAMPLE 1 Applying Properties of Real Numbers

a. $x(y - 3z + 2w) = (y - 3z + 2w)x$, by the commutative property of multiplication.

b. By the associative property of multiplication, $3(4 \cdot 5) = (3 \cdot 4)5$. Thus, the result of multiplying 3 by the product of 4 and 5 is the same as the result of multiplying the product of 3 and 4 by 5. In either case, the result is 60.

c. Show that $a(b \cdot c) \neq (ab) \cdot (ac)$

Solution: To show the negation of a general statement, it suffices to provide a *counterexample*. Here, taking $a = 2$ and $b = 1 = c$, we see that that $a(b \cdot c) = 2$ while $(ab) \cdot (ac) = 4$.

Now Work Problem 9 ◁

EXAMPLE 2 Applying Properties of Real Numbers

a. Show that $2 - \sqrt{2} = -\sqrt{2} + 2$.

Solution: By the definition of subtraction, $2 - \sqrt{2} = 2 + (-\sqrt{2})$. However, by the commutative property of addition, $2 + (-\sqrt{2}) = -\sqrt{2} + 2$. Hence, by the transitive property of equality, $2 - \sqrt{2} = -\sqrt{2} + 2$. Similarly, it is clear that, for any a and b, we have

$$a - b = -b + a$$

b. Show that $(8 + x) - y = 8 + (x - y)$.

Solution: Beginning with the left side, we have

$$
\begin{aligned}
(8 + x) - y &= (8 + x) + (-y) && \text{definition of subtraction} \\
&= 8 + (x + (-y)) && \text{associative property} \\
&= 8 + (x - y) && \text{definition of subtraction}
\end{aligned}
$$

Hence, by the transitive property of equality,

$$(8 + x) - y = 8 + (x - y)$$

Similarly, for all a, b, and c, we have

$$(a + b) - c = a + (b - c)$$

c. Show that $3(4x + 2y + 8) = 12x + 6y + 24$.

Solution: By the distributive property,

$$3(4x + 2y + 8) = 3(4x) + 3(2y) + 3(8)$$

But by the associative property of multiplication,

$$3(4x) = (3 \cdot 4)x = 12x \quad \text{and similarly} \quad 3(2y) = 6y$$

Thus, $3(4x + 2y + 8) = 12x + 6y + 24$

Now Work Problem 25 ◁

EXAMPLE 3 Applying Properties of Real Numbers

a. Show that $\dfrac{ab}{c} = a\left(\dfrac{b}{c}\right)$, for $c \neq 0$.

Solution: The restriction is necessary. Neither side of the equation is defined if $c = 0$. By the definition of division,

$$\frac{ab}{c} = (ab) \cdot \frac{1}{c} \text{ for } c \neq 0$$

But by the associative property,

$$(ab) \cdot \frac{1}{c} = a\left(b \cdot \frac{1}{c}\right)$$

However, by the definition of division, $b \cdot \dfrac{1}{c} = \dfrac{b}{c}$. Thus,

$$\frac{ab}{c} = a\left(\frac{b}{c}\right)$$

We can also show that $\dfrac{ab}{c} = \left(\dfrac{a}{c}\right)b$.

b. Show that $\dfrac{a + b}{c} = \dfrac{a}{c} + \dfrac{b}{c}$ *for* $c \neq 0$.

Solution: (Again the restriction is necessary but we won't always bother to say so.) By the definition of division and the distributive property,

$$\frac{a + b}{c} = (a + b)\frac{1}{c} = a \cdot \frac{1}{c} + b \cdot \frac{1}{c}$$

However,

$$a \cdot \frac{1}{c} + b \cdot \frac{1}{c} = \frac{a}{c} + \frac{b}{c}$$

Hence,

$$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$$

Now Work Problem 27 ◁

Finding the product of several numbers can be done by considering products of numbers taken just two at a time. For example, to find the product of x, y, and z, we could first multiply x by y and then multiply that product by z; that is, we find $(xy)z$. Alternatively, we could multiply x by the product of y and z; that is, we find $x(yz)$. The associative property of multiplication guarantees that both results are identical, regardless of how the numbers are grouped. Thus, it is not ambiguous to write xyz. This concept can be extended to more than three numbers and applies equally well to addition.

Not only should you be able to manipulate real numbers, you should also be aware of, and familiar with, the terminology involved. It will help you read the book, follow your lectures, and — most importantly — allow you to frame your questions when you have difficulties.

The following list states important properties of real numbers that you should study thoroughly. Being able to manipulate real numbers is essential to your success in mathematics. A numerical example follows each property. *All denominators are assumed to be different from zero* (but for emphasis we have been explicit about these restrictions).

Property	***Example(s)***
1. $a - b = a + (-b)$	$2 - 7 = 2 + (-7) = -5$
2. $a - (-b) = a + b$	$2 - (-7) = 2 + 7 = 9$
3. $-a = (-1)(a)$	$-7 = (-1)(7)$
4. $a(b + c) = ab + ac$	$6(7 + 2) = 6 \cdot 7 + 6 \cdot 2 = 54$
5. $a(b - c) = ab - ac$	$6(7 - 2) = 6 \cdot 7 - 6 \cdot 2 = 30$
6. $-(a + b) = -a - b$	$-(7 + 2) = -7 - 2 = -9$
7. $-(a - b) = -a + b$	$-(2 - 7) = -2 + 7 = 5$
8. $-(-a) = a$	$-(-2) = 2$
9. $a(0) = 0$	$2(0) = 0$
10. $(-a)(b) = -(ab) = a(-b)$	$(-2)(7) = -(2 \cdot 7) = 2(-7) = -14$
11. $(-a)(-b) = ab$	$(-2)(-7) = 2 \cdot 7 = 14$
12. $\frac{a}{1} = a$	$\frac{7}{1} = 7, \frac{-2}{1} = -2$
13. $\frac{a}{b} = a\left(\frac{1}{b}\right)$ for $b \neq 0$	$\frac{2}{7} = 2\left(\frac{1}{7}\right)$
14. $\frac{a}{-b} = -\frac{a}{b} = \frac{-a}{b}$ for $b \neq 0$	$\frac{2}{-7} = -\frac{2}{7} = \frac{-2}{7}$
15. $\frac{-a}{-b} = \frac{a}{b}$ for $b \neq 0$	$\frac{-2}{-7} = \frac{2}{7}$
16. $\frac{0}{a} = 0$ for $a \neq 0$	$\frac{0}{7} = 0$
17. $\frac{a}{a} = 1$ for $a \neq 0$	$\frac{2}{2} = 1, \frac{-5}{-5} = 1$
18. $a\left(\frac{b}{a}\right) = b$ for $a \neq 0$	$2\left(\frac{7}{2}\right) = 7$
19. $a \cdot \frac{1}{a} = 1$ for $a \neq 0$	$2 \cdot \frac{1}{2} = 1$
20. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ for $b, d \neq 0$	$\frac{2}{3} \cdot \frac{4}{5} = \frac{2 \cdot 4}{3 \cdot 5} = \frac{8}{15}$
21. $\frac{ab}{c} = \left(\frac{a}{c}\right)b = a\left(\frac{b}{c}\right)$ for $c \neq 0$	$\frac{2 \cdot 7}{3} = \frac{2}{3} \cdot 7 = 2 \cdot \frac{7}{3}$

Property	***Example(s)***
22. $\dfrac{a}{bc} = \dfrac{a}{b} \cdot \dfrac{1}{c} = \dfrac{1}{b} \cdot \dfrac{a}{c}$ for $b, c \neq 0$	$\dfrac{2}{3 \cdot 7} = \dfrac{2}{3} \cdot \dfrac{1}{7} = \dfrac{1}{3} \cdot \dfrac{2}{7}$
23. $\dfrac{a}{b} = \dfrac{a}{b} \cdot \dfrac{c}{c} = \dfrac{ac}{bc}$ for $b, c \neq 0$	$\dfrac{2}{7} = \left(\dfrac{2}{7}\right)\left(\dfrac{5}{5}\right) = \dfrac{2 \cdot 5}{7 \cdot 5}$
24. $\dfrac{a}{b(-c)} = \dfrac{a}{(-b)c} = \dfrac{-a}{bc} = \dfrac{-a}{(-b)(-c)} = -\dfrac{a}{bc}$ for $b, c \neq 0$	$\dfrac{2}{3(-5)} = \dfrac{2}{(-3)(5)} = \dfrac{-2}{3(5)} = \dfrac{-2}{(-3)(-5)} = -\dfrac{2}{3(5)} = -\dfrac{2}{15}$
25. $\dfrac{a(-b)}{c} = \dfrac{(-a)b}{c} = \dfrac{ab}{-c} = \dfrac{(-a)(-b)}{-c} = -\dfrac{ab}{c}$ for $c \neq 0$	$\dfrac{2(-3)}{5} = \dfrac{(-2)(3)}{5} = \dfrac{2(3)}{-5} = \dfrac{(-2)(-3)}{-5} = -\dfrac{2(3)}{5} = -\dfrac{6}{5}$
26. $\dfrac{a}{c} + \dfrac{b}{c} = \dfrac{a+b}{c}$ for $c \neq 0$	$\dfrac{2}{9} + \dfrac{3}{9} = \dfrac{2+3}{9} = \dfrac{5}{9}$
27. $\dfrac{a}{c} - \dfrac{b}{c} = \dfrac{a-b}{c}$ for $c \neq 0$	$\dfrac{2}{9} - \dfrac{3}{9} = \dfrac{2-3}{9} = \dfrac{-1}{9}$
28. $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad+bc}{bd}$ for $b, d \neq 0$	$\dfrac{4}{5} + \dfrac{2}{3} = \dfrac{4 \cdot 3 + 5 \cdot 2}{5 \cdot 3} = \dfrac{22}{15}$
29. $\dfrac{a}{b} - \dfrac{c}{d} = \dfrac{ad-bc}{bd}$ for $b, d \neq 0$	$\dfrac{4}{5} - \dfrac{2}{3} = \dfrac{4 \cdot 3 - 5 \cdot 2}{5 \cdot 3} = \dfrac{2}{15}$
30. $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \cdot \dfrac{d}{c} = \dfrac{ad}{bc}$ for $b, c, d \neq 0$	$\dfrac{\frac{2}{3}}{\frac{7}{5}} = \dfrac{2}{3} \div \dfrac{7}{5} = \dfrac{2}{3} \cdot \dfrac{5}{7} = \dfrac{2 \cdot 5}{3 \cdot 7} = \dfrac{10}{21}$
31. $\dfrac{a}{\frac{b}{c}} = a \div \dfrac{b}{c} = a \cdot \dfrac{c}{b} = \dfrac{ac}{b}$ for $b, c \neq 0$	$\dfrac{2}{\frac{3}{5}} = 2 \div \dfrac{3}{5} = 2 \cdot \dfrac{5}{3} = \dfrac{2 \cdot 5}{3} = \dfrac{10}{3}$
32. $\dfrac{\frac{a}{b}}{c} = \dfrac{a}{b} \div c = \dfrac{a}{b} \cdot \dfrac{1}{c} = \dfrac{a}{bc}$ for $b, c = 0$	$\dfrac{\frac{2}{3}}{5} = \dfrac{2}{3} \div 5 = \dfrac{2}{3} \cdot \dfrac{1}{5} = \dfrac{2}{3 \cdot 5} = \dfrac{2}{15}$

Property 23 is particularly important and could be called the **fundamental principle of fractions**. It states that *multiplying or dividing both the numerator and denominator of a fraction by the same nonzero number results in a fraction that is equal to the original fraction*. Thus,

$$\frac{7}{\frac{1}{8}} = \frac{7 \cdot 8}{\frac{1}{8} \cdot 8} = \frac{56}{1} = 56$$

By Properties 28 and 23, we have

$$\frac{2}{5}+\frac{4}{15}=\frac{2\cdot 15+5\cdot 4}{5\cdot 15}=\frac{50}{75}=\frac{2\cdot 25}{3\cdot 25}=\frac{2}{3}$$

We can also do this problem by converting $\frac{2}{5}$ and $\frac{4}{15}$ into fractions that have the same denominators and then using Property 26. The fractions $\frac{2}{5}$ and $\frac{4}{15}$ can be written with a common denominator of $5\cdot 15$:

$$\frac{2}{5}=\frac{2\cdot 15}{5\cdot 15}\quad\text{and}\quad\frac{4}{15}=\frac{4\cdot 5}{15\cdot 5}$$

However, 15 is the *least* such common denominator and is called the *least common denominator* (LCD) of $\frac{2}{5}$ and $\frac{4}{15}$. Thus,

$$\frac{2}{5}+\frac{4}{15}=\frac{2\cdot 3}{5\cdot 3}+\frac{4}{15}=\frac{6}{15}+\frac{4}{15}=\frac{6+4}{15}=\frac{10}{15}=\frac{2}{3}$$

Similarly,

$$\begin{aligned}\frac{3}{8}-\frac{5}{12}&=\frac{3\cdot 3}{8\cdot 3}-\frac{5\cdot 2}{12\cdot 2} \qquad \text{LCD}=24\\&=\frac{9}{24}-\frac{10}{24}=\frac{9-10}{24}\\&=-\frac{1}{24}\end{aligned}$$

PROBLEMS 0.2

In Problems 1–10, determine the truth of each statement.

1. Every real number has a reciprocal.

2. The reciprocal of 6.6 is 0.151515 . . . .

3. The negative of 7 is $\frac{-1}{7}$.

4. $1(x\cdot y)=(1\cdot x)(1\cdot y)$

5. $-x+y=-y+x$

6. $(x+2)(4)=4x+8$

7. $\frac{x+3}{5}=\frac{x}{5}+3$

8. $3\left(\frac{x}{4}\right)=\frac{3x}{4}$

9. $2(x\cdot y)=(2x)\cdot(2y)$

10. $x(4y)=4xy$

In Problems 11–20, state which properties of the real numbers are being used.

11. $2(x+y)=2x+2y$

12. $(x+5.2)+0.7y=x+(5.2+0.7y)$

13. $2(3y)=(2\cdot 3)y$

14. $\frac{a}{b}=\frac{1}{b}\cdot a$

15. $5(b-a)=(a-b)(-5)$

16. $y+(x+y)=(y+x)+y$

17. $\frac{5x-y}{7}=1/7(5x-y)$

18. $5(4+7)=5(7+4)$

19. $(2+a)b=2b+ba$

20. $(-1)(-3+4)=(-1)(-3)+(-1)(4)$

In Problems 21–27, show that the statements are true by using properties of the real numbers.

21. $2x(y-7)=2xy-14x$

22. $\frac{x}{y}z=x\frac{z}{y}$

23. $(x+y)(2)=2x+2y$

24. $a(b+(c+d))=a((d+b)+c)$

25. $x((2y+1)+3)=2xy+4x$

26. $(1+a)(b+c)=b+c+ab+ac$

27. Show that $(x-y+z)w=xw-yw+zw$.
[*Hint*: $b+c+d=(b+c)+d$.]

Simplify each of the following, if possible.

28. $-2+(-4)$ **29.** $-a+b$ **30.** $6+(-4)$

31. $7-2$ **32.** $\frac{3}{2^{-1}}$ **33.** $-5-(-13)$

34. $-(-a)+(-b)$ **35.** $(-2)(9)$ **36.** $7(-9)$

37. $(-1.6)(-0.5)$ **38.** $19(-1)$ **39.** $\frac{-1}{\frac{-1}{a}}$

40. $-(-6+x)$ **41.** $-7(x)$ **42.** $-3(a-b)$

43. $-(-6+(-y))$ **44.** $-3\div 3a$ **45.** $-9\div(-27)$

46. $(-a)\div(-b)$ **47.** $3+(3^{-1}9)$ **48.** $3(-2(3)+6(2))$

49. $(-a)(-b)(-1)$ **50.** $(-12)(-12)$ **51.** $X(1)$

52. $-71(x-2)$ **53.** $4(5+x)$ **54.** $-(x-y)$

55. $0(-x)$ **56.** $8\left(\frac{1}{11}\right)$ **57.** $\frac{X}{1}$

58. $\dfrac{14x}{21y}$ **59.** $\dfrac{2x}{-2}$ **60.** $\dfrac{2}{3}\cdot\dfrac{1}{x}$

61. $\dfrac{a}{c}(3b)$ **62.** $5a+(7-5a)$ **63.** $\dfrac{-aby}{-ax}$

64. $\dfrac{a}{b}\cdot\dfrac{1}{c}$ **65.** $\dfrac{2}{x}\cdot\dfrac{5}{y}$ **66.** $\dfrac{1}{2}+\dfrac{1}{3}$

67. $\dfrac{x}{3a}+\dfrac{y}{a}$ **68.** $\dfrac{3}{10}-\dfrac{7}{15}$ **69.** $\dfrac{a}{b}+\dfrac{c}{b}$

70. $\dfrac{X}{\sqrt{5}}-\dfrac{Y}{\sqrt{5}}$ **71.** $\dfrac{3}{2}-\dfrac{1}{4}+\dfrac{1}{6}$ **72.** $\dfrac{3}{7}-\dfrac{5}{9}$

73. $\dfrac{\frac{6}{x}}{y}$ **74.** $\dfrac{\frac{l}{w}}{m}$ **75.** $\dfrac{\frac{-x}{y^2}}{\frac{z}{xy}}$

76. $\dfrac{7}{0}$ **77.** $\dfrac{0}{X}$, for $X \neq 0$ **78.** $\dfrac{0}{0}$

Objective

To review positive integral exponents, the zero exponent, negative integral exponents, rational exponents, principal roots, radicals, and the procedure of rationalizing the denominator.

0.3 Exponents and Radicals

The product $x \cdot x \cdot x$ of 3 x's is abbreviated x^3. In general, for n a positive integer, x^n is the abbreviation for the product of n x's. The letter n in x^n is called the **exponent**, and x is called the **base**. More specifically, if n is a positive integer, we have

1. $x^n = \underbrace{x \cdot x \cdot x \cdot \ldots \cdot x}_{n \text{ factors}}$
2. $x^{-n} = \dfrac{1}{x^n} = \dfrac{1}{\underbrace{x \cdot x \cdot x \cdot \ldots \cdot x}_{n \text{ factors}}}$ for $x \neq 0$
3. $\dfrac{1}{x^{-n}} = x^n$ for $x \neq 0$
4. $x^0 = 1$

Some authors say that 0^0 is not defined. However, $0^0 = 1$ is a consistent and often useful definition.

EXAMPLE 1 **Exponents**

a. $\left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{16}$

b. $3^{-5} = \dfrac{1}{3^5} = \dfrac{1}{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3} = \dfrac{1}{243}$

c. $\dfrac{1}{3^{-5}} = 3^5 = 243$

d. $2^0 = 1, \pi^0 = 1, (-5)^0 = 1$

e. $x^1 = x$

Now Work Problem 5 ◁

If $r^n = x$, where n is a positive integer, then r is an nth **root** of x. Second roots, the case $n = 2$, are called square roots; and third roots, the case $n = 3$, are called cube roots. For example, $3^2 = 9$, so 3 is a square root of 9. Since $(-3)^2 = 9$, -3 is also a square root of 9. Similarly, -2 is a cube root of -8, since $(-2)^3 = -8$, while 5 is a fourth root of 625 since $5^4 = 625$.

Some numbers do not have an nth root that is a real number. For example, since the square of any real number is nonnegative: there is no real number that is a square root of -4.

The **principal nth root** of x is the nth root of x that is positive if x is positive and is negative if x is negative and n is odd. We denote the principal nth root of x by $\sqrt[n]{x}$. Thus,

$$\sqrt[n]{x} \text{ is } \begin{cases} \text{positive if } x \text{ is positive} \\ \text{negative if } x \text{ is negative and } n \text{ is odd} \end{cases}$$

For example, $\sqrt[2]{9} = 3$, $\sqrt[3]{-8} = -2$, and $\sqrt[3]{\frac{1}{27}} = \frac{1}{3}$. We define $\sqrt[n]{0} = 0$.

Although both 2 and -2 are square roots of 4, the principal square root of 4 is 2, not -2. Hence, $\sqrt{4} = 2$. For positive x, we often write $\pm\sqrt{x}$ to indicate both square roots of x, and "$\pm\sqrt{4} = \pm 2$" is a convenient short way of writing "$\sqrt{4} = 2$ and $-\sqrt{4} = -2$", but the only value of $\sqrt{4}$ is 2.

The symbol $\sqrt[n]{x}$ is called a **radical**. With principal square roots we usually write $\sqrt{x}$ instead of $\sqrt[2]{x}$. Thus, $\sqrt{9} = 3$.

If x is positive, the expression $x^{p/q}$, where p and q are integers with no common factors and q is positive, is defined to be $\sqrt[q]{x^p}$. Hence,

$$x^{3/4} = \sqrt[4]{x^3}; \; 8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$$

$$4^{-1/2} = \sqrt[2]{4^{-1}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

Here are the basic laws of exponents and radicals:

Law	***Example(s)***
1. $x^m \cdot x^n = x^{m+n}$	$2^3 \cdot 2^5 = 2^8 = 256;\ x^2 \cdot x^3 = x^5$
2. $x^0 = 1$	$2^0 = 1$
3. $x^{-n} = \frac{1}{x^n}$	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
4. $\frac{1}{x^{-n}} = x^n$	$\frac{1}{2^{-3}} = 2^3 = 8;\ \frac{1}{x^{-5}} = x^5$
5. $\frac{x^m}{x^n} = x^{m-n} = \frac{1}{x^{n-m}}$	$\frac{2^{12}}{2^8} = 2^4 = 16;\ \frac{x^8}{x^{12}} = \frac{1}{x^4}$
6. $\frac{x^m}{x^m} = 1$	$\frac{2^4}{2^4} = 1$
7. $(x^m)^n = x^{mn}$	$(2^3)^5 = 2^{15};\ (x^2)^3 = x^6$
8. $(xy)^n = x^n y^n$	$(2 \cdot 4)^3 = 2^3 \cdot 4^3 = 8 \cdot 64 = 512$
9. $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$	$\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3} = \frac{8}{27}$
10. $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n$	$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$
11. $x^{1/n} = \sqrt[n]{x}$	$3^{1/5} = \sqrt[5]{3}$
12. $x^{-1/n} = \frac{1}{x^{1/n}} = \frac{1}{\sqrt[n]{x}}$	$4^{-1/2} = \frac{1}{4^{1/2}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$
13. $\sqrt[n]{x}\sqrt[n]{y} = \sqrt[n]{xy}$	$\sqrt[3]{9}\sqrt[3]{2} = \sqrt[3]{18}$
14. $\frac{\sqrt[n]{x}}{\sqrt[n]{y}} = \sqrt[n]{\frac{x}{y}}$	$\frac{\sqrt[3]{90}}{\sqrt[3]{10}} = \sqrt[3]{\frac{90}{10}} = \sqrt[3]{9}$
15. $\sqrt[m]{\sqrt[n]{x}} = \sqrt[mn]{x}$	$\sqrt[3]{\sqrt[4]{2}} = \sqrt[12]{2}$
16. $x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$	$8^{2/3} = \sqrt[3]{8^2} = (\sqrt[3]{8})^2 = 2^2 = 4$
17. $(\sqrt[m]{x})^m = x$	$(\sqrt[8]{7})^8 = 7$

EXAMPLE 2 Exponents and Radicals

When computing $x^{m/n}$, it is often easier to first find $\sqrt[n]{x}$ and then raise the result to the mth power. Thus, $(-27)^{4/3} = (\sqrt[3]{-27})^4 = (-3)^4 = 81$.

a. By Law 1,

$$x^6 x^8 = x^{6+8} = x^{14}$$

$$a^3 b^2 a^5 b = a^3 a^5 b^2 b^1 = a^8 b^3$$

$$x^{11} x^{-5} = x^{11-5} = x^6$$

$$z^{2/5} z^{3/5} = z^1 = z$$

$$x x^{1/2} = x^1 x^{1/2} = x^{3/2}$$

b. By Law 16,

$$\left(\frac{1}{4}\right)^{3/2} = \left(\sqrt{\frac{1}{4}}\right)^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

c. $$\left(-\frac{8}{27}\right)^{4/3} = \left(\sqrt[3]{\frac{-8}{27}}\right)^4 = \left(\frac{\sqrt[3]{-8}}{\sqrt[3]{27}}\right)^4 \qquad \text{Laws 16 and 14}$$

$$= \left(\frac{-2}{3}\right)^4$$

$$= \frac{(-2)^4}{3^4} = \frac{16}{81} \qquad \text{Law 9}$$

d. $$(64a^3)^{2/3} = 64^{2/3}(a^3)^{2/3} \qquad \text{Law 8}$$

$$= (\sqrt[3]{64})^2 a^2 \qquad \text{Laws 16 and 7}$$

$$= (4)^2 a^2 = 16a^2$$

Now Work Problem 39 ◁

Rationalizing the numerator is a similar procedure.

Rationalizing the denominator of a fraction is a procedure in which a fraction having a radical in its denominator is expressed as an equal fraction without a radical in its denominator. We use the fundamental principle of fractions, as Example 3 shows.

EXAMPLE 3 Rationalizing Denominators

a. $$\frac{2}{\sqrt{5}} = \frac{2}{5^{1/2}} = \frac{2 \cdot 5^{1/2}}{5^{1/2} \cdot 5^{1/2}} = \frac{2 \cdot 5^{1/2}}{5^1} = \frac{2\sqrt{5}}{5}$$

b. $$\frac{2}{\sqrt[6]{3x^5}} = \frac{2}{\sqrt[6]{3} \cdot \sqrt[6]{x^5}} = \frac{2}{3^{1/6}x^{5/6}} = \frac{2 \cdot 3^{5/6}x^{1/6}}{3^{1/6}x^{5/6} \cdot 3^{5/6}x^{1/6}} \qquad \text{for } x \neq 0$$

$$= \frac{2(3^5x)^{1/6}}{3x} = \frac{2\sqrt[6]{3^5x}}{3x}$$

Now Work Problem 63 ◁

The following examples illustrate various applications of the laws of exponents and radicals. All denominators are understood to be nonzero.

EXAMPLE 4 Exponents

a. Eliminate negative exponents in $\dfrac{x^{-2}y^3}{z^{-2}}$ for $x \neq 0, z \neq 0$.

Solution: $$\frac{x^{-2}y^3}{z^{-2}} = x^{-2} \cdot y^3 \cdot \frac{1}{z^{-2}} = \frac{1}{x^2} \cdot y^3 \cdot z^2 = \frac{y^3z^2}{x^2}$$

By comparing our answer with the original expression, we conclude that we can bring a factor of the numerator down to the denominator, and vice versa, by changing the sign of the exponent.

b. Simplify $\dfrac{x^2y^7}{x^3y^5}$ for $x \neq 0, y \neq 0$.

Solution: $$\frac{x^2y^7}{x^3y^5} = \frac{y^{7-5}}{x^{3-2}} = \frac{y^2}{x}$$

c. Simplify $(x^5y^8)^5$.

Solution: $$(x^5y^8)^5 = (x^5)^5(y^8)^5 = x^{25}y^{40}$$

d. Simplify $(x^{5/9}y^{4/3})^{18}$.

Solution: $$(x^{5/9}y^{4/3})^{18} = (x^{5/9})^{18}(y^{4/3})^{18} = x^{10}y^{24}$$

e. Simplify $\left(\dfrac{x^{1/5}y^{6/5}}{z^{2/5}}\right)^5$ for $z \neq 0$.

Solution: $$\left(\frac{x^{1/5}y^{6/5}}{z^{2/5}}\right)^5 = \frac{(x^{1/5}y^{6/5})^5}{(z^{2/5})^5} = \frac{xy^6}{z^2}$$

f. Simplify $\dfrac{x^3}{y^2} \div \dfrac{x^6}{y^5}$ for $x \neq 0, y \neq 0$.

Solution: $$\frac{x^3}{y^2} \div \frac{x^6}{y^5} = \frac{x^3}{y^2} \cdot \frac{y^5}{x^6} = \frac{y^3}{x^3}$$

Now Work Problem 51 ◁

EXAMPLE 5 Exponents

a. For $x \neq 0$ and $y \neq 0$, eliminate negative exponents in $x^{-1} + y^{-1}$ and simplify.

Solution: $$x^{-1} + y^{-1} = \frac{1}{x} + \frac{1}{y} = \frac{y+x}{xy}$$

b. Simplify $x^{3/2} - x^{1/2}$ by using the distributive law.

Solution: $$x^{3/2} - x^{1/2} = x^{1/2}(x-1)$$

c. For $x \neq 0$, eliminate negative exponents in $7x^{-2} + (7x)^{-2}$.

Solution: $$7x^{-2} + (7x)^{-2} = \frac{7}{x^2} + \frac{1}{(7x)^2} = \frac{7}{x^2} + \frac{1}{49x^2} = \frac{344}{49x^2}$$

d. For $x \neq 0$ and $y \neq 0$, eliminate negative exponents in $(x^{-1} - y^{-1})^{-2}$.

Solution: $$(x^{-1} - y^{-1})^{-2} = \left(\frac{1}{x} - \frac{1}{y}\right)^{-2} = \left(\frac{y-x}{xy}\right)^{-2}$$
$$= \left(\frac{xy}{y-x}\right)^2 = \frac{x^2y^2}{(y-x)^2}$$

e. Apply the distributive law to $x^{2/5}(y^{1/2} + 2x^{6/5})$.

Solution: $$x^{2/5}(y^{1/2} + 2x^{6/5}) = x^{2/5}y^{1/2} + 2x^{8/5}$$

Now Work Problem 41 ◁

EXAMPLE 6 Radicals

a. Simplify $\sqrt[4]{48}$.

Solution: $$\sqrt[4]{48} = \sqrt[4]{16 \cdot 3} = \sqrt[4]{16}\,\sqrt[4]{3} = 2\sqrt[4]{3}$$

b. Rewrite $\sqrt{2+5x}$ without using a radical sign.

Solution: $$\sqrt{2+5x} = (2+5x)^{1/2}$$

c. Rationalize the denominator of $\dfrac{\sqrt[5]{2}}{\sqrt[3]{6}}$ and simplify.

Solution: $$\frac{\sqrt[5]{2}}{\sqrt[3]{6}} = \frac{2^{1/5} \cdot 6^{2/3}}{6^{1/3} \cdot 6^{2/3}} = \frac{2^{3/15}6^{10/15}}{6} = \frac{(2^3 6^{10})^{1/15}}{6} = \frac{\sqrt[15]{2^3 6^{10}}}{6}$$

d. Simplify $\dfrac{\sqrt{20}}{\sqrt{5}}$.

Solution:
$$\frac{\sqrt{20}}{\sqrt{5}} = \sqrt{\frac{20}{5}} = \sqrt{4} = 2$$

Now Work Problem 71 ◁

EXAMPLE 7 Radicals

a. Simplify $\sqrt[3]{x^6y^4}$.

Solution:
$$\sqrt[3]{x^6y^4} = \sqrt[3]{(x^2)^3y^3y} = \sqrt[3]{(x^2)^3} \cdot \sqrt[3]{y^3} \cdot \sqrt[3]{y}$$
$$= x^2y\sqrt[3]{y}$$

b. Simplify $\sqrt{\dfrac{2}{7}}$.

Solution:
$$\sqrt{\frac{2}{7}} = \sqrt{\frac{2 \cdot 7}{7 \cdot 7}} = \sqrt{\frac{14}{7^2}} = \frac{\sqrt{14}}{\sqrt{7^2}} = \frac{\sqrt{14}}{7}$$

c. Simplify $\sqrt{250} - \sqrt{50} + 15\sqrt{2}$.

Solution:
$$\sqrt{250} - \sqrt{50} + 15\sqrt{2} = \sqrt{25 \cdot 10} - \sqrt{25 \cdot 2} + 15\sqrt{2}$$
$$= 5\sqrt{10} - 5\sqrt{2} + 15\sqrt{2}$$
$$= 5\sqrt{10} + 10\sqrt{2}$$

d. If x is any real number, simplify $\sqrt{x^2}$.

Solution:
$$\sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Thus, $\sqrt{2^2} = 2$ and $\sqrt{(-3)^2} = -(-3) = 3$.

Now Work Problem 75 ◁

PROBLEMS 0.3

In Problems 1–14, simplify and express all answers in terms of positive exponents.

1. $(2^3)(2^2)$
2. x^6x^9
3. $17^5 \cdot 17^2$
4. z^3zz^2
5. $\dfrac{x^3x^5}{y^9y^5}$
6. $(x^{12})^4$
7. $\dfrac{(a^3)^7}{(b^4)^5}$
8. $\left(\dfrac{13^{14}}{13}\right)^2$
9. $(2x^2y^3)^3$
10. $\left(\dfrac{w^2s^3}{y^2}\right)^2$
11. $\dfrac{x^9}{x^5}$
12. $\left(\dfrac{2a^4}{7b^5}\right)^6$
13. $\dfrac{(y^3)^4}{(y^2)^3y^2}$
14. $\dfrac{(x^2)^3(x^3)^2}{(x^3)^4}$

In Problems 15–28, evaluate the expressions.

15. $\sqrt{25}$
16. $\sqrt[4]{81}$
17. $\sqrt[7]{-128}$
18. $\sqrt[5]{0.00243}$
19. $\sqrt[4]{\dfrac{1}{16}}$
20. $\sqrt[3]{-\dfrac{8}{27}}$
21. $(49)^{1/2}$
22. $(64)^{1/3}$
23. $81^{3/4}$
24. $(9)^{-5/2}$
25. $(32)^{-2/5}$
26. $(0.09)^{-1/2}$
27. $\left(\dfrac{1}{32}\right)^{4/5}$
28. $\left(-\dfrac{243}{1024}\right)^{2/5}$

In Problems 29–40, simplify the expressions.

29. $\sqrt{50}$
30. $\sqrt[3]{54}$
31. $\sqrt[3]{2x^3}$
32. $\sqrt{4x}$
33. $\sqrt{49u^8}$
34. $\sqrt[4]{\dfrac{x}{16}}$
35. $2\sqrt{8} - 5\sqrt{27} + \sqrt[3]{128}$
36. $\sqrt{\dfrac{3}{13}}$
37. $(9z^4)^{1/2}$
38. $(729x^6)^{3/2}$
39. $\left(\dfrac{27t^3}{8}\right)^{2/3}$
40. $\left(\dfrac{256}{x^{12}}\right)^{-3/4}$

In Problems 41–52, write the expressions in terms of positive exponents only. Avoid all radicals in the final form. For example,

$$y^{-1}\sqrt{x} = \frac{x^{1/2}}{y}$$

41. $\dfrac{a^5b^{-3}}{c^2}$
42. $\sqrt[5]{x^2y^3z^{-10}}$
43. $3a^{-1}b^{-2}c^{-3}$

44. $x + y^{-1}$ **45.** $(3t)^{-2}$ **46.** $(3 - z)^{-4}$

47. $\sqrt[5]{5x^2}$ **48.** $(X^5Y^{-7})^{-4}$ **49.** $\sqrt{x} - \sqrt{y}$

50. $\dfrac{u^{-2}v^{-6}w^3}{vw^{-5}}$ **51.** $x^2\sqrt[4]{xy^{-2}z^3}$ **52.** $\sqrt[4]{a^{-3}b^{-2}a^5b^{-4}}$

In Problems 53–58, rewrite the exponential forms using radicals.

53. $(a - b + c)^{3/5}$ **54.** $(ab^2c^3)^{3/4}$

55. $x^{-4/5}$ **56.** $2x^{1/2} - (2y)^{1/2}$

57. $3w^{-3/5} - (3w)^{-3/5}$ **58.** $((y^{-2})^{1/4})^{1/5}$

In Problems 59–68, rationalize the denominators.

59. $\dfrac{6}{\sqrt{5}}$ **60.** $\dfrac{3}{\sqrt[4]{8}}$ **61.** $\dfrac{4}{\sqrt{2x}}$

62. $\dfrac{y}{\sqrt{2y}}$ **63.** $\dfrac{1}{\sqrt[5]{3b}}$ **64.** $\dfrac{2}{3\sqrt[3]{y^2}}$

65. $\dfrac{\sqrt{12}}{\sqrt{3}}$ **66.** $\dfrac{\sqrt{18}}{\sqrt{2}}$ **67.** $\dfrac{\sqrt[5]{2}}{\sqrt[4]{a^2b}}$

68. $\dfrac{\sqrt[3]{3}}{\sqrt{2}}$

In Problems 69–86, simplify. Express all answers in terms of positive exponents. Rationalize the denominator where necessary to avoid fractional exponents in the denominator.

69. $2x^2y^{-3}x^4$ **70.** $\dfrac{3}{u^{5/2}v^{1/2}}$

71. $\dfrac{\sqrt{243}}{\sqrt{3}}$ **72.** $(((3a^3)^2)^{-5})^{-2}$

73. $\dfrac{3^0}{(3^{-4}x^{2/3}y^{-2})^3}$ **74.** $\dfrac{\sqrt{s^5}}{\sqrt[3]{s^2}}$

75. $\sqrt[3]{x^2yz^3}\sqrt[3]{xy^2}$ **76.** $(\sqrt[4]{3})^8$

77. $3^2(32)^{-2/5}$ **78.** $(\sqrt[3]{u^3v^2})^{2/3}$

79. $(2x^{-1}y^2)^2$ **80.** $\dfrac{3}{\sqrt[3]{y}\sqrt[4]{x}}$

81. $\sqrt{x}\sqrt{x^2y^3}\sqrt{xy^2}$ **82.** $\sqrt{75k^4}$

83. $\dfrac{(a^3b^{-4}c^5)^6}{(a^{-2}c^{-3})^{-4}}$ **84.** $\sqrt[3]{7(49)}$

85. $\dfrac{(x^2)^3}{x^4} \div \left(\dfrac{x^3}{(x^3)^2}\right)^2$ **86.** $\sqrt{(-6)(-6)}$

Objective

To add, subtract, multiply, and divide algebraic expressions. To define a polynomial, to use special products, and to use long division to divide polynomials.

0.4 Operations with Algebraic Expressions

If numbers, represented by symbols, are combined by any or all of the operations of addition, subtraction, multiplication, division, exponentiation, and extraction of roots, then the resulting expression is called an **algebraic expression**.

EXAMPLE 1 **Algebraic Expressions**

a. $\sqrt[3]{\dfrac{3x^3 - 5x - 2}{10 - x}}$ is an algebraic expression in the variable x.

b. $10 - 3\sqrt{y} + \dfrac{5}{7 + y^2}$ is an algebraic expression in the variable y.

c. $\dfrac{(x + y)^3 - xy}{y} + 2$ is an algebraic expression in the variables x and y.

Now Work Problem 1 ◁

The algebraic expression $5ax^3 - 2bx + 3$ consists of three **terms**: $+5ax^3$, $-2bx$, and $+3$. Some of the **factors** of the first term, $5ax^3$, are $5, a, x, x^2, x^3, 5ax$, and ax^2. Also, $5a$ is the **coefficient** of x^3, and 5 is the *numerical coefficient* of ax^3. If a and b represent fixed numbers throughout a discussion, then a and b are called **constants**.

Algebraic expressions with exactly one term are called **monomials**. Those having exactly two terms are **binomials**, and those with exactly three terms are **trinomials**. Algebraic expressions with more than one term are called **multinomials**. Thus, the multinomial $2x - 5$ is a binomial; the multinomial $3\sqrt{y} + 2y - 4y^2$ is a trinomial.

The words *polynomial* and *multinomial* should not be used interchangeably. A polynomial is a special kind of multinomial. For example, $\sqrt{x} + 2$ is a multinomial but not a polynomial. On the other hand, $x + 2$ is a polynomial and hence a multinomial.

A **polynomial** *in* x is an algebraic expression of the form

$$c_nx^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$$

where n is a nonnegative integer and the coefficients $c_0, c_1, \ldots, c_n$ are constants with $c_n \neq 0$. Here, the three dots indicate all other terms that are understood to be included in the sum. We call n the **degree** of the polynomial. So, $4x^3 - 5x^2 + x - 2$ is a polynomial in x of degree 3, and $y^5 - 2$ is a polynomial in y of degree 5. A nonzero constant is a polynomial of degree zero; thus, 5 is a polynomial of degree zero. The constant 0 is considered to be a polynomial; however, no degree is assigned to it.

In the following examples, we illustrate operations with algebraic expressions.

EXAMPLE 2 Adding Algebraic Expressions

Simplify $(3x^2y - 2x + 1) + (4x^2y + 6x - 3)$.

Solution: We first remove the parentheses. Next, using the commutative property of addition, we gather all like terms together. Like terms are terms that differ only by their numerical coefficients. In this example, $3x^2y$ and $4x^2y$ are like terms, as are the pairs $-2x$ and $6x$, and 1 and -3. Thus,

$$\begin{aligned}(3x^2y - 2x + 1) + (4x^2y + 6x - 3) &= 3x^2y - 2x + 1 + 4x^2y + 6x - 3 \\ &= 3x^2y + 4x^2y - 2x + 6x + 1 - 3\end{aligned}$$

By the distributive property,

$$3x^2y + 4x^2y = (3 + 4)x^2y = 7x^2y$$

and

$$-2x + 6x = (-2 + 6)x = 4x$$

Hence, $(3x^2y - 2x + 1) + (4x^2y + 6x - 3) = 7x^2y + 4x - 2$

Now Work Problem 3 ◁

EXAMPLE 3 Subtracting Algebraic Expressions

Simplify $(3x^2y - 2x + 1) - (4x^2y + 6x - 3)$.

Solution: Here we apply the definition of subtraction and the distributive property:

$$\begin{aligned}&(3x^2y - 2x + 1) - (4x^2y + 6x - 3) \\ &\quad = (3x^2y - 2x + 1) + (-1)(4x^2y + 6x - 3) \\ &\quad = (3x^2y - 2x + 1) + (-4x^2y - 6x + 3) \\ &\quad = 3x^2y - 2x + 1 - 4x^2y - 6x + 3 \\ &\quad = 3x^2y - 4x^2y - 2x - 6x + 1 + 3 \\ &\quad = (3 - 4)x^2y + (-2 - 6)x + 1 + 3 \\ &\quad = -x^2y - 8x + 4\end{aligned}$$

Now Work Problem 13 ◁

EXAMPLE 4 Removing Grouping Symbols

Simplify $3\{2x[2x + 3] + 5[4x^2 - (3 - 4x)]\}$.

Solution: We first eliminate the innermost grouping symbols (the parentheses). Then we repeat the process until all grouping symbols are removed—combining similar terms whenever possible. We have

$$\begin{aligned}3\{2x[2x + 3] + 5[4x^2 - (3 - 4x)]\} &= 3\{2x[2x + 3] + 5[4x^2 - 3 + 4x]\} \\ &= 3\{4x^2 + 6x + 20x^2 - 15 + 20x\} \\ &= 3\{24x^2 + 26x - 15\} \\ &= 72x^2 + 78x - 45\end{aligned}$$

Observe that properly paired parentheses are the only grouping symbols needed

$$3\{2x[2x + 3] + 5[4x^2 - (3 - 4x)]\} = 3(2x(2x + 3) + 5(4x^2 - (3 - 4x)))$$

but the optional use of brackets and braces sometimes adds clarity.

Now Work Problem 15 ◁

The distributive property is the key tool in multiplying expressions. For example, to multiply $ax + c$ by $bx + d$ we can consider $ax + c$ to be a single number and then use the distributive property:

$$(ax + c)(bx + d) = (ax + c)bx + (ax + c)d$$

Using the distributive property again, we have

$$\begin{aligned}(ax + c)bx + (ax + c)d &= abx^2 + cbx + adx + cd \\ &= abx^2 + (ad + cb)x + cd\end{aligned}$$

Thus, $(ax + c)(bx + d) = abx^2 + (ad + cb)x + cd$. In particular, if $a = 2, b = 1$, $c = 3$, and $d = -2$, then

$$\begin{aligned}(2x + 3)(x - 2) &= 2(1)x^2 + [2(-2) + 3(1)]x + 3(-2) \\ &= 2x^2 - x - 6\end{aligned}$$

We now give a list of special products that can be obtained from the distributive property and are useful in multiplying algebraic expressions.

Special Products

1. $x(y + z) = xy + xz$	distributive property
2. $(x + a)(x + b) = x^2 + (a + b)x + ab$	
3. $(ax + c)(bx + d) = abx^2 + (ad + cb)x + cd$	
4. $(x + a)^2 = x^2 + 2ax + a^2$	square of a sum
5. $(x - a)^2 = x^2 - 2ax + a^2$	square of a difference
6. $(x + a)(x - a) = x^2 - a^2$	product of sum and difference
7. $(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$	cube of a sum
8. $(x - a)^3 = x^3 - 3ax^2 + 3a^2x - a^3$	cube of a difference

EXAMPLE 5 Special Products

a. By Rule 2,

$$\begin{aligned}(x + 2)(x - 5) &= (x + 2)(x + (-5)) \\ &= x^2 + (2 - 5)x + 2(-5) \\ &= x^2 - 3x - 10\end{aligned}$$

b. By Rule 3,

$$\begin{aligned}(3z + 5)(7z + 4) &= 3 \cdot 7z^2 + (3 \cdot 4 + 5 \cdot 7)z + 5 \cdot 4 \\ &= 21z^2 + 47z + 20\end{aligned}$$

c. By Rule 5,

$$\begin{aligned}(x - 4)^2 &= x^2 - 2(4)x + 4^2 \\ &= x^2 - 8x + 16\end{aligned}$$

d. By Rule 6,

$$\begin{aligned}(\sqrt{y^2 + 1} + 3)(\sqrt{y^2 + 1} - 3) &= (\sqrt{y^2 + 1})^2 - 3^2 \\ &= (y^2 + 1) - 9 \\ &= y^2 - 8\end{aligned}$$

e. By Rule 7,

$$\begin{aligned}(3x+2)^3 &= (3x)^3 + 3(2)(3x)^2 + 3(2)^2(3x) + (2)^3 \\ &= 27x^3 + 54x^2 + 36x + 8\end{aligned}$$

Now Work Problem 19 ◁

EXAMPLE 6 **Multiplying Multinomials**

Find the product $(2t-3)(5t^2+3t-1)$.

Solution: We treat $2t-3$ as a single number and apply the distributive property twice:

$$\begin{aligned}(2t-3)(5t^2+3t-1) &= (2t-3)5t^2 + (2t-3)3t - (2t-3)1 \\ &= 10t^3 - 15t^2 + 6t^2 - 9t - 2t + 3 \\ &= 10t^3 - 9t^2 - 11t + 3\end{aligned}$$

Now Work Problem 35 ◁

In Example 3(b) of Section 0.2, we showed that $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$. Similarly, $\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$. Using these results, we can divide a multinomial by a monomial by dividing each term in the multinomial by the monomial.

EXAMPLE 7 **Dividing a Multinomial by a Monomial**

a. $\frac{x^3+3x}{x} = \frac{x^3}{x} + \frac{3x}{x} = x^2 + 3$

b. $$\begin{aligned}\frac{4z^3 - 8z^2 + 3z - 6}{2z} &= \frac{4z^3}{2z} - \frac{8z^2}{2z} + \frac{3z}{2z} - \frac{6}{2z} \\ &= 2z^2 - 4z + \frac{3}{2} - \frac{3}{z}\end{aligned}$$

Now Work Problem 47 ◁

Long Division

To divide a polynomial by a polynomial, we use so-called **long division** when the degree of the divisor is less than or equal to the degree of the dividend, as the next example shows.

EXAMPLE 8 **Long Division**

Divide $2x^3 - 14x - 5$ by $x - 3$.

Solution: Here $2x^3 - 14x - 5$ is the dividend and $x - 3$ is the divisor. To avoid errors, it is best to write the dividend as $2x^3 + 0x^2 - 14x - 5$. Note that the powers of x are in decreasing order. We have

$$\begin{array}{rl}
 & 2x^2 + 6x + 4 \leftarrow \text{quotient} \\
\text{divisor} \rightarrow x - 3 \overline{)} & 2x^3 + 0x^2 - 14x - 5 \leftarrow \text{dividend} \\
 & \underline{2x^3 - 6x^2} \\
 & \quad 6x^2 - 14x \\
 & \quad \underline{6x^2 - 18x} \\
 & \qquad 4x - 5 \\
 & \qquad \underline{4x - 12} \\
 & \qquad\quad 7 \leftarrow \text{remainder}
\end{array}$$

Note that we divided x (the first term of the divisor) into $2x^3$ and got $2x^2$. Then we multiplied $2x^2$ by $x - 3$, getting $2x^3 - 6x^2$. After subtracting $2x^3 - 6x^2$ from $2x^3 + 0x^2$, we obtained $6x^2$ and then "brought down" the term $-14x$. This process is continued until we arrive at 7, the remainder. We always stop when the remainder is 0 or is a polynomial whose degree is less than the degree of the divisor. Our answer can be written as

$$2x^2 + 6x + 4 + \frac{7}{x-3}$$

That is, the answer to the question

$$\frac{\text{dividend}}{\text{divisor}} = ?$$

has the form

$$\text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

A way of checking a division is to verify that

$$(\text{quotient})(\text{divisor}) + \text{remainder} = \text{dividend}$$

By using this equation, you should be able to verify the result of the example.

Now Work Problem 51 ◁

PROBLEMS 0.4

Perform the indicated operations and simplify.

1. $(8x - 4y + 2) + (3x + 2y - 5)$
2. $(4a^2 - 2ab + 3) + (5c - 3ab + 7)$
3. $(8t^2 - 6s^2) + (4s^2 - 2t^2 + 6)$
4. $(\sqrt{x} + 2\sqrt{x}) + (3\sqrt{x} + 4\sqrt{x})$
5. $(\sqrt{a} + 2\sqrt{3b}) - (\sqrt{c} - 3\sqrt{3b})$
6. $(3a + 7b - 9) - (5a + 9b + 21)$
7. $(7x^2 + 5xy + \sqrt{2}) - (2z - 2xy + \sqrt{2})$
8. $(\sqrt{x} + 2\sqrt{x}) - (\sqrt{x} + 3\sqrt{x})$
9. $(\sqrt[2]{2x} + \sqrt[3]{3y}) - (\sqrt[2]{2x} + \sqrt[4]{4z})$
10. $4(2z - w) - 3(w - 2z)$
11. $3(3x + 3y - 7) - 3(8x - 2y + 2)$
12. $(4s - 5t) + (-2s - 5t) + (s + 9)$
13. $5(x^2 - y^2) + x(y - 3x) - 4y(2x + 7y)$
14. $(7 + 3(x - 3) - (4 - 5x))$
15. $2(3(3(x^2 + 2) - 2(x^2 - 5)))$
16. $4(3(t + 5) - t(1 - (t + 1)))$
17. $-2(3u^2(2u + 2) - 2(u^2 - (5 - 2u)))$
18. $-(-3[2a + 2b - 2] + 5(2a + 3b) - a(2(b + 5)))$
19. $(2x + 5)(3x - 2)$
20. $(u + 2)(u + 5)$
21. $(w + 2)(w - 5)$
22. $(x - 4)(x + 7)$
23. $(2x + 3)(5x + 2)$
24. $(t^2 - 5t)(3t^2 - 7t)$
25. $(X + 2Y)^2$
26. $(2x - 1)^2$
27. $(7 - X)^2$
28. $(\sqrt{x} - 1)(2\sqrt{x} + 5)$
29. $(\sqrt{5x} - 2)^2$
30. $(\sqrt{y} - 3)(\sqrt{y} + 3)$
31. $(2s - 1)(2s + 1)$
32. $(a^2 + 2b)(a^2 - 2b)$
33. $(x^2 - 3)(x + 4)$
34. $(u - 1)(u^2 + 3u - 2)$
35. $(x^2 - 4)(3x^2 + 2x - 1)$
36. $(3y - 2)(4y^3 + 2y^2 - 3y)$
37. $t(3(t + 2)(t - 4) + 5(3t(t - 7)))$
38. $((2z + 1)(2z - 1))(4z^2 + 1)$
39. $(s - t + 4)(3s + 2t - 1)$
40. $(x^2 + x + 1)^2$
41. $(2a + 3)^3$
42. $(2a - 3)^3$
43. $(2x - 3)^3$
44. $(3a + b)^3$
45. $\dfrac{z^2 - 18z}{z}$
46. $\dfrac{2x^3 - 7x + 4}{x}$
47. $\dfrac{6u^5 + 9u^3 - 1}{3u^2}$
48. $\dfrac{(3y - 4) - (9y + 5)}{3y}$
49. $(x^2 + 7x - 5) \div (x + 5)$
50. $(x^2 - 5x + 4) \div (x - 4)$
51. $(3x^3 - 2x^2 + x - 3) \div (x + 2)$
52. $(x^4 + 3x^2 + 2) \div (x + 1)$
53. $x^3 \div (x + 2)$
54. $(8x^2 + 6x + 7) \div (2x + 1)$
55. $(3x^2 - 4x + 3) \div (3x + 2)$
56. $(z^3 + z^2 + z) \div (z^2 - z + 1)$

Objective

To state the basic rules for factoring and apply them to factor expressions.

0.5 Factoring

If two or more expressions are multiplied together, the expressions are called *factors* of the product. Thus, if $c = ab$, then a and b are both factors of the product c. The process by which an expression is written as a product of its factors is called *factoring*.

Listed next are rules for factoring expressions, most of which arise from the special products discussed in Section 0.4. The right side of each identity is the factored form of the left side.

Rules for Factoring

1. $xy + xz = x(y + z)$ — common factor
2. $x^2 + (a + b)x + ab = (x + a)(x + b)$
3. $abx^2 + (ad + cb)x + cd = (ax + c)(bx + d)$
4. $x^2 + 2ax + a^2 = (x + a)^2$ — perfect-square trinomial
5. $x^2 - 2ax + a^2 = (x - a)^2$ — perfect-square trinomial
6. $x^2 - a^2 = (x + a)(x - a)$ — difference of two squares
7. $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$ — sum of two cubes
8. $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$ — difference of two cubes

When factoring a polynomial, we usually choose factors that themselves are polynomials. For example, $x^2 - 4 = (x + 2)(x - 2)$. We will not write $x - 4$ as $(\sqrt{x} + 2)(\sqrt{x} - 2)$ unless it allows us to simplify other calculations.

Always factor as completely as you can. For example,

$$2x^2 - 8 = 2(x^2 - 4) = 2(x + 2)(x - 2)$$

EXAMPLE 1 Common Factors

a. Factor $3k^2x^2 + 9k^3x$ completely.

Solution: Since $3k^2x^2 = (3k^2x)(x)$ and $9k^3x = (3k^2x)(3k)$, each term of the original expression contains the common factor $3k^2x$. Thus, by Rule 1,

$$3k^2x^2 + 9k^3x = 3k^2x(x + 3k)$$

Note that although $3k^2x^2 + 9k^3x = 3(k^2x^2 + 3k^3x)$, we do not say that the expression is completely factored, since $k^2x^2 + 3k^3x$ can still be factored.

b. Factor $8a^5x^2y^3 - 6a^2b^3yz - 2a^4b^4xy^2z^2$ completely.

Solution: $8a^5x^2y^3 - 6a^2b^3yz - 2a^4b^4xy^2z^2 = 2a^2y(4a^3x^2y^2 - 3b^3z - a^2b^4xyz^2)$

Now Work Problem 5 ◁

EXAMPLE 2 Factoring Trinomials

a. Factor $3x^2 + 6x + 3$ completely.

Solution: First we remove a common factor. Then we factor the resulting expression completely. Thus, we have

$$\begin{aligned} 3x^2 + 6x + 3 &= 3(x^2 + 2x + 1) \\ &= 3(x + 1)^2 \qquad \text{Rule 4} \end{aligned}$$

b. Factor $x^2 - x - 6$ completely.

Solution: If this trinomial factors into the form $(x + a)(x + b)$, which is a product of two binomials, then we must determine the values of a and b. Since $(x + a)(x + b) = x^2 + (a + b)x + ab$, it follows that

$$x^2 + (-1)x + (-6) = x^2 + (a + b)x + ab$$

By equating corresponding coefficients, we want

$$a + b = -1 \quad \text{and} \quad ab = -6$$

It is not always possible to factor a trinomial, using real numbers, even if the trinomial has integer coefficients. We will comment further on this point in Section 0.8.

If $a = -3$ and $b = 2$, then both conditions are met and hence

$$x^2 - x - 6 = (x - 3)(x + 2)$$

As a check, it is wise to multiply the right side to see if it agrees with the left side.

c. Factor $x^2 - 7x + 12$ completely.

Solution: $$x^2 - 7x + 12 = (x - 3)(x - 4)$$

Now Work Problem 9 ◁

EXAMPLE 3 Factoring

The following is an assortment of expressions that are completely factored. The numbers in parentheses refer to the rules used.

a. $x^2 + 8x + 16 = (x + 4)^2$ (4)

b. $9x^2 + 9x + 2 = (3x + 1)(3x + 2)$ (3)

c. $6y^3 + 3y^2 - 18y = 3y(2y^2 + y - 6)$ (1)

$\quad = 3y(2y - 3)(y + 2)$ (3)

d. $x^2 - 6x + 9 = (x - 3)^2$ (5)

e. $z^{1/4} + z^{5/4} = z^{1/4}(1 + z)$ (1)

f. $x^4 - 1 = (x^2 + 1)(x^2 - 1)$ (6)

$\quad = (x^2 + 1)(x + 1)(x - 1)$ (6)

g. $x^{2/3} - 5x^{1/3} + 4 = (x^{1/3} - 1)(x^{1/3} - 4)$ (2)

h. $ax^2 - ay^2 + bx^2 - by^2 = a(x^2 - y^2) + b(x^2 - y^2)$ (1), (1)

$\quad = (x^2 - y^2)(a + b)$ (1)

$\quad = (x + y)(x - y)(a + b)$ (6)

i. $8 - x^3 = (2)^3 - (x)^3 = (2 - x)(4 + 2x + x^2)$ (8)

j. $x^6 - y^6 = (x^3)^2 - (y^3)^2 = (x^3 + y^3)(x^3 - y^3)$ (6)

$\quad = (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)$ (7), (8)

Now Work Problem 35 ◁

Note in Example 3(f) that $x^2 - 1$ is factorable, but $x^2 + 1$ is not. In Example 3(h), note that the common factor of $x^2 - y^2$ was not immediately evident.

Students often wonder why factoring is important. Why does the prof seem to think that the right side of $x^2 - 7x + 12 = (x - 3)(x - 4)$ is better than the left side? Often, the reason is that *if a product of numbers is* 0 *then at least one of the numbers is* 0. In symbols

$$\text{If} \quad ab = 0 \quad \text{then} \quad a = 0 \quad \text{or} \quad b = 0$$

This is a useful principle for solving equations. For example, knowing $x^2 - 7x + 12 = (x - 3)(x - 4)$ it follows that if $x^2 - 7x + 12 = 0$ then $(x - 3)(x - 4) = 0$ and from the principle above, $x - 3 = 0$ or $x - 4 = 0$. Now we see immediately that either $x = 3$ or $x = 4$. We should also remark that in the displayed principle the word "or" is use inclusively. In other words, if $ab = 0$ it *may* be that both $a = 0$ and $b = 0$.

If $ab = 0$, *at least* one of a and b is 0.

PROBLEMS 0.5

Factor the following expressions completely.

1. $5bx + 5b$

2. $6y^2 - 4y$

3. $10xy + 5xz$

4. $3x^2y - 9x^3y^3$

5. $3a^3bcd^2 - 4ab^3c^2d^2 + 2a^3bc^4d^3$

6. $5r^2st^2 + 10r^3s^2t^3 - 15r^2t^2$

7. $z^2 - 49$

8. $x^2 - x - 6$

9. $p^2 + 4p + 3$

10. $t^2 - t - 12$

11. $25y^2 - 4$

12. $x^2 + 2x - 24$

13. $a^2 + 12a + 35$

14. $4t^2 - 9s^2$

15. $y^2 + 8y + 15$

16. $t^2 - 18t + 72$

17. $5x^2 + 25x + 30$

18. $3t^2 + 12t - 15$

19. $3x^2 - 3$

20. $6x^2 + 31x + 35$

21. $5x^2 + 16x + 3$

22. $4x^2 - x - 3$

23. $12s^3 + 10s^2 - 8s$

24. $9z^2 + 30z + 25$

25. $a^{11/3}b - 4a^{2/3}b^3$

26. $4x^{6/5} - 1$

27. $2x^3 + 2x^2 - 12x$

28. $x^2y^2 - 4xy + 4$

29. $(4x + 2)^2$

30. $x^2(2x^2 - 4x^3)^2$

31. $x^3y^2 - 16x^2y + 64x$

32. $(5x^2 + 2x) + (10x + 4)$

33. $(x^3 - 4x) + (8 - 2x^2)$

34. $(x^2 - 1) + (x^2 - x - 2)$

35. $4ax^2 - ay^2 + 12bx^2 - 3by^2$

36. $t^3u - 3tu + t^2w^2 - 3w^2$

37. $b^3 + 64$

38. $x^3 - 1$

39. $x^6 - 1$

40. $64 + 27t^3$

41. $(x + 4)^3(x - 2) + (x + 4)^2(x - 2)^2$

42. $(a + 5)^3(a + 1)^2 + (a + 5)^2(a + 1)^3$

43. $P(1 + r) + P(1 + r)r$

44. $(X - 3I)(3X + 5I) - (3X + 5I)(X + 2I)$

45. $16u^2 - 81v^2w^2$

46. $256y^4 - z^4$

47. $y^8 - 1$

48. $t^4 - 4$

49. $X^4 + 4X^2 - 5$

50. $4x^4 - 20x^2 + 25$

51. $a^4b - 8a^2b + 16b$

52. $4x^3 - 6x^2 - 4x$

Objective

To simplify, add, subtract, multiply, and divide algebraic fractions. To rationalize the denominator of a fraction.

0.6 Fractions

Students should take particular care in studying *fractions*. In everyday life, numerical fractions often disappear from view with the help of calculators. However, manipulation of fractions of algebraic expressions is essential in calculus, and here most calculators are of no help.

Simplifying Fractions

By using the fundamental principle of fractions (Section 0.2), we may be able to simplify algebraic expressions that are fractions. That principle allows us to multiply or divide both the numerator and the denominator of a fraction by the same nonzero quantity. The resulting fraction will be equal to the original one. The fractions that we consider are assumed to have nonzero denominators. Thus, all the factors of the denominators in our examples are assumed to be nonzero. This will often mean that certain values are excluded for the variables that occur in the denominators.

EXAMPLE 1 Simplifying Fractions

a. Simplify $\dfrac{x^2 - x - 6}{x^2 - 7x + 12}$.

Solution: First, we completely factor both the numerator and the denominator:

$$\frac{x^2 - x - 6}{x^2 - 7x + 12} = \frac{(x - 3)(x + 2)}{(x - 3)(x - 4)}$$

Dividing both numerator and denominator by the common factor $x - 3$, we have

$$\frac{(x - 3)(x + 2)}{(x - 3)(x - 4)} = \frac{1(x + 2)}{1(x - 4)} = \frac{x + 2}{x - 4} \quad \text{for } x \neq 3$$

Usually, we just write

$$\frac{x^2 - x - 6}{x^2 - 7x + 12} = \frac{(x - 3)(x + 2)}{(x - 3)(x - 4)} = \frac{x + 2}{x - 4} \quad \text{for } x \neq 3$$

The process of eliminating the common factor $x - 3$ is commonly referred to as "cancellation." We issued a blanket statement before this example that all fractions are assumed to have nonzero denominators and that this requires excluding certain values for the variables. Observe that, nevertheless, we explicitly wrote "for $x \neq 3$". This is because the expression to the right of the equal sign, $\frac{x+2}{x-4}$, *is defined for* $x = 3$. Its value is -5 but we want to make it clear that *the expression to the left of the equal sign is not defined for* $x = 3$.

b. Simplify $\frac{2x^2 + 6x - 8}{8 - 4x - 4x^2}$.

Solution:

$$\frac{2x^2 + 6x - 8}{8 - 4x - 4x^2} = \frac{2(x^2 + 3x - 4)}{4(2 - x - x^2)} = \frac{2(x-1)(x+4)}{4(1-x)(2+x)}$$

$$= \frac{2(x-1)(x+4)}{2(2)[(-1)(x-1)](2+x)}$$

$$= \frac{x+4}{-2(2+x)} \quad \text{for } x \neq 1$$

The simplified expression is defined for $x = 1$, but since the original expression is not defined for $x = 1$, we explicitly exclude this value.

Now Work Problem 3 ◁

Multiplication and Division of Fractions

The rule for multiplying $\frac{a}{b}$ by $\frac{c}{d}$ is

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

EXAMPLE 2 Multiplying Fractions

a. $$\frac{x}{x+2} \cdot \frac{x+3}{x-5} = \frac{x(x+3)}{(x+2)(x-5)}$$

b. $$\frac{x^2 - 4x + 4}{x^2 + 2x - 3} \cdot \frac{6x^2 - 6}{x^2 + 2x - 8} = \frac{[(x-2)^2][6(x+1)(x-1)]}{[(x+3)(x-1)][(x+4)(x-2)]}$$

$$= \frac{6(x-2)(x+1)}{(x+3)(x+4)} \quad \text{for } x \neq 1, 2$$

Note that we explicitly excluded the values that make the "cancelled factors" 0. While the final expression is defined for these values, the original expression is not.

Now Work Problem 9 ◁

To divide $\frac{a}{b}$ by $\frac{c}{d}$, where $b \neq 0$, $d \neq 0$, and $c \neq 0$, we have

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c}$$

In short, to divide by a fraction we invert the divisor and multiply.

EXAMPLE 3 Dividing Fractions

a. $$\frac{x}{x+2} \div \frac{x+3}{x-5} = \frac{x}{x+2} \cdot \frac{x-5}{x+3} = \frac{x(x-5)}{(x+2)(x+3)}$$

b. $$\frac{\frac{x-5}{x-3}}{2x} = \frac{\frac{x-5}{x-3}}{\frac{2x}{1}} = \frac{x-5}{x-3} \cdot \frac{1}{2x} = \frac{x-5}{2x(x-3)}$$

c. $$\frac{\frac{4x}{x^2-1}}{\frac{2x^2+8x}{x-1}} = \frac{4x}{x^2-1} \cdot \frac{x-1}{2x^2+8x} = \frac{4x(x-1)}{[(x+1)(x-1)][2x(x+4)]}$$

$$= \frac{2}{(x+1)(x+4)} \quad \text{for } x \neq 0, 1$$

Why did we write "for $x \neq 0, 1$"?

Now Work Problem 11 ◁

Rationalizing the Denominator

Sometimes the denominator of a fraction has two terms and involves square roots, such as $2 - \sqrt{3}$ or $\sqrt{5} + \sqrt{2}$. The denominator may then be rationalized by multiplying by an expression that makes the denominator a difference of two squares. For example,

$$\frac{4}{\sqrt{5}+\sqrt{2}} = \frac{4}{\sqrt{5}+\sqrt{2}} \cdot \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}}$$

$$= \frac{4(\sqrt{5}-\sqrt{2})}{(\sqrt{5})^2-(\sqrt{2})^2} = \frac{4(\sqrt{5}-\sqrt{2})}{5-2}$$

$$= \frac{4(\sqrt{5}-\sqrt{2})}{3}$$

Rationalizing the *numerator* is a similar procedure.

EXAMPLE 4 Rationalizing Denominators

a. $$\frac{x}{\sqrt{2}-6} = \frac{x}{\sqrt{2}-6} \cdot \frac{\sqrt{2}+6}{\sqrt{2}+6} = \frac{x(\sqrt{2}+6)}{(\sqrt{2})^2-6^2}$$

$$= \frac{x(\sqrt{2}+6)}{2-36} = -\frac{x(\sqrt{2}+6)}{34}$$

b. $$\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}} = \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}} \cdot \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}}$$

$$= \frac{(\sqrt{5}-\sqrt{2})^2}{5-2} = \frac{5-2\sqrt{5}\sqrt{2}+2}{3} = \frac{7-2\sqrt{10}}{3}$$

Now Work Problem 53 ◁

Addition and Subtraction of Fractions

In Example 3(b) of Section 0.2, it was shown that $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$. That is, if we add two fractions having a common denominator, then the result is a fraction whose denominator is the common denominator. The numerator is the sum of the numerators of the original fractions. Similarly, $\frac{a}{c} - \frac{b}{c} = \frac{\mathrm{a}-\mathrm{b}}{\mathrm{c}}$.

EXAMPLE 5 Adding and Subtracting Fractions

a. $$\frac{p^2-5}{p-2} + \frac{3p+2}{p-2} = \frac{(p^2-5)+(3p+2)}{p-2}$$

$$= \frac{p^2+3p-3}{p-2}$$

b. $$\frac{x^2-5x+4}{x^2+2x-3}-\frac{x^2+2x}{x^2+5x+6}=\frac{(x-1)(x-4)}{(x-1)(x+3)}-\frac{x(x+2)}{(x+2)(x+3)}$$
$$=\frac{x-4}{x+3}-\frac{x}{x+3}=\frac{(x-4)-x}{x+3}=-\frac{4}{x+3}\quad \text{for } x\neq -2, 1$$

c. $$\frac{x^2+x-5}{x-7}-\frac{x^2-2}{x-7}+\frac{-4x+8}{x^2-9x+14}=\frac{x^2+x-5}{x-7}-\frac{x^2-2}{x-7}+\frac{-4}{x-7}$$
$$=\frac{(x^2+x-5)-(x^2-2)+(-4)}{x-7}$$
$$=\frac{x-7}{x-7}=1\quad \text{for } x\neq 2, 7$$

Why did we write "for $x \neq 2, 7$"?

Now Work Problem 29 ◁

To add (or subtract) two fractions with *different* denominators, use the fundamental principle of fractions to rewrite the fractions as fractions that have the same denominator. Then proceed with the addition (or subtraction) by the method just described.

For example, to find
$$\frac{2}{x^3(x-3)}+\frac{3}{x(x-3)^2}$$
we can convert the first fraction to an equal fraction by multiplying the numerator and denominator by $x-3$:
$$\frac{2(x-3)}{x^3(x-3)^2}$$
and we can convert the second fraction by multiplying the numerator and denominator by x^2:
$$\frac{3x^2}{x^3(x-3)^2}$$
These fractions have the same denominator. Hence,
$$\frac{2}{x^3(x-3)}+\frac{3}{x(x-3)^2}=\frac{2(x-3)}{x^3(x-3)^2}+\frac{3x^2}{x^3(x-3)^2}$$
$$=\frac{3x^2+2x-6}{x^3(x-3)^2}$$
We could have converted the original fractions into equal fractions with *any* common denominator. However, we chose to convert them into fractions with the denominator $x^3(x-3)^2$. This denominator is the **least common denominator (LCD)** of the fractions $2/(x^3(x-3))$ and $3/[x(x-3)^2]$.

In general, to find the LCD of two or more fractions, first factor each denominator completely. *The LCD is the product of each of the distinct factors appearing in the denominators, each raised to the highest power to which it occurs in any single denominator*.

EXAMPLE 6 Adding and Subtracting Fractions

a. Subtract: $\dfrac{t}{3t+2}-\dfrac{4}{t-1}$.

Solution: The LCD is $(3t+2)(t-1)$. Thus, we have
$$\frac{t}{(3t+2)}-\frac{4}{t-1}=\frac{t(t-1)}{(3t+2)(t-1)}-\frac{4(3t+2)}{(3t+2)(t-1)}$$
$$=\frac{t(t-1)-4(3t+2)}{(3t+2)(t-1)}$$
$$=\frac{t^2-t-12t-8}{(3t+2)(t-1)}=\frac{t^2-13t-8}{(3t+2)(t-1)}$$

b. Add: $\dfrac{4}{q-1} + 3$.

Solution: The LCD is $q - 1$.

$$\frac{4}{q-1} + 3 = \frac{4}{q-1} + \frac{3(q-1)}{q-1}$$

$$= \frac{4 + 3(q-1)}{q-1} = \frac{3q+1}{q-1}$$

Now Work Problem 33 ◁

EXAMPLE 7 Subtracting Fractions

$$\frac{x-2}{x^2+6x+9} - \frac{x+2}{2(x^2-9)}$$

$$= \frac{x-2}{(x+3)^2} - \frac{x+2}{2(x+3)(x-3)} \qquad [\text{LCD} = 2(x+3)^2(x-3)]$$

$$= \frac{(x-2)(2)(x-3)}{(x+3)^2(2)(x-3)} - \frac{(x+2)(x+3)}{2(x+3)(x-3)(x+3)}$$

$$= \frac{(x-2)(2)(x-3) - (x+2)(x+3)}{2(x+3)^2(x-3)}$$

$$= \frac{2(x^2-5x+6) - (x^2+5x+6)}{2(x+3)^2(x-3)}$$

$$= \frac{2x^2 - 10x + 12 - x^2 - 5x - 6}{2(x+3)^2(x-3)}$$

$$= \frac{x^2 - 15x + 6}{2(x+3)^2(x-3)}$$

Now Work Problem 39 ◁

Example 8 is important for later work. Note that we explicitly assume $h \neq 0$.

EXAMPLE 8 Combined Operations with Fractions

Simplify $\dfrac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h}$, where $h \neq 0$.

Solution: First we combine the fractions in the numerator and obtain

$$\frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} = \frac{\dfrac{x}{x(x+h)} - \dfrac{x+h}{x(x+h)}}{h} = \frac{\dfrac{x-(x+h)}{x(x+h)}}{h}$$

$$= \frac{\dfrac{-h}{x(x+h)}}{\dfrac{h}{1}} = \frac{-h}{x(x+h)h} = -\frac{1}{x(x+h)}$$

Now Work Problem 47 ◁

Percentages

In business applications fractions are often expressed as *percentages*, which are sometimes confusing. We recall that $p\%$ means $\frac{p}{100}$. Also $p\%$ *of* x simply means $\frac{p}{100} \cdot x = \frac{px}{100}$. Notice that the p in $p\%$ is not required to be a number between 0 and 100. In fact, for any real number r we can write $r = 100r\%$. Thus, there are 3100% days in January. While this might sound absurd, it is correct and reinforces understanding of the *definition*:

$$p\% = \frac{p}{100}$$

Similarly, the use of "of" when dealing with percentages really is just multiplication. If we say "5 of 7", it means "five sevens" — which is 35.

If a cost has increased by 200% it means that the cost has increased by $\frac{200}{100} = 2$. Strictly speaking, this should mean that the *increase* is 2 times the old cost so that

$$\text{new cost} = \text{old cost} + \text{increase} = \text{old cost} + 2 \cdot \text{old cost} = 3 \cdot \text{old cost}$$

but people are not always clear when they speak in these terms. If you want to say that a cost has doubled, you can say that the cost has increased by 100%.

EXAMPLE 9 Operations with Percentages

A restaurant bill comes to \$73.59 to which is added Harmonized Sales Tax (HST) of 15%. A customer wishes to leave the waiter a tip of 20%, and the restaurant's credit card machine calculates a 20% tip by calculating 20% of the *after-tax* total. How much is charged to the customer's credit card?

Solution:

$$\begin{aligned}
\text{charge} &= (1 + 20\%)(\text{after-tax total}) \\
&= \frac{120}{100}((1 + 15\%)(\text{bill})) \\
&= \left(\frac{120}{100}\right)\left(\frac{115}{100}\right)(73.59) \\
&= \frac{13800}{10000}(73.59) \\
&= 1.38(73.59) \\
&= 101.5542
\end{aligned}$$

So, the credit card charge is \$101.55.

Now Work Problem 59 ◁

PROBLEMS 0.6

In Problems 1–6, simplify.

1. $\frac{x^3 + 27}{x^2 + 3x}$ **2.** $\frac{x^2 - 3x - 10}{x^2 - 4}$ **3.** $\frac{x^2 - 9x + 20}{x^2 + x - 20}$

4. $\frac{3x^2 - 27x + 24}{2x^3 - 16x^2 + 14x}$ **5.** $\frac{15x^2 + x - 2}{3x^2 + 20x - 7}$ **6.** $\frac{6x^2 - 19x - 7}{15x^2 + 11x + 2}$

In Problems 7–48, perform the operations and simplify as much as possible.

7. $\frac{y^2}{y - 3} \cdot \frac{-1}{y + 2}$ **8.** $\frac{t^2 - 9}{t^2 + 3t} \cdot \frac{t^2}{t^2 - 6t + 9}$

9. $\frac{ax - b}{x - c} \cdot \frac{c - x}{ax + b}$ **10.** $\frac{a^2 - b^2}{a - b} \cdot \frac{a^2 - 2ab + b^2}{2a + 2b}$

11. $\frac{3x + 3}{x^2 + 3x + 2} \div \frac{x^2 - x}{x^2 + x - 2}$

12. $\frac{x^2 + 2x}{3x^2 - 18x + 24} \div \frac{x^2 - x - 6}{x^2 - 4x + 4}$

13. $\dfrac{\frac{X^2}{8}}{\frac{X}{4}}$

14. $\dfrac{\frac{3x^2}{7x}}{\frac{x}{14}}$

15. $\dfrac{\frac{15u}{v^3}}{\frac{3u}{v^4}}$

16. $\dfrac{\frac{2x+y}{x}}{\frac{2x-y}{3x}}$

17. $\dfrac{\frac{4x}{3}}{2x}$

18. $\dfrac{4x}{\frac{3}{2x}}$

19. $\dfrac{-9x^3}{\frac{x}{3}}$

20. $\dfrac{\frac{21t^5}{t^2}}{-7}$

21. $\dfrac{\frac{2x+1}{2x^2-5x-3}}{x-3}$

22. $\dfrac{\frac{x^2+6x+9}{x}}{x+3}$

23. $\dfrac{\frac{10x^3}{x^2-1}}{\frac{5x}{x+1}}$

24. $\dfrac{\frac{x^2-x-6}{x^2-9}}{\frac{x^2-4}{x^2+2x-3}}$

25. $\dfrac{\frac{x^2+8x+12}{x^2+9x+18}}{\frac{x^2-3x-10}{x^2-2x-15}}$

26. $\dfrac{\frac{(x+1)^2}{2x-1}}{\frac{4x+4}{1-4x^2}}$

27. $\dfrac{\frac{4x^2-9}{x^2+3x-4}}{\frac{2x-3}{1-x^2}}$

28. $\dfrac{\frac{6x^2y+7xy-3y}{xy-x+5y-5}}{\frac{x^3y+4x^2y}{xy-x+4y-4}}$

29. $\dfrac{x^2}{x+3}+\dfrac{5x+6}{x+3}$

30. $\dfrac{-1}{x-1}+\dfrac{x}{x-1}$

31. $\dfrac{4}{x}+\dfrac{3}{5x^2}$

32. $\dfrac{9}{X^3}-\dfrac{1}{X^2}$

33. $1-\dfrac{x^3}{x^3-1}$

34. $\dfrac{4}{s+4}+s$

35. $\dfrac{1}{3x-1}+\dfrac{x}{x+1}$

36. $\dfrac{(x+1)^3-(x-1)^3}{(x-1)(x^2+x-1)}$

37. $\dfrac{1}{x^2-2x-3}+\dfrac{1}{x^2-9}$

38. $\dfrac{4}{2x^2-7x-4}-\dfrac{x}{2x^2-9x+4}$

39. $\dfrac{4}{x-1}-3+\dfrac{-3x^2}{5-4x-x^2}$

40. $\dfrac{x+1}{2x^2+3x-2}-\dfrac{x-1}{3x^2+5x-2}+\dfrac{1}{3x-1}$

41. $(1+x^{-1})^{-1}$

42. $(x^{-1}+y^{-1})^2$

43. $(x^{-1}-y)^{-1}$

44. $(a+b^{-1})^2$

45. $\dfrac{5+\frac{2}{x}}{3}$

46. $\dfrac{\frac{x+a}{x}}{x-\frac{a^2}{x}}$

47. $\dfrac{3-\frac{1}{2x}}{x+\frac{x}{x+2}}$

48. $\dfrac{\frac{x-1}{x^2+5x+6}-\frac{1}{x+2}}{3+\frac{x-7}{3}}$

In Problems 49 and 50, perform the indicated operations, but do not rationalize the denominators.

49. $\dfrac{3}{\sqrt[3]{x+h}}-\dfrac{3}{\sqrt[3]{x}}$

50. $\dfrac{x\sqrt{x}}{\sqrt{3+x}}+\dfrac{2}{\sqrt{x}}$

In Problems 51–60, simplify, and express your answer in a form that is free of radicals in the denominator.

51. $\dfrac{1}{a+\sqrt{b}}$

52. $\dfrac{1}{1-\sqrt{2}}$

53. $\dfrac{\sqrt{2}}{\sqrt{3}-\sqrt{6}}$

54. $\dfrac{5}{\sqrt{6}+\sqrt{7}}$

55. $\dfrac{2\sqrt{3}}{\sqrt{3}+\sqrt{5}}$

56. $\dfrac{\sqrt{a}}{\sqrt{b}-\sqrt{c}}$

57. $\dfrac{3}{t+\sqrt{7}}$

58. $\dfrac{x-3}{\sqrt{x}-1}+\dfrac{4}{\sqrt{x}-1}$

59. Pam Alnwick used to live in Rockingham, NS, where the Harmonized Sales Tax, HST, was 15%. She recently moved to Melbourne, FL, where sales tax is 6.5%. When shopping, the task of comparing American prices with Canadian prices was further complicated by the fact that, at the time of her move the Canadian dollar was worth 0.75US$. After thinking about it, she calculated a number K so that a pre-tax shelf price of A US$ could be sensibly compared with a pre-tax shelf price of C CDN$, so as to take into account the different sales tax rates. Her K had the property that if $AK = C$ then the after-tax costs in Canadian dollars were the same, while if AK is less (greater) than C then, after taxes, the American (Canadian) price is cheaper. Find Pam's multiplier K.

60. Repeat the calculation assuming a US tax rate of $a\%$ and a Canadian tax rate of $c\%$, when 1 CDN$ = R US$, so that Pam can help her stepson Tom, who moved from Calgary AB to Santa Barbara CA.

Objective

To discuss equivalent equations and to develop techniques for solving linear equations, including literal equations as well as fractional and radical equations that lead to linear equations.

0.7 Equations, in Particular Linear Equations

Equations

An **equation** is a statement that two expressions are equal. The two expressions that make up an equation are called its **sides**. They are separated by the **equality sign**, $=$.

EXAMPLE 1 **Examples of Equations**

a. $x+2=3$

b. $x^2+3x+2=0$

c. $\dfrac{y}{y-4} = 6$

d. $w = 7 - z$

Now Work Problem 1 ◁

In Example 1, each equation contains at least one variable. A **variable** is a symbol that can be replaced by any one of a set of different numbers. The most popular symbols for variables are letters from the latter part of the alphabet, such as x, y, z, w, and t. Hence, Equations (a) and (c) are said to be in the variables x and y, respectively. Equation (d) is in the variables w and z. In the equation $x + 2 = 3$, the numbers 2 and 3 are called *constants*. They are fixed numbers.

Here we discuss restrictions on variables.

We *never* allow a variable in an equation to have a value for which any expression in that equation is undefined. For example, in

$$\frac{y}{y-4} = 6$$

y cannot be 4, because this would make the denominator zero; while in

$$\sqrt{x-3} = 9$$

we cannot have $x-3$ negative because we cannot take square roots of negative numbers. We must have $x - 3 \geq 0$, which is equivalent to the requirement $x \geq 3$. (We will have more to say about inequalities in Chapter 1.) In some equations, the allowable values of a variable are restricted for physical reasons. For example, if the variable q represents quantity sold, negative values of q may not make sense.

To **solve** an equation means to find all values of its variables for which the equation is true. These values are called **solutions** of the equation and are said to **satisfy** the equation. When only one variable is involved, a solution is also called a **root**. The set of all solutions is called the **solution set** of the equation. Sometimes a letter representing an unknown quantity in an equation is simply called an *unknown*. Example 2 illustrates these terms.

EXAMPLE 2 Terminology for Equations

a. In the equation $x + 2 = 3$, the variable x is the unknown. The only value of x that satisfies the equation is obviously 1. Hence, 1 is a root and the solution set is $\{1\}$.

b. -2 is a root of $x^2 + 3x + 2 = 0$ because substituting -2 for x makes the equation true: $(-2)^2 + 3(-2) + 2 = 0$. Hence -2 is an element of the solution set, but in this case it is not the only one. There is one more. Can you find it?

c. $w = 7 - z$ is an equation in two unknowns. One solution is the pair of values $w = 4$ and $z = 3$. However, there are infinitely many solutions. Can you think of another?

Now Work Problem 3 ◁

Equivalent Equations

Two equations are said to be **equivalent** if they have exactly the same solutions, which means, precisely, that the solution set of one is equal to the solution set of the other. Solving an equation may involve performing operations on it. We prefer that any such operation result in an equivalent equation. Here are three operations that guarantee equivalence:

1. Adding (subtracting) the same polynomial to (from) both sides of an equation, where the polynomial is in the same variable as that occurring in the equation.

For example, if $-5x = 5 - 6x$, then adding $6x$ to both sides gives the equivalent equation $-5x + 6x = 5 - 6x + 6x$, which in turn is equivalent to $x = 5$.

Equivalence is not guaranteed if both sides are multiplied or divided by an expression involving a variable.

2. Multiplying (dividing) both sides of an equation by the same *nonzero* constant.

For example, if $10x = 5$, then dividing both sides by 10 gives the equivalent equation $\frac{10x}{10} = \frac{5}{10}$, equivalently, $x = \frac{1}{2}$.

3. Replacing either side of an equation by an equal expression.

For example, if the equation is $x(x + 2) = 3$, then replacing the left side by the equal expression $x^2 + 2x$ gives the equivalent equation $x^2 + 2x = 3$.

We repeat: Applying Operations 1–3 guarantees that the resulting equation is equivalent to the given one. However, sometimes in solving an equation we have to apply operations other than 1–3. These operations may *not* necessarily result in equivalent equations. They include the following:

Operations That May Not Produce Equivalent Equations

4. Multiplying both sides of an equation by an expression involving the variable.

5. Dividing both sides of an equation by an expression involving the variable.

6. Raising both sides of an equation to equal powers.

Operation 6 includes taking roots of both sides.

Let us illustrate the last three operations. For example, by inspection, the only root of $x - 1 = 0$ is 1. Multiplying each side by x (Operation 4) gives $x^2 - x = 0$, which is satisfied if x is 0 or 1. (Check this by substitution.) But 0 *does not* satisfy the *original* equation. Thus, the equations are not equivalent.

Continuing, you can check that the equation $(x - 4)(x - 3) = 0$ is satisfied when x is 4 or when x is 3. Dividing both sides by $x - 4$ (Operation 5) gives $x - 3 = 0$, whose only root is 3. Again, we do not have equivalence, since in this case a root has been "lost." Note that when x is 4, division by $x - 4$ implies division by 0, an invalid operation.

Finally, squaring each side of the equation $x = 2$ (Operation 6) gives $x^2 = 4$, which is true if $x = 2$ or if $x = -2$. But -2 is not a root of the given equation.

From our discussion, it is clear that when Operations 4–6 are performed, we must be careful about drawing conclusions concerning the roots of a given equation. Operations 4 and 6 *can* produce an equation with more roots. Thus, you should check whether or not each "solution" obtained by these operations satisfies the *original* equation. Operation 5 *can* produce an equation with fewer roots. In this case, any "lost" root may never be determined. Thus, avoid Operation 5 whenever possible.

In summary, an equation can be thought of as a set of restrictions on any variable in the equation. Operations 4–6 may increase or decrease the number of restrictions, giving solutions different from those of the original equation. However, Operations 1–3 never affect the restrictions.

Linear Equations

The principles presented so far will now be demonstrated in the solution of a **linear equation**.

> **Definition**
>
> A ***linear equation*** in the variable x is an equation that is equivalent to one that can be written in the form
>
> $$ax + b = 0 \quad \textbf{(1)}$$
>
> where a and b are constants and $a \neq 0$.

A linear equation is also called a first-degree equation or an equation of degree one, since the highest power of the variable that occurs in Equation (1) is the first.

To solve a linear equation, we perform operations on it until we have an equivalent equation whose solutions are obvious. This means an equation in which the variable is isolated on one side, as the following examples show.

EXAMPLE 3 **Solving a Linear Equation**

Solve $5x - 6 = 3x$.

Solution: We begin by getting the terms involving x on one side and the constant on the other. Then we solve for x by the appropriate mathematical operation. We have

$$
\begin{aligned}
5x - 6 &= 3x \\
5x - 6 + (-3x) &= 3x + (-3x) && \text{adding } -3x \text{ to both sides} \\
2x - 6 &= 0 && \text{simplifying, that is, Operation 3} \\
2x - 6 + 6 &= 0 + 6 && \text{adding 6 to both sides} \\
2x &= 6 && \text{simplifying} \\
\frac{2x}{2} &= \frac{6}{2} && \text{dividing both sides by 2} \\
x &= 3
\end{aligned}
$$

Clearly, 3 is the only root of the last equation. Since each equation is equivalent to the one before it, we conclude that 3 must be the only root of $5x - 6 = 3x$. That is, the solution set is $\{3\}$. We can describe the first step in the solution as moving a term from one side of an equation to the other while changing its sign; this is commonly called *transposing*. Note that since the original equation can be put in the form $2x + (-6) = 0$, it is a linear equation.

Now Work Problem 21 ◁

EXAMPLE 4 **Solving a Linear Equation**

Solve $2(p + 4) = 7p + 2$.

Solution: First, we remove parentheses. Then we collect like terms and solve. We have

$$
\begin{aligned}
2(p + 4) &= 7p + 2 \\
2p + 8 &= 7p + 2 && \text{distributive property} \\
2p &= 7p - 6 && \text{subtracting 8 from both sides} \\
-5p &= -6 && \text{subtracting } 7p \text{ from both sides} \\
p &= \frac{-6}{-5} && \text{dividing both sides by } -5 \\
p &= \frac{6}{5}
\end{aligned}
$$

Now Work Problem 25 ◁

EXAMPLE 5 **Solving a Linear Equation**

Solve $\dfrac{7x + 3}{2} - \dfrac{9x - 8}{4} = 6$.

Solution: We first clear the equation of fractions by multiplying *both* sides by the LCD, which is 4. Then we use various algebraic operations to obtain a solution. Thus,

$$4\left(\frac{7x + 3}{2} - \frac{9x - 8}{4}\right) = 4(6)$$

The distributive property requires that *both* terms within the parentheses be multiplied by 4.

$$4 \cdot \frac{7x+3}{2} - 4 \cdot \frac{9x-8}{4} = 24 \qquad \text{distributive property}$$
$$2(7x+3) - (9x-8) = 24 \qquad \text{simplifying}$$
$$14x + 6 - 9x + 8 = 24 \qquad \text{distributive property}$$
$$5x + 14 = 24 \qquad \text{simplifying}$$
$$5x = 10 \qquad \text{subtracting 14 from both sides}$$
$$x = 2 \qquad \text{dividing both sides by 5}$$

Now Work Problem 29 ◁

Every linear equation has exactly one root. The root of $ax + b = 0$ is $x = -\frac{b}{a}$.

Each equation in Examples 3–5 has one and only one root. This is true of every linear equation in one variable.

Literal Equations

Equations in which some of the constants are not specified, but are represented by letters, such as a, b, c, or d, are called **literal equations**, and the letters are called **literal constants**. For example, in the literal equation $x+a = 4b$, we can consider a and b to be literal constants. Formulas, such as $I = Prt$, that express a relationship between certain quantities may be regarded as literal equations. If we want to express a particular letter in a formula in terms of the others, this letter is considered the unknown.

EXAMPLE 6 Solving Literal Equations

a. The equation $I = Prt$ is the formula for the simple interest I on a principal of P dollars at the annual interest rate of r for a period of t years. Express r in terms of I, P, and t.

Solution: Here we consider r to be the unknown. To isolate r, we divide both sides by Pt. We have

$$I = Prt$$
$$\frac{I}{Pt} = \frac{Prt}{Pt}$$
$$\frac{I}{Pt} = r \text{ so } r = \frac{I}{Pt}$$

When we divided both sides by Pt, we assumed that $Pt \neq 0$, since we cannot divide by 0. Notice that this assumption is equivalent to requiring *both* $P \neq 0$ *and* $t \neq 0$. Similar assumptions will be made when solving other literal equations.

b. The equation $S = P + Prt$ is the formula for the value S of an investment of a principal of P dollars at a simple annual interest rate of r for a period of t years. Solve for P.

Solution:

$$S = P + Prt$$
$$S = P(1+rt) \qquad \text{factoring}$$
$$\frac{S}{1+rt} = P \qquad \text{dividing both sides by } 1+rt$$

Now Work Problem 79 ◁

EXAMPLE 7 Solving a Literal Equation

Solve $(a + c)x + x^2 = (x + a)^2$ for x.

Solution: We first simplify the equation and then get all terms involving x on one side:

$$\begin{aligned} (a+c)x + x^2 &= (x+a)^2 \\ ax + cx + x^2 &= x^2 + 2ax + a^2 \\ ax + cx &= 2ax + a^2 \\ cx - ax &= a^2 \\ x(c-a) &= a^2 \\ x &= \frac{a^2}{c-a} \quad \text{for } c \neq a \end{aligned}$$

Now Work Problem 81 ◁

EXAMPLE 8 Solving the "Tax in a Receipt" Problem

We recall Lesley Griffith's problem from the opening paragraphs of this chapter. We now generalize the problem so as to illustrate further the use of literal equations. Lesley had a receipt for an amount R. She knew that the sales tax rate was $p\%$. She wanted to know the amount that was paid in sales tax. Certainly,

$$\text{price} + \text{tax} = \text{receipt} \tag{2}$$

Writing P for the price (which she did not yet know), the tax was $(p/100)P$ so that she knew

$$\begin{aligned} P + \frac{p}{100}P &= R \\ P\left(1 + \frac{p}{100}\right) &= R \\ P\left(\frac{100+p}{100}\right) &= R \\ P &= \frac{100R}{100+p} \end{aligned}$$

It follows that the tax paid was

$$R - P = R - \frac{100R}{100+p} = R\left(1 - \frac{100}{100+p}\right) = R\left(\frac{p}{100+p}\right)$$

where you should check the manipulations with fractions, supplying more details if necessary. Recall that the French tax rate was 19.6% and the Italian tax rate was 18%. We conclude that Lesley had only to multiply a French receipt by $\frac{19.6}{119.6} \approx 0.16388$ to determine the tax it contained, while for an Italian receipt she should have multiplied the amount by $\frac{18}{118}$. With the current tax rates (20% and 22%, respectively) her multipliers would be $\frac{20}{120}$ and $\frac{22}{122}$, respectively, but she doesn't have to re-solve the problem. It should be noted that working from the simple conceptual Equation (2) we have been able to avoid the *assumption* about proportionality that we made at the beginning of this chapter.

It is also worth noting that while problems of this kind are often given using percentages, the algebra may be simplified by writing $p\% = \frac{p}{100}$ as a decimal. Algebraically, we make the substitution

$$r = \frac{p}{100}$$

so that Equation (2) becomes

$$P + rP = R$$

It should be clear that $P(1+r) = R$ so that $P = \dfrac{R}{1+r}$ and the tax in R is just $Pr = \dfrac{Rr}{1+r}$. The reader should now replace r in $\dfrac{Rr}{1+r}$ with $\dfrac{p}{100}$ and check that this simplifies to $R\left(\dfrac{p}{100+p}\right)$.

Moreover, examining this problem side by side with Example 6(b) above, we see that solving the "Tax in a Receipt" problem is really the same as determining, from an investment balance R, the amount of interest earned during the most recent interest period, when the interest rate per interest period is r.

Now Work Problem 99 ◁

Fractional Equations

A **fractional equation** is an equation in which an unknown is in a denominator. We illustrate that solving such a nonlinear equation may lead to a linear equation.

EXAMPLE 9 Solving a Fractional Equation

Solve $\dfrac{5}{x-4} = \dfrac{6}{x-3}$.

Solution:

Strategy We first write the equation in a form that is free of fractions. Then we use standard algebraic techniques to solve the resulting equation.

An alternative solution that avoids multiplying both sides by the LCD is as follows:

$$\frac{5}{x-4} - \frac{6}{x-3} = 0$$

Assuming that x is neither 3 nor 4 and combining fractions gives

$$\frac{9-x}{(x-4)(x-3)} = 0$$

A fraction can be 0 only when its numerator is 0 and its denominator is not. Hence, $x = 9$.

Multiplying both sides by the LCD, $(x-4)(x-3)$, we have

$$(x-4)(x-3)\left(\frac{5}{x-4}\right) = (x-4)(x-3)\left(\frac{6}{x-3}\right)$$

$$5(x-3) = 6(x-4) \qquad \text{linear equation}$$

$$5x - 15 = 6x - 24$$

$$9 = x$$

In the first step, we multiplied each side by an expression involving the *variable* x. As we mentioned in this section, this means that we are not guaranteed that the last equation is equivalent to the *original* equation. Thus, we must check whether or not 9 satisfies the *original* equation. Since

$$\frac{5}{9-4} = \frac{5}{5} = 1 \quad \text{and} \quad \frac{6}{9-3} = \frac{6}{6} = 1$$

we see that 9 indeed satisfies the original equation.

Now Work Problem 47 ◁

Some equations that are not linear do not have any solutions. In that case, we say that the solution set is the **empty set**, which we denote by $\emptyset$. Example 10 will illustrate.

EXAMPLE 10 Solving Fractional Equations

a. Solve $\dfrac{3x+4}{x+2} - \dfrac{3x-5}{x-4} = \dfrac{12}{x^2-2x-8}$.

Solution: Observing the denominators and noting that

$$x^2 - 2x - 8 = (x+2)(x-4)$$

we conclude that the LCD is $(x+2)(x-4)$. Multiplying both sides by the LCD, we have

$$(x+2)(x-4)\left(\frac{3x+4}{x+2} - \frac{3x-5}{x-4}\right) = (x+2)(x-4)\cdot\frac{12}{(x+2)(x-4)}$$

$$(x-4)(3x+4) - (x+2)(3x-5) = 12$$

$$3x^2 - 8x - 16 - (3x^2 + x - 10) = 12$$
$$3x^2 - 8x - 16 - 3x^2 - x + 10 = 12$$
$$-9x - 6 = 12$$
$$-9x = 18$$
$$x = -2 \tag{3}$$

However, the *original* equation is not defined for $x = -2$ (we cannot divide by zero), so there are no roots. Thus, the solution set is $\emptyset$. Although -2 is a solution of Equation (3), it is not a solution of the *original* equation.

b. Solve $\dfrac{4}{x-5} = 0$.

Solution: The only way a fraction can equal zero is for the numerator to equal zero (and the denominator to not equal zero). Since the numerator, 4, is not 0, the solution set is $\emptyset$.

Now Work Problem 43 ◁

EXAMPLE 11 Literal Equation

If $s = \dfrac{u}{au + v}$, express u in terms of the remaining letters; that is, solve for u.

Solution:

Strategy Since the unknown, u, occurs in the denominator, we first clear fractions and then solve for u.

$$s = \frac{u}{au + v}$$
$$s(au + v) = u \qquad \text{multiplying both sides by } au + v$$
$$sau + sv = u$$
$$sau - u = -sv$$
$$u(sa - 1) = -sv$$
$$u = \frac{-sv}{sa - 1} = \frac{sv}{1 - sa}$$

Now Work Problem 83 ◁

Radical Equations

A **radical equation** is one in which an unknown occurs in a radicand. The next two examples illustrate the techniques employed to solve such equations.

EXAMPLE 12 Solving a Radical Equation

Solve $\sqrt{x^2 + 33} - x = 3$.

Solution: To solve this radical equation, we raise both sides to the same power to eliminate the radical. This operation does *not* guarantee equivalence, so we must check any resulting "solutions." We begin by isolating the radical on one side. Then we square both sides and solve using standard techniques. Thus,

$$\sqrt{x^2 + 33} = x + 3$$
$$x^2 + 33 = (x + 3)^2 \qquad \text{squaring both sides}$$

$$x^2 + 33 = x^2 + 6x + 9$$
$$24 = 6x$$
$$4 = x$$

You should show by substitution that 4 is indeed a root.

Now Work Problem 71 ◁

With some radical equations, you may have to raise both sides to the same power more than once, as Example 13 shows.

EXAMPLE 13 Solving a Radical Equation

Solve $\sqrt{y-3} - \sqrt{y} = -3$.

The reason we want one radical on each side is to avoid squaring a binomial with two different radicals.

Solution: When an equation has two terms involving radicals, first write the equation so that one radical is on each side, if possible. Then square and solve. We have

$$\begin{aligned} \sqrt{y-3} &= \sqrt{y} - 3 \\ y - 3 &= y - 6\sqrt{y} + 9 && \text{squaring both sides} \\ 6\sqrt{y} &= 12 \\ \sqrt{y} &= 2 \\ y &= 4 && \text{squaring both sides} \end{aligned}$$

Substituting 4 into the left side of the *original* equation gives $\sqrt{1} - \sqrt{4}$, which is -1. Since this does not equal the right side, -3, there is no solution. That is, the solution set is $\emptyset$.

Now Work Problem 69 ◁

PROBLEMS 0.7

In Problems 1–6, determine by substitution which of the given numbers, if any, satisfy the given equation.

1. $9x - x^2 = 0;\ 1,\ 0$

2. $10 - 7x = -x^2;\ 2, 4$

3. $z + 3(z-4) = 5;\ \frac{17}{4},\ 4$

4. $x^2 + x - 6 = 0;\ 2, 3$

5. $x(6+x) - 2(x+1) - 5x = 4;\ -2,\ 0$

6. $x(x+1)^2(x+2) = 0;\ 0, -1,\ 2$

In Problems 7–16, determine what operations were applied to the first equation to obtain the second. State whether or not the operations guarantee that the equations are equivalent. Do not solve the equations.

7. $2x - 3 = 4x + 12;\ 2x = 4x + 15$

8. $8x - 4 = 16;\ x - \frac{1}{2} = 2$

9. $x = 5;\ x^4 = 625$

10. $2x^2 + 4 = 5x - 7;\ x^2 + 2 = \frac{5}{2}x - \frac{7}{2}$

11. $x^2 - 2x = 0;\ x - 2 = 0$

12. $\frac{a}{x-b} + x = x^2;\ a + x(x-b) = x^2(x-b)$

13. $\frac{x^2-1}{x-1} = 3;\ x^2 - 1 = 3(x-1)$

14. $(x+2)(x+1) = (x+3)(x+1);\ x + 2 = x + 3$

15. $\frac{2x(3x+1)}{2x-3} = 2x(x+4);\ 3x + 1 = (x+4)(2x-3)$

16. $2x^2 - 9 = x;\ x^2 - \frac{1}{2}x = \frac{9}{2}$

In Problems 17–72, solve the equations.

17. $\pi x = 3.14$

18. $0.2x = 7$

19. $-8x = 12 - 20$

20. $4 - 7x = 3$

21. $5x - 3 = 9$

22. $\sqrt[3]{2}x + 2 = 11$

23. $7x + 7 = 2(x+1)$

24. $4s + 3s - 1 = 41$

25. $5(p-7) - 2(3p-4) = 3p$

26. $t = 2 - 2(2t - 3(1-t))$

27. $\frac{2x}{5} = 4x - 3$

28. $\frac{5y}{7} - \frac{6}{7} = 2 - 4y$

29. $7 + \frac{4x}{9} = \frac{x}{2}$

30. $\frac{x}{3} - 4 = \frac{x}{5}$

31. $3x + \frac{x}{5} - 5 = \frac{1}{5} + 5x$

32. $x + \frac{x}{2} + \frac{x}{3} = \frac{x}{4}$

33. $\frac{2y-3}{4} = \frac{6y+7}{3}$

34. $\frac{t}{4} + \frac{5}{3}t = \frac{7}{2}(t-1)$

35. $t + \frac{t}{3} - \frac{t}{4} + \frac{t}{36} = 10$

36. $\frac{7 + 2(x+1)}{3} = \frac{6x}{5}$

37. $\frac{7}{5}(2-x)=\frac{5}{7}(x-2)$

38. $\frac{2x-7}{3}+\frac{8x-9}{14}=\frac{3x-5}{21}$

39. $\frac{4}{3}(5x-2)=7[x-(5x-2)]$

40. $(2x-5)^2+(3x-3)^2=13x^2-5x+7$

41. $\frac{5}{x}=25$

42. $\frac{4}{x-1}=2$

43. $\frac{5}{x+3}=0$

44. $\frac{3x-5}{x-3}=0$

45. $\frac{3}{5-2x}=\frac{7}{2}$

46. $\frac{x+3}{x}=\frac{2}{5}$

47. $\frac{a}{x-b}=\frac{c}{x-d}$ for $a\neq 0$ and $c\neq 0$

48. $\frac{2x-3}{4x-5}=6$

49. $\frac{1}{x}+\frac{1}{7}=\frac{3}{7}$

50. $\frac{2}{x-1}=\frac{3}{x-2}$

51. $\frac{2t+1}{2t+3}=\frac{3t-1}{3t+4}$

52. $\frac{x-1}{x+2}=\frac{x-3}{x+4}$

53. $\frac{y-6}{y}-\frac{6}{y}=\frac{y+6}{y-6}$

54. $\frac{y-2}{y+2}=\frac{y-2}{y+3}$

55. $\frac{-5}{2x-3}=\frac{7}{3-2x}+\frac{11}{3x+5}$

56. $\frac{1}{x+1}+\frac{2}{x-3}=\frac{-6}{3-2x}$

57. $\frac{1}{x-2}=\frac{3}{x-4}$

58. $\frac{x}{x+3}-\frac{x}{x-3}=\frac{3x-4}{x^2-9}$

59. $\sqrt{x+5}=4$

60. $\sqrt{z-2}=3$

61. $\sqrt{2x+3}-4=0$

62. $3-\sqrt{2x+1}=0$

63. $\sqrt{\frac{x}{2}+1}=\frac{2}{3}$

64. $(x+6)^{1/2}=7$

65. $\sqrt{4x-6}=\sqrt{x}$

66. $\sqrt{x+1}=\sqrt{2x-3}$

67. $(x-7)^{3/4}=8$

68. $\sqrt{y^2-9}=9-y$

69. $\sqrt{y}+\sqrt{y+2}=3$

70. $\sqrt{x}-\sqrt{x+1}=1$

71. $\sqrt{a^2+2a}=2+a$

72. $\sqrt{\frac{1}{w-1}}-\sqrt{\frac{2}{3w-4}}=0$

In Problems 73–84, express the indicated symbol in terms of the remaining symbols.

73. $I=Prt;\ r$

74. $P\left(1+\frac{p}{100}\right)-R=0;\ P$

75. $p=8q-1;\ q$

76. $p=10-2q;\ q$

77. $S=P(1+rt);\ t$

78. $r=\frac{2mI}{B(n+1)};I$

79. $A=\frac{R(1-(1+i)^{-n})}{i};\ R$

80. $S=\frac{R((1+i)^n-1)}{i};\ R$

81. $S=P(1+r)^n;\ r$

82. $\frac{x-a}{x+b}=\frac{x+b}{x-a};\ x$

83. $r=\frac{2mI}{B(n+1)};\ n$

84. $\frac{1}{p}+\frac{1}{q}=\frac{1}{f};\ q$

85. Geometry Use the formula $P=2l+2w$ to find the length l of a rectangle whose perimeter P is 660 m and whose width w is 160 m.

86. Geometry Use the formula $V=\pi r^2h$ to find the radius r of an energy drink can whose volume V is 355 ml and whose height h is 16 cm.

87. Sales Tax A salesperson needs to calculate the cost of an item with a sales tax of 6.5%. Write an equation that represents the total cost c of an item costing x dollars.

88. Revenue A day care center's total monthly revenue from the care of x toddlers is given by $r=450x$, and its total monthly costs are given by $c=380x+3500$. How many toddlers need to be enrolled each month to break even? In other words, when will revenue equal costs?

89. Straight-Line Depreciation If you purchase an item for business use, in preparing your income tax you may be able to spread out its expense over the life of the item. This is called *depreciation*. One method of depreciation is *straight-line depreciation,* in which the annual depreciation is computed by dividing the cost of the item, less its estimated salvage value, by its useful life. Suppose the cost is C dollars, the useful life is N years, and there is no salvage value. Then it can be shown that the value $V(n)$ (in dollars) of the item at the end of n years is given by

$$V(n) = C\left(1 - \frac{n}{N}\right)$$

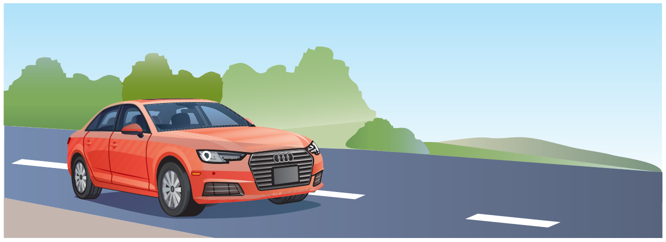
If new office furniture is purchased for \$3200, has a useful life of 8 years, and has no salvage value, after how many years will it have a value of \$2000?

90. Radar Beam When radar is used on a highway to determine the speed of a car, a radar beam is sent out and reflected from the moving car. The difference F (in cycles per second) in frequency between the original and reflected beams is given by

$$F = \frac{vf}{334.8}$$

where v is the speed of the car in miles per hour and f is the frequency of the original beam (in megacycles per second).

Suppose you are driving along a highway with a speed limit of 65 mi/h. A police officer aims a radar beam with a frequency of 2500 megacycles per second at your car, and the officer observes the difference in frequency to be 495 cycles per second. Can the officer claim that you were speeding?



91. Savings Theresa wants to buy a house, so she has decided to save one quarter of her salary. Theresa earns \$47.00 per hour and receives an extra \$28.00 a week because she declined company benefits. She wants to save at least \$550.00 each week. How many hours must she work each week to achieve her goal?

92. Predator–Prey Relation Predator–prey relations from biology also apply to competition in economics. To study a predator–prey relationship, an experiment[1] was conducted in which a blindfolded subject, the "predator," stood in front of a 3-ft-square table on which uniform sandpaper discs, the "prey," were placed. For 1 minute the "predator" searched for the discs by tapping with a finger. Whenever a disc was found, it was removed and searching resumed. The experiment was repeated for various disc densities (number of discs per 9 ft^2). It was estimated that if y is the number of discs picked up in 1 minute when x discs are on the table, then

$$y = a(1 - by)x$$

where a and b are constants. Solve this equation for y.

[1] C. S. Holling, "Some Characteristics of Simple Types of Predation and Parasitism," *The Canadian Entomologist*, XCI, no. 7 (1959), 385–98.

93. Prey Density In a certain area, the number y of moth larvae consumed by a single predatory beetle over a given period of time is given by

$$y = \frac{1.4x}{1 + 0.09x}$$

where x is the *prey density* (the number of larvae per unit of area). What prey density would allow a beetle to survive if it needs to consume 10 larvae over the given period?

94. Store Hours Suppose the ratio of the number of hours a store is open to the number of daily customers is constant. When the store is open 8 hours, the number of customers is 92 less than the maximum number of customers. When the store is open 10 hours, the number of customers is 46 less than the maximum number of customers. Write an equation describing this situation, and find the maximum number of daily customers.

95. Travel Time The time it takes a boat to travel a given distance upstream (against the current) can be calculated by dividing the distance by the difference of the speed of the boat and the speed of the current. Write an equation that calculates the time t it takes a boat moving at a speed r against a current c to travel a distance d. Solve your equation for c.

96. Wireless Tower A wireless tower is 100 meters tall. An engineer determines electronically that the distance from the top of the tower to a nearby house is 2 meters greater than the horizontal distance from the base of the tower to the house. Determine the distance from the base of the tower to the house.

97. Automobile Skidding Police have used the formula $s = \sqrt{30fd}$ to estimate the speed s (in miles per hour) of a car if it skidded d feet when stopping. The literal number f is the coefficient of friction, determined by the kind of road (such as concrete, asphalt, gravel, or tar) and whether the road is wet or dry. Some values of f are given in Table 0.1. At 85 mi/h, about how many feet will a car skid on a wet concrete road? Give your answer to the nearest foot.

Table 0.1

	Concrete	Tar
Wet	0.4	0.5
Dry	0.8	1.0

98. Interest Earned Cassandra discovers that she has \$1257 in an off-shore account that she has not used for a year. The interest rate was 7.3% compounded annually. How much interest did she earn from that account over the last year?

99. Tax in a Receipt In 2006, Nova Scotia consumers paid HST, *harmonized sales tax*, of 15%. Tom Wood traveled from Alberta, which has only federal GST, *goods and services tax,* of 7% to Nova Scotia for a chemistry conference. When he later submitted his expense claims in Alberta, the comptroller was puzzled to find that her usual multiplier of $\frac{7}{107}$ to determine tax in a receipt did not produce correct results. What percentage of Tom's Nova Scotia receipts were HST?

Objective

To solve quadratic equations by factoring or by using the quadratic formula.

0.8 Quadratic Equations

To learn how to solve certain classical problems, we turn to methods of solving *quadratic equations*.

Definition

A **quadratic equation** in the variable x is an equation that can be written in the form

$$ax^2 + bx + c = 0 \tag{1}$$

where a, b, and c are constants and $a \neq 0$.

A quadratic equation is also called a *second-degree equation* or an *equation of degree two,* since the highest power of the variable that occurs is the second. Whereas a linear equation has only one root, a quadratic equation may have two different roots.

Solution by Factoring

A useful method of solving quadratic equations is based on factoring, as the following example shows.

EXAMPLE 1 Solving a Quadratic Equation by Factoring

a. Solve $x^2 + x - 12 = 0$.

Solution: The left side factors easily:

$$(x-3)(x+4) = 0$$

Think of this as two quantities, $x - 3$ and $x + 4$, whose product is zero. *Whenever the product of two or more quantities is zero, at least one of the quantities must be zero.* (We emphasized this principle in Section 0.5 Factoring.) Here, it means that either

$$x - 3 = 0 \quad \text{or} \quad x + 4 = 0$$

Solving these gives $x = 3$ and $x = -4$, respectively. Thus, the roots of the original equation are 3 and -4, and the solution set is $\{-4, 3\}$.

b. Solve $6w^2 = 5w$.

We do not divide both sides by w (a variable) since equivalence is not guaranteed and we may "lose" a root.

Solution: We write the equation as

$$6w^2 - 5w = 0$$

so that one side is 0. Factoring gives

$$w(6w - 5) = 0$$

so we have

$$\begin{aligned} w = 0 \quad &\text{or} \quad 6w - 5 = 0 \\ w = 0 \quad &\text{or} \quad 6w = 5 \end{aligned}$$

Thus, the roots are $w = 0$ and $w = \frac{5}{6}$. Note that if we had divided both sides of $6w^2 = 5w$ by w and obtained $6w = 5$, our only solution would be $w = \frac{5}{6}$. That is, we would lose the root $w = 0$. This is in line with our discussion of Operation 5 in Section 0.7 and sheds light on the problem with Operation 5. One way of approaching the possibilities for a variable quantity, w, is to observe that *either* $w \neq 0$ *or* $w = 0$. In the first case we are free to divide by w. *In this case*, the original equation is *equivalent* to $6w = 5$, whose only solution is $w = \frac{5}{6}$. Now turning to *the other case*, $w = 0$, we are obliged to examine whether it is also a solution of the original equation—and in *this* problem it is.

Now Work Problem 3 ◁

EXAMPLE 2 Solving a Quadratic Equation by Factoring

Approach a problem like this with caution. If the product of two quantities is equal to -2, it is not true that at least one of the quantities must be -2. Why?

Solve $(3x - 4)(x + 1) = -2$.

Solution: We first multiply the factors on the left side:

$$3x^2 - x - 4 = -2$$

Rewriting this equation so that 0 appears on one side, we have

$$3x^2 - x - 2 = 0$$
$$(3x + 2)(x - 1) = 0$$
$$x = -\frac{2}{3}, 1$$

Now Work Problem 7 ◁

Some equations that are not quadratic may be solved by factoring, as Example 3 shows.

EXAMPLE 3 Solving a Higher-Degree Equation by Factoring

a. Solve $4x - 4x^3 = 0$.

Solution: This is called a *third-degree equation*. We proceed to solve it as follows:

$$4x - 4x^3 = 0$$
$$4x(1 - x^2) = 0 \qquad \text{factoring}$$
$$4x(1 - x)(1 + x) = 0 \qquad \text{factoring}$$

Do not neglect the fact that the factor x gives rise to a root.

Setting each factor equal to 0 gives $4 = 0$ (impossible), $x = 0$, $1 - x = 0$, or $1 + x = 0$. Thus,

$$x = 0 \text{ or } x = 1 \text{ or } x = -1$$

so that the solution set is $\{-1, 0, 1\}$.

b. Solve $x(x + 2)^2(x + 5) + x(x + 2)^3 = 0$.

Solution: Factoring $x(x + 2)^2$ from both terms on the left side, we have

$$x(x + 2)^2[(x + 5) + (x + 2)] = 0$$
$$x(x + 2)^2(2x + 7) = 0$$

Hence, $x = 0$, $x + 2 = 0$, or $2x + 7 = 0$, from which it follows that the solution set is $\{-\frac{7}{2}, -2, 0\}$.

Now Work Problem 23 ◁

EXAMPLE 4 A Fractional Equation Leading to a Quadratic Equation

Solve

$$\frac{y + 1}{y + 3} + \frac{y + 5}{y - 2} = \frac{7(2y + 1)}{y^2 + y - 6} \qquad \textbf{(2)}$$

Solution: Multiplying both sides by the LCD, $(y + 3)(y - 2)$, we get

$$(y - 2)(y + 1) + (y + 3)(y + 5) = 7(2y + 1) \qquad \textbf{(3)}$$

Since Equation (2) was multiplied by an expression involving the variable y, remember (from Section 0.7) that Equation (3) is not necessarily equivalent to Equation (2). After simplifying Equation (3), we have

$$2y^2 - 7y + 6 = 0 \qquad \text{quadratic equation}$$
$$(2y - 3)(y - 2) = 0 \qquad \text{factoring}$$

We have shown that *if* y satisfies the original equation *then* $y = \frac{3}{2}$ or $y = 2$. Thus, $\frac{3}{2}$ and 2 are the only *possible* roots of the given equation. But 2 cannot be a root of Equation (2), since substitution leads to a denominator of 0. However, you should check that $\frac{3}{2}$ does indeed satisfy the *original* equation. Hence, its only root is $\frac{3}{2}$.

Now Work Problem 63 ◁

EXAMPLE 5 Solution by Factoring

Solve $x^2 = 3$.

Do not hastily conclude that the solution of $x^2 = 3$ consists of $x = \sqrt{3}$ only.

Solution:

$$x^2 = 3$$
$$x^2 - 3 = 0$$

Factoring, we obtain

$$(x - \sqrt{3})(x + \sqrt{3}) = 0$$

Thus $x - \sqrt{3} = 0$ or $x + \sqrt{3} = 0$, so $x = \pm\sqrt{3}$.

Now Work Problem 9 ◁

A more general form of the equation $x^2 = 3$ is $u^2 = k$, for $k \geq 0$. In the same manner as the preceding, we can show that

$$\text{If } u^2 = k \quad \text{for } k \geq 0 \qquad \text{then} \qquad u = \pm\sqrt{k}. \tag{4}$$

Quadratic Formula

Solving quadratic equations by factoring can be difficult, as is evident by trying that method on $0.7x^2 - \sqrt{2}x - 8\sqrt{5} = 0$. However, there is a formula called the **quadratic formula** that gives the roots of any quadratic equation.

Quadratic Formula

The roots of the quadratic equation $ax^2 + bx + c = 0$, where a, b, and c are constants and $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Actually, the quadratic formula is not hard to derive if we first write the quadratic equation in the form

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

and then as

$$\left(x + \frac{b}{2a}\right)^2 - K^2 = 0$$

for a number K, as yet to be determined. This leads to

$$\left(x + \frac{b}{2a} - K\right)\left(x + \frac{b}{2a} + K\right) = 0$$

which in turn leads to $x = -\frac{b}{2a} + K$ or $x = -\frac{b}{2a} - K$ by the methods already under consideration. To see what K is, observe that we require $\left(x + \frac{b}{2a}\right)^2 - K^2 = x^2 + \frac{b}{a}x + \frac{c}{a}$ (so that the equation we just solved is the quadratic equation we started with), which leads to $K = \frac{\sqrt{b^2-4ac}}{2a}$. Substituting this value of K in $x = -\frac{b}{2a} \pm K$ gives the Quadratic Formula.

From the quadratic formula we see that the given quadratic equation has two real roots if $b^2 - 4ac > 0$, one real root if $b^2 - 4ac = 0$, and no real roots if $b^2 - 4ac < 0$.

We remarked in Section 0.5 Factoring that it is not always possible to factor $x^2 + bx + c$ as $(x - r)(x - s)$ for real numbers r and s, even if b and c are integers. This is because, for any such pair of real numbers, r and s would be roots of the equation $x^2 + bx + c = 0$. When $a = 1$ in the quadratic formula, it is easy to see that $b^2 - 4c$ can be negative, so that $x^2 + bx + c = 0$ can have no real roots. At first glance it might seem that the numbers r and s can be found by simultaneously solving

$$r + s = -b$$

$$rs = c$$

for r and s, thus giving another way of finding the roots of a general quadratic. However, rewriting the first equation as $s = -b - r$ and substituting this value in the second equation, we just get $r^2 + br + c = 0$, right back where we started.

Notice too that we can now verify that $x^2 + 1$ cannot be factored. If we try to solve $x^2 + 1 = 0$ using the quadratic formula with $a = 1$, $b = 0$, and $c = 1$ we get

$$x = \frac{-0 \pm \sqrt{0^2 - 4}}{2} = \pm\frac{\sqrt{-4}}{2} = \pm\frac{\sqrt{4}\sqrt{-1}}{2} = \pm\frac{2\sqrt{-1}}{2} = \pm\sqrt{-1}$$

and $\sqrt{-1}$ is not a real number. It is common to write $i = \sqrt{-1}$ and refer to it as the *imaginary unit*. The *Complex Numbers* are those of the form $a + ib$, where a and b are real. The Complex Numbers extend the Real Numbers, but except for Example 8 below they will make no further appearances in this book.

EXAMPLE 6 A Quadratic Equation with Two Real Roots

Solve $4x^2 - 17x + 15 = 0$ by the quadratic formula.

Solution: Here $a = 4$, $b = -17$, and $c = 15$. Thus,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-17) \pm \sqrt{(-17)^2 - 4(4)(15)}}{2(4)}$$

$$= \frac{17 \pm \sqrt{49}}{8} = \frac{17 \pm 7}{8}$$

The roots are $\frac{17+7}{8} = \frac{24}{8} = 3$ and $\frac{17-7}{8} = \frac{10}{8} = \frac{5}{4}$.

Now Work Problem 31 ◁

EXAMPLE 7 A Quadratic Equation with One Real Root

Solve $2 + 6\sqrt{2}y + 9y^2 = 0$ by the quadratic formula.

Solution: Look at the arrangement of the terms. Here $a = 9$, $b = 6\sqrt{2}$, and $c = 2$. Hence,

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6\sqrt{2} \pm \sqrt{0}}{2(9)}$$

Thus,

$$y = \frac{-6\sqrt{2} + 0}{18} = -\frac{\sqrt{2}}{3} \quad \text{or} \quad y = \frac{-6\sqrt{2} - 0}{18} = -\frac{\sqrt{2}}{3}$$

Therefore, the only root is $-\frac{\sqrt{2}}{3}$.

Now Work Problem 33 ◁

EXAMPLE 8 **A Quadratic Equation with No Real Roots**

Solve $z^2 + z + 1 = 0$ by the quadratic formula.

Solution: Here $a = 1, b = 1$, and $c = 1$. The roots are

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{-1}\sqrt{3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

Neither of the roots are real numbers. Both are complex numbers as described briefly in the paragraph preceding Example 6.

Now Work Problem 37 ◁

This describes the nature of the roots of a quadratic equation.

Examples 6–8 illustrate the three possibilities for the roots of a quadratic equation: either two different real roots, exactly one real root, or no real roots. In the last case there are two different complex roots, where if one of them is $a + ib$ with $b \neq 0$ then the other is $a - ib$.

Quadratic-Form Equation

Sometimes an equation that is not quadratic can be transformed into a quadratic equation by an appropriate substitution. In this case, the given equation is said to have **quadratic form**. The next example will illustrate.

EXAMPLE 9 **Solving a Quadratic-Form Equation**

Solve $\frac{1}{x^6} + \frac{9}{x^3} + 8 = 0$.

Solution: This equation can be written as

$$\left(\frac{1}{x^3}\right)^2 + 9\left(\frac{1}{x^3}\right) + 8 = 0$$

so it is quadratic in $1/x^3$ and hence has quadratic form. Substituting the variable w for $1/x^3$ gives a quadratic equation in the variable w, which we can then solve:

Do not assume that -8 and -1 are solutions of the *original* equation.

$$w^2 + 9w + 8 = 0$$
$$(w + 8)(w + 1) = 0$$
$$w = -8 \quad \text{or} \quad w = -1$$

Returning to the variable x, we have

$$\frac{1}{x^3} = -8 \quad \text{or} \quad \frac{1}{x^3} = -1$$

Thus,

$$x^3 = -\frac{1}{8} \quad \text{or} \quad x^3 = -1$$

from which it follows that

$$x = -\frac{1}{2} \quad \text{or} \quad x = -1$$

Checking, we find that these values of x satisfy the original equation.

Now Work Problem 49 ◁

PROBLEMS 0.8

In Problems 1–30, solve by factoring.

1. $x^2 - 4x + 4 = 0$

2. $t^2 + 3t + 2 = 0$

3. $t^2 + 4t - 21 = 0$

4. $x^2 + 3x - 10 = 0$

5. $x^2 - 2x - 3 = 0$

6. $x^2 - 16 = 0$

7. $u^2 - 13u = -36$

8. $5z^2 + 14z - 3 = 0$

9. $x^2 - 4 = 0$

10. $3u^2 - 6u = 0$

11. $t^2 - 5t = 0$

12. $x^2 + 9x = -14$

13. $4x^2 - 4x = 3$

14. $2z^2 + 9z = 5$

15. $v(3v - 5) = -2$

16. $2 + x - 6x^2 = 0$

17. $-x^2 + 3x + 10 = 0$

18. $\frac{2}{3}u^2 = \frac{5}{7}u$

19. $2p^2 = 3p$

20. $-r^2 - r + 12 = 0$

21. $x(x + 4)(x - 1) = 0$

22. $(w - 3)^2(w + 1)^2 = 0$

23. $(3t^4 - 3t^2)(t + 3) = 0$

24. $x^3 - 4x^2 - 5x = 0$

25. $6x^3 + 5x^2 - 4x = 0$

26. $(x + 1)^2 - 5x + 1 = 0$

27. $(x - 3)(x^2 - 4) = 0$

28. $5(z^2 - 3z + 2)(z - 3) = 0$

29. $p(p - 3)^2 - 4(p - 3)^3 = 0$

30. $x^4 - 3x^2 + 2 = 0$

In Problems 31–44, find all real roots by using the quadratic formula.

31. $x^2 + 2x - 24 = 0$

32. $x^2 - 2x - 15 = 0$

33. $9x^2 - 42x + 49 = 0$

34. $q^2 - 5q = 0$

35. $p^2 - 2p - 7 = 0$

36. $2 - 2x + x^2 = 0$

37. $4 - 2n + n^2 = 0$

38. $2u^2 + 3u = 7$

39. $4x^2 + 5x - 2 = 0$

40. $w^2 - 2w + 1 = 0$

41. $0.02w^2 - 0.3w = 20$

42. $0.01x^2 + 0.2x - 0.6 = 0$

43. $z^2 - z + 1 = 0$

44. $-2x^2 - 6x + 5 = 0$

In Problems 45–54, solve the given quadratic-form equation.

45. $x^4 - 5x^2 + 6 = 0$

46. $X^4 - 3X^2 - 10 = 0$

47. $\frac{3}{x^2} - \frac{7}{x} + 2 = 0$

48. $x^{-2} + x^{-1} - 2 = 0$

49. $x^{-4} - 9x^{-2} + 20 = 0$

50. $\frac{1}{x^4} - \frac{9}{x^2} + 8 = 0$

51. $(X - 5)^2 + 7(X - 5) + 10 = 0$

52. $(3x + 2)^2 - 5(3x + 2) = 0$

53. $\frac{1}{(x - 4)^2} - \frac{7}{x - 4} + 10 = 0$

54. $\frac{2}{(x + 4)^2} + \frac{7}{x + 4} + 3 = 0$

In Problems 55–76, solve by any method.

55. $x^2 = \frac{x + 3}{2}$

56. $\frac{x}{2} = \frac{7}{x} - \frac{5}{2}$

57. $\frac{3}{x - 4} + \frac{x - 3}{x} = 2$

58. $\frac{2}{2x + 1} + \frac{3}{x + 2} = 2$

59. $\frac{3x + 2}{x + 1} - \frac{2x + 1}{2x} = 1$

60. $\frac{6(w + 1)}{2 - w} + \frac{w}{w - 1} = 3$

61. $\frac{2}{r - 2} - \frac{r + 1}{r + 4} = 0$

62. $\frac{2x - 3}{2x + 5} + \frac{2x}{3x + 1} = 1$

63. $\frac{t}{t - 1} + \frac{t - 1}{t - 2} = \frac{t - 3}{t^2 - 3t + 2}$

64. $\frac{2}{x + 1} + \frac{3}{x} = \frac{4}{x + 2}$

65. $\frac{2}{x^2 - 1} - \frac{1}{x(x - 1)} = \frac{2}{x^2}$

66. $5 - \frac{3(x + 3)}{x^2 + 3x} = \frac{1 - x}{x}$

67. $\sqrt{2x - 3} = x - 3$

68. $\sqrt{x + 2} = x + 1$

69. $q + 2 = 2\sqrt{4q - 7}$

70. $x + \sqrt{4x} - 5 = 0$

71. $\sqrt{z + 3} - \sqrt{3z} - 1 = 0$

72. $\sqrt{x} - \sqrt{2x - 8} - 2 = 0$

73. $\sqrt{x + 1} - \sqrt{x} = 1$

74. $\sqrt{y - 2} + 2 = \sqrt{2y + 3}$

75. $\sqrt{x + 3} + 1 = 3\sqrt{x}$

76. $\sqrt{\sqrt{t} + 2} = \sqrt{3t - 1}$

In Problems 77 and 78, find the roots, rounded to two decimal places.

77. $0.04x^2 - 2.7x + 8.6 = 0$

78. $x^2 + (0.2)x - 0.3 = 0$

79. Geometry The area of a rectangular picture with a width 2 inches less than its length is 48 square inches. What are the dimensions of the picture?

80. Temperature The temperature has been rising X degrees per day for X days. X days ago it was 15 degrees. Today it is 51 degrees. How much has the temperature been rising each day? How many days has it been rising?

81. Economics One root of the economics equation

$$\overline{M} = \frac{Q(Q + 10)}{44}$$

is $-5 + \sqrt{25 + 44\overline{M}}$. Verify this by using the quadratic formula to solve for Q in terms of $\overline{M}$. Here Q is real income and $\overline{M}$ is the level of money supply.

82. Diet for Rats A group of biologists studied the nutritional effects on rats that were fed a diet containing 10% protein.[2] The protein was made up of yeast and corn flour. By changing the percentage P (expressed as a decimal) of yeast in the protein mix, the group estimated that the average weight gain g (in grams) of a rat over a period of time was given by

$$g = -200P^2 + 200P + 20$$

What percentage of yeast gave an average weight gain of 60 grams?

83. Drug Dosage There are several rules for determining doses of medicine for children when the adult dose has been specified. Such rules may be based on weight, height, and so on. If A is the age of the child, d is the adult dose, and c is the child's dose, then here are two rules:

Young's rule: $$c = \frac{A}{A + 12}d$$

Cowling's rule: $$c = \frac{A + 1}{24}d$$

[2] Adapted from R. Bressani, "The Use of Yeast in Human Foods," in R. I. Mateles and S. R. Tannenbaum (eds.), *Single-Cell Protein* (Cambridge, MA: MIT Press, 1968).

At what age(s) are the children's doses the same under both rules? Round your answer to the nearest year. Presumably, the child has become an adult when $c = d$. At what age does the child become an adult according to Cowling's rule? According to Young's rule? If you know how to graph functions, graph both $Y(A) = \dfrac{A}{A+12}$ and $C(A) = \dfrac{A+1}{24}$ as functions of A, for $A \geq 0$, in the same plane. Using the graphs, make a more informed comparison of Young's rule and Cowling's rule than is obtained by merely finding the age(s) at which they agree.

84. Delivered Price of a Good In a discussion of the delivered price of a good from a mill to a customer, DeCanio[3] arrives at and solves the two quadratic equations

$$(2n-1)v^2 - 2nv + 1 = 0$$

and

$$nv^2 - (2n+1)v + 1 = 0$$

where $n \geq 1$.

(a) Solve the first equation for v.
(b) Solve the second equation for v if $v < 1$.

85. Motion Suppose the height h of an object thrown straight upward from the ground is given by

$$h = 39.2t - 4.9t^2$$

where h is in meters and t is the elapsed time in seconds.

(a) After how many seconds does the object strike the ground?
(b) When is the object at a height of 68.2 m?

Chapter 0 Review

Important Terms and Symbols

Section	Important Terms and Symbols	Examples
Section 0.1	**Sets of Real Numbers**	
	set integers rational numbers real numbers coordinates	
Section 0.2	**Some Properties of Real Numbers**	
	commutative associative identity inverse reciprocal distributive	Ex. 3, p. 22
Section 0.3	**Exponents and Radicals**	
	exponent base principal nth root radical	Ex. 2, p. 27
Section 0.4	**Operations with Algebraic Expressions**	
	algebraic expression term factor polynomial	Ex. 6, p. 34
	long division	Ex. 8, p. 34
Section 0.5	**Factoring**	
	common factoring perfect square difference of squares	Ex. 3, p. 37
Section 0.6	**Fractions**	
	multiplication and division	Ex. 3, p. 39
	addition and subtraction	Ex. 5, p. 40
	rationalizing denominators	Ex. 4, p. 40
Section 0.7	**Equations, in Particular Linear Equations**	
	equivalent equations linear equations	Ex. 5, p. 47
	fractional equations	Ex. 9, p. 50
	radical equations	Ex. 3, p. 47

[3] S. J. DeCanio, "Delivered Pricing and Multiple Basing Point Equilibria: A Reevalution," *Quarterly Journal of Economics*, XCIX, no. 2 (1984), 329–49.

Section 0.8 **Quadratic Equations**

	solved by factoring	Ex. 2, p. 56
	quadratic formula	Ex. 8, p. 59

Summary

There are certainly basic *formulas* that have to be remembered when reviewing algebra. It is often a good exercise, while attempting this memory work, to find the formulas that are basic *for you*. For example, the list of properties, each followed by an example, near the end of Section 0.2, has many redundancies in it, but all those *formulas* need to be part of your own mathematical tool kit. However, Property 2, which says $a - (-b) = a + b$, will probably jump out at you as $a - (-b) = a + (-(-b)) = a + b$. The first equality here is just the definition of subtraction, as provided by Property 1, while the second equality is Property 8 applied to b. If this is obvious to you, you can strike Property 2 from *your list* of formulas that *you* personally need to memorize. Try to do your own treatment of Property 5, perhaps using Property 1 (twice) and Properties 4 and 10. If you succeed, continue working through the list, striking off what you don't need to memorize. All of this is to say that you will remember faster what you need to know if you work to shorten your personal list of formulas that require memory. In spite of technology, this is a task best done with pencil and paper. Mathematics is not a spectator sport.

The same comments apply to the list of formulas in Section 0.3 and the *special products* in Section 0.4. *Long division* of polynomials (Section 0.3) is a skill that comes with practice (and only with practice) as does *factoring* (Section 0.4). A *linear equation* in the variable x is one that can be written in the form $ax + b = 0$, where a and b are constants and $a \neq 0$. Be sure that you understand the derivation of its solution, which is $x = -\frac{b}{a}$. Even though the subject matter of this book is Applied Mathematics, particularly as it pertains to Business and Economics, it is essential to understand how to solve *literal equations* in which the coefficients, such as a and b in $ax+b=0$, are not presented as particular numbers. In Section 0.7 there are many equations, many but not all of which reduce to linear equations.

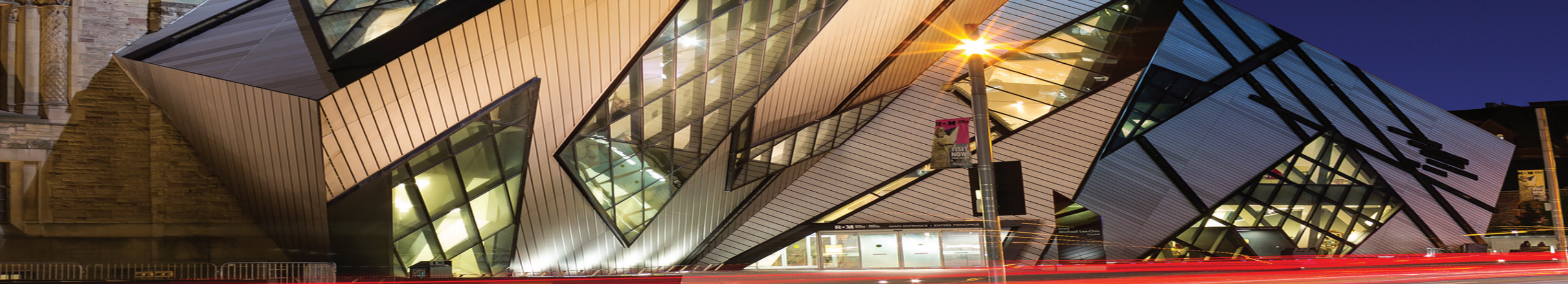
The general quadratic equation in the variable x is $ax^2+bx+c=0$, where a, b, and c are constants with $a \neq 0$, and it forms the subject of Section 0.8. Its roots (solutions) are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{5}$$

although, if the coefficients a, b, and c are simple integers, these roots *may* be more easily found by factoring. We recommend simply memorizing Equation 5, which is known as the Quadratic Formula. The radical in the Quadratic Formula tells us, at a glance, that the nature of the roots of a quadratic are determined by $b^2 - 4ac$. If $b^2 - 4ac$ is positive, the equation has two real roots. If $b^2 - 4ac$ is zero, it has one real root. If $b^2 - 4ac$ is negative, there are no real roots.

Review Problems

1. Rewrite $\sqrt[5]{a^{-5}b^{-3}c^2}b^4c^3$ without radicals and using only positive exponents.
2. Rationalize the denominator of $\frac{\sqrt{5}}{\sqrt[7]{13}}$.
3. Rationalize the *numerator* of $\frac{\sqrt{x+h}-\sqrt{x}}{h}$.
4. Calculate $(3x^3 - 4x^2 + 3x + 7) \div (x - 1)$.
5. Simplify $\dfrac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$.
6. Solve $S = P(1 + r)^n$ for P.
7. Solve $S = P(1 + r)^n$ for r.
8. Solve $x + 2\sqrt{x} - 15 = 0$ by treating it as an equation of quadratic-form.
9. **Interest Earned** Emily discovers that she has \$5253.14 in a bank account that has been untouched for two years, with interest earned at the rate of 3.5% compounded annually. How much of the current amount of \$5253.14 was interest earned? [Hint: If an amount P is invested for 2 years at a rate r (given as a real number), compounded annually, then the value of the investment after 2 years is given by $S = P(1 + r)^2$.]
10. We looked earlier at the economics equation $\overline{M} = \frac{Q(Q+10)}{44}$, where $\overline{M}$ is the level of money supply and Q is real income. We verified that one of its roots is given by $Q = -5 + \sqrt{25 + 44\overline{M}}$. What is the other root and does it have any significance?



1 Applications and More Algebra

In this chapter, we will apply equations to various practical situations. We will also do the same with inequalities, which are statements that one quantity is less than ($<$), greater than ($>$), less than or equal to ($\leq$), or greater than or equal to ($\geq$) some other quantity.

Here is an example of the use of inequalities in the regulation of sporting equipment. Dozens of baseballs are used in a typical major league game and it would be unrealistic to expect that every ball weigh exactly $5\frac{1}{8}$ ounces. But it is reasonable to require that each one weigh no less than 5 ounces and no more than $5\frac{1}{4}$ ounces, which is how 1.09 of the Official Rules of Major League Baseball reads. (See http://mlb.mlb.com/ and look up "official rules".) Note that *no less than* means the same thing as *greater than or equal to* while *no more than* means the same thing as *less than or equal to*. In translating English statements into mathematics, we recommend avoiding the negative wordings as a first step. Using the mathematical symbols we have

$$\text{ball weight} \geq 5 \text{ ounces} \quad \text{and} \quad \text{ball weight} \leq 5\frac{1}{4} \text{ ounces}$$

which can be combined to give

$$5 \text{ ounces} \leq \text{ball weight} \leq 5\frac{1}{4} \text{ ounces}$$

and nicely displays the ball weight *between* 5 and $5\frac{1}{4}$ ounces (where *between* here includes the extreme values).

Another inequality applies to the sailboats used in the America's Cup race. The America's Cup Class (ACC) for yachts was defined until 30 January, 2009, by

$$\frac{L + 1.25\sqrt{S} - 9.8\sqrt[3]{DSP}}{0.686} \leq 24.000 \text{ m}$$

The "$\leq$" signifies that the expression on the left must come out as less than or equal to the 24.000 m on the right. The L, S, and DSP were themselves specified by complicated formulas, but roughly, L stood for length, S for sail area, and DSP for displacement (the hull volume below the waterline).

The ACC formula gave yacht designers some latitude. Suppose a yacht had $L = 20.2$ m, $S = 282 \text{ m}^2$, and $DSP = 16.4 \text{ m}^3$. Since the formula is an inequality, the designer could reduce the sail area while leaving the length and displacement unchanged. Typically, however, values of L, S, and DSP were used that made the expression on the left as close to 24.000 m as possible.

In addition to applications of linear equations and inequalities, this chapter will review the concept of absolute value and introduce sequences and summation notation.

Objective

To model situations described by linear or quadratic equations.

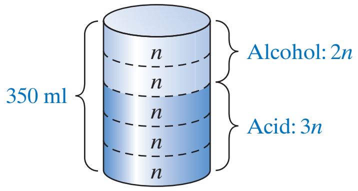


FIGURE 1.1 Chemical solution (Example 1).

Note that the solution to an equation is not necessarily the solution to the problem posed.

1.1 Applications of Equations

In most cases, the solution of practical problems requires the translation of stated relationships into mathematical symbols. This is called *modeling*. The following examples illustrate basic techniques and concepts.

EXAMPLE 1 Mixture

A chemist needs to prepare 350 ml of a chemical solution made up of two parts alcohol and three parts acid. How much of each should be used?

Solution: Let n be the number of milliliters in each *part*. Figure 1.1 shows the situation. From the diagram, we have

$$\begin{aligned} 2n + 3n &= 350 \\ 5n &= 350 \\ n &= \frac{350}{5} = 70 \end{aligned}$$

But $n = 70$ is *not* the answer to the original problem. Each *part* has 70 ml. The amount of alcohol is $2n = 2(70) = 140$, and the amount of acid is $3n = 3(70) = 210$. Thus, the chemist should use 140 ml of alcohol and 210 ml of acid. This example shows how helpful a diagram can be in setting up a word problem.

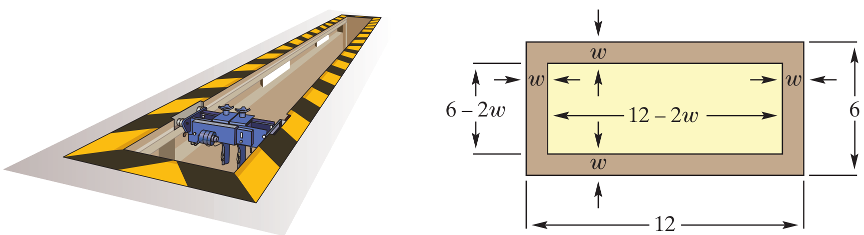
Now Work Problem 5 ◁

EXAMPLE 2 Vehicle Inspection Pit

A vehicle inspection pit is to be built in a commercial garage. [See Figure 1.2(a).] The garage has dimensions 6 m by 12 m. The pit is to have area 40 m^2 and to be centered in the garage so that there is a uniform walkway around the pit. How wide will this walkway be?

Solution: A diagram of the pit is shown in Figure 1.2(b). Let w be the width (in meters) of the walkway. Then the pit has dimensions $12 - 2w$ by $6 - 2w$. Since its area must be 40 m^2, where area = (length)(width), we have

$$\begin{aligned} (12 - 2w)(6 - 2w) &= 40 \\ 72 - 36w + 4w^2 &= 40 && \text{multiplying} \\ 4w^2 - 36w + 32 &= 0 \\ w^2 - 9w + 8 &= 0 && \text{dividing both sides by 4} \\ (w - 8)(w - 1) &= 0 \\ w &= 8, \ 1 \end{aligned}$$



(a) (b)

FIGURE 1.2 Pit walkway (Example 2).

Although 8 is a solution of the equation, it is *not* a solution to our problem, because one of the dimensions of the garage itself is only 6 m. Thus, the only possible solution is that the walkway be 1 m wide.

Now Work Problem 7 ◁

The key words introduced here are *fixed cost, variable cost, total cost, total revenue*, and *profit*. This is the time to gain familiarity with these terms because they recur throughout the book.

In the next example, we refer to some business terms relative to a manufacturing firm. **Fixed cost** is the sum of all costs that are independent of the level of production, such as rent, insurance, and so on. This cost must be paid whether or not output is produced. **Variable cost** is the sum of all costs that are dependent on the level of output, such as labor and material. **Total cost** is the sum of variable cost and fixed cost:

$$\text{total cost} = \text{variable cost} + \text{fixed cost}$$

Total revenue is the money that the manufacturer receives for selling the output:

$$\text{total revenue} = (\text{price per unit})\,(\text{number of units sold})$$

Profit is total revenue minus total cost:

$$\text{profit} = \text{total revenue} - \text{total cost}$$

EXAMPLE 3 Profit

The Acme Company produces a product for which the variable cost per unit is \$6 and fixed cost is \$80,000. Each unit has a selling price of \$10. Determine the number of units that must be sold for the company to earn a profit of \$60,000.

Solution: Let q be the number of units that must be sold. (In many business problems, q represents quantity.) Then the variable cost (in dollars) is $6q$. The *total* cost for the business is therefore $6q + 80{,}000$. The total revenue from the sale of q units is $10q$. Since

$$\text{profit} = \text{total revenue} - \text{total cost}$$

our model for this problem is

$$60{,}000 = 10q - (6q + 80{,}000)$$

Solving gives

$$\begin{aligned} 60{,}000 &= 10q - 6q - 80{,}000 \\ 4q &= 140{,}000 \\ q &= 35{,}000 \end{aligned}$$

Thus, 35,000 units must be sold to earn a profit of \$60,000.

Now Work Problem 9 ◁

EXAMPLE 4 Pricing

Sportcraft manufactures denim clothing and is planning to sell its new line of jeans to retail outlets. The cost to the retailer will be \$60 per pair of jeans. As a convenience to the retailer, Sportcraft will attach a price tag to each pair. What amount should be marked on the price tag so that the retailer can reduce this price by 20% during a sale and still make a profit of 15% on the cost?

Solution: Here we use the fact that

Note that price = cost + profit.

$$\text{selling price} = \text{cost per pair} + \text{profit per pair}$$

Let p be the tag price per pair, in dollars. During the sale, the retailer actually receives $p - 0.2p$. This must equal the cost, \$60, plus the profit, (0.15)(60). Hence,

$$\begin{aligned}\text{selling price} &= \text{cost} + \text{profit}\\ p - 0.2p &= 60 + (0.15)(60)\\ 0.8p &= 69\\ p &= 86.25\end{aligned}$$

Sportcraft should mark the price tag at \$86.25.

Now Work Problem 13 ◁

EXAMPLE 5 Investment

A total of \$10,000 was invested in two business ventures, A and B. At the end of the first year, A and B yielded returns of 6% and $5\frac{3}{4}\%$, respectively, on the original investments. How was the original amount allocated if the total amount earned was \$588.75?

Solution: Let x be the amount (in dollars) invested at 6%. Then $10{,}000 - x$ was invested at $5\frac{3}{4}\%$. The interest earned from A was $(0.06)(x)$ and that from B was $(0.0575)(10{,}000 - x)$ with a total of 588.75. Hence,

$$\begin{aligned}(0.06)x + (0.0575)(10{,}000 - x) &= 588.75\\ 0.06x + 575 - 0.0575x &= 588.75\\ 0.0025x &= 13.75\\ x &= 5500\end{aligned}$$

Thus, \$5500 was invested at 6%, and $\$10,000 - \$5500 = \$4500$ was invested at $5\frac{3}{4}\%$.

Now Work Problem 11 ◁

EXAMPLE 6 Bond Redemption

The board of directors of Maven Corporation agrees to redeem some of its bonds in two years. At that time, \$1,102,500 will be required. Suppose the firm presently sets aside \$1,000,000. At what annual rate of interest, compounded annually, will this money have to be invested in order that its future value be sufficient to redeem the bonds?

Solution: Let r be the required annual rate of interest. At the end of the first year, the accumulated amount will be \$1,000,000 plus the interest, $1{,}000{,}000r$, for a total of

$$1{,}000{,}000 + 1{,}000{,}000r = 1{,}000{,}000(1 + r)$$

Under compound interest, at the end of the second year the accumulated amount will be $1{,}000{,}000(1 + r)$ plus the interest on this, which is $1{,}000{,}000(1 + r)r$. Thus, the total value at the end of the second year will be

$$1{,}0000{,}000(1 + r) + 1{,}000{,}000(1 + r)r$$

This must equal \$1,102,500:

$$1{,}000{,}000(1 + r) + 1{,}000{,}000(1 + r)r = 1{,}102{,}500 \tag{1}$$

Since $1{,}000{,}000(1 + r)$ is a common factor of both terms on the left side, we have

$$\begin{aligned}1{,}000{,}000(1 + r)(1 + r) &= 1{,}102{,}500\\ 1{,}000{,}000(1 + r)^2 &= 1{,}102{,}500\\ (1 + r)^2 &= \frac{1{,}102{,}500}{1{,}000{,}000} = \frac{11{,}025}{10{,}000} = \frac{441}{400}\\ 1 + r &= \pm\sqrt{\frac{441}{400}} = \pm\frac{21}{20}\\ r &= -1 \pm \frac{21}{20}\end{aligned}$$

Thus, $r = -1 + (21/20) = 0.05$, or $r = -1 - (21/20) = -2.05$. Although 0.05 and -2.05 are roots of Equation (1), we reject -2.05 since we require that r be positive. Hence, $r = 0.05 = 5\%$ is the desired rate.

Now Work Problem 15 ◁

At times there may be more than one way to model a word problem, as Example 7 shows.

EXAMPLE 7 Apartment Rent

A real estate firm owns the Parklane Garden Apartments, which consist of 96 apartments. At \$550 per month, every apartment can be rented. However, for each \$25 per month increase, there will be three vacancies with no possibility of filling them. The firm wants to receive \$54,600 per month from rents. What rent should be charged for each apartment?

Solution:

Method I. Suppose r is the rent (in dollars) to be charged per apartment. Then the increase over the \$550 level is $r - 550$. Thus, the number of \$25 increases is $\dfrac{r-550}{25}$. Because each \$25 increase results in three vacancies, the total number of vacancies will be $3\left(\dfrac{r-550}{25}\right)$. Hence, the total number of apartments rented will be $96 - 3\left(\dfrac{r-550}{25}\right)$. Since

$$\text{total rent} = (\text{rent per apartment})(\text{number of apartments rented})$$

we have

$$\begin{aligned}
54{,}600 &= r\left(96 - \frac{3(r-550)}{25}\right) \\
54{,}600 &= r\left(\frac{2400 - 3r + 1650}{25}\right) \\
54{,}600 &= r\left(\frac{4050 - 3r}{25}\right) \\
1{,}365{,}000 &= r(4050 - 3r)
\end{aligned}$$

Thus,

$$3r^2 - 4050r + 1{,}365{,}000 = 0$$

By the quadratic formula,

$$\begin{aligned}
r &= \frac{4050 \pm \sqrt{(-4050)^2 - 4(3)(1{,}365{,}000)}}{2(3)} \\
&= \frac{4050 \pm \sqrt{22{,}500}}{6} = \frac{4050 \pm 150}{6} = 675 \pm 25
\end{aligned}$$

Hence, the rent for each apartment should be either \$650 or \$700.

Method II. Suppose n is the number of \$25 increases. Then the increase in rent per apartment will be $25n$ and there will be $3n$ vacancies. Since

$$\text{total rent} = (\text{rent per apartment})(\text{number of apartments rented})$$

we have

$$\begin{aligned}
54{,}600 &= (550 + 25n)(96 - 3n) \\
54{,}600 &= 52{,}800 + 750n - 75n^2 \\
75n^2 - 750n + 1800 &= 0 \\
n^2 - 10n + 24 &= 0 \\
(n-6)(n-4) &= 0
\end{aligned}$$

Thus, $n = 6$ or $n = 4$. The rent charged should be either $550 + 25(6) = \$700$ or $550 + 25(4) = \$650$. However, it is easy to see that the real estate firm can receive \$54,675 per month from rents by charging \$675 for each apartment and that \$54,675 is the *maximum* amount from rents that it can receive, given existing market conditions. In a sense, the firm posed the wrong question. A considerable amount of our work in this book focuses on a better question that the firm might have asked.

Now Work Problem 29 ◁

PROBLEMS 1.1

1. Fencing A fence is to be placed around a rectangular plot so that the enclosed area is 800 ft^2 and the length of the plot is twice the width. How many feet of fencing must be used?

2. Geometry The perimeter of a rectangle is 300 ft, and the length of the rectangle is 3 ft more than twice the width. Find the dimensions of the rectangle.

3. Tent Caterpillars One of the most damaging defoliating insects is the tent caterpillar, which feeds on foliage of shade, forest, and fruit trees. A homeowner lives in an area in which the tent caterpillar has become a problem. She wishes to spray the trees on her property before more defoliation occurs. She needs 145 oz of a solution made up of 4 parts of insecticide A and 5 parts of insecticide B. The solution is then mixed with water. How many ounces of each insecticide should be used?

4. Concrete Mix A builder makes a certain type of concrete by mixing together 1 part Portland cement (made from lime and clay), 3 parts sand, and 5 parts crushed stone (by volume). If 765 ft^3 of concrete are needed, how many cubic feet of each ingredient does he need?

5. Homemade Ice Cream Online recipes claim that you can make no-churn ice cream using 7 parts of sweetened condensed milk and 8 parts of cold, heavy whipping cream. How many millilitres of whipping cream will you need to make 3 litres of ice cream?

6. Forest Management A lumber company owns a forest that is of rectangular shape, 1 mi by 2 mi. If the company cuts a uniform strip of trees along the outer edges of this forest, how wide should the strip be if $\frac{3}{4}$ sq mi of forest is to remain?

7. Garden Pavement A 10-m-square plot is to have a circular flower bed of 60 m^2 centered in the square. The other part of the plot is to be paved so that the owners can walk around the flower bed. What is the minimum "width" of the paved surface? In other words, what is the smallest distance from the flower bed to the edge of the plot?

8. Ventilating Duct The diameter of a circular ventilating duct is 140 mm. This duct is joined to a square duct system as shown in Figure 1.3. To ensure smooth airflow, the areas of the circle and square sections must be equal. To the nearest millimeter, what should the length x of a side of the square section be?

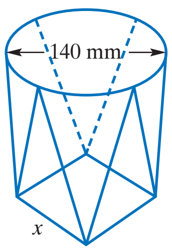


FIGURE 1.3 Ventilating duct (Problem 8).

9. Profit A toy manufacturing company produces teddy bears at a variable cost of €5 per bear. If fixed costs are €15,000 per month and each teddy bear sells for €15, how many teddy bears must be sold each month for the company to have a monthly profit of €25,000?

10. Sales Chen, the owner of a UK-based chocolate shop, would like to know the total sales units that are required for her company to earn a profit of £2000. The following data are available: unit selling price of £7, variable cost per unit of £4, total fixed cost of £300. From these data, determine the required sales units.

11. Investment Sue wants to invest \$50,000 in two different bonds that yield annual returns of 14.5% and 9.5%, respectively. How much must Sue invest in each bond to earn a total income of \$5000 per year?

12. Investment A bank offers two different types of mutual funds that yield returns of 13% and 15.5% over three years. A customer wants to invest \$20,000 in the two funds. How much should she invest in each to yield a 14% rate at the end of three years?

13. Pricing The cost of a piece of equipment is €8.50. How much should it be sold for to make a profit of 15% on the selling price?

14. Bond Retirement A student in Japan invests ¥50,000 in some bonds offered by a bank that requires ¥54,080 to retire the bonds in two years. What annual rate of interest, compounded annually, should the student receive to retire the bonds?

15. Expansion Program Calvin is going to retire in five years. He has decided to invest his savings of £50,000 now so that the total value of the investment will exceed £100,000 in five years. What is the annual rate of interest, compounded annually, that Calvin must receive to achieve his goal?

16. Business A manufacturing company determines that with a fixed cost of \$900 and a variable cost of \$1.50 per unit, it will have a total revenue of $75\sqrt{q}$ (in dollars) by manufacturing and selling q units. What is the value of q if the company's profit is zero?

17. Overbooking A company with 160 employees wants to treat its staff by taking them all to a movie. However, only 80% of the staff are going for the movie. How many tickets should the company purchase to avoid overbooking?

18. Poll A group of school students were polled to change the current student council. Twenty students, or 25%, had been appointed as members of the council. How many students were polled?

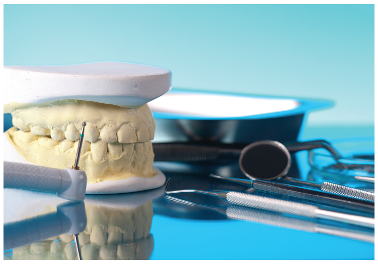
19. Salary A general clerk of Bär & Caputo Law earns 60% (or CHf800) a month less than the law firm's general manager. Find the general manager's annual salary. Give your answer to the nearest Swiss franc (the currency of Switzerland, CHf).

20. Fast Food The wages of employees at a fast food restaurant in Germany increases to €15 an hour from €7.25 per hour. However, this causes the cost of a burger meal to increase from €7.00 to €7.31. By what percentage will the cost of the meal increase if the fast food employees receive €22 per hour?

21. Break Even A French bakery incurs monthly fixed costs of €500 and the variable cost of baking an entire cake is €14. If the bakery sells each cake for €60, how many cakes must be sold in a month for the bakery to break even, that is, in order that total revenue equals total cost?

22. Investment Dev bought shares of stock in company A for \$25,000. The stock yields 2.5% per year. He also bought shares of stock in company B. The stock sells at \$5 per share and earns a dividend of \$1.25 per share per year. How many shares should Dev buy so that his total investment in both stocks yields 4% per year?

23. Dental Coverage A dental insurance policy bought by a company for its employees provides coverage of first \$100 on an employee's dental expenses and 90% of subsequent dental expenses up to a maximum of \$500 per year. Find the total coverage provided by this policy.



24. Quality Control A manufacturer of LED bulbs found that on average 1.4% of the LED bulbs produced daily are defective.

(a) If c bulbs are produced per month, how many defective bulbs are there in a month?

(b) The manufacturer receives an order of 300,000 LED bulbs, which it has to deliver to its customer in a month. Approximately how many LED bulbs must be manufactured if the defective bulbs are taken into consideration?

25. Business The price of a good is $(200 - q)$, where q is the quantity. How many units of the good must be sold to earn a revenue of \$10,000?

26. Investment In how many years will a simple interest investment double with a rate of 10.5% per year? [*Hint*: See Example 6(a) of Section 0.7, and express 10.5% as 0.105.]

27. Business Kate is setting up a fruit juice booth. She estimates that the total cost in setting up her business is \$50,000. The cost of renting a booth is \$1200 per month and Kate incurs an additional cost of \$25,000 for setting up the booth. Each cup of juice sells for \$2.50. How many cups of juice should Kate sell in the first month to cover the total amount spent on her fruit juice booth?

28. Land Area Steve, a sheep farmer in New Zealand, owns a farm that is 144 ft long and 96 ft wide. Due to an increase in the number of sheep, he decides to double the farmland area by increasing equal length on both sides (length and width) of his property. Determine the length needed to be increased by Steve on both sides of his farm.

29. Sales A property agent is helping a developer to sell 140 new launching units. All the units can be sold for \$400,000 per unit. However, for every \$20,000 increase in the selling price for each unit, there will be 5 units that will remain unsold. The developer wants to receive a total \$57.5 million from the sales of the units. The property agent has to determine the selling price of each unit. What should the property agent state?

30. Purchase Susan is going to purchase a new blouse and a skirt for an event. She visits a local boutique and picks a blouse that costs \$45 with a 10% discount and a skirt that costs \$89 and does not have a discount. The boutique also offers customers an additional discount of 5% on the total bill if two items are purchased. What is the price of the blouse after the discounts?

31. Revenue The revenue function of a company is given by $R = 12q + 0.05q^2$, where q is the quantity of products sold. How many products must the company sell to earn a revenue of €10,000?

32. Price-Earnings Ratio The *price-earnings ratio*, P/E, is the relationship between a company's stock price and earnings per share. Suppose stock A has a share price of \$30 per share with earnings \$2 per share, and stock B has a share price of \$20 per share with earnings \$0.67 per share. Which stock is lower priced and by what percent?

33. Market Equilibrium A company is analyzing the market for chocolate in South Korea. The quantity demand of chocolate is $-10 + 2p$ and the quantity supply of chocolate is $20 - 2p$, where p is the price of chocolate (in South Korean Won, KRW). Find the value of p when the South Korean chocolate market is said to be in equilibrium (for which supply equals demand).

34. Market Equilibrium Repeat Problem 33 with the quantity demand of chocolate in South Korea is $10p - 0.1p^2$ and the quantity supply of chocolate is $-75 - p$.

35. Fencing For safety purposes, the management of a company decides to add a fence around the car park, which is bounded by the office building on one side. The car park is a rectangular area of 12,400 m^2 and the bounding side of the building is 350 m long. The company needs 450 m of fencing to cover the three sides of the car park. Find the dimensions of the car park.

We reject 385.7016 m because the building is only 350 m long. So, the car park follows the dimensions of the rectangular area are 64.2984 m by 192.8508 m.

36. Volume Don is trying to recreate a pyramid structure that he came across in Egypt. His model is half the volume of the original structure. He reduces the same amount for both high and base of the structure (See Figure 1.4). The original pyramid structure has a height of 30 cm and a base of 5 cm. Find the dimensions for Don's model.

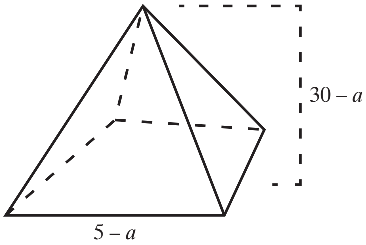


FIGURE 1.4 Reduce volume (Problem 36).

37. Swimming Pool David is building a swimming pool that is 10 ft long, 9 ft wide, and 4 ft deep. He feels the swimming pool is not deep enough and is trying to make it deeper. While keeping the width of the swimming pool the same, David decides to increase the depth and the length by equal amounts. By doing this, he estimates that the swimming pool will increase 40% in volume. What will be the length and depth of David's new swimming pool?

38. Product Design A lollipop factory makes spherical lollipops. To reduce the cost, the factory decides to reduce the radius of each lollipop, which will reduce the volume of each lollipop by 5%. The original radius of each lollipop is 12 millimeters. By how much is the radius of each lollipop reduced?

39. Compensating Balance *Compensating balance* refers to the minimum balance that a customer must maintain in a bank account, used to offset the cost incurred by a bank to set up a loan. For example, if a firm takes out a $10,000 loan that requires a compensating balance of 10%, it will have to leave $1000 on deposit and would have the use of $9000. A factory needs $150,000 to invest in new equipment. A bank requires a compensating balance of 5%. How much the factory supposed to loan (to the nearest thousand dollars) from the bank? Now derive the general equation in determining the amount L of a loan that is needed to handle expenses E with p% compensating balance is required.

40. Incentive Plan Dimond Water is selling household water filter. The company provides a commission of $150 to each salesperson for the first 10 water filter machines they sell. In addition, they receive a commission of $50 for each water filter sold. How many water filters, in the nearest whole number, should a salesperson sell to earn $10,000?

41. Selling Products Kitty purchased various products from Korea for $5500. She decided to sell these products once she returned home. On average, after selling all except 10 products, she found that for 10 products she earned $50 more than the respective cost price. How many products did Kitty bring back from Korea?

42. Marginal Profit The *margin of profit* of a company is the net income divided by the total sales. Dora owns a bakery and wants to know her margin of profit for the slices of cake sold. She finds that her margin of profit increased by 1% in comparison to the previous year. Last year she sold one slice of cake for €10 and had a net income of €550. This year she increased the price of each slice by €2 and managed to sell 500 slices more, earning a net income of €1440. Dora's bakery has never had a margin of profit greater than 5%. How many slices of cake were sold last year and how many did she sell this year?

43. Business A bulb manufacturer produces two types of bulbs: the compact fluorescent lamps (CFL) and light emitting diode (LED). The cost of producing each LED bulb is £9 more than that of producing each CFL bulb. The cost of production for LED and CFL bulbs are £1326 and £300, respectively. The manufacturer produced 60 more units of LED bulbs than CFL bulbs. How many of each type are produced?

Objective

To solve linear inequalities in one variable and to introduce interval notation.

1.2 Linear Inequalities

Suppose a and b are two points on the real-number line. Then either a and b coincide, or a lies to the left of b, or a lies to the right of b. (See Figure 1.5.)

If a and b coincide, then $a = b$. If a lies to the left of b, we say that a is less than b and write $a < b$, where the *inequality symbol* "$<$" is read "is less than." On the other hand, if a lies to the right of b, we say that a is greater than b, written $a > b$. The statements $a > b$ and $b < a$ are equivalent. (If you have trouble keeping these symbols straight, it may help to notice that $<$ looks somewhat like the letter L for *left* and that we have $a < b$ precisely when a lies to the *left* of b.)

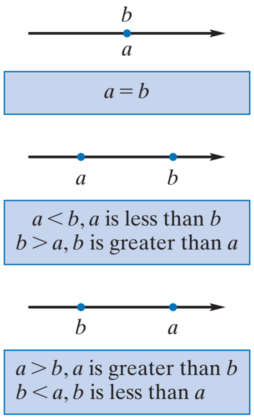


FIGURE 1.5 Relative positions of two points.

Another inequality symbol "$\leq$" is read "is less than or equal to" and is defined as follows: $a \leq b$ if and only if $a < b$ or $a = b$. Similarly, the symbol "$\geq$" is defined as follows: $a \geq b$ if and only if $a > b$ or $a = b$. In this case, we say that a is greater than or equal to b.

We often use the words *real numbers* and *points* interchangeably, since there is a one-to-one correspondence between real numbers and points on a line. Thus, we can speak of the points -5, -2, 0, 7, and 9 and can write $7 < 9$, $-2 > -5$, $7 \leq 7$, and $7 \geq 0$. (See Figure 1.6.) Clearly, if $a > 0$, then a is positive; if $a < 0$, then a is negative.

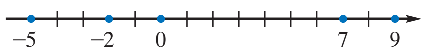


FIGURE 1.6 Points on a number line.

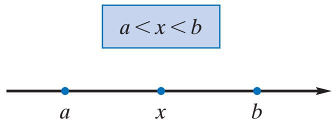


FIGURE 1.7 $a < x$ and $x < b$.

Suppose that $a < b$ and x is between a and b. (See Figure 1.7.) Then not only is $a < x$, but also, $x < b$. We indicate this by writing $a < x < b$. For example, $0 < 7 < 9$. (Refer back to Figure 1.6.)

> **Definition**
>
> An **inequality** is a statement that one quantity is less than, or greater than, or less than or equal to, or greater than or equal to, another quantity.

Of course, we represent inequalities by means of inequality symbols. If two inequalities have their inequality symbols pointing in the same direction, then the inequalities are said to have the *same sense*. If not, they are said to be *opposite in sense*. Hence, $a < b$ and $c < d$ have the same sense, but $a < b$ has the opposite sense of $c > d$.

Solving an inequality, such as $2(x-3) < 4$, means finding all values of the variable for which the inequality is true. This involves the application of certain rules, which we now state.

Rules for Inequalities

1. If the same number is added to or subtracted from both sides of an inequality, then another inequality results, having the same sense as the original inequality. Symbolically,

Keep in mind that the rules also apply to $\leq$, $>$, and $\geq$.

$$\text{If } a < b, \text{ then } a + c < b + c \text{ and } a - c < b - c.$$

For example, $7 < 10$ so $7 + 3 < 10 + 3$.

2. If both sides of an inequality are multiplied or divided by the same *positive* number, then another inequality results, having the same sense as the original inequality. Symbolically,

$$\text{If } a < b \text{ and } c > 0, \text{ then } ac < bc \text{ and } \frac{a}{c} < \frac{b}{c}.$$

For example, $3 < 7$ and $2 > 0$ so $3(2) < 7(2)$ and $\frac{3}{2} < \frac{7}{2}$.

3. If both sides of an inequality are multiplied or divided by the same *negative* number, then another inequality results, having the opposite sense of the original inequality. Symbolically,

$$\text{If } a < b \text{ and } c < 0, \text{ then } ac > bc \text{ and } \frac{a}{c} > \frac{b}{c}.$$

Multiplying or dividing an inequality by a negative number gives an inequality of the opposite sense.

For example, $4 < 7$ and $-2 < 0$, so $4(-2) > 7(-2)$ and $\frac{4}{-2} > \frac{7}{-2}$.

4. Any side of an inequality can be replaced by an expression equal to it. Symbolically,

$$\text{If } a < b \text{ and } a = c, \text{ then } c < b.$$

For example, if $x < 2$ and $x = y + 4$, then $y + 4 < 2$.

5. If the sides of an inequality are either both positive or both negative and reciprocals are taken on both sides, then another inequality results, having the opposite sense of the original inequality. Symbolically,

$$\text{If } 0 < a < b \text{ or } a < b < 0, \text{ then } \frac{1}{a} > \frac{1}{b}.$$

For example, $2 < 4$ so $\frac{1}{2} > \frac{1}{4}$ and $-4 < -2$ so $\frac{1}{-4} > \frac{1}{-2}$.

6. If both sides of an inequality are positive and each side is raised to the same positive power, then another inequality results, having the same sense as the original inequality. Symbolically,

$$\text{If } 0 < a < b \text{ and } n > 0, \text{ then } a^n < b^n.$$

For n a positive integer, this rule further provides

$$\text{If } 0 < a < b, \text{ then } \sqrt[n]{a} < \sqrt[n]{b}.$$

For example, $4 < 9$ so $4^2 < 9^2$ and $\sqrt{4} < \sqrt{9}$.

A pair of inequalities will be said to be **equivalent inequalities** if when either is true then the other is true. When any of Rules 1–6 are applied to an inequality, it is easy to show that the result is an equivalent inequality.

Expanding on the terminology in Section 0.1, a number a is *positive* if $0 < a$ and *negative* if $a < 0$. It is often useful to say that a is *nonnegative* if $0 \leq a$.

Observe from Rule 1 that $a \leq b$ is equivalent to "$b - a$ is nonnegative." Another simple observation is that $a \leq b$ is equivalent to "there exists a nonnegative number s such that $a + s = b$." The s which does the job is just $b - a$ but the idea is useful when one side of $a \leq b$ contains an unknown.

This idea allows us to replace an inequality with an equality—at the expense of introducing a variable. In Chapter 7, the powerful simplex method builds on replacement of inequalities $a \leq b$ with equations $a + s = b$, for nonnegative s. In this context, s is called a *slack variable* because it takes up the "slack" between a and b.

We will now apply Rules 1–4 to a *linear inequality*.

The definition also applies to $\leq$, $>$, and $\geq$.

Definition

A **linear inequality** in the variable x is an inequality that can be written in the form

$$ax + b < 0$$

where a and b are constants and $a \neq 0$.

We should expect that the inequality will be true for some values of x and false for others. To *solve* an inequality involving a variable is to find all values of the variable for which the inequality is true.

APPLY IT ►

1. A salesman has a monthly income given by $I = 200 + 0.8S$, where S is the number of products sold in a month. How many products must he sell to make at least \$4500 a month?

EXAMPLE 1 Solving a Linear Inequality

Solve $2(x - 3) < 4$.

Solution:

Strategy We will replace the given inequality by equivalent inequalities until the solution is evident.

$$
\begin{aligned}
2(x-3) &< 4 & \\
2x - 6 &< 4 & \text{Rule 4} \\
2x - 6 + 6 &< 4 + 6 & \text{Rule 1} \\
2x &< 10 & \text{Rule 4} \\
\frac{2x}{2} &< \frac{10}{2} & \text{Rule 2} \\
x &< 5 & \text{Rule 4}
\end{aligned}
$$

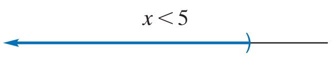


FIGURE 1.8 All real numbers less than 5.

All of the foregoing inequalities are equivalent. Thus, the original inequality is true for *all* real numbers x such that $x < 5$. For example, the inequality is true for $x = -10, -0.1, 0, \frac{1}{2}$, and 4.9. We can write our solution simply as $x < 5$ and present it geometrically by the colored half-line in Figure 1.8. The parenthesis indicates that 5 *is not included* in the solution.

Now Work Problem 9 ◁

APPLY IT ►

2. A zoo veterinarian can purchase four different animal foods with various nutrient values for the zoo's grazing animals. Let x_1 represent the number of bags of food 1, x_2 represent the number of bags of food 2, and so on. The number of bags of each food needed can be described by the following equations:

$$x_1 = 150 - x_4$$

$$x_2 = 3x_4 - 210$$

$$x_3 = x_4 + 60$$

Assuming that each variable must be nonnegative, write four inequalities involving x_4 that follow from these equations.

In Example 1 the solution consisted of a set of numbers, namely, all real numbers less than 5. It is common to use the term **interval** to describe such a set. In the case of Example 1, the set of all x such that $x < 5$ can be denoted by the *interval notation* $(-\infty, 5)$. The symbol $-\infty$ is not a number, but is merely a convenience for indicating that the interval includes all numbers less than 5.

There are other types of intervals. For example, the set of all real numbers x for which $a \le x \le b$ is called a **closed interval** and includes the numbers a and b, which are called **endpoints** of the interval. This interval is denoted by $[a, b]$ and is shown in Figure 1.9(a). The square brackets indicate that a and b *are included* in the interval. On the other hand, the set of all x for which $a < x < b$ is called an **open interval** and is denoted by (a, b). The endpoints *are not included* in this set. [See Figure 1.9(b).] Extending these concepts and notations, we have the intervals shown in Figure 1.10. Just as $-\infty$ is not a number, so ∞ is not a number but (a, ∞) is a convenient notation for the set of all real numbers x for which $a < x$. Similarly, $[a, \infty)$ denotes all real x for which $a \le x$. It is a natural extension of this notation to write $(-\infty, \infty)$ for the set of *all* real numbers and we will do so throughout this book.

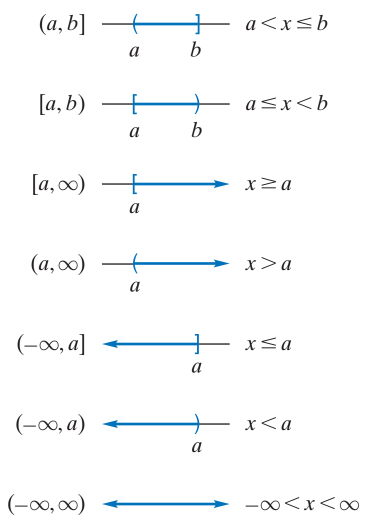


FIGURE 1.10 Intervals.

Closed interval $[a, b]$ (a) Open interval (a, b) (b)

FIGURE 1.9 Closed and open intervals.

EXAMPLE 2 Solving a Linear Inequality

Solve $3 - 2x \le 6$.

Solution:

$$
\begin{aligned}
3 - 2x &\le 6 & \\
-2x &\le 3 & \text{Rule 1} \\
x &\ge -\frac{3}{2} & \text{Rule 3}
\end{aligned}
$$

Dividing both sides by -2 *reverses* the sense of the inequality.

The solution is $x \geq -\frac{3}{2}$, or, in interval notation, $[-\frac{3}{2}, \infty)$. This is represented geometrically in Figure 1.11.

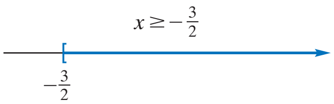


FIGURE 1.11 The interval $[-\frac{3}{2}, \infty)$.

Now Work Problem 7 ◁

EXAMPLE 3 Solving a Linear Inequality

Solve $\frac{3}{2}(s-2)+1 > -2(s-4)$.

Solution:

$$\frac{3}{2}(s-2)+1 > -2(s-4)$$

$$2\left(\frac{3}{2}(s-2)+1\right) > 2(-2(s-4)) \qquad \text{Rule 2}$$

$$3(s-2)+2 > -4(s-4)$$

$$3s-4 > -4s+16$$

$$7s > 20 \qquad \text{Rule 1}$$

$$s > \frac{20}{7} \qquad \text{Rule 2}$$

The solution is $(\frac{20}{7}, \infty)$; see Figure 1.12.

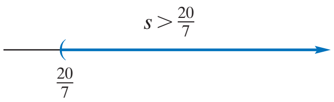


FIGURE 1.12 The interval $(\frac{20}{7}, \infty)$.

Now Work Problem 19 ◁

EXAMPLE 4 Solving Linear Inequalities

a. Solve $2(x-4)-3 > 2x-1$.

Solution:

$$2(x-4)-3 > 2x-1$$

$$2x-8-3 > 2x-1$$

$$-11 > -1$$

Since it is never true that $-11 > -1$, there is no solution, and the solution set is $\emptyset$ (the set with no elements).

b. Solve $2(x-4)-3 < 2x-1$.

Solution: Proceeding as in part (a), we obtain $-11 < -1$. This is true for all real numbers x, so the solution is $(-\infty, \infty)$; see Figure 1.13.

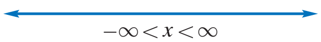


FIGURE 1.13 The interval $(-\infty, \infty)$.

Now Work Problem 15 ◁

PROBLEMS 1.2

In Problems 1–34, solve the inequalities. Give your answer in interval notation, and indicate the answer geometrically on the real-number line.

1. $3x > 21$

2. $4x < -2$

3. $5x - 11 \leq 9$

4. $5x \leq 0$

5. $-4x \geq 2$

6. $2z + 5 > 0$

7. $5 - 7s > 3$

8. $4s - 1 < -5$

9. $3 < 2y + 3$

10. $4 \leq 3 - 2y$

11. $t + 4 \leq 3 + 2t$

12. $-3 \geq 8(2 - x)$

13. $3(2 - 3x) > 4(1 - 4x)$

14. $8(x + 1) + 1 < 3(2x) + 1$

15. $2(4x - 2) > 4(2x + 1)$

16. $7 - (x + 3) \leq 3(3 - x)$

17. $x + 2 < \sqrt{3} - x$

18. $\sqrt{2}(x + 2) > \sqrt{8}(3 - x)$

19. $\frac{5}{6}x < 40$

20. $-\frac{2}{3}x > 6$

21. $\frac{3y+1}{2} \leq 5y - 1$

22. $\frac{3y-2}{3} \geq \frac{1}{4}$

23. $-3x + 1 \leq -3(x - 2) + 1$

24. $0x \leq 0$

25. $\frac{1-t}{2} < \frac{3t-7}{3}$

26. $\frac{3(2t+2)}{2} > \frac{t-3}{4} + \frac{t}{3}$

27. $2x + 13 \geq \frac{1}{3}x - 7$

28. $3x - \frac{1}{3} \leq \frac{5}{2}x$

29. $\frac{2}{3}r < \frac{5}{6}r$

30. $\frac{7}{4}t > -\frac{8}{3}t$

31. $2y + \frac{y}{5} < \frac{y}{2} + \frac{y}{3}$

32. $9 - 0.1x \leq \frac{2 - 0.01x}{0.2}$

33. $0.1(0.03x + 4) \geq 0.02x + 0.434$

34. $\dfrac{3y - 1}{-3} < \dfrac{5(y + 1)}{-3}$

35. Expenditure Every month Allen spends more than €200, but not more than €500, on healthcare products. If E represents Allen's yearly expenditure on healthcare products, describe E by using inequalities.

36. Distance Using inequalities, show that the average monthly distance d traveled by a salesperson is at least 500 kilometers (km) and not more than 850 km.

37. Geometric Shapes The area of a rectangle is less than 100 cm^2 and its length is 4 times longer than its width. Determine the length of the rectangle.

38. Spending Daren receives \$4 everyday as his pocket money. At his school canteen, one raisin bun costs \$0.60 and one bottle of milk costs \$1.20. If Daren buys a bottle of milk, find the number of buns he can buy each day.

Objective

To model real-life situations in terms of inequalities.

1.3 Applications of Inequalities

Solving word problems may sometimes involve inequalities, as the following examples illustrate.

EXAMPLE 1 Profit

For a company that manufactures aquarium heaters, the combined cost for labor and material is \$21 per heater. Fixed costs (costs incurred in a given period, regardless of output) are \$70,000. If the selling price of a heater is \$35, how many must be sold for the company to earn a profit?

Solution:

Strategy Recall that

$$\text{profit} = \text{total revenue} - \text{total cost}$$

We will find total revenue and total cost and then determine when their difference is positive.

Let q be the number of heaters that must be sold. Then their cost is $21q$. The total cost for the company is therefore $21q + 70{,}000$. The total revenue from the sale of q heaters will be $35q$. Now,

$$\text{profit} = \text{total revenue} - \text{total cost}$$

and we want profit > 0. Thus,

$$\begin{aligned} \text{total revenue} - \text{total cost} &> 0 \\ 35q - (21q + 70{,}000) &> 0 \\ 14q &> 70{,}000 \\ q &> 5000 \end{aligned}$$

Since the number of heaters must be a nonnegative integer, we see that at least 5001 heaters must be sold for the company to earn a profit.

Now Work Problem 1 ◁

EXAMPLE 2 Renting versus Purchasing

A builder must decide whether to rent or buy an excavating machine. If he were to rent the machine, the rental fee would be \$3000 per month (on a yearly basis), and the daily cost (gas, oil, and driver) would be \$180 for each day the machine is used. If he were to buy it, his fixed annual cost would be \$20,000, and daily operating and maintenance costs would be \$230 for each day the machine is used. What is the least number of days each year that the builder would have to use the machine to justify renting it rather than buying it?

Solution:

Strategy We will determine expressions for the annual cost of renting and the annual cost of purchasing. We then find when the cost of renting is less than that of purchasing.

Let d be the number of days each year that the machine is used. If the machine is rented, the total yearly cost consists of rental fees, which are (12)(3000), and daily charges of $180d$. If the machine is purchased, the cost per year is $20{,}000 + 230d$. We want

$$\text{cost}_{\text{rent}} < \text{cost}_{\text{purchase}}$$
$$12(3000) + 180d < 20{,}000 + 230d$$
$$36{,}000 + 180d < 20{,}000 + 230d$$
$$16{,}000 < 50d$$
$$320 < d$$

Thus, the builder must use the machine at least 321 days to justify renting it.

Now Work Problem 3 ◁

EXAMPLE 3 Current Ratio

The **current ratio** of a business is the ratio of its **current assets** (such as cash, merchandise inventory, and accounts receivable) to its **current liabilities** (such as short-term loans and taxes payable).

After consulting with the comptroller, the president of the Ace Sports Equipment Company decides to take out a short-term loan to build up inventory. The company has current assets of \$350,000 and current liabilities of \$80,000. How much can the company borrow if the current ratio is to be no less than 2.5? (Note: The funds received are considered as current assets and the loan as a current liability.)

Solution: Let x denote the amount the company can borrow. Then current assets will be $350{,}000 + x$, and current liabilities will be $80{,}000 + x$. Thus,

$$\text{current ratio} = \frac{\text{current assets}}{\text{current liabilities}} = \frac{350{,}000 + x}{80{,}000 + x}$$

We want

$$\frac{350{,}000 + x}{80{,}000 + x} \geq 2.5$$

Although the inequality that must be solved is not apparently linear, it is equivalent to a linear inequality.

Since x is positive, so is $80{,}000 + x$. Hence, we can multiply both sides of the inequality by $80{,}000 + x$ and the sense of the inequality will remain the same. We have

$$350{,}000 + x \geq 2.5(80{,}000 + x)$$
$$150{,}000 \geq 1.5x$$
$$100{,}000 \geq x$$

Consequently, the company may borrow as much as \$100,000 and still maintain a current ratio greater than or equal to 2.5.

Now Work Problem 8 ◁

EXAMPLE 4 Crime Statistics

In the television show *The Wire*, the detectives declared a homicide to be *cleared* if the case was solved. If the number of homicides in a month was $H > 0$ and C were cleared then the *clearance rate* was defined to be C/H. The section boss, Rawls, fearing a smaller clearance rate, is angered when McNulty adds 14 new homicides to the responsibility of his section. However, the 14 cases are related to each other, so solving one case will solve them all. If all 14 new cases are solved, will the clearance rate change, and if so will it be for the better or worse?

Solution: The question amounts to asking about the relative size of the fractions $\frac{C}{H}$ and $\frac{C+14}{H+14}$. For nonnegative numbers a and c and positive numbers b, and d, we have

$$\frac{a}{b} < \frac{c}{d} \qquad \text{if and only if} \qquad ad < bc$$

We have $0 < H$ and $0 \leq C \leq H$. If $C = H$, a perfect clearance rate, then $C + 14 = H + 14$ and the clearance rate is still perfect when 14 new cases are both added and solved.

But if $C < H$ then $14C < 14H$ and $CH + 14C < CH + 14H$ shows that $C(H + 14) < H(C + 14)$ and hence

$$\frac{C}{H} < \frac{C+14}{H+14}$$

giving a *better* clearance rate when 14 new cases are both added and solved.

Now Work Problem 13 ◁

Of course there is nothing special about the positive number 14 in the last example. Try to formulate a general rule that will apply to Example 4.

PROBLEMS 1.3

1. Profit F&G, a multinational consumer goods corporation, estimates a profit of at least £450,000 for the coming year. The corporation's fixed costs are £100,000 and it has a unit cost of £2. Every unit of F&G's product has a selling price of £10. Find the quantity F&G should sell to earn its estimated profit.

2. Cost Don is going to open a bread stall. He plans his fixed monthly expense to be no more than \$10,000. He will hire an assistant for a monthly salary of \$500. The rent for a stall is \$1800 and the monthly utility cost for a stall is \$200. The variable cost of producing a loaf of bread is \$2.50. Find the number of loaves that Don can produce.

3. Fisherman The variable costs incurred by a fisherman depends on the level of fishing activity and the fixed cost is normally equal to the fisherman's capital value. The fixed monthly expense of a fisherman is \$5000. The amount he spends on fuel is a variable cost, which is \$600 per ton of fish caught. The fish sells for \$1100 per ton. What is the minimum ton of fish the fisherman needs to catch to avoid incurring a loss?

4. Cookies Florence is preparing cookies for a charity event and has to ensure that she earns profit to contribute to the charity. Each packet she sells consists of 10 cookies. Her fixed cost is €60 and the material cost for her cookies is €4.1N with a packaging cost of €0.5N, in which N is the number of packs for every 10 cookies. Find N if Florence sells each packet for €6.50.

5. Packing Harvey Lombard, a Swedish art company, is preparing anniversary special box set as a part of its 25$^{\text{th}}$ anniversary celebrations. The cost of each set is 118 kr (note: krona, kr, is the official currency of Sweden). The art supplies in each set costs 98 kr and the packaging of each set costs 2.50 kr. An order of more than 5000 sets will receive a discount of 20% of the price of the set. Find the least number of sets that Harvey Lombard produces profitably, if 85% of them are sold.

6. Part-time Workers A factory receives an order for which it needs to hire part-time workers to meet the due date. The factory's total labor cost for the order is \$10,000. The factory has 150 regular workers and pays them \$50 per day. It will pay the part-time workers \$65 per day. How many part-time workers can the factory hire?

7. Investment Elizabeth has £25,000 to invest in Bank A and B's mutual fund. Bank A and B offer her an annual return rates of 5.2% and 4.25%, respectively. How much should Elizabeth invest in Bank A's mutual fund if she wants an annual yield of not less than 5.15% from her total investment?

8. Current Ratio Shawn is going to apply for a loan for his company. His current assets are \$550,000 with a current ratio of 2.2. Find Shawn's current liabilities. What is the maximum loan amount Shawn can apply for if the current ratio is to be no less than 1.8? (See Example 3 for an explanation of current ratio.)

9. Business Plans A company offers two types of plans for its sales representatives. Plan A offers a basic pay of €500 and a 10% of commission on an individual's sales. Plan B offers a monthly salary of €2100. How much sales would a sales representative need to make in Plan A to get not less than Plan B's monthly salary?

10. Revenue A shop is trying to increase its revenue by increasing the price of its product by $\frac{140}{q} + 15$ dollars per unit, where q is the number of units purchased by the consumers. How many products must be sold in order that its revenue is greater than \$25,000?

11. Hourly Rate A company is hiring data entry operators who are responsible for entering data into the company's databases and maintaining accurate records of the company information. The operator may receive an hourly wage of \$15 or may be paid on a per-job basis of \$250 plus \$4 for each hour less than 20 if they complete a job in less than 20 hours. For what values of $t(t < 20)$ is the hourly rate better than the job-basis rate?

12. Earnings Joanne works part-time and makes \$55 per hour. During winter, she requires her suit to be dry cleaned every day for \$45 and the babysitter she hired charges \$6 per hour for babysitting and an additional hour for travel. Excluding her expenses, Joanne wants to earn more than \$250 a day. For at least how many hours must she work in a day?

13. Fractions If $0 < b < a$ and $0 < d < c$, show that $\frac{a}{d} > \frac{b}{c}$.

14. Purchasing Jack is saving money to buy a pair of sports shoes that will cost \$190 in three weeks' time. So far, he has saved \$45. How much must he save every week to have enough money to purchase the pair of sports shoes? Give your answer to the nearest dollar.

Objective

To solve equations and inequalities involving absolute values.

1.4 Absolute Value

Absolute-Value Equations

On the real-number line, the **distance** of a number x from 0 is called the **absolute value** of x and is denoted by $|x|$. For example, $|5| = 5$ and $|-5| = 5$ because both 5 and -5 are 5 units from 0. (See Figure 1.14.) Similarly, $|0| = 0$. Notice that $|x|$ can never be negative; that is, $|x| \geq 0$.

If x is positive or zero, then $|x|$ is simply x itself, so we can omit the vertical bars and write $|x| = x$. On the other hand, consider the absolute value of a negative number, like $x = -5$.

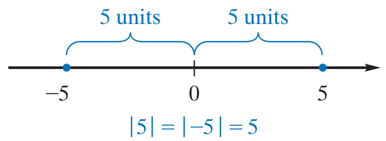


FIGURE 1.14 Absolute value.

$$|x| = |-5| = 5 = -(-5) = -x$$

Thus, if x is negative, then $|x|$ is the positive number $-x$. The minus sign indicates that we have changed the sign of x. The geometric definition of absolute value as a distance is equivalent to the following:

Definition

The **absolute value** of a real number x, written $|x|$, is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$\sqrt{x^2}$ is not necessarily x but $\sqrt{x^2} = |x|$. For example, $\sqrt{(-2)^2} = |-2| = 2$, not -2.

Observe that $|-x| = |x|$ follows from the definition.

Applying the definition, we have $|3| = 3$, $|-8| = -(-8) = 8$, and $|\frac{1}{2}| = \frac{1}{2}$. Also, $-|2| = -2$ and $-|-2| = -2$.

Also, $|-x|$ is not necessarily x and, thus, $|-x-1|$ is not necessarily $x+1$. For example, if we let $x = -3$, then $|-(-3)| \neq -3$, and

$$|-(-3) - 1| \neq -3 + 1$$

EXAMPLE 1 Solving Absolute-Value Equations

a. Solve $|x - 3| = 2$.

Solution: This equation states that $x - 3$ is a number 2 units from 0. Thus, either

$$x - 3 = 2 \quad \text{or} \quad x - 3 = -2$$

Solving these equations gives $x = 5$ or $x = 1$. See Figure 1.15.

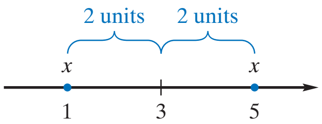


FIGURE 1.15 The solution of $|x - 3| = 2$ is 1 or 5.

b. Solve $|7 - 3x| = 5$.

Solution: The equation is true if $7 - 3x = 5$ or if $7 - 3x = -5$. Solving these equations gives $x = \frac{2}{3}$ or $x = 4$.

c. Solve $|x - 4| = -3$.

Solution: The absolute value of a number is never negative, so the solution set is $\varnothing$.

Now Work Problem 19 ◁

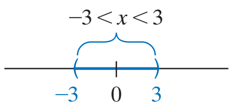
We can interpret $|a - b| = |-(b - a)| = |b - a|$ as the distance between a and b. For example, the distance between 5 and 9 can be calculated via

$$\begin{aligned} \text{either} \quad |9 - 5| &= |4| = 4 \\ \text{or} \quad |5 - 9| &= |-4| = 4 \end{aligned}$$

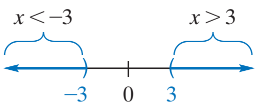
Similarly, the equation $|x - 3| = 2$ states that the distance between x and 3 is 2 units. Thus, x can be 1 or 5, as shown in Example 1(a) and Figure 1.16.

Absolute-Value Inequalities

Let us turn now to inequalities involving absolute values. If $|x| < 3$, then x is less than 3 units from 0. Hence, x must lie between -3 and 3; that is, on the interval $-3 < x < 3$. [See Figure 1.16(a).] On the other hand, if $|x| > 3$, then x must be greater than 3 units from 0. Hence, there are two intervals in the solution: Either $x < -3$ or $x > 3$. [See Figure 1.16(b).] We can extend these ideas as follows: If $|x| \le 3$, then $-3 \le x \le 3$; if $|x| \ge 3$, then $x \le -3$ or $x \ge 3$. Table 1.1 gives a summary of the solutions to absolute-value inequalities.



(a) Solution of $|x| < 3$



(b) Solution of $|x| > 3$

FIGURE 1.16 Solutions of $|x| < 3$ and $|x| > 3$.

Table 1.1

Inequality ($d > 0$)	Solution
$\lvert x\rvert < d$	$-d < x < d$
$\lvert x\rvert \le d$	$-d \le x \le d$
$\lvert x\rvert > d$	$x < -d$ or $x > d$
$\lvert x\rvert \ge d$	$x \le -d$ or $x \ge d$

EXAMPLE 2 Solving Absolute-Value Inequalities

a. Solve $|x - 2| < 4$.

Solution: The number $x - 2$ must be less than 4 units from 0. From the preceding discussion, this means that $-4 < x - 2 < 4$. We can set up the procedure for solving this inequality as follows:

$$\begin{aligned} &-4 < x - 2 < 4 \\ &-4 + 2 < x < 4 + 2 \qquad \text{adding 2 to each member} \\ &-2 < x < 6 \end{aligned}$$

Thus, the solution is the open interval $(-2, 6)$. This means that all numbers between -2 and 6 satisfy the original inequality. (See Figure 1.17.)

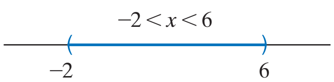


FIGURE 1.17 The solution of $|x - 2| < 4$ is the interval $(-2, 6)$.

b. Solve $|3 - 2x| \leq 5$.

Solution:

$$\begin{array}{rcll} -5 \leq & 3 - 2x & \leq 5 & \\ -5 - 3 \leq & -2x & \leq 5 - 3 & \text{subtracting 3 throughout} \\ -8 \leq & -2x & \leq 2 & \\ 4 \geq & x & \geq -1 & \text{dividing throughout by } -2 \\ -1 \leq & x & \leq 4 & \text{rewriting} \end{array}$$

Note that the sense of the original inequality was *reversed* when we divided by a negative number. The solution is the closed interval $[-1, 4]$.

Now Work Problem 29 ◁

EXAMPLE 3 Solving Absolute-Value Inequalities

a. Solve $|x + 5| \geq 7$.

Solution: Here $x + 5$ must be *at least* 7 units from 0. Thus, either $x + 5 \leq -7$ *or* $x + 5 \geq 7$. This means that either $x \leq -12$ *or* $x \geq 2$. Thus, the solution consists of two intervals: $(-\infty, -12]$ and $[2, \infty)$. We can abbreviate this collection of numbers by writing

$$(-\infty, -12] \cup [2, \infty)$$

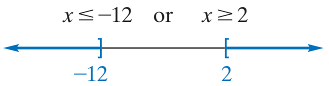


FIGURE 1.18 The union $(-\infty, -12] \cup [2, \infty)$.

where the connecting symbol $\cup$ is called the *union* symbol. (See Figure 1.18.) More formally, the **union** of sets A and B is the set consisting of all elements that are in either A or B (or in both A and B).

The inequalities $x \leq -12$ or $x \geq 2$ in (a) and $x < 1$ *or* $x > \frac{5}{3}$ in (b) do not give rise to a single interval as in Examples 2a and 2b.

b. Solve $|3x - 4| > 1$.

Solution: Either $3x - 4 < -1$ *or* $3x - 4 > 1$. Thus, either $3x < 3$ *or* $3x > 5$. Therefore, $x < 1$ *or* $x > \frac{5}{3}$, so the solution consists of all numbers in the set $(-\infty, 1) \cup (\frac{5}{3}, \infty)$.

Now Work Problem 31 ◁

APPLY IT ▶

3. Express the following statement using absolute-value notation: The actual weight w of a box of cereal must be within 0.3 oz of the weight stated on the box, which is 22 oz.

EXAMPLE 4 Absolute-Value Notation

Using absolute-value notation, express the following statements:

a. x is less than 3 units from 5.

Solution: $$|x - 5| < 3$$

b. x differs from 6 by at least 7.

Solution: $$|x - 6| \geq 7$$

c. $x < 3$ and $x > -3$ simultaneously.

Solution: $$|x| < 3$$

d. x is more than 1 unit from -2.

Solution:
$$|x - (-2)| > 1$$
$$|x + 2| > 1$$

e. x is less than σ (a Greek letter read "sigma") units from μ (a Greek letter read "mu").

Solution: $$|x - \mu| < \sigma$$

f. x is within ϵ (a Greek letter read "epsilon") units from a.

Solution: $$|x - a| < \epsilon$$

Now Work Problem 11 ◁

Properties of the Absolute Value

Five basic properties of the absolute value are as follows:

1. $|ab| = |a| \cdot |b|$
2. $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$
3. $|a - b| = |b - a|$
4. $-|a| \leq a \leq |a|$
5. $|a + b| \leq |a| + |b|$

For example, Property 1 states that the absolute value of the product of two numbers is equal to the product of the absolute values of the numbers. Property 5 is known as *the triangle inequality*.

EXAMPLE 5 Properties of Absolute Value

a. $|(-7) \cdot 3| = |-7| \cdot |3| = 21$

b. $|4 - 2| = |2 - 4| = 2$

c. $|7 - x| = |x - 7|$

d. $\left|\dfrac{-7}{3}\right| = \dfrac{|-7|}{|3|} = \dfrac{7}{3}; \left|\dfrac{-7}{-3}\right| = \dfrac{|-7|}{|-3|} = \dfrac{7}{3}$

e. $\left|\dfrac{x-3}{-5}\right| = \dfrac{|x-3|}{|-5|} = \dfrac{|x-3|}{5}$

f. $-|2| \leq 2 \leq |2|$

g. $|(-2) + 3| = |1| = 1 \leq 5 = 2 + 3 = |-2| + |3|$

Now Work Problem 5 ◁

PROBLEMS 1.4

In Problems 1–10, evaluate the absolute value expression.

1. $|-13|$ **2.** $|2^{-1}|$ **3.** $|3 - 5|$

4. $|(-3 - 5)/2|$ **5.** $|2(-\frac{7}{2})|$ **6.** $|3 - 5| - |5 - 3|$

7. $|x| < 4$ **8.** $|x| < -1$ **9.** $|3 - \sqrt{10}|$

10. $|\sqrt{5} - 2|$

11. Using the absolute-value symbol, express each fact.
(a) x is less than 3 units from 7.
(b) x differs from 2 by less than 3.
(c) x is no more than 5 units from 7.
(d) The distance between 7 and x is 4.
(e) $x + 4$ is less than 2 units from 0.
(f) x is between -3 and 3, but is not equal to 3 or -3.
(g) $x < -6$ or $x > 6$.
(h) The number x of hours that a machine will operate efficiently differs from 105 by less than 3.
(i) The average monthly income x (in dollars) of a family differs from 850 by less than 100.

12. Use absolute-value notation to indicate that $f(x)$ and L differ by less than ϵ.

13. Use absolute-value notation to indicate that the prices p_1 and p_2 of two products differ by at least 5 dollars.

14. Find all values of x such that $|x - \mu| < 3\sigma$.

In Problems 15–36, solve the given equation or inequality.

15. $|x| = 7$ **16.** $|-x| = 2$ **17.** $\left|\dfrac{x}{5}\right| = 7$

18. $\left|\dfrac{3}{x}\right| = 7$ **19.** $|x - 5| = 9$ **20.** $|4 + 3x| = 6$

21. $|5x - 2| = 0$ **22.** $|7x + 3| = x$ **23.** $|3 - 5x| = 2$

24. $|5 - 3x| = 7$ **25.** $|x| < M$ for $M > 0$

26. $|-x| < 3$ **27.** $\left|\dfrac{x}{4}\right| > 2$ **28.** $\left|\dfrac{x}{2}\right| > \dfrac{1}{3}$

29. $|x + 7| < 3$ **30.** $|2x - 17| < -4$ **31.** $\left|x - \dfrac{1}{2}\right| > \dfrac{1}{2}$

32. $|1 - 3x| > 2$ **33.** $|3 - 2x| \leq 2$ **34.** $|3x - 2| \geq 0$

35. $\left|\dfrac{3x-8}{2}\right| \geq 4$ **36.** $\left|\dfrac{x-7}{3}\right| \leq 5$

In Problems 37–38, express the statement using absolute-value notation.

37. In a science experiment, the measurement of a distance d is 35.2 m, and is accurate to ± 20 cm.

38. The difference in temperature between two chemicals that are to be mixed must be at least 5 degrees and at most 10 degrees.

39. Statistics In the test statistic value, the difference between the observed values, O_i, and expected values, E_i, in which one variable is associated with fewer than half of the other variable, is

$$\chi^2 = \sum_{i=1}^{n} \frac{(|O_i - E_i| - 0.5)^2}{E_i}$$

Find the association for $|O_i - E_i| - 0.5 < 0$.

40. Measurement Error We seldom can get precise measurements. The greatest possible error of a measurement is one-half of the measuring unit. If you use a ruler to measure the length of a product to be 2.6 cm, then the greatest possible error is one-half of one tenth, or 0.05 cm. Express the error of measurements for a ruler using the absolute-value symbol.

Objective

To write sums in summation notation and evaluate such sums.

1.5 Summation Notation

There was a time when school teachers made their students add up all the positive integers from 1 to 105 (say), perhaps as punishment for unruly behavior while the teacher was out of the classroom. In other words, the students were to find

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + \cdots + 104 + 105 \tag{1}$$

A related exercise was to find

$$1 + 4 + 9 + 16 + \cdots + 81 + 100 + 121 \tag{2}$$

The three dots notation is supposed to convey the idea of continuing the task, using the same pattern, until the last explicitly given terms have been added, too. With this notation there are no hard and fast rules about how many terms at the beginning and end are to be given explicitly. The custom is to provide as many as are needed to ensure that the intended reader will find the expression unambiguous. This is too imprecise for many mathematical applications.

Suppose that for any positive integer i we define $a_i = i^2$. Then, for example, $a_6 = 36$ and $a_8 = 64$. The instruction, "Add together the numbers a_i, for i taking on the integer values 1 through 11 inclusive" is a precise statement of Equation (2). It would be precise regardless of the formula defining the values a_i, and this leads to the following:

Definition

If, for each positive integer i there is given a unique number a_i, and m and n are positive integers with $m \leq n$, then **the sum of the numbers a_i, with i successively taking on all the integer values in the interval $[m, n]$, is denoted**

$$\sum_{i=m}^{n} a_i$$

Thus,

The $\sum$-notation on the left side of (3) eliminates the imprecise dots on the right.

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_n \tag{3}$$

The $\sum$ is the Greek capital letter read "sigma", from which we get the letter S. It stands for "sum" and the expression $\sum_{i=m}^{n} a_i$, can be read as the the sum of all numbers a_i, i ranging from m to n (through positive integers being understood). The description of a_i may be very simple. For example, in Equation (1) we have $a_i = i$ and

$$\sum_{i=1}^{105} i = 1 + 2 + 3 + \cdots + 105 \tag{4}$$

while Equation (2) is

$$\sum_{i=1}^{11} i^2 = 1 + 4 + 9 + \cdots + 121 \tag{5}$$

We have merely defined a notation, which is called **summation notation**. In Equation (3), i is the **index of summation** and m and n are called the **bounds of summation**. It is important to understand from the outset that the name of the index of summation can be replaced by any other so that we have

$$\sum_{i=m}^{n} a_i = \sum_{j=m}^{n} a_j = \sum_{\alpha=m}^{n} a_\alpha = \sum_{N=m}^{n} a_N$$

for example. In each case, replacing the index of summation by the positive integers m through n successively, and adding gives

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

We now illustrate with some concrete examples.

EXAMPLE 1 Evaluating Sums

Evaluate the given sums.

a. $\sum_{n=3}^{7}(5n-2)$

Solution:

$$\begin{aligned}\sum_{n=3}^{7}(5n-2) &= [5(3)-2]+[5(4)-2]+[5(5)-2]+[5(6)-2]+[5(7)-2]\\ &= 13+18+23+28+33\\ &= 115\end{aligned}$$

b. $\sum_{j=1}^{6}(j^2+1)$

Solution:

$$\begin{aligned}\sum_{j=1}^{6}(j^2+1) &= (1^2+1)+(2^2+1)+(3^2+1)+(4^2+1)+(5^2+1)+(6^2+1)\\ &= 2+5+10+17+26+37\\ &= 97\end{aligned}$$

Now Work Problem 5 ◁

EXAMPLE 2 Writing a Sum Using Summation Notation

Write the sum $14 + 16 + 18 + 20 + 22 + \cdots + 100$ in summation notation.

Solution: There are many ways to express this sum in summation notation. One method is to notice that the values being added are $2n$, for $n = 7$ to 50. The sum can thus be written as

$$\sum_{n=7}^{50} 2n$$

Another method is to notice that the values being added are $2k + 12$, for $k = 1$ to 44. The sum can thus also be written as

$$\sum_{k=1}^{44}(2k+12)$$

Now Work Problem 9 ◁

Since summation notation is used to express the addition of terms, we can use the properties of addition when performing operations on sums written in summation notation. By applying these properties, we can create a list of properties and formulas for summation notation.

By the distributive property of addition,

$$ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n)$$

So, in summation notation,

$$\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i \qquad \textbf{(6)}$$

Note that c must be constant with respect to i for Equation (6) to be used.

By the commutative property of addition,

$$a_1 + b_1 + a_2 + b_2 + \cdots + a_n + b_n = a_1 + a_2 + \cdots + a_n + b_1 + b_2 + \cdots + b_n$$

So we have

$$\sum_{i=m}^{n}(a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i \qquad \textbf{(7)}$$

Sometimes we want to change the bounds of summation.

$$\sum_{i=m}^{n} a_i = \sum_{i=p}^{p+n-m} a_{i+m-p} \qquad \textbf{(8)}$$

A sum of 37 terms can be regarded as the sum of the first 17 terms plus the sum of the next 20 terms. The next rule generalizes this observation.

$$\sum_{i=m}^{p-1} a_i + \sum_{i=p}^{n} a_i = \sum_{i=m}^{n} a_i \qquad \textbf{(9)}$$

In addition to these four basic rules, there are some other rules worth noting.

$$\sum_{i=1}^{n} 1 = n \qquad \textbf{(10)}$$

This is because $\sum_{i=1}^{n} 1$ is a sum of n terms, each of which is 1. The next follows from combining Equation (6) and Equation (10).

$$\sum_{i=1}^{n} c = cn \qquad \textbf{(11)}$$

Similarly, from Equations (6) and (7) we have

$$\sum_{i=m}^{n}(a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i \qquad \textbf{(12)}$$

Formulas (14) and (15) are best established by a proof method called mathematical induction, which we will not demonstrate here.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \textbf{(13)}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \textbf{(14)}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \qquad \textbf{(15)}$$

However, there is an easy derivation of Formula (13). If we add the following equations, "vertically," term by term,

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$$

$$\sum_{i=1}^{n} i = n + (n-1) + (n-2) + \cdots + 1$$

we get

$$2\sum_{i=1}^{n} i = (n+1) + (n+1) + (n+1) + \cdots + (n+1)$$

and since there are n terms on the right, we have

$$2\sum_{i=1}^{n} i = n(n+1)$$

and, finally

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Observe that if a teacher assigns the task of finding

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + \cdots + 104 + 105$$

as a *punishment* and if he or she knows the formula given by Formula (13), then a student's work can be checked quickly by

$$\sum_{i=1}^{105} i = \frac{105(106)}{2} = 105 \cdot 53 = 5300 + 265 = 5565$$

EXAMPLE 3 Applying the Properties of Summation Notation

Evaluate the given sums.

a. $\displaystyle\sum_{j=30}^{100} 4$ **b.** $\displaystyle\sum_{k=1}^{100} (5k+3)$ **c.** $\displaystyle\sum_{k=1}^{200} 9k^2$

Solution:

a.

$$\sum_{j=30}^{100} 4 = \sum_{j=1}^{71} 4 \quad \text{by Equation (8)}$$

$$= 4 \cdot 71 \quad \text{by Equation (11)}$$

$$= 284$$

b.

$$\sum_{k=1}^{100} (5k+3) = \sum_{k=1}^{100} 5k + \sum_{k=1}^{100} 3 \quad \text{by Equation (7)}$$

$$= 5\left(\sum_{k=1}^{100} k\right) + 3\left(\sum_{k=1}^{100} 1\right) \quad \text{by Equation (6)}$$

$$= 5\left(\frac{100 \cdot 101}{2}\right) + 3(100) \quad \text{by Equations (13) and (10)}$$

$$= 25{,}250 + 300$$

$$= 25{,}550$$