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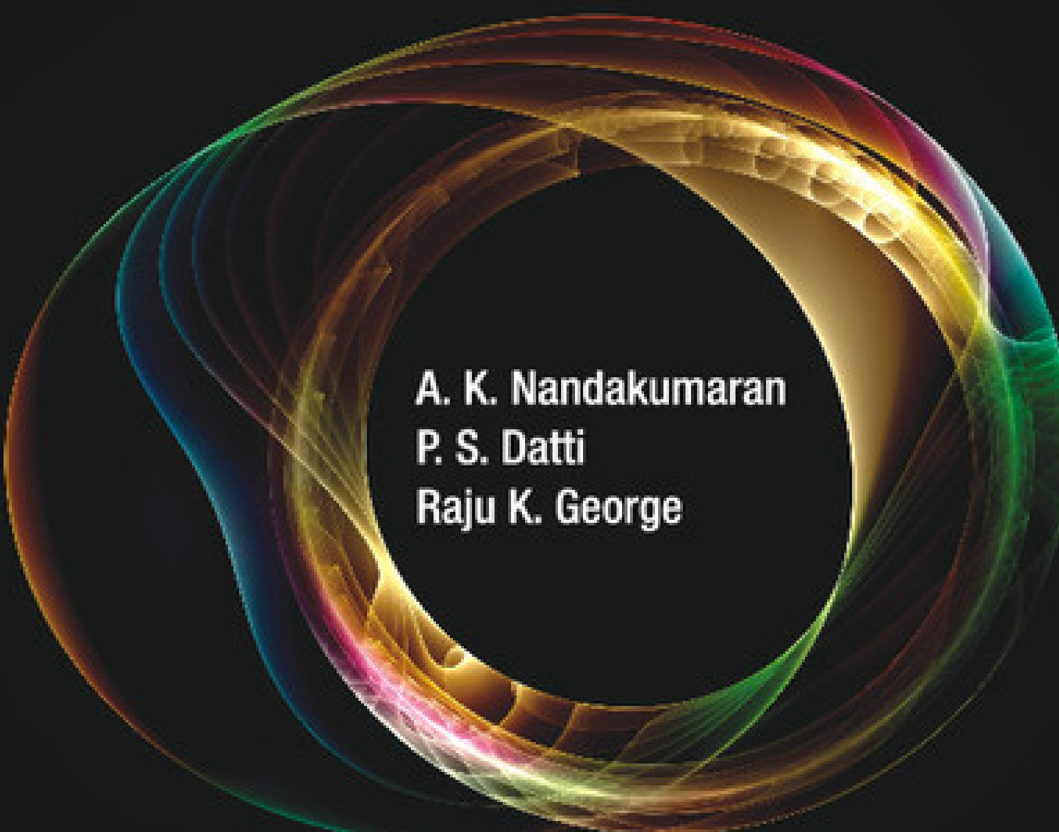
Ordinary Differential Equations

Principles and Applications



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Ordinary Differential Equations

Many interesting and important real life problems in the field of mathematics, physics, chemistry, biology, engineering, economics, sociology and psychology are modelled using the tools and techniques of ordinary differential equations (ODEs). This book offers detailed treatment on fundamental concepts of ordinary differential equations. Important topics including first and second order linear equations, initial value problems and qualitative theory are presented in separate chapters. The concepts of physical models and first order partial differential equations are discussed in detail. The text covers two-point boundary value problems for second order linear and nonlinear equations. Using two linearly independent solutions, a Green's function is also constructed for given boundary conditions.

The text emphasizes the use of calculus concepts in justification and analysis of equations to get solutions in explicit form. While discussing first order linear systems, tools from linear algebra are used and the importance of these tools is clearly explained in the book. Real life applications are interspersed throughout the book. The methods and tricks to solve numerous mathematical problems with sufficient derivations and explanations are provided.

The first few chapters can be used for an undergraduate course on ODE, and later chapters can be used at the graduate level. Wherever possible, the authors present the subject in a way that students at undergraduate level can easily follow advanced topics, such as qualitative analysis of linear and nonlinear systems.

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We would like to dedicate the book to our parents
who brought us to this wonderful world.

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Preface

Many interesting and important real life problems are modeled using ordinary differential equations (ODE). These include, but are not limited to, physics, chemistry, biology, engineering, economics, sociology, psychology etc. In mathematics, ODE have a deep connection with geometry, among other branches. In many of these situations, we are interested in understanding the future, given the present phenomenon. In other words, we wish to understand the time evolution or the dynamics of a given phenomenon. The subject field of ODE has developed, over the years, to answer adequately such questions. Yet, there are many important intriguing situations, where complete answers are still awaited. The present book aims at giving a good foundation for a beginner, starting at an undergraduate level, without compromising on the rigour.

We have had several occasions to teach the students at the undergraduate and graduate level in various universities and institutions across the country, including our own institutions, on many topics covered in the book. In our experience and the interactions we have had with the students, we felt that many students lack a clear notion of ODE including the simplest integral calculus problem. For other students, a course on ODE meant learning a few tricks to solve equations. In India, in particular, the books which are generally prescribed, consist of a few tricks to solve problems, making ODE one of the most uninteresting subject in the mathematical curriculum. We are of the opinion that many students at the beginning level do not have clarity about the essence of ODE, compared to other subjects in mathematics.

While we were still contemplating to write a book on ODE, to address some of the issues discussed earlier, we got an opportunity to present a video course on ODE, under the auspices of the National Programme for Technology Enhanced Learning (NPTEL), Department of Science and Technology (DST), Government of India, and our course is freely available on the NPTEL website (see www.nptel.ac.in/courses/111108081). In this video course, we have presented several topics. We have also tried to address many of the doubts that students may have at the beginning level and the misconceptions some other students may possess.

Many in the academic fraternity, who watched our video course, suggested that we write a book. Of course, writing a text book, that too about a classical subject at a beginning level, meant a much bigger task than a video course, involving choosing and presenting the material in a very systematic way. In a way, the video course may supplement the book as it gives a flavour of a classroom lecture. We hope that in this way, students in remote areas and/or places where there is lack of qualified teachers, benefit from the book and the video course, making good use of the modern technology available through the Internet. The teachers of undergraduate courses can also benefit, we hope, from this book in fine tuning their skills in ODE.

We have written the present book with the hope that it can also be used at the undergraduate level in universities everywhere, especially in the context of Indian universities, with appropriately chosen topics in Chapters 1, 2 and 3. As the students get more acquainted with basic analysis and linear algebra, the book can be introduced at the graduate level as well and even at the beginning level of a research programme.

We now briefly describe the contents of the book. The book has a total of ten chapters and one appendix.

Chapter 1 describes some important examples from real life situations in the field of physics to biology to engineering. We thought this as a very good motivation for a beginner to undertake the study of ODE; in a rigorous course on ODE, often a student does not see a good reason to study the subject. We have observed that this has been one of the major concerns faced by students at a beginning level.

As far as possible, we have kept the prerequisite to a minimum: a good course on calculus. With this in mind, we have collected, in Chapter 2, a number of important results from analysis and linear algebra that are used in the main text. Wherever possible, we have provided proofs and simple presentations. This makes the book more or less self contained, though a deeper knowledge in analysis and linear algebra will enhance the understanding of the subject.

First and second order equations are dealt with in Chapter 3. This chapter also contains the usual methods of solutions, but with sufficient mathematical explanation, so that students feel that there is indeed rigorous mathematics behind these methods. The concept behind the exact differential equation is also explained. Second order linear equations, with or without constant coefficients, are given a detailed treatment. This will make a student better equipped to study linear systems, which are treated in Chapter 5.

Chapter 4 deals with the hard theme of existence, non-existence, uniqueness etc., for a single equation and also a system of first order equations. We have tried to motivate the reader to wonder why these questions are important and how to deal with them. We have also discussed other topics such as continuous dependence on initial data, continuation of solutions and the maximal interval of existence of a solution.

Linear systems are studied in great detail in Chapter 5. We have tried to show the power of linear algebra in obtaining the phase portrait of 2×2 and general systems. We have also included a brief discussion on Floquet theory, which deals with linear systems with periodic coefficients.

In the case of a second order linear equation with variable coefficients, it is not possible in general, to obtain a solution in explicit form. This has been discussed at length in Chapter 3. Chapter 6 deals with a class of second order linear equations, whose solutions may be written explicitly, although in the form of an infinite series. This method is attributed to Frobenius.

Chapter 7 deals with the regular Sturm–Lioville theory. This theory is concerned with boundary value problems associated with linear second order equations with smooth coefficients, in a compact interval on the real, involving a parameter. We, then, show the existence of a countable number of values of the parameter and associated non-trivial solutions of the differential equation satisfying the boundary conditions. There are many similarities with the existence of eigenvalues and eigenvectors of a matrix, though we are now in an infinite dimensional situation.

The qualitative theory of nonlinear systems is the subject of Chapter 8. The contents may be suitable for a senior undergraduate course or a beginning graduate course. This chapter does demand for more prerequisites and these are described in Chapter 2. The main topics of the chapter are equilibrium points or solutions of autonomous systems and their stability analysis; existence of periodic orbits in a two-dimensional system. We have tried to make a presentation of these important notions so that it can be easily understood by any student at a senior undergraduate level. The proofs of two important theorems on the existence of periodic orbits are given in the Appendix.

Chapter 9 considers the study of two point boundary value problems for second order linear and nonlinear equations. The first dealing with linear equations fully utilises the theory developed in Chapter 3. Using two linearly independent solutions, a Green's function is constructed for given boundary conditions. This is similar to an integral calculus problem. For nonlinear equations, we no longer have the luxury of two linearly independent solutions. A result which gives a taste of delicate analysis is proved. It is also seen through some examples how phase plane analysis can help in deciding whether a given boundary value problem has a solution or not.

In Chapter 10, we have attempted to show how the methods of ODE are used to find solutions of first order partial differential equations (PDE). We essentially describe the method of characteristics for solving general first order PDE. As very few books on ODE deal with this topic, we felt like including this, as a student gets some benefit of studying PDE and (s)he can later pursue a course on PDE.

We have followed the standard notations. Vectors in Euclidean faces and matrices are in boldface.

Acknowledgement

We wish to express our sincere appreciation to Gadadhar Misra and others at the IISc Press for suggesting to publish our book through the joint venture of IISc Press and Cambridge University Press. We also would like to thank Gadadhar Misra for all the help in this regard. We wish to acknowledge the support we received from our respective institutions and the moral support from our colleagues, during the preparation of the manuscript. We thank our academic fraternity, who have made valuable suggestions after reading through the various parts of the book. We would like to thank the students who attended our lectures at various places and contributed in a positive way. Over the years, we have had the opportunity to deliver talks in various lecture programs conducted by National Programme in Differential Equations (NPDE), India and the Indian Science Acadamies; our sincere thanks to them. We also wish to thank the anonymous referees for their constructive criticism and suggestions, which have helped us in improving the presentation. The illustrations have been drawn using the freely available software packages tikz and circuitikz. We are also thankful to the CUP team for their coordination from the beginning and their excellent production. Last but not the least, we wish to thank our family members for their patience and support during the preparation of this book.

1

Introduction and Examples: Physical Models

1.1 A Brief General Introduction

The beginning of the study of ordinary differential equations (ODE) could perhaps be attributed to Newton and Leibnitz, the inventors of differential and integral calculus. The theory began in the late 17th century with the early works of Newton, Leibnitz and Bernoulli. As was customary then, they were looking at the fundamental problems in geometry and celestial mechanics. There were also important contributions to the development of ODE, in the initial stages, by great mathematicians – Euler, Lagrange, Laplace, Fourier, Gauss, Abel, Hamilton and others. As the modern concept of function and analysis were not developed at that time, the aim was to obtain solutions of differential equations (and in turn, solutions to physical problems) in terms of elementary functions. The earlier methods in this direction are the concepts of *integrating factors* and method of separation of variables.

In the process of developing more systematic procedures, Euler, Lagrange, Laplace and others soon realized that it is hopeless to discover methods to solve differential equations. Even now, there are only a handful of sets of differential equations, that too in a simpler form, whose solutions may be written down in explicit form. It is in this scenario that the *qualitative analysis – existence, uniqueness, stability properties, asymptotic behaviour* and so on – of differential equations became very important. This qualitative analysis depends on the development of other branches of mathematics, especially analysis. Thus, a second phase in the study of differential equations started from the beginning of the 19th century based on a more rigorous approach to calculus via the mathematical analysis. We remark that the first existence theorem for first order differential equations is due to Cauchy in 1820. A class of differential equations known as *linear differential equations*, is much easier to handle. We will analyse linear equations and linear systems in more detail and see the extensive use of linear algebra; in particular, we will see how the nature of eigenvalues of a given matrix influences the stability of solutions.

After the invention of differential calculus, the question of the existence of antiderivative led to the following question regarding differential equation: Given a function f, does there exist a function g such that $\dot{g}(t) = f(t)$? Here, $\dot{g}(t)$ is the derivative of g with respect to t. This was the beginning of integral calculus and we refer to this problem as an *integral calculus problem*. In fact, Newton's second law of motion describing the motion of a particle having mass m states that the rate change of momentum equals the applied force. Mathematically, this is written as $\frac{d}{dt}(mv) = -F$, where v is the velocity of the particle. If $x = x(t)$ is the position of the particle at time t, then $v(t) = \dot{x}(t)$. In general, the applied force F is a function of t, x and v. If we assume F is a function of t, x, we have a second order equation for x given by $m\ddot{x} = -F(t,x)$. If F is a function of x alone, we obtain a conservative equation which we study in Chapter 8. If on the other hand, F is a function of t alone, then the second law leads to two integral calculus problems: namely, first solve for the momentum $p = mv$ by $\dot{p} = -F(t)$ and then solve for the position using $m\dot{x} = p$. This also suggests that one of the best ways to look at a differential equation is to view it as a *dynamical system*; namely, the motion of some physical object. Here t, the *independent variable* is viewed as time and x is the unknown variable which depends on the independent variable t, and is known as the *dependent variable*.

A large number of physical and biological phenomena can be modelled via differential equations. Applications arise in almost all branches of science and engineering–radiation decay, aging, tumor growth, population growth, electrical circuits, mechanical vibrations, simple pendulum, motion of artificial satellites, to mention a few.

In summary, real life phenomena together with physical and other relevant laws, observations and experiments lead to mathematical models (which could be ODE). One would like to do mathematical analysis and computations of solutions of these models to simulate the behaviour of these physical phenomena for better understanding.

Definition 1.1.1

An ODE is an equation consisting of an independent variable t, an unknown function (dependent variable) $y = y(t)$ and its derivatives up to a certain order. Such a relation can be written as

$$f\left(t, y, \frac{dy}{dt}, \cdots, \frac{d^n y}{dt^n}\right) = 0. \tag{1.1.1}$$

Here, n is a positive integer, known as the order of the differential equation.

For example, first and second order equations, respectively, can be written as

$$f\left(t, y, \frac{dy}{dt}\right) = 0 \text{ and } f\left(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right) = 0. \tag{1.1.2}$$

We will be discussing some special cases of these two classes of equations. It is possible that there will be more than one unknown function and in that case, we will have a system of differential equations. A higher order differential equation in one unknown function may be reduced into a system of first order differential equations. On the other hand, if there are more than one independent variable, we end up with partial differential equations (PDEs).

1.2 Physical and Other Models

We begin with a few mathematical models of some real life problems and present solutions to some of these problems. However, methods of obtaining such solutions will be introduced in Chapter 3, and so are the terminologies like linear and nonlinear equations.

1.2.1 Population growth model

We begin with a linear model. If $y = y(t)$, represents the population size of a given species at time t, then the rate of change of population $\dfrac{dy}{dt}$ is proportional to $y(t)$ if there is no other species to influence it and there is no net migration. Thus, we have a simple linear model [Bra78]

$$\frac{dy}{dt} = ry(t), \tag{1.2.1}$$

where r denotes the difference between birth rate and death rate. If $y(t_0) = y_0$ is the population at time t_0, our problem is to find the population for all $t > t_0$. This leads to the so-called *initial value problem (IVP)* which will be discussed in Chapter 3. Assuming that r is a constant, the solution is given by

$$y(t) = y_0 e^{r(t-t_0)} \tag{1.2.2}$$

Note that, if $r > 0$, then as $t \to \infty$, the population $y(t) \to \infty$. Indeed, this linear model is found to be accurate when the population is small and for small time. But it cannot be a good model as no population, in reality, can grow indefinitely. As and when the population becomes large, there will be competition among the population entities for the limited resources like food, space etc.

This suggests that we look for a more realistic model which is given by the following logistic nonlinear model. The statistical average of the number of encounters of two members per unit time is proportional to y^2. Thus, a better model would be

$$\frac{dy}{dt} = ay - by^2, \; y(t_0) = y_0. \tag{1.2.3}$$

Here a, b are positive constants. The negative sign in the quadratic term represents the competition and reduces the growth rate. This is known as the *logistic law of population growth*. It was introduced by the Dutch mathematical biologist Verhulst in 1837. It is also known as the Malthus law.

Practically, b is small compared to a. Thus, if y is not too large, then by^2 will be negligible compared to ay and the model behaves similar to the linear model. However, when y becomes large, the term by^2 will have a considerable influence on the growth of y, as can be seen from the following discussion.

The solution of (1.2.3) is given by[1]

$$\frac{1}{a} \log \frac{|y|}{|y_0|} \left| \frac{a - by_0}{a - by} \right| = t - t_0, \; t > t_0. \tag{1.2.4}$$

Note that $y \equiv 0$ and $y \equiv \frac{a}{b}$ are solutions to the nonlinear differential equation in (1.2.3) with the initial condition $y(t_0) = 0$ and $y(t_0) = \frac{a}{b}$,

[1] The reader, after getting familiarised with the methods of solutions in Chapter 3, should work out the details for this and the other examples in this chapter.

respectively. Hence, if the initial population y_0 satisfies $0 < y_0 < \frac{a}{b}$, then the solution will remain in the same interval for all time. This follows from the existence and uniqueness theory, which will be developed in Chapter 4. A simplification of (1.2.4) gives

$$y(t) = \frac{ay_0}{by_0 + (a - by_0)e^{-a(t-t_0)}}. \tag{1.2.5}$$

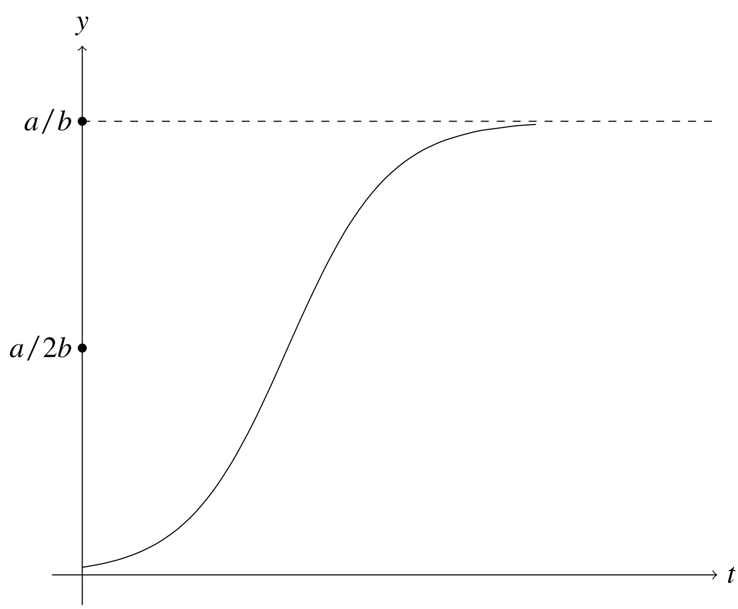

Fig. 1.1 Logistic map

In case $0 < y_0 < \dfrac{a}{b}$, the curve $y(t)$ is depicted as in Fig. 1.1. This curve is called the *logistic* curve; it is also called an *S-shaped curve*, because of its shape. Note that $\dfrac{a}{b}$ is the limiting population, also known as capacity of the ecological environment. In this case, the rate of population $\dfrac{dy}{dt}$ is positive and hence, y is an increasing function. Since $\dfrac{d^2y}{dt^2} = (a - 2by)\dfrac{dy}{dt}$, we immediately see that it is positive if the population is between 0 and half the limiting population, namely, $\dfrac{a}{2b}$, whereas, it is negative when the population crosses the half way mark $\frac{a}{2b}$. This indicates that if the initial population is less than half the limiting population, then there is an accelerated growth $\left(\frac{dy}{dt} > 0, \frac{d^2y}{dt^2} > 0\right)$, but after reaching half the population, the population still grows $\left(\frac{dy}{dt} > 0\right)$, but it has now a decelerated growth $\left(\frac{d^2y}{dt^2} < 0\right)$.

When we analyse the case where the initial population is bigger than the limiting population, we observe that $\frac{dy}{dt} < 0$ and $\frac{d^2y}{dt^2} < 0$. Thus, the population *decreases with a decelerated growth* to the limiting population.

Remark 1.2.1

The estimation of the vital coefficients a and b in a particular population model is indeed an important issue which has to be updated in a period of time as they are influenced by other parameters like pollution, sociological trends, etc. In a more realistic model, one needs to consider more than one species, their interactions, unforeseen issues like epidemics, natural disasters, etc., which may lead to more complicated equations.

1.2.2 An atomic waste disposal problem

The dumping of tightly sealed drums containing highly concentrated radioactive waste in the sea below a certain depth (say 300 feet) from the surface is a very sensitive issue as it could be environmentally hazardous. The drums could break due to the impact of their velocity exceeding a certain limit, say 40 ft/sec. Our problem is to compute the velocity by using Newton's second law of motion and assess the level of safety involved in the process. Let $y(t)$ denote the position, at time t, of the object, the drum, (considered as a particle) measured from the sea surface (indicating $y = 0$) as a positive quantity. The total force acting on the object is given by

$$F = W - B - D,$$

where the weight $W = mg$ is the force due to gravity, B is the buoyancy force of water acting against the forward movement and $D = cV$ is the drag exerted by water (it is a kind of resistance), where $V = \dfrac{dy}{dt}$, the velocity of the object and $c > 0$ is a constant of proportionality. Thus, we have the differential equation

$$\frac{d^2y}{dt^2} = \frac{1}{m}F = \frac{1}{m}(W - B - cV) = \frac{g}{W}(W - B - cV),\ y(0) = 0. \tag{1.2.6}$$

Equivalently,

$$\frac{dV}{dt} + \frac{cg}{W}V = \frac{g}{W}(W - B),\ V(0) = 0. \tag{1.2.7}$$

Equation (1.2.7) can be solved to get

$$V(t) = \frac{W - B}{c}\left(1 - e^{-\frac{cg}{W}t}\right). \tag{1.2.8}$$

Thus, $V(t)$ is increasing and tends to $\frac{W-B}{c}$ as $t \to \infty$ and the value (practically) of $\dfrac{W - B}{c} \approx 700$.

The limiting value 700 ft/sec of velocity is far above the permitted critical value. Thus, it remains to ensure that $V(t)$ does not reach 40 ft/sec by the time it reaches the sea bed. But it is not possible to compute t at which time the drum hits the sea bed and one needs to do further analysis.

Analysis: The idea is to view the velocity $V(t)$ not as a function of time, but as a function of position y. Let $v(y)$ be the velocity at height y measured from the surface of the sea downwards. Then, clearly, $V(t) = v(y(t))$ so that $\dfrac{dV}{dt} = \dfrac{dv}{dy}\dfrac{dy}{dt} = v\dfrac{dv}{dy}$. Hence, (1.2.7) becomes

$$\begin{cases} \dfrac{v}{W - B - cv}\dfrac{dv}{dy} = \dfrac{g}{W}, \\ v(0) = 0. \end{cases} \tag{1.2.9}$$

This is a first order non-homogeneous nonlinear equation for the velocity v. Indeed, the equation is more difficult, but it is in a variable separable form and can be integrated easily. We can solve this equation to obtain the solution in the form

$$\frac{gy}{W} = -\frac{v}{c} - \frac{W - B}{c^2}\log\frac{W - B - cv}{W - B}. \tag{1.2.10}$$

Of course, v cannot be explicitly expressed in terms of y as it is a nonlinear equation. However, it is possible to obtain accurate estimates for the velocity $v(y)$ at height y and it is estimated that $v(300) \approx 45$ ft/sec and hence, the drum could break at a depth of 300 feet.

Tail to the Tale: This problem was initiated when environmentalists and scientists questioned the practice of dumping waste materials by the Atomic Energy Commission of USA. After the study, the dumping of atomic waste was forbidden, in regions of sea not having sufficient depths.

1.2.3 Mechanical vibration model

The fundamental mechanical model, namely *spring-mass-dashpot system (SMD)* has applications in shock absorbers in automobiles, heavy guns, etc. An object of mass m is attached to an elastic spring of length l which is suspended from a rigid horizontal body. This is a spring–mass system. Elastic spring has the property that when it is stretched or compressed by a small length Δl, it will exert a force of magnitude proportional to Δl, say $k\Delta l$ in the opposite direction of stretching or compressing. The positive constant k is called *spring constant* which is a measure of stiffness of the spring. We then obtain an SMD system when this spring–mass is immersed in a medium like oil which will also resist the motion of the spring–mass. In a simple situation, we may assume that the force exerted by the medium on the spring–mass is proportional to the velocity of the mass and in the opposite direction of the movement of mass. It is also similar to a seismic instrument used to obtain a seismograph to detect the motion of the earth's surface.

Let $y(t)$ denote the position of mass at time t, $y = 0$ being the position of the mass at equilibrium and let us take the downward direction as positive. There are four forces acting on the system, that is, $F = W + R + D + F_0$, where $W = mg$, the force due to gravity; $R = -k(\Delta l + y)$, the restoring force; D, the damping or drag force and F_0, the external applied force, if any. Drag force is the kind of resistance force which the medium exerts on the mass and hence, it will be negative. It is usually proportional to the velocity, that is, $D = -c\dfrac{dy}{dt}$. At equilibrium, the spring has been stretched a length Δl and so $k\Delta l = mg$. Applying Newton's second law, we get

$$m\frac{d^2y}{dt^2} = -ky - c\frac{dy}{dt} + F_0(t). \tag{1.2.11}$$

That is,

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F_0(t),\ m,c,k \geq 0. \tag{1.2.12}$$

This is a second order non-homogeneous linear equation with constant coefficients and we study such equations in detail in Chapter 3. Such a system also arises in electrical circuits, which we discuss next.

1.2.4 Electrical circuit

A basic LCR electrical circuit is shown in Fig. 1.2, and is described as follows:

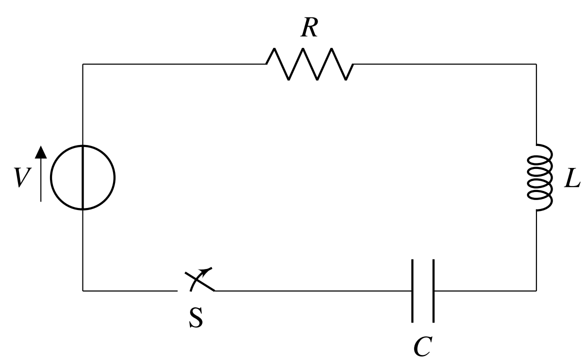

Fig. 1.2 A basic LCR circuit

By Kirchoff's second law, the impressed voltage in a closed circuit equals the sum of the voltage drops in the rest of the circuit. Let $E(t)$ be the source of electro motive force (emf), say a battery, $I = \frac{dQ}{dt}$ be the current flow, $Q(t)$ the charge on the capacitor at time t. Then, the voltage drops across inductance (L), resistance (R) and capacitance (C), respectively, are given by $L\frac{dI}{dt} = L\frac{d^2Q}{dt^2}$, $RI = R\frac{dQ}{dt} + \frac{Q}{c}$. Thus, we obtain a similar equation for Q as in (1.2.12):

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{c} = E(t). \tag{1.2.13}$$

More often, the current $I(t)$ is the physical quantity of interest; by differentiating (1.2.13) with respect to t, the equation satisfied by I is

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{c}I=\frac{dE}{dt}(t). \tag{1.2.14}$$

Mathematically, the equation is exactly same as the equation obtained in the spring–mass–dashpot system. We can also see the similarity between various quantities: inductance corresponding to mass, resistance corresponding to damping constant and so on.

1.2.5 Satellite problem

Consider an artificial satellite of mass m orbiting the earth. We assume that the satellite has thrusting capacity with radial thrust u_1 and a thrust u_2 which is applied in a direction perpendicular to the radial direction. The thrusters u_1 and u_2 are considered as the external force F or control inputs applied to the satellite.

The satellite can be considered as a particle P moving around the earth in the equatorial plane. If $(x,y)$is the rectangular coordinate of the particle P of mass m, then by Newton's law, the equations of motion along the rectangular coordinate axes are given by

$$m\ddot{x}=F_x, \quad m\ddot{y}=F_y \tag{1.2.15}$$

where, F_x and F_y denote the components of the force F in the directions of the axes (see Fig. 1.3). It will be convenient to represent the motion in polar coordinates (r,θ), where,

$$x=r\cos\theta, \;\; y=r\sin\theta$$

We will resolve the velocity, acceleration and force of the particle into components along the radial direction and the direction perpendicular to it. Denote by u, v; a_1, a_2 and F_r, F_θ the components of velocity, acceleration and force, respectively in the new coordinate system. The resultant of u and v is also equal to the resultant of the components of $\dot{x}$ and $\dot{y}$. Therefore, by resolving parallel to the x-axis, we get

$$\dot{x} \;\; = \;\; u\cos\theta - v\sin\theta \tag{1.2.16}$$

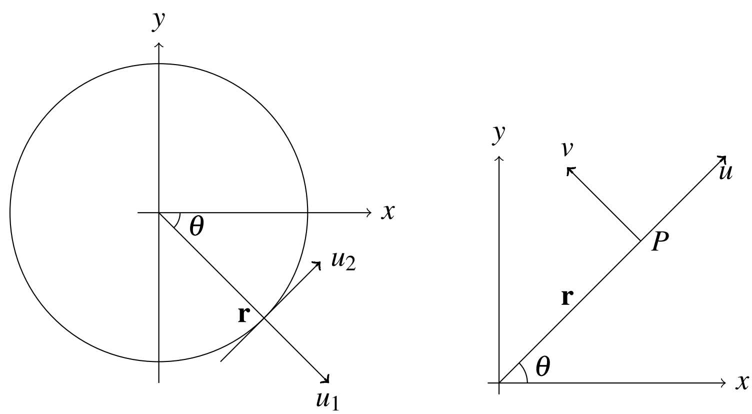

Fig. 1.3 Satellite problem

Since $x = r\cos\theta$, differentiating with respect to time t,

$$\dot{x} \;=\; \dot{r}\cos\theta - r(\sin\theta)\dot{\theta} \tag{1.2.17}$$

From (1.2.16) and (1.2.17), we have

$$u\cos\theta - v\sin\theta \;=\; \dot{r}\cos\theta - r(\sin\theta)\dot{\theta} \tag{1.2.18}$$

Comparing coefficients of $\cos\theta$ and $\sin\theta$ from (1.2.18), we have

$$u = \dot{r}, \quad v = r\dot{\theta}. \tag{1.2.19}$$

Then, by resolving the acceleration parallel to the x- and y-axes, we get

$$\ddot{x} = a_1\cos\theta - a_2\sin\theta.$$

By differentiating (1.2.17), we obtain

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d}{dt}(\dot{r}\cos\theta - r(\sin\theta)\dot{\theta}) \tag{1.2.20}$$

$$= \ddot{r}\cos\theta - \dot{r}(\sin\theta)\dot{\theta} - \dot{r}(\sin\theta)\dot{\theta} - r(\cos\theta)\dot{\theta}^2 - r(\sin\theta)\ddot{\theta} \tag{1.2.21}$$

$$= (\ddot{r} - r\dot{\theta}^2)\cos\theta - (2\dot{r}\dot{\theta} + r\ddot{\theta})\sin\theta \tag{1.2.22}$$

Equating the expressions of $\ddot{x}$ obtained here, we get

$$a_1\cos\theta - a_2\sin\theta \;=\; (\ddot{r} - r\dot{\theta}^2)\cos\theta - (2\dot{r}\dot{\theta} + r\ddot{\theta})\sin\theta \tag{1.2.23}$$

Comparing coefficients of $\cos\theta$ and $\sin\theta$ from (1.2.23), we get the components of the acceleration as

$$a_1 = \ddot{r} - r\dot{\theta}^2, \quad a_2 = 2\dot{r}\dot{\theta} + r\ddot{\theta} \tag{1.2.24}$$

Therefore, the equations of motion of the particle P reduce to

$$m(\ddot{r} - r\dot{\theta}^2) = F_r, \quad 2m\dot{r}\dot{\theta} + mr\ddot{\theta} = F_\theta, \tag{1.2.25}$$

The force F is called a *central force* if $F_\theta = 0$. In this case, the force is always directed towards a fixed point. Take this point as the origin, where the earth is placed. The central force, by Newton's law of gravitation is proportional to the product of the mass M of earth, mass m of the satellite and inversely proportional to the square of the distance between them. Thus, $F_r = -\frac{GMm}{r^2}$, where G is the gravitational constant. Let $k = GMm$. Now, the equations of motion are given by

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{k}{r^2} \tag{1.2.26}$$

$$mr\ddot{\theta} + 2\dot{r}\dot{\theta}m = 0. \tag{1.2.27}$$

Newton derived Kepler's laws of planetary motion using these equations. The interested reader can refer to [Sim91]. Note that $r(t) = \sigma$, $\theta(t) = \omega t$, where σ, ω are appropriate constants, is a special solution to the aforementioned equations which corresponds to a circular orbit.

Assume that the mass is equipped with the ability to exert a thrust u_1 in the radial direction and u_2 in the direction perpendicular to the radial direction. Then, under the presence of these external forces (known as controls), the equations of motion become

$$\begin{aligned} m\ddot{r} - mr\dot{\theta}^2 + \frac{k}{r^2} &= u_1 \\ 2m\dot{r}\dot{\theta} + r\ddot{\theta}m &= u_2. \end{aligned} \tag{1.2.28}$$

By scaling the time variable, we may assume that the mass m of the satellite is 1. Then, the motion of the satellite is described by a pair of second order nonlinear differential equations:

$$\frac{d^2r}{dt^2} = r(t)\left(\frac{d\theta}{dt}\right)^2 - \frac{k}{r^2(t)} + u_1(t). \tag{1.2.29}$$

$$\frac{d^2\theta}{dt^2} = -\frac{2}{r(t)}\frac{d\theta}{dt}\frac{dr}{dt} + \frac{u_2(t)}{r(t)}. \tag{1.2.30}$$

In applications, when a satellite is injected into an orbit, it usually drifts from its prescribed orbit due to the influence of other cosmic forces. The thrusters (controls) are activated to maintain the desired orbit of the satellite.

1.2.6 Flight trajectory problem

We consider an aeroplane which departs from an airport located at point $(a,0)$ and intends to reach an airport located at $(0,0)$ in the western direction from the departure airport. Assume that the constant wind velocity in the northern direction is w and the plane travels with constant speed v_0 relative to the wind. Assume that the plane's pilot maintains its heading directly towards the origin $(0,0)$.

The ground velocities of the plane in the direction of the x-axis and the y-axis are given by

$$\begin{aligned}\frac{dx}{dt} &= -v_0\cos\theta = -\frac{v_0 x}{\sqrt{x^2+y^2}}\\ \frac{dy}{dt} &= -v_0\sin\theta + w = -\frac{v_0 y}{\sqrt{x^2+y^2}} + w.\end{aligned}$$

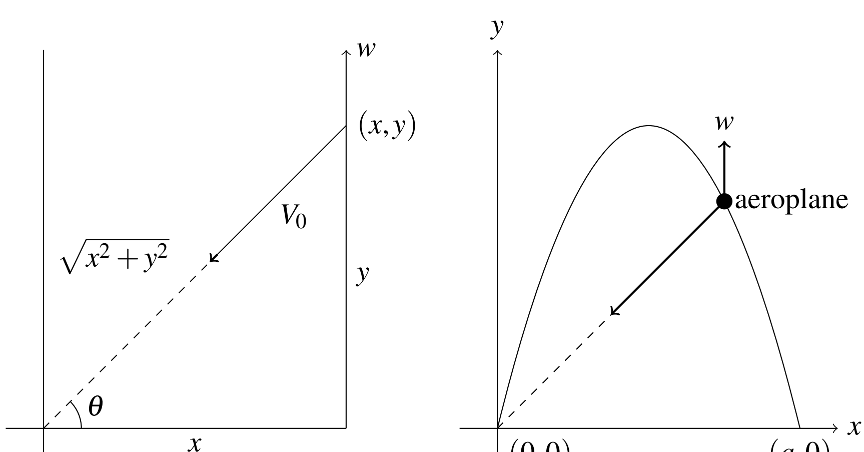

Fig. 1.4 Flight trajectory

The path $\{(x(t),y(t)),t \geq 0\}$ is called the orbit or trajectory of the aircraft in the xy plane (Figure 1.4). These equations can be implicitly written as

$$\frac{dy}{dx} = \frac{1}{v_0 x}(v_0 y - w\sqrt{x^2+y^2}).$$

1.2.7 Other examples

Example 1.2.2

[Unforced Duffing equation or oscillator] This is a second order equation, named after Georg Duffing, and is given by

$$\ddot{x} - \alpha x + \beta x^3 + \delta \dot{x} = 0. \tag{1.2.31}$$

Here α, β are nonzero real numbers and $\delta \geq 0$. This equation, referred to as a nonlinear oscillator, is a perturbation of the usual linear oscillator, namely (1.2.31) with $\beta = \delta = 0$ and $\alpha < 0$. The nonlinear equation (1.2.31) has a cubic nonlinearity and a linear damping term. It models more complicated dynamics of a spring pendulum whose spring stiffness does not exactly obey Hooke's law. The case of $\delta = 0$ and a periodic forcing term was extensively studied by Duffing.

By dilating the variables x to ax and t to bt, for suitable constants a, b, we can write (1.2.31) in the following standard form

$$\ddot{x} - x + x^3 + \delta \dot{x} = 0, \tag{1.2.32}$$

if $\alpha\beta > 0$ and

$$\ddot{x} + x + x^3 + \delta \dot{x} = 0, \tag{1.2.33}$$

if $\alpha\beta < 0$.

Example 1.2.3

[Unforced van der Pol equation or oscillator] This is also a second order nonlinear equation given by

$$\ddot{x} - \mu(x^2 - 1)\dot{x} + x = 0, \mu \in \mathbb{R}. \tag{1.2.34}$$

This equation, apparently first introduced in 1896 by Lord Rayleigh, was extensively studied both theoretically and experimentally using electrical