

# Mathematical Economics and the Dynamics of Capitalism

Goodwin's legacy continued

*Edited by*  
**Peter Flaschel** and  
**Michael Landesmann**

Routledge Frontiers of Political Economy

# Mathematical Economics and the Dynamics of Capitalism

Richard Goodwin was a pioneer in the use of mathematical tools to understand the dynamics of capitalist economies. This book contains contributions which focus on the rigorous extension of Goodwin's modelling of macro-dynamics and the micro-structures underlying them, and also research with a wider perspective related to Goodwin's vision of an integrated Marx-Keynes-Schumpeter (M-K-S) system of the dynamics of capitalist economies.

The variety of approaches in this book range from detailed business cycle analyses to Schumpeterian processes of creative destruction. They include

- thorough theoretical analysis of delayed dynamical systems;
- empirical studies of Goodwin's classical growth cycle model and the integration of Keynesian aspects of effective demand and of financial mechanisms that impact the real macro-economy;
- micro-economic structural analysis;
- expectations driven aspects of micro-founded business cycle modelling.

Providing both theoretical analysis and empirical evidence for approaches in the tradition of Goodwin's M-K-S system, this book represents a significant step forward in this type of research agenda. The volume is useful for students and researchers in the areas of nonlinear macro and micro-economic modelling, in business cycle modelling and in frontier research regarding classical, Schumpeterian and Keynesian modelling of the dynamics of capitalist economies.

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# **Mathematical Economics and the Dynamics of Capitalism**

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**Edited by Peter Flaschel  
and Michael Landesmann**

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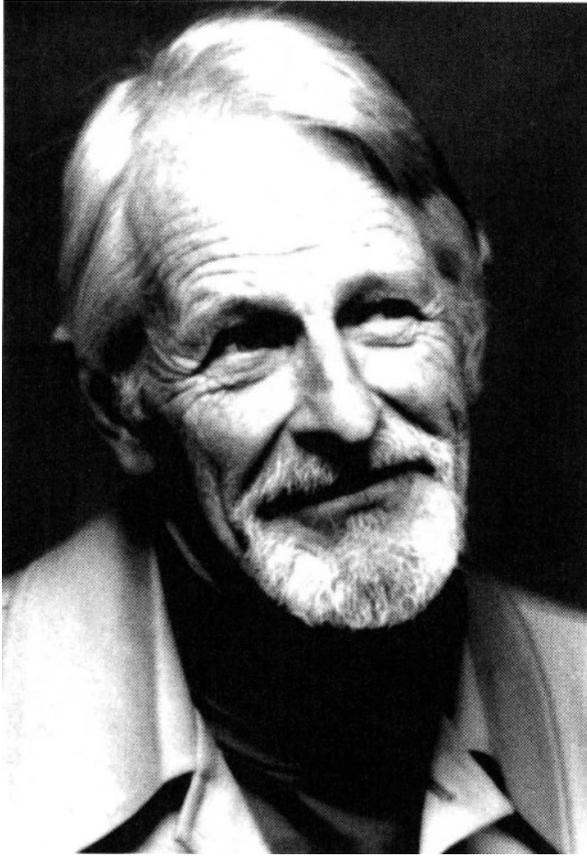
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Richard Murphey Goodwin (1913–1996)

Photograph by Dorothy Hahn

# Introduction

## Mathematical economics and the dynamics of capitalism

*Peter Flaschel and Michael Landesmann*

The capitalist system is of great interest and importance in view of its outstanding dynamism relative to that of other systems tried in the past century. Yet the established body of economic theory – intertemporal, information-theoretic and game-theoretic – does not incorporate key elements of the capitalist dynamic: business innovation as distinct from technological advance and the contributions of entrepreneurs and financiers to the innovation process. As a consequence, established theory cannot capture the core of the dynamism. In fact it contradicts the existence of such dynamism: Capitalism is an evolving, unruly, open-ended system while the theory implies a deterministic future however buffeted it is by stochastic shocks <sup>1</sup>

The editors of this book believe that – going back in history – an ideal candidate for supporting and for actively and significantly contributing to the objectives expressed in the above quotation would have been Richard Goodwin. He was a student of Josef Alois Schumpeter and a colleague of Paul Samuelson, and he experienced his formative period as a researcher at Harvard, Oxford and Cambridge at a time when many new ideas for understanding and modeling capitalism were born and tested. His vision of the workings of capitalism is exemplified and summarized in his article on ‘The M-K-S system. Functioning and Evolution of Capitalism’ (Goodwin, 1989). There he sketched an analytic system consisting of selected elements, suitably combined, of the concepts of three great economists, Marx, Keynes and Schumpeter. The dynamics of capitalism was surely at the center of Marx’s as well as Schumpeter’s theory, while Keynes’ General Theory only sketches in a closing chapter the implications of his approach for economic dynamics. Of course, not all results in Marx’s, Schumpeter’s and Keynes’s main oeuvre are consistent with each other, but suitably combined concepts from these three great economists can be viewed as providing a research agenda which not only guided Goodwin’s research but will continue to provide an anchor to the understanding of the dynamics of capitalism.

Richard Murphey Goodwin (1913–1996) was a pioneer in the use of mathematical tools to understand the dynamics of capitalist economies (see the selected references below). Goodwin’s first productive period was a period in which exciting new techniques of nonlinear dynamics were taken up both in

## 2 *Peter Flaschel and Michael Landesmann*

neoclassical and in heterodox economics. From the early 1950s he developed and applied nonlinear dynamic modeling tools to understand problems of cyclical growth and distributional dynamics. At the same time, he was also interested in changes in the structure of economies, in the composition of activities and of techniques of production, following the traditions of von Neumann and Leontief. Investigating these two research lines led him to innovative formulations to combine decomposition techniques with tools which allow the tracking of macroeconomic dynamics. With respect to the latter, he not only applied interesting modeling techniques to represent (regular) cyclical patterns, but also explored in a number of ways how the irregularity inherent in economic fluctuations can be modeled.

After the Second World War, Richard Goodwin was in a very fertile phase of pursuing and synthesizing a range of contributions to the theory of economic fluctuations, distributional dynamics and economic growth. He also always worked on structural, disaggregated features of capitalist dynamics, thus attempting to bring Schumpeterian features into the analysis of macroeconomic dynamics of capitalist economies. His research program was influenced by economists as diverse as Marx and Schumpeter, Keynes, Hicks and Harrod, Leontief and Sraffa, as well as physicists and mathematicians such as Poincare, LeCorbeiller, Smale, Haken and Roessler. It was persistently shaped by Goodwin into a consistent and rich picture of the dynamics of capitalist economies. His ambitious oeuvre will remain a motivating point of departure for current and future generations of researchers. In particular, Goodwin's seminal contribution, his famous growth cycle model, Goodwin (1967), much admired by Nobel Laureate Robert Solow (1990), had a very strong impact on younger scholars in his fields of research.

Goodwin retired from the University of Cambridge in 1980 where he had been teaching since 1950 and took up a professorship at the University Siena. During the last two decades of his life, Richard Goodwin continued to do research both in Italy and in Cambridge, and he was a focal point for a sizable group of young economists who had taken up his ambitious research programme to advance the study of macroeconomic and structural dynamics.

Richard Goodwin died in 1996. His inspirations continue to exert a significant influence in many respects, as Richard Goodwin managed to inspire a succession of generations of young scholars to continue pursuing his innovative research agenda. In 2005, the editors of this book decided to organize a conference to honor Goodwin's work on the occasion of the tenth anniversary of his death. The conference aimed at bringing together senior as well as younger economists who were actively pursuing research in the areas pushed forward by Goodwin's life-long contributions. The conference was well attended with numerous contributions based on Goodwin's macroeconomic as well as his structural and microeconomic contributions. The contributions in this book, drawn from this conference,<sup>2</sup> as well as the recently published books by Chiarella *et al.* (2000, 2005) and Taylor (2004), see also Barbosa-Filho and Taylor (2006), testify to the continued influence of Goodwin's work. Goodwin's ambitious oeuvre will remain a motivating point of departure for current and future generations of researchers in mathematical

economics and the dynamics of capitalism, a combination in which Richard Goodwin was truly a pioneer.

The introductory quotation points to research lines that are somewhat lacking in mainstream research programs. The quotation represents a vast and demanding research agenda to which the chapters in this book contribute in specific ways, concentrating on theories of fluctuating growth and its interaction with financial markets. Mostly this is put in the context of heterogeneity of economic agents at the micro-level. In contrast to most of the standard approaches to macroeconomic dynamics which are based on representative agents, market clearing and rational expectations, they aim to investigate rigorous, macro-dynamic theory which avoids these assumptions and relate the theory to quantitative and empirical insights and hence provide a foundation for economic and social policy analysis.

In brief, the edited book, which can be considered as a continuation of the research contributions contained in the collected contributions in Velupillai (1990) and Punzo (2001), consists of three parts which significantly extend work in the tradition of Goodwin's research program. Part I studies from various perspectives major, but also some more technical, components of the Marx-Keynes-Schumpeter vision of Richard Goodwin (including also a Kaleckian contribution). Part II provides empirical applications of Goodwin's original macro-dynamics, its extension with financial features as well as its application to monetary policy debates. Part III contains chapters which focus on structural and micro-questions of the research programme.

Starting from the well known fact that Goodwin's 1951 business cycle model gives rise to cyclical oscillations when its stationary point is unstable, Aoki Matsumoto and Mami Suzuki show in Chapter 1 the coexistence of multiple closed orbits; namely, besides a stable stationary point, an unstable limit cycle and a stable limit cycle in the case where the stationary point is locally stable. The authors demonstrate that the model is globally stable in the sense that the trajectories are bounded for any disturbances but its dynamics are different for different disturbances. From these findings the authors derive two properties for Goodwin's (1951) model: first, the existence of robust cyclical oscillations regardless of the local dynamic properties of the steady state and, second, the existence of corridor stability in the locally unstable case.

Richard Day and Chengyu Yang specify in Chapter 2 a Keynesian out-of-equilibrium model of aggregate demand and supply and derive the conditions under which it converges to a Solow growth model. By establishing the conditions for existence of and convergence to steady-state growth – which in turn implies the existence of a supporting social utility function – Day and Yang encompass the independently formulated major macroeconomic theories of Keynes, Solow, and the optimal growth school.

The main topic of Chapter 3 is the role of the welfare state in post-war industrialized economies. Using a Kaleckian framework, Peter Flaschel, Reiner Franke and Willi Semmler consider an economy where investment depends positively on the rate of return on capital and negatively on the rate of employment.

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This simple assumption allows the authors to study Kalecki's (1943) analysis of the economic and political aspects of full employment.

Flaschel *et al.* show that even though there is a balanced growth path solution for the resulting model, it is likely to be locally unstable. The authors then show that, in contrast to the conflict-driven macroeconomy where profit-led goods demand in combination with declining real wages (enforced by mass unemployment and labor market reforms) may account for lower turning points during long-phased depressions, as Kalecki (1943) perceived it, the business leaders and policy makers could pursue a consensus driven macroeconomic strategy stressing a more collaborative and long-term approach which will take the economy on a stable growth path. Starting from the lack of micro-foundations in the original Goodwin (1951) business cycle model, Glombowski and Krüger propose in Chapter 4 a modified theoretical framework based on game theoretical considerations which allows labor unions and employers' organizations to behave more flexibly in the course of time. The authors establish a non-cooperative cyclical path of the economy à la Goodwin, and they allow for the possibility of cooperative behavior between workers and employers, which gives rise to higher employment rates resulting from wage restraints and high investment rates. These two 'regimes' may appear both in a more or less regular pattern, depending on a variety of social, political and institutional circumstances. Glombowski and Krüger do not work out deterministic patterns, but rather refer to stochastic mechanisms giving rise to different sequences of regimes and, therefore, different time paths of the variables under consideration. As the authors show, this procedure generates developments characterized by strong social partnership relations, and, likewise, situations in which workers and capitalists getting into conflict whenever the economic circumstances offer a chance in their favor.

In Chapter 5, Peter Flaschel, Alfred Greiner, Sigrid Luchtenberg and Edward Nell start from the finding that the main stabilizing device in Goodwin's (1967) growth cycle model, the Marxian reserve army mechanism, does not represent a process of social reproduction that can be considered an adequate socio-economic foundation for a democratic society in the long-run. They then derive a basic macro-dynamic framework where this distinct form of cyclical growth and social reproduction is overcome by an employer of 'first' resort, added to an economic reproduction process that is highly competitive and thus not of the type of the past Eastern socialism. There is high labor and capital mobility (concerning 'hiring' and 'firing' in particular) where fluctuations of employment in the private sector are made socially acceptable through a second labor market where all remaining workers get employment and income. The resulting socio-economic system is closely related to the flexicurity model developed and applied in Denmark in particular. The authors show that this economy exhibits a balanced growth path that is globally attracting. Moreover, pension-fund financed investment can be added to this model without disturbing the prevailing situation of stable full capacity growth. The closing section in this chapter shows that a paper credit formulation of this process can lead to Keynesian effective demand constraints and thus Keynesian business fluctuations.

Simon Mohun and Roberto Veneziani provide in Chapter 6 empirical support for an interpretation of the Goodwin growth cycle as identifying the main forces underlying distributive conflicts. However, this occurs in a fragile symbiotic manner as it generates endogenous forces that modify the balance of class power. Goodwin cycles can be seen as the shorter-run cycles that appear around a long-run motion which results from structural change. The authors describe long-run trends in the state variables of the Goodwin model for the US corporate economy from 1948 to 2004; these exhibit both a sharp break at the beginning of the 1980s, and no long-run cycles. Mohun and Veneziani also identify short-run de-trended Goodwin cycles which broadly coincide in period and timing with the NBER dating of (the troughs of) business cycles.

New empirical evidence for the existence of a long-phased cycle in the wage share and the employment rate as described by the Goodwin model and its successors is provided by Peter Flaschel, Gangolf Groh, Göran Kauermann and Timo Teuber in Chapter 7. Their investigation is based not only on new and recent data, now comprising a period of 50 years, but is also based on advanced econometric techniques aimed at a proper separation between trend and business cycles. Their results confirm not only the existence of a long-period clockwise loop with regard to the magnitudes mentioned, but also a similar result for the cyclical interaction between unemployment and inflation.

In Chapter 9, Christopher Malikane and Willi Semmler formulate a Keynesian framework for practical policy analysis in small open economies. At the core of their model is a Goodwin growth cycle process which, when the model is subjected to shocks, produces instability in the macro aggregates. The model features two Phillips curves; one for prices and the other for nominal wages, an aggregate demand bloc with explicit consideration of income distribution effects, a dynamic multiplier output adjustment process, an investment-based version of Okun's law that links the goods market and the labor market, financial markets that exhibit Blanchard and Dornbusch type accelerators, and a Taylor type rule to describe the behavior of the central bank. The model is estimated using South African quarterly data from 1960:1–2006:4. The authors find macroeconomic stability when the central bank responds with sufficient strength to asset prices and when it targets a demand pressure term which is connected to the prevailing rigidities in the economy.

Peter Flaschel, Reiner Franke and Christian Proaño set up in Chapter 10 a Keynes-Goodwin framework to study the interaction between income distribution and the dynamics of wage and price inflation in a closed economy. The specification of labor productivity growth as an endogenous and not only a residual variable of the system allows the authors to specify a law of motion for the wage share which is determined by developments in both the labor and the goods markets. Together with an advanced specification of the inflationary expectations (the inflation climate in the economy) which accounts for central bank credibility and a simple Taylor rule, a semi-structural model is obtained which is rich in various macroeconomic channels that are of great importance in industrialized economies. Through eigen-value analysis of the resulting dynamical framework it is shown that both excessive

wage and price flexibility might be destabilizing for the dynamic behavior of the economy.

Srinivas Raghavendra studies in Chapter 11 the world income distribution using a Richard Stone-Richard Goodwin type of decomposition method. The author's objective is to collate the social accounting matrices of a group of nations (with the Rest of the World treated exogenously) and to decompose the transactions matrix, expressed in a common unit of account, to delineate effects of intra- ('direct effects') country income disparity from inter- ('cross effects') group income disparities. As Raghavendra points out, this decomposition is pertinent from the point of view of understanding whether the inter-regional income disparities are mainly mediated through the intra-regional income disparities implied by their structure of production and thereby assessing the impact of various growth policies on the macro structure of the economies.

Following Goodwin (1947), in Chapter 12 Nicolás Garrido explores the effects of interdependencies between markets. First, Garrido analyzes a basic building block consisting of a Cobweb model with inventories and budget constraints. Second, these markets are interrelated vertically through their demands and supplies. Using simulation analysis, all types of vertical couplings that may exist among four markets are explored and, using the patterns produced by each topology, equivalence classes that generate a partition in the space of all possible topologies are created. Finally, the author analyzes the properties of five types of vertical couplings, identified in the Japanese industrial district of Ohta by Nakanao and White (2006).

In Chapter 13, Reiner Franke considers the formation of an average opinion index in a microfounded framework where the agents switch between two kinds of sentiments with certain transition probabilities. The index can thus represent a general business climate, or the famous animal spirits assumption of Keynes. Circumventing elaborated tools of statistical mechanics that are usually applied in this context, Franke puts forward a more elementary argument that allows one to derive a macroeconomic adjustment equation for the climate variable, which is also shown to contain a global self-stabilization mechanism. Combining this building block with a simple multiplier and a real wage Phillips curve, the author obtains a structurally stable model of Goodwinian growth cycles with a significant herding component.

Finally, in Chapter 14, Landesmann and Stehrer pursue work originally started with Richard Goodwin in the 1990s. In that phase, Goodwin attempted to synthesize, on the one hand, Schumpeterian and Keynesian elements of his various research lines and, on the other hand, to combine the analysis of macro-dynamics with disaggregated structural modeling. In this chapter, the authors introduce a complex model of interdependent economies and analyze the effects of catching-up processes of developing ('Southern') countries. In the first instance, the focus is the changing patterns of international trade specialization and, in the second place, the relative attractiveness of foreign direct investment (in 'Northern' and 'Southern' economies respectively). The authors show that periods in which there is a high level of structural change dynamic (such as that induced through catching-up of

sizable ‘Southern economies’ such as China and India) are also periods which have a higher potential for effective demand failures. A particular issue which the authors highlight in their chapter, given many of the complex features of catching-up processes, are the additional effects which ‘outsourcing’ or a higher scope for global ‘fragmentation’ (of production processes) might induce in a global world of interdependencies.

Summing up, this book collects recent research papers in the tradition of Goodwin’s work on Marx, Keynes and Schumpeter. It contains, on the one hand, contributions that focus on rigorous extensions of Goodwin’s modeling of macro-dynamics and the structural features underlying them. On the other hand, and to a greater extent, it also provides research with a wider perspective, related to Goodwin’s vision of an integrated Marx-Keynes-Schumpeter system. With its selection of theoretical macro and micro models and empirical investigations, this collection of papers therefore contributes to a deeper understanding and a continuation of the research agenda initiated by Richard Goodwin after the Second World, concerning cycles, growth and structural change.

Finally, we want to thank, in the first place, Wolfram Elsner and EAEPE for the possibility to organize the Goodwin conference within the larger framework of the annual conference of the European Association for Evolutionary Political Economy in 2005. We are also very grateful to Terry Clague and Sarah Hastings for all the editorial help we needed during the gestation period of this book. Finally, our thanks go to Christian Proaño for his invaluable assistance in the preparation of the final manuscript.

Peter Flaschel and Michael Landesmann  
Bielefeld and Vienna, September 2007

## Notes

- 1 From *Capitalism and Society. A Journal of the Center on Capitalism and Society, Aims and Scope*. [www.bepress.com/cas/aimsandscope.html](http://www.bepress.com/cas/aimsandscope.html)
- 2 See the Special Issue ‘Richard Murphey Goodwin (1913–1996): His Legacy Continued’ of the journal *Structural Change and Economic Dynamics* (2006:4), edited by Peter Flaschel and Michael Landesmann, for further contributions from the Goodwin conference.

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**Part I**

**Nonlinear macrodynamics:  
theory**

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# 1 Coexistence of multiple business cycles in Goodwin's 1951 model

*Akio Matsumoto and Mami Suzuki*

## Introduction

The contribution of Goodwin (1951) is reconsidered in this study. Goodwin developed a nonlinear accelerator business cycle model and showed that it could generate a stable limit cycle when a stationary point was locally unstable. Considerable effort has been devoted to investigate the dynamic structure of Goodwin's model since then. However, in the existing literature, not much has yet been revealed with respect to the circumstances under which the stationary point is locally stable. In particular, it is not yet known whether cyclical global dynamics may appear in the stable case. We draw attention to this unexplored case and exhibit the coexistence of multiple limit states, namely a stable stationary point, an unstable limit cycle and a stable limit cycle. Since very few explorations have been made in the global dynamics of the stable case, this study is intended as an investigation of an unexplored aspect of Goodwin's nonlinear business cycle model.

Goodwin proposed five different versions of his business cycle model. The first version assumes a piecewise linear function with three levels of investment, which can be thought as the crudest or simplest of the nonlinear accelerator. This is a textbook model that can give a simple exhibition of how nonlinearities give rise to endogenous cycles without relying on structurally unstable parameters, exogenous shocks, etc. The second version replaces the piecewise linear investment function with a smooth nonlinear investment function. Although persistent cyclical oscillations of output are shown to exist, the second version includes a unfavorable phenomenon, namely, discontinuous investment jumps, which is not observed in the real economic world. "In order to come close to reality" (Goodwin 1951: 11), a production lag is introduced in the third version. However, no dynamic considerations are given to this third version by Goodwin. The existence of a stable limit cycle is examined in the fourth version, which is a linear approximation of the third version with respect to the production lag. Goodwin's final modification makes the amount of autonomous expenditure alter over time. This fifth version is recently reconsidered by Lorenz (1987) as a forced oscillator system in which the emergence of chaotic motion is demonstrated. More recently Sasakura (1996) gave an elegant proof of the stability and uniqueness of Goodwin's cycle for the fourth version. Thus it has been confirmed that Goodwin's nonlinear accelerator model possesses a unique stable limit cycle. Since all these results are obtained when the stationary point is locally unstable, we can ask a basic question: *Is cyclical behavior robust under locally stable circumstances?*

The main result of this study is to provide a positive answer to this question. For this purpose, we augment Goodwin's fourth version by introducing a nonlinear investment function of arctangent type and demonstrate the coexistence of multiple cycles. We

will combine the Poincaré-Bendixson theorem with the Hopf bifurcation theorem and characterize the global dynamics when the stationary state is locally stable. The coexistence of multiple cycles is also shown for Kaldor's business cycle model in Grasman and Wentzel (1994) and for a Metzlerian inventory cycle model in Matsumoto (1996) using an approach which we continue here.

We also examine the dynamics of the third version with an unstable stationary point. Only limited efforts has been devoted to this version in the past. Since it is a nonlinear differential equation, attempts at analytical solution seem fruitless. Hence, we perform numerical simulations to find what effects the production lag produces on the characteristics (i.e. length and amplitude) of cyclical oscillations of the output.

The following section "Goodwin business cycle model" overviews three versions (i.e. the second, third and fourth versions) of Goodwin's business cycle model and reveals the characteristics of Goodwin's cycle. The section "Coexistence of multiple cycles" presents the new results which shows that the linear approximated version exhibits corridor stability in which the solution is stable for small shocks but is unstable and generates multiple limit cycles for large shocks. Concluding remarks are made in the final section.

## Goodwin business cycle model

This section is divided into three parts. Each of three versions of Goodwin's model are reviewed in each subsection. In particular, we recapitulate the basic elements of the second version and numerically simulate the model to see what dynamics it can generate in "Basic model". We then introduce a production lag into the second version to get the third version and perform, again, numerical simulations to find out how the lag affects the characteristics (i.e. the length of a period and the amplitude) of endogenous cycles in "Delayed model". Finally, we derive the most popular version, the fourth version, by expanding the third version with respect to the lag and reveal its stability condition in "Approximated model".

### *Basic model*

The second model, which we call the *basic model*, is summarized as follows.

$$\begin{cases} \varepsilon \dot{y}(t) = \dot{k}(t) - (1 - \alpha)y(t), \\ \dot{k}(t) = \varphi(\dot{y}(t)). \end{cases} \quad (1.1)$$

Here  $k$  is the capital stock,  $y$  the income,  $\alpha$  the marginal propensity to consume which is positive and less than unity, and the reciprocal of  $\varepsilon$  is the adjustment coefficient and positive. The first equation of (1.1) defines an adjustment process of national income in a such a way that national income rises or falls if investment is greater or less than savings. The second is an adjustment process of capital stock based on the acceleration principle, according to which investment depends on the rate of changes in national income. On the other hand, we depart from Goodwin's non-essential assumption of positive autonomous expenditure and will work with zero autonomous expenditure in the interest of simplicity. A direct consequence of this assumption is that an equilibrium solution or a stationary point of the basic model is  $y(t) = \dot{y}(t) = 0$  for all  $t$ . Inserting the second equation of (1.1) into the first and arranging terms gives dynamics equation of the national income,

$$\varepsilon \dot{y}(t) - \varphi(\dot{y}(t)) + (1 - \alpha)y(t) = 0. \quad (1.2)$$