Life Cycle Costing

B.S. DHILLON

Gordon and Breach Science Publishers

LIFE CYCLE COSTING

LIFE CYCLE COSTING

Techniques, Models and Applications

by

B.S. Dhillon
Department of Mechanical Engineering
University of Ottawa
Ontario, Canada

GORDON AND BREACH SCIENCE PUBLISHERS
New York London Paris Montreux Tokyo Melbourne

Gordon and Breach Science Publishers

Post Office Box 786
Cooper Station
New York, New York 10276
United States of America

Post Office Box 161
1820 Montreux 2
Switzerland

Post Office Box 197
London WC2E 9PX
England

Private Bag 8
Camberwell,
Victoria 3124
Australia

58, rue Lomond
75005 Paris
France

3-14-9, Okubo
Shinjuku-ku, Tokyo
Japan

Library of Congress Cataloging-in-Publication Data

Dhillon, B.S.
Life cycle costing.

Includes bibliographies and index.
1. Engineering economy. 2. Cost control.
3. Product life cycle. I. Title.
TA177.7. D35 1988 658.1′552 88-21238
ISBN 2-88124-302-9

CONTENTS

PREFACE

Due to various reasons, the equipment procurement process, in general, is experiencing a new trend. This trend is based on the principle that a newly procured product must be supported for its total life years. In several cases, the product acquisition cost is less than the support cost over the life cycle years. Depending on the equipment type, the ownership cost over the life span may vary from 10 to 100 times the acquisition cost. Therefore, in the purchase of new major equipment, procurement management, particularly in the military, examines the total life cycle cost of the product rather than just its initial cost.

Over the years, many advances in life cycle costing have resulted in a large amount of published literature—witness the appendix of this book. This book covers these developments and recent related progress in the field. Topics of current interest are treated in such a manner that the reader needs no previous knowledge to understand the contents. Four chapters of the book provide background information on life cycle costing and on related areas. The source of most of the material presented is given in references for those readers who want to delve deeper into a specific area. The text cites over 1150 references, most listed in the Appendix, as well as some at the end of each chapter. In addition, the book contains several examples and their corresponding solutions. Throughout the text, emphasis is made on the structure of concepts rather than on minute details.

The book contains twelve chapters and the Appendix. Chapter 1 briefly discusses life cycle costing history, definitions, and scope of the book. It lists related periodicals, conference proceedings, and books. Chapter 2 reviews basic economic concepts useful for life cycle costing. Life cycle costing basics are described in Chapter 3.

Chapter 4 is devoted to life cycle models. These models are separated into two major categories: general life cycle cost models and specific life cycle cost models. Ten general and thirteen specific life cycle cost models are presented. Chapter 5 presents several cost estimation techniques. Fifteen techniques are described and sources for cost data are listed.

Chapter 6 is divided into three parts: manufacturing, quality, and maintenance costs. All three topics are covered in considerable depth. Chapter 7 is devoted to warranties. This chapter covers topics such as general warranties, reliability improvement warranties, and warranty-related cost models. Cost estimation models are discussed in Chapter 8. This chapter contains a total of twenty-five models ranging from estimating the equipment development cost to cost of centrifugal pumps.

Computer hardware and software costing is discussed in Chapter 9. Basically, the chapter is divided into two major parts: computer hardware costing and computer software costing. A list of software cost databases is given. The next two chapters, Chapters 10 and 11, deal with life cycle costing in the aircraft industry and life cycle costing of energy systems and on-surface vehicles, respectively. Chapter 10 covers topics such as cost drivers, aircraft life cycle cost, and models employed in estimating life cycle cost for United States Air Force aircraft systems, and Chapter 11 presents several life cycle cost models concerned with energy systems ranging from circuit breakers to water heaters. In addition, life cycle cost models for urban rail, automobile, cargo ship, etc., are presented.

Chapter 12 is completely devoted to reliability. Several important areas of reliability directly or indirectly useful for life cycle costing are covered. Some of those are reliability networks, maintainability measures, and availability.

This book should be useful to procurement, maintenance, reliability, design, maintainability, systems, chemical, electrical, computer, mechanical, civil, electronics, quality control, and aeronautical engineers, as well as senior students of engineering and business administration. Administrators, cost analysts, researchers, academics, and others should find it a valuable reference source.

ACKNOWLEDGMENTS

I wish to thank MCB University Press for granting permission to duplicate some of the references listed in the Appendix from an article by Y. Gupta and W.S. Chow, "Twenty-Five Years Life Cycle Costing—Theory and Applications: A Survey", *The International Journal of Quality & Reliability Management*, Vol. 2, 1985, pp. 51–76. I am indebted to my relatives and friends for their interest and encouragement during moments of need. I thank my wife, Rosy, for typing the original manuscript and for her ever present assistance during the preparation of the manuscript.

CHAPTER 1

Introduction

1.1 Brief Life Cycle Costing History

It was almost a quarter century ago when wide dissemination of the term life cycle costing was given in a report [1] entitled "Life Cycle Costing in Equipment Procurement". This document was the result of a study conducted by the Logistics Management Institute, Washington, D.C. for the Assistant Secretary of Defense for Installations and Logistics. As a result of this report, a series of three guidelines [2–4] for life cycle costing procurement was published by the United States Department of Defense. These guidelines were entitled "Life Cycle Costing Procurement Guide (interim)", "Life Cycle Costing in Equipment Procurement-Casebook", and "Life Cycle Costing Guide for System Acquisitions (interim)". In 1971, the Directive 5000.1 [5] entitled "Acquisition of Major Defense Systems" was issued by the Department of Defense. This directive established the requirement for life cycle costing procurement for major defense systems acquisitions.

A project entitled "Life cycle budgeting and costing as an aid in decision making" was initiated by the United States Department of Health, Education, and Welfare in 1975 [6]. In 1974, the state of Florida formally adopted life cycle costing and in 1978, the United States Congress established the National Energy Conservation Policy Act. This Act requires every new federal building be life cycle cost effective.

Since 1974, many states in the United States have passed legislations making life cycle cost analysis mandatory in the planning, design, and construction of state buildings. Some of these states are New Mexico, Alaska, Maryland, North Carolina, and Texas. Over the years many other events have taken place in the history of life cycle costing and a large amount of published literature on the subject have appeared. References [7, 8] list a large volume of published literature on life cycle costing. An extensive list of publications on life cycle costing and related areas is presented in the Appendix.

1.2 Periodicals and Conference Proceedings

Information on life cycle costing can be found in various journals, conference proceedings, books, and reports. To the best of author's knowledge there are no periodicals or annual conferences specifically on the subject of life cycle costing. However, there are several journals and conference proceedings which time to time publish articles on the subject in question. Some of those journals and conference proceedings are listed below:

Journals

i) Microelectronics and Reliability: An International Journal
ii) IEEE Transactions on Reliability
iii) Defense Management Journal

Conference Proceedings

i) Proceedings of the Annual Reliability and Maintainability Symposium, USA.
ii) Proceedings of the Annual International Reliability, Availability, and Maintainability (Inter-RAM) Conference for the Electric Power Industry
iii) Proceedings of the Annual American Society for Quality Control (ASQC) Conference

1.3 Books

So far there has been only a limited number of books written on life cycle costing. Those books are listed below:

i) M.R. Seldon, Life Cycle Costing: A Better Method of Government Procurement, Westview Press, Boulder, Colorado, 1979.
ii) R.J. Brown, R.R. Yanuck, Life Cycle Costing: A Practical Guide for Energy Managers, The Fairmont Press, Inc., Atlanta, Georgia, 1980.
iii) B.S. Blanchard, Design and Manage to Life Cycle Cost, M/.A Press, Portland, Oregon, 1978.
iv) M.E. Earles, Factors, Formulas, and Structures for Life Cycle Costing, Second Edition, Eddins-Earles, Privately Published, Concord, Massachusetts, 1981.

v) A.J. Dell'isola, S.J. Kirk, Life Cycle Costing for Design Professionals, McGraw-Hill Book Company, New York, 1981.

A chapter on life cycle costing is given in the following books:

i) B.S. Dhillon, Reliability Engineering in Systems Design and Operation, Van Nostrand Reinhold Company, New York, 1983, pp. 210–238.
ii) B.S. Dhillon, H. Reiche, Reliability and Maintainability Management, Van Nostrand Reinhold Company, New York, 1985, pp. 214–235.
iii) H. Reiche, Life Cycle Cost, in Reliability and Maintainability of Electronic Systems, Edited by J.E. Arsenault and J.A. Roberts, Computer Science Press, Potomac, Maryland, 1980, pp. 3–23.

1.4 Definitions

This section presents selected definitions directly or indirectly related to life cycle costing [9–14].

Life Cycle Cost: the sum of all costs incurred during the life time of an item, i.e., the total of procurement and ownership costs.

Procurement Cost: the total of investment or acquisition costs (recurring and non-recurring).

Ownership Cost: the sum of all costs other than the procurement cost during the life time of an item.

Recurring Cost: the cost which recurs periodically during the life of a project.

Reliability: the probability that an item will carry out its mission satisfactorily for the desired period when used according to specified conditions.

Failure: the termination of the ability of an item to carry out its stated mission or function.

Maintenance Cost: the labor and materials expense required to maintain item(s) in suitable use condition.

Manufacturing Cost: the sum of fixed and variable costs chargeable to the production of a specified item.

Maintainability: the probability that a failed item will be restored to its satisfactory operational state within a specified total downtime when maintenance action is started according to stated conditions.

Redundancy: the existence of more than one means for carrying out a stated function.

Cost Model: an approach, based on technical and programmatic parameters, for computing concerned costs.

Nonrecurring Cost: the cost that is not repeated.

Maintenance: all scheduled and unscheduled actions appropriate for keeping an item in a serviceable condition or restoring it to serviceability. It includes inspection, repair, modification, servicing, remove and replace, etc.

Annuity: a series of equal payments, at equal intervals.

Cost: the amount of money paid or payable for the acquirement of services, materials, or property.

Downtime: the total time during which the product (or item) is not in a condition to carry out its stated function or mission.

Failure Rate: the number of failures of a product per unit measure of life (e.g. hours).

Active Redundancy: the term used when all redundant items are operating simultaneously.

Battery-Limits: a geographical designation for the process area of the project proposed.

Cost Estimating Relationship: an equation relating cost as the dependent variable to one or more independent variables.

Mean Time to Repair: the mean time needed to repair a product (or an item).

Repair Cost: the cost of restoring an item or a facility to its original condition or performance.

Compound Amount: the future value of money invested or loaned at compound interest.

1.5 Scope of the Book

Nowadays, life cycle costing is receiving increasing attention in government setups, industry, etc. Most of the information on life cycle costing is available in the form of technical papers or reports and there are only a limited number of books specifically dealing with life cycle costing. All of those books appeared during the period from 1978–1981. Since 1981, to the best of author's knowledge not a single text appeared on the topic but there have been many articles and reports. Professionals and others involved in life cycle costing need up to date information on the subject in question and generally face a great deal of difficulty. This book is an attempt to fulfill the current need. The book is written after reviewing the available literature on life cycle costing.

Therefore, every attempt was made to cover important past and current issues in life cycle costing. Previous knowledge is not generally necessary to comprehend the contents, since two chapters on basic economics and reliability theory are provided to give sufficient background. This book will find use in many disciplines and will be useful to reliability, maintainability, and quality control engineers, managers, procurement specialists, cost analysts, project, maintenance, design, system, computer hardware and software, chemical, electrical, electronic, civil, and mechanical engineers, researchers and university level teachers, and senior students of engineering and business administration.

1.6 Summary

This chapter presented a brief history of life cycle costing. A list of journals and conference proceedings publishing articles on life cycle costing is given along with books concerned with the topic in question. Several definitions directly or indirectly related to life cycle costing are presented. The scope of the text is discussed.

1.7 Problems

1. Write an essay on the history of life cycle costing.
2. Define the terms given below:

    i) Life cycle costing
    ii) Interest rate
    iii) Disposal cost
    iv) Economy
    v) Development cost
    vi) Mean time between failures
    vii) Cost drivers
    viii) Warranty
    ix) Design to cost
    x) Direct cost

1.8 References

1. "Life Cycle Costing in Equipment Procurement", Report No. LMI Task 4C-5, Prepared by Logistics Management Institute (LMI), Washington, D.C., April, 1965.

2. "Life Cycle Costing Procurement Guide (interim)", Department of Defense Guide No. LCC-1, United States Department of Defense, Washington, D.C., July 1970.
3. "Life Cycle Costing in Equipment Procurement-Casebook", Department of Defense Guide No. LCC-2, United States Department of Defense, Washington, D.C., July 1970.
4. "Life Cycle Costing Guide for System Acquisitions (interim)", Department of Defense Guide No. LCC-3, United States Department of Defense, Washington, D.C., January 1973.
5. "Acquisition of Major Defense Systems", Department of Defense Directive No. 5000.1, United States Department of Defense, Washington, D.C., July, 1971.
6. M.E. Earles, Factors, Formulas, and Structures for Life Cycle Costing, Published by Eddins-Earles, 89 Lee Drive, Concord, Mass. 01742.
7. B.S. Dhillon, Life Cycle Cost: A Survey, Microelectronics and Reliability, Vol. 21, 1981, pp. 495–511.
8. B.S. Dhillon, Reliability Engineering in Systems Design and Operation, Van Nostrand Reinhold Company, New York, 1983, pp. 210–238.
9. K.K. Humphreys, P. Wellman, Basic Cost Engineering, Marcel Dekker, Inc., New York, 1987.
10. R.D. Stewart, R.M. Wyskida, Cost Estimator's Reference Manual, John Wiley & Sons, New York, 1987.
11. M.R. Seldon, Life Cycle Costing: A Better Method of Government Procurement, Westview Press, Boulder, Colorado, 1979.
12. R.J. Brown, R.R. Yanuck, Life Cycle Costing, The Fairmont Press, Inc., Atlanta, Georgia, 1980.
13. K.K. Humphreys, Project and Cost Engineers' Handbook, Marcel Dekker, Inc., New York, 1984.
14. "Aircraft Engine Life Cycle Cost", Document No. SP-721, Society of Automotive Engineers, Inc., Warrendale, Pennsylvania, 1987.

CHAPTER 2

Life Cycle Costing Economics

2.1 Introduction

The discipline of economics plays an important role in life cycle costing. In order to calculate life cycle cost of items various types of economics related information is required.

Life cycle costing requires that future costs have to be calculated by taking into consideration the time value of money. This is due to the fact that same sum of money received or spent at various different points in time will have different values. More specifically, the sum of money in today's dollars, usually, will not have the same value a year later and thereon. Taking this one step further, one may state that the future value of present dollars will be greater because of earned interest or smaller because of inflation. Similarly, the present value of a future sum of money would be generally less. In life cycle costing, the future costs such as operation and maintenance have to be converted to their appropriate values before adding them to the item's procurement cost. There are a number of formulas developed in economics to convert money from one point of time to another. In life cycle costing studies such formulas are indispensible.

This chapter presents several aspects of economics useful for life cycle costing.

2.2 Early History of Interest

The concept of interest is not new as many of us may believe. Just like taxes its history goes back thousands of years. The early history of man reveals that it existed, in Babylon, two thousand years before the birth of Jesus Christ (i.e. 2000 B.C.). In those days interest was paid on borrowed commodities (e.g. grain) in form of grain or other means. By the year 575 B.C., the concept of interest was so well developed that led it to the formation of a company of international bankers with home offices in Babylon [1]. The firm derived its income by charging high interest rates on its money used for financing international trade.

2.3 Simple and Compound Interest

The simple interest is the simplest form of interest. What it means is that the interest is paid only on the original sum of money borrowed, and not on the accrued interest. In the case of simple interest, the total interest earned or paid is given by

$$TI = (PA)(j)(m) \tag{2.1}$$

where

TI is the total interest
m is the interest periods (e.g. years)
j is the interest rate (per specified period)
PA is the principal amount (i.e. borrowed or lent)

The total amount, TA, of money at the end of m years can be obtained from the following equation:

$$TA = PA + TI \tag{2.2}$$

Substituting Equation (2.1) into Equation (2.2) yields

$$\begin{aligned} TA &= PA + (PA)(j)(m) \\ &= PA(1 + jm) \end{aligned} \tag{2.3}$$

Example 2.1

A person loaned \$2,000 for the period of 4 years at an annual simple interest rate of 15%. Calculate the total amount of money at the end of 4 years.

Using Equation (2.1) and the given data, the total interest earned is

$$\begin{aligned} TI &= (2{,}000)(0.15)(4) \\ &= \$1{,}200 \end{aligned}$$

Adding the above result to the original amount \$2,000, the Equation (2.3) yields

$$\begin{aligned} TA &= 2{,}000 + 1{,}200 \\ &= \$3{,}200 \end{aligned}$$

Thus, the total amount of money at the end of 4 years will be \$3,200.

In the case of compound interest, the interest earned during each interest period is added (at the end of the period) to the principal amount and thereafter it starts earning interest itself for the remaining term of the loan or investment. In order to develop a formula for the total amount, we assume that present or principal amount, PA, is invested or loaned at compound interest rate i per interest period (usually year). In this case the total amount, TA_1, of money at the end of first interest period (e.g. year) will be

$$\begin{aligned} TA_1 &= PA + (PA)i \\ &= PA(1+i) \end{aligned} \tag{2.4}$$

The total amount, TA_2, of money at the end of second interest period (e.g. year) will be

$$\begin{aligned} TA_2 &= (\text{original principal}) + \begin{bmatrix} \text{interest for period} - 1 \\ \text{(e.g. first year)} \end{bmatrix} \\ &\quad + \begin{bmatrix} \text{original principal} \\ \text{plus interest} \\ \text{for period} - 1 \end{bmatrix} (\text{interest rate}) \\ &= PA + PA(i) + (TA_1)i \end{aligned} \tag{2.5}$$

Substituting Equation (2.4) into Equation (2.5) leads to

$$\begin{aligned} TA_2 &= PA + (PA)(i) + PA(1+i)i \\ &= PA + (PA)(i) + (PA)(i) + (PA)(i)^2 \\ &= (PA)[1 + 2i + i^2] \\ &= (PA)(1+i)^2 \end{aligned} \tag{2.6}$$

Similarly, the total amount, TA_3, of money at the end of third interest period (e.g. year) will be

$$TA_3 = TA_2 + (TA_2)i \tag{2.7}$$

Substituting Equations (2.6) into Equation (2.7) results in

$$\begin{aligned} TA_3 &= (PA)(1+i)^2 + (PA)(1+i)^2 i \\ &= PA(1+i)^2[1+i] \\ &= (PA)(1+i)^3 \end{aligned} \tag{2.8}$$

Thus, the total amount, TA_m, of money at the end of mth interest period (e.g. year) will be

$$TA_m = (PA)(1+i)^m \tag{2.9}$$

where m is the number of interest periods. From the above equation the total amount, TA_{m-1}, of money at the beginning of mth interest period (e.g. year) is

$$TA_{m-1} = (PA)(1+i)^{m-1} \tag{2.10}$$

The interest for the mth period (e.g. year) is

$$I_m = (1+i)^{m-1}(PA)(i) \tag{2.11}$$

where I_m is the interest for the mth period (e.g. year). In cases when the interest is compounded more than once in a year, the given yearly interest rate, say i, must be divided by the number of times the compounding is taking place during the year. To take this factor into account, Equation (2.9) is modified to the following form [2]:

$$TA_m = (PA)\left(1+\frac{i}{k}\right)^{mk} \tag{2.12}$$

where

k is the number of times the interest is compounded per year
i is the annual interest rate
m is the number of years
PA is the principal amount
TA_m is the total amount of money at the end of year m

Example 2.2

Assume that six thousand dollars are deposited in a bank. The annual interest rate is 12%. If the interest is compounded quarterly during a year, calculate the total amount of money at the end of the 4th year.

In this example we have $PA = \$6{,}000$, $i = 12\%$, $k = 4$, and $m = 4$ years. Thus, substituting these data into Equation (2.12) yields

$$\begin{aligned} TA_4 &= (6{,}000)\left[1+\frac{0.12}{4}\right]^{(4)(4)} \\ &= \$9{,}628.24 \end{aligned}$$

Thus, the total amount of money after 4 years will be \$9,628.24.

Example 2.3

A person deposited \$4,000 in a bank at annual interest rate of 9%, compounded annually. Calculate the total amount of money at the end of the 10th year, the interest earned during the 10th year, and the total amount at the beginning of the 10th year.

In this example, the following data are specified:

$m = 10$ years
$i = 9\%$ per year
$PA = \$4,000$

Using these data in Equation (2.9) yields

$$TA_{10} = (4{,}000)(1 + 0.09)^{10}$$
$$= \$9{,}469.455$$

Thus, the total amount of money at the end of the 10th year will be \$9,469.455.

From Equation (2.11), the interest earned during the 10th year is

$$I_{10} = (1 + 0.09)^{10-1}(4{,}000)(0.09)$$
$$= \$781.882$$

Finally, the total amount at the beginning of the 10th year from Equation (2.10) is

$$TA_9 = (4{,}000)(1 + 0.09)^{10-1}$$
$$= \$8{,}687.573$$

2.4 Nominal and Effective Interest Rates

The nominal interest rate may simply be described as an annual interest rate without considering the effect of any compounding during the year.

On the other hand, the effective yearly interest rate may be described as the true annual interest rate, considering the effect of all compounding during the year. The effective annual interest rate can be estimated from the following equation:

$$(1 + i_{ef}) = \left(1 + \frac{i}{k}\right)^k \qquad (2.13)$$

The above equation is developed by reasoning that the effective interest rate compounded once in a year generates the same amount of interest as a nominal interest rate compounded k times in a year.

Rearranging Equation (2.13) yields

$$i_{ef} = \left(1 + \frac{i}{k}\right)^k - 1 \tag{2.14}$$

The symbols used in Equations (2.13) and (2.14) are defined below:

i_{ef} is the effective annual interest rate
i is the nominal interest rate (per year)
k is the number of interest periods in a year

From Equation (2.14), it is to be noted that at $k = 1$, $i_{ef} = i$. It means the nominal (yearly) interest rate is same as the effective (annual) interest rate.

Nowadays, a number of financial institutions offer several interest periods per year. One example is daily compounding. The interest compounds continuously as k approaches infinity. In limit, if interest were compounded continuously, from Equation (2.14) we get

$$i_{ef} = \lim_{k \to \infty} \left(1 + \frac{i}{k}\right)^k - 1 \tag{2.15}$$

The right hand term is rewritten in the following form:

$$\left(1 + \frac{i}{k}\right)^k - 1 = \left[\left(1 + \frac{i}{k}\right)^{k/i}\right]^i - 1 \tag{2.16}$$

Substituting Equation (2.16) into Equation (2.15) yields

$$i_{ef} = \lim_{k \to \infty} \left[\left(1 + \frac{i}{k}\right)^{k/i}\right]^i - 1 \tag{2.17}$$

Since $\lim_{k \to \infty} (1 + i/k)^{k/i} = 2.71828 = e$, the above equation leads to

$$i_{ef} = e^i - 1 \tag{2.18}$$

Taking natural logarithms of Equation (2.18) and rearranging yields

$$i = \ln(i_{ef} + 1) \tag{2.19}$$

Example 2.4

Assume that a person deposited 15,000 dollars in a bank at a nominal interest rate of 12% compounded daily, for one year. Calculate the effective annual interest rate.

In this example, we have $i = 12\%$, $k = 365$ days, and $PA = \$15{,}000$. Using these data into Equation (2.14) results in

$$i_{ef} = \left(1 + \frac{0.12}{365}\right)^{365} - 1 = 0.127475 \text{ or } 12.7475\%$$

Alternatively, from Equation (2.18) we get

$$i_{ef} = e^{(0.12)} - 1 = 0.127497 \text{ or } 12.7497\%$$

In this case Equations (2.14) and (2.18) yielded very similar results. The effective annual interest rate may be taken as 12.75%.

2.5 Formulas for Life Cycle Cost Analysis

This section presents several formulas considered useful for performing life cycle cost analysis.

2.5.1 Formula for Finding Future Worth (Single Payment)

This formula was developed earlier in the chapter and the future worth (compound-amount) from Equation (2.9) is

$$FW = TA_m = (PA)(1 + i)^m \tag{2.20}$$

where

FW is the future worth (i.e., principal plus interest due)
PA is the principal
i is the compound interest rate per period
m is the number of interest periods

2.5.2 Formula for Finding Present Worth (Single Payment)

From Equation (2.20), the present worth of a future sum of money is

$$PW = PA = \frac{FW}{(1 + i)^m} \tag{2.21}$$

where PW is the present worth.

Example 2.5

Assume that the total maintenance cost of an equipment at the end of its seven year operation will be \$80,000. More specifically, this is the amount which will occur seven years from now. The estimated annual compound interest rate is 10%. Compute the present worth of the 80,000 maintenance dollars.

Using Equation (2.21) for $i = 10\%$, $m = 7$ years, and $FW =$ \$80,000, we get

$$PW = \frac{80{,}000}{(1 + 0.1)^7}$$
$$= \$41{,}052.65$$

Thus, the present value of the total maintenance cost is \$41,052.65.

2.5.3 Formula for Finding Future Worth (Uniform Periodic Payments)

This formula is used to determine future worth at the end of m interest periods (in this formula an interest period is assumed one year) of equal payments made at the end of each of those m interest periods (years). The payments are deposited at interest rate i per year, compounded annually. This formula is developed as follows:

First Year

At the end of the first year, the first payment is made and the amount will be

$$AM_1 = PY \tag{2.22}$$

where

AM_1 is the amount at the end of the first year
PY is the payment made at the end of a year

Second Year

At the end of the second year, the second payment is made, as well as, interest is earned on AM_1, the amount will be

$$AM_2 = PY + AM_1(1 + i) \tag{2.23}$$

where

AM_2 is the amount at the end of the second year
i is the annual compound interest rate

Substituting Equation (2.22) into Equation (2.23) yields

$$AM_2 = PY + PY(1 + i) \tag{2.24}$$

Third Year

At the end of the third year, the third payment is made, as well as, interest is earned on AM_2, the amount will be

$$AM_3 = PY + AM_2(1 + i) \tag{2.25}$$

Substituting Equation (2.24) into Equation (2.25) leads to

$$\begin{aligned} AM_3 &= PY + [PY + PY(1 + i)](1 + i) \\ &= PY + PY(1 + i) + PY(1 + i)^2 \end{aligned} \tag{2.26}$$

Fourth Year

At the end of the fourth year, the fourth payment is made, as well as, interest is earned on AM_3, the amount will be

$$AM_4 = PY + AM_3(1 + i) \tag{2.27}$$

Substituting Equation (2.26) into Equation (2.27) results in

$$\begin{aligned} AM_4 &= PY + [PY + PY(1 + i) + PY(1 + i)^2](1 + i) \\ &= PY + PY(1 + i) + PY(1 + i)^2 + PY(1 + i)^3 \end{aligned} \tag{2.28}$$

Now, it is obvious that the above expression can be generalized.

***m*th Year**

At the end of the mth year, the mth payment is made, as well as, interest is earned on AM_{m-1}, the amount will be

$$AM_m = PY + PY(1 + i) + \cdots + PY(1 + i)^{m-2} + PY(1 + i)^{m-1} \tag{2.29}$$

The above expression is a geometric series which can be summed as follows:

Multiply both sides of Equation (2.29) by $(1 + i)$ to get

$$(1 + i)AM_m = PY(1 + i) + PY(1 + i)^2 + \cdots + PY(1 + i)^{m-1} + PY(1 + i)^m \tag{2.30}$$

Subtracting Equation (2.29) from Equation (2.30) gives

$$(1 + i)AM_m - AM_m = PY(1 + i)^m - PY \tag{2.31}$$

Rearranging Equation (2.31) yields

$$FW = AM_m = \frac{PY[(1 + i)^m - 1]}{i} \tag{2.32}$$

where FW is the future worth.

Example 2.6

Assume that for next 10 years, a person will deposit $5,000 at the end of each of those years. The interest rate per year will be 10% compounded yearly. Find the total sum of money at the end of the specified 10-year period.

In this example, we have $PY = \$5{,}000$, $i = 10\%$, and $m = 10$ years. Using these data in Equation (2.32) will give

$$FW = AM_{10} = \frac{5{,}000[(1 + 0.1)^{10} - 1]}{0.1}$$
$$= \$79{,}687.123$$

Thus, the total sum of money at the end of the 10-year period will be $79,687.123.

2.5.4 Formula for Finding Present Worth (Uniform Periodic Payments)

This formula is concerned with determining the present value of equal payments made at the end of each of m interest periods (in this formula an interest period is assumed one year). All payments are deposited at annual interest rate i compounded annually. The present worth formula is developed below:

First Year

At the end of the first year, the first payment is made and its present worth from Equation (2.21) will be

$$PW_1 = \frac{PY}{(1 + i)} \tag{2.33}$$

where

PW_1 is the present worth of the payment, PY, made at the end of the first year
i is the annual compound interest rate

Second Year

At the end of the second year, the second payment is made and its present worth from Equation (2.21) will be

$$PW_2 = \frac{PY}{(1 + i)^2} \tag{2.34}$$

where PW_2 is the present worth of the payment, PY, made at the end of the second year.

Third Year

At the end of the third year, the third payment is made and its present worth from Equation (2.21) will be

$$PW_3 = \frac{PY}{(1+i)^3} \tag{2.35}$$

where PW_3 is the present worth of the payment, PY, made at the end of the third year.

From Equations (2.33), (2.34), and (2.35), it is quite clear that, similarly, we can write down a present value equation for the mth payment.

*m*th Year

At the end of the mth year, the mth payment is made and its present worth from Equation (2.21) will be

$$PW_m = \frac{PY}{(1+i)^m} \tag{2.36}$$

From Equations (2.33)–(2.36) the present worth, PWA, of all payments is

$$\begin{aligned} PWA &= PW_1 + PW_2 + PW_3 + \cdots + PW_m \\ &= \frac{PY}{(1+i)} + \frac{PY}{(1+i)^2} + \frac{PY}{(1+i)^3} + \cdots + \frac{PY}{(1+i)^m} \end{aligned} \tag{2.37}$$

This is a geometric series and can be summed as follows:

Multiply both sides of Equation (2.37) by $1/(1+i)$ to get

$$\frac{PWA}{(1+i)} = \frac{PY}{(1+i)^2} + \frac{PY}{(1+i)^3} + \frac{PY}{(1+i)^4} + \cdots + \frac{PY}{(1+i)^{m+1}} \tag{2.38}$$

Subtracting Equation (2.37) from Equation (2.38) yields

$$\frac{PWA}{(1+i)} - PWA = \frac{PY}{(1+i)^{m+1}} - \frac{PY}{(1+i)} \tag{2.39}$$

Rearranging Equation (2.39) results in

$$PWA = PY\left[\frac{1-(1+i)^{-m}}{i}\right] \tag{2.40}$$

The correctness of the above equation can easily be checked using Equation (2.32). The present value of the right-hand side of Equation (2.32) must be the same as the right-hand side of Equation (2.40). This is demonstrated below:

$$PY\left[\frac{(1+i)^m - 1}{i}\right]\frac{1}{(1+i)^m} = PY\left[\frac{1-(1+i)^{-m}}{i}\right]$$

$$PY\left[\frac{1-(1+i)^{-m}}{i}\right] = PY\left[\frac{1-(1+i)^{-m}}{i}\right] \tag{2.41}$$

Example 2.7

Use the data given in Example 2.6 to find present worth of all payments.

Thus, using the data specified in Example 2.6 in Equation (2.40) yields

$$PWA = 5{,}000\left[\frac{1-(1+0.1)^{-10}}{0.1}\right]$$
$$= \$30{,}722.836$$

The present worth of all payment is \$30,722.836.

2.5.5 Formula for Finding the Value of Annuity Payments When the Future Worth of the Annuity is Known

An annuity may be described as a series of equal payments, at equal intervals. Rearranging Equation (2.32), we get the value of annuity payments expressed as

$$PY = \frac{FW(i)}{(1+i)^m - 1} \tag{2.42}$$

PY is the value of annuity payments
FW is the future worth of the annuity
i is the annual compound interest rate
m is the number of interest periods (years)

Example 2.8

A company desires to purchase a facility at the end of next six years. The estimated cost of the facility is \$250,000 and the company has decided to make deposits of equal sum of money at the end of each of next six years so that the total sum accumulates to \$250,000. If the annual interest rate is 12% compounded annually, calculate the amount of money to be deposited at the end of each year.

In this example, we have $FW = \$250{,}000$, $i = 12\%$, and $m = 6$ years. Substituting these data into Equation (2.42) results in

$$PY = \frac{(250{,}000)(0.12)}{[(1+0.12)^6 - 1]}$$

$$= \$30{,}806.43$$

Thus, the amount of money to be deposited at the end of each year is \$30,806.43.

2.5.6 Formula for Finding the Value of Annuity Payments When the Present Worth of the Annuity is Known

Rearranging Equation (2.40), the value of annuity payments is

$$PY = \frac{(PWA)(i)}{1-(1+i)^{-m}} \tag{2.43}$$

where

PY is the value of annuity payments
PWA is the present worth of the annuity
m is the number of interest periods (years)
i is the compound interest rate per year

Example 2.9

Assume that in Equation (2.43), we have $PWA = \$80{,}000$, $i = 10\%$, and $m = 10$ years. Calculate the value of annuity payments. Utilizing these data in Equation (2.43) yields

$$PY = \frac{(80{,}000)(0.1)}{[1-(1+0.1)^{-10}]}$$

$$= \$13{,}019.63$$

Thus, the value of annuity payments is \$13,019.63.