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Symmetry, Phase Modulation and Nonlinear Waves

Thomas J. Bridges

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Symmetry, Phase Modulation and Nonlinear Waves

THOMAS J. BRIDGES
University of Surrey

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To Margot Dance and Christopher Justin

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Introduction

In the theory of dynamical systems, normal form theory is a strategy for producing simplified models near particular parameter values of ordinary differential equations (ODEs). In the theory of nonlinear waves, where partial differential equations (PDEs) are considered on the real line or the plane, a generalization of normal forms is modulation equations.

"Modulation" is one of the most widely used words in the theory of linear and nonlinear waves, and can refer to a range of concepts. In linear wave theory modulation is normally the process of varying the envelope of a signal. In electronics this concept is expanded further to include digital modulation, analogue modulation, pulse modulation, frequency modulation, phase modulation, and so on. In the theory of nonlinear waves it is used to describe "modulation equations" which typically are nonlinear PDEs governing the envelope of a wave, although the term is much more widely used now, with any equation on a slow time and space scale called a modulation equation. The interest in this book is in *phase modulation* of nonlinear waves in conservative systems.

In conservative systems, phase modulation of nonlinear waves is most closely associated with Whitham modulation theory. Within this theory, a given basic state, dependent on a phase and a parameter or parameters, is treated as a slowly varying function of space and time, and governing equations are derived for the slowly varying parameters. The backbone of Whitham modulation theory is *conservation of wave action* and *conservation of waves*.

In this book Whitham theory will be reformulated by allowing greater flexibility in the phase, scaling and use of singularities, generating new modulation equations with nonlinearity and dispersion. A central example is a new theory for the emergence of the Korteweg–de Vries (KdV) equation in the form

$$(\mathscr{A}_k + \mathscr{B}_\omega)q_T + \mathscr{B}_{kk}qq_X + \mathscr{K}\,q_{XXX} = 0\,, \tag{1.1}$$

obtained by morphing the conservation of wave action in the presence of singularity, with q a wavenumber modulation. Further details of the origin

and derivation of this form of KdV will unfold. The important features at this point are that the coefficients are universal, they do not depend on particular PDEs but on a general class, and that the KdV equation arises in general via modulation in the universal form (1.1), connected explicitly with a conservation law.

Phase modulation applies equally well to ODEs, PDEs in $1+1$, or PDEs in $2+1$, or indeed any number of space dimensions, but it is the case of $1+1$ (one space dimension and time) which will feature most prominently in this book. The starting point for phase modulation is an underlying phase symmetry of the governing equations, which naturally arises in the case of periodic orbits of ODEs, and periodic travelling waves of PDEs, but this class will be enlarged in this book to include *relative equilibria*.

A canonical example illustrating phase modulation in conservative systems is perturbation of periodic orbits of finite-dimensional autonomous Hamiltonian systems. Consider a Hamiltonian system

$$\mathbf{J}Z_t = \nabla H(Z)\,, \quad Z \in \mathbb{R}^{2n}\,, \tag{1.2}$$

where $\mathbf{J}$ is a unit symplectic operator, $H : \mathbb{R}^{2n} \to \mathbb{R}$ is a given smooth Hamiltonian function, and $\nabla H(Z)$ is the gradient with respect to the standard inner product on $\mathbb{R}^{2n}$. Suppose there exists a periodic orbit satisfying (1.2) with frequency ω,

$$\widehat{Z}(\theta,\omega)\,, \quad \text{with} \quad \theta = \omega t + \theta_0 \quad \text{and} \quad \widehat{Z}(\theta+2\pi,\omega) = \widehat{Z}(\theta,\omega)\,.$$

Since the system (1.2) is autonomous, for all $\phi \in S^1$,

$$\widehat{Z}(\theta+\phi,\omega) \text{ is a solution of (1.2) whenever } \widehat{Z}(\theta,\omega) \text{ is a solution.} \tag{1.3}$$

This does not mean that $\widehat{Z}(\theta+\phi,\omega) = \widehat{Z}(\theta,\omega)$ for all ϕ, since that is true for $\phi = 2\pi$ only. It means that initial data of (1.2) $\widehat{Z}(\theta+\phi,\omega)\big|_{t=0}$ leads to the same periodic solution for any ϕ, but with a different starting value of the periodic orbit; the starting value is shifted by a phase. This symmetry is just a recognition that a periodic solution is a closed orbit in the phase space with no natural origin.

The phase shift in the choice of initial data in (1.3) is a constant. The theory of phase modulation starts by allowing the phase shift to depend on a slow time variable, say $\phi(T,\varepsilon)$ with $T = \varepsilon^\alpha t$, for some rational number α, with ε a small parameter measuring the distance of the perturbation field from the periodic orbit.

To study the perturbation of a periodic orbit the strategy is to insert $\phi(T,\varepsilon)$ into the solution to capture changes tangent to the periodic orbit as well as

normal perturbations. This strategy is based on an ansatz of the following form for the perturbed periodic orbit:

$$Z(t) = \widehat{Z}(\theta + \varepsilon^a \phi(T,\varepsilon), \omega) + \varepsilon^b W(\theta, T, \varepsilon)\,, \quad T = \varepsilon^\alpha t\,, \tag{1.4}$$

with a, b rational numbers to be determined or imposed, with $b \geq 1$. The remainder W accounts for the fact that the modulated $\widehat{Z}$, with T-dependent ϕ, will not in general be an exact solution of (1.2). The strategy in phase modulation is to substitute the ansatz into the governing equation, expand all terms in power series in ε, and solve order by order. This ansatz approach to modulation is very effective, and applies equally well to both conservative and non-conservative ODEs and PDEs. On the other hand, conservative systems have the additional property that the period and energy, or frequency and action, have a dual relationship and so modulating the frequency brings in a connection with conserved quantities. This connection will be important in the theory as it is used to generate coefficients in the resulting modulation equations. Including frequency modulation, the revised ansatz for solutions in a neighbourhood of a periodic orbit is

$$Z(t) = \widehat{Z}(\theta + \varepsilon^a \phi(T,\varepsilon), \omega + \varepsilon^c \Omega(T,\varepsilon)) + \varepsilon^b W(\theta, T, \varepsilon)\,, \quad T = \varepsilon^\alpha t\,. \tag{1.5}$$

The rational numbers a, b, c and α are problem dependent, and different choices will produce different modulation equations in general. The frequency modulation, $\Omega(T,\varepsilon)$, and phase modulation, $\phi(T,\varepsilon)$, can be solved independently or they can be related by $\Omega = \phi_T$, which puts a constraint, $a + \alpha = c$, on the exponents.

Once the exponents are fixed in (1.5) the ansatz is substituted into the governing equation (1.2), all terms are expanded in a Taylor series in ε resulting in a sequence of inhomogeneous ODEs at each order. Taking into account that $\widehat{Z}(\theta, \omega)$ is an exact solution, a solvability condition arises, and the outcome is an ODE for the perturbation frequency $\Omega(T,\varepsilon)$.

An example of an equation for $\Omega(T,\varepsilon)$, resulting from the modulation ansatz (1.5) with $a = 1$, $\alpha = 1$, $b = 3$ and $c = 2$, is the following modulation equation, which is derived in Chapter 4, starting with the abstract Hamiltonian system (1.2),

$$\kappa \Omega \Omega_T + \mathscr{K} \Omega_{TTT} = 0\,, \quad \Omega = \phi_T\,, \text{ and } T = \varepsilon t\,. \tag{1.6}$$

A theory for the coefficients κ, which is a curvature, and $\mathscr{K}$, which is deduced from a symplectic Jordan chain, is developed in Chapter 4. The equation (1.6) is a derivation via modulation of a normal form that is well known in dynamical systems theory, but a curiosity is that by replacing T with X in (1.6) it is the

steady KdV equation, and moreover the coefficient κ has a geometric interpretation in terms of the curvature of a conserved quantity. This ODE modulation will feed into the travelling wave modulation leading to the KdV equation (1.1) in Chapter 8.

Relative equilibria, which are more general than periodic solutions, will be the class of candidate solutions for phase modulation. Relative equilibria arise as natural solutions of *symmetric Hamiltonian systems* (see MONTALDI [147] and references therein). The system (1.2) is symmetric when it is equivariant with respect to the action of a Lie group, and relative equilibria are solutions which are aligned with a group orbit. The group orbit can be interpreted as a phase and so these solutions can be phase modulated. Moreover, periodic orbits of non-symmetric but autonomous Hamiltonian systems can also be characterized as relative equilibria, and so relative equilibria are abundant in both symmetric and non-symmetric Hamiltonian ODEs. The theory of relative equilibria of Hamiltonian ODEs is developed in Chapter 2. The combination of symmetric Hamiltonian systems, one-parameter Lie groups, conservation laws and relative equilibria will form the backbone of the theory of phase modulation for ODEs.

The theory of phase modulation extends in a natural way to Hamiltonian PDEs. Hamiltonian PDEs, equivariant with respect to a one-parameter Lie group, can have familes of relative equilibria and the phase shift associated with the relative equilibria is a starting point for modulation. Similarly, any periodic travelling wave of a non-symmetric Hamiltonian system can be characterized as a relative equilibrium, and concomitantly modulated. There are three new features in going from ODEs to PDEs. The phase-dependent basic states, $\widehat{Z}(\theta, k, \omega)$, depend on both a frequency and a wavenumber, $\theta = kx+\omega t+\theta_0$, conservation laws replace conserved quantities, and phase modulation will depend on a slow time and space scale: $\phi(X, T, \varepsilon)$, where $T = \varepsilon^\alpha t$ and $X = \varepsilon^\beta x$, thereby generating a modulation wavenumber, $q = \phi_X$, in addition to the modulation frequency, $\Omega = \phi_T$. The generalization of the modulation ansatz (1.5) to PDEs in one space dimension and time is

$$Z(x,t) = \widehat{Z}(\theta + \varepsilon^a\phi, k + \varepsilon^d q, \omega + \varepsilon^c\Omega) + \varepsilon^b W(\theta, X, T, \varepsilon)\,, \tag{1.7}$$

with ϕ, q, Ω dependent on X, T, ε. When the constraints $\Omega = \phi_T$ and $q = \phi_X$ are imposed, conservation of waves arises naturally by cross-differentiation

$$\Omega = \phi_T \text{ and } q = \phi_X \;\Rightarrow\; q_T - \Omega_X = 0\,. \tag{1.8}$$

To implement the ansatz (1.7) a class of Hamiltonian PDEs needs to be identified. The natural starting point for conservative PDEs is a Lagrangian functional, but there is insufficient geometric structure for a complete modulation

theory, and so transformation to symplectic and multisymplectic Hamiltonian structure is of interest.

On the other hand, many of the key features of conservative PDEs, like conservation of waves and conservation of wave action, can be seen in the Lagrangian formulation. In Chapter 5 Whitham theory is re-appraised, starting with a Lagrangian

$$\mathcal{L}(u) = \int_{t_1}^{t_2} \int_{x_1}^{x_2} L(u_t, u_x, u)\, \mathrm{d}x\mathrm{d}t\,, \tag{1.9}$$

for a scalar field, say $u(x,t)$, on $[x_1, x_2]\times[t_1, t_2]$. Instead of using the Whitham strategy, the modulation equations are derived starting with an ansatz of the form (1.7) in a pure Lagrangian setting. The outcome is conservation of waves (1.8) coupled to conservation of wave action:

$$\frac{\partial}{\partial T}\left(\mathscr{A}(\omega+\Omega, k+q)\right) + \frac{\partial}{\partial X}\left(\mathscr{B}(\omega+\Omega, k+q)\right) = 0\,, \tag{1.10}$$

for the two unknowns Ω and q, with ω, k considered fixed. The time and space scales are $X = \varepsilon x$, $T = \varepsilon t$, and $\mathscr{A}, \mathscr{B}$ are the components of the conservation of wave action evaluated on a family of basic states (relative equilibria, periodic travelling waves, etc.). The *Whitham modulation equations* in one space dimension and time consist of conservation of waves (1.8) coupled to conservation of wave action (1.10). The Whitham equations are a dispersionless nonlinear first-order system of PDEs which may be hyperbolic, elliptic or degenerate. There is now a large literature on the nonlinear Whitham modulation equations and a review is given in Chapter 7.

Expand (1.10) in a Taylor series in Ω and q to leading order:

$$\mathscr{A}_\omega \Omega_T + (\mathscr{A}_k + \mathscr{B}_\omega) q_T + \mathscr{B}_k q_X + \cdots\,.$$

When the third coefficient has a singularity,

$$\mathscr{B}_k(\omega, k) = 0\,, \tag{1.11}$$

on some curve in the (ω, k)-plane, the leading order Whitham equations break down. But adjustment of the scaling in (1.7) to

$$\alpha = 3\,, \quad \beta = 1\,, \quad a = 1\,, \quad b = 2\,, \quad c = 4\,, \quad d = 3\,,$$

re-substitution and re-expansion morph the Whitham equations into (1.1) exposing the $\mathscr{B}_{kk}$ nonlinear term and evoking dispersion.

Adjustment of scale starts with changing from a "fast phase" ansatz to a "slow phase" ansatz. The conservation of wave action (1.10) is based on $a = -1$ in the ansatz (1.7),

$$\theta \mapsto \theta + \frac{1}{\varepsilon}\phi(X, T, \varepsilon)\,,$$

which will be called a *fast phase*. The modulation ansatz with a fast phase produces dispersionless but nonlinear modulation equations. The limitations imposed by a fast phase are relaxed by focusing attention on a slow phase where $a \geq 0$ in (1.7). It is this change, from a fast phase to an order-one or slow phase, which will be central to the modulation theory in this book. The slow phase will lead to modulation equations with nonlinearity and dispersion. In addition to the limitations imposed by fast phase and dispersionless modulation equations, the Whitham theory is also limited by the emphasis on modulation of periodic travelling waves, when in fact any basic state with a phase is a candidate for modulation, and the most important class is relative equilibria.

A Lagrangian formulation of PDEs will be the starting point for the theory, but the theory rapidly becomes complex without additional structure in the Lagrangian. A Legendre transform can be used to convert it to a Hamiltonian system, but even this added structure will be insufficient. The strategy is to partition the Lagrangian using a multisymplectic structure. A multisymplectic structure generates a symplectic operator for each space direction and time. This latter partition will also give a geometric formulation of symmetry and the conservation of wave action. The additional structure in the Lagrangian, which is central to the derivation of (1.1), is the generalization of (1.2) to a multisymplectic Hamiltonian PDE:

$$\mathbf{M}Z_t + \mathbf{J}Z_x = \nabla S(Z)\,, \quad Z \in \mathbb{R}^{2n}\,, \tag{1.12}$$

where $\mathbf{M}$ and $\mathbf{J}$ are skew-symmetric operators and $S : \mathbb{R}^{2n} \to \mathbb{R}$ is a given smooth generalized Hamiltonian function. The theory of multisymplectic Hamiltonian PDEs, which can be interpreted as a geometric reformulation of the Lagrangian, is developed in Chapter 6.

The modulation theory for PDEs proceeds similarly to the ODE case. Suppose there exists a periodic travelling wave satisfying (1.12) of frequency ω and wavenumber k,

$$\widehat{Z}(\theta, \omega, k)\,, \quad \text{with} \quad \theta = kx + \omega + \theta_0 \quad \text{and} \quad \widehat{Z}(\theta + 2\pi, \omega, k) = \widehat{Z}(\theta, \omega, k)\,.$$

The phase symmetry (1.3) generalizes to

$$\widehat{Z}(\theta + \phi, \omega, k) \text{ is a solution of (1.12) whenever } \widehat{Z}(\theta, \omega, k) \text{ is a solution.} \tag{1.13}$$

The phase shift is replaced by a slowly varying phase $\phi(X, T, \varepsilon)$ as in the ansatz (1.7), substituted into equation (1.12), expanded in powers of ε and solved order by order. Solvability conditions and the conservation of

wave action then provide the geometric and universal characterization of the coefficients in (1.1). This theory is developed in detail in Chapter 8.

The modulation ansatz (1.7) is a form of the *method of multiple scales* but the embedding of the phase modulation in the basic state differs from the strategy of the classical method of multiple scales. Given a basic state, dependent on a phase and a parameter, represented by $\widehat{Z}(\theta, k, \omega)$, with $\theta = kx + \omega t + \theta_0$, satisfying a given PDE, a classical multiple scales perturbation is of the form

$$Z(x,t) = \widehat{Z}(\theta, k, \omega) + \varepsilon^d \widetilde{W}(\theta, X, T, \varepsilon)\,, \tag{1.14}$$

with slow space and time scales $T = \varepsilon t, X = \varepsilon x$. For example, this is the approach that is used in the justification of Whitham modulation theory in [179], and is widely used in fluid mechanics [85]. However, in this approach it is not immediately clear how to generalize it to different scalings, and how to track singularities in the mapping $(k, \omega) \mapsto (\mathscr{A}(\omega, k), \mathscr{B}(\omega, k))$. On the other hand, the use of classical multiple scales is useful for the case where the basic state is not necessarily a relative equilibrium.

The advantage of the ansatz (1.7) over the classical multiple scales ansatz (1.14) is that modulation is embedded in the basic state, with the phase modulation explicit, and the frequency and wavenumber also explicitly modulated. In principle the two expressions (1.14) and (1.7) are equivalent: expand the first term in (1.7) in a Taylor series in ε and absorb into W to form $\widetilde{W}$. However, the separation and embedding of the perturbation of θ, k, ω and W in (1.7) gives more information that feeds into the reduced modulation equations for $\phi(X, T, \varepsilon)$ and $q(X, T, \varepsilon)$.

The idea of adding dispersion to the Whitham modulation equations has been introduced before (e.g. section 16.5 of WHITHAM [180], section 5.3.3 of INFELD & ROWLANDS [105], and references [57, 182, 81, 72, 166, 167, 168]). However, in these cases the dispersion is introduced for *weakly nonlinear waves*. Here, via scaling and singularity, dispersion will emerge in a general way for *fully nonlinear* basic states.

To summarize the case of one space dimension and time, the starting point is a class of PDEs generated by a Lagrangian with a one-parameter symmetry group. Relative equilibria on the symmetry are modulated using (1.7), and different scalings, combined with singularities, generate reduced equations such as the Whitham equations and the KdV equation. The modulation approach to producing the KdV equation is interesting for a number of reasons. Firstly, the modulation approach produces equations with coefficients that are universal and easy to calculate using properties of the family of relative equilibria. Secondly, the modulation equations are obtained as perturbations of *finite-amplitude basic states*. Thirdly, it gives a new way to find examples in

applications where the KdV equation is the right model, by searching for singularities of relative equilibria, rather than searching for an appropriate dispersion relation. Fourthly, by experimenting with different singularities and scalings in (1.7) other key PDEs such as the dual KdV equation (space and time reversed), the Kadomtsev–Petviashvili (KP) equation (and its variants), and a family of two-way Boussinesq equations emerge. For example, it is shown in Chapter 12 that the generalization of (1.1) in 2+1 is the KP equation

$$(\mathscr{A}_k + \mathscr{B}_\omega)\, q_T + \mathscr{B}_{kk}\, qq_X + \mathscr{K}\, q_{XXX} + \mathscr{C}_\ell p_Y = 0\,, \quad p_X = q_Y\,. \tag{1.15}$$

In 2+1 the conservation of wave action evaluated on the basic state has three components $(\mathscr{A}, \mathscr{B}, \mathscr{C})$. In this case a remarkable feature is that the transverse *dispersion coefficient* $\mathscr{C}_\ell$ is determined by the properties of the family of basic states. Fifthly, modulation of relative equilibria is the "right mechanism" for emergence of KdV, KP, Whitham equations and many others. Singularity in families of relative equilibria captures the physical mechanism for emergence of KdV in a purely mathematical construction.

The purely mathematical construction leading to the KdV equation (1.1) may give the impression that the emergence of KdV in the form (1.1) is "yet another way" for the KdV equation to emerge. This mechanism appears to be completely unrelated to the way the KdV equation emerges in the theory of water waves, since the KdV equation is normally derived in shallow water hydrodynamics from the trivial solution and so no modulation appears to be possible. However, there is a hidden phase and hidden relative equilibrium structure, which when modulated shows that the KdV equation in the theory of water waves emerges in precisely the form (1.1). The only difference is that the conservation of wave action which generates (1.1) is replaced by the conservation of mass, which is mathematically equivalent but is quite different physically. Even in the case of water waves, the generation of the KdV equation using the formula (1.1) gives new information in that the coefficient of the nonlinear term is related to the curvature of the mass flux as a function of the uniform flow.

Before presenting the connection between phase modulation and the KdV equation in shallow water hydrodynamics, the governing equations and background physics for water waves are introduced in Chapters 9 and 10. Uniform flows are characterized as relative equilibria, and criticality in hydrodynamics is shown to be equivalent to the singularity (1.11), thereby generating the KdV equation in shallow water hydrodynamics in the form (1.1). An additional feature arising from the new characterization of the emergent KdV equation is that it is now clear that the KdV equation is not just a model for shallow water, it is much more pervasive in the theory of water waves. Indeed, an example is

given in Chapter 10 showing that the KdV equation can also be a model for deep water waves.

The theory of phase modulation generalizes to PDEs in 2+1 (two space dimensions and time), with the key new features being an additional wavenumber in space and an additional component in the conservation laws. Otherwise, the theory extends in a straightforward way. First the classical 2+1 Whitham theory, based on a fast phase, is reviewed, then extended to the case of a slow phase in Chapter 11. The slow phase and appropriate singularities lead to the emergence of the KP equation via modulation. The 2+1 modulation theory with nonlinearity and multi-dimensional dispersion is developed in Chapter 12. It is shown with model equations and the full water wave problem how the KP equation emerges in the theory of water waves via phase modulation in Chapters 13 and 14. The conditions for emergence of KP are quite general and it is shown that the emergence of KP is much more pervasive than previously thought.

One of the first examples of the Whitham theory was the application to bifurcation of planforms by HOWE [97, 98]. He used the steady Whitham modulation equations to study pattern changes in the plane such as phase jumps. In chapter 12.4 of [180], the steady Whitham modulation equations are used to study a finite-amplitude version of the theory for the Kelvin wake behind ships. All this theory is based on the fast-phase approach. This theory is generalized in Chapter 15 using the ansatz and slow-phase approach giving a wider range of modulation equations for steady planforms, including KdV planforms.

The chapter on planforms provides a segue into the parallel development of phase modulation theory for periodic solutions of non-conservative systems, which is widely used in pattern formation. Non-conservative systems are not considered in this book in general, but a review of the theory is given in Chapter 17, with particular attention to features that resonate with the conservative case. There are two main strands in the theory of phase modulation for non-conservative systems: pattern formation PDEs and reaction–diffusion equations. The cornerstones of phase modulation in pattern formation are the *phase diffusion equation* of POMEAU & MANNEVILLE [156] and its nonlinear generalization, the *Cross–Newell equation* [60]. Starting with a steady periodic pattern, the phase-diffusion equation is obtained by modulation using an order-one phase, leading to a linear diffusion equation. The nonlinear Cross–Newell equation is obtained by using a fast phase. In reaction–diffusion the cornerstone is phase modulation of periodic travelling waves leading generically to Burgers' equation. This approach started with the Howard–Kopell theory [123] and was comprehensively generalized including validity results by DOELMAN ET AL. [67]. When the basic state is a periodic travelling wave,

rather than a stationary wave, then nonlinearity is generated in the phase diffusion equation by a group velocity, thereby generating a nonlinear Burgers equation. When the coefficient of dissipation is small, the Burgers equation is modified to the Kuramoto–Sivashinsky equation [126, 125]. A brief overview of phase modulation for reaction–diffusion equations and pattern formation is given in Chapter 17.

This book just scratches the surface of the potential of the theory of phase modulation for conservative systems. By removing the restrictions of a fast-phase approach, introducing the role of singularities, and using a general ansatz (1.7), new reduced models are found, and even familiar equations such as the KdV equation are viewed afresh. Remarkably, all the theory in this book is based on conservative PDEs with a single one-parameter group of symmetries. Naturally, PDEs arising in applications may have higher dimensional symmetry groups, and so there is a vast area for generalization of the one-phase theory, including bifurcation of multi-phase wavetrains and coupling with mean flow. Multi-dimensional groups can also be non-abelian, which brings in new complexity and geometry, requiring new theory. Another direction that is not considered but has important implications is systems with inhomogeneities, where the Lagrangian depends explicitly on space and time. Some speculation about these other directions is given in Chapter 18.

2

Hamiltonian ODEs and Relative Equilibria

A large class of solutions of Hamiltonian ODEs which are candidates for phase modulation are *relative equilibria*, a class which includes periodic orbits. Relative equilibria are solutions that are aligned with the orbit of a symmetry group. In this chapter the background needed for the introduction of relative equilibria is developed, including symplectic structures, Hamiltonian ODEs, symmetry and Lie groups, symplectic Noether theory, and the geometry of the linearization about relative equilibria.

Classically, Hamiltonian systems were obtained from a Lagrangian formulation as the Lagrangian formulation can be deduced from physics: the kinetic and potential energies and constraints. Consider a classical Lagrangian in standard form:

$$\mathcal{L}(U) = \int_{x_1}^{x_2} L(U_x, U)\,\mathrm{d}x\,, \tag{2.1}$$

for some field $U(x) \in \mathbb{R}^n$, $n \geq 1$, defined on the interval $x \in [x_1, x_2]$, and smooth Lagrangian density L. The Lagrangian is assumed to be non-degenerate, $L_{11} \neq 0$, where

$$\big\langle L_{11}V, V\big\rangle = \frac{\mathrm{d}^2}{\mathrm{d}s^2} L(U_x + sV, U)\bigg|_{s=0}, \quad V \in \mathbb{R}^n \tag{2.2}$$

and $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^n$. Here and throughout this chapter the time-like direction will be denoted by x in preparation for the later PDE analysis where the symplectic structure in space will be an organizing centre.

A Hamiltonian system is obtained by Legendre transform. Let

$$V = \frac{\delta L}{\delta U_x}\,, \tag{2.3}$$

and define the functional $H(U, V)$ by

$$H(U, V) := \big\langle V, U_x\big\rangle - L(U, U_x)\,,$$

where U_x is considered a function of V by inverting (2.3). Back substitution into (2.1) then gives

$$\widehat{\mathcal{L}}(U,V) = \int_{x_1}^{x_2} \left[\langle V, U_x\rangle - H(U,V)\right] \,\mathrm{d}x\,. \tag{2.4}$$

This functional is the same as (2.1) but with the Lagrangian density partitioned into two parts: a term involving a symplectic operator and a Hamiltonian function which does not contain any derivatives. The symplectic structure can be more easily seen by introducing the operator

$$\mathbf{J} = \begin{pmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \tag{2.5}$$

where $\mathbf{I}$ is the identity on $\mathbb{R}^n$. Then the functional in (2.4) can be written in the form

$$\widetilde{\mathcal{L}}(Z) = \int_{x_1}^{x_2} \left[\tfrac{1}{2}\langle \mathbf{J}Z_x, Z\rangle - H(Z)\right] \,\mathrm{d}x\,, \quad Z = \begin{pmatrix} U \\ V \end{pmatrix} \in \mathbb{R}^{2n}\,, \tag{2.6}$$

where $\langle\cdot,\cdot\rangle$ is now an inner product on $\mathbb{R}^{2n}$, and $\widetilde{\mathcal{L}}(Z)$ is the same $\mathcal{L}$ in different coordinates. The operator $\mathbf{J}$ defines a symplectic form

$$\mathsf{sym}(Z,W) = \langle \mathbf{J}Z, W\rangle\,, \quad Z, W \in \mathbb{R}^{2n}\,, \tag{2.7}$$

and the first variation of the functional in (2.6) gives the Hamiltonian system

$$\mathbf{J}Z_x = \nabla H(Z)\,, \quad Z \in \mathbb{R}^{2n}\,. \tag{2.8}$$

In general a Hamiltonian system does not have to come from a Lagrangian. It can be constructed abstractly from a symplectic vector space or manifold. A symplectic vector space is a pair $(\mathcal{P}, \mathsf{sym})$, where $\mathcal{P}$ is a vector space (the phase space) and sym is a non-degenerate closed two-form (the symplectic form, (2.7)). Here and throughout the vector space $\mathcal{P}$ will be taken to be $\mathbb{R}^{2n}$ with standard inner product

$$\langle Z, W\rangle := Z_1W_1 + \cdots + Z_{2n}W_{2n}\,, \quad Z, W \in \mathcal{P}\,. \tag{2.9}$$

A Hamiltonian system consists of the triple $(\mathcal{P}, \mathsf{sym}, H)$, where $H : \mathcal{P} \to \mathbb{R}$ is a given smooth function, and the flow of the Hamiltonian system is defined by (2.8).

2.1 Symmetry

A symmetric Hamiltonian system is one which is equivariant with respect to the action of a group G. For the purposes of this book the simplest