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Polynomial Methods and Incidence Theory

ADAM SHEFFER

POLYNOMIAL METHODS AND INCIDENCE THEORY

The past decade has seen numerous major mathematical breakthroughs for topics such as the finite field Kakeya conjecture, the cap set conjecture, Erdös's distinct distances problem, the joints problem, as well as others, thanks to the introduction of new polynomial methods. There has also been significant progress on a variety of problems from additive combinatorics, discrete geometry, and more. This book gives a detailed yet accessible introduction to these new polynomial methods and their applications, with a focus on incidence theory.

Based on the author's own teaching experience, the text requires a minimal background, allowing graduate and advanced undergraduate students to get to grips with an active and exciting research front. The techniques are presented gradually and in detail, with many examples, warm-up proofs, and exercises included. An appendix provides a quick reminder of basic results and ideas.

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To Liora, Daniel, and Amanda.

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Introduction

Algebra is the offer made by the devil to the mathematician. The devil says: I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine.

Michael Atiyah (2005).

In his famous essay on how to write mathematics, Paul Halmos (1970) states, "Just as there are two ways for a sequence not to have a limit (no cluster points or too many), there are two ways for a piece of writing not to have a subject (no ideas or too many)." The book that you are now starting has two main subjects, which is hopefully a reasonable amount. These two subjects, *the polynomial method* and *incidence theory*, are tied together and difficult to separate.

Geometric incidences are a family of problems that have existed in discrete geometry for many decades. Starting around 2009, these problems have been experiencing a renaissance. New and interesting connections between incidences and other parts of mathematics are constantly being exposed. Incidences already have a variety of applications in harmonic analysis, theoretical computer science, model theory, number theory, and more. At the same time, significant progress is being made on long-standing open incidence problems. The study of geometric incidences is currently an active and exciting research field. One purpose of this book is to survey this field, the recent developments in it, and its connections to other fields.

What are incidences? Consider a set of points $\mathcal{P}$ and a set of lines $\mathcal{L}$ in the plane $\mathbb{R}^2$. An *incidence* is a pair $(p, \ell) \in \mathcal{P} \times \mathcal{L}$ such that the point p is on the line ℓ. For example, see Figure 1. One fundamental incidence result states that n points and n lines in $\mathbb{R}^2$ form at most $2.5n^{4/3}$ incidences. While the exponent $4/3$ cannot be improved, it is possible that the coefficient 2.5 could be replaced with a slightly smaller one.

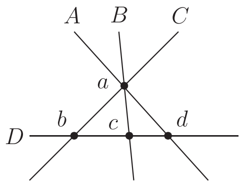

Figure 1 A configuration of four points, four lines, and nine incidences. For example, the point a forms an incidence with each of the lines A, B, and C.

In other incidence problems, we replace the lines with circles, parabolas, or other types of curves. Additional variants include incidences with higher-dimensional objects in $\mathbb{R}^d$, incidences with semi-algebraic sets, incidences with complex objects in $\mathbb{C}^d$, in spaces over finite fields, o-minimal structures, and more. In most of these cases, finding the maximum possible number of incidences remains an open problem.

An incidence result of a different flavor states that there exists a positive constant $c \in \mathbb{R}$ that satisfies the following. For every sufficiently large n, every set of n points in $\mathbb{R}^2$ satisfies at least one of the following statements:

- There exists a line that is incident to at least cn of the points.
- There exist at least cn^2 lines that are incident to at least two of the points.

Sylvester (1868) studied incidence problems back in the 1860s. The earliest incidence problem that we are aware of appears in a book of riddles (Jackson, 1821). This book contains 10 problems of the form that is presented in Figure 2. In modern English, the problem in Figure 2 asks for the following: Place points in the plane, such that the number of lines that contain exactly three points is at least the number of points.

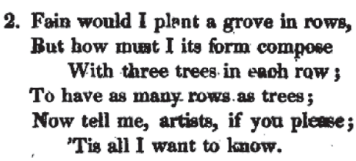
2. Fain would I plant a grove in rows,
But how must I its form compose
With three trees in each row;
To have as many rows as trees;
Now tell me, artists, if you please;
'Tis all I want to know.

Figure 2 A riddle from the 1821 book *Rational Amusement for Winter Evenings, Or, A Collection of Above 200 Curious and Interesting Puzzles and Paradoxes Relating to Arithmetic, Geometry, Geography.*

Most of the recent progress in incidence theory is due to new algebraic techniques. One may describe the philosophy behind these techniques as

Collections of objects that exhibit extremal behavior often have hidden algebraic structure. This algebraic structure can be exploited to gain a better understanding of the original problem.

For example, in a point-line configuration with many incidences, we might expect the points to form a lattice structure. Intuitively, we expose the algebraic structure by defining polynomials according to the problem, and then studying properties of these polynomials. In an incidence problem, we might study a polynomial that vanishes on all the points. This approach is called *the polynomial method*. In this book, we explore a wide variety of such polynomial proofs. We use these techniques to study incidence bounds, the finite field Kakeya problem, the cap set problem, distinct distances problems, the joints problem, and more.

Polynomial methods have existed for several decades. One well-known polynomial method is Alon's Combinatorial Nullstellensatz, as described in Alon (1999). As long ago as 1970, Rédei introduced an elegant polynomial proof. This book is focused on the new wave of polynomial methods that started to appear around 2009. These methods are quite different from the preceding ones.

This book aims to be an accessible introduction to the new polynomial methods and to incidence theory. For that reason, the book includes many examples, warm-up proofs, figures, and intuitive ways of thinking about tricky ideas. Many techniques are presented gradually and in detail. Readers who wish to dig deeper into a particular topic can find references in the relevant chapter.

Incidence theory and the polynomial methods are still developing. There are many interesting open problems, and, in some sense, the foundations are not completely established yet. For that reason, most of the chapters of this book end with an open problems section. These sections focus mostly on long-standing difficult problems. Their goal is to illustrate the current research fronts and the main difficulties that researchers are currently facing.

Several sections are defined as *optional*. Some sections, such as Section 7.3, are optional because they consist of standard technical proofs that may not provide any new insights. Other sections require familiarity with a topic that is orthogonal to the topics of this book. For example, the optional Section 9.3 requires basic familiarity with differential topology, which does not appear anywhere else in the book.

Two other good sources for polynomial methods in discrete geometry are the book *Polynomial Methods in Combinatorics* (Guth, 2016) and the survey

"Incidence theorems and their applications" (Dvir, 2012). While these sources and the current book study similar topics, the overlap between them is smaller than one might expect.

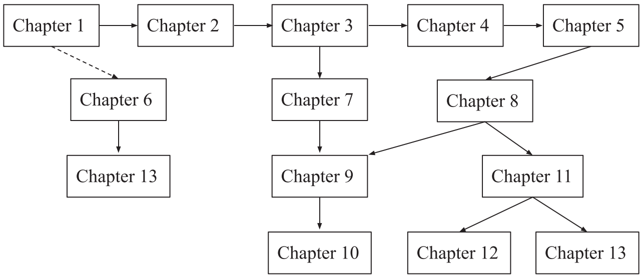

Figure 3 Chapter dependencies. The dashed edge marks a dependency that is recommended but not necessary.

How to Read This Book

Throughout this book, we rely heavily on asymptotic notation such as $x = O(y)$. The appendix contains an introduction to asymptotic notation, together with exercises. This appendix also briefly surveys basic graph theory notation and the Cauchy–Schwarz inequality.

There are many ways to read this book, depending on the goal of the reader. One way is to start from the beginning and read the chapters consecutively. The beginning of the book contains more introductory material. The end of the book contains mostly optional advanced topics. Figure 3 illustrates the chapter dependencies. Some reading options are:

- **A brief introduction to discrete geometry.** For an introduction to problems and techniques from classical discrete geometry, read Chapter 1. This chapter does not involve polynomial methods.
- **An introduction to polynomial partitioning.** To learn how to prove incidence results by using polynomial methods, read Chapters 1–3. Chapter 2 is a minimal introduction to algebraic curves in the real plane. Chapter 3 consists of the basics of the *polynomial partitioning* technique, and how to use this technique to prove incidence bounds.
- **A variety of polynomial methods in combinatorics.** To see a variety of polynomial methods in combinatorics, read Chapters 1–6. In addition to the polynomial partitioning technique, Chapters 5 and 6 contain several other polynomial breakthroughs. Chapter 4 introduces basic concepts from

real algebraic geometry, and can be quickly skimmed by a reader who does not intend to read beyond Chapter 6. Chapter 5 contains the polynomial proof of the joints theorem. Chapter 6 contains polynomial proofs for problems in finite fields, such as the finite field Kakeya problem and the cap set problem.

- **The distinct distances theorem.** To understand the distinct distances theorem of Guth and Katz, read Chapters 1–5 and 7–10. Chapter 7 reduces the distinct distances problem to an incidence problem in $\mathbb{R}^3$. Chapter 8 introduces the *constant-degree polynomial-partitioning* technique and uses it to prove incidence bounds in the complex plane. Chapter 9 extends this technique and uses it to prove the distinct distances theorem. Chapter 10 studies a few variants of the distinct distances problem.
- **Incidences and polynomial methods over finite fields.** To study incidences and polynomial methods over finite fields, read Chapters 6 and 13. You might wish to first read Chapter 1, but this is not necessary. Chapter 13 studies point-line incidences over finite fields.
- **Incidences in $\mathbb{R}^d$.** To understand advanced incidence techniques in $\mathbb{R}^d$, read Chapters 1–5, 8, 11, 12, and 14. Chapter 11 studies more advanced techniques for deriving incidence bounds in $\mathbb{R}^d$. Chapter 12 consists of applications for such incidence bounds. Chapter 14 introduces more advanced tools for studying incidences and related problems. In particular, this final chapter studies properties of ruled surfaces.

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1

Incidences and Classical Discrete Geometry

My most striking contribution to geometry is, no doubt, my problem on the number of distinct distances. This can be found in many of my papers on combinatorial and geometric problems.

Paul Erdős, in a survey of his favorite contributions to mathematics, compiled for the celebration of his 80th birthday (Erdős, 1993).

1.1 Introduction to Incidences

We begin our study of geometric incidences by surveying the field and deriving a few first bounds. In this chapter we only discuss classical discrete geometry from before the discovery of the new polynomial methods. This makes the current chapter rather different from the rest of the book (outrageously, it even includes some graph theory). We also learn basic tricks that are used throughout the book, such as double counting, applying the Cauchy–Schwarz inequality, and dyadic decomposition. These techniques are presented in full detail in this chapter, while some details are omitted in the following chapters.

Consider a set $\mathcal{P}$ of points and a set $\mathcal{L}$ of lines, both in $\mathbb{R}^2$. An *incidence* is a pair $(p, \ell) \in \mathcal{P} \times \mathcal{L}$ such that the point p is contained in the line ℓ. We denote the number of incidences in $\mathcal{P} \times \mathcal{L}$ as $I(\mathcal{P}, \mathcal{L})$. For example, Figure 1 (in the Introduction) depicts a configuration with nine incidences. For any m and n, Erdős constructed a set $\mathcal{P}$ of m points and a set $\mathcal{L}$ of n lines with $\Theta(m^{2/3}n^{2/3} + m + n)$ incidences. Erdős (1985) conjectured that no point-line configuration has an asymptotically larger number of incidences. This conjecture was proved by Szemerédi and Trotter in 1983.

Theorem 1.1 (The Szemerédi–Trotter theorem) *Let $\mathcal{P}$ be a set of m points and let $\mathcal{L}$ be a set of n lines, both in $\mathbb{R}^2$. Then $I(\mathcal{P}, \mathcal{L}) = O(m^{2/3}n^{2/3} + m + n)$.*

The original proof of the Szemeredi–Trotter theorem is rather involved. In this chapter we present a later elegant proof by Székely (1997). A more general algebraic proof is presented in Chapter 3.

Finding the maximum number of point-line incidences in $\mathbb{R}^2$ is one of the simplest incidence problems. It is also one of very few incidence problems that are solved asymptotically. Other problems involve incidences with circles or other types of curves, incidences with varieties in $\mathbb{R}^d$, with semi-algebraic objects in $\mathbb{R}^d$, in complex spaces $\mathbb{C}^d$, in spaces over finite fields, and much more. In each of these problems, we wish to find the maximum number of incidences between a set of points and a set of geometric objects. If you ever need to snub a discrete geometer, try pointing out how they can barely solve any of these problems after decades of work.

One reason for studying incidence problems is that they are natural combinatorial problems. Throughout this chapter, we start to see additional reasons for studying incidence problems, including:

- *Incidence problems are not purely combinatorial, but also require an understanding of the underlying geometry.* One example of this appears in Section 1.5, where we introduce the unit distances problem. This problem involves studying properties that distinguish the Euclidean metric from almost all other distance metrics.
- *Incidence results are also useful for problems that may not seem related to geometry.* In Section 1.8, we use incidences to study the sum-product problem. This problem started as a number-theoretic problem that does not involve any geometry.

1.2 First Proofs

We now develop some initial intuition about incidences. We begin by deriving our first bound for an incidence problem. This is a weak bound, but it is still useful in some cases.

Lemma 1.2 *Let $\mathcal{P}$ be a set of m points and let $\mathcal{L}$ be a set of n lines, both in $\mathbb{R}^2$. Then $I(\mathcal{P}, \mathcal{L}) = O(m\sqrt{n} + n)$ and $I(\mathcal{P}, \mathcal{L}) = O(n\sqrt{m} + m)$.*

Why do we say that Lemma 1.2 is weaker than Theorem 1.1? For some intuition, consider the case where $m = n$. In this case, Theorem 1.1 leads to the bound $O(n^{4/3})$, while Lemma 1.2 only gives $O(n^{3/2})$.

Proof of Lemma 1.2 We only derive $I(\mathcal{P}, \mathcal{L}) = O(m\sqrt{n} + n)$. The other bound is obtained in a symmetric manner. Consider the set of triples

$$T = \left\{(a,b,\ell) \in \mathcal{P}^2 \times \mathcal{L} \ : \ a \text{ and } b \text{ are both incident to } \ell\right\}.$$

Note that T also contains triples (a,b,ℓ) where $a = b$.

Let m_j be the number of points of $\mathcal{P}$ that are incident to the jth line of $\mathcal{L}$. Then the number of triples of T that include the jth line of $\mathcal{L}$ is m_j^2. This implies that $|T| = \sum_{j=1}^n m_j^2$. Also, note that $I(\mathcal{P},\mathcal{L}) = \sum_{j=1}^n m_j$. We apply the Cauchy–Schwarz inequality (Theorem A.1). We present this first application of the inequality in full detail. Throughout the rest of the book, we skip the intermediary steps. For $1 \le j \le n$, we set $a_j = m_j$ and $b_j = 1$. The Cauchy–Schwarz inequality implies that

$$\sum_{j=1}^n m_j \le \left(\sum_{j=1}^n m_j^2\right)^{1/2} \left(\sum_{j=1}^n 1\right)^{1/2} = \left(\sum_{j=1}^n m_j^2\right)^{1/2} \cdot n^{1/2}.$$

Squaring both sides and rearranging leads to

$$|T| = \sum_{j=1}^n m_j^2 \ge \frac{\left(\sum_{j=1}^n m_j\right)^2}{n} = \frac{I(\mathcal{P},\mathcal{L})^2}{n}. \tag{1.1}$$

The number of triples $(a,b,\ell) \in T$ with $a = b$ is $I(\mathcal{P},\mathcal{L})$. The number of triples $(a,b,\ell) \in T$ with $a \neq b$ is at most $\binom{m}{2}$, since each pair of distinct a, $b \in \mathcal{P}$ is contained in at most one line of $\mathcal{L}$. Thus, $|T| \le \binom{m}{2} + I(\mathcal{P},\mathcal{L})$. Combining this with Equation (1.1) gives

$$\frac{I(\mathcal{P},\mathcal{L})^2}{n} \le \binom{m}{2} + I(\mathcal{P},\mathcal{L}). \tag{1.2}$$

When $\binom{m}{2} \ge I(\mathcal{P},\mathcal{L})$, rearranging Equation (1.2) leads to $I(\mathcal{P},\mathcal{L}) = O(mn^{1/2})$. Otherwise, rearranging Equation (1.2) leads to $I(\mathcal{P},\mathcal{L}) = O(n)$. $\square$

To prove Lemma 1.2, we used a common combinatorial method called *double counting*. In this method, we bound some quantity X in two different ways and then compare the two bounds. This leads to new information that does not involve X. In the proof of Lemma 1.2, we derived upper and lower bounds for the size of T. By comparing these two bounds, we obtained a bound for the number of incidences. Double counting is ubiquitous in this book.

In the proof of Lemma 1.2, we did not use any geometry beyond observing that two points are contained in one line. This implies that the proof still holds after removing all the other geometric properties of the problem. That is, when replacing the lines with abstract sets of points, such that every two sets have at most one common element. For example, instead of the lines in Figure 1 (in the Introduction), we can consider the sets

$$A = \{a, d\}, \quad B = \{a, c\}, \quad C = \{a, d\}, \quad D = \{b, c, d\}.$$

In this abstract setting, the bounds of Lemma 1.2 are asymptotically tight. There exist n subsets of m elements with the above property and $\Theta(mn^{1/2})$ incidences (or $\Theta(nm^{1/2})$). Thus, to derive a stronger upper bound for point-line incidences, we must rely on additional geometric properties of lines.

We now consider an asymptotically tight lower bound for Theorem 1.1. Instead of Erdős's original construction, we present a simpler construction due to Elekes (2001).

Claim 1.3 *For every m and n there exist a set $\mathcal{P}$ of m points and a set $\mathcal{L}$ of n lines, both in $\mathbb{R}^2$, such that $I(\mathcal{P}, \mathcal{L}) = \Theta(m^{2/3}n^{2/3} + m + n)$.*

Proof The term m dominates the bound $\Theta(m^{2/3}n^{2/3}+m+n)$ when $m = \Omega(n^2)$. In this case we can simply take m points on a single line to obtain m incidences. Similarly, the term n dominates the bound when $n = \Omega(m^2)$. In this case we take n lines that pass through a single point to obtain n incidences. It remains to construct a configuration with $\Theta(m^{2/3}n^{2/3})$ incidences when $m = O(n^2)$ and $n = O(m^2)$.

Let $r = (m^2/4n)^{1/3}$ and $s = (2n^2/m)^{1/3}$ (for simplicity, instead of taking the ceiling function of s and r, we assume that these are integers). We set

$$\mathcal{P} = \{\ (i, j)\ :\ 1 \le i \le r \quad \text{and} \quad 1 \le j \le 2rs\ \},$$

and

$$\mathcal{L} = \{\ y = ax + b\ :\ 1 \le a \le s \quad \text{and} \quad 1 \le b \le rs\ \}.$$

Note that $\mathcal{P}$ is a rectangular section of the integer lattice. The slopes and y-intercepts of the lines of $\mathcal{L}$ also form such a lattice. Figure 1.1 depicts an example configuration rotated by 90°. We also have that

$$|\mathcal{P}| = 2r^2s = 2 \cdot \frac{m^{4/3}}{(4n)^{2/3}} \cdot \frac{(2n^2)^{1/3}}{m^{1/3}} = m,$$

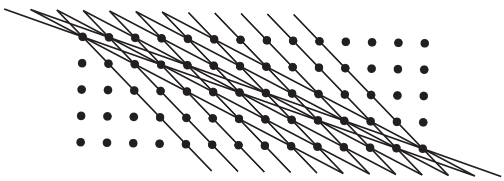


Figure 1.1 Elekes's construction, rotated by 90°.

and

$$|\mathcal{L}| = rs^2 = \frac{m^{2/3}}{(4n)^{1/3}} \cdot \frac{(2n^2)^{2/3}}{m^{2/3}} = n.$$

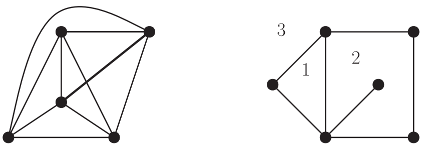
Consider a line $\ell \in \mathcal{L}$ that is defined by the equation $y = ax + b$. For any $x \in \{1, \ldots, r\}$, there exists $y \in \{1, \ldots, 2rs\}$ such that the point (x, y) is incident to ℓ. That is, every line of $\mathcal{L}$ is incident to exactly r points of $\mathcal{P}$, which in turn implies that

$$I(\mathcal{P}, \mathcal{L}) = r \cdot |\mathcal{L}| = \frac{m^{2/3}}{(4n)^{1/3}} \cdot n = 2^{-2/3} m^{2/3} n^{2/3}. \qquad \square$$

1.3 The Crossing Lemma

One elegant proof of Theorem 1.1 is based on the *crossing lemma*. We study this proof in Section 1.4. Here, we first go over some required preliminaries. For a brief review of graph theory notation, see Section A.2.

The *crossing number* of a graph $G = (V, E)$, denoted $\text{cr}(G)$, is the smallest integer k such that we can draw G in the plane with k edge crossings. Figure 1.2(a) depicts a drawing of K_5 with a single crossing. Since K_5 cannot be drawn without crossings, we have that $\text{cr}(K_5) = 1$. Intuitively, we expect a graph with a lot more edges than vertices to have a large crossing number. Given a graph $G = (V, E)$, we are interested in a lower bound for $\text{cr}(G)$ with respect to $|V|$ and $|E|$.



(a) (b)

Figure 1.2 (a) A drawing of K_5 with a single crossing. (b) A graph with two bounded faces and one unbounded face.

A graph G is *planar* if $\text{cr}(G) = 0$. We consider a connected planar graph $G = (V, E)$ with v vertices and e edges. More specifically, we consider a drawing of G in the plane with no crossings. The *faces* of this drawing are the maximal two-dimensional connected regions that are bounded by the edges. This includes one outer, infinitely large region. For an example, see Figure 1.2(b). Denote by f the number of faces in the drawing of G. Then *Euler's formula* states that

$$v + f = e + 2. \tag{1.3}$$

For planar graphs that are not connected, we instead have that $v + f > e + 2$.

Every edge of G is either on the boundary of two faces or has both of its sides on the boundary of the same face. Moreover, the boundary of every face of G consists of at least three edges. Thus, we have $2e \geq 3f$. Plugging this into Equation (1.3) yields

$$e \leq v + f - 2 \leq v + \frac{2e}{3} - 2.$$

That is, for any planar graph $G = (V, E)$, we have that

$$|E| \leq 3|V| - 6. \tag{1.4}$$

The above leads to our first lower bound on $\mathrm{cr}(G)$.

Lemma 1.4 *For any graph $G = (V, E)$, we have* $\mathrm{cr}(G) \geq |E| - 3|V| + 6$.

Proof Consider a drawing of G in the plane that minimizes the number of crossings. Let $E' \subset E$ be a maximum subset of the edges such that no two edges of E' intersect in the drawing. By Equation (1.4), we have that $|E'| \leq 3|V| - 6$. Since every edge of $E \backslash E'$ intersects at least one edge of E', and since $|E \backslash E'| \geq |E| - 3|V| + 6$, there are at least $|E| - 3|V| + 6$ crossings in the drawing. □

Since K_5 has 5 vertices and 10 edges, Lemma 1.4 gives the correct value $\mathrm{cr}(K_5) = 1$. However, in general the bound of this lemma is rather weak. For example, it is known that $\mathrm{cr}(K_n) = \Theta(n^4)$, while Lemma 1.4 only implies that $\mathrm{cr}(K_n) = \Omega(n^2)$. We can amplify the lower bound of Lemma 1.4 by combining it with a probabilistic argument. The following lemma was originally derived in Ajtai et al. (1982); Leighton (1983), with different proofs.

Lemma 1.5 (The crossing lemma) *Let $G = (V, E)$ be a graph with $|E| \geq 4|V|$. Then* $\mathrm{cr}(G) = \Omega(|E|^3/|V|^2)$.

Proof Consider a drawing of G with $\mathrm{cr}(G)$ crossings. Set $p = \frac{4|V|}{|E|}$. The assumption of the lemma implies that $0 < p \leq 1$. We remove every vertex of V from the drawing with probability $1 - p$ (together with the edges adjacent to the vertex). Let $G' = (V', E')$ denote the resulting subgraph. Let c' denote the number of crossings in the drawing of G that have both of their edges in E'.

To avoid confusion with the edge set E, we denote expectation of a random variable as $\mathbb{E}[\cdot]$. Since every vertex remains with probability p, we have that $\mathbb{E}[|V'|] = p|V|$. Since every edge remains if and only if its two endpoints remain, we have that $\mathbb{E}[|E'|] = p^2|E|$. Finally, since each crossing remains if and only if the two corresponding edges remain, we have that $\mathbb{E}[c'] = p^4\mathrm{cr}(G)$. By linearity of expectation,

$$\begin{aligned}\mathbb{E}[c' - |E'| + 3|V'|] &= p^4\mathrm{cr}(G) - p^2|E| + 3p|V| \\ &= \frac{4^4|V|^4}{|E|^4}\mathrm{cr}(G) - \frac{4^2|V|^2}{|E|^2} \cdot |E| + 3 \cdot \frac{4|V|}{|E|} \cdot |V| \\ &= \frac{4^4|V|^4}{|E|^4}\mathrm{cr}(G) - \frac{4|V|^2}{|E|}.\end{aligned}$$

Since this is the expected value, there exists a subgraph $G^* = (V^*, E^*)$ with c^* crossings remaining from the drawing of G, such that

$$c^* - |E^*| + 3|V^*| \leq \frac{4^4|V|^4}{|E|^4}\mathrm{cr}(G) - \frac{4|V|^2}{|E|}. \tag{1.5}$$

By Lemma 1.4, we have $c^* \geq |E^*| - 3|V^*| + 6$. Combining this with Inequality (1.5) implies

$$0 < 6 \leq c^* - |E^*| + 3|V^*| \leq \frac{4^4|V|^4}{|E|^4}\mathrm{cr}(G) - \frac{4|V|^2}{|E|}.$$

That is, $\frac{4|V|^2}{|E|} < \frac{4^4|V|^4}{|E|^4}\mathrm{cr}(G)$. Tidying up this inequality leads to the required bound. □

Lemma 1.5 implies the asymptotically tight bound $\mathrm{cr}(K_n) = \Omega(n^4)$.

1.4 Szemerédi–Trotter via the Crossing Lemma

We are now ready to prove Theorem 1.1. We first restate this theorem.

Theorem 1.1 *Let $\mathcal{P}$ be a set of m points and let $\mathcal{L}$ be a set of n lines, both in $\mathbb{R}^2$. Then $I(\mathcal{P}, \mathcal{L}) = O(m^{2/3}n^{2/3} + m + n)$.*

Proof We write $\mathcal{L} = \{\ell_1, \ldots, \ell_n\}$ and denote by m_j the number of points of $\mathcal{P}$ that are on ℓ_j. Notice that $I(\mathcal{P}, \mathcal{L}) = \sum_{j=1}^n m_j$. We may remove any line ℓ_j that satisfies $m_j = 0$, since this would not change the number of incidences.

We build a graph $G = (V, E)$ as follows. Every vertex of V corresponds to a point of $\mathcal{P}$. For $v, u \in V$, we add (v, u) to E if v and u correspond to consecutive points along a line of $\mathcal{L}$. For an example, see Figure 1.3. A line ℓ_j contributes exactly $m_j - 1$ edges of E. Thus, we have $|V| = m$ and $|E| = \sum_{j=1}^n (m_j - 1) = I(\mathcal{P}, \mathcal{L}) - n$.

If $|E| < 4|V|$ then $I(\mathcal{P}, \mathcal{L}) = O(m + n)$, which completes the proof. We may thus assume that $|E| \geq 4|V|$. Then, Lemma 1.5 leads to

$$\mathrm{cr}(G) = \Omega\left(\frac{(I(\mathcal{P}, \mathcal{L}) - n)^3}{m^2}\right). \tag{1.6}$$

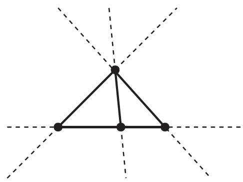


Figure 1.3 (Solid segment) The edges of the graph. (Dashed segment) The portions of the lines ℓ_j that do not form graph edges.

We draw G according to the point-line configuration: Every vertex is at the corresponding point and every edge is the corresponding line segment. Every crossing in this drawing is an intersection of two lines of $\mathcal{L}$. Since every two lines intersect at most once, we have that $\text{cr}(G) \leq \binom{n}{2} = O(n^2)$. Combining this with Equation (1.6) implies that

$$\frac{(I(\mathcal{P},\mathcal{L})-n)^3}{m^2} = O(n^2).$$

Rearranging this equation gives $I(\mathcal{P},\mathcal{L}) = O(m^{2/3}n^{2/3}+n)$. $\square$

The proof of Theorem 1.1 is another example of the double counting method. We counted $\text{cr}(G)$ in two different ways. By combining the two resulting bounds, we obtained a bound on the number of incidences.

In the proof of Theorem 1.1, we used the geometric property that two lines intersect at most once. This is similar to the observation that any two points are contained in one line,[1] which was used in the proof of Lemma 1.2. In the proof of Theorem 1.1 we used a second geometric property when stating that the line ℓ_j corresponds to exactly $m_j - 1$ edges of E. This statement relies on the observation that a line consists of a single connected component and does not intersect itself. When replacing the lines with other curves that satisfy the same geometric properties, the proof of Theorem 1.1 remains valid.

1.5 The Unit Distances Problem

The *unit distances problem* is one of the main open problems in discrete geometry. While it is extremely difficult to solve this problem, it easy to state:

In a set of n points in the plane, what is the maximum possible number of pairs of points at distance 1 from each other?

[1] These two geometric properties are equivalent when studying point-line incidences, due to point-line duality. We discuss this concept in Section 1.10.

We denote this maximum number of pairs as $u(n)$. By taking a set of n points equally spaced on a line, we immediately obtain that $u(n) \geq n-1$. Erdős (1946) introduced the problem, while also deriving the bounds $u(n) = O(n^{3/2})$ and $u(n) = \Omega(n^{1+c/\log\log n})$, for some constant c. While many mathematicians have studied this problem, the lower bound for $u(n)$ has not been improved since 1946 and the upper bound was last improved in 1984. That was when Spencer et al. (1984) derived the bound $u(n) = O(n^{4/3})$.

Consider a set $\mathcal{P} \subset \mathbb{R}^2$ of n points such that the number of unit distances between pairs of points of $\mathcal{P}$ is $u(n)$. We draw a unit circle (a circle of radius one) around each point of $\mathcal{P}$, and denote the set of these n circles as C. Every two points $p, q \in \mathcal{P}$ that determine a unit distance correspond to two incidences in $\mathcal{P} \times C$: The circle around p is incident to q and vice versa. See Figure 1.4 for an example. Thus, to bound $u(n)$ it suffices to bound the maximum number of incidences between n points and n unit circles (it is not difficult to show that this maximum number of incidences is asymptotically equivalent to $u(n)$).

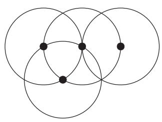


Figure 1.4 Every two points that are at a unit distance correspond to two point-circle incidences.

Theorem 1.6 *Let $\mathcal{P}$ be a set of n points and let C be a set of n unit circles, both in $\mathbb{R}^2$. Then $I(\mathcal{P}, C) = O(n^{4/3})$.*

Theorem 1.6 immediately implies the current best bound $u(n) = O(n^{4/3})$.

Proof of Theorem 1.6 We imitate the proof of Theorem 1.1. Let $C = \{c_1, \ldots, c_n\}$ and let m_j denote the number of points of $\mathcal{P}$ on c_j. Note that $I(\mathcal{P}, C) = \sum_{j=1}^{n} m_j$. We may remove any circle c_j that satisfies $m_j < 3$, since this reduces the number of incidences by at most $2n$.

We build a graph $G = (V, E)$ as follows. Every vertex of V corresponds to a point of $\mathcal{P}$. For $v, u \in V$, the edge (v, u) is in E if v and u are consecutive points along at least one circle of C. A circle c_j corresponds to exactly m_j edges of E, and every edge originates from at most two unit circles. Note that $|V| = n$ and $|E| \geq \left(\sum_{j=1}^{n} m_j\right)/2 = I(\mathcal{P}, C)/2$.

If $|E| < 4|V|$ then $I(\mathcal{P}, C) = O(n)$, which completes the proof. We may thus assume that $|E| \geq 4|V|$. By Lemma 1.5, we have that

$$\mathrm{cr}(G) = \Omega\left(\frac{I(\mathcal{P}, C)^3}{n^2}\right). \tag{1.7}$$