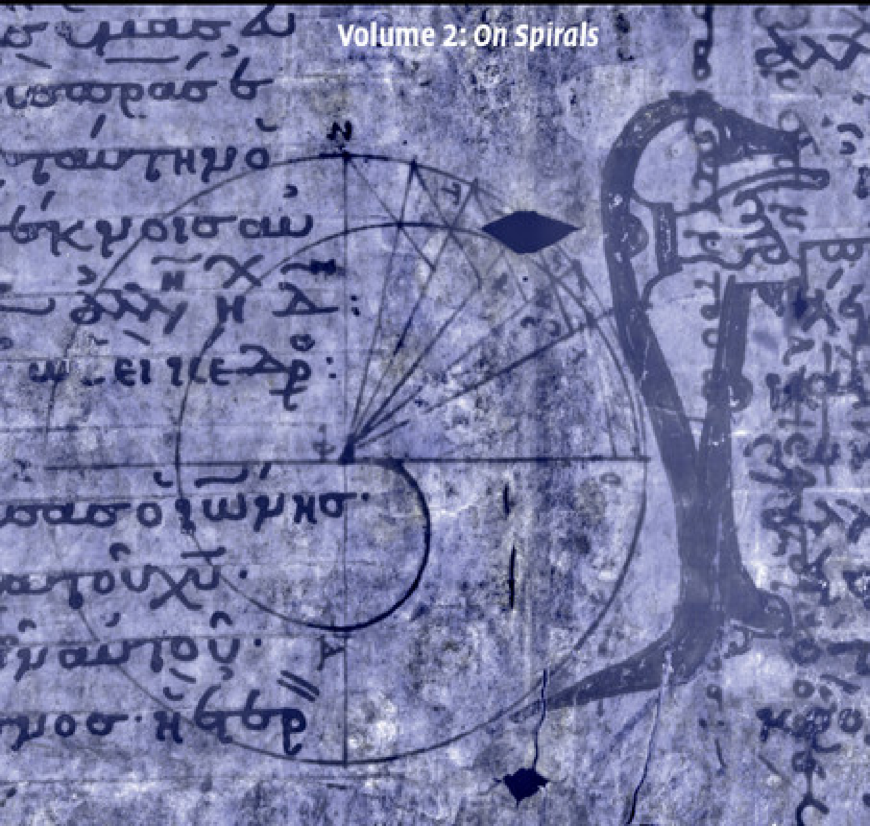
Reviel Netz

The Works of Archimedes

Translation and Commentary

Volume 2: *On Spirals*



The Works of Archimedes
Translation and Commentary

Volume II: *On Spirals*

This is the second volume of the first fully fledged English translation of the works of Archimedes – antiquity's greatest scientist and one of the most important scientific figures in history. It covers *On Spirals* and is based on a reconsideration of the Greek text and diagrams, now made possible through new discoveries from the Archimedes palimpsest. *On Spirals* is one of Archimedes' most dazzling geometrical *tours de force*, suggesting a manner of "squaring the circle" and, along the way, introducing the attractive geometrical object of the spiral. The form of argument, no less than the results themselves, is striking, and Reviel Netz contributes extensive and insightful comments that focus on Archimedes' scientific style, making this volume indispensable for scholars of classics and the history of science, and of great interest for the scientists and mathematicians of today.

REVIEL NETZ is Patrick Suppes Professor of Greek Mathematics and Astronomy at Stanford University, and is the leading scholar of Archimedes today. He has published numerous articles and books, many of which have led to new directions in the study of the history of science, including *The Shaping of Deduction in Greek Mathematics* (Cambridge University Press, 1999), *The Transformation of Mathematics in the Early Mediterranean World* (Cambridge University Press, 2004), and *Ludic Proof* (Cambridge University Press, 2009). He is also producing a complete new translation of and commentary on the works of Archimedes.

THE WORKS OF
ARCHIMEDES

Translated into English, with commentary, and critical edition of the diagrams

REVIEL NETZ
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Volume II
On Spirals

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To Maya, Darya and Tamara

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Eut. *SC* I	Eutocius' commentary on the above
SC II	The second book *On the Sphere and the Cylinder*
Eut. *SC* II	Eutocius' commentary on the above
SL	*On Spiral Lines*
CS	*On Conoids and Spheroids*
DC	*Measurement of the Circle (Dimensio Circuli)*
Eut. *DC*	Eutocius' commentary on the above
Aren.	*The Sand-Reckoner (Arenarius)*
PE I, II	*Planes in Equilibrium*
Eut. *PE* I, II	Eutocius' commentary on the above
QP	*Quadrature of the Parabola*
Meth.	*The Method*
CF I	The first book *On Floating Bodies (de Corporibus Fluitantibus)*
CF II	The second book *On Floating Bodies (de Corporibus Fluitantibus)*
Bov.	*The Cattle Problem (Problema Bovinum)*
Stom.	*Stomachion*

ABBREVIATIONS OF CITED MANUSCRIPTS AND PRE-HEIBERG EDITIONS

Manuscripts

A (reconstructed source manuscript of DEGH4)
B (Moerbeke's Latin translation, partly of A, partly of a different manuscript) Ottob. Lat. 1850
C (The Archimedes Palimpsest)
D Laur. XXVIII.4
E Marc. 305
G Par. Gr. 2360
H Par. Gr. 2361
4 Vat. Gr. Pii II.16

Editions

Commandino
Commandino, F. 1558. *Archimedes Opera non nulla*. Venice.

Torelli
Torelli, J. 1792. *Archimedis quae supersunt Omnia*. Oxford.

Nizze
Nizze, E. 1824. *Archimedes von Syrakus vorhandene Werke*. Stralsund.

ACKNOWLEDGMENTS

Work on this volume proceeded alongside the publication of *The Archimedes Palimpsest* (Cambridge University Press, 2011). In many ways, this translation would have been impossible without the palimpsest. Indeed, the palimpsest grounds the new text, in particular that of the diagrams. It is a pleasure to have the opportunity to thank again the wonderful team that made work on the palimpsest possible – and a pleasure. I thank my co-editor Nigel Wilson, curator of manuscripts Abigail Quandt, and the three lead imagers – Roger Easton, Bill Christens-Barry and Keith Knox. Above all I thank my co-author and curator of manuscripts, now Director of the Schoenberg Institute for Manuscript Studies at the University of Pennsylvania, William Noel.

This is the second volume of my translation of Archimedes. The first volume got published in 2004, and I have been receiving e-mails for several years now asking where the second volume was! I apologize. Delay was due to various technical reasons, mostly outside of the author's sway, and it is reasonably hoped that future volumes will be published at a faster clip. Meanwhile, I have accumulated many debts: to my publishers at Cambridge University Press, in particular the Classics Editor Michael Sharp, copy-editor Christopher Jackson, and the ever patient Emma Collison, to my colleagues among the wider community of the history of science and mathematics, as well as here at the Stanford departments of Classics and the History and Philosophy of Science and Technology, and especially to two students who have helped me substantially with the detail of the book: Amy Carlow and Dr. Johannes Wietzke.

As the years have passed, my titles have evolved, and I have now metamorphosed into the Patrick Suppes Professor for Greek Astronomy and Mathematics at Stanford. I had revered Pat Suppes well before I met him, and then I was greatly privileged to co-teach with him – effectively, therefore, to be his student – in the philosophy of science. Pat Suppes always insisted, above all, that the way to make real progress in the philosophical understanding of science is through the precise understanding of the detail of a scientific practice. I am grateful to have had Pat Suppes as my inspiration as I was translating Archimedes, and I hope to share this sense, now, with new readers of Archimedes.

INTRODUCTION

I THE STRUCTURE OF ARCHIMEDES' ON SPIRALS

One may be forgiven for considering this, *On Spirals*,[1] to be Archimedes' finest. The figures bend and balance as the argument reaches – effortlessly, quickly, and yet, how, one cannot quite grasp – towards several magnificent results. These suggest no less than the squaring of the circle: first, a certain line (defined by a tangent to the circle) is equal to the circumference of the circle; second, a certain area is equal to the circle's third.

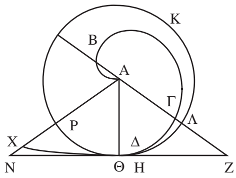
We are witnesses to Archimedes in action, as he engaged in a campaign of publications. At some early date, we are told in this treatise, he sent out via his mathematician friend Conon a complex geometrical challenge containing many claims. He had gradually discharged this challenge. Previously, he had sent to Dositheus the two books *On the Sphere and the Cylinder* (following on the *Quadrature of the Parabola*, which contained results independent from the original challenge sent via Conon). Now, he sends out *On Spirals*. This, once again, is sent to Dositheus. Archimedes once again proves some of the claims contained in that letter to Conon; he also reflects, briefly, on that geometrical challenge as a whole.

In this treatise, Archimedes promises to find not two, but four results. One of them is the result on the tangent mentioned above (being equal to the circumference of the circle). The result on the area of the spiral (being one-third the circle enclosing it) is proved and then further expanded to two extra, inherently interesting results, showing the ratios between the entire shells of spirals enclosing each other as well as the ratios of fragments of shells enclosing each other. The main results, then, are:

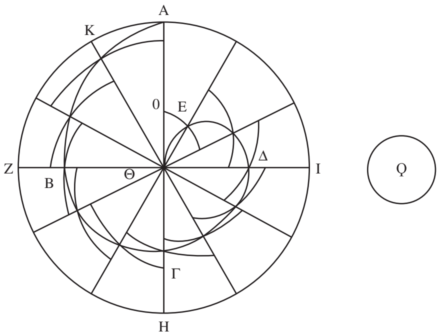
18. The line AZ is equal to the circumference of the circle HΘK.

---

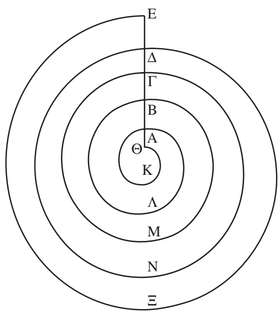
[1] *On Spirals* translates the title transmitted through the manuscript tradition, περι ελικων. A slightly expanded version, "*Spiral Lines*," is the one most often used by previous English discussions of Archimedes, and is implied by my own abbreviation to the title, SL.



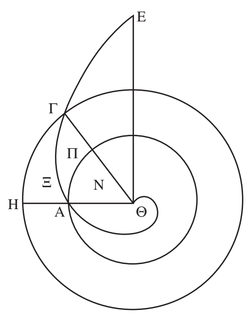
24. The spiral area ΑΒΓΔΕΘΑ is one-third the circle.



27. In the series of shells ΛΜΝΞ, Μ is twice Λ, Ν is three times Μ, Ξ is four times Ν, etc.



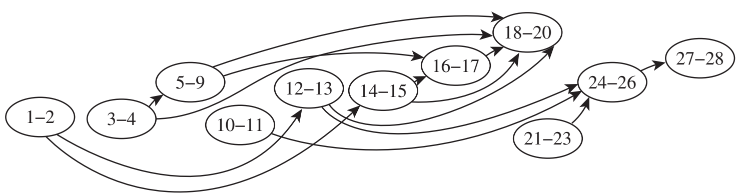
28. The shell fragment Ξ is to the shell fragment Π as (ΑΘ+$\frac{2}{3}$ΗΑ):(ΑΘ+⅓ΗΑ).



The deductive flow of the propositions in this treatise may be summed up as a table of dependence:

Proposition	Relies on
1	
2	1
3	
4	3
5	3
6–9	
10	
11	10
12	1
13	12
14	2
15	2, 14
16	5, 14
17	5, 16
18	4, 7, 8, 13, 14, 15, 16
19	4, 7, 13, 15, 17,
20	4, 7, 13, 14, 16
21	
22	
23	
24	10, 12, 21
25	11, 12, 22
26	11, 12, 23
27	24, 25
28	26

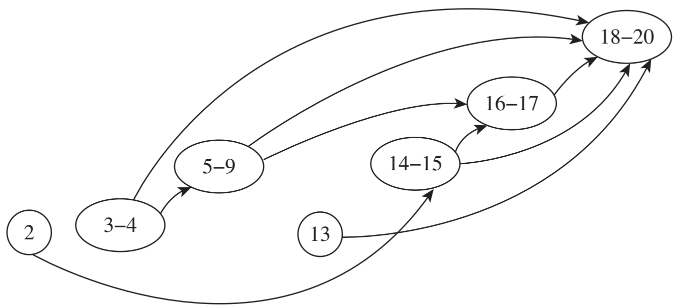
It is apparent that results cluster together in pairs and triplets, and it is perhaps best to visualize the logical flow as a chart based on such clusters:



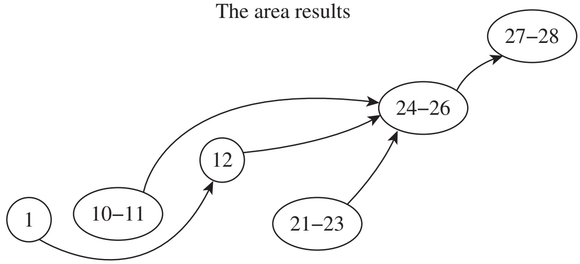
The immediate observation is how "shallow" the structure is. There is limited recursion (the top results are at level "4": 1–2 leads to 14–15 leads to 16–17 leads to 18–20, and 1–2 leads to 12–13 leads to 24–26 leads to 27–28). A substantial fraction of the treatise is at the elementary level where one directly applied widely known results (1–2, 3–4, 10–11, 21–23: notice that some of this "elementary" level is very complex). Instead of vertical recursion, we see the horizontal bringing-together of unrelated strands at two key moments of the treatise: 18–20, bringing together 3–4, 5–9, 12–13, 14–15 and 16–17; and 24–26, bringing together 10–11, 12–13 and 21–23.

Indeed, there are two such moments because there are two separate lines of reasoning. I set out the logical flow now for each strand apart (for this purpose, I distinguish 1 from 2, 12 from 13):

The tangent results



The area results



We find that the treatise cleaves nearly in half (sixteen propositions serve in the tangent results, twelve in the area results).[2] And cleave it does: the paths to the tangent results, and to the area results, are essentially independent. The one complication is the set of results 1–2, 12–13, where:

1 leads to 2

1 leads to 12 leads to 13

2, 13 are used in the tangent results

12 is used in the areas results

It is apparent that the one link shared between the two strands is proposition 12 – which is in the nature of an alternative definition of the spiral line (that lines drawn from the start on the spiral line differ from each other in the ratio of the

---

[2] Merely counting propositions is misleading, however, if we measure propositions by logical size – for the sake of the exercise, by the number of Steps in the proof: we find 183 Steps used in the tangent results, 195 in the area results: the area results are fewer but on the whole more complex (indeed, the tangent results appear to be slightly padded, with propositions 6, 9 seemingly unmotivated; they take 25 Steps).

angles they make with each other). Archimedes did make a choice to present this as a theorem,[3] so the two strands do hang together, if by the thinnest of threads. For indeed Archimedes also made the choice not to display the cleavability of the treatise. Adding to his bivalence of propositions 12–13, Archimedes inserted the pair 10–11 before them and, in between, inserted a passage of definitions. The result is a long passage composed of 10–11, definitions, 12–13, which cannot be read as leading at all, or strictly, to the tangent results. As for the area results, those are broken much more powerfully into two segments, 10–11 (as well as 12), and then the main sequence from 21 onwards.

Archimedes could easily have positioned proposition 12 as a definition or as a consequence obtained directly from the definitions, and then divided his treatise into two parts (two books?), one for each set of results. The complex pattern in which the two strands are brought together serves to maximize the distance between tools and results, indeed to obscure, at first reading, the very identity of the tools required for the results obtained.

This, however, somewhat misrepresents Archimedes' choice as an author. It is not as if Archimedes was provided with a pile of twenty-eight propositions which he had to arrange is some form. Rather, he was looking for interesting things to say about spirals. Considered in this way, his basic choice is seen to be saying two things or, more precisely, dual-and-more: essentially, one result for tangents; and then one result for areas, which, however, is expanded to produce further results (this is seen in the logical flow in the segment 27–28, which derives directly from 24–26). "Dual-and-more" is a repeated pattern of this treatise, seen also in the way in which almost all propositions are presented in pairs or triplets (indeed, since many propositions carry brief corollaries, even the results that come in pairs display, in fact, the structure of dual-and-more). The architecture of the treatise as a whole can be derived from these two principles: a desire to maximize the distance between tools and their applications; and a repeated pattern of "dual-and-more." Hence the elegant, combined pattern of strands within strands. Thus results which are obtained quickly and effortlessly still make one gasp with wonder: how did we get there?

2 CONVENTIONS AND GOALS OF THE TRANSLATION

The translation follows the same conventions as in the first volume, *On the Sphere and the Cylinder*, and I should repeat here the account of the conventions in Netz 2004b: 3–8. I stated there that "There are many possible barriers to the reading of a text written in a foreign language, and the purpose of a scholarly translation as I understand it is to remove all barriers having to do with the foreign language itself, leaving all other barriers intact." This entails a more-than-usual literal translation. The following conventions of my translation – and of Greek mathematics itself – should therefore be explained. (To aid further in the reading, a glossary was added to this volume, so that when less familiar

[3] Even so, it takes a mere six Steps to accomplish this result, and even these are not so much argument as explication: see the comments on the theorem.

terms are introduced for the first time, they are accounted for. A marginal note refers to the Glossary, which is located at the end of the volume.)

1. Greek word order is much freer than English word order, and so, selecting from among the wider set of options, Greek authors can choose one word order over another to emphasize a certain idea. Thus, for instance, instead of writing "A is equal to B," Greek authors might write "to B is equal A." This would stress that the main information concerns B, not A – word order would make B, not A, the focus. (For instance, we may have been told something about B, and now we are being told the extra property of B, that it is equal to A.) Generally speaking, such word order cannot be kept in the English, but I try to note it when it is of special significance, usually in a footnote.

2. The summation of objects is often done in Greek through ordinary conjunction. Thus "the squares ABGD and EZHQ" will often stand for what we may call "the square ABGD plus the square EZHQ." As an extension of this, the ordinary plural form can serve, as well, to represent summation: "the squares ABGD, EZHQ" (even without the "and" connector!) will then mean "the square ABGD plus the square EZHQ." In such cases, the sense of the expression is in itself ambiguous (the following predicate may apply to the sum of the objects, or it may apply to each individually), but such expressions are, generally speaking, easily disambiguated in context. Note also that while such "implicit" summations are very frequent, summation is often more explicit and may be represented by such connectors as "together with," "taken together" or simply "with."

3. Greek has certain pairs of particles that do not merely govern their own clause, but also attach to each other to form a single, conjoint clause out of two separate phrases. One of those conjoint particles becomes nearly technical in Greek mathematics: *te*. . . *kai*. . . (conveyed most idiomatically in English by: both. . . and. . .). Thus, in expressions such as

the area contained by: *te* the line AB, *kai* the spiral AGB

the two elements of the expression, the line and the spiral, are not merely listed in order, but instead are understood to be conjoined so as to form, together, the border of a single figure.

To express this technical, somewhat unidiomatic meaning, I translate this combination into the somewhat unidiomatic English "both. . . as well as. . ."

4. The main expression of Greek mathematics is that of proportion:

"As A is to B, so is C to D."

(A, B, C and D being some mathematical objects). This expression is often represented symbolically, in modern texts, by:

A:B::C:D

and I will use such symbolism in my footnotes and commentary. In the main text I will translate, of course, the original non-symbolic form. Note especially that this expression may become even more concise, e.g.:

"As A is to B, C to D," "As A to B, C to D."

And that it may have more complex syntax, especially:

"A has to B the same ratio as C has to D," "A has to B a greater ratio than C has to D."

The last example involves an obvious extension of proportion, to ratio-inequalities, i.e. A:B>C:D. More concisely, this may be expressed by:

"A has to B a greater ratio than C to D."

A ratio can also be "duplicate another": this means, in terms more transparent to us today, that it is its square (the ratio of 9 to 4 is duplicate the ratio of 3 to 2); "triplicate" is, in the same sense, a cube (27 to 8 is triplicate 3 to 2).

5. Greek mathematical propositions have, in many cases, the following six parts:

- *Enunciation*, in which the claim of the proposition is made, in general terms, without reference to the diagram. It is important to note that, generally speaking, the enunciation is equivalent to a conditional statement that *if* x is the case, *then* so is y.
- *Setting-out*, in which the antecedent of the claim is restated, in particular terms referring to the diagram (with the example above, x is restated in particular reference to the diagram).
- *Definition of goal*, in which the consequent of the claim is restated, as an exhortation addressed by the author to himself: "I say that...," "it is required to prove that...," again in the particular terms of the diagram (with the same example, we can say that y is restated in particular reference to the diagram).
- *Construction*, in which added mathematical objects (beyond those required by the setting-out) may be introduced.
- *Proof*, in which the particular claim is proved.
- *Conclusion*, in which the conclusion is reiterated for the general claim from the enunciation.

Some of these parts will be missing in most Archimedean propositions, but the scheme remains a useful analytic tool, and I will use it as such in my commentary. The reader should be prepared in particular for the following difficulty. It is often very difficult to follow the enunciations as they are presented. Since they do not refer to the particular diagram, they use completely general terms, and since they aspire to great precision, they may have complex qualifications and combinations of terms. I wish to exonerate myself: this is not a problem of my translation, but of Greek mathematics. Most modern readers find that they can best understand such enunciations by reading, first, the setting-out and the definition of the goal, with the aid of the diagram. Having read this, a better sense of the *dramatis personae* is gained, and the enunciation may be deciphered. In all probability the ancients did the same.

6. The main "<...>" policy: Greek mathematical proofs always refer to concrete objects, realized in the diagram. Because Greek has a definite article with a rich morphology, it can elide the reference to the objects, leaving the definite article alone. Thus the Greek may contain such expressions as

“The by the AB, BG”

whose reference is

“The <rectangle contained> by the <lines> AB, BG”

(the morphology of the word “the” determines, in the original Greek, the identity of the elided expressions, given of course the expectations created by the genre).

In this translation, most such elided expressions are added inside pointed brackets, so as to make it possible for the reader to appreciate the radical concision of the original formulation and the concreteness of reference – while allowing me to represent the considerable variability of elision (very often, expressions have only partial elision). This variability, of course, will be seen in the fluctuating positions of pointed brackets:

“The <rectangle contained> by the <lines> AB, BG,” as against, e.g., “The <rectangle> contained by the <lines> AB, BG”

(Notice that I do not at all strive at consistency *inside* pointed brackets. Inside pointed brackets I put whatever seems to me, in context, most useful to the reader; the duties of consistency are limited to the translation proper, outside pointed brackets.)

The main exception to my general pointed-brackets policy concerns points and lines. These are so frequently referred to in the text that to insist, always, upon a strict representation of the original, with such expressions as

“The <point> A,” “The <line> AB”

would be tedious, while serving little purpose. I thus usually write, simply,

A, AB

and, in the less common cases of a non-elliptic form,

“The point A,” “The line AB”

The price paid for this is that (relatively rarely) it is necessary to stress that the objects in question are points or lines, and while the elliptic Greek expresses this through the definite article, my elliptic “A,” “AB” does not. Hence I need to introduce, here and there, the expressions

“The <point> A,” “The <line> AB”

but notice that these stand for precisely the same as

A, AB.

I avoid distinguishing, typically, between ευθεια and γραμμη. The precise translation of ευθεια is “straight <line>,” while the precise translation of γραμμη, when a straight line is intended, is “<straight> line.” Not wishing to split such hairs, I have decided to make both simply a “line.” In this treatise one may often compare straight and curved lines, and it would therefore