The Child's Conception of Number

Jean Piaget

Jean Piaget: Selected Works

Volume II

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The Child's Conception of Number

Jean Piaget

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FOREWORD

In our earlier books (*The Language and Thought of the Child, Judgment and Reasoning in the Child, The Child's Conception of the World*, and *The Child's Conception of Physical Causality*), we analysed various verbal and conceptual aspects of the child's thought. Later on, we examined the beginnings of thought on the practical and sensory-motor planes (*La Naissance de l'Intelligence* and *La Construction du Réel chez l'Enfant*). It now remains, in order to discover the mechanisms that determine thought, to investigate how the sensory-motor schemata of assimilating intelligence are organized in operational systems on the plane of thought. Beyond the child's verbal constructions, and in line with his practical activity, we now have to trace the development of the operations which give rise to number and continuous quantities, to space, time, speed, etc., operations which, in these essential fields, lead from intuitive and egocentric pre-logic to rational co-ordination that is both deductive and inductive.

In dealing with these new problems, appropriate methods must be used. We shall still keep our original procedure of free conversation with the child, conversation which is governed by the questions put, but which is compelled to follow the direction indicated by the child's spontaneous answers. Our investigation of sensory-motor intelligence has, however, shown us the necessity for actual manipulation of objects. In *The Child's Conception of Physical Causality*, we saw, though it was not possible to take full advantage of the fact, that conversation with the child is much more reliable and more fruitful when it is related to experiments made with adequate material, and when the child, instead of thinking in the void, is talking about actions he has just performed. As far as the study of number is concerned, this is an essential condition, and the gifts of Mlle Szeminska have made it possible to discover techniques adapted to the various problems which needed to be solved and analysed separately. In another volume, written with the collaboration of Mlle Inhelder, the same methods will be used in the description of continuous quantities as the product of quantification of physical qualities (weight, volume, etc.).

In the present volume, it has not been possible to include all that we should have wished to say on the subject of the evolution of number. In particular, there is an inexhaustible mine of information, on which we have deliberately not drawn, in the observations collected at the Maison des Petits by Mlles Audemars and Lafendel, who constructed original material which they have been using for more than twenty years. It is to be hoped that these gifted teachers will shortly publish their findings concerned with the beginnings of arithmetic in the school. We have, of course, greatly benefited from the spirit of their research, and we are also greatly indebted to many works on the arithmetic of the child, particularly to those of K. Bühler, Decroly, Mlle Descœudres, and others. We have not entered into a detailed discussion of existing works because we have here deliberately restricted ourselves to the problem of the construction of number in relation to logical operations.

Our hypothesis is that the construction of number goes hand-in-hand with the development of logic, and that a pre-numerical period corresponds to the pre-logical level. Our results do, in fact, show that number is organized, stage after stage, in close connection with the gradual elaboration of systems of inclusions (hierarchy of logical classes) and systems of asymmetrical relations (qualitative seriations), the sequence of numbers thus resulting from an operational synthesis of classification and seriation. In our view, logical and arithmetical operations therefore constitute a single system that is psychologically natural, the second resulting from generalization and fusion of the first, under the two complementary headings of inclusion of classes and seriation of relations, quality being disregarded. When the child applies this operational system to sets that are defined by the qualities of their elements, he is compelled to consider separately classes (which depend on the qualitative equivalence of the elements) and asymmetrical relations (which express the seriable differences). Hence the dualism of logic of classes and logic of asymmetrical relations. But when the same system is applied to sets irrespective of their qualities, the fusion of inclusion and seriation of the elements into a single operational totality takes place, and this totality constitutes the sequence of whole numbers, which are indissociably cardinal and ordinal.

Although the facts recorded in this volume lead to this conclusion almost without any attempt at interpretation, its very simplicity seemed to us a cause for doubt. Discussion as to the relationship between number and logic has, as we know, been endless. The logisticians, with Russell, have tried to reduce cardinal number to the notion of 'class of classes', and ordinal number, dissociated from cardinal number, to the notion of 'class of relationships', while their opponents maintained, with Poincaré and Brunschvicg, that the whole number is essentially synthetic and irreducible. Our hypothesis seems to obviate the necessity for this alternative, for if number is at the same time both class and asymmetrical relation, it does not derive from one or other of the logical operations, but from their union, continuity thus being reconciled with irreducibility, and the relationships between logic and arithmetic being regarded not as unilateral but as reciprocal. Nevertheless, the connections established in the field of experimental psychology needed to be verified in the field of logistics, and we proceeded to attempt this verification.

In studying the literature on the subject, we were surprised to find to what extent the usual point of view was 'realist' rather than 'operational', with the exception of the interesting work of A. Reymond. This fact accounts for the connections, many of them artificial, established by Russell, which forcibly separated logistic investigation from psychological analysis, whereas each should be a support for the other in the same way as mathematics and experimental physics.

If, on the contrary, we construct a logistics based on the reality of operations as such, in accordance with, and not in opposition to, the psychogenetic processes, we discover that the natural psychological systems of thought, such as simple and multiple classifications, simple and multiple seriations, nesting of symmetrical relations, etc., correspond from the logistic point of view to operational structures closely akin to mathematical 'groups', and which we have called 'groupings'. The laws of these groupings, once formulated, proved to be of constant help in our psychological analysis.

Translator's Note

While keeping as closely as possible to the French text, we have, with the author's permission, used a certain freedom on occasion, but only when it seemed desirable in the interests of clarity and when no essential idea was involved. In particular, we have omitted the logistic algorism introduced by the author in Chapters III and X to which reference can be made in the original text.

[1] Since this book was published (Geneva, 1941), this problem has been considerably developed in two further volumes, *Classes, relations et nombres* (Vrin, Paris, 1942) and *Traité de Logique* (A. Colin, Paris, 1950). (Translator's note).

The first chapter of this volume appeared in 1939 in the *Journal de Psychologie*, and the first paragraphs of Chapter VII form part of an article published in 1937 in the *Recueil de travaux de l'Université de Lausanne, publié à l'occasion du IVme Centenaire de la fondation de l'Université.*

In the *Compte rendu des séances de la Société de Physique et d'Histoire naturelle de Geneve* (vol. 58, 1941) the theory of groupings appeared for the first time, in condensed form.

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PART ONE

CONSERVATION OF QUANTITIES AND INVARIANCE OF WHOLES

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CHAPTER I

CONSERVATION OF CONTINUOUS QUANTITIES

EVERY notion, whether it be scientific or merely a matter of common sense, presupposes a set of principles of conservation, either explicit or implicit. It is a matter of common knowledge that in the field of the empirical sciences the introduction of the principle of inertia (conservation of rectilinear and uniform motion) made possible the development of modern physics, and that the principle of conservation of matter made modern chemistry possible. It is unnecessary to stress the importance in every-day life of the principle of identity; any attempt by thought to build up a system of notions requires a certain permanence in their definitions. In the field of perception, the schema of the permanent object[1] presupposes the elaboration of what is no doubt the most primitive of all these principles of conservation. Obviously conservation, which is a necessary condition of all experience and all reasoning, by no means exhausts the representation of reality or the dynamism of the intellectual processes, but that is another matter. Our contention is merely that conservation is a necessary condition for all rational activity, and we are not concerned with whether it is sufficient to account for this activity or to explain the nature of reality.

This being so, arithmetical thought is no exception to the rule. A set or collection is only conceivable if it remains unchanged irrespective of the changes occurring in the relationship between the elements. For instance, the permutations of the elements in a given set do not change its value. A number is only intelligible if it remains identical with itself, whatever the distribution of the units of which it is composed. A continuous quantity such as a length or a volume can only be used in reasoning if it is a permanent whole, irrespective of the possible arrangements of its parts. In a word, whether it be a matter of continuous or discontinuous qualities, of quantitative relations perceived in the sensible universe, or of sets and numbers conceived by thought, whether it be a matter of the child's earliest contacts with number or of the most refined axiomatizations of any intuitive system, in

[1] *La Construction du Réel chez l'Enfant*, chapter i.

each ana every case the conservation of something is postulated as a necessary condition for any mathematical understanding.

From the psychological point of view, the need for conservation appears then to be a kind of functional *a priori* of thought. But does this mean that arithmetical notions acquire their structure because of this conservation, or are we to conclude that conservation precedes any numerical or quantifying activities, and is not only a function, but also an *a priori* structure, a kind of innate idea present from the first awareness of the intellect and the first contact with experience? It is experiment that will provide the answer, and we shall try to show that the first alternative is the only one that is in agreement with the facts.

§1. *Technique and general results*

This chapter and the one that follows will be devoted to experiments made simultaneously with continuous and discontinuous quantities. It seemed to us essential to deal with the two questions at the same time, although the former are not arithmetical and were to be treated separately in a special volume,[1] since it was desirable to ascertain that the results obtained in the case of discontinuous sets were general.

The child is first given two cylindrical containers of equal dimensions (A1 and A2) containing the same quantity of liquid (as is shown by the levels). The contents of A2 are then poured into two smaller containers of equal dimensions (B1 and B2) and the child is asked whether the quantity of liquid poured from A2 into (B1 + B2) is still equal to that in A1. If necessary, the liquid in B1 can then be poured into two smaller, equal containers (C1 and C2), and in case of need, the liquid in B2 can be poured into two other containers C3 and C4 identical with C1 and C2. Questions as to the equality between (C1 + C2) and B2, or between (C1 + C2 + C3 + C4) and A1, etc., are then put. In this way, the liquids are subdivided in a variety of ways, and each time the problem of conservation is put in the form of a question as to equality or non-equality with one of the original containers. Conversely, as a check on his answers, the child can be asked to pour into a glass of a different shape a quantity of liquid approximately the same as that in a given glass, but the main problem is still that of conservation as such.

The results obtained seem to prove that continuous quantities are not at once considered to be constant, and that the notion of conservation is gradually constructed by means of an intellectual

[1] J. Piaget and B. Inhelder, *Le Dévelopment des Quantités chez l'Enfant* (*Conservation et Atomisme*), 1941.

mechanism which it is our purpose to explain. By grouping the answers to the various questions, it is possible to distinguish three stages. In the first, the child considers it natural for the quantity of liquid to vary according to the form and dimensions of the containers into which it is poured. Perception of the apparent changes is therefore not corrected by a system of relations that ensures invariance of quantity. In the second stage, which is a period of transition, conservation gradually emerges, but although it is recognized in some cases, of which we shall attempt to discover the characteristics, it is not so in all. When he reaches the third stage, the child at once postulates conservation of the quantities in each of the transformations to which they are subjected. Naturally this does not mean that this generalization of constancy extends at this stage beyond the limits of the field studied here.

In our interpretation of these facts, we can start from the following hypotheses, some of which directed the research of this chapter while others arose in the course of our experiments. The question to be considered is whether the development of the notion of conservation of quantity is not one and the same as the development of the notion of quantity. The child does not first acquire the notion of quantity and then attribute constancy to it; he discovers true quantification only when he is capable of constructing wholes that are preserved. At the level of the first stage, quantity is therefore no more than the asymmetrical relations between qualities, i.e., comparisons of the type 'more' or 'less' contained in judgements such as 'it's higher', 'not so wide', etc. These relations depend on perception, and are not as yet relations in the true sense, since they cannot be co-ordinated one with another in additive or multiplicative operations. This co-ordination begins at the second stage and results in the notion of 'intensive' quantity, i.e., without units, but susceptible of logical coherence. As soon as this intensive quantification exists, the child can grasp, before any other measurement, the proportionality of differences, and therefore the notion of extensive quantity. This discovery, which alone makes possible the development of number, thus results from the child's progress in logic during these stages.

§2. *Stage I: Absence of conservation*

For children at the first stage, the quantity of liquid increases or diminishes according to the size or number of the containers. The reasons given for this non-conservation vary from child to child, and from one moment to the next, but in every case the child thinks that the change he sees involves a change in the total value of the liquid. Here we have some examples:

Blas (4;0). 'Have you got a friend?—*Yes, Odette.*—Well look, we're giving you, Clairette, a glass of orangeade (A1, $\frac{3}{4}$ full), and we're giving Odette a glass of lemonade (A2, also $\frac{3}{4}$ full). Has one of you more to drink than the other?—*The same.*—This is what Clairette does: she pours her drink into two other glasses (B1 and B2, which are thus half filled). Has Clairette the same amount as Odette?—*Odette has more.*—Why?—*Because we've put less in* (She pointed to the levels in B1 and B2, without taking into account the fact that there were two glasses).—(Odette's drink was then poured into B3 and B4.) *It's the same.*—And now (pouring Clairette's drink from B1 + B2 into L, a long thin tube, which is then almost full)?—*I've got more.*—Why?—*We've poured it into that glass* (pointing to the level in L), *and here* (B3 and B4) *we haven't.*—But were they the same before?—*Yes.*—And now?—*I've got more.*' Clairette's orangeade was then poured back from L into B1 and B2: 'Look, Clairette has poured hers like Odette. So, is all the lemonade (B3 + B4) and all the orangeade (B1 + B2) the same?—*It's the same* (with conviction).—Now Clairette does this (pouring B1 into C1 which is then full, while B2 remains half full). Have you both the same amount to drink?—*I've got more.*—But where does the extra come from?—*From in there* (B1).—What must we do so that Odette has the same?—*We must take that little glass* (pouring part of B3 into C2).—And is it the same now, or has one of you got more?—*Odette has more.*—Why?—*Because we've poured it into that little glass* (C2).—But is there the same amount to drink, or has one got more than the other?—*Odette has more to drink.*—Why?—*Because she has three glasses* (B3 almost empty, B4 and C2, while Clairette has C1 full and B2).'

A moment later, a new experiment. Clairette was again shown glasses A1 and A2, $\frac{3}{4}$ full, one with orangeade for herself and the other with lemonade for Odette. 'Are they exactly the same?—*Yes* (verifying the levels).—Well, Odette is going to pour hers (A2) into all those (C1, C2, C3, C4, which were thus about half full). Have you both the same amount?—*I've got more. She has less. In the glasses there's less* (looking carefully at the levels).—But before, you both had the same?—*Yes.*—And now?—*Here* (pointing to the level in A1) *it's more, and here* (indicating the 4 glasses C) *it's less.*'

Finally she was given only the big glass A1 almost full of orangeade: 'Look, Clairette does this: she pours it like that (into B1 and B2, which are then $\frac{4}{5}$ full). Is there more to drink now than before, or less, or the same?—*There's less* (very definitely).—Explain to me why.—*When you poured it out, it made less.*—But don't the little glasses together make the same?—*It makes less.*'

Sim (5;0). She was shown A1 and A2 half full. 'There's the same amount in the glasses, isn't there?—(She verified it) *Yes.*—Look, Renée, who has the lemonade, pours it out like this (pouring A1 into B1 and B2, which were thus about $\frac{3}{5}$ full). Have you both still the same amount to drink?—*No. Renée has more because she has two glasses.*—What could you do to have the same amount?—*Pour mine into two glasses.* (She poured A2 into B3 and B4.)—Have you both got the same now?—(She looked for a long time at the 4 glasses *Yes.*—Now Madeleine (herself) is going to pour her two glasses into three (B3 and B4 into C1, C2 and C3). Are they the same now?—*No.*—Who has more to drink?—*Madeleine, because she has three glasses. Renée must pour hers too into three glasses.* (Renée's B1 and B2 were poured into C5, C6 and C7). There. —*It's the same.*—But now Madeleine pours hers into a fourth glass (C4, which was filled with a little from C1, C2 and C3). Have you both the same amount?—*I've got more.*—Is there more of the lemonade (C5, C6 and C7) or of the orangeade (C1, C2, C3 and C4)?—*The orangeade.*—(The two big glasses A1 and A2 were then put before her.) Look, we're going to pour back all the lemonade into this one (A1) as it was before, and all the orangeade into that one. Where will the lemonade come up to?—(She indicated a certain level)—And the orangeade?—(She indicated a higher level.)—Will the orangeade be higher than the lemonade?—*Yes, there's more orangeade* (pointing to the level she had indicated) *because there's more orangeade here* (pointing to C1, C2, C3 and C4).—You think it will come up to here?—*Yes.*—(This level was marked by an elastic band and she herself poured in the liquid and was delighted to find that it came up to the band. But when she poured the lemonade into A1 she was very much surprised to find that it reached the same level.) *It's the same!*—How's that?—*I think we've put a little back, and now it's the same.*'

It is clear that so far the child had thought that there were changes in quantity when the number of glasses varied, but with the next question the level intervenes: 'Look, Madeleine pours the orangeade into that glass (A2 was poured into L, which was longer and narrower. L was then $\frac{3}{4}$ full, whereas the lemonade in A1 came only half way up.)—*There's more orangeade, because it's higher.*—Is there more to drink, or does it just look as if there is?—*There's more to drink.*—And now (pouring the lemonade into B1 and B2 and the orangeade into D1 and D2 which were wide, low glasses)?—*It's the orangeade that's, more, because there* (in D1 and D2) *there's a lot.*—So if we pour the lemonade and the orangeade back here (A2 and A1), will the orangeade come up higher or will they be the same?—*Higher.*' She poured D1 and D2 back into A2, and B1 and B2 back into A1, and was again much surprised to see that the levels were the same.

Lac (5;6). 'Here are two glasses (A1 half full of orangeade and A2 slightly less full of lemonade.) The orangeade is for you and the lemonade for Lucien. Lucien is cross because he has less. He pours his drink into these two glasses (pouring A2 into B1 and B2). Who has more?—(Lac looked at the levels) *Me.*—Now you pour your drink into these two glasses (B3 and B4, the levels being thus slightly higher than in B1 and B2). Who has more?—*Me.*—And now Lucien takes this glass (B1) and divides it between these two (C1 and C2, which are then full, whereas B2 remains half-full). Who has more?—(Lac compared the levels and pointed to glasses C) *Lucien.*—Why?—*Because the glasses get smaller* (and therefore the levels rise).—But how did that happen? Before it was you who had more and now it's Lucien?—*Because there's a lot.*—But how did it happen?—*We took some.*—But where?—. . .—And how?— . . . —Has one of you got more?—*Yes, Lucien* (very definitely).

—And if I pour all the orangeade and all the lemonade into the two big glasses (A1 and A2) who will have more?—*I shall* (thus showing that he remembered the original position).—Then where has the extra you had gone to?— . . . —What could you do to have the same amount as Lucien? You can use any of the glasses.—Lac then took B3 and poured some of it into C3, an empty glass. He filled it, and put it opposite Lucien's C1 and C2. Then he compared B3 to Lucien's B2 and saw that there was less in B3 than in B2. He then took C3 again, poured it back into B3, and then, showing great disappointment, cried: '*But why was it quite full there* (C3), *and now* (B3) *it isn't full any longer?*'

Mus (5;0). This child, like those quoted earlier, relied on the number of glasses or the level, but in her case as in several others there was also a new factor, the size of the glasses. Nevertheless she followed three successive lines of thought:

I. *Size of the containers.*—She was given A1 and A2, $\frac{3}{4}$ full: 'Is there the same amount in both of them?—*Yes.*—Olga pours hers out like this (A2 into B1 and B2, almost full). Has she still the same amount?—*No.* —Who has more to drink?—*Gertrude* (A1).—Why?—*Because she has a bigger glass.*—How is it that Olga has less?— . . . —And if I pour these (B1 and B2) back into that one (A2) how will it be?—*The same amount* (as in A1).—(I did so.) And if Olga pours it back again like this (A2 into B1 and B2, almost full) is it the same?—*No.*—Why?—*It makes less.*'

II. *Level.*—'Now Gertrude pours hers like this (A1 into C1 and C2, almost filling C1 and C2 and leaving A1 $\frac{1}{3}$ full). Who has more, Gertrude with those (A1 + C1 + C2), or Olga with those (B1 and B2)? —(She looked at the levels, which were about equal) *Both the same.*—Olga pours some of hers into another glass (B3, thus lowering the general level in her glasses).—*Gertrude will have more. Olga will have less.* —Olga pours again into these glasses (B1 and B2 into C3 and C4, which were then full).—*She will have more* (level).—But before she had less; has she more now?—*Yes.*—Why?—*Because we put back here* (C3 and C4) *what was in the big glasses* (B1 and B2).' The reasoning here was thus just the opposite of what it was in I.

III. *Number of glasses and level.*—'If I give you some coffee in one cup, will it still be the same if you pour it into two glasses?—*I'll have a little more.*—Where?—*In the two glasses of course.*—Mummy gives you two glasses of coffee (B1 and B2). Then you pour that one (B2) into those (C1 and C2).—*There's more there* (C1 and C2): *there are two glasses quite full. There, there's only one.*—And of those (B1 and the 4 C) which would you rather have, that one (B1) or all those (4 C)?—*The big one* (B1).—Why?—*Because there's more: the glass is big.*'

Such then are the earliest reactions of the child confronted with the problem of conservation of quantities. He is not prepared to believe that a given quantity of liquid remains the same irrespective of changes in shape when it is poured from one container to another.

It might of course be argued that the child may not really have grasped the question. Does he always understand that it refers to the total quantity, or does he think he is merely being asked about changes in the number, level or size of the glasses? But the problem is precisely to discover whether the child is capable of grasping a quantity as being a whole, as a result of the co-ordination of the various relationships he perceives. The fact that these children isolated one of these relationships may therefore be due as much to lack of understanding of the notions in question as to failure to grasp the verbal question.

On the other hand, it might be suggested that when the liquid is poured from one container to another before the eyes of the child there are certain illusions of perception that counteract his judgement as to conservation.[1] We are well aware that perception of the quantifiable qualities such as length, weight, etc., leads to systematic distortions, and that the child finds it extremely difficult to perceive the constancy of these qualities. Hence, when the constancy is directly perceived, there is no problem as far as we are concerned. Our only problem is to discover by what means the mind succeeds in constructing the notion of constant quantity in spite of the indications to the contrary provided by immediate perception. Judgement comes into play precisely when perception proves inadequate, and only then. For instance, the discovery that a given quantity of liquid does not vary when poured from a container A into one or two containers B of a different shape, requires on the part of the child an effort of intellectual understanding which will be the greater and the more easily analysable the more deceptive the immediate perception. We are therefore not concerned to discover why this perception is deceptive, but why children at a certain level accept it without question, whereas others correct it by the use of intelligence. Moreover, either perception must be studied 'from the angle of the object', in which case intelligence will in the final resort be the origin of the constancy, or else perception presupposes an organization which elaborates the constancy on its own plane, in which case the functioning and the successive structures of perception imply a sensory-motor activity that is intelligent from the start. If the latter is the case, the development of the notion of invariant quantities (like that of 'object') would be a continuation, on a new, abstract plane, of the work already undertaken by sensory-motor intelligence in the field of conservation of the object.

We shall attempt to interpret the examples given above from this second point of view. What is most striking at this first stage is the inadequate quantification of the perceived qualities, and the lack of co-ordination between the quantitative relations involved

[1] E. Brunswik, *Wahrnehmung und Gegenstandwelt*. Leipzig u. Wien, 1939.

in the perception. For example, Blas (4;0) begins by thinking that the quantity of liquid diminishes when the contents of a large glass three-quarters full are poured into two smaller glasses, but that it increases when poured from these small glasses into a long, narrow tube. It is therefore only the level and not the number or the cross section of the glasses which seems to be Blas's criterion. But a moment later he thinks there is more liquid in three small glasses than in two medium-sized ones filled with the same quantity. There are two noteworthy features in this reaction. In the first place, the child continually contradicts himself. At one moment he thinks there is more orangeade than lemonade, at another he thinks the opposite, and yet it does not occur to him to question his previous assumption. Obviously, if it is accepted that a liquid is capable of expansion or contraction and has no constancy, there is no contradiction. The real contradiction lies in the fact that the child attempts to justify his opposing statements by resorting to explanations that he cannot co-ordinate one with another, and that lead to incompatible statements. Thus Blas sometimes finds his evidence in the level of the liquid and thinks that the quantity diminishes when it is poured from a large glass into several small ones: sometimes he bases his statement on the number of glasses involved, in which case the same operation is thought to imply an increase in quantity. Alternatively, the child will use the cross section of the containers in his estimate of the change, disregarding the number of glasses and the level, and will then take one of these factors into account and arrive at the opposite conclusion. This brings us to the second feature of the reaction: the child behaves as though he had no notion of a multi-dimensional quantity and could only reason with respect to one dimension at a time without co-ordinating it with the others. What has been said is true not only of Blas, but of all the children quoted above.

The reactions of this stage can therefore be interpreted in the following way. We must first look for the principle of differentiation between quantity and quality, and this from the first perceptual contact with the object. In every case, perception and concrete judgement attribute qualities to objects, but they cannot grasp these qualities without thereby relating them one to another. These relations can only be of two kinds: symmetrical relations expressing resemblances, and asymmetrical relations expressing differences. Now resemblances between qualities can only result in their classification (e.g., glasses C1, C2, C3 . . . are 'equally small'), whereas asymmetrical differences imply 'more' and 'less' and thus indicate the beginnings of quantification (e.g., 'A1 is larger than B1' or 'A1 is narrower than D'). In its primitive form, therefore, quantity is given at the same time as quality, since it is constituted by the asymmetrical relations that necessarily link any qualities one to another. Qualities *per se* do not in fact exist; they are always compared and differentiated, and this differentiation, since it includes asymmetrical relations of differences, is the germ of quantity. From this point of view, the judgements characteristic of the first stage are obviously already quantitative in this sense. For instance, when Sim says: 'There's more orangeade because it's higher', she is merely expressing in terms of quantity a perceptual relation of difference between two qualities (the heights of the liquids).

At this first level, however, which we shall call that of 'gross quantity', quantification is restricted to the immediate perceptual relationships, just as 'gross quality' (i.e., directly perceived quality) is incapable of giving rise to a complete classification. The relationship of similarity between qualities will of course eventually result in a system of classes, but this only becomes possible with the elaboration of sequences of hierarchical inclusions involving the whole logic of classes and asymmetrical relations. As for the relations of difference, with which alone we are for the moment concerned, they will lead to a systematic quantification whose stages we shall study in subsequent chapters. But before this is achieved they must be able to satisfy two conditions that are lacking at this level, which accounts for the absence of measurable quantity and conservation.

The first of these conditions is that, from being mere perceptual relationships, they shall become true relations, thus giving rise to systems of graduations or 'intensive quantities'. (See Glossary.) Obviously a perceptual relationship does not as such constitute a relation. The criterion for the psychological existence of relations is the possibility of their composition, or in other words, the construction of their logical transitivity (or, if they cannot become transitive, the justification for their non-transitivity). The main characteristic of the perceptual relationships of gross quantity used by the child at this first level is that they cannot be composed one with another either additively or multiplicatively. When the child thinks that the quantity increases because the level rises, he is disregarding the cross section, and when he takes the cross section into account he disregards the level, and so on.[1]

The following experiment makes this plain. The child is given

[1] By addition of asymmetrical relations we mean their actual or virtual seriation and the resulting graduation of the seriated terms. By multiplication of these relations we mean their seriation from two or more points of view simultaneously. In the examples quoted above, the simple series do not appear, but the children had to compare two quantities from several points of view, height, cross-section, number of glasses, etc., which constitutes multiplication of relations.

two containers A and L of equal height, A being wide and L narrow. A is filled to a certain height (one-quarter or one-fifth) and the child is asked to pour the same quantity of liquid into L. The dimensions are such that the level will be four times as high in L as in A for the same amount of liquid. In spite of this striking difference in the proportions, the child at this stage proves incapable of grasping that the smaller diameter of L will require a higher level of liquid. Those children who are clearly still at this stage are satisfied that there is 'the same amount to drink' in L when they have filled it to the same level as A.

Blas (4;0): 'Look, your mummy has poured out a glass of lemonade for herself (A) and she gives you this glass (L). We want you to pour into your glass as much lemonade as your mummy has in hers.—(She poured rather quickly and exceeded the level equal to that in A that she was trying to achieve.)—Will you both have the same like that?—*No.*—Who will have more?—*Me.*—Show me where you must pour to so that you both have the same.—(She poured up to the same level.)—Will you and mummy have the same amount to drink like that?—*Yes.*—Are you sure?—*Yes.*—Now watch what I'm doing (putting L′ next to L). I'm going to pour that one (A) into this one (L′). Will that make the same here (L′) as there (L)?.—*Yes.*—(When I did so, the child laughed): *Mummy has more.*—Why?—. . .'

Mus (5;0): 'Look (same story as for Blas). Show me with your finger how far I must pour.—*There* (indicating the same level in L as in A).—(I filled it slightly higher). Will there be the same amount to drink?—*You've put too much. There's a little more there* (in L). *I've a little more to drink* —What could you do to see if it's the same? (putting L′ next to L).— . . . —Where will it come up to if we pour that one (A) into this one (L′)?—*To there* (pointing to the same level as in A).—(I did so.).—*Mummy has more* (with great surprise).—How did that happen?—*Because the glass* (L′) *is smaller.* (Mus thus appeared to have grasped the relation height × cross-section, but it was only a momentary glimpse, as we shall see.)—And if I pour this (L′) back into that (A), which will have the most?—*Both a little, both the same.*—(I poured it back). Who has more to drink?—*Both less.*'

These reactions show that the child at the first stage is unable to reckon simultaneously with the height and cross section of the liquids he has to compare. It is not that he fails to notice the width of glass A when circumstances oblige him to make the comparison (e.g. Mus, when A is poured into L′), but when he merely has to estimate the quantities in A and L′ he takes into account only the height.

The child at this stage has therefore not yet acquired the notion of multi-dimensional quantity, owing to lack of composition between the relationships of differences. For him the quantity of liquid does not depend on the combination of the various relations of height, cross section, number of glasses, etc., since each of these relations is considered separately, as though independent of the others. Each relation therefore constitutes merely a 'gross quantity' that is essentially uni-dimensional. Even when the child uses terms such as 'big' or 'large' this quality is still, as the case of Mus shows, merely perceptual data not susceptible of composition with others, and therefore again constituting a uni-dimensional 'gross quantity'.

The second condition to be satisfied before these perceptual relations can lead to true quantification, namely that there shall be partition into equal units, is even more impossible of fulfilment at this first stage than is intensive graduation. Before there can be acceptance of the notion of conservation of the liquid, and therefore construction of the notion of extensive quantity (see Glossary), there must be understanding that every increase in height is compensated by a diminution in width, these two qualities being inversely proportional. Yet even in the very simple problem of the increase in the number of glasses, children at this stage show clearly that they are unable to grasp the fact that a quantity of liquid poured from one glass into two or three smaller glasses remains the same. Composition by partition is therefore as impossible as by relations.

§3. *Stage II. Intermediary reactions*

Between the children who fail to grasp the notion of conservation of quantity and those who assume it as a physical and logical necessity, we find a group showing an intermediary behaviour (not necessarily found in all children) which will characterize our second stage. Two at least of these transitional reactions are worthy of note. The first of these shows that the child is capable of assuming that the quantity of liquid will not change when it is poured from glass A into two glasses B1 and B2, but when three or more glasses are used he falls back on to his earlier belief in non-conservation. The second reaction is that of the child who accepts the notion of conservation when the differences in level, cross section, etc., are slight, but is doubtful when they are greater. Here we have some examples of the first type:

Edi (6;4): 'Is there the same in these two glasses (A1 and A2)?—*Yes.*—Your mummy says to you: Instead of giving you your milk in this glass (A1), I give it to you in these two (B1 and B2), one in the morning and one at night. (It is poured out.) Where will you have most to drink, here (A2) or there (B1 + B2)?—*It's the same.*—That's right. Now, instead of giving it to you in these two (B1 and B2), she gives it to you in three (pouring A2 into C1, C2 and C3), one in the morning, one at lunch-time and one at night. Is it the same in the two as in the three, or not?—*It's the same in 3 as in 2 . . . No, in 3 there's more.*—Why?— . . . —(B1 and B2 were poured back into A1.) And if you pour the three (C1 + C2 + C3) back into that one (A2) how far up will it come?—(He pointed to a level higher than that in A1.)—And if we pour these 3 into 4 glasses (doing so into C1 + C2 + C3 + C4, with a consequent lowering of the level) and then pour it all back into the big one (A2), how far up will it come?—(He pointed to a still higher level.)—And with 5?—(He showed a still higher level.)—And with 6?—*There wouldn't be enough room in the glass.*'

Pie (5;0): 'Is there the same amount here (A1) and there (A2)?—(He tested the levels.) *Yes.*—(A1 was poured into B1 + B2). Is there the same amount to drink in these two together as in the other?—(He examined the levels in B1 and B2, which were higher than in A1.) *There's more here.*—Why?—*Oh yes, it's the same.*—And if I pour the two glasses (B1 and B2) into these three (C1 + C2 + C3), is it the same?—*There's more in the 3.*—And if I pour it back into the 2?—*Then there'll be the same* (B1 + B2) *as there* (A2).'

Here is an example of the second type:

Fried (6;5) agreed that A1 = A2. A1 was poured into B1 + B2. 'Is there as much lemonade as orangeade?—*Yes.*—Why?—*Because those* (B1 + B2) *are smaller than that* (A2).—And if we pour the orangeade (A2) as well into two glasses (doing so into B3 + B4, but putting more in B3 than in B4), is it the same?—*There's more orangeade than lemonade.*'—(B3 + B4 thus seemed to him more than B1 + B2).

A minute later he was given A1 half full, and A2 only a third full. 'Are they the same?—*No, three's more here* (A1).—(A1 was then poured into several glasses C.) *It's the same now as there* (A2).' He finally decided, however: '*No, it doesn't change, because it's the same drink* (i.e. A1 = C1 + C2 + C3 + C4 and A1 < A2).'

These two types of intermediary reactions are important and enable us to dismiss an objection that doubtless occurred to the reader in §2. Instead of concluding that the notion of conservation has its origin in quantification properly so called (itself the result of progressive co-ordination of the relations involved), could we not explain the absence of the notion as being due merely to failure to understand the question as referring to the quantity as a whole? The child might simply be comparing one level with another or one width with another, without considering the total quantity of liquid, but that would not necessarily prove that he was incapable of so doing. If this were so, as soon as the idea of the whole quantity made its appearance, the child would suddenly discover conservation; he would at once understand that the liquid remains the same since nothing is added to or subtracted from it. And indeed, when Edi and other children state, when first questioned, that (A2) and (B1 + B2) are 'the same', they give the impression that the difference between them and the children in §2 is due merely to the fact that they interpret the question differently. The correct solution would then be the result of a kind of immediate identification and there would be no need for a complex process of quantification. But the intermediary reactions of this second stage make it clear that this too simple interpretation is not valid. If the child hesitates, if he gives a correct answer when the variations are slight but does not assume conservation when the variation in shape is greater, it is obvious that he understands the question but is not convinced *a priori* of the constancy of the whole quantity.

This being so, how are we to interpret the progress shown by children at the second stage? The two conditions laid down in §2 as defining the transition from 'gross quantity' to true quantification are beginning to be fulfilled.

At this stage the child is attempting to co-ordinate the perceptual relations involved and thus to transform them into true, operational relations. Whereas the child at the first stage is satisfied that two quantities of liquid are equal if the two levels are the same, irrespective of the width of the containers, the child at the second stage tries to take the two relations into account simultaneously, but without success, hesitating continually between this attempt at co-ordination and the influence of the perceptual illusions. This reaction is already apparent in the most advanced children of the first stage, but generally speaking it is typical of the second period. Here we have some examples, of which the first belongs to the earlier stage:

Lac (5;6): 'Your brother Lucien has this orangeade (A, $\frac{1}{5}$ full). Pour the same amount for yourself into this glass (L).—(He filled L. to a higher level than that in A.) *No, I've got too much* (he poured some back so that L was $\frac{1}{5}$ full, i.e. the same level as in A).—Are they the same?—*No* (bringing L nearer to A and saying to himself): *Who has the most?*—Yes, who has the most?—(He pointed to A): *It's that one, because it's bigger.*—But you must have as much as Lucien.—(He added a little to L and compared the two levels.) *It's too much.* (He poured back the contents of L and began again. He gave himself the same level as in A, then added a little more so that L was about $\frac{2}{5}$ full.) *Oh! it's too much! It's not the same.* (In order to arrive at an equal quantity in L and A he then made the levels the same.)—You think you have the same amount to drink like that?—*Yes.*—(A was then poured into L'.)—*Oh! it's more!* (greatly astonished.)' Lac thus showed that he was still at the first stage, although his first reactions suggested the second stage.

Edi (6;4). Glass A was $\frac{1}{5}$ filled. 'Pour as much orangeade into this one (L) as there is there (A).—(He filled L to the same level as that in A.)—Is there the same amount to drink?—*Yes.*—Exactly the same?—*No.*—Why not?—*That one* (A) *is bigger.*—What must you do to have the