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Equivariant Topology and Derived Algebra

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Preface

The chapters of the present volume cover a range of topics in equivariant topology and derived algebra, chosen to connect with major themes from John Greenlees' vast mathematical career.

Conference

The catalyst for these proceedings was a week-long conference held at NTNU (Trondheim) between the 29^{th} of July and the 2^{nd} of August 2019. This conference, entitled *Equivariant Topology and Derived Algebra*, was held in honor of John Greenlees' 60^{th} birthday. The conference consisted of 15 invited talks, 11 contributed talks, and 13 shorter *gong show* style talks, and was attended by a diverse group of over 90 international participants. The mathematical content was enhanced by a customary hiking excursion and a hearty conference dinner with beautiful scenic views.

Summary of the chapters

We briefly outline the chapters appearing in these proceedings, while also taking the opportunity to connect them to the work of John Greenlees, which at the date of writing spans more than 90 papers and four research monographs [32, 45, 57, 73].

Comparing Dualities in the $K(n)$-local Category

P. G. Goerss and M. J. Hopkins

Duality is a recurring theme through the work of John Greenlees; the starting place is perhaps Spanier–Whitehead duality. In modern language, the Spanier–Whitehead dual of a spectrum X is the function spectrum $DX = F(X, \mathbb{S})$, which arises from the commutative monoidal structure of the stable homotopy category. A common calculation is to show that the dual of the Moore spectrum for $\mathbb{Z}/2$ is simply a shift of that spectrum. A detailed examination of functional duals and Moore spectra is the subject of Greenlees' first published work, [93]. It is natural to look for generalisations of (Spanier–Whitehead) duality, for example [86] and [88] consider duality in the equivariant stable homotopy category, while [31] and [41] look more generally at questions of duality.

The first chapter of this volume takes up this theme and investigates duality in the $K(n)$-local stable homotopy category, giving a full and detailed proof of a result relating $K(n)$-local Spanier–Whitehead duality to the more computable notion of Brown–Comenetz duality.

Axiomatic Representation Theory of Finite Groups by way of Groupoids

I. Dell'Ambrogio

A second major theme in the work of John Greenlees is representation theory, and in particular, the use and study of Mackey functors. The most immediate way Mackey functors appear in the work of Greenlees is via equivariant cohomology theories. These are a generalisation of cohomology theories that have G-spaces as input, and take G-Mackey functors as coefficients. The category of Mackey functors is also a rich and interesting category in its own right, as demonstrated in [40, 60, 73, 82, 85]. Indeed, three chapters of this volume consider Mackey functors at length.

This chapter considers very general notions of Mackey functors and gives a common conceptual framework for several different versions. It provides relations between these different versions and connects the theory to 2-categories and bisets.

Chromatic Fracture Cubes

O. Antolín-Camarena and T. Barthel

A sizable portion of John Greenlees' work is dedicated to the development of algebraic models for rational G-spectra, where G is a compact Lie group. Algebraic models for several groups have been established, including all finite groups, the circle, tori of arbitrary dimension, $O(2)$ and $SO(3)$, see [73, 57, 16, 11, 59, 52]. Greenlees has conjectured that an algebraic model (satisfying a list of key properties) exists for every compact Lie group G. A key tool for this project is an isotropy separation of the sphere spectrum in rational G-spectra. This separation is a pullback square similar to the arithmetic pullback square or the chromatic fracture square. As the sphere spectrum is the monoidal unit, the isotropy separation extends to a decomposition of the (homotopy) category of rational G-spectra into simpler building blocks. Recent work of Greenlees abstracts this machinery to the setting of axiomatic stable homotopy theory [1].

This chapter generalises the familiar chromatic fracture square in the $E(n)$-local stable homotopy category to a chromatic fracture cube. This cube categorifies to provide a combinatorial decomposition of the category into monochromatic pieces. The $E(n)$-local stable homotopy category can be reconstructed by taking a homotopy limit of these monochromatic pieces over a certain diagram of diagrams.

An Introduction to Algebraic Models for Rational G-Spectra

D. Barnes and M. Kędziorek

As mentioned above, a major project of Greenlees is the development of algebraic models for rational G-spectra, where G is a compact Lie group, see [73, 57, 16, 11, 59, 52]. The initial case is where G is a finite group, here the algebraic model for rational G-spectra is a finite product over conjugacy classes of subgroups $H \leq G$ of graded $\mathbb{Q}[W_G H]$-modules ($W_G H$ is the Weyl group of H in G). One of the ways to prove this result uses the idempotent splitting of rational G-Mackey functors, see Appendix A of [73]. There are many papers building upon that work, such as constructing an algebraic model for naive-commutative ring G-spectra [9].

This chapter gives an introduction to rational Mackey functors and summarises the main techniques used to obtain algebraic models for rational G-spectra, concentrating on the case of a finite group G. It discusses the topological and algebraic parallels of using idempotents to split the category of rational G-spectra and rational G-Mackey functors. It also briefly mentions the techniques to obtain algebraic models when G is not finite.

Monoidal Bousfield Localizations and Algebras over Operads
D. White

Commutative monoidal structures appear throughout John Greenlees' work, occurring with algebraic origins [63, 72, 74], topological origins [4, 9] and bridging the divide between algebra and topology: [13, 36, 37, 75]. Moreover, the construction of algebraic models for rational G-spectra often depends upon making use of (commutative) monoidal structures in both topology and algebra. For example, the isotropy separation arguments require that certain localizations of the sphere spectrum are still commutative monoids. This property is not automatic, even under suitable cofibrancy conditions.

This chapter characterizes those Bousfield localizations that respect (commutative) monoidal structures, and moreover proves that these localizations preserve algebras over cofibrant operads. This general machinery can be used to retrieve many classical results which have repeatedly been used in the work of Greenlees.

Stratification and Duality for Unipotent Finite Supergroup Schemes
D. Benson, S. B. Iyengar, H. Krause and J. Pevtsova

A recent direction in the work of Greenlees is the study of tensor-triangulated categories, triangulated categories with compatible symmetric monoidal product and function object. A central example is the stable homotopy category, arising from homotopy (co)fibre sequences and the smash product and function spectrum. The equivariant stable homotopy category for a compact Lie group G is an even richer example, see [1], [3] and [8]. An important problem in such contexts is to classify the tensor ideal localising subcategories, and also the tensor ideal thick subcategories of compact objects, and to study questions related to duality as in [33].

The purpose of this chapter is to survey some methods developed to address these problems, and to illustrate them by establishing such classifications and duality statements for the stable module category of a unipotent finite supergroup scheme.

Bi-incomplete Tambara Functors

A. J. Blumberg and M. A. Hill

One of the recent themes in equivariant homotopy theory is to understand commutative ring objects in G-spaces and G-spectra. This is reflected in John Greenlees' work through the research on commutativity described above and more directly, as in [16, 9]. The subtlety and complications of equivariant commutativity can be described using a certain class of G-operads, called N_∞ operads. Algebras over an N_∞ operad $\mathcal{O}$ in G-topological spaces correspond, roughly speaking, to a G-spectrum with transfers determined by $\mathcal{O}$. Thus, one might think of $\mathcal{O}$ as governing the additive structure of a G-spectrum. Algebras over an N_∞ operad $\mathcal{O}$ in G-spectra (as opposed to G-spaces) correspond, roughly speaking, to $\mathcal{O}$-commutative ring G-spectra, that is, ring G-spectra with norm maps on homotopy groups determined by $\mathcal{O}$. Thus, in this case one might think of $\mathcal{O}$ as governing the multiplicative ring structure of a G-spectrum. The natural question is: how one can mix the various additive and multiplicative structures?

This chapter investigates the compatibility conditions between incomplete additive transfers and incomplete multiplicative norms in the algebraic setting of G-Tambara functors and provides a full description of the possible interactions of these two classes of maps.

Homotopy Limits of Model Categories, Revisited

J. E. Bergner

A key observation of the paper [1] is that the algebraic models for rational G-equivariant spectra can be described as homotopy limits of diagrams of model categories. This observation developed from homotopy pullback constructions in [11] based on isotropy separation, building on machinery of Greenlees–Shipley [20, 21, 25]. Homotopy limits also occur in the chapter of Antolín-Camarena–Barthel (in the setting of $(\infty, 1)$-categories) further demonstrating their ubiquity.

The final chapter of this volume provides a comprehensive outline of the machinery required for constructing homotopy limits of diagrams of Quillen model categories and left Quillen functors between them, collecting previous work of the author. Moreover the chapter provides a wealth of important examples of this homotopy limit construction. The chapter also provides some warning on working with diagrams which come with a mix of left and right Quillen functors.

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John Greenlees' published work to date

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1

Comparing Dualities in the $K(n)$-local Category

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Abstract

In their work on the period map and the dualizing sheaf for Lubin-Tate space, Gross and the second author wrote down an equivalence between the Spanier-Whitehead and Brown-Comenetz duals of certain type n-complexes in the $K(n)$-local category at large primes. In the culture of the time, these results were accessible to educated readers, but this seems no longer to be the case; therefore, in this note we give the details. Because we are at large primes, the key result is algebraic: in the Picard group of Lubin-Tate space, two important invertible sheaves become isomorphic modulo p.

For John Greenlees, the master of duality.

Introduction

Fix a prime p and and an integer $n \geq 0$, and let $K(n)$ denote the nth Morava K-theory at the prime p. If $n \geq 1$, the $K(n)$-local stable homotopy category has two dualities. First, there is $K(n)$-local Spanier-Whitehead duality $D_n(-)$. This behaves very much like Spanier-Whitehead duality in the ordinary stable category: it has good formal properties, but it can be very hard to compute. Second, there is Brown-Comenetz duality $I_n(-)$, which behaves much like a Serre-Grothendieck duality and, in many ways, is much more computable. One of the key features of the

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$K(n)$-local category is that under some circumstances the two dualities are closely related.

Recall that a finite spectrum X is of type n if $K(m)_*X = 0$ for $m < n$. By [22], any type n spectrum has a $v_n^{p^k}$-self map; that is, there is an integer k and map

$$\Sigma^{2p^k(p^n-1)} X \to X$$

which induces multiplication by $v_n^{p^k}$ in $K(n)_*$. In their papers on the period map and the dualizing sheaf for Lubin-Tate space, Gross and the second author [20] wrote down the following result. Suppose X is a type n-spectrum with a $v_n^{p^k}$-self map and suppose further that p times the identity map of X is zero. Then if $2p > \max\{n^2+1, 2n+2\}$ there is an equivalence in the $K(n)$-local category[1]

$$I_n X \simeq \Sigma^{2p^{nk}r(n)+n^2-n} D_n X \tag{1.1}$$

where $r(n) = (p^n-1)/(p-1) = p^{n-1} + \cdots + p + 1$. This equivalence gives a conceptual explanation for many of the self-dual patterns apparent in the amazing computations of Shimomura and his coauthors. See, for example, [32], [31], [5], and [26].

The point of this note is to write down a linear narrative with this result at the center. In some sense, there is nothing new here, as the key ideas can be found scattered through the literature, and other authors have obliquely touched on this topic. A rich early example is in §5 of the paper [8] by Devinatz and the second author, and the important paper of Dwyer, Greenlees, and Iyengar [10] embeds many of the ideas here into a far-reaching and beautiful theory. In another sense, however, there is quite a bit to say, as there are any number of key technical ideas we need to access, some of which have not quite made it into print and others buried in ways that make them hard to uncover. In any case, the result is of enough importance that it deserves specific memorialization.

Here is a little more detail. We fix p and n and let $E = E_n$ be Morava E-theory for n and p. This represents a complex oriented cohomology theory with formal group law a universal deformation of the Honda formal group law H_n of height n. See §1 for more details. As always we write

$$E_*X = \pi_* L_{K(n)}(E \wedge X).$$

The E_*-module E_*X is a graded Morava module: it has a continuous

[1] The bound on p is very slightly different than in [20]; see Proposition 1.9.

and twisted action of the Morava stabilizer group $\mathbb{G}_n = \mathrm{Aut}(H_n, \mathbb{F}_{p^n})$. See Remark 1.5.

There are two key steps to the equivalence (1.1). We have a $K(n)$-local equivalence $I_nX \simeq I_n \wedge D_nX$ where $I_n = I_n(S^0)$; thus, the first step is the identification of the homotopy type of I_n, at least for p large with respect to n. This is also due to Gross and the second author, with details laid out in [33]. The key fact is that I_n is dualizable in the $K(n)$-local category; by [21] this is equivalent to the statement that E_*I_n is an invertible graded Morava module and, indeed, the main result of [33] (interpreting [20]) is that there is an isomorphism of Morava modules

$$E_*I_n \cong E_*S^{n^2-n}[\det]$$

where $S^0[\det] = S[\det]$ is a determinant twisted sphere in the $K(n)$-local category; see Remark 1.26. The number $r(n)$ in (1.1) is an artifact of the determinant; see (1.3.1).

The second key step is an analysis of the $K(n)$-local Picard group $\mathrm{Pic}_{K(n)}$ of equivalence classes of invertible objects in the $K(n)$-local category. As mentioned, we know that a $K(n)$-local spectrum X is invertible if and only if E_*X is an invertible graded Morava module. We also know that the group of invertible graded Morava modules concentrated in even degrees is isomorphic to the continuous cohomology group $H^1(\mathbb{G}_n, E_0^\times)$, where $E_0^\times$ is the group of units in the ring E_0. Hence, if we write $\mathrm{Pic}^0_{K(n)} \subseteq \mathrm{Pic}_{K(n)}$ for the subgroup of objects X with E_*X in even degrees, we get a map

$$e : \mathrm{Pic}^0_{K(n)} \longrightarrow H^1(\mathbb{G}_n, E_0^\times).$$

The map is an injection under the hypothesis $2p > \max\{n^2+1, 2n+2\}$. See Proposition 1.9. This is the origin for the hypothesis on p and n in the equivalence of (1.1): it reduces that equivalence to an algebraic calculation.

It is an observation of [21] that the map $\mathbb{Z} \to \mathrm{Pic}^0_{K(n)}$ sending k to S^{2k} extends to an inclusion of the completion of the integers

$$\mathfrak{Z}_\mathfrak{n} \overset{\mathrm{def}}{=} \lim_k \mathbb{Z}/(p^k(p^n-1)) \to \mathrm{Pic}^0_{K(n)};$$

that is, for any $a \in \mathfrak{Z}_\mathfrak{n}$ we have a sphere S^{2a}. (The phrase "p-adic sphere" is common here, but misleading: $\mathfrak{Z}_\mathfrak{n}$ is not the p-adic integers. See Remark 1.23.) Now let $\lambda = \lim_k\ p^{nk}r(n) \in \mathfrak{Z}_\mathfrak{n}$. The key algebraic result can now be deduced from Proposition 1.30 below: under the composition

$$\mathrm{Pic}^0_{K(n)} \xrightarrow{e} H^1(\mathbb{G}_n, E_0^\times) \to H^1(\mathbb{G}_n, (E_0/p)^\times)$$

the spectra $S[\det]$ and $S^{2\lambda}$ map to the same element. The equivalence (1.1) follows once we observe that if X is type n and has a $v_n^{p^k}$-self map, then there is $K(n)$-local equivalence

$$S^{2\lambda} \wedge X \simeq \Sigma^{2p^{nk}r(n)} X.$$

See Theorems 1.42 and 1.43.

It is worth emphasizing that the algebraic result Proposition 1.30 only requires $p > 2$; it is the topological applications which require the more stringent restrictions on the prime. In fact, the equivalence of dualities in (1.1) can be false if the prime is small. See Remark 1.45.

The plan of this note is as follows: in the first section we give some homotopy theoretic and algebraic background, in the second section we give a discussion of the Picard group, lingering long enough to give details of the structure of $\mathrm{Pic}^0_{K(n)}$ as a profinite $\mathfrak{Z}_\mathfrak{n}$-module. See Proposition 1.18. In Section 3 we discuss the determinant and prove the key Proposition 1.30. In Section 4 we give some discussion of how Spanier-Whitehead and Brown-Comenetz duality behave in the Adams-Novikov Spectral Sequence. In the final section, we give the homotopy theoretic applications.

Acknowledgements

This project began as an attempt to find a conceptual computation of the self-dual patterns apparent in the Shimomura-Yabe calculation [32] of $\pi_* L_{K(2)} V(0)$ at $p > 3$. Then in a conversation with Tobias Barthel it emerged that there was no straightforward argument in print to prove the equivalence of dualities in (1.1). Later, as Guchuan Li was working on his thesis [27] it became apparent that he needed these results and, more, there were constructions once present in the general culture that were no longer easily accessible. Thus a sequence of notes, begun at MSRI in Spring of 2014, have evolved into this chapter. Many thanks for Agnès Beaudry and Vesna Stojanoska for reading through an early draft. Others have surely written down proofs for themselves as well; for example, Hans-Werner Henn once remarked that "the determinant essentially disappears mod p," which is a very succinct summary of Proposition 1.30.

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1.1 Some background

In this section we gather together the basic material used in later sections. All of this is thoroughly covered in the literature and collected here only for narrative continuity.

1.1.1 The $K(n)$-local category

For an in-depth study of the technicalities in the $K(n)$-local category, see Hovey and Strickland [24]. Other introductions can be found in almost any paper on chromatic homotopy theory. We were especially thorough in [3] §2.

Fix a prime p and an integer $n > 0$. In order to be definite we define the nth Morava K-theory $K(n)$ to be the 2-periodic complex oriented cohomology theory with coefficients $K(n)_* = \mathbb{F}_{p^n}[u^{\pm 1}]$ with u in degree -2. The associated formal group law over $K(n)_0 = \mathbb{F}_{p^n}$ is the unique p-typical formal group law H_n with p-series $[p]_{H_n}(x) = x^{p^n}$. This is, of course, the nth Honda formal group law. For H_n we have

$$v_n = u^{1-p^n} \in K(n)_{2(p^n-1)}.$$

The $K(n)$-local category is the category of $K(n)$-local spectra.

We also have $K(0) = H\mathbb{Q}$, the rational Eilenberg-MacLane spectrum, and $K(0)$-local spectra are the subject of rational stable homotopy theory.

We define $\mathbb{G}_n = \mathrm{Aut}(H_n, \mathbb{F}_{p^n})$ to be the group of automorphisms of the pair $(H_n, \mathbb{F}_{p^n})$. Since H_n is defined over $\mathbb{F}_p$, there is a splitting

$$\mathrm{Aut}(H_n, \mathbb{F}_{p^n}) \cong \mathrm{Aut}(H_n/\mathbb{F}_{p^n}) \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$$

where the normal subgroup is the isomorphisms of H_n as a formal group law over $\mathbb{F}_{p^n}$. We write $\mathbb{S}_n = \mathrm{Aut}(H_n/\mathbb{F}_{p^n})$ for this subgroup.

To get a Landweber exact homology theory which captures more than Morava K-theory, we use the Morava (or Lubin-Tate) theory $E = E_n$. This theory has coefficients

$$E_* = \mathbb{W}[[u_1, \ldots, u_{n-1}]][u^{\pm 1}]$$

where again u is in degree -2 but the power series ring is in degree 0. The ring $\mathbb{W} = W(\mathbb{F}_{p^n})$ is the Witt vectors of $\mathbb{F}_{p^n}$.

Note that E_0 is a complete local ring with maximal ideal $\mathfrak{m}$ generated by the regular sequence $\{p, u_1, \ldots, u_{n-1}\}$. We choose the formal group law G_n over E_0 to be the unique p-typical formal group law with p-series

$$[p]_{G_n}(x) = px +_{G_n} u_1 x^p +_{G_n} \cdots +_{G_n} u_{n-1} x^{p^{n-1}} +_{G_n} x^{p^n}. \qquad (1.1.1)$$

Thus $v_i = u_i u^{1-p^i}$, $1 \leq i \leq n-1$, $v_n = u^{1-p^n}$ and $v_i = 0$ if $i > n$. Note that G_n reduces to H_n modulo $\mathfrak{m}$.

We define $E_*X = (E_n)_*X$ by

$$E_*X = \pi_* L_{K(n)}(E \wedge X).$$

While not quite a homology theory, as it does not take wedges to sums, it is by far our most sensitive algebraic invariant in $K(n)$-local homotopy theory. The group $\mathbb{G}_n$ acts continuously on E_*X making E_*X into a *Morava module.* We will be more precise on this notion below in Remark 1.5.

A basic computation gives

$$E_0E = \pi_0 L_{K(n)}(E \wedge E) \cong \mathrm{map}^c(\mathbb{G}_n, E_0)$$

where map^c denotes the continuous maps. See Lemma 10 of [33] for a proof. The $K(n)$-local E_n-based Adams-Novikov Spectral Sequence now reads

$$H^s(\mathbb{G}_n, E_tX) \Longrightarrow \pi_{t-s} L_{K(n)}X. \qquad (1.1.2)$$

Cohomology here is continuous cohomology.

Remark 1.1 (Lubin-Tate theory) The pair (G_n, E_0) has an important universal property which is useful for understanding the action of $\mathbb{G}_n$.

Consider a complete local ring $(S, \mathfrak{m}_S)$ with $S/\mathfrak{m}_S$ of characteristic p. Define the groupoid of deformations $\mathrm{Def}_{H_n}(S)$ to be the category with objects (i, G) where $i : \mathbb{F}_{p^n} \to S/\mathfrak{m}_S$ is a morphism of fields and G is a formal group law over S with $q_*G = i_*H_n$. Here $q : S \to S/\mathfrak{m}_S$ is the quotient map. There are no morphisms $\psi : (i, G) \to (j, H)$ if $i \neq j$ and a morphism $(i, G) \to (i, H)$ is an isomorphism of formal groups laws $\psi : G \to H$ so that $q_*\psi$ is the identity. These are the $\star$-isomorphisms. By a theorem of Lubin and Tate [28] we know that if two deformations are $\star$-isomorphic, then there is a unique $\star$-isomorphism between them.

Put another way, the groupoid $\mathrm{Def}_{H_n}(S)$ is discrete. Furthermore, E_0 represents the functor of $\star$-isomorphism classes of deformations:

$$\mathrm{Hom}^c_{\mathbb{W}}(E_0, S) \cong \pi_0 \mathrm{Def}_{H_n}(S).$$

Here $\mathrm{Hom}^c_{\mathbb{W}}$ is the set of continuous $\mathbb{W}$-algebra maps. As a universal deformation we can and do choose the formal group law G_n over E_0 to be the p-typical formal group law defined above in (1.1.1).

Remark 1.2 (**The action of the Morava stabilizer group**) We use Lubin-Tate theory to get an action of $\mathbb{G}_n$ on E_0. This exposition follows [18] §3.

Let $g = g(x) \in \mathbb{F}_{p^n}[[x]]$ be an element in $\mathbb{S}_n$. Choose any lift of $g(x)$ to $h(x) \in E_0[[x]]$ and let G_h be the unique formal group law over E_0 so that

$$h : G_h \to G_n$$

is an isomorphism. Since $g : H_n \to H_n$ is an isomorphism over $\mathbb{F}_{p^n}$, the pair (id, G_h) is a deformation of H_n. Hence there is a unique $\mathbb{W}$-algebra map $\phi = \phi_g : E_0 \to E_0$ and a unique $\star$-isomorphism $f : \phi_* G_n \to G_h$. Let ψ_g be the composition

$$\begin{tikzcd} \phi_* G_n \arrow[r, "f"] \arrow[rr, bend right, "\psi_g"'] & G_h \arrow[r, "h"] & G_n \end{tikzcd} . \tag{1.1.3}$$

Note that while G_h depends on choices, the map ϕ_g and the isomorphism ψ_g do not. The map $\mathbb{S}_n \to \mathrm{Aut}(E_0)$ sending g to ϕ_g defines the action of $\mathbb{S}_n$ on E_0. The Galois action on $\mathbb{W} \subseteq E_0$ extends this to an action of all of $\mathbb{G}_n$ on E_0. The action can be extended to all of E_* be noting that $E_2 \cong \tilde{E}^0 S^2 \cong \tilde{E}^0 \mathbb{C}\mathrm{P}^1$ is isomorphic to the module of invariant differentials on the universal deformation G_n. See (1.1.4) below for an explicit formula.

Remark 1.3 (**Formulas for the action**) We make the action of $\mathbb{S}_n$ a bit more precise. By (1.1.3) we have an isomorphism $\psi_g : \phi_* G_n \to G_n$ of p-typical formal group laws over E_0. This can be written

$$\psi_g(x) = t_0(g) +_{G_n} t_1(g) x^p +_{G_n} t_2(g) x^{p^2} +_{G_n} \cdots .$$

This formula defines continuous functions $t_i : \mathbb{S}_n \to E_0$. As in Section 4.1 of [18] we have

$$g_* u = t_0(g) u. \tag{1.1.4}$$

The function t_0 is a crossed homomorphism $t_0 : \mathbb{S}_n \to E_0^\times$; that is,

$$t_0(gh) = [gt_0(h)]t_0(g).$$

Since the Honda formal group is defined over $\mathbb{F}_p$ we can choose the class u to be invariant under the action of the Galois group; hence t_0 extends to crossed homomorphism $t_0 : \mathbb{G}_n \to E_0^\times$ sending $(g, \phi) \in \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{G}_n$ to $t_0(g)$.

Remark 1.4 We record here some basic useful facts about the $K(n)$-local Adams-Novikov Spectral Sequence (1.1.2) which we will use later.

The first two statements are standard and are proved using the action of the center of $Z(\mathbb{G}_n) \subseteq \mathbb{G}_n$ on $E_* = E_*S^0$. There is an isomorphism $\mathbb{Z}_p^\times \cong Z(\mathbb{G}_n)$ sendings $a \in \mathbb{Z}_p^\times$ to the a-series $[a]_{H_n}(x)$ of the Honda formal group. The action of $Z(\mathbb{G}_n)$ on E_0 is trivial and the action on E_* is then determined by the fact that $t_0(a) = a$; that is, a acts on $u \in E_{-2}$ by multiplication by a.

1.) **Sparseness:** If $t \not\equiv 0$ modulo $2(p-1)$, then $H^*(\mathbb{G}_n, E_t) = 0$. If $p = 2$ this is not new information. If $p > 2$ let $C \subseteq Z(\mathbb{G}_n)$ be the cyclic subgroup of Teichmüller lifts of $\mathbb{F}_p^\times$. Then $E_t^C = 0$ and hence

$$H^*(\mathbb{G}_n, E_t) \cong H^*(\mathbb{G}_n/C, E_t^C) = 0.$$

2.) **Bounded torsion:** Suppose $p > 2$ and suppose

$$2t = 2p^k m(p-1) \neq 0$$

with m not divisible by p. Then we have

$$p^{k+1}H^*(\mathbb{G}_n, E_{2t}) = 0.$$

If $p = 2$ write $2t = 2^k(2m+1)$. Then we have

$$2H^*(\mathbb{G}_n, E_{2t}) = 0 \qquad \text{if } k = 1,$$

and

$$2^{k+1}H^*(\mathbb{G}_n, E_{2t}) = 0 \qquad \text{if } k > 1.$$

To get these bounds, first suppose $p > 2$. Let $K = 1 + p\mathbb{Z}_p \subseteq Z(\mathbb{G}_n)$ be the torsion-free subgroup and let $x \in K$ be a topological generator; for example, $x = 1 + p$. The choice of x defines an isomorphism $\mathbb{Z}_p \cong K$. Thus, there is an exact sequence

$$0 \to H^0(K, E_{2t}) \longrightarrow E_{2t} \xrightarrow{x^k - 1} E_{2t} \longrightarrow H^1(K, E_{2t}) \to 0.$$

Thus we see that $p^{k+1}H^1(K, E_{2t}) = 0$ and $H^q(K, E_{2t}) = 0$ if $q \neq 1$. Now use the Lyndon-Hochschild-Serre Spectral Sequence

$$H^p(\mathbb{G}_n/K, H^q(K, E_{2t})) \Longrightarrow H^{p+q}(\mathbb{G}_n, E_{2t})$$

to deduce the claim. At the prime 2 let $x \in \mathbb{Z}_2^\times$ be an element of infinite order which reduces to -1 modulo 4 – for example, $x = 3$ – and let K be the subgroup generated by x. The proof then proceeds in the same fashion.

Note that the arguments for parts (1) and (2) apply not only to $\mathbb{G}_n$, but also for any closed subgroup $G \subseteq \mathbb{G}_n$ which contains the center. In fact, for part (1) we need only have $C = \mathbb{F}_p^\times \subseteq G$.

3.) **There is a uniform and horizontal vanishing line at E_∞:** there is an integer N, depending only on n and p, so that in the Adams-Novikov Spectral Sequence (1.1.2) for any spectrum X

$$E_\infty^{s,*} = 0, \qquad s > N.$$

This can be found in the literature in several guises; for example, it can be put together from the material in Section 5 of [9], especially Lemma 5.11. See [3] §2.3 for references and explanation. See also [2] for even further explanation. If $p - 1 > n$, there is often a horizontal vanishing line at E_2. See Proposition 1.6 below.

1.1.2 Some local homological algebra.

Because E_0 is a complete local ring with maximal ideal $\mathfrak{m}$ generated by a regular sequence, we have a variety of tools from homological algebra. The classic paper here is Greenlees and May [15], but see also [24], Appendix A for direct connections to $E_*(-)$. Tensor product below will mean the $\mathfrak{m}$-completed tensor product. This is one place where the notation E_0 gets out of hand; thus we write $R = E_0$ in this subsection.

Let $u_0 = p$ and define a cochain complex $\Gamma_\mathfrak{m}$ by

$$\Gamma_\mathfrak{m} = \left(R \to R[\frac{1}{u_0}]\right) \otimes_R \left(R \to R[\frac{1}{u_1}]\right) \otimes_R \cdots \otimes_R \left(R \to R[\frac{1}{u_{n-1}}]\right)$$

and more generally we set

$$\Gamma_m(M) = M \otimes_R \Gamma_\mathfrak{m}.$$

Then $H^0_\mathfrak{m}(M) \overset{\text{def}}{=} H^0\Gamma_m(M)$ is the sub-module of $\mathfrak{m}$-torsion and we see that

$$H^s\Gamma_\mathfrak{m}(M) \overset{\text{def}}{=} H^s_\mathfrak{m}(M)$$

is the sth right derived functor of the $\mathfrak{m}$-torsion functor and thus independent of the choices. These are the local cohomology groups. If M is $\mathfrak{m}$-torsion, there is a composite functor spectral sequence

$$\mathrm{Ext}_R^p(M, H^q_{\mathfrak{m}}(N)) \Longrightarrow \mathrm{Ext}_R^{p+q}(M, N). \tag{1.1.5}$$

In the case $N = R$, this spectral sequence simplifies considerably. Note that $H^s_{\mathfrak{m}}(R) = 0$ unless $s = n$ and

$$H^n_{\mathfrak{m}}(R) \stackrel{\mathrm{def}}{=} R/\mathfrak{m}^\infty \stackrel{\mathrm{def}}{=} R/(p^\infty, u_1^\infty, \ldots, u_{n-1}^\infty)\ . \tag{1.1.6}$$

The R-module $R/\mathfrak{m}^\infty$ is an injective R-module and, in fact the injective hull of $R/\mathfrak{m}$. This is a consequence of Matlis duality for $(R, \mathfrak{m})$; see §12.1 of [7], especially Definition 12.1.2 and Remark 12.1.3.

Combining this observation with the spectral sequence (1.1.5) we have

$$\mathrm{Ext}_R^{p+n}(M, R) \cong \mathrm{Ext}_R^p(M, R/\mathfrak{m}^\infty) \cong \begin{cases} \mathrm{Hom}_R(M, R/\mathfrak{m}^\infty), & p = 0; \\ 0, & p \neq 0. \end{cases} \tag{1.1.7}$$

The module $R/\mathfrak{m}^\infty$ also arises in the theory of derived functors of completion. The completion functor

$$M \longmapsto \lim_k \ \left[M \otimes_R R/\mathfrak{m}^k\right]$$

is neither left nor right exact; however, it still has left derived functors $L^{\mathfrak{m}}_s(M)$. These vanish if $s > n$ and there is an isomorphism

$$\begin{aligned} L^{\mathfrak{m}}_n(M) &\cong \lim\ \mathrm{Tor}^R_n(M, R/\mathfrak{m}^k) \\ &\cong \lim\ \mathrm{Hom}_R(R/\mathfrak{m}^k, M) \\ &\cong \mathrm{Hom}_R(R/\mathfrak{m}^\infty, M). \end{aligned}$$

From this it follows that

$$L^{\mathfrak{m}}_s(M) \cong \mathrm{Ext}_R^{n-s}(R/\mathfrak{m}^\infty, M).$$

Remark 1.5 (**Morava modules**) If X is a spectrum we defined

$$E_*X = \pi_* L_{K(n)}(E \wedge X).$$

By [24], Proposition 8.4, the E_*-module E_*X is $L^{\mathfrak{m}}$-complete; that is, the map

$$E_*X \longrightarrow L^{\mathfrak{m}}_0(E_*X)$$

is an isomorphism. In particular, E_*X is equipped with the $\mathfrak{m}$-adic topology.

The action of $\mathbb{G}_n$ on E determines a continuous action of $\mathbb{G}_n$ on E_tX. This action is twisted in the sense that if $g \in \mathbb{G}_n$, $a \in E_0$ and $x \in E_tX$, then $g_*(ax) = g_*(a)g_*(x)$. We will call an $L^{\mathfrak{m}}$-complete E_0-module with a continuous and twisted $\mathbb{G}_n$ action a *Morava module*. Many (if not all) of our Morava modules will actually be $\mathfrak{m}$-complete; that is, the natural maps

$$M \longrightarrow L_0^{\mathfrak{m}} M \longrightarrow \lim M/\mathfrak{m}^k M$$

are all isomorphisms. For example, if M is $\mathfrak{m}$-complete, so is the induced module of continuous map $\mathrm{map}^c(\mathbb{G}_n, M)$. Hence the continuous cohomology of $\mathfrak{m}$-complete Morava modules can be constructed entirely in the category of $\mathfrak{m}$-complete Morava modules.

The graded Morava module module E_*X is determined by E_0X, E_1X, and the isomorphism of $\mathbb{G}_n$-modules, for $n \in \mathbb{Z}$,

$$E_{t+2n}X \cong E_2^{\otimes n} \otimes_{E_0} E_tX.$$

The $\mathbb{G}_n$-action is the diagonal action. If $n \geq 0$, $E_2^{\otimes n} = E_2 \otimes_{E_0} \cdots \otimes_{E_0} E_2$ is a free of rank 1 over E_0. If $n < 0$, then $E_2^{\otimes n}$ is the dual $\mathbb{G}_n$-module to $E_2^{\otimes -n}$. This discussion gives an evident category of graded Morava modules.

We say a graded Morava module M_* is finitely generated if it is finitely generated as a graded E_*-module or, equivalently, if M_0 and M_1 are finitely generated as E_0-modules. We also say a graded Morava module M_* is finite if M_0 and M_1 are finite. If X is a finite CW spectrum then E_*X is finitely generated. More generally, X is dualizable in the $K(n)$-local category if and only if E_*X is finitely generated. See Theorem 8.6 of [24]. If X is also of type n, then E_*X is finite.

Here is a key fact about Morava modules which we use often. The argument owes quite a good deal to the proof of Lemma 5 of [33].

Proposition 1.6 *Let $p-1 > n$ and let M be an $\mathfrak{m}$-complete Morava module. Then for all $s > n^2$*

$$H^s(\mathbb{G}_n, M) = 0.$$

Proof Let $S_n \subseteq \mathbb{S}_n$ be the subgroup of automorphisms $g = g(x)$ of the Honda formal group H_n so that $g'(0) = 1$. Then S_n is a compact p-adic analytic group of dimension n^2 by §3.1.2 of [17]. Under the assumption $p - 1 > n$ the group S_n is torsion-free; see Theorem 3.2.1 of [17]. By a theorem of Lazard (combine Theorems 4.4.1 and 5.1.9. of [34]) we may conclude S_n is a Poincaré duality group of dimension n^2 and of cohomological dimension n^2. Since the index of S_n in $\mathbb{S}_n$ is finite and prime to p, the cohomological dimension of $\mathbb{S}_n$ is also n^2. So far we have not used the hypothesis on M.

Let $\mathrm{Gal} = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. If M is an $\mathfrak{m}$-complete Morava module then $M^{\mathbb{S}_n}$ is a p-complete twisted $\mathbb{W}$-Gal-module; that is, if $g \in \mathrm{Gal}$ and $x \in M^{\mathbb{S}_n}$, then $g(ax) = g(a)g(x)$. Now we use a version of Galois descent – see Lemma 1.7 below – to conclude

$$H^*(\mathbb{G}_n, M) \cong H^*(\mathbb{S}_n, M)^{\mathrm{Gal}}$$

and we have the vanishing we need. □

Lemma 1.7 *Let* $\mathrm{Gal} = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$*. Let M be a p-complete twisted* $\mathbb{W}$-Gal-*module. Then the inclusion* $M^{\mathrm{Gal}} \to M$ *of the invariants extends to an isomorphism of twisted* $\mathbb{W}$-Gal-*modules*

$$\mathbb{W} \otimes_{\mathbb{Z}_p} M^{\mathrm{Gal}} \cong M.$$

The functor $M \mapsto M^{\mathrm{Gal}}$ *from p-complete twisted* $\mathbb{W}$-Gal-*modules to p-complete modules is exact.*

Proof This can be proved using standard descent theory, but here is a completely explicit argument.

We are using the completed tensor product

$$\mathbb{W} \otimes_{\mathbb{Z}_p} N = \lim(\mathbb{W} \otimes_{\mathbb{Z}_p} N)/p^k \cong \lim(\mathbb{W}/p^k \otimes_{\mathbb{Z}/p^k} N/p^k N).$$

First, since inverse limits commute with invariants, we have

$$M^{\mathrm{Gal}} \cong (\lim M/p^k M)^{\mathrm{Gal}} \cong \lim (M/p^k M)^{\mathrm{Gal}}.$$

Next, the map $M^{\mathrm{Gal}} \to (M/p^k M)^{\mathrm{Gal}}$ factors as

$$M^{\mathrm{Gal}} \longrightarrow M^{\mathrm{Gal}}/p^k M^{\mathrm{Gal}} \longrightarrow (M/p^k M)^{\mathrm{Gal}}$$

with the second map an inclusion. This yields isomorphisms

$$M^{\mathrm{Gal}} \cong \lim (M^{\mathrm{Gal}}/p^k M^{\mathrm{Gal}}) \cong \lim (M/p^k M)^{\mathrm{Gal}}.$$

Since $\mathbb{W}$ is a finitely generated free $\mathbb{Z}_p$ module

$$\begin{aligned} \mathbb{W} \otimes_{\mathbb{Z}_p} M^{\mathrm{Gal}} &\cong \lim(\mathbb{W}/p^k \otimes_{\mathbb{Z}_{p^k}} M^{\mathrm{Gal}}/p^k M^{\mathrm{Gal}}) \\ &\cong \lim(\mathbb{W}/p^k \otimes_{\mathbb{Z}_{p^k}} (M/p^k M)^{\mathrm{Gal}}). \end{aligned}$$

Finally, $\mathbb{Z}/p^k \to \mathbb{W}/p^k$ is Galois with Galois group Gal we have

$$\mathbb{W}/p^k \otimes_{\mathbb{Z}/p^k} (M/p^k M)^{\mathrm{Gal}} \cong M/p^k M.$$

The exactness statement follows from the fact that $\mathbb{W}$ is a free and finitely generated $\mathbb{Z}_p$-module, so $N \mapsto \mathbb{W} \otimes_{\mathbb{Z}_p} N$ is exact. □

1.2 Picard groups

The point of this section is to develop enough technology to pave the way for the key Proposition 1.30.

1.2.1 Some basics

Let $\mathrm{Pic}_{K(n)}$ denote the $K(n)$-local Picard group of weak equivalence classes of invertible elements. Here is an observation from [21]. If $X \in \mathrm{Pic}_{K(n)}$, then $K(n)_*X$ is an invertible $K(n)_*$-module and, since $K(n)_*$ is a graded field, it follows that $K(n)_*X$ is of rank 1 over $K(n)_*$. From this it follows from Proposition 8.4 of [24] that E_*X is also free of rank 1 over E_*.

Remark 1.8 Let $\mathrm{Pic}^0_{K(n)} \subseteq \mathrm{Pic}_{K(n)}$ be the subgroup of index 2 generated by the elements X with E_*X in even degrees. Then E_0X is free of rank 1 over E_0. If we choose a generator $a \in E_0X$ then we can define a crossed homomorphism $\phi : \mathbb{G}_n \to E_0^\times$ by the formula

$$ga = \phi(g)a, \qquad g \in \mathbb{G}_n.$$

This defines a homomorphism

$$e : \mathrm{Pic}^0_{K(n)} \longrightarrow H^1(\mathbb{G}_n, E_0^\times) \tag{1.2.1}$$

to the algebraic Picard group of invertible Morava modules. We write κ_n for the kernel of e; this is the subgroup of *exotic elements* in the Picard group.

Notation: Both $\mathrm{Pic}_{K(n)}$ and $H^1(\mathbb{G}_n, E_0^\times)$ are abelian groups where the group operation is written as multiplicatively; thus $e(X \wedge Y) = e(X)e(Y)$.

The following result explains the hypothesis on the prime in the equivalence of dualities (1.1). This appears in the literature in various guises; the exact criterion on the prime depends on the setting, but the proof is always the same as in Theorem 5.4 of [23].

Proposition 1.9 *Suppose $2p > \max\{n^2+1, 2n+2\}$. Then $\kappa_n = 0$ and the map e is an injection.*

Proof Suppose X and Y are two invertible spectra so $E_0(X) \cong E_0(Y)$ as Morava modules. Let $D_nY = F(Y, L_{K(n)}S^0)$ be the $K(n)$-local Spanier-Whitehead dual of Y. Then D_nY is the inverse of Y in $\mathrm{Pic}^0_{K(n)}$; hence, E_0D_nY is the inverse of E_0Y as an invertible Morava module. It follows that $E_*(X \wedge D_nY) \cong E_*S^0$ as Morava modules and we need only show that the class

$$\iota \in H^0(\mathbb{G}_n, E_0(X \wedge D_nY))$$

determined by this isomorphism is a permanent cycle. The differentials will lie in subquotients of

$$H^{s+1}(\mathbb{G}_n, E_s), \quad s \geq 1.$$

Under the hypotheses here, we can now apply the sparseness result of part (1) of Remark 1.4 and the horizontal vanishing line of Proposition 1.6. The second of these requires $p-1 > n$ and the first requires $2(p-1)+1 > n^2$. Combined they imply that all differentials on ι land in zero groups. □

Remark 1.10 A more sophisticated variation of the argument used to prove Proposition 1.9 will also show that e is surjective under the same hypotheses on p and n. See [29] for details. Here is an outline.

Let M be an invertible graded Morava module and let

$$M^\vee = \mathrm{Hom}_{E_0}(M, E_0)$$

with conjugation $\mathbb{G}_n$-action; see Remark 1.4.1 on why this action arises.The essential idea is to use a Toda-style obstruction theory with successively defined obstructions

$$\theta_s \in H^{s+2}(\mathbb{G}_n, E_s \otimes_{E_0} M \otimes_{E_0} M^\vee) \cong H^{s+2}(\mathbb{G}_n, E_s), \quad s \geq 1,$$

to finding such an X with $E_0X \cong M$. Such an obstruction theory can be constructed using Toda's techniques [35] or a linearized version of the vastly more complex obstruction theory of [14]. These obstruction groups will vanish if $2p > \max\{n^2+1, 2n+2\}$.

The basic example of an element in $\mathrm{Pic}^0_{K(n)}$ is the localized 2-sphere $L_{K(n)}S^2$. Since $E_0S^2 \cong E_{-2}$ we can choose $u \in E_{-2}$ as the generator and the associated crossed homomorphism is $t_0 : \mathbb{G}_n \to E_0^\times$. See (1.1.4).

We next explore the underlying algebraic structure of the Picard group $\mathrm{Pic}^0_{K(n)}$; in particular, it is a profinite abelian group and continuous module over the rather unusual completion of the integers

$$\mathfrak{Z}_n \overset{\mathrm{def}}{=} \lim_k \ \mathbb{Z}/p^k(p^n-1)\ . \qquad (1.2.2)$$

The canonical isomorphisms $\mathbb{Z}/p^k(p^n-1) \cong \mathbb{Z}/p^k \times \mathbb{Z}/(p^n-1)$ assemble to give a continuous isomorphism of rings

$$\mathfrak{Z}_{\mathfrak{n}} \cong \mathbb{Z}_p \times \mathbb{Z}/(p^n-1).$$

See Remark 1.23 for more thoughts on the ring $\mathfrak{Z}_{\mathfrak{n}}$.

The number p^n-1 appears in a number of ways in $K(n)$-local homotopy theory; for example, the element $v_n = u^{-(p^n-1)} \in E_{2(p^n-1)}$. We explore that observation more in Remark 1.24 below. In this context, however, the ring $\mathfrak{Z}_{\mathfrak{n}}$ arises for a much more basic reason.

Lemma 1.11 *Let $(S, \mathfrak{m}_S)$ be a complete local ring with residue field $\mathbb{F}_{p^n} \cong S/\mathfrak{m}_s$. The abelian group structure on the group of units $S^\times$ extends to a continuous $\mathfrak{Z}_{\mathfrak{n}}$-module structure in the topology given by the isomorphism $S^\times \cong \lim(S/\mathfrak{m}^k)^\times$.*

Proof For any $a \in \mathfrak{Z}_{\mathfrak{n}}$ and any $x \in S^\times$ we must define an element $x^a \in S^\times$. Furthermore if $a = n \in \mathbb{Z}$, then we need $x^a = x^n$.

Since $(S/\mathfrak{m})^\times \cong \mathbb{F}_{p^n}^\times$, any $x \in S^\times$ has the property that $x^{p^n-1} \equiv 1$ modulo $\mathfrak{m}_S$ and, hence, that

$$x^{p^k(p^n-1)} \equiv 1 \mod \mathfrak{m}^{k+1}.$$

Let $a \in \lim_k \ \mathbb{Z}/p^k(p^n-1) \in \mathfrak{Z}_{\mathfrak{n}}$. For each integer $k \geq 0$ choose an integer a_k so that $a_k \equiv a \in \mathbb{Z}/p^k(p^n-1)$. Then the elements

$$x^{a_k} \in (S/\mathfrak{m}^{k+1})^\times$$

define an element $x^a \in S^\times \cong \lim(S/\mathfrak{m}^{k+1})^\times$ as needed. □

The basic application of Lemma 1.11 is to the ring $S = E_0$. This further implies that the continuous cohomology group $H^1(\mathbb{G}_n, E_0^\times)$ is a continuous module over $\mathfrak{Z}_{\mathfrak{n}}$.

It turns out we can show $\mathrm{Pic}^0_{K(n)}$ is also a profinite module over $\mathfrak{Z}_{\mathfrak{n}}$ and the evaluation map

$$e : \mathrm{Pic}^0_{K(n)} \longrightarrow H^1(\mathbb{G}_n, E_0^\times)$$

is a continuous map of $\mathfrak{Z}_{\mathfrak{n}}$ modules. This can be deduced from Proposition 14.3.d of [24], which in turn depends heavily on [21]. The argument given here is essentially the same, but packaged to emphasize the role of the group κ_n of exotic elements in the Picard group and the cohomology group $H^1(\mathbb{G}_n, E_0^\times)$. We'll give a proof in Proposition 1.18 below.

Remark 1.12 Using nilpotence technology derived from [22] and working as in [24] §4 we can choose a sequence of ideals $J(i) \subseteq \mathfrak{m} \subseteq E_0$ and spectra $S/J(i)$ with the following properties:

1. $J(i+1) \subseteq J(i)$ and $\cap\, J(i) = 0$;
2. $E_0/J(i)$ is finite;
3. $E_0(S/J(i)) \cong E_0/J(i)$ and there are maps $q : S/J(i+1) \to S/J(i)$ realizing the quotient map $E_0/J(i+1) \to E_0/J(i)$;
4. there are maps $\eta = \eta_i : S^0 \to S/J(i)$ inducing the quotient map $E_0 \to E_0/J(i)$ and $q\eta_{i+1} = \eta_i : S/J(i) \to S/J(i)$;
5. if X a finite type n-spectrum, then the map $X \to \operatorname{holim} X \wedge S/J(i)$ induced by the maps η is an equivalence; and,
6. the $S/J(i)$ are μ-spectra; that is, there are maps

$$\mu : S/J(i) \wedge S/J(i) \to S/J(i)$$

so that $\mu(\eta \wedge 1) = 1 : S/J(i) \to S/J(i)$.

They also prove that items (1)-(5) characterize the tower $\{S/J(i)\}$ up to equivalence in the pro-category of towers under S^0. See Proposition 4.22 of [24].

Hovey and Strickland choose the $J(i)$ with the property that there are positive integers $a_0, a_1, \ldots, a_{n-1}$ (depending on i) so that

$$J(i) = (p^{a_0}, u_1^{a_1}, \ldots, u_{n-1}^{a_{n-1}}).$$

They don't quite say it explicitly, but in their construction they choose the a_i, $i \geq 1$, to be powers of p.

Remark 1.13 Let $G(i) \subseteq \mathrm{Pic}^0_{K(n)}$ be the set of equivalence classes X which can be given a $K(n)$-local equivalence

$$X \wedge S/J(i) \simeq S/J(i).$$

Item (6) of Remark 1.12 is used to show $G(i+1) \subseteq G(i)$. By Proposition 14.2 of [24] $G(i)$ is a finite index subgroup and $\cap\, G(i) = \{L_{K(n)}S^0\}$; thus the subgroups $G(i)$ define a separated profinite topology on $\mathrm{Pic}^0_{K(n)}$.

Lemma 1.14 *The evaluation map*

$$e : \mathrm{Pic}^0_{K(n)} \longrightarrow H^1(\mathbb{G}_n, E_0^\times)$$

is a continuous homomorphism of profinite abelian groups.