London Mathematical Society
Lecture Note Series 381

Symmetries and Integrability of Difference Equations

Edited by
Decio Levi, Peter Olver, Zora Thomova
and Pavel Winternitz

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London Mathematical Society Lecture Note series: 381

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town,
Singapore, São Paulo, Delhi, Tokyo, Mexico City

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521136587

First published 2011

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data
Symmetries and integrability of difference equations / edited by Decio Levi . . . [et al.].
p. cm. – (London Mathematical Society lecture note series ; 381)
Includes bibliographical references.
ISBN 978-0-521-13658-7 (pbk.)
1. Difference equations. 2. Symmetry (Mathematics) 3. Integrals. I. Levi, D. (Decio)
QA431.S952 2011
515′.625 – dc22 2011006852

ISBN 978-0-521-13658-7 Paperback

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Preface

This book is based upon lectures delivered during the Summer School on Symmetries and Integrability of Difference Equations at the Université de Montréal, Canada, June 8, 2008–June 21, 2008. The lectures are devoted to methods that have been developed over the last 15–20 years for discrete equations. They are based on either the inverse spectral approach or on the application of geometric and group theoretical techniques. The topics covered in this volume can be summarized in the following categories:

- Integrability of difference equations
- Discrete differential geometry
- Special functions and their relation to continuous and discrete Painlevé functions
- Discretization of complex analysis
- General aspects of Lie group theory relevant for the study of difference equations. Specifically, two such subjects are treated: 1. Cartan's method of moving frames 2. Lattices in Euclidean space, symmetrical under the action of semisimple Lie groups
- Lie point symmetries and generalized symmetries of discrete equations

Twelve distinct lecture series were presented at the Summer School of which eleven are included in this volume. Close to 50 registered graduate students and researchers from twelve different countries participated.

The Summer School, Séminaire de mathématiques supérieures, is a yearly event at the Département de Mathématiques, Université de Montréal. The organizing committee for the year 2008 consisted of Pavel Winternitz (Université de Montréal, Canada), Vladimir Dorodnitsyn (Keldysh Institute of Applied Mathematics, Russian Academy of Sciences), Decio Levi (Universitá degli Studi Roma Tre, Italy) and Peter Olver (University of Minnesota, USA). The two scientific directors were Pavel Winternitz and Vladimir Dorodnitsyn. The

financial support for the Summer School was provided by Université de Montréal, the National Foundation Grant No. 0737765, the Centre de Recherches Mathématiques, and the Institut de Sciences Mathématiques.

The editors would like to thank André Montpetit (CRM) for technical help in preparing this volume. We also acknowledge the support of David Tranah, Suzanne Parker and Clare Dennison from Cambridge University Press.

Introduction

The concept of integrability of Hamiltonian systems goes back at least to the 19th century. The idea of integrability in classical mechanics was formalised by J. Liouville. A finite-dimensional Hamiltonian system with n degrees of freedom is called "Liouville integrable" or "completely integrable" if it allows n functionally independent integrals of motion that are well defined functions on phase space and are in involution. In classical mechanics the equations of motion for a Liouville integrable system can be, at least in principle, reduced to quadratures. A completely integrable system in quantum mechanics is defined similarly. It should allow n commuting integrals of motion (including the Hamiltonian) that are well defined operators in the enveloping algebra of the Heisenberg algebra, or some generalization of this enveloping algebra. In quantum mechanics complete integrability does not guarantee that the spectral problem for the Schrödinger operator can be solved explicitly, or even that the energy levels can be calculated algebraically.

An n-dimensional integrable Hamiltonian system that admits more than n integrals of motion is called "superintegrable". Systems with $2n-1$ integrals, with at least one subset of n of them in involution, are "maximally superintegrable". Such systems, namely the Kepler-Coulomb system and the harmonic oscillator, played a pivotal role in the development of physics and mathematics. Trajectories in classical maximally superintegrable systems can at least in principle be calculated algebraically (without using any calculus). Ironically, calculus was invented in order to calculate orbits in the Kepler system. In quantum mechanics these systems are exactly solvable: their energy spectra can be calculated algebraically, their wave functions expressed in terms of polynomials (in appropriate variables) multiplied by a known function. The $2n-1$ integrals of motion generate (under Lie or Poisson commutation) a finite-dimensional nonabelian algebra that is usually an associative algebra rather than a Lie algebra. Superintegrability has also been called "nonabelian integrability".

The theory of nonlinear infinite dimensional integrable systems went through a rapid development since the creation of “soliton theory” in the famous 1967 paper by C.S. Gardner, J.M. Green, M.D. Kruskal, and R.M. Miura. This paper introduced the inverse scattering transform as a method of solving certain nonlinear partial differential equations by essentially linear techniques. It rapidly became clear that there exists an infinite number of nonlinear partial differential equations that can be solved in this manner. These equations are usually called “integrable” nonlinear partial differential equations. They can actually be considered to be infinite-dimensional analogues of superintegrable finite dimensional systems. Indeed, they not only allow infinitely many integrals of motion but these integrals form nonabelian algebras. The usual infinite families of commuting flows actually form infinite dimensional abelian subalgebras of these larger non-abelian ones. Moreover, the corresponding soliton equations are exactly solvable in the sense that the inverse scattering transform provides exact solutions for large classes of initial data.

The use of group theory to solve ordinary and partial differential equations also has a long history going back to S. Lie. Lie group methods are applicable to a much wider class of equations than methods based on integrability. Whether a partial differential equation is integrable or not, Lie theory allows one to reduce the number of independent variables and to obtain special exact analytical solutions. For an ordinary differential equation admitting a symmetry group, Lie group methods enable one to decrease the order of the equation and, under appropriate solvability conditions, obtain the general solution. When both integrability and symmetry methods are applicable, they interact and complement each other fruitfully. In particular, group theory provides criteria of integrability.

A vigorous application of the ideas of integrability and of symmetry to discrete equations started much later, around 1990. Pioneering work on the integrability of difference and differential-difference equations was done 20 years earlier by M.J. Ablowitz and J.F. Ladik, R. Hirota, and others. The first applications of Lie group theory to difference equations are due to S. Maeda already in 1980.

This volume is devoted to recent developments in the theory of integrability and symmetries of discrete equations of all types: difference equations, q-difference equations, differential-difference equations, ultradiscrete equations and others. The contributions are ordered alphabetically by authors although by content they could be subdivided into several overlapping themes.

The first chapter, by V. Dorodnitsyn and R. Kozlov, is devoted to a specific aspect of the application of continuous Lie point symmetries to difference systems involving one discrete independent variable. An ordinary difference scheme consists of two difference equations, determining both the lattice and the actual difference equation. The authors develop discrete Lagrangian and Hamiltonian formalisms. They then use them to investigate the relation between continuous symmetries and conservation laws and first integrals for discrete Hamiltonian and Euler-Lagrange equations.

Chapter 2, by B. Grammaticos and A. Ramani, shifts the focus to the field of integrability and constitutes a comprehensive review of the Painlevé equations and the properties of their solutions. The authors give parallel derivations of continuous and discrete Painlevé equations and emphasize their shared integrability properties. The discrete equations considered are difference equations, q-difference equations and ultra-discrete ones. Two descriptions are presented. A "top-down" approach, starting from a Hamiltonian formulation and an isomonodromy deformation problem. The complementary "bottom-up" approach consists of applying certain integrability criteria to chosen classes of equations. For discrete systems the criteria selected in this chapter are singularity confinement and algebraic entropy.

By alphabetic coincidence, Chapter 3, written by J. Hietarinta, is closely related to Chapter 2 and presents different definitions of integrability and integrability criteria for difference equations. The author considers both ordinary and partial difference equations (with two discrete independent variables) and provides algorithmic tools for deciding whether a discrete equation is integrable, partially integrable, or chaotic. A section is devoted to conserved quantities, i.e. constants of motion. Singularity confinement and algebraic entropy are presented as algorithmic tools. When applying the algebraic entropy criterion, linear, polynomial, and exponential growth of complexity are associated with linearizable, integrable and chaotic equations, respectively. Finally, the author shows how the "consistency-around-a cube" criterion discussed by Yu.B. Suris in Chapter 10 can be applied to equations on square lattices to obtain Lax pairs and multisoliton solutions.

Chapters 4 and 5 of this book are related in that they both deal with orthogonal polynomials and their relation to discrete and continuous Painlevé functions. In both cases the orthogonal polynomials satisfy three term linear recurrence relations, i.e. second order linear difference equations. The coefficients in the recursion relations satisfy discrete or continuous Painlevé equations.

Chapter 4, written by M.E.H. Ismail, considers in particular the case when the polynomials are orthogonal with respect to an exponential measure and the recursion coefficients satisfy the discrete Painlevé I equation. The emphasis is on the spectral theory of orthogonal polynomials and on applications.

In Chapter 5, written by A. Its, the emphasis is on the connection between orthogonal polynomials, integrable systems and random matrices. The Riemann-Hilbert formalism for orthogonal polynomials is explained and used to introduce discrete Painlevé equations systematically. This setting is also used to perform a global asymptotic analysis of the solutions of discrete Painlevé equations.

Chapter 6, by D. Levi and R. Yamilov, is one of the three chapters in the book specifically devoted to Lie symmetries of difference equations. More specifically it deals with generalized symmetries. The authors consider generalized symmetries of partial difference equations with two independent variables on fixed non-transforming lattices. They make use of the formalism of evolutionary vector fields, acting on the dependent variables only. A method of constructing generalized symmetries for integrable multivariable difference equations or differential-difference equations is presented. It makes use of integrability properties of the equations, in particular recursion operators. A subclass of generalized symmetries is identified that in the continuous limit "contracts" to point symmetries. A section in Chapter 6 is devoted to how formal symmetries provide an integrability criterion for equations on lattices.

In Chapter 7, S.P. Novikov reviews an ambitious program that amounts to a discretization of complex analysis. After emphasizing the role of linear operators and their factorization properties for continuous nonlinear integrable systems, the author proceeds to their discretization. This is done on square lattices for hyperbolic equations and on equilateral triangular lattices for elliptic ones. The concept of triangle equations is introduced as well as that of GL_n connections. The discretization of complex analysis on square lattices was introduced by Ferrand in 1944 and has been used for discrete integrable systems by many authors. The approach of S.P. Novikov and his collaborators is instead based on the properties of an equilateral triangle lattice. The discretization is carried out both on flat and hyperbolic planes.

Chapters 8 and 9 are devoted to some rather general aspects of Lie group theory that are relevant, in particular, to the study of difference equations.

In Chapter 8, P.J. Olver gives an exposition of the method of moving frames. The modern development of this method goes back to Élie Cartan. The chapter starts with a definition of a moving frame as an equivariant map from a manifold M to a transformation group G. This definition turns the method into an algorithm applicable to very general group actions. The treatment is restricted to finite dimensional Lie groups (though the author refers to his work on moving frames for pseudogroups as well). Many aspects and applications of the method, and obtained differential invariants, joint invariants and joint differential invariants are discussed. The most relevant application from the point of view of this volume is Section 8.6 on invariant numerical approximations, more specifically on symmetry preserving approximations. An example is given in which Runge-Kutta schemes are compared. This use of moving frames provides an example of geometric numerical integration techniques for ordinary and partial differential equations.

Chapter 9, by J. Patera, provides an algorithm for constructing n-dimensional lattices in real Euclidean space E_n that are symmetrical with respect to the action of a compact semi-simple Lie group of rank n. A symmetric lattice is first constructed in a finite region of the space. The symmetry of the lattice is given by the symmetry of the weight lattice of the chosen Lie group and the density of points can be chosen a priori. The action of the affine Weyl group then extends the lattice to an infinite one on the entire space E_n. The motivation provided is the construction of functions that are orthogonal on the lattices. These in turn are needed in the treatment of digital data on lattices. The construction of symmetric lattices can also be related to the construction of the discrete integrable systems on other lattices than simple rectangular ones.

Chapter 10, by Yu.B. Suris, is on discrete differential geometry, a new subject emerging on the border between differential and discrete geometry. Discrete differential geometry is not only related to the topic of integrability of difference equations, but it actually provides new insights into the concept of integrability, both for discrete and continuous equations. It also leads to new integrability criteria. The author introduces basic notions like that of discrete nets, Q-nets and circular nets. The concept of integrability for discrete systems is introduced in terms of a multidimensional consistency principle. Namely, the discretization of surfaces, coordinate systems and all related concepts should be extendable to multidimensional consistent nets. The usual fundamental attributes of integrable systems like the existence of Lax pairs, Bäcklund transformations, permutability theorems, infinite families of commuting flows here appear as consequences of multidimensional consistency requirements.

The last chapter, Chapter 11, by P. Winternitz, concentrates on point symmetries of difference and differential-difference equations. It is thus related to Chapter 1 by V. Dorodnitsyn and R. Kozlov, Chapter 6 by D. Levi and R. Yamilov and partially to Chapter 8 by P.J. Olver. Sections 11.1-4 are on the symmetry preserving discretization of ordinary differential equations on symmetry adapted lattices. In particular, Section 11.3 discusses examples of geometric integration methods. It is shown that symmetry adapted numerical methods (so far for ordinary differential equations) provide qualitatively superior solutions, specially in the neighbourhood of singularities. Sections 11.5 and 11.6 are devoted to Lie point symmetries of differential-difference equations. The discrete independent variables in these sections are defined on uniform non-transforming lattices.

In summary, the papers in this volume provide a comprehensive overview of the current state of the art in integrability and symmetry for discrete equations. Our hope is that it will inspire the reader to further develop these fascinating and important theories and their applications.

The Editors

1

Lagrangian and Hamiltonian Formalism for Discrete Equations: Symmetries and First Integrals

Vladimir Dorodnitsyn and Roman Kozlov

Abstract

In this chapter the relation between symmetries and first integrals of discrete Euler–Lagrange and discrete Hamiltonian equations is considered. These results are built on those for continuous Euler–Lagrange and canonical Hamiltonian equations. First, the well-known Noether theorem which provides conservation laws for continuous Euler–Lagrange equations is reviewed. Then, its discrete analog is presented. Further, it is mentioned that continuous and discrete Hamiltonian equations can be obtained by the variational principle from action functionals. This is used to develop Noether-type theorems for canonical Hamiltonian equations and their discrete counterparts (discrete Hamiltonian equations). The approach based on symmetries of the discrete action functionals provides a simple and clear way to construct first integrals of discrete Euler–Lagrange and discrete Hamiltonian equations by means of differentiation of discrete Lagrangian (or Hamiltonian) and algebraic manipulations. It can be used to conserve structural properties of underlying differential equations under a discretization procedure that is useful for numerical implementation. The results are illustrated by a number of examples.

1.1 Introduction

It has been known since E. Noether's fundamental work that conservation laws of differential equations are connected with their symmetry properties [28]. For convenience we present here some well-known results (see also, for example, [1, 3, 18]) for the Lagrangian approach to conservation laws (first integrals). We restrict ourselves to the case with one independent variable.

Let us consider the functional

$$\mathbb{L}(\mathbf{u}) = \int_{t_1}^{t_2} L(t, \mathbf{u}, \dot{\mathbf{u}})\, \mathrm{d}t, \tag{1.1}$$

where t is the independent variable, $\mathbf{u} = (u^1, u^2, \ldots, u^n)$ are dependent variables, $\dot{\mathbf{u}} = (\dot{u}^1, \dot{u}^2, \ldots, \dot{u}^n)$ are first-order derivatives and $L(t, \mathbf{u}, \dot{\mathbf{u}})$ is a *r st-order* Lagrangian. The functional (1.1) achieves its extremal values when $\mathbf{u}(t)$ satisfies the Euler–Lagrange equations

$$\frac{\delta L}{\delta u^i} = \frac{\partial L}{\partial u^i} - D\left(\frac{\partial L}{\partial \dot{u}^i}\right) = 0, \qquad i = 1, \ldots, n, \tag{1.2}$$

where

$$D = \frac{\partial}{\partial y} + \dot{u}^k \frac{\partial}{\partial u^k} + \ddot{u}^k \frac{\partial}{\partial \dot{u}^k} + \cdots$$

is the total differentiation operator. Here and below we assume summation over repeated indexes. Note that (1.2) are second-order ODEs.

We consider a Lie point transformation group G generated by the infinitesimal operator

$$X = \xi(t, \mathbf{u})\frac{\partial}{\partial t} + \eta^i(t, \mathbf{u})\frac{\partial}{\partial u^i} + \cdots, \tag{1.3}$$

where dots mean an appropriate prolongation of the operator to derivatives [5, 21, 29, 30]. The group G is called a variational symmetry of the functional $\mathbb{L}(\mathbf{u})$ if and only if the Lagrangian satisfies [28]

$$X(L) + LD(\xi) = 0, \tag{1.4}$$

where X is the first prolongation, i.e., the prolongation of the vector field X to the first derivatives $\dot{\mathbf{u}}$. We will actually need a weaker invariance condition than given by (1.4). The vector field X is a divergence symmetry of the functional $\mathbb{L}(\mathbf{u})$ if there exists a function $V(t, \mathbf{u}, \dot{\mathbf{u}})$ such that [4] (see also [5, 21, 29])

$$X(L) + LD(\xi) = D(V). \tag{1.5}$$

Generally, (1.5) should hold on the solutions of the Euler–Lagrange equations (1.2).

Noether's theorem [28] can be based on the following Noether-type identity [21], which holds for any vector field and any smooth function $L(t, \mathbf{u}, \dot{\mathbf{u}})$:

$$X(L) + LD(\xi) \equiv (\eta^i - \xi\dot{u}^i)\frac{\delta L}{\delta u^i} + D\left(\xi L + (\eta^i - \xi\dot{u}^i)\frac{\partial L}{\partial \dot{u}^i}\right). \tag{1.6}$$

The theorem states that for a Lagrangian satisfying the condition (1.4) there exists a first integral of the Euler–Lagrange equations (1.2):

$$I = \xi L + (\eta^i - \xi \dot{u}^i)\frac{\partial L}{\partial \dot{u}^i}. \tag{1.7}$$

This result can be generalized [4]: If X is a divergence symmetry of the functional $\mathbb{L}(\mathbf{u})$, i.e., (1.5) is satisfied, then there exists a conservation law

$$I = \xi L + (\eta^i - \xi \dot{u}^i)\frac{\partial L}{\partial \dot{u}^i} - V \tag{1.8}$$

of the corresponding Euler–Lagrange equations.

The strong version of Noether's theorem [21] states that there exists a conservation law of the Euler–Lagrange equations (1.2) in the form (1.7) if and only if the condition (1.4) is satisfied on the solutions of (1.2).

The goal of this chapter is to extend the results presented above to discrete equations in the Lagrangian and Hamiltonian frameworks. We will need to consider canonical Hamiltonian equations before we start to treat their discrete counterparts. It is known that the preservation of first integrals (conservation laws) in numerics is of great importance (see, for example, [19, 31]). Therefore, there is a strong motivation to establish discrete analogs of the conservation properties of the continuous Euler–Lagrange and Hamiltonian equations.

In the next section we will comment on invariance of the Euler–Lagrange equations. In Section 1.3 we will present the Lagrangian formalism for second-order difference equations, which are a discrete analog of the second-order ordinary differential equations. Canonical Hamiltonian equations are considered in Section 1.4. We will develop an analog of Noether's theorem which is based on invariance properties of the action functional, generating canonical Hamiltonian equations. The discrete Hamiltonian equations and their conservation properties are treated in Section 1.5. Section 1.6 presents applications of the theoretical results to a number of examples. Finally Section 1.7 contains concluding remarks.

1.2 Invariance of Euler–Lagrange equations

There exists a relation between the invariance of the Lagrangian function and invariance of the corresponding Euler–Lagrange equations:

Theorem 1.1 ([21, 29]) *If the Lagrangian L is invariant with respect to operator* (1.3)*, i.e., condition* (1.4) *is satis ed, then the Euler–Lagrange equations* (1.2) *are also invariant.*

Remark 1.2 If the Lagrangian L is divergence invariant, i.e., satisfies the condition (1.5), then the Euler–Lagrange equations (1.2) are also invariant. This follows from the fact that full divergences belong to the kernel of variational operators.

Thus, if X is a variational or divergence symmetry of the functional $\mathbb{L}(\mathbf{u})$, it is also a symmetry of the corresponding Euler–Lagrange equations (1.2). The symmetry group of the Euler–Lagrange equations can of course be larger than the group generated by variational and divergence symmetries of the Lagrangian.

It is interesting to establish the necessary and sufficient condition for invariance of the Euler–Lagrange equations. We will need the following lemma:

Lemma 1.3 *For any smooth function* $L(t, \mathbf{u}, \dot{\mathbf{u}})$ *the following identity holds*

$$\frac{\delta}{\delta u^j}(X(L) + LD(\xi)) \equiv X\left(\frac{\delta L}{\delta u^j}\right) + \left(\frac{\partial \eta^i}{\partial u^j} + \delta_{ij} D(\xi) - \frac{\partial \xi}{\partial u^j}\dot{u}^i\right)\frac{\delta L}{\delta u^i}, \qquad j = 1, \ldots, n, \tag{1.9}$$

where the notation δ_{ij} *stands for the Kronecker symbol.*

Proof The result can be established by a direct computation. □

Theorem 1.1 and Remark 1.2 follow from this lemma. The lemma also provides the *necessary and sufficient* condition for the invariance of the Euler–Lagrange equations:

Theorem 1.4 *The Euler–Lagrange equations* (1.2) *are invariant with respect to a symmetry* (1.3) *if and only if the following conditions are true (on the solutions of the equations):*

$$\left.\frac{\delta}{\delta u^j}(X(L) + LD(\xi))\right|_{\delta L/\delta u^1 = \cdots = \delta L/\delta u^n = 0} = 0, \qquad j = 1, \ldots, n. \tag{1.10}$$

Proof The statement follows from the identities of Lemma 1.3. □

Example 1.5 Equation

$$\ddot{u} = \frac{1}{u^2} \tag{1.11}$$

is the Euler–Lagrange equation for the Lagrangian function

$$L(t, u, \dot{u}) = \frac{\dot{u}^2}{2} - \frac{1}{u}.$$

The equation admits symmetries

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = 3t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u}.$$

The operator X_1 is a symmetry of Lagrangian L and, consequently, a symmetry of (1.11). The symmetry X_2 is not a symmetry of the Lagrangian:

$$X_2(L) + LD(\xi_2) = L.$$

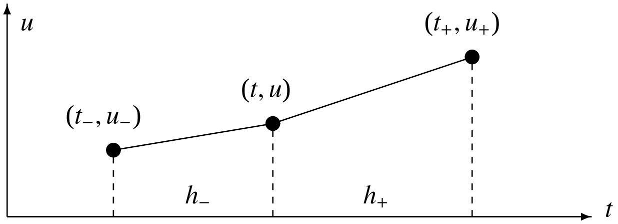
However, it is a symmetry of the equation as follows from Theorem 1.4:

$$\frac{\delta}{\delta u}(X_2(L) + LD(\xi_2))\bigg|_{\delta L/\delta u=0} = \frac{\delta L}{\delta u}\bigg|_{\delta L/\delta u=0} = 0.$$

In the next section we will develop discrete analogs of these results.

1.3 Lagrangian formalism for second-order difference equations

Let us present the results concerning the variational formulation of discrete Euler–Lagrange equations [9–11, 13, 14]. The notations are clear from the following picture:



We consider a finite-difference functional

$$\mathbb{L}_h = \sum_{\Omega} \mathcal{L}(t, t_+, \mathbf{u}, \mathbf{u}_+)h_+, \tag{1.12}$$

defined on some one-dimensional lattice Ω with steplength $h_+ = t_+ - t$. Generally, the lattice can depend on the solution, for example, as

$$\Omega(t, t_-, t_+, \mathbf{u}, \mathbf{u}_-, \mathbf{u}_+) = 0. \tag{1.13}$$

Functional (1.12) must be considered together with lattice (1.13). On different lattices it can have different continuous limits.

Let us take a variation of the difference functional (1.12) along some curve $u^i = \phi_i(t)$, $i = 1, \ldots, n$ at some point $(t, \mathbf{u})$. The variation will affect only two terms in the sum (1.12):

$$\mathbb{L}_h = \cdots + \mathcal{L}(t_-, t, \mathbf{u}_-, \mathbf{u})h_- + \mathcal{L}(t, t_+, \mathbf{u}, \mathbf{u}_+)h_+ + \cdots. \tag{1.14}$$

Thus, we get the following expression for the variation of the difference functional

$$\delta \mathbb{L}_h = \frac{\delta \mathcal{L}}{\delta u^i} \delta u^i + \frac{\delta \mathcal{L}}{\delta t} \delta t, \tag{1.15}$$

where $\delta u^i = \phi_i' \delta t$, $i = 1, \ldots, n$ and

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta u^i} &= h_+ \frac{\partial \mathcal{L}}{\partial u^i} + h_- \frac{\partial \mathcal{L}^-}{\partial u^i}, \qquad i = 1, \ldots, n, \\ \frac{\delta \mathcal{L}}{\delta t} &= h_+ \frac{\partial \mathcal{L}}{\partial t} + h_- \frac{\partial \mathcal{L}^-}{\partial t} + \mathcal{L}^- - \mathcal{L}, \end{aligned} \tag{1.16}$$

with $\mathcal{L} = \mathcal{L}(t, t_+, \mathbf{u}, \mathbf{u}_+)$ and $\mathcal{L}^- = \underset{-h}{S}(\mathcal{L}) = \mathcal{L}(t_-, t, \mathbf{u}_-, \mathbf{u})$. For convenience we will use the following total left and right shift operators

$$\underset{-h}{S} f(t, \mathbf{u}) = f(t_-, \mathbf{u}_-), \qquad \underset{+h}{S} f(t, \mathbf{u}) = f(t_+, \mathbf{u}_+)$$

and left and right total difference operators

$$\underset{+h}{D} = \frac{\underset{+h}{S} - 1}{h_+}, \qquad \underset{-h}{D} = \frac{1 - \underset{-h}{S}}{h_-}.$$

Thus, for an arbitrary curve the stationary value of the difference functional is given by a solution of the $n + 1$ equations

$$\frac{\delta \mathcal{L}}{\delta u^i} = 0, \quad i = 1, \ldots, n, \qquad \frac{\delta \mathcal{L}}{\delta t} = 0, \tag{1.17}$$

called *global extremal* equations. These equations represent the entire difference scheme and could be called "the discrete Euler–Lagrange system." They can be interpreted as a three-point difference scheme of the form

$$\begin{aligned} &F_i(t, t_-, t_+, \mathbf{u}, \mathbf{u}_-, \mathbf{u}_+) = 0, \qquad i = 1, \ldots, n, \\ &\Omega(t, t_-, t_+, \mathbf{u}, \mathbf{u}_-, \mathbf{u}_+) = 0. \end{aligned}$$

Here the first n equations are approximations of differential equations (1.2) and the last equation provides a lattice, on which these approximations are considered. In the continuous limit the lattice equation vanishes (turns into an identity like $0 = 0$). Given two points, for instance $(t, \mathbf{u})$ and $(t_-, \mathbf{u}_-)$, we can calculate $(t_+, \mathbf{u}_+)$.

Note that the variational equations (1.17) can be obtained by the action of discrete variational operators

$$\frac{\delta}{\delta u^i} = \frac{\partial}{\partial u^i} + \underset{-h}{S} \frac{\partial}{\partial u^i_+}, \qquad i = 1, \ldots, n, \tag{1.18}$$

$$\frac{\delta}{\delta t} = \frac{\partial}{\partial t} + \underset{-h}{S} \frac{\partial}{\partial t_+} \tag{1.19}$$

on the discrete elementary action $\mathcal{L}(t, t_+, \mathbf{u}, \mathbf{u}_+) h_+$.

Now let us consider a variation of the functional (1.12) along the orbit of a group generated by the operator (1.3). Then, we have $\delta t = \xi\delta a$, $\delta u^i = \eta^i\delta a$, $i = 1, \ldots, n$, where δa is the variation of the group parameter. A stationary value of the difference functional (1.12) along the flow generated by this vector field is given by the equation

$$\eta^i \frac{\delta \mathcal{L}}{\delta u^i} + \xi \frac{\delta \mathcal{L}}{\delta t} = 0, \tag{1.20}$$

which depends explicitly on the coefficients of the generator. This equation is called a *quasiextremal* equation. If we have a Lie algebra of vector fields of dimension $n + 1$ or more, then the stationary value of difference functional (1.12) along the entire flow will be achieved on the intersection of the solutions of all quasiextremal equations of the type (1.20), i.e., the system of equations (1.17).

Remark 1.6 Sometimes it is convenient to consider the variational equations (1.17) in a modified form

$$\begin{aligned} &\frac{\partial \mathcal{L}}{\partial u^i} + \frac{h_-}{h_+}\frac{\partial \mathcal{L}^-}{\partial u^i} = 0, \qquad i = 1, \ldots, n, \\ &\frac{\partial \mathcal{L}}{\partial t} + \frac{h_-}{h_+}\frac{\partial \mathcal{L}^-}{\partial t} - \underset{+h}{D}(\mathcal{L}^-) = 0, \end{aligned} \tag{1.21}$$

obtained on dividing by h_+.

Let us consider a Lie group of point transformations, generated by a vector field (1.3). When acting on discrete equations and functionals, a vector field must be prolonged to variables at other points of the lattice. The prolongation is obtained by shifting the coefficients to the corresponding points. For three-point schemes we have

$$\begin{aligned} X = \xi\frac{\partial}{\partial t} + \xi_-\frac{\partial}{\partial t_-} + \xi_+\frac{\partial}{\partial t_+} + \eta^i\frac{\partial}{\partial u^i} + \eta^i_-\frac{\partial}{\partial u^i_-} + \eta^i_+\frac{\partial}{\partial u^i_+} \\ + (\xi_+ - \xi)\frac{\partial}{\partial h_+} + (\xi - \xi_-)\frac{\partial}{\partial h_-}, \end{aligned} \tag{1.22}$$

where coefficients are given as follows

$$\xi_- = \xi(t_-, \mathbf{u}_-), \qquad \eta^i_- = \eta^i(t_-, \mathbf{u}_-), \qquad \xi_+ = \xi(t_+, \mathbf{u}_+), \qquad \eta^i_+ = \eta^i(t_+, \mathbf{u}_+).$$

The infinitesimal invariance condition for the functional (1.12) on the lattice (1.13) is given by two equations [9–11, 14]:

$$X(\mathcal{L}) + \mathcal{L}\underset{+h}{D}(\xi)\Big|_{\Omega=0} = 0, \qquad X(\Omega)\Big|_{\Omega=0} = 0, \tag{1.23}$$

which are valid on the lattice (1.13). Generally, the lattice is provided by the global extremal equations (1.17). Therefore, we need to require their invariance to consider the invariance of the functional.

A useful operator identity, valid for any Lagrangian $\mathcal{L}(t, t_+, \mathbf{u}, \mathbf{u}_+)$ and any vector field X is [9, 11]

$$X(\mathcal{L}) + \mathcal{L}\underset{+h}{D}(\xi) \equiv \xi\left(\frac{\partial \mathcal{L}}{\partial t} + \frac{h_-}{h_+}\frac{\partial \mathcal{L}^-}{\partial t} - \underset{+h}{D}(\mathcal{L}^-)\right)$$
$$+ \eta^i\left(\frac{\partial \mathcal{L}}{\partial u^i} + \frac{h_-}{h_+}\frac{\partial \mathcal{L}^-}{\partial u^i}\right) + \underset{+h}{D}\left(h_-\eta^i\frac{\partial \mathcal{L}^-}{\partial u^i} + h_-\xi\frac{\partial \mathcal{L}^-}{\partial t} + \xi\mathcal{L}^-\right). \quad (1.24)$$

The identity is a discrete analog of Noether identity (1.6) and can be called *the discrete Noether identity*. From this relation we obtain the following discrete analog of Noether's theorem.

Theorem 1.7 ([9, 11, 14]) *The global extremal equations* (1.17)*, invariant under the Lie group G of local point transformations generated by vector elds X of the form* (1.3)*, possess a r st integral*

$$\mathcal{I} = h_-\eta^i\frac{\partial \mathcal{L}^-}{\partial u^i} + h_-\xi\frac{\partial \mathcal{L}^-}{\partial t} + \xi\mathcal{L}^- \quad (1.25)$$

if and only if the Lagrangian density $\mathcal{L}$ is invariant with respect to the same group on the solutions of (1.17)*.*

Remark 1.8 If the Lagrangian density $\mathcal{L}$ is divergence invariant under Lie group G of local point transformations, i.e.,

$$X(\mathcal{L}) + \mathcal{L}\underset{+h}{D}(\xi) = \underset{+h}{D}(V) \quad (1.26)$$

for some function $V(t, \mathbf{u})$, then each element X of the Lie algebra corresponding to group G provides us with a first integral of the global extremal equations (1.17), namely

$$\mathcal{I} = h_-\eta^i\frac{\partial \mathcal{L}^-}{\partial u^i} + h_-\xi\frac{\partial \mathcal{L}^-}{\partial t} + \xi\mathcal{L}^- - V. \quad (1.27)$$

Remark 1.9 In a particular case when the discrete Lagrangian is invariant with respect to time translations, i.e., $\mathcal{L} = \mathcal{L}(h_+, \mathbf{q}, \mathbf{q}_+)$, where $h_+ = t_+ - t$ is the step size, there is a conservation of energy

$$\mathcal{E} = -\mathcal{L}^- - h_-\frac{\partial \mathcal{L}^-}{\partial h_-} = -\mathcal{L} - h_+\frac{\partial \mathcal{L}}{\partial h_+}.$$

In this case we get symplectic-momentum-energy preserving variational integrators [22].

It has been shown elsewhere [9–11], that if the functional (1.12) is invariant or divergence invariant under some group G, then the global extremal equations (1.17) are also invariant with respect to G:

Theorem 1.10 *If the Lagrangian $\mathcal{L}$ is invariant with respect to the operator* (1.3)*, then the global extremal equations* (1.17) *are also invariant.*

Remark 1.11 If the Lagrangian $\mathcal{L}$ is divergence invariant, then the global extremal equations (1.17) are also invariant. This follows from the fact that total finite differences belong to the kernel of discrete variational operators.

As in the continuous case, the global extremal equations can be invariant with respect to a larger group than the corresponding Lagrangian.

Now we are in a position to establish the necessary and sufficient condition for the invariance of global extremal equations. We will obtain new identities and a new theorem.

Lemma 1.12 *The following identities hold for any smooth function $\mathcal{L}(t, t_+, \mathbf{u}, \mathbf{u}_+)$:*

$$\frac{\delta}{\delta u^j}\left(\left(X(\mathcal{L}) + \mathcal{L}\underset{+h}{D}(\xi)\right)h_+\right) \equiv X\left(\frac{\delta \mathcal{L}}{\delta u^j}\right) + \frac{\partial \eta^i}{\partial u^j}\frac{\delta \mathcal{L}}{\delta u^i} + \frac{\partial \xi}{\partial u^j}\frac{\delta \mathcal{L}}{\delta t}, \qquad j = 1, \ldots, n, \tag{1.28}$$

$$\frac{\delta}{\delta t}\left(\left(X(\mathcal{L}) + \mathcal{L}\underset{+h}{D}(\xi)\right)h_+\right) \equiv X\left(\frac{\delta \mathcal{L}}{\delta t}\right) + \frac{\partial \eta^i}{\partial t}\frac{\delta \mathcal{L}}{\delta u^i} + \frac{\partial \xi}{\partial t}\frac{\delta \mathcal{L}}{\delta t}. \tag{1.29}$$

Proof The identities can be verified directly. □

The lemma allows us to obtain not only the sufficient (Theorem 1.10) but also the necessary and sufficient condition for the invariance of the global extremal equations.

Theorem 1.13 *The global extremal equations* (1.17) *are invariant with respect to a symmetry* (1.3) *if and only if the following conditions are true (on the solutions of the equations):*

$$\frac{\delta}{\delta u^j}\left(\left(X(\mathcal{L}) + \mathcal{L}\underset{+h}{D}(\xi)\right)h_+\right)\Bigg|_{(1.17)} = 0, \qquad j = 1, \ldots, n, \tag{1.30}$$

$$\frac{\delta}{\delta t}\left(\left(X(\mathcal{L}) + \mathcal{L}\underset{+h}{D}(\xi)\right)h_+\right)\Bigg|_{(1.17)} = 0. \tag{1.31}$$

Proof The statement follows from identities of Lemma 1.12. □

Many examples of applications of the discrete version of Noether's theorem in Lagrangian framework can be found in [14]. It should be noted that the discrete Lagrangian formalism and the corresponding Noether's theorem are not restricted to ordinary equations. They can also be used for discretizations of partial differential equations [6].

We note that there exists an alternative approach to conservation laws of discrete equations on fixed meshes based on direct methods [20].

1.4 Hamiltonian formalism for differential equations

In this chapter we will also present the Hamiltonian formalism for discrete Hamiltonian equations. Before that we consider the canonical Hamiltonian equations

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \qquad i = 1, \dots, n \tag{1.32}$$

and rewrite results concerning their invariance and conservation properties in a nonstandard way, that provides us with a simple "translation" of the Lagrangian formalism into the Hamiltonian one. We also present a new criterion (Theorem 1.23) for the invariance of the Hamiltonian equations.

1.4.1 Canonical Hamiltonian equations

It is well known that canonical Hamiltonian equations (1.32) can be obtained by the variational principle from the action functional

$$\delta \int_{t_1}^{t_2} (p_i \dot{q}^i - H(t, \mathbf{q}, \mathbf{p}))\, \mathrm{d}t = 0 \tag{1.33}$$

in the phase space $(\mathbf{q}, \mathbf{p})$, where $\mathbf{q} = (q^1, q^2, \dots, q^n)$ and $\mathbf{p} = (p_1, p_2, \dots, p_n)$ [17, 27]. Let us note that the canonical Hamiltonian equations (1.32) can be derived by action of the variational operators

$$\frac{\delta}{\delta p_i} = \frac{\partial}{\partial p_i} - D\frac{\partial}{\partial \dot{p}_i}, \qquad i = 1, \dots, n, \tag{1.34}$$

$$\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - D\frac{\partial}{\partial \dot{q}^i}, \qquad i = 1, \dots, n, \tag{1.35}$$

where D is the operator of total differentiation with respect to time

$$D = \frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k} + \dot{p}_k \frac{\partial}{\partial p_k} + \cdots,$$

on the function

$$p_i \dot{q}^i - H(t, \mathbf{q}, \mathbf{p}).$$

As an analog of Lagrangian elementary action $L\,\mathrm{d}t$ [21, 29] we consider Hamiltonian elementary action [12], namely

$$p_i\, \mathrm{d}q^i - H(t, \mathbf{q}, \mathbf{p})\, \mathrm{d}t, \tag{1.36}$$

and investigate its invariance with respect to a point transformation group generated by an operator

$$X = \xi(t, \mathbf{q}, \mathbf{p})\frac{\partial}{\partial t} + \eta^i(t, \mathbf{q}, \mathbf{p})\frac{\partial}{\partial q^i} + \zeta_i(t, \mathbf{q}, \mathbf{p})\frac{\partial}{\partial p_i}. \tag{1.37}$$

It should be noted that such point symmetry operators might correspond to nonpoint symmetries in the Lagrangian framework.

Definition 1.14 A Hamiltonian function is called invariant with respect to a symmetry operator (1.37) if the elementary action (1.36) is an invariant of the group generated by this operator.

This definition makes it possible to develop the following proposition.

Theorem 1.15 ([12]) *A Hamiltonian is invariant with respect to a group generated by the operator* (1.37) *if and only if the following condition holds*

$$\zeta_i \dot{q}^i + p_i D(\eta^i) - X(H) - HD(\xi) = 0. \tag{1.38}$$

The basic identity, stated in [12], relates conservation properties of the canonical Hamiltonian equations to the invariance of the Hamiltonian function:

$$\begin{aligned}\zeta_i \dot{q}^i + p_i D(\eta^i) - X(H) - HD(\xi) \equiv \xi\left(D(H) - \frac{\partial H}{\partial t}\right)& \\ - \eta^i\left(\dot{p}_i + \frac{\partial H}{\partial q^i}\right) + \zeta_i\left(\dot{q}^i - \frac{\partial H}{\partial p_i}\right) + D[p_i\eta^i - \xi H].&\end{aligned} \tag{1.39}$$

This identity, called *the Hamiltonian identity*, is the well-known Noether identity rewritten for the Hamiltonian function. It allows us to state the following result.

Theorem 1.16 ([12]) *The canonical Hamiltonian equations* (1.32) *possess a r st integral of the form*

$$J = p_i\eta^i - \xi H \tag{1.40}$$

if and only if the Hamiltonian function is invariant with respect to operator (1.37) *on the solutions of the equations.*

Theorem 1.16 corresponds to the strong version of the Noether theorem (i.e., necessary and sufficient condition) for invariant Lagrangians and Euler–Lagrange equations [21].

Remark 1.17 Theorem 1.16 can be generalized to the case of divergence invariance of the Hamiltonian action

$$\zeta_i \dot{q}^i + p_i D(\eta^i) - X(H) - HD(\xi) = D(V), \tag{1.41}$$

where $V = V(t, \mathbf{q}, \mathbf{p})$ is some function. If this condition holds on the solutions of the canonical Hamiltonian equations (1.32), then there is a first integral

$$J = p_i\eta^i - \xi H - V. \tag{1.42}$$