

Geometry as Objective Science in Elementary School Classrooms

Mathematics in the Flesh

Wolff-Michael Roth



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Preface

This book is about early learning experiences in and with geometry. I exhibit the origins of geometry as *objective* science in and out of the intuitively given everyday lifeworlds of children. These lifeworlds are pre-geometric; but they are not without model objects that denote and come to anchor geometric idealities that the children will understand at later points in their lives. That is, my analyses show how geometry, an objective science, arises anew from the pre-scientific but nevertheless methodic actions of children in a structured world always already shot through with significations. More importantly, though, I present a way of understanding knowing and learning in mathematics that differs from other current approaches, although it shares aspects with other theories, most notably enactivism and embodiment approaches.¹ I develop this new way of understanding with materials collected during a three-week unit of geometry in a second-grade elementary classroom. I use these materials to both articulate the contradictions and incongruences of other theories—Immanuel Kant, Jean Piaget, and more recent forms of (radical, social) constructivism, embodiment theories, and enactivism. I show how *material phenomenology* fused with *phenomenological sociology* provides answers to the problems that these other paradigms do not answer.

Projects such as this book do not emerge from a vacuum, but are based on years of developing understandings. In my case, there are more than 25 years of intense study of concrete data on learning, collected under controlled and field conditions (classroom, everyday work). In the course of this analytic work, numerous theoretical frameworks that I have come to know and have worked through did not hold up to the empirical evidence, encouraging me to seek new ways of theorizing and understanding the data sources I collected. Among those theoretical paradigms that I worked extensively with are Piagetian and neo-Piagetian theories, information processing theory, artificial neural networks, radical constructivism, social constructivism, situated/distributed cognition, embodiment/enactivist theories, and cultural-historical activity theory. The theories that have stuck with me, and which I contributed to developing, are characterized by two aspects: (a) they are phenomenological, view the world from the perspective of the

actor (which tends not to be one of theoretical reflection on the ongoing action) and (b) they focus on the social/societal aspects of life, emphasizing that all consciousness, as the etymology of the term suggests, is *knowing with* and *for* the other. It is impossible to trace one's understanding back to everything one has read or encountered. But across this book, the main philosophical trends that have shaped my thinking are recognized in the references—but the same reference is not likely to appear repeatedly unless there is a specific need (e.g., paraphrase) or purpose (quote).

Some chapters developed from the original versions—sometimes vastly longer drafts—of three published papers. In the historically first of the three entitled “Bodily Experience and Mathematical Conceptions: From Classical Views to a Phenomenological Reconceptualization” (Roth & Thom, 2009a)², I developed a new theory of mathematical concepts and conceptions; I subsequently realized that this framework needed to be pushed even further. In “The Emergence of 3D Geometry from Children’s (Teacher-guided) Classification Tasks” (Roth & Thom, 2009b)³, I developed the idea of how geometry as objective science can be thought of as emerging from the non-geometrical practices that children bring to school. Both papers were, taking up a comment from Hans Freudenthal concerning his own work, “a bit phenomenologically tainted.” Subsequent to writing these two papers, I became familiar with the work of the French philosopher Michel Henry and his commentaries on another French philosopher, Pierre Maine de Biran. Together with the writings of three further French philosophers—Jean-Luc Nancy, Jacques Derrida, Jean-Luis Chrétien—on the role of the body generally and the sense of touch (tact) more specifically to knowing led me to propose a radicalization of theories of mathematical knowing. In “Incarnation: Radicalizing the Embodiment of Mathematics” (Roth, 2010)⁴, I formulated mathematical knowing for the first time in terms of the material phenomenological approach of Maine de Biran and Michel Henry.

I am grateful to the three publishers for allowing me the use of materials from the articles, which I de- and recomposed into different chapters for the present purposes. In addition, I was able to develop my early ideas concerning this approach in two papers that my colleague Jennifer Thom first-authored. To her I am particularly indebted for providing me with a space for developing theoretical issues to understand phenomena to which she tends to take the enactivist approach of Humberto Maturana and Francisco Varela.

Throughout this book I use my own translations of original French and German texts, because the licensed translations constitute interpretations, which often have epistemologies embedded that are opposite to the one I develop here. Moreover, translations frequently do not render what is so special about the texts in their native language. For example, I use a quote that in my translation reads “[A concept therefore is] an incorporeal, even though it incarnates or effectuates itself in bodies” (Deleuze & Guattari, 1991/2005, p. 26). The official translation, however, suggests, “The

concept is an incorporeal, even though it is incarnated or effectuated in bodies” (p. 21). In this translation, the reflexivity of the concept—which incarnates itself almost despite the person—is lost, though this passivity is precisely what we need to retain in a proper account of learning and that is highlighted in the work of Michel Henry. As I show in this book, we do not intend our intentions/intentionality; we receive them/it as a gift. This passivity, a central issue in phenomenological philosophy of the 20th century, is absent in the Anglo-Saxon literature but is central to my own approach—which follows such philosophers as Martin Heidegger, Emmanuel Levinas, Michel Henry, Jean-Luc Marion, and Jean-Luc Nancy. Many translations of philosophical books into English include a translator’s preface in which the difficulties of translation are articulated and in which the partiality of the decisions by the translators are made salient. I personally tend to carry the multiple senses of a word into English, separating/adjoining them by using a slash—e.g., Heidegger’s “die Hand reicht” becomes “the hand reaches/extends” because of the double entendre underlying the German verb—rather than choosing one or the other translation.

This research was made possible by two research grants from the Social Sciences and Humanities Research Council of Canada and an institutionally administered small grant from the same Council. I am grateful to Jennifer Thom, with whom I held two of these grants; to Mijung Kim, who was involved in the data collection as a postdoctoral fellow; and to Lilian Pozzer-Ardenghi, who at the time was a doctoral student in my research team. Jennifer had taken a central role in the design of the curriculum. Mijung and Lilian helped in the videotaping. My thanks also go to the teacher and all her children for their participation in this project.

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But the craft of the hand is richer than we tend to think. The hand does not only grasp and catch, does not only squeeze and push. The hand reaches/extends, receives/welcomes—and not just things: the hand extends itself, and receives/welcomes itself in the hand of others. The hand holds. The hand carries. The hand draws/signs, presumably because man is a sign. . . . But the gestures of the hand run everywhere through language and precisely then with their most perfect purity when man speaks by being silent. And only in as far as man speaks does he think; not the other way around, as metaphysics still tends to think. Every motion of the hand in every one of its works carries itself through the element of thinking; every hand gesture bears itself in that element.

(Heidegger, 1954, p. 51)

Introduction

Of Hands, Flesh, and Mind

Perception is not a sort of beginning science, and a first exercise in intelligence; we have to find again an exchange with the world and *a presence in the world much older than intelligence*. (Merleau-Ponty, 1996, p. 66, emphasis added)

“We have to find again,” writes Maurice Merleau-Ponty, “a presence in the world much older than intelligence.” This quote, found after the first draft of this book had already been completed, succinctly summarizes my intention. Already before finding this quote I had remembered an instance in my life, the story of which I have sometimes told as an example of embodiment but that now is taking on new significance. I begin this introduction/book with this story, because it is exhibiting the nature of the phenomena at the heart of my present concerns: the hand, the quintessential organ of tact, as a synecdoche¹ for the *flesh*, which is the foundation of all senses, and, therefore, of all sense that we can make. The analytical category of the *flesh* refers us to the experience of the body proper in as far as it is incarnated, a living sensuous body in which cognition is but one of the forms of consciousness and where the different forms of consciousness—affect, tact, rhythm, and other non-cognitive forms—exist side-by-side. The category of the flesh (Ger. *Leib*, Fr. *chair*) is a familiar one in European phenomenology, where it has been developed as a distinction to the material body (Ger. *Körper*, Fr. *corps*) in a history of European thought that ranges from Maine de Biran to Edmund Husserl, Merleau-Ponty, Emmanuel Levinas, Didier Franck, Jacques Derrida, Jean-Luc Nancy, Jean-Luis Chrétien, and Michel Henry. Unlike the body, “flesh is not matter in the sense of corpuscles of being that are added or follow one another to form beings” (Merleau-Ponty, 1964, p. 181), it is a modality of life. Flesh, as I show in this book, is at the origin of intentions, bodily skill, and the emergence of mathematics. In particular, my concern is a rethinking of embodiment in the way it has come to us through a line of philosophical and cognitive science research on the body, the mind, and metaphors/metonymies that are said to allow complex forms of cognition to emerge from basic experiences in the world. This following instance from my life takes us right into the world of the hand that Heidegger writes about in the frontispiece, the intelligent hand that traverses language, the hand that has an intelligence of its own, an intelligence that is older and deeper than the one that comes with and from language.

IMMEMORIAL MEMORY—IN/OF THE HAND

Some 20 years ago—I had graduated a few years earlier and I am already teaching at a private high school outside Toronto—I decide to call my doctoral supervisor. But as I turn to the telephone, I realize that I have forgotten her telephone number. I try remembering it, but—it may appear evident but seldom attended to in the learning science literature—precisely because I have forgotten it, I cannot *aim at* retrieving it in the way I would do if I wanted to retrieve a bean jar from the top of my kitchen shelf when I make a bean salad. I decide to call the telephone directory assistance in the state of Mississippi where she lives. But as I sit down and orient my fingers toward the touch pad, they dial the area code I remember but then continue and, as my fingers push the keys almost despite myself, a familiar melody begins to emerge from the receiver. The phone on the other side rings, and then her voice tells me not only “hello” but also that my hand indeed has found the forgotten number.

In this episode, my fingers remember a telephone number that I have forgotten and cannot recall: “I” have both forgotten and not forgotten the number. Given that the telephone number is a sign of the form 1-601-###-####, it might have been possible to employ another sign to recall it, like some people make a knot in their handkerchief, an arbitrary sign constructed to recall something else just as a computer uses pointers (i.e., a deictic sign, an address) to retrieve the information that the central processor needs at the moment. But this is not the case here. In the present situation, my fingers remember, but they do so in an unmediated way, without a sign that points to something else, the memory, to retrieve the information. They remember in moving; the movement *is* their memory. The memory in the fingers therefore is unmediated: it is in the finger themselves, in the movement that they produce. It is here that we can find this memory. Because there are no signs involved, it is a special form of memory, a form of *immemorial* memory that has arisen from the movement itself (Maine de Biran, 2006). Or rather, the memory has arisen from the self-affection of the *flesh*, which, through previous dialing, has developed the capacity to dial this number without being told so by the conscious mind. In fact any intention to dial already presupposes the capacity to move the arm, hand, and fingers, to reach out, and to push keys, and to dial. Thus, beneath the intentional act of my dialing, which requires a knowing consciousness of the object, there is an operative intentionality that makes the conscious dialing of the number possible. Without this capacity, without the underlying unmediated “I can dial,” there would not be an intention. That is, intention is a consequence of the capacities that my fingers have for movement rather than their origin and driver.

In the ear, something special also has happened. The ear recognizes the melody that comes from dialing the particular number as a familiar one, even though at the moment I cannot remember (have forgotten) the number.

How can the ear recognize a tone sequence as familiar even though the mind has forgotten the numbers that produce the sequence? The answer is this: memory realizes itself in the act of hearing. The recognition, therefore, somehow precedes itself. In fact, “recognition always precedes itself when we want to derive it from any content whatsoever” (Merleau-Ponty, 1945, p. 473). Recognition is not a passive event; it is an active form of consciousness. It requires that the sound striking my ear resonate with the capacity to hear; it strikes a form of memory that I have not recalled. It requires a connection with the past, a trace of the past, that is, a past in its own domain. Cognition of the melody is wound up with its recognition. Recognition requires some form of comparison between the contents of a memory and the current content of imagination: it is a fundamental mode of consciousness where the “I” is conscious of the fact that what it does is the re-production of a previous perception (Husserl, 2001). But the active form of this memory has gone lost, being evoked not in and by itself—as in the case of the fingers that dial—but by the original melody. Recognition and remembering are wound up into one and the same process; the difference between them has become undecidable, that is, we do not know where one begins and the other ends. “The *non-intended* effort that constitutes the vigil also constitutes the enduring of the *I*, or the identical person, and makes remembering possible” (Maine de Biran, 1859b, p. 439).

The episode of my hand that remembers dialing a phone number may be discarded all too easily, as a story of rote recall of little value to mathematical knowing. But while writing this book, other episodes of my life returned to my mind, episodes, for example, in which my hands continue to write in English while I am talking to my wife who has come by my office to say goodbye for the day as she goes to work. My hands have been walking and working the keys, but this time they have not done so because of rote memory. They have been producing text all the while I am orienting to my wife and having one of the typical conversations that we have at such occasions. The hands have continued to write, productively creating text, not just repeating age-old tunes but contributing new text to whatever I am working on at the moment. How can we theorize this knowledge of the hands to write and produce text even though the mind apparently focuses on something else? In fact, the hand knows much more and tends to be ahead of myself, and when there is an error, such as an error of typing, my hand may have already finished the subsequent words before my intellectual mind realizes that that there is something wrong. And it is not so much that my hand articulates what my mind has already prefigured: My hand does in writing what my mouth does in speaking: It is ahead of the game. In each case, the articulation is faster than the intellectual mind that comes to realize what has just been said or written only after the fact. That is, the expression produced by mouth or hand *is* the thinking that the mind subsequently claims for itself.² This expression is expression of itself, not expression of something else—e.g., mind, thought—that pre-exists it.

4 Geometry as Objective Science in Elementary School Classrooms

The first movements of an organ leave traces. These traces are the result of an auto-affectation, the moving aspects of the flesh affecting themselves prior to any sensory capacity, and constitute memory before memory, immemorial memory. We observe such immemorial (i.e., unmediated) memory in the two cases described above, the fingers that remember the telephone number and the ear that recognizes the melody as a familiar one. This auto-affectation is possible only in the flesh but not in the body (generally): The flesh is the locus of the originary sense-giving intuition because it is endowed with sense and affect (Henry, 2003). That is, this auto-affectation *precedes* the sensorimotor capacities and their schemata on which constructivist and embodiment approaches ground their theories of knowing and learning: “When we begin, there already is an absolute antecedence” (Nancy, 2006, p. 59). The coming into presence, into consciousness, is at least partially enabled by incarnation:

The other version of the coming calls itself incarnation. If I say *verbum caro factum est (logos sarx egeneto)*³, I say that in one sense that *caro* makes the glory of the veritable coming of the *verbum*. But I immediately say, in a very different sense, that *verbum (logos)* makes the veritable presence and the sense of *caro (sarx)*. (p. 58)

In his extended reflection on knowing and the body, the French philosopher Jean-Luc Nancy shows how the word, consciousness, and the flesh are intimately intertwined. In the tropes of tact and touch lies the unity of sense and the senses, inside and outside, material bodies and sense, subject and object, or subjectivity and objectivity.

OF BODIES AND FLESH

Some readers may be tempted to attribute my writing to a constructing and constructivist mind. But, to call the text a result of my mind denies fingers “the intelligence of the integrated knowing hand, which guides as it is guided, singing from place to place, making melodies in a network of spatial contexts that are grasped and tactilely appreciated in the most intimate and still mysterious ways” (Sudnow, 1979, p. 11). The knowing hand is an improvisatory one, which “handles the keyboard by finding itself in good positions at all times to move ahead. It knows the terrain as routes to be taken for speaking a language” (p. 12). And it does so with its own rhythm, pacing, and melody, accelerating when there is a lot to say and slowing down for taking a breath. It is precisely this rhythmization, the performative of making music, writing, and talking/gesturing/doing mathematics that we need to theorize rather than engaging in stale formalisms that expel the most distinguishing features that separate the human body from all other animate and inanimate bodies. In this book, I show that—because rhythm

and pacing of expressive communication, among others, constitutes a form of consciousness very different from intellectualizing consciousness—the difference between body and mind becomes undecidable. Thus, “rhythmization, just like the apprehension of a melody, includes an act radically different from a simple encounter with the rhythm-object—an essentially creative act” (Abraham, 1995, p. 73). Each creative act, “synthesizing the successive emergences, sights a phenomenon irreducible to either the mere perception of these emergences or their mechanical production: it is precisely this phenomenon that we call *rhythm*” (p. 73). Especially in Chapter 7, I show how rhythmicity underlies mathematical knowing, learning, and teaching.

But the body I refer to is not the body theorized in the work of embodiment theorists Mark Johnson, George Lakoff, or Rafael Núñez. Theirs is an intentional body, the intention of which remains unexplained. How would their bodies intend their own intentionality? Their body is not the body that I am after. Mine is the one capable of the caress, which is not driven by “an intentionality of disclosure/unveiling, but of searching: a movement in and unto the invisible” (Levinas, 1971, p. 235). It is precisely the flesh and not the materiality of the body that is capable of the caress. Whereas two bodies may be in contact, it is only in and through the flesh that there is an excess from that which I touch and that which touches me; this excess precisely attests itself in the caress (Chrétien, 1992). It is in and through the flesh—because we are present *in flesh and blood*—that we come to face a fellow human being rather than another material body. The living body inhabited by flesh is “a felt body, a body that is seen, that can be touched, render a sound when it is hit, that has an odor, this sweet odor of honey of a piece of wax, that is smooth or rough, cold or hot, dry or moist, hard or soft” (Henry, 2000, p. 157).

These living bodies are very different from machines, including the living, autopoietic ones that Maturana and Varela (1980) conceive. The flesh, the incarnated body, is different from the body-thing of the physiologist, the lived but purely intentional body, the one embodied in the autopoietic living machine, or the body purely considered as expression. “[T]he problem with a living machine is basically no different from the problem of an inanimate one; both need a helmsman, a director, a master, a conductor” (Sheets-Johnstone, 2009, p. 18). Maturana and Varela’s bodies relate to the ones I am after as sheet music relates to a live performance of *improvisational* jazz or as dance instructions to dancing. Theirs is a body already endowed with schemata that have linguistic structure and that is the material to be transformed and proliferated by metonymic and metaphoric processes in embodiment theory.⁴ My living body, or, as I refer to it here, the *flesh*, is always in performance; its knowing exists nowhere else but in performance, and with different forms of consciousness among which linguistic consciousness is only one (Sheets-Johnstone, 2009). This consciousness includes the rhythmization and melodies that Nicolas Abraham articulates and the jazz improvisations, writing on the typewriter keyboard that David

Sudnow has written about, and the dancing that is at the heart of the matter in Maxine Sheets-Johnstone's work. In the same way as the hand sings with the piano, the hand thinks with the computer keyboard, the voice produces the scanting rhythms denoted by the classical meters, so my category of the flesh is used to theorize mathematical knowing, or rather, geometrical thinking. There is not thought *behind* expressive performance, thought which drives the body. The flesh does the thinking in and through its expression; and this expression is not an expression of something else, some intellectual phenomenon behind. Any expression is that of a living, sensing, feeling, moving body itself constituted by the flesh.

The bodies in embodiment and enactivist theories do not seem to have affect, emotion, but they are characterized by formal structures, schemas, without flesh and blood. These schemas constitute the cold cognition that the various constructivist paradigms have offered to us. What I am concerned with here are real people, *in flesh and blood*, with strengths, weaknesses, feelings, shortcomings, powers, and infirmities. In a word, I am concerned with mathematical knowing of real, living, incarnate and therefore *human* beings. As such, human beings are subject to/of forms of experience not theorized in any other learning theory: radical passivity, a passivity that we do not choose, but which comes with life (being) itself. I could not *intentionally* dial my supervisor's telephone number, the number gave itself in the movements of my hand; even the intention to call her was given to me and not intended by another intention. This radical passivity exists because bodies resist to movements—otherwise they would be moving on their own such as the uncontrolled and unwanted movements in certain illnesses (e.g., Parkinson's)—and from the fact that we are subject to the constraints of the social and material worlds. This passivity is not theorized at all in the enactivist, embodiment, and constructivist literatures, but it underlies the phenomenon of *entrainment* that is central to the phenomenological sociology that I articulate in Chapters 3 and 7 to explain how the expressions of the organic body always are expressions of a socialized body. Unlike in the constructivist, embodiment, and enactivist theories, the living organic body I theorize here does not exhibit subjective knowing but always already collective knowing, knowing-with.

CHILDREN'S HANDS IN GEOMETRY, GEOMETRY IN CHILDREN'S HANDS

In the previous section, I articulate forms of knowing in, of, and with the hand. I do so for three reasons. First, the hand is the most important organ in the evolution of knowing, as may be evident from the frontispiece. The hand, with its capacity to do work, touch (sense), and express (gesture) is more foundational to knowing than the eye (and associated mirrors) that has come to underlie the metaphors of knowing throughout the history

of metaphysics.⁵ It is in and through the hand that the sense of the body becomes the body of sense in both a material phenomenological and cultural-historical activity theoretic approach.⁶ We can (learn to) function without sight or hearing, but without tact, the capacity of touching, we lose all sense of (and with) reality. Second, the hand serves me in a synecdochical function: It is a part of the living body that stands for all other parts of the living body that are capable of tact, contact, and contingency. I describe what students and teachers do with their hands, well knowing that it is not the hand that operates on its own. The hand is part of an integral capacity to act, an integral “I can,” the integrality of which comes from the fact that “from an enactive viewpoint, any mental act is characterized by the concurrent participation of several functionally distinct and topographically distributed regions of the brain and their sensorimotor embodiment” (Varela, 1999, p. 272). It is to this central capacity that all specific capacities are subordinated functionally; and this central capacity coordinates the specific activities as well. Thought, then, is but one phenomenon of expression among many others that can be isolated analytically from the expression of the body as a whole.

In this book, the children’s hands, which build, work on, sense, and gesture with/out objects, know mathematics in the ways that Sudnow’s hands know in playing the piano, which orient toward a situation rather than playing notes. They take a stance, a position, and this stance expresses itself in the way the hands move. What I pursue in this book is a way of understanding what the children do in early geometry, which is more like getting to the melody of their mathematics rather than to the structures of the transcriptions, the musical scores and sensorimotor schemata that theoreticians put on paper. It is mathematics as performance that I am interested in describing and theorizing, the rhythms, the melodies, the hums, which always resonate when people make mathematics. Children learn to do mathematics in the way those who follow the “Suzuki method” learn to play an instrument by playing before learning to read sheet music—they learn to play by ear, in the way they learn to speak their mother tongue. In the process their bodies become mathematical, because they can feel (sense) when it (mathematics) makes sense. My hand knows the telephone number even when I remember it in the form of mathematical digits—and the recall of numbers, even my own, is tied to rhythmic and musical performance that allows me to reproduce the number that I so seldom call.

The literature on knowing and learning mathematics in school classrooms—or, for that matter, science, history, and any other subjects—is too much concerned with the intellectual aspect of life. This literature forgets that in and through the children, life gives itself a mind rather than the other way around. From the Kantian gesture of focusing on the mind at the expense of everything else, knowing and learning mathematics—as it appears in many accounts of mathematics education—looks more like what we know about computers rather than what we know about how

human beings conduct their life and mobilize/enact knowing as they go. Thus, this existing mathematics education literature focuses on “interpretation” of the talk of others, on “taking things as shared,” when in fact much of our lives does not require us to interpret, or “to make meaning.” In communications with others, in written or oral form, it is irrelevant that we somehow become “aware of the meaning of the expression by doing concurrent thinking about its reference, even though, while saying the words with greater emphasis, there is the sense that what they stand for is being sought” (Sudnow, 1979, p. 52). What really matters is not that we somehow “make meaning” but that we seek “the melody of the text. In experience, this seems to reside precisely in a special manner of heightened and more extensive bodily absorption with the finest particulars of the unfolding sound places” (p. 52). *Being* means understanding; *mathematically being* means understanding mathematics in a practical way. Thematic interpretation is but a derivative mode, always already based in practical understanding of the world.⁷ But thematic interpretation develops understanding—not leading to something different, a different understanding. Rather, through interpretation, understanding comes to understand itself in an explicit way.

In this book, I am precisely after this practical understanding of the world that children bring to and develop in the mathematics classroom. I am interested in understanding how this practical understanding undergirds learning and knowing, and how knowing and learning are never independent of this practical understanding. In fact, this prior, practical understanding undergirds and allows mathematics to emerge even and precisely when this understanding is inconsistent with formal mathematics. That is, other than Kantian, Piagetian, or constructivist theories, where there is a gap between knowing *one’s way around* the world and knowledge *of* the world, the approach outlined here never severs the feeling, moving, sensing person inhabiting its world in flesh and blood from the “knowledge” or “cognitions” that it is said to “have.”

Even though I suggest that the essence of mathematical performance is not captured by the schemata that Piaget, enactivist, and embodiment theorists propose in varying ways as the basis of mathematical knowing, I do not mean to suggest that this essence is ineffable. My case is similar to the one in musical theory of the *nuance*, which cannot be captured by schema theory and Chomsky-like generative grammar (Roholt, 2010) but constitutes an aspect in which practical musical consciousness makes the difference. Thus, “hearing a B-flat(17) is not to perceive that pitch as an isolated property . . . [but] what is truly *musical* about a B-flat(1) is what it does in a perceived musical context” (p. 7). Rather, what the ear hears are changes that musicians characterize for each other by means of metaphors such as an F-sharp (13) or F-sharp(14) that are “bright,” “shimmering,” or “radiant” in different ways and that are further characterized by comparisons with recorded or live performances.

Mathematics generally and geometry specifically emerge from the children's hands—here serving as synecdoches of the living bodies to which they belong and which they constitute—much as music emerges from the knowing hands on the piano keyboard and much in the way my hands write seemingly on their own using a computer keyboard while my eyes are fixed on the monitor. These hands learn to act mathematically as they move, gather, or shape objects; and they learn as the children actively search to sense the characteristics of specific geometrical forms—such as cubes, rectangular parallelepipeds (also called cuboids or rectangular prisms),⁸ spheres, cones, or pyramids—and the everyday objects with which they come to be associated—such as dice, pizza boxes, Post-it pads, toothpaste boxes, balls, ice creams, and reproductions of Egyptian tombs.

Most importantly, what children do with their living body generally and with their hands specifically cannot be represented appropriately by means of language. Throughout this book I use drawings to show events from the position of the person operating the camera. The images should not be looked at with a linguistic consciousness, for this would reduce knowing-in-action to linguistic consciousness. Rather, each image should be treated as an instruction for *doing* what can be seen, much like the texts and images in a cookbook are instructions for the user to enact certain steps to end up with some meal. Merely looking at the images is like merely looking at a cookbook. Without actually doing what instructions tell us, the very modes of consciousness required for doing mathematics/cooking are not mobilized unless we do it ourselves. Repeatedly, I invite readers to engage in investigations (e.g., in Chapter 1). Readers should do these investigations, because the forms of consciousness I want them to experience would not be mobilized otherwise. Watching a soccer game, golf, or ice hockey on television or reading about such games is very different than playing soccer, golf, or ice hockey: The sense is different when we play audience then when we play the game. It is precisely the sense of the game that matters, because it underpins the sense we make. The sense of the game derives from and exists in and as of the living body (flesh); and this is so not only for doing sports, dancing, playing music but also and especially for doing mathematics. The sense of mathematics emerges from a developing sense of the game for doing mathematics.

STRUCTURE OF THIS BOOK

This book presents a way of understanding the emergence of mathematics generally and geometry specifically from elementary children's engagement with cultural-historically marked objects that have their specific place in geometrical science. Whereas I am positively inclined toward enactivism—which I consider the best approach to knowing, though its implications have not yet worked out to its fullest in the mathematics education literature—I

show two important lacunae in this theory: (a) the source of intention and memory that underlie mathematical cognition and (b) the source of the inherently cultural-historical—and therefore shared—nature of knowing that any individual incarnate agent does evolve. Among those mathematics educators employing the theory, too many focus on intellectual consciousness and too few on all the other forms of consciousness that constitute integral parts of knowing. In their approach, the very distinction between body and mind is presupposed and maintained by the role of intentionality. In my approach articulated here, this integral knowing is only partially and one-sidedly represented by intellectual consciousness.

This book is divided into three parts, each prefaced by an introduction of its own, and each preparing what is to come or building on what has been developed before. Thus, Part A is devoted to the articulation of theoretical issues. More specifically, using a concrete lesson fragment that features a student in the process of articulating an answer concerning his insights about geometrical objects generally and cubes particularly—for which I provide an initial reading in Chapter 1—I articulate the reigning theoretical frameworks that have been used in the philosophy of mathematics and mathematics education over the last several decades. Thus, in Chapter 2, I comment on and make reference to the positions of Immanuel Kant, Jean Piaget, Maurice Merleau-Ponty, enactivism (as articulated by Humberto Maturana and Francisco Varela), embodiment (as articulated by Rafael Núñez and his collaborators), (radical, social) constructivism (as articulated by Ernst von Glasersfeld and Paul Cobb, respectively), Hans Freudenthal, and Pierre van Hiele. In the third chapter, I develop an approach entirely grounded in material life, presenting the perspectives of material phenomenology and its emphasis on the *flesh* as opposed to the body, phenomenological sociology and its emphasis of the socialization of the body, and cultural-historical activity theory, which incorporates both phenomenological and sociological thinking.

In Part B, “Stories of Mathematics in the Flesh,” I present detailed analyses of learning in a second-grade mathematics classroom, where children are in the process of doing their first unit on geometry. In particular, I use lesson fragments in which cubes somehow come into play—as objects of activity, comparative objects, fashioned objects, gestured objects, or *objets trouvés*—which allows us to see similarities and differences in knowing across the episodes. The four chapters of this Part B exhibit a development, from fragments in which children articulate issues out of their personal experiences with the world and out of their intuitions, which often do not coincide with the geometrical knowing that the teachers attempt to foster. Yet these experiences, as we can see in the unfolding sequence of the four chapters, constitute the very material from which formal geometry emerges, such as the use of normative measurement for deciding what one has to do to make a cube from a pizza box. Chapter 5, for example, presents a lesson fragment in which a student comes to realize that the object

that she had modeled as a cube based on what she felt by touching the object without seeing it really is a rectangular prism with three different pairs of faces. In the final chapter of Part B, I show how rhythmic phenomena in the expressive articulations of teachers and students encourage us to employ the proposed approach of *mathematics in the flesh*. These rhythmic phenomena arise from a self-affection (of movement) that is central to my radicalization of embodiment, on the one hand, and is central to a phenomenological sociological approach in which *entrainment* ascertains the coordination of knowing across all individuals of a collectivity on the other hand. This chapter, therefore, explicitly shows where and how both embodiment/enactivist and social constructivist theories fail in contrast to the approach presented here.

In Part C I anticipate the questions traditionally trained mathematics educators may have. Specifically, this third part of the book is concerned with (a) how formal geometry—here exemplified by acts of geometrical sorting—arises from the everyday methods of sorting; (b) how formal geometry as an *objective* science emerges from the children's ethno- (i.e., everyday) methods of sorting; and (c) how to rethink the nature of mathematical concepts and conceptions. As the entirety of Part B, two of the three chapters employ lesson fragments that focuses on children's classification of cubes, whereas the final chapter, as a sort of extension into objects other than parallelepipeds (cuboids, cubes, rectangular prisms), uses an episode featuring the classification of cylinders.

I conclude this book with an ever-so-brief epilogue, in which I take a panoptic view back over the main aspects of the argument for theorizing mathematical knowing as incarnated, as a form of *mathematics in the flesh*. In the Appendix, I provide a description of the transcription conventions used throughout this book.

Part A

Toward a Theory of Mathematics in the Flesh

Introduction to Part A

Spatiality may be the projection of the extension of the psychic apparatus. No other derivation probable. Instead of Kant's a priori, conditions of our psychic apparatus. *Psyche is spread out, does not know thereof.* (Freud, 1999, p. 152, emphasis added)

During the final days of his life, Sigmund Freud, who had spent his entire career researching psyche and the unconscious, suggests that the human psyche, heretofore thought as something immaterial, actually is spread out.¹ “‘Psyche’ is body,” Nancy (2006, p. 22) notes, “and it is precisely that which escapes it.” It is precisely this escape, the process of the escape that constitutes the psyche. What is it in or of the living human body that might play this central role in who we are and what we know? Numerous phenomenological philosophers of the late 20th century discuss the flesh as the foundation and medium of tact, itself the foundation of all senses.² Others, often coming from the arts, theater, and dance, have joined in the celebration of the living body and aliveness as the fundamental aspect of being that enables all forms of knowing, including the intellectual forms (e.g., Sheets-Johnstone, 2009). Tact relates human beings to the world, as apparent in all those concepts that are based on the same Latin root word *tangēre*, to touch: contact, contingency, contagion, contagious, contiguous, contiguity, contaminate, touching, tact (beat), tactile, tangent (line, distance, point, length), tangent (function), or co-tangent. Tact is the general sense. It is synonymous with sensibility itself, thereby constituting the sense of all the senses, any sense we can make, and the sense of sense itself. Psyche, the flesh, the senses, and mathematics come to be irremediably and intricately woven into a complex lacework. And it is precisely the connateness of tact with contact, contingency, contagion, and musical tact that makes our experience inherently embodied, shared, and intersubjective rather than metaphysical, monadic, subjective, and singular.

As the end result of transcendence, the Kantian, constructivist mind becomes metaphysical, no longer is present in the world, but withdraws into the netherworld of its representations and constructions. Kant and all constructivist theories that ensue are concerned with the mind, which, as its etymological origin in the Proto-Indo-Germanic root *men-* (to think, to be busy mentally) suggests, is concerned with theorizing that which is conscious and accessible to reflection. But reflective activity makes up only part of our lives, in fact, is subordinate to a more integrative minimal unit: collectively motivated activity. That is, activity, which always realizes

collective object/motives, sublates (i.e., keeps, suspends, supersedes) and therefore integrates, the generally separate phenomena of intellect and affect. Karl Marx and Friedrich Engels, on whose theory Lev Vygotsky and Alexei Leontjew built their dialectical materialist, societal psychological theories of knowing, articulated the integration of materialist and dialectical idealist approaches into one of lived and living human praxis.³ In this approach, all higher-order cognitive functions are understood to be the result of interaction rituals in the context of concrete, sympractical work designed to fulfill collective and individual needs.

In Western scholarship, however, neither phenomenology nor cultural-historical activity theory has been central to the discussions of knowing and learning in philosophy, education, and the learning sciences (cognitive science, cognitive psychology, artificial intelligence). Rather, if at all, the phenomenological and subject-centered Marxist theories that did affect the learning sciences are rooted in Western scholarship—including the phenomenological approaches of Edmund Husserl, Martin Heidegger, and Maurice Merleau-Ponty or the cultural-historical approach articulated by Yrjö Engeström—have influenced a broad range of research in cognitive science, artificial intelligence, sociology, poetics, and even mathematics.

In Western cognitive science, first phenomenology then the embodied cognition research showed how there could be no cognition without the human body. There is something unsatisfying and lacking, however, in the concept of the body, which undermines the very effort to ground (mathematical) knowing differently than in the private cogitations of the isolated mind that moves and processes ephemeral representations. There is a difference between the German *Körper* (body) and *Leib* (body), between the French *corps* (body) and *chair* (body) that is not thought by the concept of the body. Thus, “with respect to the *body* [English in the original] (in all places where it obsesses, for example, the American culture and academy), whom will we make believe without laughter that it is a trustworthy equivalent of that which we call *corps*, *corps propre*, or *chair* [flesh]” (Derrida, 2000, p. 79)?

In this Part A, I shift the discourse from the body to the flesh to present a far more radical approach to the conceptualization of mathematical knowledge that is grounded in a material and sociological phenomenology than that provided by the embodiment and enactivist literatures on the topic. This approach is further developed in Part B and Part C of this book. In this first part, though, I present the central ideas underlying this book from dialectical materialist psychology (as developed by Lev Vygotsky) to materialist phenomenology (as developed by Maine de Biran and Michel Henry) and to phenomenological sociology (Bourdieu). The close relation between cognition and the world is possible only when there is flesh—which is able to self-affect and to remember without representation. In contrast, material bodies constitute an insufficient condition for mind to emerge.

In the brief Chapter 1, I provide a fragment from a lesson in a second-grade classroom and enact a careful reading thereof. I subsequently use this lesson fragment and my reading to articulate different theories, their strengths and shortcomings with respect to the task of providing us with an understanding of the exhibited mathematical knowing and its conditions. Specifically, in Chapter 2 I articulate the different approaches to mathematics generally and to geometrical knowing specifically that have been taken in a history of theorizing mathematical thought from Kant to modern day embodiment and enactivist theories. I draw on philosophical analyses of experiences with and knowledge of cubes, wherever I could find them, as a way of exhibiting where these theoretical frameworks fall short to account for geometrical knowing in real time. In Chapter 3, then, I present a way of understanding geometrical knowing, *mathematics in the flesh*, and how its cultural dimensions are acquired in and through participation in everyday human interactions generally and in those that occur in schools more specifically. That is, unlike the enactivist approach, in which cognitive and development is dealt with as a “strictly *subject*-dependent creative process” (Maturana & Varela, 1980, p. 49, my emphasis), phenomenological sociological and cultural-historical approaches recognize the essentially passive aspects of human experience that lead the flesh to produce and exhibit cultural-historical, that is, intersubjective forms of knowing rather than forms that are singular and subjective.

1 What Makes a Cube a Cube?

A Phenomenological Overture

Can you go out for a game of tennis, make love, repair a roof, or plant a garden short of being in the flesh? Cheshire cats might be able to accomplish such feats . . . (Sheets-Johnstone, 2009, p. 20)

I stepped straightly into phenomenology, which was a bit didactically tainted. (Freudenthal, 1983, p. 210)

In the course of reading this book, readers discover that even the first elementary school tasks involving children with three-dimensional shapes allow the objective nature of geometry to emerge from the events in an elementary classroom. But this objective nature of geometry emerges each and every time from the flesh, much like playing a game of tennis, making love, repairing a roof, or planting a garden. The purpose of this book is to push—in style and content a tribute to Hans Freudenthal—how we think and think about the “embodiment” of mathematical knowledge as something that only an incarnate being (i.e., being in the flesh) can accomplish. To get the point of this chapter you have to *enact* the following task.¹ Through this exploration, I intend us to focus on the phenomenology of geometrical and spatial experience—which is very different for non-Western cultures.

Find (what you have learned to be) a cube and take it in your hand. Look at the object. Do you see a cube? Of course not: you see aspects (parts) of a cube, and the nature of the aspect depends on your current perspective and the orientation of the object. But how do you know that these are aspects of a *cube*? To know this, you need to know that a cube expresses itself in the aspects you see given perspective and orientation. Close your eyes and feel the object. Do you feel a *cube*? Of course you do not. Even if the object is small so that you can enclose it entirely in one or both palms, you do not feel the twelfoldedness of the edges, the sixfoldedness of the sides, and the eightfoldedness, the smoothness/roughness of the surfaces, the color, and so on; and especially, you do not feel all of these aspects of a *cube* simultaneously. If you still were to think you felt a cube, then you are mistaken, for you only hold a token of a cube rather than a type.² This distinction is important, as geometry is dealing only in *ideal* types rather than concrete tokens (Husserl, 1997a), not in the least because the exactness required by geometry cannot be achieved in nature given Heisenberg’s uncertainty principle. That is, we never *perceive* a cube, be it by means of