

# Mathematical Economics and the Dynamics of Capitalism

Goodwin's legacy continued

*Edited by*  
**Peter Flaschel** and  
**Michael Landesmann**

Routledge Frontiers of Political Economy

# Mathematical Economics and the Dynamics of Capitalism

Richard Goodwin was a pioneer in the use of mathematical tools to understand the dynamics of capitalist economies. This book contains contributions which focus on the rigorous extension of Goodwin's modelling of macro-dynamics and the micro-structures underlying them, and also research with a wider perspective related to Goodwin's vision of an integrated Marx-Keynes-Schumpeter (M-K-S) system of the dynamics of capitalist economies.

The variety of approaches in this book range from detailed business cycle analyses to Schumpeterian processes of creative destruction. They include

- thorough theoretical analysis of delayed dynamical systems;
- empirical studies of Goodwin's classical growth cycle model and the integration of Keynesian aspects of effective demand and of financial mechanisms that impact the real macro-economy;
- micro-economic structural analysis;
- expectations driven aspects of micro-founded business cycle modelling.

Providing both theoretical analysis and empirical evidence for approaches in the tradition of Goodwin's M-K-S system, this book represents a significant step forward in this type of research agenda. The volume is useful for students and researchers in the areas of nonlinear macro and micro-economic modelling, in business cycle modelling and in frontier research regarding classical, Schumpeterian and Keynesian modelling of the dynamics of capitalist economies.

**Peter Flaschel** is Professor of Economics at the Department of Economics and Business Administration, Bielefeld University, Bielefeld, Germany

**Michael Landesmann** is Scientific Director of the Vienna Institute for International Economic Studies and Professor and Chair of the Department of Economics at the Johannes Kepler University in Linz, Austria.

## Routledge Frontiers of Political Economy

- 1 Equilibrium Versus Understanding**  
Towards the rehumanization of economics within social theory  
*Mark Addleson*
- 2 Evolution, Order and Complexity**  
*Edited by Elias L. Khalil and Kenneth E. Boulding*
- 3 Interactions in Political Economy**  
Malvern after ten years  
*Edited by Steven Pressman*
- 4 The End of Economics**  
*Michael Perelman*
- 5 Probability in Economics**  
*Omar F. Hamouda and Robin Rowley*
- 6 Capital Controversy, Post Keynesian Economics and the History of Economics**  
Essays in honour of Geoff Harcourt, Volume 1  
*Edited by Philip Arestis, Gabriel Palma and Malcolm Sawyer*
- 7 Markets, Unemployment and Economic Policy**  
Essays in honour of Geoff Harcourt, Volume 2  
*Edited by Philip Arestis, Gabriel Palma and Malcolm Sawyer*
- 8 Social Economy**  
The logic of capitalist development  
*Clark Everling*
- 9 New Keynesian Economics/Post Keynesian Alternatives**  
*Edited by Roy J. Rotheim*
- 10 The Representative Agent in Macroeconomics**  
*James E. Hartley*
- 11 Borderlands of Economics**  
Essays in honour of Daniel R. Fusfeld  
*Edited by Nahid Aslanbeigui and Young Back Choi*
- 12 Value, Distribution and Capital**  
Essays in honour of Pierangelo Garegnani  
*Edited by Gary Mongiovi and Fabio Petri*
- 13 The Economics of Science**  
Methodology and epistemology as if economics really mattered  
*James R. Wible*
- 14 Competitiveness, Localised Learning and Regional Development**  
Specialisation and prosperity in small open economies  
*Peter Maskell, Heikki Eskelinen, Ingjaldur Hannibalsson, Anders Malmberg and Eirik Vatne*
- 15 Labour Market Theory**  
A constructive reassessment  
*Ben J. Fine*
- 16 Women and European Employment**  
*Jill Rubery, Mark Smith, Colette Fagan, Damian Grimshaw*

- 17 Explorations in Economic Methodology**  
From Lakatos to empirical philosophy of science  
*Roger Backhouse*
- 18 Subjectivity in Political Economy**  
Essays on wanting and choosing  
*David P. Levine*
- 19 The Political Economy of Middle East Peace**  
The impact of competing trade agendas  
*Edited by J.W. Wright, Jnr*
- 20 The Active Consumer**  
Novelty and surprise in consumer choice  
*Edited by Marina Bianchi*
- 21 Subjectivism and Economic Analysis**  
Essays in memory of Ludwig Lachmann  
*Edited by Roger Koppl and Gary Mongiovi*
- 22 Themes in Post-Keynesian Economics**  
Essays in honour of Geoff Harcourt, Volume 3  
*Edited by Claudio Sardoni and Peter Kriesler*
- 23 The Dynamics of Technological Knowledge**  
*Cristiano Antonelli*
- 24 The Political Economy of Diet, Health and Food Policy**  
*Ben J. Fine*
- 25 The End of Finance**  
Capital market inflation, financial derivatives and pension fund capitalism  
*Jan Toporowski*
- 26 Political Economy and the New Capitalism**  
*Edited by Jan Toporowski*
- 27 Growth Theory**  
A philosophical perspective  
*Patricia Northover*
- 28 The Political Economy of the Small Firm**  
*Edited by Charlie Dannreuther*
- 29 Hahn and Economic Methodology**  
*Edited by Thomas Boylan and Paschal F O'Gorman*
- 30 Gender, Growth and Trade**  
The miracle economies of the postwar years  
*David Kucera*
- 31 Normative Political Economy**  
Subjective freedom, the market and the state  
*David Levine*
- 32 Economist with a Public Purpose**  
Essays in honour of John Kenneth Galbraith  
*Edited by Michael Keaney*
- 33 Involuntary Unemployment**  
The elusive quest for a theory  
*Michel De Vroey*
- 34 The Fundamental Institutions of Capitalism**  
*Ernesto Screpanti*
- 35 Transcending Transaction**  
The search for self-generating markets  
*Alan Shipman*
- 36 Power in Business and the State**  
An historical analysis of its concentration  
*Frank Bealey*
- 37 Editing Economics**  
Essays in honour of Mark Perlman  
*Hank Lim, Ungsuh K. Park and Geoff Harcourt*

- 38 Money, Macroeconomics and Keynes**  
Essays in honour of Victoria Chick,  
Volume 1  
*Philip Arestis, Meghnad Desai and Sheila Dow*
- 39 Methodology, Microeconomics and Keynes**  
Essays in honour of Victoria Chick,  
Volume 2  
*Philip Arestis, Meghnad Desai and Sheila Dow*
- 40 Market Drive and Governance**  
Reexamining the rules for economic  
and commercial contest  
*Ralf Boscheck*
- 41 The Value of Marx**  
Political economy for contemporary  
capitalism  
*Alfredo Saad-Filho*
- 42 Issues in Positive Political Economy**  
*S Mansoob Murshed*
- 43 The Enigma of Globalisation**  
A journey to a new stage of capitalism  
*Robert Went*
- 44 The Market**  
Equilibrium, stability, mythology  
*S N Afriat*
- 45 The Political Economy of Rule  
Evasion and Policy Reform**  
*Jim Leitzel*
- 46 Unpaid Work and the Economy**  
*Edited by Antonella Picchio*
- 47 Distributional Justice**  
Theory and measurement  
*Hilde Bojer*
- 48 Cognitive Developments in  
Economics**  
*Edited by Salvatore Rizzello*
- 49 Social Foundations of Markets,  
Money and Credit**  
*Costas Lapavistas*
- 50 Rethinking Capitalist Development**  
Essays on the economics of Josef  
Steindl  
*Edited by Tracy Mott and Nina  
Shapiro*
- 51 An Evolutionary Approach to Social  
Welfare**  
*Christian Sartorius*
- 52 Kalecki's Economics Today**  
*Edited by Zdzislaw L. Sadowski and  
Adam Szeworski*
- 53 Fiscal Policy from Reagan to Blair**  
The left veers right  
*Ravi K. Roy and Arthur T. Denzau*
- 54 The Cognitive Mechanics of  
Economic Development and  
Institutional Change**  
*Bertin Martens*
- 55 Individualism and the Social Order**  
The social element in liberal thought  
*Charles R. McCann Jnr.*
- 56 Affirmative Action in the United  
States and India**  
A comparative perspective  
*Thomas E. Weisskopf*
- 57 Global Political Economy and the  
Wealth of Nations**  
Performance, institutions, problems  
and policies  
*Edited by Phillip Anthony O'Hara*
- 58 Structural Economics**  
*Thijs ten Raai*
- 59 Macroeconomic Theory and  
Economic Policy**  
Essays in honour of Jean-Paul Fitoussi  
*Edited by K. Vela Velupillai*
- 60 The Struggle Over Work**  
The 'end of work' and employment  
alternatives in post-industrial societies  
*Shaun Wilson*
- 61 The Political Economy of Global  
Sporting Organisations**  
*John Forster and Nigel Pope*

- 62 The Flawed Foundations of General Equilibrium Theory**  
Critical essays on economic theory  
*Frank Ackerman and Alejandro Nadal*
- 63 Uncertainty in Economic Theory**  
Essays in honor of David Schmeidler's 65th birthday  
*Edited by Itzhak Gilboa*
- 64 The New Institutional Economics of Corruption**  
*Edited by Johann Graf Lambsdorff, Markus Taube and Matthias Schramm*
- 65 The Price Index and its Extension**  
A chapter in economic measurement  
*S.N. Afriat*
- 66 Reduction, Rationality and Game Theory in Marxian Economics**  
*Bruce Philp*
- 67 Culture and Politics in Economic Development**  
*Volker Bornschier*
- 68 Modern Applications of Austrian Thought**  
*Edited by Jürgen G. Backhaus*
- 69 Ordinary Choices**  
Individuals, incommensurability, and democracy  
*Robert Urquhart*
- 70 Labour Theory of Value**  
*Peter C. Dooley*
- 71 Capitalism**  
*Victor D. Lippit*
- 72 Macroeconomic Foundations of Macroeconomics**  
*Alvaro Cencini*
- 73 Marx for the 21st Century**  
*Edited by Hiroshi Uchida*
- 74 Growth and Development in the Global Political Economy**  
Social structures of accumulation and modes of regulation  
*Phillip Anthony O'Hara*
- 75 The New Economy and Macroeconomic Stability**  
A neo-modern perspective drawing on the complexity approach and Keynesian economics  
*Teodoro Dario Togati*
- 76 The Future of Social Security Policy**  
Women, work and a citizens basic income  
*Ailsa McKay*
- 77 Clinton and Blair**  
The political economy of the third way  
*Flavio Romano*
- 78 Marxian Reproduction Schema**  
Money and aggregate demand in a capitalist economy  
*A.B. Trigg*
- 79 The Core Theory in Economics**  
Problems and solutions  
*Lester G. Telser*
- 80 Economics, Ethics and the Market**  
Introduction and applications  
*Johan J. Graafland*
- 81 Social Costs and Public Action in Modern Capitalism**  
Essays inspired by Karl William Kapp's theory of social costs  
*Edited by Wolfram Elsner, Pietro Frigato and Paolo Ramazzotti*
- 82 Globalization and the Myths of Free Trade**  
History, theory and empirical evidence  
*Edited by Anwar Shaikh*
- 83 Equilibrium in Economics: Scope and Limits**  
*Edited by Valeria Mosini*
- 84 Globalization**  
State of the art and perspectives  
*Edited by Stefan A. Schirm*
- 85 Neoliberalism**  
National and regional experiments with global ideas  
*Edited by Ravi K. Roy, Arthur T. Denzau and Thomas D. Willett*

- 86 Post-Keynesian Macroeconomic Economics**  
Essays in honour of Ingrid Rima  
*Edited by Mathew Forstater, Gary Mongiovi and Steven Pressman*
- 87 Consumer Capitalism**  
*Anastasios S. Korkotsides*
- 88 Remapping Gender in the New Global Order**  
*Edited by Marjorie Griffin Cohen and Janine Brodie*
- 89 Hayek and Natural Law**  
*Eric Angner*
- 90 Race and Economic Opportunity in the Twenty-First Century**  
*Edited by Marlene Kim*
- 91 Renaissance in Behavioural Economics**  
Harvey Leibenstein's impact on contemporary economic analysis  
*Edited by Roger Frantz*
- 92 Human Ecology Economics**  
A new framework for global sustainability  
*Edited by Roy E. Allen*
- 93 Imagining Economics Otherwise**  
Encounters with identity/difference  
*Nitasha Kaul*
- 94 Reigniting the Labor Movement**  
Restoring means to ends in a democratic labor movement  
*Gerald Friedman*
- 95 The Spatial Model of Politics**  
*Norman Schofield*
- 96 The Economics of American Judaism**  
*Carmel Ullman Chiswick*
- 97 Critical Political Economy**  
*Christian Arnsperger*
- 98 Culture and Economic Explanation**  
Economics in the US and Japan  
*Donald W. Katzner*
- 99 Feminism, Economics and Utopia**  
Time travelling through paradigms  
*Karin Schönpflug*
- 100 Risk in International Finance**  
*Vikash Yadav*
- 101 Economic Policy and Performance in Industrial Democracies**  
Party governments, central banks and the fiscal-monetary policy mix  
*Takayuki Sakamoto*
- 102 Advances on Income Inequality and Concentration Measures**  
*Edited by Gianni Betti and Achille Lemmi*
- 103 Economic Representations**  
Academic and everyday  
*Edited by David F. Ruccio*
- 104 Mathematical Economics and the Dynamics of Capitalism**  
Goodwin's legacy continued  
*Edited by Peter Flaschel and Michael Landesmann*

# **Mathematical Economics and the Dynamics of Capitalism**

Goodwin's legacy continued

**Edited by Peter Flaschel  
and Michael Landesmann**



First published 2008  
by Routledge

Published 2014 by Routledge  
2 Park Square, Milton Park, Abingdon, Oxfordshire OX14 4RN  
Simultaneously published in the USA and Canada  
by Routledge  
711 Third Avenue, New York, NY 10017

First issued in paperback 2014  
*Routledge is an imprint of the Taylor & Francis Group, an informa  
business*

© 2008 Editorial matter and selection, Peter Flaschel and Michael  
Landesmann; individual chapters, the contributors

Typeset in Times New Roman by  
HWA Text and Data Management, London

All rights reserved. No part of this book may be reprinted or reproduced or  
utilised in any form or by any electronic, mechanical, or other means, now  
known or hereafter invented, including photocopying and recording, or in  
any information storage or retrieval system, without permission in writing  
from the publishers.

The publisher makes no representation, express or implied, with regard to  
the accuracy of the information contained in this book and cannot accept  
any legal responsibility or liability for any efforts or omissions that may  
be made.

*British Library Cataloguing in Publication Data*  
A catalogue record for this book is available from the British Library

*Library of Congress Cataloging-in-Publication Data*  
A catalog record for this book has been requested

ISBN 978-0-415-45145-1 (hbk)  
ISBN 978-0-415-76201-4 (pbk)  
ISBN 978-0-203-92664-2 (ebk)

# Contents

<i>List of figures</i>	<i>xi</i>
<i>List of tables</i>	<i>xiv</i>
<i>List of contributors</i>	<i>xvi</i>

Introduction: mathematical economics and the dynamics of capitalism	1
PETER FLASCHEL AND MICHAEL LANDESMANN	

## PART I

<b>Nonlinear macrodynamics: theory</b>	<b>9</b>
--	----------

1 Coexistence of multiple business cycles in Goodwin's 1951 model	11
AKIO MATSUMOTO AND MAMI SUZUKI	

2 An encompassing theory of macroeconomic growth and fluctuation	24
RICHARD DAY AND CHENGYU YANG	

3 Kaleckian investment and employment cycles in post-war industrialized economies	35
PETER FLASCHEL, REINER FRANKE AND WILLI SEMMLER	

4 A Goodwin cycle with changing regimes of industrial relations	66
JÖRG GLOMBOWSKI AND MICHAEL KRÜGER	

5 Varieties of capitalism: the flexicurity model	76
PETER FLASCHEL, ALRED GREINER, SIGRID LUCHTENBERG AND EDWARD NELL	

**PART II**

**Nonlinear macrodynamics: empirical analysis 105**

- 6 Goodwin cycles and the US economy, 1948–2004 107

SIMON MOHUN AND ROBERTO VENEZIANI

- 7 The classical growth cycle after fifteen years of new observations 131

PETER FLASCHEL, GANGOLF GROH, GÖRAN KAUEMANN AND TIMO TEUBER

- 8 Debt–equity cycles in the twentieth century: empirical evidence and a dynamic Keynesian model 145

LANCE TAYLOR AND CODRINA RADA VON ARNIM

- 9 Monetary policy rules in small open economies: a Keynesian perspective 163

CHRISTOPHER MALIKANE AND WILLI SEMMLER

- 10 Semi-structural Keynes–Goodwin modeling: elaboration and calibration of a baseline theoretical framework 178

REINER FRANKE, PETER FLASCHEL AND CHRISTIAN R. PROAÑO

**PART III**

**Multisectoral and microfounded approaches 205**

- 11 Decomposition methods for analyzing intra-regional and inter-regional income distribution 206

SRINIVAS RAGHAVENDRA

- 12 Computational explorations of vertically coupled markets 218

NICOLÁS GARRIDO

- 13 Microfounded animal spirits and Goodwinian income distribution dynamics 239

REINER FRANKE

- 14 Goodwin’s structural economic dynamics in the context of globalization 255

MICHAEL LANDESMANN AND ROBERT STEHRER

- Index* 284

# Figures

1.1	Endogenous cycle with discontinuous jumps	14
1.2	Cyclical oscillations for different $\theta$ parameters	15
1.3	Limit cycles with different lags	16
1.4	Division of the parameter region	18
1.5	Coexistence of limit cycles	20
1.6	Bifurcation diagram	21
2.1	The influence of real interest on investment	26
2.2	Long-run demand-generated output in efficiency units	29
2.3	The pay-as-you-go fiscal policy	31
3.1	Damped fluctuations in Kaleckian investment and employment dynamics	55
3.2	Eigenvalue diagrams for selected speeds of adjustment	56
3.3	Eigenvalue diagrams for selected benchmark values of the model	57
4.1	Inward spiralling trajectory: basic model	68
4.2	Simulation results I	72
4.3	Simulation results II	73
5.1	Goodwinian wage share/employment rate dynamics	77
5.2	Goodwin-type long-phased wage share/employment dynamic	78
5.3	UK income distribution cycles 1870–2004	79
6.1	Long-run trends: profit rate, profit share and capital productivity, corporate sector, USA, 1948–2004	116
6.2	National employment rate and corporate wage share in corporate net value added, long-run trends, USA, 1948–2004	117
6.3	Detrended cycles, national employment rate and corporate sector wage share, USA, annual data, 1948–2004	118
6.4	Detrended corporate sector cycles, USA, quarterly data, 1949–1980	119
6.5	Detrended corporate sector cycles, USA, quarterly data, 1980–2001	120
6.6	Employment rate and wage shares, USA, 1948–2002	121
6.7	Non-supervisory employees, USA, 1951–1991	122
6.8	Non-supervisory employees, USA, 1991(1994)–2001	123
6.9	Employment rate and all employee wage share, non-farm private industry, trends, USA, 1948–2002	124
7.1	Basic ingredients for persistent and attracting growth cycle dynamics	135

*xii List of figures*

7.2	Time series phase plots: the Goodwin growth cycle and the inflation–unemployment nexus	136
7.3	The Goodwin growth cycle: two-stage least-squares estimation	137
7.4	The Goodwin growth cycle: HP-trend visualization	138
7.5	Exploring US income distribution cycles with bivariate loops using penalized spline regression	140
7.6	Exploring US inflation cycles with bivariate loops using penalized spline regression	141
7.7	UK Income distribution cycles 1855–1965	142
8.1	Post-war debt–equity cycles in the US	147
8.2	Post-war debt–equity cycles in the UK	147
8.3	NIPA-based financial needs of the government, foreign sector, household and the business sectors	148
8.4	Pre-war debt–equity cycles in the US	149
8.5	A post-war debt–equity cycle in Japan	150
8.6	Dynamics with an equity-accelerated debt ratio and debt-led growth	156
8.7	Dynamics with an equity-accelerated debt ratio and debt-burdened growth	156
8.8	“Fast” dynamics and bifurcations in the bull–bear cycles	157
9.1	The Goodwin cycle in a partial framework: the role of policy	173
9.2	Macro-instability under temporary expansionary monetary shock	174
9.3	Macro-instability with more aggressive response to stock price fluctuations	174
10.1	Response of the discrete-time model to a demand shock	188
10.2	Response of the model to a demand shock	192
10.3	VAR impulse-response functions to a demand shock	193
10.4	Adverse real wage effects.	194
10.5	Parameter diagrams under profit-led and wage-led conditions.	195
10.6	Response of the model to a sudden decrease of $v^o$	201
12.1	Transition with and without backlog	222
12.2	Simulation with budget constraints evolution and different costs	224
12.3	Dynamics of the simulation with one period credit and unitary costs higher than equilibrium price	225
12.4	Simulation of coupling market	229
12.5	Price evolution for the 64 economies	231
12.6	First row $g_9 = \{13, 19, 35\}$ and second row $g_7 = \{15, 23, 39\}$	233
12.7	First row $\{13, 19, 35\}$ and second row $\{15, 23, 39\}$	235
12.8	Economies $e_{13}$ and $e_{15}$ with dominance of market 1	235
13.1	Phase diagrams of system (13.13) and (13.14)	247
14.1	Inputs per unit of output by sectors and countries	274
14.2	Price levels and relative prices	274
14.3	Output levels and relative output	275
14.4	Labour demand levels	276
14.5	Unemployment rate (in percent) and wage rate	276
14.6	Skill premium	277
14.7	Labor demand shares by industry	277

14.8	Retained earnings	278
14.9	Output levels and relative output	278
14.10	Labour demand levels	279
14.11	Skill premium	280
14.12	Retained earnings	280

# Tables

1.1	Numerical results	14
3.1	Four types of real wage adjustment processes	36
3.2	The feedback structure of the KMGS model	47
3.3	Four types of market economies	63
4.1	Notation	67
5.1	Firms: production and income account	80
5.2	Households I and II (primary and secondary labor market)	81
5.3	Income account (retired households)	82
5.4	The government income account	83
5.5	Four types of market economies	84
5.6	Firms: production and income account	94
5.7	Investment and credit account	94
5.8	Firms' net worth	95
5.9	Pension funds and credit (stocks)	95
5.10	Pension funds and credit (flows)	95
5.11	Households of types I and II (primary and secondary labor market: income account)	96
5.12	Scheme to fund excess employment and pensioners	96
5.13	Firms: production and income account	100
5.14	Income account: Households I (primary labor market) and Retired Households	100
6.1	Compensation of employees by sector and legal form of organization as a percentage of total compensation of employees, 1948–2004	114
6.2	Cycle date comparisons	123
7.1	Four types of real wage or wage share feedback mechanisms	134
8.1	Balance sheets underlying the debt–equity model	150
8.2	Four Possible Scenarios for the two differential equation system	155
9.1	Estimated parameters	172
10.1	Quarterly data used in the econometric estimations	187
10.2	Alternative econometric estimations of wage and price Phillips curves	190
10.3	Characterization of parameters as stabilizing and destabilizing the steady state position	196
10.4	Standard deviations of the time series	198

10.5	Variabilities in the stochastic model	199
10.6	Characterization of parameters as stabilizing and destabilizing in the stochastic economy	200
11.1	Structure of the inter-regional social accounting matrix (SAM)	215
12.1	Partition of $E$ according to $\mathbf{p}_i = \mathbf{p}_j$	232
12.2	Number of economies with the same composition of commodity markets	233
14.1	Technology parameters	270
14.2	Parameters for consumption behavior	271
14.3	Equilibrium values	272
14.4	Parameter values for adjustment dynamics	273



# Contributors

**Richard Day** Professor Emeritus, University of Southern California, USA.

**Peter Flaschel** Professor of Economics, Bielefeld University, Germany.

**Reiner Franke** Lecturer, Department of Economics, Kiel University, Germany.

**Nicolás Garrido** Associate Professor of Economics, Universidad Católica del Norte, Chile.

**Jörg Glombowski** Professor of Economics, Osnabrück University, Germany.

**Alfred Greiner** Professor of Economics, Bielefeld University, Germany.

**Gangolf Groh** One-Year-Substitute (Chair Professor Peter Flaschel), Bielefeld University, Germany.

**Göran Kauermann** Professor Statistics, Bielefeld University, Germany.

**Michael Krüger** Professor of Economics and Finance, Touro College, Berlin.

**Michael Landesmann** Scientific Director, The Vienna Institute for International Economic Studies (wiiw) and Professor of Economics, Johannes Kepler University Linz, Austria.

**Sigrid Luchtenberg** Professor of Education, University of Duisburg–Essen, Germany.

**Christopher Malikane** Lecturer, Department of Economics, University of Witwatersrand, Johannesburg, South Africa.

**Akio Matsumoto** Professor for Economics, Chuo University, Japan.

**Simon Mohun** Professor of Political Economy, School of Business and Management, Queen Mary, University of London, UK.

**Edward Nell** Professor of Economics, New School for Social Research, New School University, USA.

**Christian R. Proaño** PhD Candidate, Bielefeld University and Associate Researcher, Macroeconomic Policy Institute (IMK), Germany.

**Codrina Rada von Arnim** Assistant Professor of Economics, University of Utah, USA.

**Srinivas Raghavendra** Lecturer, Department of Economics, National University of Ireland, Galway.

**Willi Semmler** Professor of Economics, New School for Social Research, New School University, USA.

**Robert Stehrer** Senior Researcher, The Vienna Institute for International Economic Studies (WIIW), Austria.

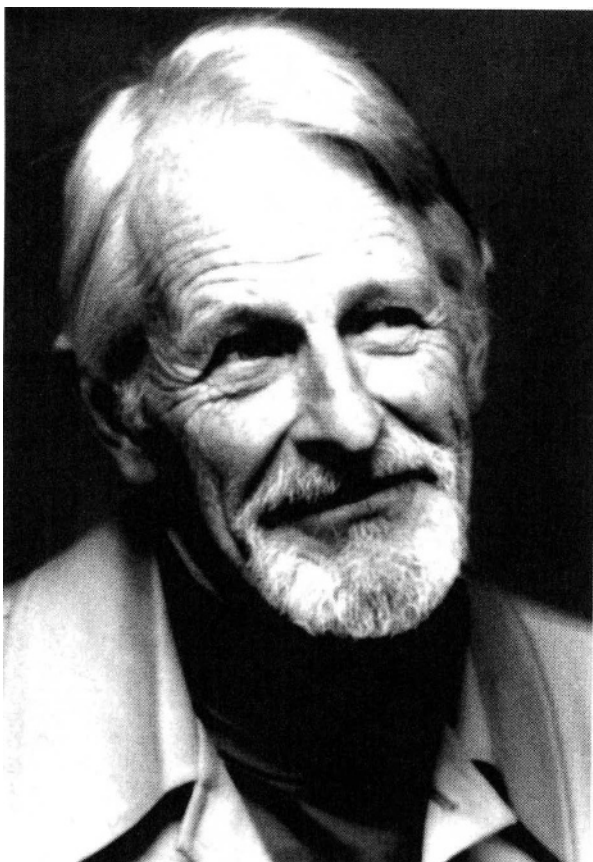
**Mami Suzuki** Department of Mathematics, Obirin University.

**Lance Taylor** Professor of Economics and Director of the Schwartz Center for Economic Policy Analysis, New School for Social Research, New School University, USA.

**Timo Teuber** PhD Candidate, Bielefeld University, Germany.

**Roberto Veneziani** Lecturer, Department of Economics, Queen Mary, University of London, UK.

**Chengyu Yang** Assistant Professor of Business Economics, Central Florida University, USA.



Richard Murphey Goodwin (1913–1996)  
Photograph by Dorothy Hahn

# Introduction

## Mathematical economics and the dynamics of capitalism

*Peter Flaschel and Michael Landesmann*

The capitalist system is of great interest and importance in view of its outstanding dynamism relative to that of other systems tried in the past century. Yet the established body of economic theory – intertemporal, information-theoretic and game-theoretic – does not incorporate key elements of the capitalist dynamic: business innovation as distinct from technological advance and the contributions of entrepreneurs and financiers to the innovation process. As a consequence, established theory cannot capture the core of the dynamism. In fact it contradicts the existence of such dynamism: Capitalism is an evolving, unruly, open-ended system while the theory implies a deterministic future however buffeted it is by stochastic shocks <sup>1</sup>

The editors of this book believe that – going back in history – an ideal candidate for supporting and for actively and significantly contributing to the objectives expressed in the above quotation would have been Richard Goodwin. He was a student of Josef Alois Schumpeter and a colleague of Paul Samuelson, and he experienced his formative period as a researcher at Harvard, Oxford and Cambridge at a time when many new ideas for understanding and modeling capitalism were born and tested. His vision of the workings of capitalism is exemplified and summarized in his article on ‘The M-K-S system. Functioning and Evolution of Capitalism’ (Goodwin, 1989). There he sketched an analytic system consisting of selected elements, suitably combined, of the concepts of three great economists, Marx, Keynes and Schumpeter. The dynamics of capitalism was surely at the center of Marx’s as well as Schumpeter’s theory, while Keynes’ General Theory only sketches in a closing chapter the implications of his approach for economic dynamics. Of course, not all results in Marx’s, Schumpeter’s and Keynes’s main oeuvre are consistent with each other, but suitably combined concepts from these three great economists can be viewed as providing a research agenda which not only guided Goodwin’s research but will continue to provide an anchor to the understanding of the dynamics of capitalism.

Richard Murphey Goodwin (1913–1996) was a pioneer in the use of mathematical tools to understand the dynamics of capitalist economies (see the selected references below). Goodwin’s first productive period was a period in which exciting new techniques of nonlinear dynamics were taken up both in

## 2 *Peter Flaschel and Michael Landesmann*

neoclassical and in heterodox economics. From the early 1950s he developed and applied nonlinear dynamic modeling tools to understand problems of cyclical growth and distributional dynamics. At the same time, he was also interested in changes in the structure of economies, in the composition of activities and of techniques of production, following the traditions of von Neumann and Leontief. Investigating these two research lines led him to innovative formulations to combine decomposition techniques with tools which allow the tracking of macroeconomic dynamics. With respect to the latter, he not only applied interesting modeling techniques to represent (regular) cyclical patterns, but also explored in a number of ways how the irregularity inherent in economic fluctuations can be modeled.

After the Second World War, Richard Goodwin was in a very fertile phase of pursuing and synthesizing a range of contributions to the theory of economic fluctuations, distributional dynamics and economic growth. He also always worked on structural, disaggregated features of capitalist dynamics, thus attempting to bring Schumpeterian features into the analysis of macroeconomic dynamics of capitalist economies. His research program was influenced by economists as diverse as Marx and Schumpeter, Keynes, Hicks and Harrod, Leontief and Sraffa, as well as physicists and mathematicians such as Poincare, LeCorbeiller, Smale, Haken and Roessler. It was persistently shaped by Goodwin into a consistent and rich picture of the dynamics of capitalist economies. His ambitious oeuvre will remain a motivating point of departure for current and future generations of researchers. In particular, Goodwin's seminal contribution, his famous growth cycle model, Goodwin (1967), much admired by Nobel Laureate Robert Solow (1990), had a very strong impact on younger scholars in his fields of research.

Goodwin retired from the University of Cambridge in 1980 where he had been teaching since 1950 and took up a professorship at the University Siena. During the last two decades of his life, Richard Goodwin continued to do research both in Italy and in Cambridge, and he was a focal point for a sizable group of young economists who had taken up his ambitious research programme to advance the study of macroeconomic and structural dynamics.

Richard Goodwin died in 1996. His inspirations continue to exert a significant influence in many respects, as Richard Goodwin managed to inspire a succession of generations of young scholars to continue pursuing his innovative research agenda. In 2005, the editors of this book decided to organize a conference to honor Goodwin's work on the occasion of the tenth anniversary of his death. The conference aimed at bringing together senior as well as younger economists who were actively pursuing research in the areas pushed forward by Goodwin's life-long contributions. The conference was well attended with numerous contributions based on Goodwin's macroeconomic as well as his structural and microeconomic contributions. The contributions in this book, drawn from this conference,<sup>2</sup> as well as the recently published books by Chiarella *et al.* (2000, 2005) and Taylor (2004), see also Barbosa-Filho and Taylor (2006), testify to the continued influence of Goodwin's work. Goodwin's ambitious oeuvre will remain a motivating point of departure for current and future generations of researchers in mathematical

economics and the dynamics of capitalism, a combination in which Richard Goodwin was truly a pioneer.

The introductory quotation points to research lines that are somewhat lacking in mainstream research programs. The quotation represents a vast and demanding research agenda to which the chapters in this book contribute in specific ways, concentrating on theories of fluctuating growth and its interaction with financial markets. Mostly this is put in the context of heterogeneity of economic agents at the micro-level. In contrast to most of the standard approaches to macroeconomic dynamics which are based on representative agents, market clearing and rational expectations, they aim to investigate rigorous, macro-dynamic theory which avoids these assumptions and relate the theory to quantitative and empirical insights and hence provide a foundation for economic and social policy analysis.

In brief, the edited book, which can be considered as a continuation of the research contributions contained in the collected contributions in Velupillai (1990) and Punzo (2001), consists of three parts which significantly extend work in the tradition of Goodwin's research program. Part I studies from various perspectives major, but also some more technical, components of the Marx-Keynes-Schumpeter vision of Richard Goodwin (including also a Kaleckian contribution). Part II provides empirical applications of Goodwin's original macro-dynamics, its extension with financial features as well as its application to monetary policy debates. Part III contains chapters which focus on structural and micro-questions of the research programme.

Starting from the well known fact that Goodwin's 1951 business cycle model gives rise to cyclical oscillations when its stationary point is unstable, Aoki Matsumoto and Mami Suzuki show in Chapter 1 the coexistence of multiple closed orbits; namely, besides a stable stationary point, an unstable limit cycle and a stable limit cycle in the case where the stationary point is locally stable. The authors demonstrate that the model is globally stable in the sense that the trajectories are bounded for any disturbances but its dynamics are different for different disturbances. From these findings the authors derive two properties for Goodwin's (1951) model: first, the existence of robust cyclical oscillations regardless of the local dynamic properties of the steady state and, second, the existence of corridor stability in the locally unstable case.

Richard Day and Chengyu Yang specify in Chapter 2 a Keynesian out-of-equilibrium model of aggregate demand and supply and derive the conditions under which it converges to a Solow growth model. By establishing the conditions for existence of and convergence to steady-state growth – which in turn implies the existence of a supporting social utility function – Day and Yang encompass the independently formulated major macroeconomic theories of Keynes, Solow, and the optimal growth school.

The main topic of Chapter 3 is the role of the welfare state in post-war industrialized economies. Using a Kaleckian framework, Peter Flaschel, Reiner Franke and Willi Semmler consider an economy where investment depends positively on the rate of return on capital and negatively on the rate of employment.

#### 4 *Peter Flaschel and Michael Landesmann*

This simple assumption allows the authors to study Kalecki's (1943) analysis of the economic and political aspects of full employment.

Flaschel *et al.* show that even though there is a balanced growth path solution for the resulting model, it is likely to be locally unstable. The authors then show that, in contrast to the conflict-driven macroeconomy where profit-led goods demand in combination with declining real wages (enforced by mass unemployment and labor market reforms) may account for lower turning points during long-phased depressions, as Kalecki (1943) perceived it, the business leaders and policy makers could pursue a consensus driven macroeconomic strategy stressing a more collaborative and long-term approach which will take the economy on a stable growth path. Starting from the lack of micro-foundations in the original Goodwin (1951) business cycle model, Glombowski and Krüger propose in Chapter 4 a modified theoretical framework based on game theoretical considerations which allows labor unions and employers' organizations to behave more flexibly in the course of time. The authors establish a non-cooperative cyclical path of the economy à la Goodwin, and they allow for the possibility of cooperative behavior between workers and employers, which gives rise to higher employment rates resulting from wage restraints and high investment rates. These two 'regimes' may appear both in a more or less regular pattern, depending on a variety of social, political and institutional circumstances. Glombowski and Krüger do not work out deterministic patterns, but rather refer to stochastic mechanisms giving rise to different sequences of regimes and, therefore, different time paths of the variables under consideration. As the authors show, this procedure generates developments characterized by strong social partnership relations, and, likewise, situations in which workers and capitalists getting into conflict whenever the economic circumstances offer a chance in their favor.

In Chapter 5, Peter Flaschel, Alfred Greiner, Sigrid Luchtenberg and Edward Nell start from the finding that the main stabilizing device in Goodwin's (1967) growth cycle model, the Marxian reserve army mechanism, does not represent a process of social reproduction that can be considered an adequate socio-economic foundation for a democratic society in the long-run. They then derive a basic macro-dynamic framework where this distinct form of cyclical growth and social reproduction is overcome by an employer of 'first' resort, added to an economic reproduction process that is highly competitive and thus not of the type of the past Eastern socialism. There is high labor and capital mobility (concerning 'hiring' and 'firing' in particular) where fluctuations of employment in the private sector are made socially acceptable through a second labor market where all remaining workers get employment and income. The resulting socio-economic system is closely related to the flexicurity model developed and applied in Denmark in particular. The authors show that this economy exhibits a balanced growth path that is globally attracting. Moreover, pension-fund financed investment can be added to this model without disturbing the prevailing situation of stable full capacity growth. The closing section in this chapter shows that a paper credit formulation of this process can lead to Keynesian effective demand constraints and thus Keynesian business fluctuations.

Simon Mohun and Roberto Veneziani provide in Chapter 6 empirical support for an interpretation of the Goodwin growth cycle as identifying the main forces underlying distributive conflicts. However, this occurs in a fragile symbiotic manner as it generates endogenous forces that modify the balance of class power. Goodwin cycles can be seen as the shorter-run cycles that appear around a long-run motion which results from structural change. The authors describe long-run trends in the state variables of the Goodwin model for the US corporate economy from 1948 to 2004; these exhibit both a sharp break at the beginning of the 1980s, and no long-run cycles. Mohun and Veneziani also identify short-run de-trended Goodwin cycles which broadly coincide in period and timing with the NBER dating of (the troughs of) business cycles.

New empirical evidence for the existence of a long-phased cycle in the wage share and the employment rate as described by the Goodwin model and its successors is provided by Peter Flaschel, Gangolf Groh, Göran Kauermann and Timo Teuber in Chapter 7. Their investigation is based not only on new and recent data, now comprising a period of 50 years, but is also based on advanced econometric techniques aimed at a proper separation between trend and business cycles. Their results confirm not only the existence of a long-period clockwise loop with regard to the magnitudes mentioned, but also a similar result for the cyclical interaction between unemployment and inflation.

In Chapter 9, Christopher Malikane and Willi Semmler formulate a Keynesian framework for practical policy analysis in small open economies. At the core of their model is a Goodwin growth cycle process which, when the model is subjected to shocks, produces instability in the macro aggregates. The model features two Phillips curves; one for prices and the other for nominal wages, an aggregate demand bloc with explicit consideration of income distribution effects, a dynamic multiplier output adjustment process, an investment-based version of Okun's law that links the goods market and the labor market, financial markets that exhibit Blanchard and Dornbusch type accelerators, and a Taylor type rule to describe the behavior of the central bank. The model is estimated using South African quarterly data from 1960:1–2006:4. The authors find macroeconomic stability when the central bank responds with sufficient strength to asset prices and when it targets a demand pressure term which is connected to the prevailing rigidities in the economy.

Peter Flaschel, Reiner Franke and Christian Proaño set up in Chapter 10 a Keynes-Goodwin framework to study the interaction between income distribution and the dynamics of wage and price inflation in a closed economy. The specification of labor productivity growth as an endogenous and not only a residual variable of the system allows the authors to specify a law of motion for the wage share which is determined by developments in both the labor and the goods markets. Together with an advanced specification of the inflationary expectations (the inflation climate in the economy) which accounts for central bank credibility and a simple Taylor rule, a semi-structural model is obtained which is rich in various macroeconomic channels that are of great importance in industrialized economies. Through eigen-value analysis of the resulting dynamical framework it is shown that both excessive



wage and price flexibility might be destabilizing for the dynamic behavior of the economy.

Srinivas Raghavendra studies in Chapter 11 the world income distribution using a Richard Stone-Richard Goodwin type of decomposition method. The author's objective is to collate the social accounting matrices of a group of nations (with the Rest of the World treated exogenously) and to decompose the transactions matrix, expressed in a common unit of account, to delineate effects of intra- ('direct effects') country income disparity from inter- ('cross effects') group income disparities. As Raghavendra points out, this decomposition is pertinent from the point of view of understanding whether the inter-regional income disparities are mainly mediated through the intra-regional income disparities implied by their structure of production and thereby assessing the impact of various growth policies on the macro structure of the economies.

Following Goodwin (1947), in Chapter 12 Nicolás Garrido explores the effects of interdependencies between markets. First, Garrido analyzes a basic building block consisting of a Cobweb model with inventories and budget constraints. Second, these markets are interrelated vertically through their demands and supplies. Using simulation analysis, all types of vertical couplings that may exist among four markets are explored and, using the patterns produced by each topology, equivalence classes that generate a partition in the space of all possible topologies are created. Finally, the author analyzes the properties of five types of vertical couplings, identified in the Japanese industrial district of Ohta by Nakanao and White (2006).

In Chapter 13, Reiner Franke considers the formation of an average opinion index in a microfounded framework where the agents switch between two kinds of sentiments with certain transition probabilities. The index can thus represent a general business climate, or the famous animal spirits assumption of Keynes. Circumventing elaborated tools of statistical mechanics that are usually applied in this context, Franke puts forward a more elementary argument that allows one to derive a macroeconomic adjustment equation for the climate variable, which is also shown to contain a global self-stabilization mechanism. Combining this building block with a simple multiplier and a real wage Phillips curve, the author obtains a structurally stable model of Goodwinian growth cycles with a significant herding component.

Finally, in Chapter 14, Landesmann and Stehrer pursue work originally started with Richard Goodwin in the 1990s. In that phase, Goodwin attempted to synthesize, on the one hand, Schumpeterian and Keynesian elements of his various research lines and, on the other hand, to combine the analysis of macro-dynamics with disaggregated structural modeling. In this chapter, the authors introduce a complex model of interdependent economies and analyze the effects of catching-up processes of developing ('Southern') countries. In the first instance, the focus is the changing patterns of international trade specialization and, in the second place, the relative attractiveness of foreign direct investment (in 'Northern' and 'Southern' economies respectively). The authors show that periods in which there is a high level of structural change dynamic (such as that induced through catching-up of

sizable ‘Southern economies’ such as China and India) are also periods which have a higher potential for effective demand failures. A particular issue which the authors highlight in their chapter, given many of the complex features of catching-up processes, are the additional effects which ‘outsourcing’ or a higher scope for global ‘fragmentation’ (of production processes) might induce in a global world of interdependencies.

Summing up, this book collects recent research papers in the tradition of Goodwin’s work on Marx, Keynes and Schumpeter. It contains, on the one hand, contributions that focus on rigorous extensions of Goodwin’s modeling of macro-dynamics and the structural features underlying them. On the other hand, and to a greater extent, it also provides research with a wider perspective, related to Goodwin’s vision of an integrated Marx-Keynes-Schumpeter system. With its selection of theoretical macro and micro models and empirical investigations, this collection of papers therefore contributes to a deeper understanding and a continuation of the research agenda initiated by Richard Goodwin after the Second World, concerning cycles, growth and structural change.

Finally, we want to thank, in the first place, Wolfram Elsner and EAEPE for the possibility to organize the Goodwin conference within the larger framework of the annual conference of the European Association for Evolutionary Political Economy in 2005. We are also very grateful to Terry Clague and Sarah Hastings for all the editorial help we needed during the gestation period of this book. Finally, our thanks go to Christian Proaño for his invaluable assistance in the preparation of the final manuscript.

Peter Flaschel and Michael Landesmann  
Bielefeld and Vienna, September 2007

## Notes

- 1 From *Capitalism and Society. A Journal of the Center on Capitalism and Society, Aims and Scope*. [www.bepress.com/cas/aimsandscope.html](http://www.bepress.com/cas/aimsandscope.html)
- 2 See the Special Issue ‘Richard Murphey Goodwin (1913–1996): His Legacy Continued’ of the journal *Structural Change and Economic Dynamics* (2006:4), edited by Peter Flaschel and Michael Landesmann, for further contributions from the Goodwin conference.

## References

### ***Selected work by Richard Goodwin and co-authors***

- Goodwin, R.M. (1946) Innovations and the irregularity of economic cycles. *Review of Economics and Statistics*, 28, 95–104.
- Goodwin, R.M. (1947) Dynamical Coupling with Especial Reference to Markets Having Production Lags. *Econometrica*, 15, 181–204.
- Goodwin, R.M. (1951) The nonlinear accelerator and the persistence of business cycles. *Econometrica*, 19, 1–17.
- Goodwin, R.M. (1967) A Growth Cycle. In C.H. Feinstein, (ed.) *Socialism, Capitalism and Economic Growth*. Cambridge: Cambridge University Press. Also in Goodwin (1982)

## 8 Peter Flaschel and Michael Landesmann

- Goodwin, R.M. (1982) *Essays in Economic Dynamics*. London: Macmillan.
- Goodwin, R.M. (1983) *Essays in Linear Economic Structures*. London: Macmillan.
- Goodwin, R., M. Krüger and A. Vercelli (eds) *Nonlinear Models of Fluctuating Growth*. Berlin: Springer.
- Goodwin, R.M. (1989) The M-K-S system. Functioning and evolution of capitalisms. In R. Goodwin *Essays in Nonlinear Economic Dynamics*. Bern: Peter Lang, ch.6.
- Goodwin, R.M. (1990) *Chaotic Economic Dynamics*. Oxford: Clarendon Press.
- Goodwin, R.M. and L.F. Punzo (1987) *The Dynamics of a Capitalist Economy. A Multisectoral Approach*. Oxford and Boulder, CO: Polity Press and Westview.

### ***Quoted work on Goodwin's approach to fluctuating growth and structural change***

- Barbosa-Filho, N. and L. Taylor (2006) Distributive and demand cycles in the US economy: a structuralist Goodwin model. *Metroeconomica*, (57):3, 389–411.
- Chiarella, C. and P. Flaschel (2000): *The Dynamics of Keynesian Monetary Growth. Macrofoundations*. Cambridge, UK: Cambridge University Press.
- Chiarella, C., P. Flaschel and R. Franke (2005): *Foundations of a Disequilibrium Theory of the Business Cycle*. Cambridge: Cambridge University Press.
- Kalecki, M. (1943) Political aspects of full employment. Reprinted in M. Kalecki (1971), *Selected Essays on the Dynamics of the Capitalist Economy*. Cambridge: Cambridge University Press,.
- Nakano, T. and D.R. White (2006) *The Visible Hand in a Production-Chain Market: A Market Equilibrium from Network Analytical Perspective*. SFI Working Paper.
- Punzo, L.F. (ed.) (2001) *Cycles, Growth and Structural Change*. London: Routledge.
- Solow, R. (1990) Goodwin's growth cycle: reminiscence and rumination. In Velupillai, K. (ed.) *Nonlinear and Multisectoral Macrodynamics*. London: Macmillan.
- Taylor, L. (2004) *Reconstructing Macroeconomics: Structuralists Proposals and Critiques of the Mainstream*. Cambridge, MA: Harvard University Press.
- Velupillai, K., ed. (1990): *Nonlinear and Multisectoral Macrodynamics*. London: Macmillan.

**Part I**

# **Nonlinear macrodynamics: theory**

This page intentionally left blank

# 1 Coexistence of multiple business cycles in Goodwin's 1951 model

*Akio Matsumoto and Mami Suzuki*

## Introduction

The contribution of Goodwin (1951) is reconsidered in this study. Goodwin developed a nonlinear accelerator business cycle model and showed that it could generate a stable limit cycle when a stationary point was locally unstable. Considerable effort has been devoted to investigate the dynamic structure of Goodwin's model since then. However, in the existing literature, not much has yet been revealed with respect to the circumstances under which the stationary point is locally stable. In particular, it is not yet known whether cyclical global dynamics may appear in the stable case. We draw attention to this unexplored case and exhibit the coexistence of multiple limit states, namely a stable stationary point, an unstable limit cycle and a stable limit cycle. Since very few explorations have been made in the global dynamics of the stable case, this study is intended as an investigation of an unexplored aspect of Goodwin's nonlinear business cycle model.

Goodwin proposed five different versions of his business cycle model. The first version assumes a piecewise linear function with three levels of investment, which can be thought as the crudest or simplest of the nonlinear accelerator. This is a textbook model that can give a simple exhibition of how nonlinearities give rise to endogenous cycles without relying on structurally unstable parameters, exogenous shocks, etc. The second version replaces the piecewise linear investment function with a smooth nonlinear investment function. Although persistent cyclical oscillations of output are shown to exist, the second version includes a unfavorable phenomenon, namely, discontinuous investment jumps, which is not observed in the real economic world. "In order to come close to reality" (Goodwin 1951: 11), a production lag is introduced in the third version. However, no dynamic considerations are given to this third version by Goodwin. The existence of a stable limit cycle is examined in the fourth version, which is a linear approximation of the third version with respect to the production lag. Goodwin's final modification makes the amount of autonomous expenditure alter over time. This fifth version is recently reconsidered by Lorenz (1987) as a forced oscillator system in which the emergence of chaotic motion is demonstrated. More recently Sasakura (1996) gave an elegant proof of the stability and uniqueness of Goodwin's cycle for the fourth version. Thus it has been confirmed that Goodwin's nonlinear accelerator model possesses a unique stable limit cycle. Since all these results are obtained when the stationary point is locally unstable, we can ask a basic question: *Is cyclical behavior robust under locally stable circumstances?*

The main result of this study is to provide a positive answer to this question. For this purpose, we augment Goodwin's fourth version by introducing a nonlinear investment function of arctangent type and demonstrate the coexistence of multiple cycles. We

will combine the Poincaré-Bendixson theorem with the Hopf bifurcation theorem and characterize the global dynamics when the stationary state is locally stable. The coexistence of multiple cycles is also shown for Kaldor's business cycle model in Grasman and Wentzel (1994) and for a Metzlerian inventory cycle model in Matsumoto (1996) using an approach which we continue here.

We also examine the dynamics of the third version with an unstable stationary point. Only limited efforts have been devoted to this version in the past. Since it is a nonlinear differential equation, attempts at analytical solution seem fruitless. Hence, we perform numerical simulations to find what effects the production lag produces on the characteristics (i.e. length and amplitude) of cyclical oscillations of the output.

The following section "Goodwin business cycle model" overviews three versions (i.e. the second, third and fourth versions) of Goodwin's business cycle model and reveals the characteristics of Goodwin's cycle. The section "Coexistence of multiple cycles" presents the new results which shows that the linear approximated version exhibits corridor stability in which the solution is stable for small shocks but is unstable and generates multiple limit cycles for large shocks. Concluding remarks are made in the final section.

## Goodwin business cycle model

This section is divided into three parts. Each of three versions of Goodwin's model are reviewed in each subsection. In particular, we recapitulate the basic elements of the second version and numerically simulate the model to see what dynamics it can generate in "Basic model". We then introduce a production lag into the second version to get the third version and perform, again, numerical simulations to find out how the lag affects the characteristics (i.e. the length of a period and the amplitude) of endogenous cycles in "Delayed model". Finally, we derive the most popular version, the fourth version, by expanding the third version with respect to the lag and reveal its stability condition in "Approximated model".

### *Basic model*

The second model, which we call the *basic model*, is summarized as follows.

$$\begin{cases} \varepsilon \dot{y}(t) = \dot{k}(t) - (1 - \alpha)y(t), \\ \dot{k}(t) = \varphi(\dot{y}(t)). \end{cases} \quad (1.1)$$

Here  $k$  is the capital stock,  $y$  the income,  $\alpha$  the marginal propensity to consume which is positive and less than unity, and the reciprocal of  $\varepsilon$  is the adjustment coefficient and positive. The first equation of (1.1) defines an adjustment process of national income in a such a way that national income rises or falls if investment is greater or less than savings. The second is an adjustment process of capital stock based on the acceleration principle, according to which investment depends on the rate of changes in national income. On the other hand, we depart from Goodwin's non-essential assumption of positive autonomous expenditure and will work with zero autonomous expenditure in the interest of simplicity. A direct consequence of this assumption is that an equilibrium solution or a stationary point of the basic model is  $y(t) = \dot{y}(t) = 0$  for all  $t$ . Inserting the second equation of (1.1) into the first and arranging terms gives dynamics equation of the national income,

$$\varepsilon \dot{y}(t) - \varphi(\dot{y}(t)) + (1 - \alpha)y(t) = 0. \quad (1.2)$$