Noise-Induced Phenomena in the Environmental Sciences

Luca Ridolfi
Paolo D'Odorico
Francesco Laio

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NOISE-INDUCED PHENOMENA IN THE ENVIRONMENTAL SCIENCES

Randomness is ubiquitous in nature. Random drivers are generally considered a source of disorder in environmental systems. However, the interaction between noise and nonlinear dynamics may lead to the emergence of a number of ordered behaviors (in time and space) that would not exist in the absence of noise. This counterintuitive effect of randomness may play a crucial role in environmental processes. For example, seemingly "random" background events in the atmosphere can grow into larger instabilities that have great effects on weather patterns. This book presents the basics of the theory of stochastic calculus and its application to the study of noise-induced phenomena in environmental systems. It will be an invaluable reference text for ecologists, geoscientists, and environmental engineers interested in the study of stochastic environmental dynamics.

LUCA RIDOLFI is Professor in the Department of Water Engineering at the Polytechnic of Turin. His research focuses on ecohydrology, fluvial morphodynamics, biogeography, river water quality, and hydrodynamic instabilities.

PAOLO D'ODORICO is Professor in the Department of Environmental Sciences at the University of Virginia. His research focuses on ecohydrology, soil-moisture dynamics, global environmental change, nonlinear ecosystem dynamics, and desertification.

FRANCESCO LAIO is Associate Professor in the Department of Water Engineering at the Polytechnic of Turin. His research focuses on ecohydrology, stochastic hydrology, surface-water hydrology, and extreme value statistics.

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LUCA RIDOLFI
Politecnico di Torino

PAOLO D'ODORICO
University of Virginia

FRANCESCO LAIO
Politecnico di Torino

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Preface

Noise-induced phenomena are characterized by the ability of noise to induce order (either in space or in time) in dynamical systems. These phenomena are caused by the randomness of external drivers, and they would not exist in the absence of noise. The ability of noise to create order is counterintuitive. In fact, until recently, noise was generally associated with disordered random fluctuations around the steady states of the underlying deterministic dynamics. However, in the past few years the scientific community has become aware that noise can also have a more fundamental effect, in that it can determine new states and new dynamical patterns.

The speculative "beauty" of these dynamical behaviors, as well as the ubiquitous occurrence of random drivers in a number of natural and engineered systems, explains the great attention that has been recently paid to the study of noise-induced phenomena. A number of recent contributions have shown that the emergence of order and patterns in nature may result as an effect of the noise inherent in environmental variability. A typical example is climate fluctuations and their ability to induce dynamical behaviors that would not exist in the absence of random climate variability.

The main reason for writing this book is that there is a rich body of literature on noise-induced phenomena in the environmental sciences, and it has become difficult to keep track of the main theories, methods, and findings that have been presented in a number of research articles spread throughout the physics, mathematics, geoscience, and ecology journals. After working for a few years in this research field, we have become aware of the need for a book that (1) describes the main mechanisms of noise-induced order in space and in time; (2) presents rigorous mathematical tools addressing a relatively broad readership of environmental scientists, who are not necessarily familiar with the theory of stochastic processes; (3) focuses on applications to the environmental sciences; and (4) reviews a number of recent studies on noise-induced phenomena in environmental dynamics.

The goal of this book is to provide a synthesis of theories and methods for the study of noise-induced phenomena in the environment and to draw the attention of the earth and environmental science communities toward this fascinating and challenging research area. Through a number of examples of noise-induced phenomena we stress how in the natural environment random fluctuations are the rule and interesting behaviors may emerge from the interactions between the deterministic and stochastic components of environmental dynamics.

This book is not intended to be a comprehensive treatise on noise-induced phenomena. This relatively vast and fast-moving research field is enriched every day with new studies appearing in the literature. It would not be possible to contain in this volume an exhaustive review of all the existing theories of noise-induced order and their application to the environmental sciences. This book tries to provide an organized synthesis of the main contributions to this subject, drawing from material that is currently spread through a number of journals and other publications.

The completion of this book would have not been possible without the help, motivation, and support of a few collaborators and colleagues. We thank Stefania Scarsoglio and Fabio Borgogno (Politecnico di Torino) for providing invaluable help in performing the numerical simulations and contributing to the analysis of the results on noise-induced pattern formation (Chapters 5 and 6). We are grateful to Ignacio Rodriguez-Iturbe (Princeton University), Amilcare Porporato (Duke University), and Andrea Rinaldo (Ecóle Polytechnique Federale de Lausanne) for their unfailing encouragement and support through years of continued collaboration and companionship. We also acknowledge René Lefever (Université Libre de Bruxelles), whose work has inspired our research on noise-induced phenomena. We are also indebted to our institutions, the Polytechnic of Turin (Dipartimento di Idraulica, Trasporti e Infrastrutture Civili) and the University of Virginia (Department of Environmental Sciences) for providing high-quality academic environments that constantly stimulate our work.

Luca Ridolfi   Paolo D'Odorico   Francesco Laio

1

Introduction

1.1 Noise-induced phenomena

Most environmental dynamics are affected by a number of random drivers. This randomness typically results from the uncertainty inherent to the temporal or spatial variability of the driving processes. For example, if we consider the temperature record measured at a certain meteorological station, we can easily notice some obvious deterministic components of climate variability associated with the daily rotation of the Earth or with the annual seasonal cycle. At longer time scales we might recognize some patterns of interannual climate variability (e.g., the El Niño Southern Oscillation or the North Atlantic Oscillation) associated with temporally and spatially coherent anomalies in the atmospheric and oceanic circulations. These anomalies exhibit a certain degree of regularity in addition to unpredictable random fluctuations. However, besides these daily, annual, and interannual oscillations (and other deterministic signals), the temperature record will also exhibit some disorganized fluctuations that are typically ascribed to *environmental randomness*. In stochastic models of environmental dynamics this randomness is commonly expressed as *noise*.

Random environmental drivers are ubiquitous in nature. The occurrence of rainfall, sea storms, droughts, fires, or insect outbreaks are typical examples of random environmental processes. The noise underlying these processes is an important cause of environmental variability. What is the effect of this noise on the dynamics of environmental systems? Systems forced by random drivers are commonly expected to exhibit random fluctuations in their state variables. Thus the effect of noise is typically associated with the emergence of disorganized random fluctuations in the state of the system about its stable state(s). However, this trivial effect of noise is not the only possible way in which random drivers can affect a dynamical system. In the physics literature it was reported that noise can have a more fundamental role (e.g., Horsthemke and Lefever, 1984; Cross and Hohenberg, 1993; Garcia-Ojalvo and Sancho, 1999; Sagues et al., 2007). In fact it can induce new ordered states and new

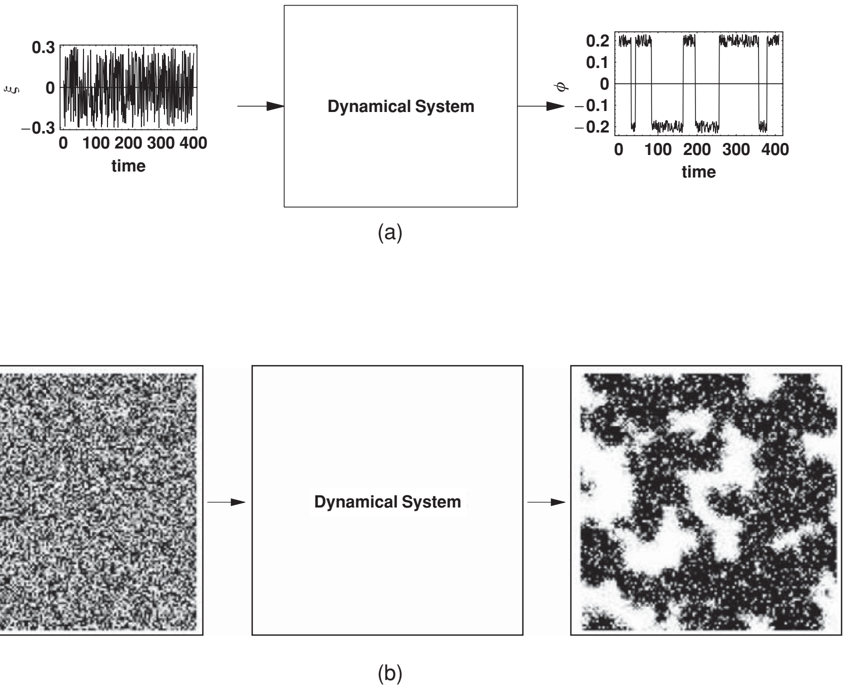

Figure 1.1. Schematic representation of noise-induced phenomena in (a) time and (b) space. Nonlinear systems forced by random drivers (left-hand panels) may lead to the emergence of ordered states in both time and space (right-hand panels).

bifurcations that would not exist in the deterministic counterpart of these systems. Noise can also modify the stability and resilience of deterministic states and induce coherence in the spatial and temporal variabilities of the state variables, including the emergence of periodic oscillations or the formation of spatial patterns. The ability of noise to induce order and organization (the so-called *constructive effect of noise*) is a quite counterintuitive effect that has seldom been investigated in the environmental science literature (e.g., May, 1972; Benzi et al., 1982a; Rodriguez-Iturbe et al., 1991; Katul et al., 2007). This book concentrates on this constructive effect of noise and on its ability to induce dynamical behaviors [i.e., states, bifurcations, spatial or temporal coherence (or both)] that do not exist in the underlying deterministic dynamics. We generically refer to these behaviors as *noise-induced phenomena*.

Figure 1.1 shows a schematic representation of some of these noise-induced behaviors: A nonlinear dynamical system forced by disordered random fluctuations either in space or in time may lead to the emergence of different forms of coherence, including for example noise-induced bistable dynamics [e.g., Fig. 1.1(a)] or morphogenesis [e.g., Fig. 1.1(b)]. In all of these cases, noise-induced behaviors appear when the noise intensity exceeds a critical level, whereas they disappear when it tends to zero.

The purpose of this book is to provide conceptual and mathematical tools that allow environmental scientists to familiarize themselves with the notion of noise-induced phenomena and with the idea that environmental noise (e.g., random climate fluctuations) is not necessarily a mere source of disturbance in environmental systems. The existence of a more fundamental role of noise should also be recognized in that it could have a crucial role in the way these systems respond to changes in environmental variability.

An example of the relevance of these constructive effects of noise can be found in the study of ecosystems' response to climate variability. Research in the field of ecosystem and population ecology has been investigating the effect of climate and land-use changes on ecosystem structure and function. In most cases the focus has been on how ecosystems respond to changes in the mean values of environmental parameters (e.g., mean annual precipitation or temperature) whereas the impact of changes in the variance has seldom been studied. However, recent climate-change studies indicate that, in addition to trends in the mean values of climate variables, interannual variability is also increasing (Katz and Brown, 1992; Easterling et al., 2000a, 2000b). It becomes therefore important to understand how this increase in the variance of environmental parameters will affect the dynamics of natural systems.

1.2 Time scales and noise models

The dynamics investigated in this book include four major components, namely, (i) a dynamical system, (ii) the external environment, (iii) a stochastic forcing, and (iv) some possible feedbacks between the state of the system and its environment or stochastic drivers. The first element is the deterministic *dynamical system* of interest. This system is a conceptually separate "entity" from a much more complex dynamical system, called *the environment*. The dynamical system generally involves a limited number of physical variables. We model it with a minimalist approach, which captures only the fundamental features of the dynamics. The physical variables representing the state of the system (e.g., plant biomass, soil moisture, soil thickness) are usually referred to as *state variables*. In this book we focus on systems that we can investigate by considering only one state variable, though we also consider the case of noise-induced phenomena that can emerge only in multivariate systems. The focus on univariate dynamics is motivated by their conceptual simplicity, the possibility of investigating them with analytical mathematical models, and the fact that their study allows us to show how noise-induced behaviors may emerge even without invoking complex interactions among a number of environmental variables. We express these univariate dynamics by using a first-order differential model,

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = f(\phi), \tag{1.1}$$

where $\phi(t)$ is the state variable characterizing the state of the system, t is time, and $f(\phi)$ is a deterministic algebraic function of the state variable. A spatially extended version of Eq. (1.1) would include also a term representing the effects on the dynamics of the values of ϕ in the neighboring sites; spatially extended systems are considered in the second half of this book, but the spatial coupling is neglected here to speculate more easily on the role of noise in the dynamics of ϕ.

The environment from which dynamical system (1.1) is extracted is generally much bigger than the dynamical system itself and is also called *external environment* to stress the fact that it is external to the dynamical system. Because the external environment is often too "large," complex, and also partially unknown to be modeled deterministically, its action on the dynamical system is represented as a stochastic process. Therefore we account for the randomness of environmental conditions through a stochastic forcing, which is modeled as noise $\xi(t)$. Thus the dynamics of the state variable read

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = f(\phi) + g(\phi)\xi(t), \tag{1.2}$$

where the (linear or nonlinear) algebraic function $g(\phi)$ accounts for the possibility that the effect of the random forcing on the system is modulated by the state of the system itself. The noise is *additive* when $g(\phi) = \text{const}$, whereas it is *multiplicative* otherwise.

Because the dynamical system is much smaller than the external environment, it is generally unable to affect its environmental drivers. However, in some dynamics the impact of the system on its environment can be important. In these cases a *feedback* exists between the state of the system and environmental conditions. This book discusses some examples of feedbacks relevant to the biogeosciences. Feedbacks with random environmental drivers are typically expressed either through the multiplicative function $g(\phi)$ or through a state dependency in the stochastic forcing [i.e., $\xi(t, \phi)$].

One of the most crucial issues in the representation of these stochastic dynamics arises from the modeling of the random forcing. To address this point, we need to consider two time scales underlying the dynamics, namely, the time scale τ_s of the deterministic dynamical system and the time scale τ_n of the random forcing. The former describes the response time of the deterministic system after a displacement from its steady state(s) ϕ_s. In other words, τ_s expresses how slowly or quickly the system will converge to its stable state(s). For example, τ_s can be expressed as inversely proportional to the first-order derivative of $f(\phi)$ calculated in ϕ_s, $\tau_s \simeq 1/|f'(\phi_s)|$.

The time scale τ_n of the random forcing is a function of its autocorrelation, which expresses the interrelations existing within the noise signal, i.e., how the values of random forcing at different times depend on their temporal separation (a formal definition is provided in Chapter 2). The time scale τ_n can be expressed as the integral of the autocorrelation function, which represents the (linear) temporal memory of the noise. In other words, values of $\xi(t)$ calculated at two different times, t_1 and t_2, are significantly interrelated if the temporal separation, $|t_2 - t_1|$, is less than τ_n. The stochastic forcing is modeled in different ways, depending on the ratio, τ_s/τ_n.

- **Case with $\tau_s/\tau_n \gg 1$:** In this case the dynamical system is very slow with respect to the temporal variability of its random drivers. Thus, because the overall dynamics are not able to "perceive" the autocorrelation of the random forcing, this autocorrelation can be reasonably neglected. The random forcing can be therefore modeled as *white noise* (i.e., uncorrelated noise). Despite its being an idealization and a mathematical singularity, white noise is a cornerstone of the theory of stochastic processes in that it lends itself to analytical mathematical solutions. Therefore, even though it suffers from physically unrealistic behaviors (e.g., noncontinuous-noise realizations), white noise is very often used to simulate stochastic forcing in environmental systems. In this case the dynamics of the state variable $\phi(t)$, driven by a white noise, do not need to be analyzed at the τ_n scale – at which the nonphysical behaviors of the noise would emerge [e.g., nondifferentiable realizations of $\phi(t)$] – but should be investigated at the time scale τ_s of the deterministic system. At this scale the predictions of theories based on the use of white noise are indistinguishable – for all practical purposes – from the behavior of the system forced by autocorrelated noise with $\tau_n \ll \tau_s$. The combination of analytical tractability and success in providing a realistic description of stochastic dynamics explains the widespread usage of the white-noise approximation, when $\tau_s/\tau_n \gg 1$.

  It is worth noticing that our ability to obtain analytical results for the dynamics of $\phi(t)$ depends also on the Markovian character of the $\phi(t)$ process forced by the white noise. We recall that a stochastic process is called Markovian when its future evolution depends on only the present state of the process. Most of the exact analytical results in the theory of stochastic processes were obtained in the case of Markovian processes.

  In the modeling of the stochastic forcing, the high dimension of the phase space of the external environment and the absence of significant correlations are very often invoked in order to apply the central-limit theorem and assume that the noise is Gaussian. Thus Gaussian white noise is commonly adopted. However, it is important to understand that whiteness and Gaussianity are two distinct noise properties, and white non-Gaussian noises are often important drivers of dynamical systems. For example, in the following chapters we show how in the biogeosciences a number of non-Gaussian, intermittent stochastic processes can be conveniently modeled as white shot noise.
- **Case with $\tau_s/\tau_n \simeq 1$:** In this case the dynamics are slow enough to be sensitive to the autocorrelation of the random forcing. Thus the white-noise approximation would not provide an appropriate representation of the stochastic driver, and therefore autocorrelated (or *colored*) noises should be used.

  It should be stressed that the process $\phi(t)$ driven by a colored noise is not Markovian. In fact, at any time the noise component depends on the past through the autocorrelation of the noise term. This non-Markovianity introduces great complications that limit our ability to obtain exact analytical results. To make the representation of the dynamics mathematically more tractable, we can assume that at least the colored noise is Markovian, i.e., it is generated by a Markovian process as in the case of dichotomous Markov noise and Gaussian colored noise (i.e., the so-called Ornstein–Uhlenbeck process) presented in the next chapter. With this assumption the bivariate system composed of the state variable ϕ and the noise ξ becomes Markovian, and some analytical representations of the stochastic dynamics can be obtained.

- **Case with $\tau_s/\tau_n \ll 1$:** In this case the system responds very quickly to the noise forcing, thereby adjusting (almost) instantaneously to the random forcing. In other words, the state variable ϕ is always in equilibrium with the noise term [i.e., $\mathrm{d}\phi/\mathrm{d}t \simeq 0$]. In these conditions we can use the so-called *adiabatic elimination* of the ϕ variable, whereby the dynamics of ϕ are described as $f(\phi) + g(\phi)\xi = 0$ and the probabilistic properties of ϕ are derived from those of the noise.

In the following chapters we consider different types of noise, including the case of both white and colored noise as well as continuous and intermittent noises. We review the major properties of each type of noise as well as their possible use in the development of stochastic models of environmental systems. To this end, we use a number of examples and case studies to show the possible impact of both additive and multiplicative noise on environmental dynamics.

2
Noise-driven dynamical systems

2.1 Introduction

We consider dynamical systems that can be represented through a stochastic differential equation in the form

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = f(\phi) + g(\phi)\xi(t), \tag{2.1}$$

where ϕ is the state variable, $f(\phi)$ and $g(\phi)$ are deterministic functions of ϕ, and $\xi(t)$ is a noise term accounting for the random external fluctuations forcing the dynamics of ϕ.

The solution of Eq. (2.1) requires that the noise term $\xi(t)$ be suitably specified. The scope of this chapter is to describe the main features of the noise term and of the resulting dynamics of ϕ in four cases, which are particularly interesting in the environmental sciences: We model $\xi(t)$ as (i) dichotomous Markov noise (DMN), (ii) white shot noise (WSN), (iii) white Gaussian noise, and (iv) Markovian colored Gaussian noise. These representations of $\xi(t)$ are very well suited for investigating the role of the random drivers typically found in the biogeosciences, and they are simple enough to allow for the analytical (probabilistic) solution of Eq. (2.1).

We first consider (in Section 2.2) the case of dichotomous noise because it is more general in that both WSN and white Gaussian noise can be obtained as limit cases of the dichotomous noise. For this reason, these two white noises are described in detail right after the case of DMN noise (i.e., in Sections 2.3 and 2.4); colored Gaussian noise is presented in Section 2.5.

2.2 Dichotomous noise

2.2.1 Definition and properties

The dichotomous Markov process is a stochastic process described by a state variable $\xi_{\mathrm{dn}}(t)$ that can take only two values, namely $\xi_{\mathrm{dn}} = \Delta_1$ and $\xi_{\mathrm{dn}} = \Delta_2$, with transition

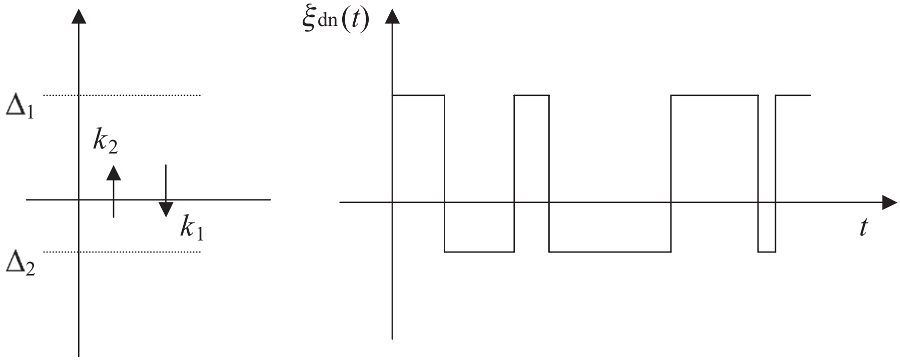

Figure 2.1. Parameters of the dichotomous noise and representation of a typical realization.

rate k_1 for the transition $\Delta_1 \to \Delta_2$ and k_2 for $\Delta_2 \to \Delta_1$. A realization of the process is shown in Fig. 2.1. The path of the noise is a step function with instantaneous jumps between Δ_1 and Δ_2 and random permanence times, t_1 and t_2, in these two states. The mean permanence times in the two states are $\langle t_1 \rangle = \tau_1 = 1/k_1$ and $\langle t_2 \rangle = \tau_2 = 1/k_2$. Moreover, when the transition rates k_1 and k_2 are constant in time, the permanence times are exponentially distributed random variables (e.g., Bena, 2006). If $\Delta_1 = |\Delta_2|$, the noise is called *symmetric* DMN; otherwise it is called *asymmetric* DMN. This type of noise was first introduced in information theory under the name of *random telegraph noise* or *Poisson square wave* (e.g., McFadden, 1959; Pawula, 1967); this process, studied in detail by physicists (Hongler, 1979; Kitahara et al., 1980), is called a two-state Markov process or DMN.

The probability $P_1(t)$ that the process is in the state Δ_1 at time t obeys the kinetic equation

$$\frac{\mathrm{d}P_1(t)}{\mathrm{d}t} = k_2 P_2(t) - k_1 P_1(t), \tag{2.2}$$

which includes a gain term $k_2 P_2(t)$ accounting for the probability of being in $\xi_{dn} = \Delta_2$ and jumping to $\xi_{dn} = \Delta_1$ and a loss term $-k_1 P_1(t)$ that accounts for the probability of escaping from the state $\xi_{dn} = \Delta_1$. Analogously, for the probability $P_2(t)$ that the process is in the state $\xi_{dn} = \Delta_2$ at time t, we have

$$\frac{\mathrm{d}P_2(t)}{\mathrm{d}t} = k_1 P_1(t) - k_2 P_2(t). \tag{2.3}$$

We obtain the steady solutions by neglecting the temporal derivatives on the left-hand side of Eqs. (2.2) and (2.3):

$$P_1 = \frac{k_2}{k_1 + k_2}, \qquad P_2 = \frac{k_1}{k_1 + k_2}. \tag{2.4}$$

The steady-state probability distribution of the state variable ξ_{dn} is then a discrete-valued distribution that can assume only two values, Δ_1 and Δ_2, with probability P_1 and P_2, respectively. The steady-state moment-generating function (see, for example, van Kampen, 1992, for a definition) is then

$$M_{\mathrm{dn}}(v) = \sum_{i=1}^{2} e^{v\Delta_i} P_i = \frac{k_2 e^{v\Delta_1} + k_1 e^{v\Delta_2}}{k_1 + k_2}, \tag{2.5}$$

and the corresponding cumulant-generating function (see van Kampen, 1992, for a definition) is

$$K_{\mathrm{dn}}(v) = \log[M_{\mathrm{dn}}(v)] = \log[k_2 e^{v\Delta_1} + k_1 e^{v\Delta_2}] - \log[k_1 + k_2]. \tag{2.6}$$

By definition of the cumulant-generating function, the steady-state cumulant of the order of m is obtained as the mth derivative of $K_{\mathrm{dn}}(v)$ with respect to v, calculated in $v = 0$. Therefore the mean of the process, $\kappa_{1\mathrm{dn}}$, is

$$\langle \xi_{\mathrm{dn}} \rangle = \kappa_{1\mathrm{dn}} = \left.\frac{\mathrm{d}K_{\mathrm{dn}}(v)}{\mathrm{d}v}\right|_{v=0} = \frac{k_2\Delta_1 + k_1\Delta_2}{k_1 + k_2}. \tag{2.7}$$

Because the DMN is used as a noise term in Eq. (2.1), it can be useful to consider a zero-average process. If this is the case, using Eq. (2.7) we have

$$\Delta_1 k_2 + \Delta_2 k_1 = \frac{\Delta_1}{\tau_2} + \frac{\Delta_2}{\tau_1} = 0. \tag{2.8}$$

In this case the (stationary) dichotomous Markov process is characterized by three independent parameters. For example, we can choose (i) the two transition rates k_1 and k_2 (or the mean durations τ_1 and τ_2) and (ii) assign the value of one of the states of ξ_{dn}, say Δ_1, and obtain the other value (i.e., Δ_2) by using Eq. (2.8). Unless explicitly stated otherwise, in what follows we refer to the case of zero-mean [Eq. (2.8)] DMN.

The variance of the dichotomous process is

$$\langle (\xi_{\mathrm{dn}} - \kappa_{1\mathrm{dn}})^2 \rangle = \kappa_{2\mathrm{dn}} = \left.\frac{\mathrm{d}^2 K_{\mathrm{dn}}(v)}{\mathrm{d}v^2}\right|_{v=0} = \frac{k_1 k_2\,(\Delta_2 - \Delta_1)^2}{(k_1 + k_2)^2} = -\Delta_1\Delta_2, \tag{2.9}$$

and the autocovariance function is

$$\langle \xi_{\mathrm{dn}}(t)\xi_{\mathrm{dn}}(t') \rangle = \frac{k_1 k_2 (\Delta_2 - \Delta_1)^2}{(k_1 + k_2)^2} e^{-|t-t'|(k_1+k_2)} = -\Delta_1\Delta_2 e^{-|t-t'|(k_1+k_2)}, \tag{2.10}$$

as demonstrated in Box 2.1, Eq. (B2.1-5). The structure of the autocovariance function shows that the dichotomous noise is a colored noise, i.e., it is autocorrelated. This is an important characteristic that explains why this type of noise is commonly used to mimic natural processes in the biogeosciences (see Subsection 2.2.2). A typical temporal scale of a correlated process is the integral scale $\mathcal{I}$, defined as the ratio between the area subtended by the autocovariace function (i.e., the integral of the autocovariance function with respect to the lag) and the variance of the process. The

Box 2.1: Transient dynamics of the dichotomous Markov process

Some considerations of the transient dynamics of the dichotomous Markov process can be of interest. Equations (2.2) and (2.3) can be solved to give

$$P_1(t) = \tau_c k_2 \left(1 - e^{-t/\tau_c}\right) + P_1(0)e^{-t/\tau_c}, \tag{B2.1-1}$$

$$P_2(t) = \tau_c k_1 \left(1 - e^{-t/\tau_c}\right) + P_2(0)e^{-t/\tau_c}, \tag{B2.1-2}$$

where $P_1(0)$ and $P_2(0) = 1 - P_1(0)$ are the initial conditions and

$$\tau_c = \frac{1}{k_1 + k_2} = \frac{\tau_1 \tau_2}{\tau_1 + \tau_2} \tag{B2.1-3}$$

is the characteristic relaxation time of the process.

It can be argued from Eqs. (B2.1-1) and (B2.1-2) that the joint probability for ξ_{dn} at time t and t' is

$$\begin{aligned} p_{i,j} &= \mathrm{prob}\left[\xi_{\mathrm{dn}}(t) = \Delta_i \, , \xi_{\mathrm{dn}}(t') = \Delta_j\right] \\ &= (1 - k_i \tau_c)(1 - k_j \tau_c)\tau_c^2 \left(1 - e^{-\frac{|t-t'|}{\tau_c}}\right) \\ &\quad + \delta_{i,j}(1 - k_i \tau_c)e^{-\frac{|t-t'|}{\tau_c}}, \end{aligned} \tag{B2.1-4}$$

where $\delta_{i,j}$ is the Kronecker delta function and $i,\, j = 1, 2$.

The steady-state autocorrelation function is then

$$\begin{aligned} \langle \xi_{\mathrm{dn}}(t)\xi_{\mathrm{dn}}(t')\rangle &= \sum_{i,j=1}^{2} p_{i,j}\Delta_i \Delta_j = \tau_c(\Delta_1^2 k_2 + \Delta_2^2 k_1)e^{-\frac{|t-t'|}{\tau_c}} \\ &= -\Delta_1 \Delta_2 e^{-\frac{|t-t'|}{\tau_c}} = \frac{s_{\mathrm{dn}}}{\tau_c} e^{-\frac{|t-t'|}{\tau_c}}, \end{aligned} \tag{B2.1-5}$$

where Eq. (2.8) has been repeatedly used. The term

$$s_{\mathrm{dn}} = k_1 k_2 \tau_c^3 (\Delta_2 - \Delta_1)^2 = -\Delta_1 \Delta_2 \tau_c \tag{B2.1-6}$$

in Eq. (B2.1-5) represents the noise *amplitude* or *intensity*.

integral scale is generally interpreted as a measure of the memory of the process, and in the case of dichotomous noise it is

$$\mathcal{I} = \frac{1}{k_1 + k_2} = \tau_c. \tag{2.11}$$

Some generalization of dichotomous noise were proposed in the literature. Notable examples include the so-called *trichotomous* noise (Mankin et al., 1999), characterized by a three-valued state space and its further generalization, multivalued noise (Weiss et al., 1987); compound dichotomous noise (van den Broeck, 1983), in which the value assumed in one of the two states is a random variable; *complicated* DMN (Li, 2007) in which both states are (Gaussian) random variables; the gamma and McFadden dichotomous noise (Pawula et al., 1993), in which the distribution of the permanence times in the two states follows a gamma or a McFadden probability distribution rather than an exponential distribution, as in the classical DMN.

2.2.2 Dichotomous noise in the environmental sciences

Dichotomous noise can be encountered in a wide variety of physical and mathematical models for two main reasons. First, dichotomous noise is a simple and analytically tractable form of colored noise; in fact, it is possible to obtain exact analytical solutions for a stochastic differential equation driven by DMN in steady-state conditions. Thus DMN can be used to investigate the effect of an autocorrelated random driver on a dynamical system. We define this approach as the *functional* usage of the DMN because of its function as a tool to conveniently represent a correlated (i.e., colored) random forcing. In this case (functional usage) the starting point is a given deterministic system, say $\mathrm{d}\phi/\mathrm{d}t = f(\phi)$, and DMN is typically used to investigate the effect of a zero-mean correlated random driver in this system. There are several examples of processes in which the autocorrelation is one of the key characteristics of the external forcing. For example, consider the variety of biogeochemical processes that are affected by (random) daily temperature or the case of fluvial processes forced by river flow. In these processes the autocorrelation of the random forcing is relevant, and it cannot be neglected. Dichotomous noise is one of the two main mathematical tools available for the study of the effects of colored noise on dynamical systems. Colored Gaussian noise, described in Section 2.5, is another type of autocorrelated noise, which is often used in dynamical models with analytical solutions. The functional usage of DMN can be also motivated by the fact that both white Poisson noise and white Gaussian noise can be recovered from the dichotomous noise by taking suitable limits, as shown in Subsections 2.3.2 and 2.4.2.

Dichotomous noise is commonly used also for its ability to model a broad class of systems that randomly switch between two dynamical states. This approach is called the *mechanistic* usage of DMN, in which DMN is used to represent a dynamical behavior, i.e., the mechanism of random switching between two states.

The distinction between the functional and the mechanistic use of DMN is crucial in the stochastic modeling of a process. The mechanistic approach is frequently used for a class of processes characterized by the following three components: (i) the dynamical system, whose state is expressed by one state variable, $\phi(t)$; (ii) a random driver $q(t)$; (iii) a threshold value θ of $q(t)$, marking the transition between conditions favorable to growth or to decay of ϕ. For example, the variable ϕ could represent vegetation biomass in semiarid environments (D'Odorico et al., 2005) or riparian vegetation along a river (Camporeale and Ridolfi, 2006); correspondingly, q could represent random rainfall fluctuations that determine the occurrence of water-limited conditions or of flooded or unflooded states, respectively. Thus the stochastic driver determines the random alternation between stressed and unstressed conditions for the ecosystem.

The two alternating dynamics of ϕ involve growth and decay and can be modeled by two functions, $f_1(\phi)$ and $f_2(\phi)$, respectively,

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \begin{cases} f_1(\phi) & \text{if} \quad q(t) \geq \theta & \text{(2.12a)} \\ f_2(\phi) & \text{if} \quad q(t) < \theta & \text{(2.12b)} \end{cases},$$

whre $f_1(\phi) > 0$ and $f_2(\phi) < 0$. Equations (2.12a) and (2.12b) are written assuming that q is a resource, in that values of q exceeding the threshold are associated with unstressed conditions (in the sense that ϕ grows). However, the general results do not change when the random driver is a stressor. In this case the conditions in (2.12a) and (2.12b) are reversed, i.e., growth or decay occurs when q is below or above the threshold, respectively.

The class of processes defined by (2.12a) and (2.12b) can be conveniently represented through a suitable dichotomous Markov process, thereby leading to a mechanistic usage of DMN. Thus the process is random and switches between two possible states: "success" (or "no stress") when q is above the threshold or "failure" (or "stress") when q is below the threshold [see Fig. 2.2(a)]. This is by definition a dichotomous process. If we further suppose that q is uncorrelated, the driving noise is the outcome of a Bernoulli trial with probability of success $k_2 = 1 - P_Q(\theta)$, where $P_Q(\theta)$ is the cumulative probability distribution of q, evaluated in $q = \theta$. The residence time in the "above-threshold" state is then an integer number n_1 with a geometric probability distribution of $p_{N_1}(n_1) = k_2^{n_1-1}(1 - k_2), n_1 = 1, \ldots, \infty$, with average $\langle n_1 \rangle = 1/(1 - k_2)$. Analogously, the residence time n_2 in the "below-threshold" state is distributed as $p_{N_2}(n_2) = (1 - k_2)^{n_1-1} k_2$, $n_1 = 1, \ldots, \infty$, with average $\langle n_1 \rangle = 1/k_2$. The DMN (in its mechanistic interpretation) is obtained as the continuous-time approximation of this driving process [see Fig. 2.2(b)]. In fact, in continuous time the residence time in each state becomes exponentially distributed (the exponential distribution is the continuous counterpart of the geometric distribution: e.g. Kendall and Stuart, 1977), which is a basic property of DMN (see Subsection 2.2.1).

The overall dynamics of the variable ϕ can then be expressed by a stochastic differential equation forced by DMN $\xi_{\mathrm{dn}}(t)$, assuming (constant) values Δ_1 and Δ_2 (see Fig. 2.1):

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = f(\phi) + g(\phi)\xi_{\mathrm{dn}}(t), \tag{2.13}$$

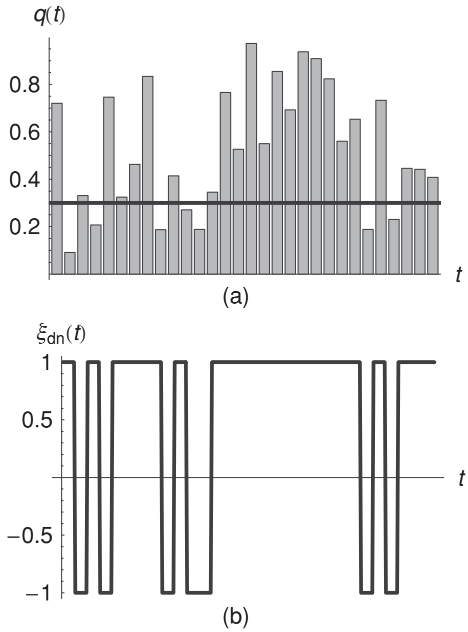

Figure 2.2. (a) The behavior of an uncorrelated random variable $q(t)$ around a threshold value θ (continuous horizontal line), (b) the path of the dichotomous noise that mimics the threshold effect on the dynamics of $q(t)$.

with

$$f(\phi) = -\frac{\Delta_2 f_1(\phi) - \Delta_1 f_2(\phi)}{\Delta_1 - \Delta_2}, \qquad g(\phi) = \frac{f_1(\phi) - f_2(\phi)}{\Delta_1 - \Delta_2}. \tag{2.14}$$

The transition rates are defined by $k_1 = P_Q(\theta)$ and $k_2 = 1 - k_1 = 1 - P_Q(\theta)$. As for the values of Δ_1 and Δ_2, in the mechanistic approach, DMN is used as a tool to randomly switch between $f_1(\phi)$ and $f_2(\phi)$. The only mechanistically relevant characteristics of DMN are in this case the switching rates k_1 and k_2, whereas the other noise characteristics, including its mean $\Delta_1 k_2 + \Delta_2 k_1$ and variance $-\Delta_1\Delta_2$, are not relevant to the representation of the dynamics of ϕ. In fact, in this case ϕ switches between two dynamics [$f_1(\phi)$ and $f_2(\phi)$] that are independent of Δ_1 and Δ_2. As a consequence, Δ_1 and Δ_2 may assume arbitrary values, and it is important to assign values of the switching rates k_1 and k_2 that are consistent with the fluctuations of $q(t)$ across the threshold.

The functional interpretation of the DMN, in contrast, is commonly used to simply investigate how an autocorrelated random forcing would affect the dynamics of a system. Thus the dynamical model has two components, namely (i) the deterministic dynamics $d\phi/dt = f(\phi)$ and (ii) an autocorrelated random forcing $\xi(t)$. The effect of $\xi(t)$ on the dynamics can be in general modulated by a function $g(\phi)$ of the state variable. The temporal dynamics are therefore modeled by the stochastic differential equation $d\phi/dt = f(\phi) + g(\phi)\xi(t)$. The functional usage of the DMN consists in approximating the colored noise, $\xi(t)$, as a DMN, i.e., $\xi(t) = \xi_{dn}(t)$. In this case none of the parameters k_1, k_2, Δ_1, and Δ_2 has an arbitrary value. In fact, these parameters need to be determined by adapting the DMN to the characteristics of the driving noise (i.e., for example, by matching the mean, variance, skewness, and correlation scale). Moreover, the functions $f(\phi)$ and $g(\phi)$ are in this case assigned a priori, whereas $f_1(\phi)$ and $f_2(\phi)$ are obtained from (2.14) and depend on the noise characteristics

$$f_1(\phi) = f(\phi) + g(\phi)\Delta_1, \qquad f_2(\phi) = f(\phi) + g(\phi)\Delta_2. \tag{2.15}$$

To summarize, the functional or mechanistic usage of the dichotomous noise corresponds to two distinct approaches to the stochastic modeling of processes driven by DMN. The differences may be relevant, in particular when dealing with noise-induced transitions (see Chapter 3). Once the approach that is suitable for the study of a specific problem is selected, the dichotomous noise provides a useful modeling framework with a number of applications to the environmental sciences, as shown in Chapter 4. Thus in the following subsection we present some probabilistic methods to solve stochastic equations driven by dichotomous noise.

2.2.3 Stochastic processes driven by dichotomous noise

2.2.3.1 General framework

Consider the stochastic process $\phi(t)$ driven by multiplicative dichotomous noise,

$$\frac{d\phi}{dt} = f(\phi) + g(\phi)\xi_{dn}(t), \tag{2.16}$$

where $f(\phi)$ and $g(\phi)$ are two deterministic functions of the state variable ϕ. We assume that both $f(\phi)$ and $g(\phi)$ are continuous for any value of ϕ. Depending on the approach or interpretation used for the DMN, the functions $f(\phi)$ and $g(\phi)$ are assigned with a direct physical meaning (functional approach) or are obtainable from functions $f_{1,2}(\phi)$ by use of Eqs. (2.14) (mechanistic approach).

Some examples can help us understand the role of the driving force in the dynamics of $\phi(t)$. We consider four simple cases. The first three examples refer to the mechanistic usage of DMN, and the fourth case considers the functional usage:

- **Example 2.1:** $\phi(t)$ exponentially increases (decreases) when the noise is in the Δ_1 (Δ_2) state:

$$f_1(\phi) = 1 - \phi, \qquad f_2(\phi) = -\phi. \tag{2.17}$$

  An example is shown in Fig. 2.3(a).
- **Example 2.2:** $\phi(t)$ linearly increases (decreases) when the noise is in the Δ_1 (Δ_2) state:

$$f_1(\phi) = 1, \qquad f_2(\phi) = -1. \tag{2.18}$$

  An example of the resulting dynamics of $\phi(t)$ is shown in Fig. 2.3(b).

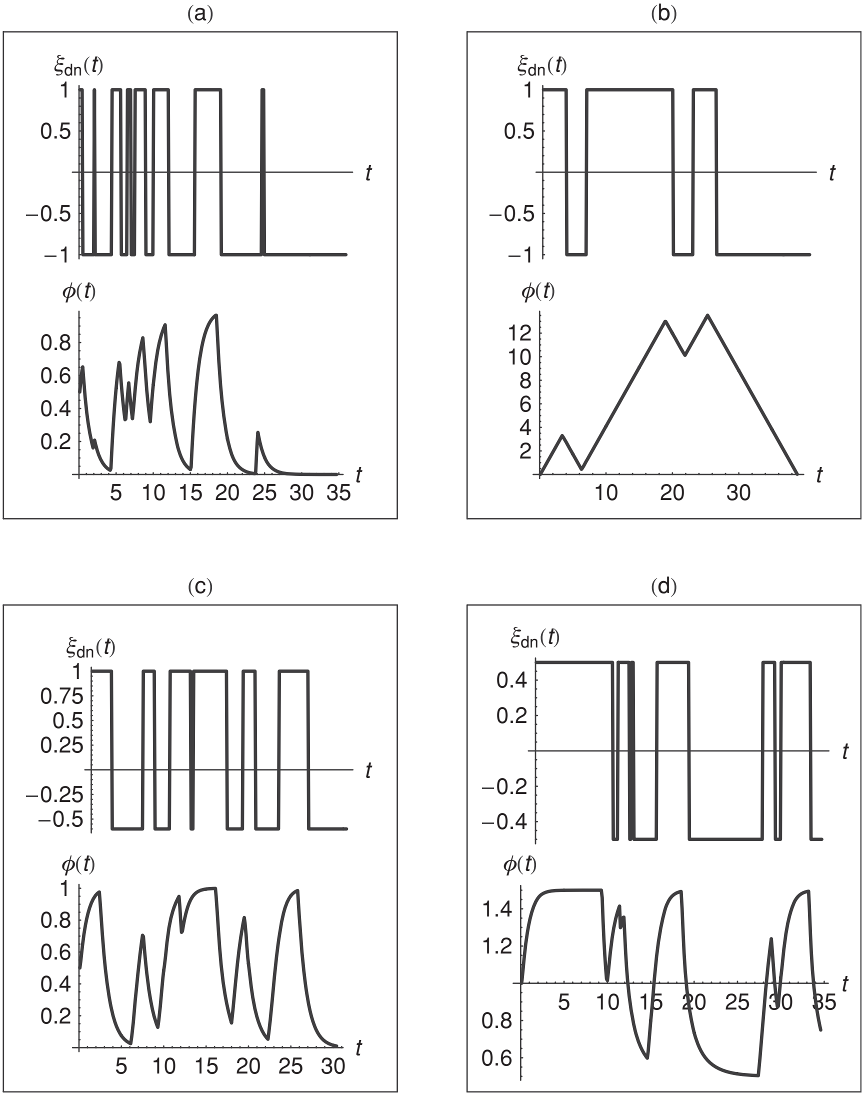

Figure 2.3. The four panels (a)–(d) show the noise path and the corresponding evolution of the $\phi(t)$ variable from Eq. (2.16): (a) Example 2.1, Example 2.2, (c) Example 2.3, (d) Example 2.4.

- **Example 2.3:** $\phi(t)$ increases following a (shifted) logistic law (see Subsection 3.2.1.1) when the noise is in the Δ_1 state, whereas it decreases exponentially in the Δ_2 state:

$$f_1(\phi) = (\phi - a)(1 - \phi), \qquad f_2(\phi) = -\phi, \tag{2.19}$$

where a is a constant. An example for $a = -0.5$ is shown in Fig. 2.3(c).

- **Example 2.4:** $\phi(t)$ follows logistic-type deterministic dynamics $f(\phi)$, perturbed by a dichotomous noise modulated by a linear $g(\phi)$:

$$f(\phi) = \phi(\beta - \phi), \qquad g(\phi) = \phi, \tag{2.20}$$

where $\beta > 0$ is a parameter. An example, calculated with $\beta = 1$, is shown in Fig. 2.3(d).

2.2.3.2 *Derivation of the steady-state probability density function*

This subsection is devoted to obtaining the steady-state probability density function (pdf) for the process described by Langevin equation (2.16). The standard procedure typically followed to address this task involves (i) deriving the master equations for the process, i.e., the forward differential equations that relate the state probabilities at different points in time; (ii) taking the limit as $t \to \infty$ in the master equation to attain statistically steady-state conditions; and (iii), solving the resulting forward differential equation to find the steady-state pdf. The detailed derivation of the steady-state probability distribution of ϕ following this approach is described in Box 2.2. In this subsection, we describe a simpler approach in a way that the nonexpert reader can more easily follow how the solution of Langevin equation (2.16) is determined.

Consider the probability that, at time $t + \Delta t$, the state variable takes a value contained within the interval $[\phi, \phi + \mathrm{d}\phi]$ and the noise is in state $\xi_{\mathrm{dn}} = \Delta_1$. These conditions may be attained either when $\xi_{\mathrm{dn}} = \Delta_1$ at time t, no jumps of ξ_{dn} occur in $[t, t + \Delta t]$, and the value of ϕ at time t is $\phi - f_1(\phi)\Delta t$, or when at time t we have $\xi_{\mathrm{dn}} = \Delta_2$, the random variable ξ_{dn} shifts from Δ_2 to Δ_1 in the interval $[t, t + \Delta t]$, and the state variable at time t is $\phi - f^*(\phi)\Delta t$, where $f^*(\phi)$ is a suitable combination of $f_1(\phi)$ and $f_2(\phi)$ to account for the fact that in the interval $[t, t + \Delta t]$ both functions contribute to determine the trajectory of ϕ. Thus the joint probability that $\xi_{\mathrm{dn}} = \Delta_1$ and ϕ is comprised within $[\phi, \phi + \Delta\phi]$ can be expressed as

$$\begin{aligned}\mathcal{P}[\phi, \Delta_1; t + \Delta t]\mathrm{d}\phi = {} & (1 - k_1\Delta t)\mathcal{P}[\phi - f_1(\phi)\Delta t, \Delta_1; t]\mathrm{d}[\phi - f_1(\phi)\Delta t] \\ & + k_2\Delta t\mathcal{P}[\phi - f^*(\phi)\Delta t, \Delta_2; t]\mathrm{d}[\phi - f^*(\phi)\Delta t]. \end{aligned} \tag{2.21}$$

The first term on the right-hand-side of Eq. (2.21) is the product of three factors: (i) the probability that the noise ξ_{dn} remains in the state $\xi_{\mathrm{dn}} = \Delta_1$ in the interval $(t, t + \Delta t)$. This probability is 1 minus the probability $k_1\Delta t$ that a jump occurs from Δ_1 to Δ_2 in the same interval; (ii) the joint probability $\mathcal{P}$ that at time t noise is equal to Δ_1 and the state variable is at $\phi - \Delta\phi$, where $\Delta\phi = f_1(\phi)\Delta t$ from Eqs. (2.15); and (iii) the infinitesimal amplitude of the interval $\mathrm{d}[\phi - f_1(\phi)\Delta t]$. Similarly, the second term represents the probability that $\xi_{\mathrm{dn}} = \Delta_2$, the state variable is equal to $\phi - \Delta\phi$ at time t, and a jump from Δ_2 to Δ_1 occurs during the interval $(t, t + \Delta t)$. This jump in ξ_{dn} occurs with probability $k_2\Delta t$. Note that, because the jump may occur at any time during $(t, t + \Delta t)$, in this case $\Delta\phi$ is expressed as $\Delta\phi = f^*(\phi)\Delta t$, where $f^*(\phi)$ is a combination of $f_1(\phi)$ and $f_2(\phi)$ (see Horsthemke and Lefever, 1984, Eq. 9.22).

The probability of occurrence of two or more jumps can be neglected in Eq. (2.21) because it is supposed that Δt is small.

Using a Taylor's expansion truncated to the first order we have[1]

$$\begin{aligned}
&\mathcal{P}[\phi, \Delta_1; t + \Delta t]\mathrm{d}\phi \\
&\quad = k_2 \Delta t \mathcal{P}[\phi, \Delta_2; t]\mathrm{d}\phi \\
&\qquad + (1 - k_1 \Delta t)\left(\mathcal{P}[\phi, \Delta_1; t] - \frac{\partial \mathcal{P}[\phi, \Delta_1; t]}{\partial \phi} f_1(\phi)\Delta t\right)\left(1 - \frac{\partial f_1(\phi)}{\partial \phi}\Delta t\right)\mathrm{d}\phi.
\end{aligned} \tag{2.22}$$

Notice how Eq. (2.22) is independent of the form of the function $f^*(\phi)$ describing the trajectory of ϕ in correspondence to jump occurrences. Rearranging the terms, dividing by $\mathrm{d}\phi$ and Δt, and taking the limit for $\Delta t \to 0$, we finally obtain the forward Kolmogorov equation:

$$\frac{\partial \mathcal{P}(\phi, \Delta_1, t)}{\partial t} = -\frac{\partial}{\partial \phi}[\mathcal{P}(\phi, \Delta_1, t) f_1(\phi)] - \mathcal{P}(\phi, \Delta_1, t)k_1 + \mathcal{P}(\phi, \Delta_2, t)k_2. \tag{2.23}$$

Analogously, we can write for the probability that at time $t + \Delta t$ the state variable is within $(\phi, \phi + \mathrm{d}\phi)$ and $\xi_{\mathrm{dn}} = \Delta_2$,

$$\begin{aligned}
\mathcal{P}[\phi, \Delta_2; t + \Delta t]\mathrm{d}\phi &= k_1 \Delta t \mathcal{P}[\phi - f^{**}(\phi)\Delta t, \Delta_1; t]\mathrm{d}[\phi - f^{**}(\phi)\Delta t] \\
&\quad + (1 - k_2 \Delta t)\mathcal{P}[\phi - f_2(\phi)\Delta t, \Delta_2; t]\mathrm{d}[\phi - f_2(\phi)\Delta t],
\end{aligned} \tag{2.24}$$

where $f^{**}(\phi)$ describes the trajectory of ϕ in the interval $(t, t + \Delta t)$ in the case in which ξ_{dn} switches from Δ_1 to Δ_2 in that interval. After a Taylor expansion for $\Delta t \to 0$ we obtain the second forward Kolmogorov equation:

$$\frac{\partial \mathcal{P}(\phi, \Delta_2, t)}{\partial t} = -\frac{\partial}{\partial \phi}[\mathcal{P}(\phi, \Delta_2, t) f_2(\phi)] - \mathcal{P}(\phi, \Delta_2, t)k_2 + \mathcal{P}(\phi, \Delta_1, t)k_1. \tag{2.25}$$

We refer the interested reader to Box 2.2 for the derivation of the full master equation in the time-dependent case. Here we concentrate on steady-state solutions of (2.16).

[1] $\mathcal{P}[\phi - f_1(x)\Delta t, \Delta_1; t]$ can be expanded in a Taylor's series around $\Delta t = 0$:

$$\begin{aligned}
\mathcal{P}[\phi - f_1(\phi)\Delta t, \Delta_1; t] &= \mathcal{P}[\phi, \Delta_1; t] + \left.\frac{\partial \mathcal{P}[\phi - f_1(\phi)\Delta t, \Delta_1; t]}{\partial \Delta t}\right|_{\Delta t = 0} \Delta t \\
&= \mathcal{P}[\phi, \Delta_1; t] + \left.\frac{\partial \mathcal{P}[z, \Delta_1; t]}{\partial z}\right|_{z=\phi} \left.\frac{\partial z}{\partial \Delta t}\right|_{\Delta t = 0} \Delta t \\
&= \mathcal{P}[\phi, \Delta_1; t] - \frac{\partial \mathcal{P}[\phi, \Delta_1; t]}{\partial \phi} f_1(\phi)\Delta t,
\end{aligned}$$

where the series has been truncated to the first order and $z = \phi - f_1(\phi)\Delta t$.

Box 2.2: Master equation of a stochastic process driven by dichotomous Markov noise

In this box, we show the key steps to determine the nonsteady-state master equation of the stochastic process described by (2.16); further details can be found in Horsthemke and Lefever (1984).

We introduce the two quantities

$$\mathcal{P}(\phi, t) = \mathcal{P}(\phi, \Delta_1, t) + \mathcal{P}(\phi, \Delta_2, t), \tag{B2.2-1}$$

$$q(\phi, t) = k_2\mathcal{P}(\phi, \Delta_2, t) - k_1\mathcal{P}(\phi, \Delta_1, t), \tag{B2.2-2}$$

where $q(\phi, t)$ is an auxiliary function, and $\mathcal{P}(\phi, t)$ expresses the time-dependent probability distribution of ϕ independently of the state of noise.

Adding Eq. (2.23) to Eq. (2.25) and using zero-average condition (2.8), we obtain

$$\frac{\partial \mathcal{P}}{\partial t} = -\frac{\partial}{\partial \phi}[\mathcal{P} f(\phi)] - \frac{\Delta_2 - \Delta_1}{k_1 + k_2}\frac{\partial}{\partial \phi}[q g(\phi)], \tag{B2.2-3}$$

and, if Eqs. (2.23) and (2.25) are multiplied by k_1 and k_2, respectively, and then (2.23) is subtracted from (2.25), we obtain

$$\begin{aligned}\frac{\partial q}{\partial t} &= \frac{\partial}{\partial \phi}\left\{\left[f(\phi) + \frac{k_2\Delta_2 - k_1\Delta_1}{k_1 + k_2} + (k_1 + k_2)\right] q\right\} \\ &\quad - \frac{(\Delta_2 - \Delta_1)k_1 k_2}{k_1 + k_2}\frac{\partial}{\partial x}[g(\phi)\mathcal{P}].\end{aligned} \tag{B2.2-4}$$

Using the independent variable

$$\eta = \int \frac{\mathrm{d}\phi}{f(\phi) + \frac{k_2\Delta_2 - k_1\Delta_1}{k_1 + k_2} g(\phi)}, \tag{B2.2-5}$$

we can reduce differential equation (B2.2-4) to a form that can be analytically integrated (Polyanin et al., 2002), leading to

$$\begin{aligned}q(\phi, t) &= \int_{-\infty}^{t} e^{-\left\{\frac{\partial}{\partial x}\left[f(\phi) + \frac{k_2\Delta_2 - k_1\Delta_1}{k_1 + k_2} g(\phi)\right] + k_1 + k_2\right\}(t - t')} \\ &\quad \times \frac{(\Delta_2 - \Delta_1)k_1 k_2}{k_1 + k_2}\frac{\partial}{\partial \phi}[g(\phi)\mathcal{P}(\phi, t')]\mathrm{d}t',\end{aligned} \tag{B2.2-6}$$

where the statistical independence between the noise ξ_{dn} and the process $\phi(t)$ at $t \to -\infty$ has been assumed as the initial condition.

Equation (B2.2-6) can be substituted into (B2.2-3) to obtain the master equation:

$$\begin{aligned}\frac{\partial \mathcal{P}(\phi, t)}{\partial t} &= -\frac{\partial}{\partial \phi}\left[f(\phi) + \frac{k_2\Delta_2 - k_1\Delta_1}{k_1 + k_2} g(\phi)\right]\mathcal{P}(\phi, t) \\ &\quad + \frac{k_1 k_2(\Delta_2 - \Delta_1)^2}{(k_1 + k_2)^2}\frac{\partial g(\phi)}{\partial \phi} \\ &\quad \times \int_{-\infty}^{t} e^{-\left\{\frac{\partial}{\partial \phi}\left[f(x) + \frac{k_2\Delta_2 - k_1\Delta_1}{k_1 + k_2} g(\phi)\right] + k_1 + k_2\right\}(t - t')} \\ &\quad \times \frac{(\Delta_2 - \Delta_1)k_1 k_2}{k_1 + k_2}\frac{\partial}{\partial \phi}[g(\phi)\mathcal{P}(\phi, t')]\mathrm{d}t'.\end{aligned} \tag{B2.2-7}$$

This rather intricate integrodifferential equation shows that, in general, $\phi(t)$ is not a Markovian process. In fact, the probability distribution of ϕ at time t depends on the integral between $-\infty$ and t of a function of ϕ. Equation (B2.2-7) can be analytically solved in only very simple cases (Bena, 2006). An important case is the so-called persistent diffusion on a line ($f(\phi) = 0$ and $g(\phi) = 1$) that has interesting applications in chemistry and physics (van den Broeck, 1990; Bena, 2006).

In steady-state conditions the temporal derivatives in Eqs. (2.23) and (2.25) are equal to zero and the forward Kolmogorov equations become

$$\frac{\partial}{\partial\phi}[\mathcal{P}(\phi,\Delta_1)f_1(\phi)] + \mathcal{P}(\phi,\Delta_1)k_1 - \mathcal{P}(\phi,\Delta_2)k_2 = 0,$$
$$\frac{\partial}{\partial\phi}[\mathcal{P}(\phi,\Delta_2)f_2(\phi)] + \mathcal{P}(\phi,\Delta_2)k_2 - \mathcal{P}(\phi,\Delta_1)k_1 = 0. \tag{2.26}$$

By summing up Eqs. (2.26) and integrating with respect to ϕ, we obtain

$$\mathcal{P}(\phi,\Delta_2) = -\mathcal{P}(\phi,\Delta_1)\frac{f_1(\phi)}{f_2(\phi)}, \tag{2.27}$$

where the integration constant is set to zero. Equation (2.27) inserted into the first of Eqs. (2.26) leads to

$$\frac{\partial}{\partial\phi}[\mathcal{P}(\phi,\Delta_1)f_1(\phi)] + \mathcal{P}(\phi,\Delta_1)k_1 + \mathcal{P}(\phi,\Delta_1)\frac{f_1(\phi)}{f_2(\phi)}k_2 = 0. \tag{2.28}$$

The integration of (2.28) provides the probability distribution

$$\mathcal{P}(\phi,\Delta_1) = \frac{C}{f_1(\phi)}\exp\left\{-\int_\phi\left[\frac{k_1}{f_1(\phi')} + \frac{k_2}{f_2(\phi')}\right]\mathrm{d}\phi'\right\}, \tag{2.29}$$

where C is an integration constant. Equation (2.29) can be set in (2.27) to obtain

$$\mathcal{P}(\phi,\Delta_2) = -\frac{C}{f_2(x)}\exp\left\{-\int_\phi\left[\frac{k_1}{f_1(\phi')} + \frac{k_2}{f_2(\phi')}\right]\mathrm{d}\phi'\right\}. \tag{2.30}$$

We now use these two joint distributions to determine the marginal steady-state pdf $p_\Phi(\phi)$ for the state variable ϕ, as $p_\Phi(\phi) = \mathcal{P}(\phi,\Delta_1) + \mathcal{P}(\phi,\Delta_2)$ (Pawula, 1977; Kitahara et al., 1980; van den Broeck, 1983):

$$p_\Phi(\phi) = C\left[\frac{1}{f_1(\phi)} - \frac{1}{f_2(\phi)}\right]\exp\left\{-\int_\phi\left[\frac{k_1}{f_1(\phi')} + \frac{k_2}{f_2(\phi')}\right]\mathrm{d}\phi'\right\}. \tag{2.31}$$