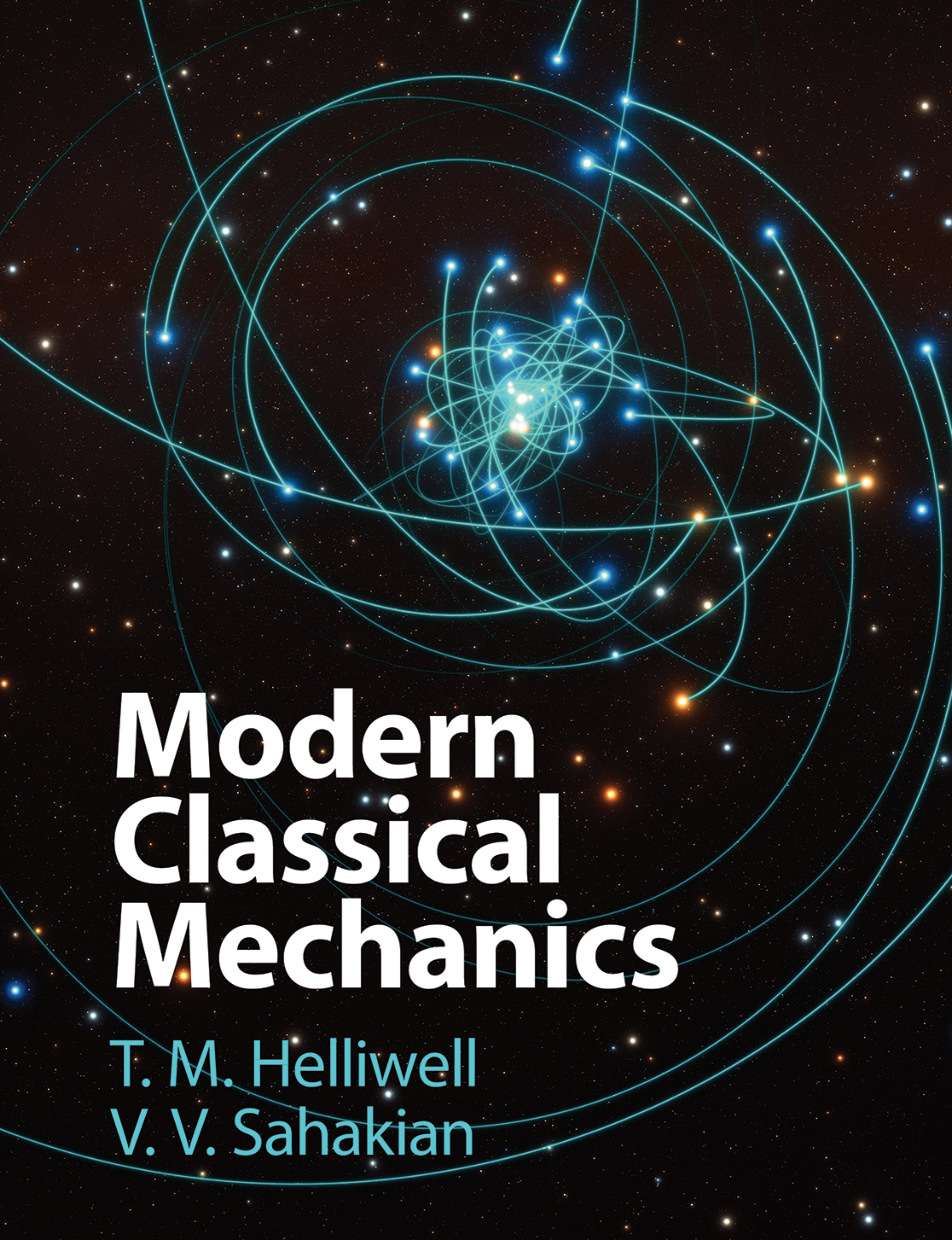

Modern Classical Mechanics

T. M. Helliwell
V. V. Sahakian

Modern Classical Mechanics

In this modern and distinctive textbook, Helliwell and Sahakian present classical mechanics as a thriving and contemporary field with strong connections to cutting-edge research topics in physics. Each part of the book concludes with a capstone chapter describing various key topics in quantum mechanics, general relativity, and other areas of modern physics, clearly demonstrating how they relate to advanced classical mechanics, and enabling students to appreciate the central importance of classical mechanics within contemporary fields of research. Numerous and detailed examples are interleaved with theoretical content, illustrating abstract concepts more concretely. Extensive problem sets at the end of each chapter further reinforce students' understanding of key concepts, and provide opportunities for assessment or self-testing. A detailed online solutions manual and lecture slides accompany the text for instructors. Often a flexible approach is required when teaching advanced classical mechanics, and, to facilitate this, the authors have outlined several paths instructors and students can follow through the book, depending on background knowledge and the length of their course.

T. M. Helliwell taught at Harvey Mudd College for more than 40 years, as well as serving in several administrative roles, including Physics Department Chair. He received two National Science Foundation fellowships, was awarded the Henry T. Mudd Prize for service to the college in 1997, and served as a consultant at the Jet Propulsion Laboratory. He is the author or co-author of more than 40 research papers on general relativity, cosmology, and quantum theory, and the author of two textbooks on special relativity.

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Modern Classical Mechanics

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Dedicated to Bonnie, Gaia, and Yuliya

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Part III

Preface

The branch of physics known as “classical mechanics” originated in the seventeenth century, but wasn’t called that until the discovery of quantum mechanics in the 1920s. It was quantum mechanics that most profoundly changed our understanding of how and why particles move as they do, and even what a particle *is*. Quantum mechanics was so completely different that the word “classical” had to be added to the older theory to make it clear which mechanics was meant. At the same time, quantum mechanics was heavily inspired and influenced by the formulations of classical mechanics by Lagrange and Hamilton dating back to the eighteenth and nineteenth centuries.

Einstein’s theories of special relativity (1905) and general relativity (1915) also had important impacts on classical mechanics, changing the laws of motion primarily by revolutionizing our understanding of the spacetime arena in which physics takes place. These theories have been viewed as either introducing a new “relativistic mechanics” or more modestly as *completing* classical mechanics, making it useful even for particles moving close to the speed of light and for particles moving in strong gravitational fields.

Quantum mechanics, special relativity, and general relativity stand together as the three pillars of modern physics. Classical mechanics integrates with all three as a robust approximation framework that is both useful in practice for problem solving – but also as an inspirational venue for developing basic intuition about physics.

In the title of the book we have endowed our exposition of classical mechanics with the word “modern,” because it is a modern approach in several ways. First, we focus on the Lagrangian and Hamiltonian formulations of mechanics almost from the outset, modern of course only relative to Newton’s formulation. Throughout we emphasize the connections of these newer approaches to the development of quantum mechanics – through contact with Feynman’s path-integral formulation of quantum mechanics and the relations of Hamilton–Jacobi theory to Schrödinger’s approach to wave mechanics. We also develop the subject of mechanics with relativity in mind early on, integrating modern differential geometry notation in the narrative and motivating the variational principle through arguments that come naturally from special relativity. In particular, immediately after a compact review of Newtonian particle mechanics in Chapter 1, special relativity is introduced already in Chapter 2. Finally, the exposition is modern in that we use a tone and physics mindset that is contemporary, often with an emphasis on the role of symmetry as a guiding principle, and we draw on many examples from modern subjects and applications such as black holes, cosmology, atomic physics, particle physics, magnetic trapping, orbital mechanics, and spaceflight.

Modern classical mechanics also stands strong on its own as a useful approximation framework that addresses physics problems in regimes where quantum mechanics and/or relativity come in as sub-leading effects. In many situations, using quantum mechanics and/or relativity to study a physical system would be tantamount to shooting a fly with a catapult. Roughly speaking, classical mechanics works very well (i.e., agrees with experiments) for macroscopic objects that are moving at speeds much less than the speed of light, and where gravity is not too strong – and also where our experimental measurements are not too precise.

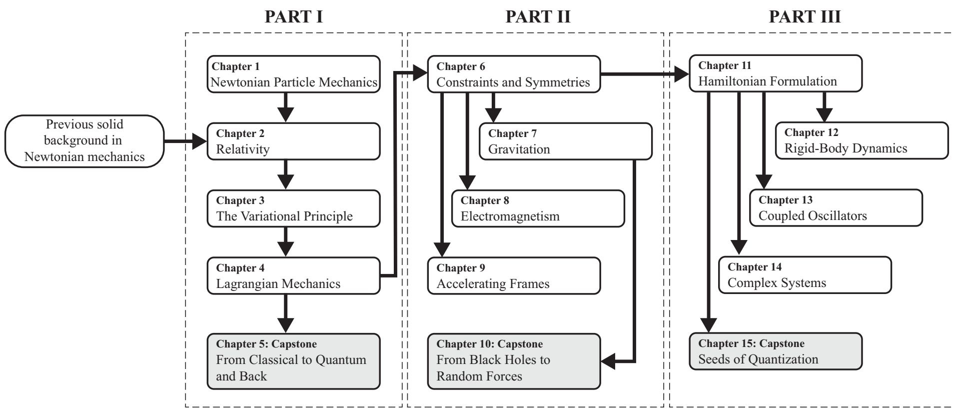
Take the motions of planets around the sun and moons around their planets, for example. Motions within the solar system were the most important testing ground for classical mechanics in the first place, and for nearly all purposes classical mechanics in this domain works as well now as it ever did. We still use it to plot the motion of spacecraft on their way to distant planets, for example – it would be completely unnecessary to tackle a problem like that using the full apparatus of quantum mechanics. The same can be said for the use of special and general relativity, except for tiny but nevertheless important effects like the precession of the planet Mercury's perihelion or the rate of atomic clocks in Global Positioning System (GPS) satellites around the earth.

Our book is first and foremost a textbook on classical mechanics and its many uses, while also showing where its limitations lie – limitations as defined by quantum mechanics as well as the relativity theories, and emphasizing the inspirational role the subject played in the development of modern physics. To accomplish these goals, the book is divided into three main parts. There are five chapters in each part, where the fifth chapter is a "capstone chapter," a special unit that elaborates further on the boundaries of classical mechanics as presented in the preceding four chapters and its connections to the three pillars of modern physics.

In broad strokes, the first part of the book is about the Lagrangian formulation of mechanics; the second part is about the various forces and symmetries that present themselves on the mechanics stage; and the third part is about the Hamiltonian formulation. The capstone chapter of the first part is a pedagogical exposition of Feynman's path-integral formulation of quantum mechanics and its connections to modern classical mechanics; the capstone chapter of the second part discusses general relativity and its relations to relativistic mechanics; and the capstone chapter of the third part is about Hamilton–Jacobi theory, phase space, and the connections to the wavefunction formulation of quantum mechanics.

This layout allows for different pathways through the book, depending on the time available for a given class and the background preparation of the students. The following diagram illustrates the conceptual connections and dependencies between the various sections.

Based on this, we can identify several possible pathways that can be adopted in a typical 15-week-long class.



- **Basic mechanics:** For students who have had a basic calculus-based mechanics course and are looking for a basic second course. Chapters 1, 2, 3, 4, 6.1 and 6.2, 7.1 to 7.4, 9.1 to 9.4.
- **Lagrangian approach plus a bit more:** For students who have had a robust calculus-based mechanics course and are looking for a more sophisticated second course. Chapters 2, 3, 4, 6, 7, 8, and 9.
- **Traditional Lagrangian and Hamiltonian mechanics:** For students who have had a robust calculus-based mechanics course and are looking for a rather traditional course on Lagrangian and Hamiltonian mechanics. Chapters 2, 3, 4, 6, 7, 9, 11, 12 or 13.
- **Advanced mechanics:** For students who have had a robust calculus-based mechanics course and are looking for an advanced modern exposure to mechanics. Chapters 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, one of 12, 13, or 14, optional reading 15.

In a 20-week timeframe, one can cover most if not all the chapters. This can also be done naturally in a 14-week graduate-level course. Chapters 5, 10, and 12 to 15 can also serve as excellent directed reading material for students who complete their second mechanics course but want to learn more advanced topics.

A Note about Notation

Throughout the book, we have attempted to accord, as much as possible, with notational conventions that are commonly used in similar textbooks. However, there is one place we have decided to adopt a notation that is instead more consistent with more advanced graduate-level textbooks: components of vectors are labeled by superscripts instead of subscripts. For example, the components of a velocity vector $\mathbf{v}$ in spherical coordinates are written as $\mathbf{v} = (\mathrm{v}^r, \mathrm{v}^\phi, \mathrm{v}^\theta)$; similarly, the components of a four-velocity vector $\boldsymbol{u}$ in Cartesian coordinates take the form $\boldsymbol{u} = (u^t, u^x, u^y, u^z)$. This notation is conventional in differential geometry and graduate-level textbooks so as to distinguish vectors from co-vectors – such as the momentum co-vector and the gauge potential co-vector in electromagnetism.

Given that our modern approach to the subject of mechanics incorporates the language of special relativity from the outset, it is indeed natural to adopt the "correct" differential geometry notation from the start. This also helps the reader later on in transitioning to graduate-level coursework and research-level literature. One pitfall of this notation is that it does require a bit of an initial learning curve as the superscript might be confused with raising a variable to a power. We have addressed this issue by choosing a different font for superscripts that represent components – and generally making sure that we point out potential confusion whenever the context does not make the interpretation obvious. Because of this, we recommend that all users of this book are at least encouraged to *read* Chapter 2, which covers special relativity, even if they are already familiar with the subject. This chapter establishes the notation clearly and gets readers used to it quickly. We have tested this in the classroom over many years and found that the adoption of the notation can be rather smooth and seamless. One bonus advantage of the new notation is that subscripts can be reserved to label particles or degrees of freedom, needs that are very common in the subject of classical mechanics; and indeed, we do so throughout the book. When all is said and done, we believe it is worthwhile to introduce readers to the newer notation, and that it pays off quickly.

Each chapter ends with a list of problems arranged in the order that the topics they cover appear in the chapter. And each problem is labeled by one, two, or three stars, indicating the level of difficulty – one star being easiest and three being hardest.

Acknowledgments and Credits

Over the many years we have taught classical mechanics at Harvey Mudd College, we have benefited greatly from the questions and enthusiasm of the many students in our classes. Teaching is a learning process for the teacher and we have tried to relay this experience in the pages of this textbook. We have also benefited from suggestions and help from many of our colleagues, especially Peter Saeta, Brian Shuve, and John Townsend; also from Sami Gara and several anonymous reviewers. Their support of our work has meant a great deal.

Image of earth used throughout the text: “The Blue Marble” photograph of the earth, taken by the Apollo 17 mission on December 7, 1972 at a distance of about 29,000 km. Taken by either Harrison Schmitt or Ron Evans.

Some artwork in figures is from Wikipedia (wikipedia.org) as material in the public domain.

Notation and Conventions

$\mathbf{v}$	three-vector
$\boldsymbol{v}$	four-vector
$\hat{\mathbf{r}}$	unit vector
v^a	three-vector component
v^μ	four-vector component
$\hat{\mathcal{R}}$	matrix
r, θ	polar coordinates
ρ, φ, z	cylindrical coordinates
r, ϕ, θ	spherical coordinates: radial, azimuthal, latitude
T	kinetic energy
U	potential energy
$-+++$	spacetime signature

Useful Relations

$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$	Cartesian
$T = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2\right)$	cylindrical
$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\sin^2\theta\dot{\phi}^2 + r^2\dot{\theta}^2\right)$	spherical
$T = \frac{1}{2}m\mathbf{v}_{\mathrm{rot}}^2 + m\mathbf{v}_{\mathrm{rot}}\cdot(\boldsymbol{\omega}\times\mathbf{r}) + \frac{1}{2}m\omega^2 r^2 - \frac{1}{2}m\left(\boldsymbol{\omega}\cdot\mathbf{r}\right)^2$	non-inertial
$\mathbf{F}_{\mathrm{rot}} = \mathbf{F}_{\mathrm{in}} \underbrace{-m\,\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})_{\mathrm{rot}}}_{\text{centrifugal}} \underbrace{-2\,m\,(\boldsymbol{\omega}\times\mathbf{v}_{\mathrm{rot}})_{\mathrm{rot}}}_{\text{Coriolis}} \underbrace{-m\,(\dot{\boldsymbol{\omega}}\times\mathbf{r})_{\mathrm{rot}}}_{\text{Euler}}$	fictitious forces
$S = -m\,c^2\int dt\sqrt{1-\frac{v^2}{c^2}} + Q\int dt\,\left(-\phi + \mathbf{A}\cdot\frac{\mathbf{v}}{c}\right)$	charged particle
$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \lambda_l a_{lk}, \quad a_{lk}\dot{q}_k + a_{lt} = 0$	equations of motion
$\delta S = \int dt\,\left(\frac{\partial L}{\partial q_k}\Delta q_k + \frac{\partial L}{\partial\dot{q}_k}\frac{d}{dt}\left(\Delta q_k\right) + \frac{d}{dt}(L\,\delta t)\right)$	transformation
$Q \equiv \frac{\partial L}{\partial\dot{q}_k}\Delta q_k + L\,\delta t$	Noether charge
$H = \frac{\partial L}{\partial\dot{q}_k}\dot{q}_k - L$	Hamiltonian
$\dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial q_k}, \quad \frac{\partial L}{\partial t} = -\frac{dH}{dt}$	Hamiltonian equations
$r = \frac{\ell^2/G\,M\,m^2}{1+\epsilon\cos\phi}$	gravitational orbits
$a = -\frac{G\,M\,m}{2\,E}, \quad \epsilon = \sqrt{1+\frac{2\,E\,\ell^2}{G^2M^2m^3}}$	orbit relations

Vector identities

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\,\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\,\mathbf{C}$$

$$\nabla\cdot(f\,\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$
$$\nabla\times(f\,\mathbf{A}) = f(\nabla\times\mathbf{A}) + (\nabla f)\times\mathbf{A}$$
$$\nabla\,(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\,\mathbf{B} + (\mathbf{B}\cdot\nabla)\,\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})$$
$$\nabla\cdot(\mathbf{A}\cdot\mathbf{B}) = (\nabla\times\mathbf{A})\cdot\mathbf{B} - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$
$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\,(\nabla\cdot\mathbf{B}) - \mathbf{B}\,(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\,\mathbf{A} - (\mathbf{A}\cdot\nabla)\,\mathbf{B}$$
$$\nabla\times(\nabla\times\mathbf{A}) = \nabla\,(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

PART I

1 Newtonian Particle Mechanics

We begin our journey of discovery by reviewing the well-known laws of Newtonian mechanics. We set the stage by introducing inertial frames of reference and the Galilean transformation that translates between them, and then present Newton's celebrated three laws of motion for both single particles and systems of particles. We review the three conservation laws of momentum, angular momentum, and energy, and illustrate how they can be used to provide insight and greatly simplify problem solving. We end by discussing the fundamental forces of nature and which of them are encountered in classical mechanics. All this is a preview to a relativistic treatment of mechanics in the following chapter.

1.1 Inertial Frames and the Galilean Transformation

Classical mechanics begins by analyzing the motion of *particles*. Classical particles are idealizations: they are point-like, with no internal degrees of freedom like vibrations or rotations. But by understanding the motion of these ideal "particles" we can also understand a lot about the motion of *real* objects, because we can often ignore what is going on inside of them. The concept of "classical particle" can in the right circumstances be used for objects all the way from electrons to baseballs to stars to entire galaxies.

In describing the motion of a particle, we first have to choose a frame of reference in which an observer can make measurements. Many reference frames could be used, but there is a special set of frames, the *non-accelerating*, **inertial frames**, in which the physics is particularly simple. Picture a set of three orthogonal meter sticks defining a set of Cartesian coordinates drifting through space with no forces applied. An inertial observer drifts with the coordinate system and uses it to make measurements of physical phenomena. This inertial frame and inertial observer are not unique, however: having established one inertial frame, any other frame moving at constant velocity relative to it is also inertial, as illustrated in Figure 1.1.

Two of these inertial observers, along with their personal coordinate systems, are depicted in Figure 1.2: observer $\mathcal{O}$ describes positions of objects through a Cartesian system labeled (x, y, z), while observer $\mathcal{O}'$ uses a system labeled (x', y', z').

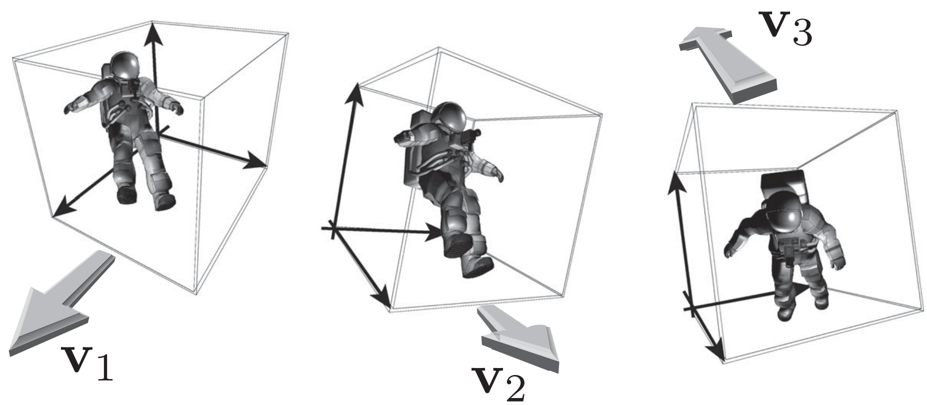


Fig. 1.1 Various inertial frames in space. If one of these frames is inertial, any other frame moving at constant velocity relative to it is also inertial.

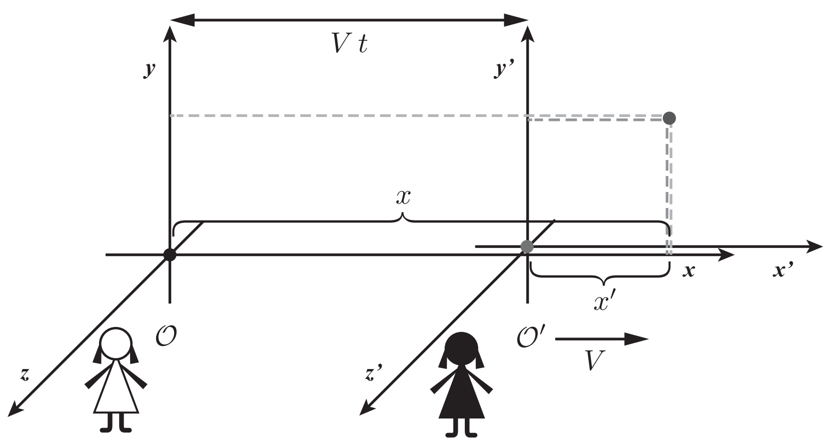


Fig. 1.2 Two inertial frames, $\mathcal{O}$ and $\mathcal{O}'$, moving relative to one another along their mutual x or x' axes.

An **event** of interest to an observer is characterized by the position in space at which the measurement is made – but also by the instant in time at which the observation occurs, according to clocks at rest in the observer's inertial frame. For example, an event could be a snapshot in time of the position of a particle along its trajectory. Hence, the event is assigned four numbers by observer $\mathcal{O}$: x, y, z, and t for time, while observer $\mathcal{O}'$ labels the same event x', y', z', and t'.

Without loss of generality, observer $\mathcal{O}$ can choose her x axis along the direction of motion of $\mathcal{O}'$, and then the x' axis of $\mathcal{O}'$ can be aligned with that axis as well, as shown in Figure 1.2. It seems intuitively obvious that the coordinates of the event are related by

$$x = x' + Vt', \quad y = y', \quad z = z', \quad t = t', \tag{1.1}$$

where we assume that the origins of the two frames coincide at time $t' = t = 0$. This is known as a **Galilean transformation**. Note that the only difference in the coordinates is in the x direction, corresponding to the distance between the two origins as each system moves relative to the other. This transformation – in spite of being highly intuitive – will turn out to be incorrect, as we shall see in the next chapter. But for now, we take it as good enough for our Newtonian purposes.

If the coordinates represent the instantaneous position of a particle, we can write

$$x(t) = x'(t') + Vt', \quad y(t) = y'(t'), \quad z(t) = z'(t'), \quad t = t'. \tag{1.2}$$

We then differentiate this transformation with respect to $t = t'$ to obtain the transformation laws of velocity and acceleration. Differentiating once gives

$$v_x = v'_x + V, \quad v_y = v'_y, \quad v_z = v'_z, \tag{1.3}$$

where, for example, $v_x \equiv dx/dt$ and $v'_x \equiv dx'/dt'$, and differentiating a second time gives

$$a_x = a'_x, \quad a_y = a'_y, \quad a_z = a'_z. \tag{1.4}$$

That is, the velocity components of a particle differ by the relative frame velocity in each direction, while the acceleration components are the same in every inertial frame. Therefore one says that the acceleration of a particle is **Galilean invariant**.

Henceforth, we assert that all statements of physics that we write are expressed from the perspective of inertial observers – unless explicitly stated otherwise. For this purpose, *any* inertial observer has a valid perspective and is no more privileged than any other. This implies that all fundamental laws of physics we will write should be unchanged between the perspectives of different inertial observers. This equivalence of physics amongst inertial frames is called the **principle of relativity**.

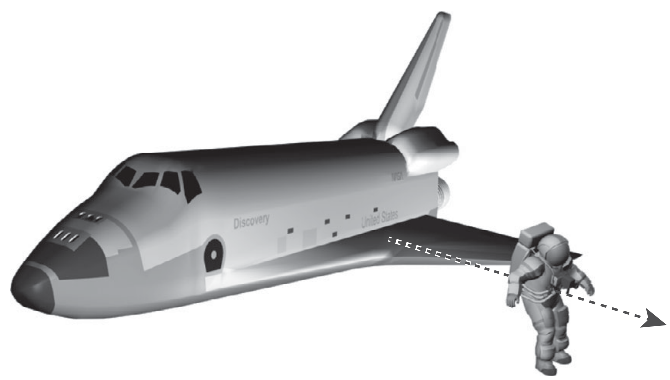
1.2 Newton's Laws of Motion

In his *Principia* of 1687, Newton presented his famous three laws. The first of these is the **law of inertia**:

> **I.** If there are no forces on an object, then if the object starts at rest it will stay at rest, or if it is initially set in motion, it will continue moving in the same direction in a straight line at constant speed. ■

Since this is a statement of physics – made by definition from the perspective of any inertial observer – it should be compatible with the principle of relativity: all inertial observers can write this same statement. On the contrary, using the Galilean velocity transformation, we see that if a particle has constant velocity in one inertial frame then it has constant velocity in all inertial frames. Hence, to assure that this statement can be written by any inertial observer and is hence compatible with the principle of relativity, we use the Galilean transformations to connect the perspectives of inertial reference frames.[1] In practice, we can henceforth use this first law of Newton to *test* whether or not our frame is inertial: if we remove all interactions from a particle under observation, and if we then notice that when set at rest the particle stays put and if tossed in any direction it keeps moving in that direction with constant speed, we can conclude that the law of inertia is obeyed and our frame is inertial.

An astronaut set adrift from her spacecraft in outer space, far from earth, or the sun, or any other gravitating object, will move off in a straight line at constant speed when viewed from an inertial frame. So if her spaceship is drifting without power and is not rotating, the spaceship frame is inertial and onboard observers will see her move away in a straight line. But if her spaceship is rotating, for example, observers on the ship will see her move off in a curved path – the frame inside a rotating spaceship is not *inertial.*



Now consider an inertial observer who observes a particle to which a force $\mathbf{F}$ is applied. Then **Newton's second law** states that

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \tag{1.5}$$

[1] Alternatively, we can think of inertial frames as some yet-undefined set of reference frames for the principle of relativity, then use this first law of Newton to *define* what inertial reference frames must be, along with the associated Galilean transformations that connect them. A curious fact is that having identified an inertial frame as one in which Newton's first law is valid, which can be accomplished by purely local observations of the motion of test particles, one finds that inertial frames are also those which are neither accelerating nor rotating relative to the distant stars! It is hard to believe this is mere coincidence, but the reasons for it are not universally agreed upon.

where the momentum of the particle is $\mathbf{p} = m\mathbf{v}$, the product of its mass and velocity. That is:

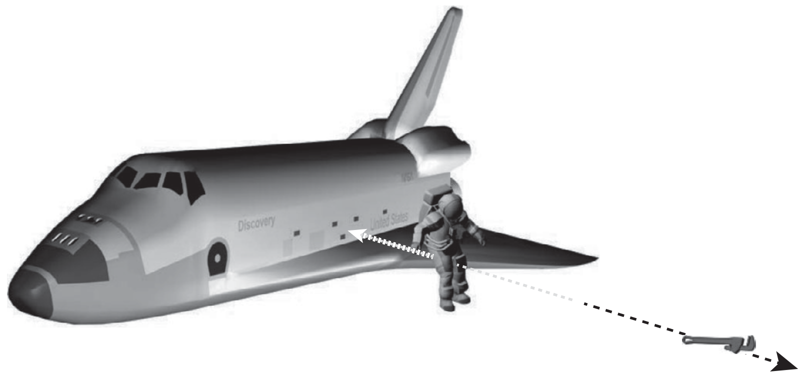
> **II.** The time rate of change of a particle's momentum is equal to the net force on that particle. ■

Newton's second law tells us that if the momentum of a particle changes, there must be a net force causing that change. Note that the second law gives us the means to identify and quantify the effect of forces and interactions. By conducting a series of measurements of the rate of change of momenta of a selection of particles, we explore the forces acting on them in their environment. Once we understand the nature of these forces, we can use this knowledge to predict the motion of other particles in a wider range of circumstances – this time by deducing the effect of such forces on rate of change of momentum.

Note also that $d\mathbf{p}/dt = md\mathbf{v}/dt = m\mathbf{a}$, so Newton's second law can also be written in the form $\mathbf{F} = m\mathbf{a}$, where $\mathbf{a}$ is the acceleration of the particle. The particle is taken to have a fixed mass, independent of its position or velocity. The law therefore implies that if we remove all forces from an object, neither its momentum nor its velocity will change: it will remain at rest if started at rest, and move in a straight line at constant speed if given an initial velocity. But that is just Newton's *first* law, so it might seem that the first law is just a special case of the second law! However, the second law is not true in all frames of reference. An accelerating observer will see the momentum of an object changing, even if there is no net force on it. In fact, it is only inertial observers who can use Newton's second law, so the first law is not so much a special case of the second as a means of specifying those observers for whom the second law is valid. Put differently, Newton uses the first law to implicitly *define* the concept of inertial reference frames.

Newton's second law is the most famous fundamental law of classical mechanics, and it must also be Galilean invariant according to our principle of relativity. We have already shown that the acceleration of a particle is invariant and we also take the mass of a particle to be the same in all inertial frames. So if $\mathbf{F} = m\mathbf{a}$ is to be a fundamental law, which can be used by observers at rest in any inertial frame, we must insist that the force on a particle is likewise Galilean invariant. Newton's second law itself does not specify which forces exist, but in classical mechanics any force on a particle (due to a spring, gravity, friction, or whatever) must be the same in all inertial frames.

If the drifting astronaut is carrying a wrench, by throwing it away (say) in the forward direction she exerts a force on it. During the throw the momentum of the wrench changes, and after it is released it travels in some straight line at constant speed.



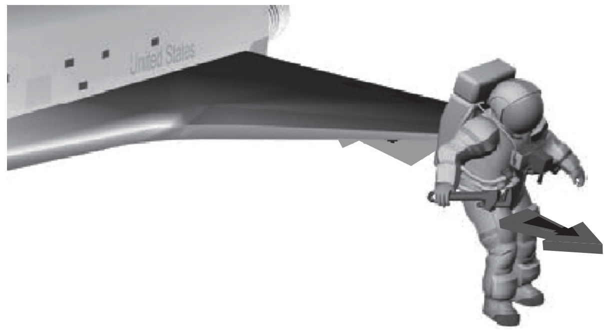
Of course, it is one thing to *know* Newton's second law; it is quite another thing to solve it to find a particle's motion in a particular case, which may range from easy to quite challenging. At the easy end of the spectrum is the case of an object of mass m moving under the influence of a constant force, such as the gravitational force $\mathbf{F} = m\mathbf{g}$ on a particle in a uniform gravitational field $\mathbf{g}$. If that is the only force, the particle's acceleration $\mathbf{a}$ will be constant, so its velocity $\mathbf{v}(t)$ can be found by integrating $\mathbf{a}$ over time, and then its position $\mathbf{r}(t)$ can be found by integrating $\mathbf{v}(t)$ over time. All this leads to the familiar equations of projectile motion.

Finally, Newton's *third* law states that

> **III.** "Action equals reaction." If one particle exerts a force on a second particle, the second particle exerts an equal but opposite force back on the first particle. ■

We have already stated that any force acting on a particle in classical mechanics must be the same in all inertial frames, so it follows that Newton's third law is also Galilean invariant: a pair of equal and opposite forces in a given inertial frame transform to the same equal and opposite pair in another inertial frame.

While the astronaut, drifting away from her spaceship, is exerting a force on the wrench, at each instant the wrench is exerting an equal but opposite force back on the astronaut. This causes the astronaut's momentum to change as well, and if the change is large enough her momentum will be reversed, allowing her to drift back to her spacecraft in a straight line at constant speed when viewed in an inertial frame.



1.3 One-Dimensional Motion: Drag Forces

Before discussing the full rich possibilities of the three-dimensional motion of a particle, we will begin with the simpler case of one-dimensional motion. In fact, if the total force acting on a particle pulls or pushes it in one linear direction, say in the x direction, and if the particle begins at rest or with some initial velocity that happens also to be in this same x direction, then the particle will continue to move in the x direction.

In general, the net force on a particle moving in one dimension might depend upon the particle's position, or its velocity, or time, or any combination of these variables. In this section we will suppose that the net force on a particle depends only upon its velocity, and not its position in space or the time. Then Newton's second law takes the form

$$\mathbf{F}(\mathbf{v}) = m\mathbf{a} \equiv m\frac{d\mathbf{v}}{dt} \tag{1.6}$$

which is a *first-order* differential equation. This often makes the problem much simpler than for position-dependent forces, which lead to second-order differential equations.

Drag forces are prime examples of one-dimensional velocity-dependent forces. They include air resistance on dropped baseballs, raindrops, and skydivers; they also include the horizontal motion of automobiles or airplanes and water drag on fish or submarines. By definition, drag forces act in opposition to an object's velocity through the fluid. For small objects moving sufficiently slowly, fluid flows around an object smoothly in what is called **laminar flow**, giving rise to "viscous drag," where the drag force is proportional to the **viscosity** of the fluid, a measure of how much of the fluid is pulled along with the object as it moves. An example would be dropping a small ball into a vat of honey or molasses, both highly viscous fluids. The viscous drag force is *linear* in the velocity, so has the form $F_{\text{drag}} = -bv$, where b is a constant.

Example 1.1

A Bacterium with a Viscous Drag Force

The most important force on a non-swimming bacterium in a fluid is the viscous drag force $F = -bv$, where v is the velocity of the bacterium relative to the fluid and b is a constant that depends on the size and shape of the bacterium and the viscosity of the fluid – the minus sign means that the drag force is opposite to the direction of motion. If the bacterium, as illustrated in Figure 1.3, gains a velocity v_0 and then stops swimming, what is its subsequent velocity as a function of time?

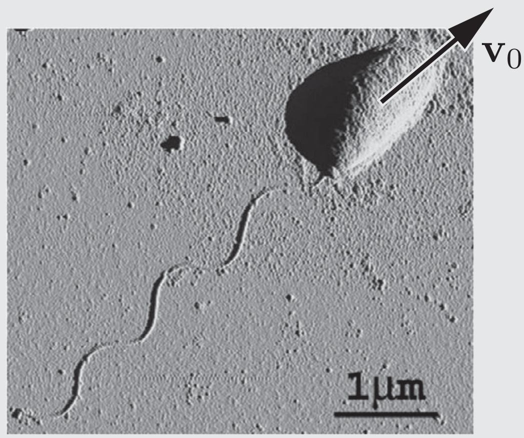


Fig. 1.3 A bacterium in a fluid. What is its motion if it begins with velocity v_0 and then stops swimming? Reprinted figure with permission from Guanglai Li, Lick-Kong Tam, and Jay X. Tang, Amplified effect of Brownian motion in bacterial near-surface swimming, PNAS, November 17, 2008 (Figure 1b). Copyright (2008) by the American Physical Society. Figure 1b. DOI: https://doi.org/10.1103/PhysRevE.84.041932

Let us assume that the fluid defines an inertial reference frame. Newton's second law then leads to the ordinary differential equation

$$m\frac{dv}{dt} = -b\,v \Rightarrow m\,\ddot{x} = -b\,\dot{x}, \tag{1.7}$$

where $\dot{x} \equiv dx/dt$ and $\ddot{x} \equiv d^2x/dt^2$. So Newton's second law is a second-order differential equation in position and time, but a particularly simple one that can be integrated at once to give a first-order differential equation in v and t. Separating variables and integrating:

$$\int_{v_0}^{v} \frac{dv}{v} = -\frac{b}{m}\int_0^t dt, \tag{1.8}$$

which gives $\ln(v) - \ln(v_0) = \ln(v/v_0) = -(b/m)t$. Exponentiating both sides:

$$v = v_0 e^{-(b/m)t} \equiv v_0 e^{-t/\tau} \quad \Rightarrow \quad a = \frac{dv}{dt} = -\frac{v_0}{\tau}e^{-t/\tau}, \tag{1.9}$$

where $\tau \equiv m/b$ is called the "time constant" of the exponential decay. In a single time constant, *i.e.*, when $t = \tau$, the velocity decreases to $1/e$ of its initial value; therefore τ is a measure of how quickly the bacterium slows down. The bigger the drag force (or the smaller the mass), the greater the deceleration.

An alternate way to solve the differential equation is to note that it is linear with constant coefficients, so the exponential form $v(t) = Ae^{\alpha t}$ is bound to work, for an *arbitrary* constant A and a *particular* constant α. In fact, the constant $\alpha = -1/\tau$, found by substituting $v(t) = Ae^{\alpha t}$ into the differential equation and requiring that it be a solution. In this first-order equation the constant A is the single required arbitrary constant. It can be determined by imposing the initial condition $v = v_0$ at $t = 0$, which tells us that $A = v_0$.

Now we can integrate once more to find the bacterium's position $x(t)$. If we choose the x direction as the $\mathbf{v}_0$ direction, then $v = dx/dt$, so

$$x(t) = v_0 \int_0^t e^{-t/\tau}\, dt = v_0 \tau \left(1 - e^{-t/\tau}\right). \tag{1.10}$$

The second integration constant is fixed by the bacterium's starting position, $x(0) = 0$. As $t \to \infty$, we see that its position x asymptotically approaches the value $v_0\tau$. Note that given a starting position and an initial velocity, the path of a bacterium is determined by the forces exerted on it. Figure 1.4 shows $x(t)$ and $v(t)$ for the bacterium.

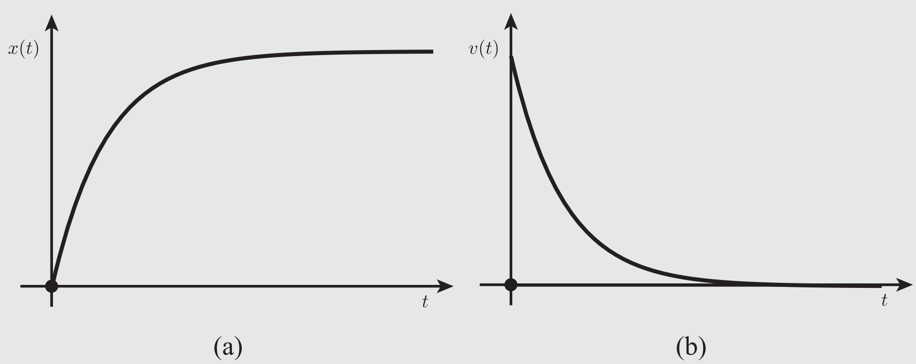


Fig. 1.4 Position (a) and velocity (b) versus time for the bacterium. ■

For larger and more quickly moving objects there comes a point where the fluid no longer flows smoothly around the object, but becomes turbulent, churning around and shedding swirling eddies and vortices. The drag force is then approximately proportional to the *square* of the object's velocity through the fluid. This is sometimes called **inertial drag** or **Newtonian drag**. Air in front of a fast-moving baseball has no time to flow smoothly out of the way, but becomes turbulent and retains this turbulence after the ball has already passed by. This is the type of drag that normally happens all around us, including the drag force on cars and airplanes moving at typical speeds. Doubling their velocity increases the drag force by a factor of four, so in the case of automobiles, for example, designers are motivated to reduce the drag force by streamlining the shape of cars to minimize the turbulence. This helps increase fuel efficiency and also the top speed attainable.

Example 1.2 A ball of mass m and radius r is dropped from the top of a skyscraper. Find the height of the skyscraper if the ball reaches the ground at a time t later.

In this case the drag force is quadratic over essentially the entire trip, so the equation of motion is

$$m\frac{dv}{dt} = mg - cv^2, \tag{1.11}$$

where c is the drag constant and we have taken the positive direction to be downward. Note that the net force on the ball goes to zero as $v \to \sqrt{mg/c}$, so there is a terminal velocity $v_T = \sqrt{mg/c}$ which the ball never quite reaches as it falls. Initially, when v is small, the ball has downward acceleration $a \simeq g$, and then $a \to 0$ as $v \to v_T = \sqrt{mg/c}$. It is the existence of a terminal velocity that helps some cats survive when they leap out of open windows in tall apartment buildings hoping to catch a bird, or even a very few people among those whose parachutes have failed to open, or in one case a soldier who jumped without a parachute from a plane in flames, preferring to take his chances in free fall rather than getting burned alive. Thanks to the terminal velocity, the impact velocity of an object at the ground stays nearly the same no matter how high up the object begins, assuming of course that the initial altitude is sufficiently great. Using the result $v_T^2 = mg/c$, the v and t variables in $F = ma$ can be separated to give

$$gt = g\int_0^t dt = \int_0^v \frac{dv}{1 - v^2/v_T^2}. \tag{1.12}$$

A particularly simple way to carry out the integration is to use the technique of partial fractions, beginning with the identity

$$\frac{1}{1-z^2} = \frac{1}{2}\left(\frac{1}{1+z} + \frac{1}{1-z}\right). \tag{1.13}$$

So if we let $z = v/v_T$, it follows that

$$gt = \frac{v_T}{2}\left[\ln(1+z) - \ln(1-z)\right] = \frac{v_T}{2}\ln\left(\frac{1+z}{1-z}\right), \tag{1.14}$$

which gives t in terms of v, since $v = v_T z$. We can invert this equation to find v as a function of t. The result is

$$v = v_T\left[\frac{e^{gt/v_T} - e^{-gt/v_T}}{e^{gt/v_T} + e^{-gt/v_T}}\right] = v_T \tanh(gt/v_T) \tag{1.15}$$

in terms of a hyperbolic tangent function. From this result we can verify that $v \simeq gt$ for small t, using the series expansion for exponentials $e^x = 1 + x + (1/2)x^2 + ...$ for small x. We can also verify that $v \to v_T$ for large t, since then $e^{-gt/v_T} \to 0$.

So far so good. Now we can find how far the ball falls in a given time by integrating the last result over time, and letting y be the distance fallen. That is:

$$y = \int dy = v_T \int_0^t dt \tanh(gt/v_T) = \frac{v_T^2}{g} \int dq \tanh q = \frac{v_T^2}{g} \int dq \left(\frac{\sinh q}{\cosh q} \right), \tag{1.16}$$

where we have defined $q = gt/v_T$ and used the identity $\tanh q = \sinh q / \cosh q$, where sinh and cosh are the hyperbolic sine and cosine functions. The differential of $\cosh q$ is $\sinh q dq$, so the integral is just the natural logarithm of $\cosh q$. So finally:

$$y = \left(\frac{v_T^2}{g} \right) \ln(\cosh q) = \left(\frac{v_T^2}{g} \right) \ln(\cosh gt/v_T). \tag{1.17}$$

This is how far the ball has fallen as a function of time. One can also invert this equation to find how long it takes the ball to reach the ground in terms of its initial height. ■

1.4 Oscillation in One-Dimensional Motion

The drag forces we have used so far are purely velocity-dependent forces in which Newton's second law becomes a first-order differential equation. In contrast, a simple harmonic oscillator consists of a mass m attached to one end of a Hooke's-law spring exerting force $F = -kx$, where k (a positive constant) is the force constant of the spring and x is the spring stretch. For such position-dependent forces, Newton's second law becomes a *second-order* differential equation. The minus sign indicates that if x is positive, when the spring has been stretched, it will *pull* the particle back toward equilibrium, and if x is negative, the spring has been compressed, and it will *push* the particle back toward equilibrium. The importance of this linear force extends far beyond the force exerted by an actual spring, because very often it is a spring-like linear restorative force that is exerted when a particle is displaced slightly from equilibrium under the influence of a wide variety of forces. We will return to this point when we discuss energy a bit later.

If the only force on a particle moving in one dimension is due to a Hooke's-law spring, the equation of motion is

$$m\ddot{x} = -kx \quad \text{or} \quad m\ddot{x} + kx = 0, \tag{1.18}$$

where each overdot means a time derivative, a notation due to Newton himself. This is the famous simple harmonic oscillator (SHO) equation, a linear second-order differential equation in x and t.

There are several ways to solve the equation. One way is to note that we require a couple of linearly independent functions whose second derivatives are the negatives of themselves, apart from constants; this suggests *sines* and *cosines*. The general solution can then be written

$$x(t) = A\cos(\omega t) + B\sin(\omega t) \quad \text{or} \quad x(t) = C\cos(\omega t + \varphi), \tag{1.19}$$

where $\omega = \sqrt{k/m} = 2\pi\nu$ with ν the frequency of oscillation, and where A and B (or C and φ) are the two arbitrary constants needed in solutions of the second-order differential equation. The constants C and φ can be found in terms of A and B, or vice versa, using the trig identity $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$. The second form of the solution is depicted in Figure 1.5, illustrating the meaning of the constants C, ω, and φ.

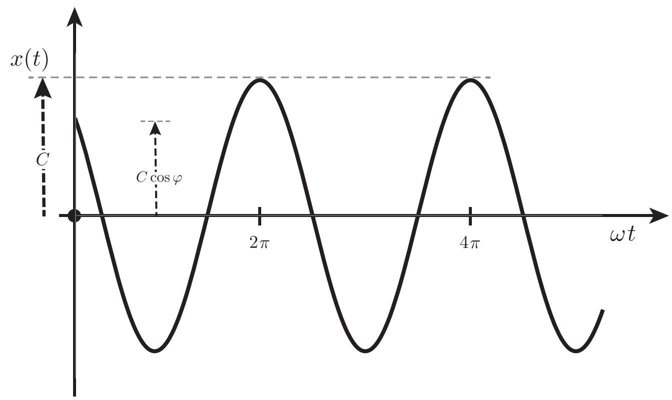


Fig. 1.5 A simple harmonic oscillation $x(t) = C\cos(\omega t + \varphi)$ for phase angle $\varphi = \pi/4$. Shown is the amplitude C. The period of oscillations is $P = 2\pi/\omega$, and $\omega = 2\pi\nu$, where $\nu = 1/P$ is the frequency and ω is the angular frequency of oscillation.

Another method of solving the SHO equation is more formal but also provides more insight. We can solve the equation in stages, integrating once to get a first-order differential equation, called a “first integral of motion,” and then integrating a second time to get the final solution $x(t)$. This first integration can be carried out by first multiplying the equation by a so-called “integrating factor” $\dot{x}$, giving

$$m\dot{x}\ddot{x} + kx\dot{x} = 0 \quad \text{or} \quad \frac{1}{2}m\frac{d\dot{x}^2}{dt} + kx\frac{dx}{dt} = 0. \tag{1.20}$$

Multiplying by dt, we have $(1/2)m\, d(\dot{x}^2) + kxdx = 0$, which is directly integrable because each term contains only a single variable. Integrating this last equation:

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E, \tag{1.21}$$

where E is the constant of integration. We recognize this as a conservation of energy equation for the particle, the sum of its kinetic and potential energies. The kinetic energy $T = (1/2)m\dot{x}^2$ depends on the particle’s velocity but not its position, and the potential energy $U = (1/2)kx^2$ depends on the particle’s position but not its velocity. The sum is the total energy, a constant of the motion.

Now we can separate the remaining variables x and t and integrate once more:

$$\int dt = \sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{E - (1/2)kx^2}}, \tag{1.22}$$

again with only a single variable in each term. Substituting $x = \sqrt{2E/k}\cos\theta$ and integrating gives $t = -\sqrt{m/k}\,\theta +$ constant and then rearranging and using the fact that $\cos(-\theta) = \cos(\theta)$ it follows that

$$x(t) = \sqrt{\frac{2E}{k}}\cos(\omega t + \varphi), \tag{1.23}$$

where $\omega = \sqrt{k/m}$ and E and φ are the two necessary arbitrary constants. In addition to showing that energy conservation is the first integral of motion, we have found the amplitude of oscillation in terms of the energy E and force constant k.

Damped Oscillations

The simple harmonic oscillator is not damped. According to the solutions, once excited it will oscillate forever. However, real oscillations eventually die out, which means they must have additional forces exerted on them that cause them to decrease their amplitude with time. A realistic force that does this in most situations is the quadratic damping force $F_{\mathrm{drag}} = -cv^2$, where c is a constant. It will continually reduce the oscillator's amplitude.

Adding this force to the oscillating object leads to the equation $m\ddot{x} = -kx - c\dot{x}^2$, which is still a second-order differential equation, but with a new $\dot{x}^2$ term that is *nonlinear*. Unfortunately, this nonlinearity makes the equation impossible to solve in terms of elementary functions, so the tradition is to replace quadratic damping with linear damping, which makes the full equation linear and easy to solve. Even though linear damping is usually unrealistic, it at least leads to decaying oscillations, which is more realistic than no damping at all.

A particular linearly damped oscillator consists of a mass m confined to move in the x direction attached at one end to a Hooke's-law spring of force constant k, and which is also subject to the damping force $-b\,v$ where b is a constant. That is, we assume that the damping force is linearly proportional to the velocity of the mass and in the direction opposite to its motion.

Newton's second law then gives $F = -kx - b\dot{x} = m\ddot{x}$, a second-order linear differential equation equivalent to

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0, \tag{1.24}$$

where we let $\beta \equiv b/2m$ and $\omega_0 \equiv \sqrt{k/m}$ to simplify the notation. Mathematically, we are guaranteed a solution once we fix *two* initial conditions. These can be, for example, the initial position $x(0) = x_0$ and velocity $v(0) = \dot{x}(0) = v_0$. Hence, our solution will depend on two constants to be specified in the particular problem. In general, each dynamical variable we track through Newton's second law will generate a single second-order differential equation, and so will require two initial conditions. This is the sense in which Newton's laws provide us with predictive power: fix a few constants using initial conditions, and physics will tell us the future evolution of the system. For the example at hand, Eq. (1.24) is a *linear* differential equation with constant coefficients, which can be solved by setting $x \propto e^{\alpha t}$ for some α. Substituting this form into Eq. (1.24) gives the quadratic equation

$$\alpha^2 + 2\beta\alpha + \omega_0^2 = 0 \quad \text{with solutions} \quad \alpha = -\beta \pm \sqrt{\beta^2 - \omega_0^2}. \tag{1.25}$$

There are now three possibilities: (1) $\beta > \omega_0$, the "overdamped" solution; (2) $\beta = \omega_0$, the "critically damped" solution; and (3), $\beta < \omega_0$, the "underdamped" solution, all as illustrated in Figure 1.5.

(1) In the overdamped case the exponent α is *real* and *negative*, and so the position of the mass as a function of time is

$$x(t) = A_1 e^{\gamma_1 t} + A_2 e^{\gamma_2 t}, \tag{1.26}$$

where $\gamma_1 = -\beta + \sqrt{\beta^2 - \omega_0^2}$ and $\gamma_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}$. Here A_1 and A_2 are arbitrary constants. The two terms are the expected linearly independent solutions of the second-order differential equation, and the coefficients A_1 and A_2 can be determined from the initial position x_0 and initial velocity v_0 of the mass. Figure 1.6(a) shows a plot of $x(t)$.

(2) In the critically damped $\beta = \omega_0$ case the two solutions of Eq. (1.25) merge into the single solution $x(t) = A\,e^{-\beta t}$. However, a second-order differential equation has two linearly independent solutions, so we need one more. This additional solution is $A' t\, e^{-\beta t}$ for an arbitrary coefficient A', as can be seen by substituting this form into Eq. (1.24). The general solution for the critically damped case is therefore

$$x = (A + A't)e^{-\beta t}, \tag{1.27}$$

which has the two independent constants A and A' determined from the initial position x_0 and velocity v_0. Figure 1.6(b) shows a plot of $x(t)$ in this case.

(3) In the underdamped case, the quantity $\sqrt{\beta^2 - \omega_0^2} = i\sqrt{\omega_0^2 - \beta^2}$ is purely *imaginary*, so

$$x(t) = e^{-\beta t}\text{Re}\,(A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t}), \tag{1.28}$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ and we take only the *real* part of the solution, as indicated by "Re." It is mathematically legal to take only the real part of the solution since the differential equation is real and linear in x: if the complex function $x(t)$ solves the differential equation, so will the real and imaginary parts of $x(t)$ separately.[1] We can use Euler's identity

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.29}$$

[1] You can convince yourself of this by plugging $x(t) = x_R(t) + i\,x_I(t)$ into the differential equation and extracting two identical equations for $x_R(t)$ and $x_I(t)$ from the real and imaginary parts, respectively.

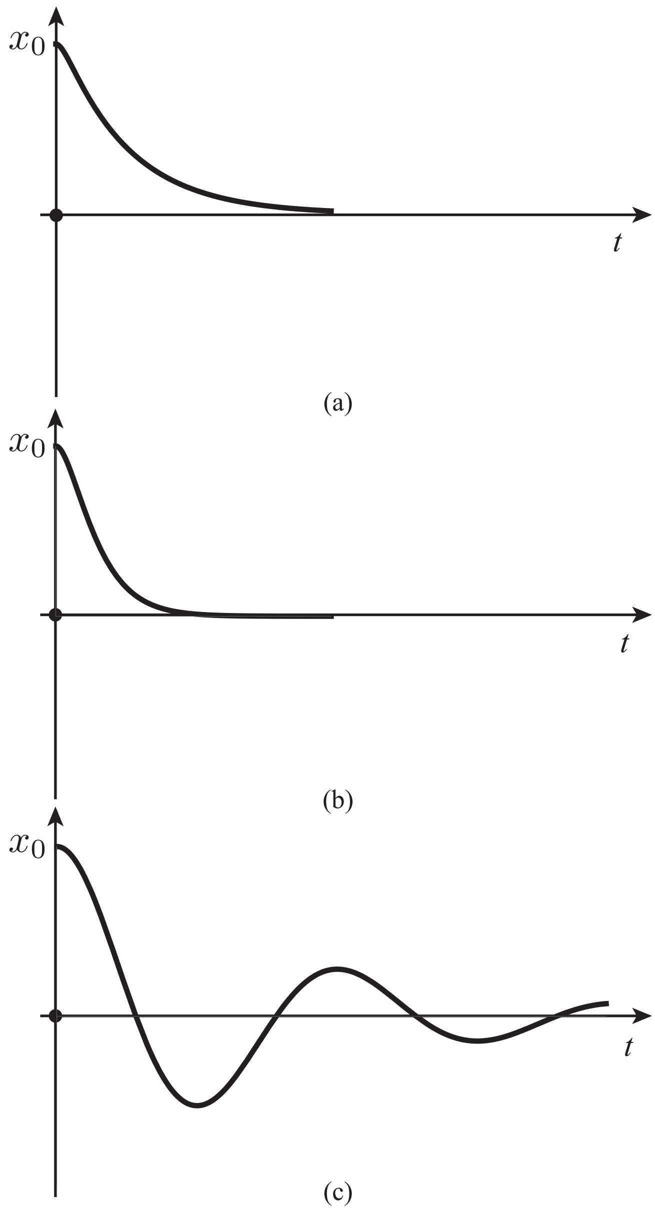


Fig. 1.6 Motion of an oscillator if it is (a) overdamped, (b) critically damped, or (c) underdamped, for the special case where the oscillator is released from rest ($v_0 = 0$) at some position x_0.

to write x in terms of purely real functions:

$$x(t) = e^{-\beta t}(\bar{A}_1 \cos \omega_1 t + \bar{A}_2 \sin \omega_1 t), \tag{1.30}$$

where $\bar{A}_1 = A_1 + A_2$ and $\bar{A}_2 = i(A_1 - A_2)$ are real coefficients. We can also use the identity $\cos(\theta + \varphi) = \cos\theta \cos\varphi - \sin\theta \sin\varphi$ to write Eq. (1.30) in the form

$$x(t) = A\,e^{-\beta t}\cos(\omega_1 t + \varphi), \tag{1.31}$$

where $A = \sqrt{\bar{A}_1^2 + \bar{A}_2^2}$ and $\varphi = \tan^{-1}(-\bar{A}_2/\bar{A}_1)$. That is, the underdamped solution corresponds to a decaying oscillation with amplitude $A\,e^{-\beta t}$. The arbitrary constants A and φ can be determined from the initial position x_0 and velocity v_0 of the mass. Figure 1.6(c) shows a plot of $x(t)$ in this case. If there is no damping at all, we have $b = \beta = 0$ (and the oscillator is obviously "underdamped"). The original Eq. (1.24) becomes the SHO equation $\ddot{x} + \omega_0^2 x = 0$ whose most general solution is

$$x(t) = A\cos(\omega_0 t + \varphi). \tag{1.32}$$

This gives away the meaning of ω_0: it is the angular frequency of oscillation of a simple harmonic oscillator, related to the oscillation frequency ν in cycles/second by $\omega_0 = 2\pi\nu$. Note that $\omega_1 < \omega_0$; *i.e.*, the damping reduces the oscillation frequency in addition to damping the amplitude.

Whichever solution applies, it is clear that the motion of the particle is determined by (a) the initial position $x(0)$ and velocity $\dot{x}(0)$, and (b) the forces acting on it throughout its motion.

1.5 Resonance

If we "drive" a lightly damped spring–mass system with an oscillating force at the right frequency we observe the phenomenon of *resonance*. Repeated small stimulations of an oscillating system at its natural frequency of oscillation can cause the oscillation amplitude to become large, especially if the damping is small. In particular, consider adding a sinusoidal driving force $F = F_0 \sin\omega t$ to the spring force and the damping force acting upon a spring–mass system. Then Newton's law becomes

$$m\ddot{x} = F_{\text{spring}} + F_{\text{damping}} + F_{\text{driving}} = -kx - b\dot{x} + F_0 \sin\omega t. \tag{1.33}$$

We can change the driving frequency ω arbitrarily. So now we have three important frequencies, the "natural" frequency $\omega_0 = \sqrt{k/m}$ of an undamped spring–mass system; the linear damped frequency $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$, where $\beta = b/2m$; and the new driving frequency ω. There are various ways to apply this sinusoidal driving force. One way is to hold the end of the spring which is *not* connected to the mass m, and move it back and forth sinusoidally in the x direction, so its position on a frictionless table as a function of time is $X = A\sin\omega t$. Then the length of the spring at any time is not x, but $(x - X)$, so the force it exerts on m is $F_{\text{spring}} = -k(x - X) = -k(x - A\sin\omega t)$. Newton's second law then gives

$$m\ddot{x} = -k(x - A\sin\omega t) - bv = -kx - b\dot{x} + kA\sin\omega t, \tag{1.34}$$

so

$$m\ddot{x} + b\dot{x} + kx = F_0 \sin \omega t \qquad \text{or} \qquad \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \sin \omega t, \tag{1.35}$$

where $F_0 \equiv kA$, $\beta \equiv b/2m$ is the *damping constant*, and $f_0 \equiv F_0/m$. This is the equation of a driven, linearly damped harmonic oscillator. Mathematically speaking, the differential equation is still linear and of second order, but it has been changed from a homogeneous to an inhomogeneous equation, due to the driving force term on the right. The solution of this inhomogeneous equation is the sum of the general (or "characteristic") solution $x_c(t)$ of the homogeneous equation (*i.e.*, the equation without the driving term on the right) and a particular solution $x_p(t)$ of the full inhomogeneous equation

$$x(t) = x_c(t) + x_p(t). \tag{1.36}$$

We have already found the general solution of the homogeneous equation. It is

$$x_c(t) = Ae^{-\beta t} \cos(\omega_1 t + \varphi_0), \tag{1.37}$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ and the amplitude A and phase angle φ_0 are the requisite number of arbitrary constants for the second-order differential equation. Note that this homogeneous term $x_c(t)$ gradually dies out, so it is often called the "transient" solution, as illustrated in Figure 1.7.

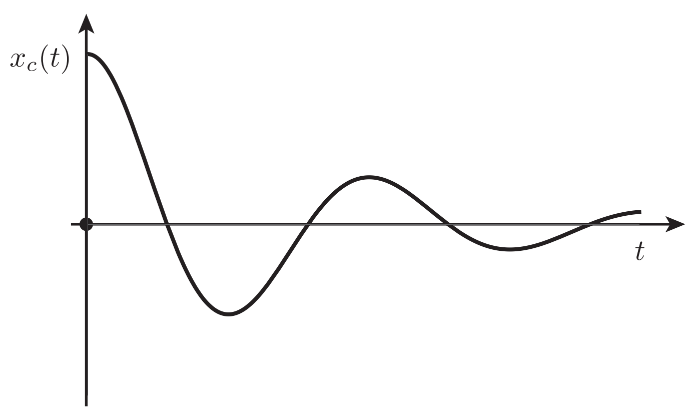


Fig. 1.7 Transient solution of a forced, damped harmonic oscillator.

It is the other, "particular" solution $x_p(t)$ that wins out in the end, and it is called the "steady state" solution

$$x_p(t) = C \sin(\omega t + \delta), \tag{1.38}$$

where C and δ are constants to be determined. The complete solution is the sum of the steady-state solution and the transient (characteristic) solution:

$$x_p(t) = Ae^{-\beta t} \cos(\omega_1 t + \varphi_0) + C \sin(\omega t - \delta). \tag{1.39}$$

The first term, the transient solution, dies away as time goes on, leaving the steady-state solution with amplitude C.

How did we know the form of $x_p(t)$? We could first try $x_p = C \sin \omega t$ for some constant C, in which the mass oscillates in synchrony with the driving force. However, that cannot work, because the first-derivative term in the differential equation converts the sine to a cosine, while every other term in the equation is the sine, so there is no value of C for which the trial solution works. Another possibility is to try the phase-shifted sine function $x_p(t) = C \sin(\omega t - \delta)$, which oscillates at the driving frequency but is phase-shifted by the angle δ.

Substituting this trial solution $x_p(t) = C \sin(\omega t - \delta)$ into the differential equation gives

$$C[(\omega_0^2 - \omega^2)\sin(\omega t - \delta) + 2\beta\omega \cos \omega t - \delta] = f_0 \sin \omega t. \tag{1.40}$$

Using the trig identities

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b \quad \text{and} \quad \cos(a \pm b) = \cos a \cos b \mp \sin a \sin b \tag{1.41}$$

we write

$$\begin{aligned} C[(\omega_0^2 - \omega^2)&(\sin \omega t \cos \delta - \cos \omega t \sin \delta) \\ &+ 2\beta\omega(\cos \omega t \cos \delta + \sin \omega t \sin \delta)] = f_0 \sin \omega t, \end{aligned} \tag{1.42}$$

which must hold at all times. Orthogonality of the sine and cosine functions implies that the coefficients of each should independently vanish. For example, at times t such that $\omega t = 0, \pi, 2\pi$, etc., the $\sin \omega t$ terms all vanish, so the $\cos \omega t$ terms alone must satisfy the equation. That is:

$$C[(-\omega_0^2 - \omega^2)\sin \delta + 2\beta\omega \cos \delta] = 0 \tag{1.43}$$

at any one of the times mentioned above. But all of these quantities are independent of time, so this expression must *always* be zero. Therefore the quantity inside the square brackets vanishes. That is:

$$\tan \delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2} = \frac{(2\beta/\omega_0)(\omega/\omega_0)}{1 - (\omega/\omega_0)^2}. \tag{1.44}$$

Notice that if the damping $\beta \to 0$, it follows that $\tan \delta \to 0$, so that the phase angle $\delta \to 0$ as well. Then the mass moves back and forth in phase with the driving force. This is also true for very low applied frequencies ω; as $\omega \to 0$, the phase angle $\delta \to 0$. This means that if the driving force causes the spring to oscillate very slowly back and forth, the mass on the other end of the spring will move back and forth in phase with the driving force.

Figure 1.8 is a graph of the phase angle δ as a function of ω/ω_0, the ratio of the driving frequency to the natural frequency of the undamped spring for a particular value of $2\beta/\omega_0$. Note that the response of the system is $\pi/2$ out of phase with the driving frequency if $\omega = \omega_0$, and out of phase by the angle π if $\omega \gg \omega_0$. It is straightforward to work out the changes in shape of this graph depending upon the value of $2\beta/\omega_0$. Equation (1.42) must also be correct for all times such that

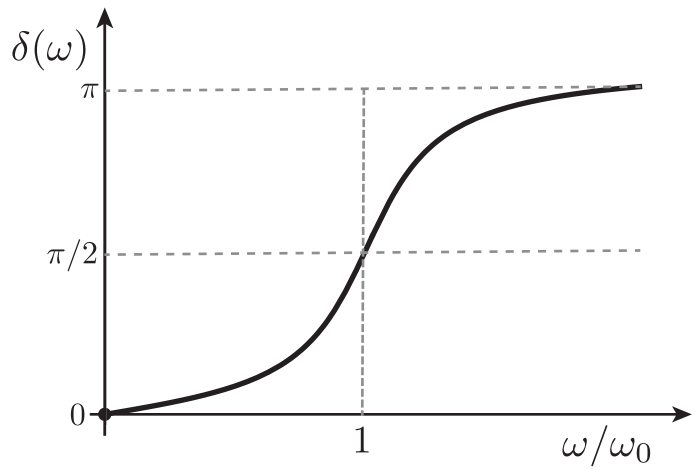


Fig. 1.8 Graph of the phase angle δ between the driving frequency and the response frequency of the oscillator, drawn for a particular value of the parameter $2\beta/\omega_0$. As the parameter is made larger, the slope of the graph becomes steeper near $\omega/\omega_0 = 1$.

$\omega t = \pi/2, 3\pi/2, 5\pi/2$, etc., when each $\cos\omega t$ term is zero. Only the $\sin\omega t$ terms survive, so it follows that

$$C[(\omega_0^2 - \omega^2)\cos\delta + 2\beta\omega\sin\delta] = f_0, \tag{1.45}$$

so the constant C is

$$C = \frac{f_0}{(\omega_0^2 - \omega^2)\cos\delta + 2\beta\omega\sin\delta}. \tag{1.46}$$

We have already found $\tan\delta$, so noting that

$$\tan\delta = \frac{\sin\delta}{\cos\delta} = \frac{\sin\delta}{\sqrt{1-\sin^2\delta}}, \tag{1.47}$$

we get

$$\sin\delta = \frac{2\beta\omega}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}}. \tag{1.48}$$

Substituting these into the previous equation for C, the final result for the amplitude C as a function of the driving frequency ω is

$$C(\omega) = \frac{f_0}{(\omega_0^2 - \omega^2)\cos\delta + 2\beta\omega\sin\delta} = \frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}}. \tag{1.49}$$

Now we have found both the amplitude and the phase angle of the "particular" (steady-state) solution $x_p(t)$ of the forced, damped oscillator:

$$x_p(t) = C(\omega)\sin(\omega t - \delta(\omega)), \tag{1.50}$$

where $C(\omega)$ is given by Eq. (1.49) and the phase angle δ by Eq. (1.44).

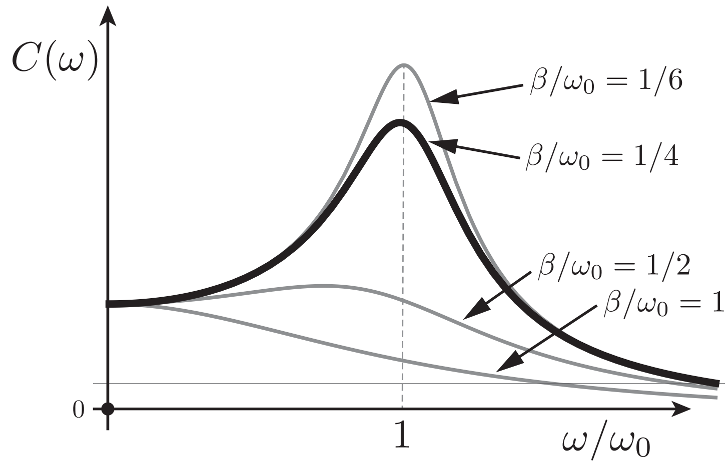


Fig. 1.9 Shape of the oscillation amplitude response of the system as a function of ω/ω_0 for various damping constants.

The shape of $C(\omega)$, the amplitude response, is especially interesting; it is displayed in Figure 1.9 as a function of the ratio ω/ω_0, for various damping constants, as characterized by the ratio β/ω_0. The curves show a resonance peak at a frequency near, but not quite at, the natural frequency ω_0 of the undamped spring–mass system. Note that the curves are sharper for small damping than for large damping. If the driving force frequency ω is close to the natural frequency ω_0 the response is large, especially if the drag is small. This is the resonance phenomenon. The resonance frequency ω_R is the frequency corresponding to the maximum in the response curve $C(\omega)$. It is then given by

$$\omega_R = \sqrt{\omega_0^2 - 2\beta^2}, \tag{1.51}$$

which is easily found by setting $dC(\omega)/d\omega = 0$. It is then easy to show that the oscillation amplitude at resonance is

$$C_R = \frac{F_0}{2m\beta\omega_1}, \tag{1.52}$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ is the frequency of the damped, undriven oscillator. Note that C_R is large if the damping β is small.

Resonance can be observed by repeated small pushes on a child on a swing at his or her natural frequency of oscillation; or by driving a car at just the right speed on a washboard road, especially when the car has no shock absorbers to damp out the motion; or when tuning a radio, where incoming radio waves striking the antenna can excite large oscillations in the radio's electrical circuits if the frequency is just right, but not otherwise, so you hear only the station you tuned for.

1.6 Motion in Two or Three Dimensions

So far all of our examples have been restricted to one-dimensional motion. When the motion is in two or three dimensions, the first step is to select an appropriate coordinate system that fits the problem. For two-dimensional motion there are Cartesian or plane polar coordinates, for example, and for three-dimensional motion there are Cartesian, spherical, or cylindrical coordinates, the most common choices among many others.

Having chosen a coordinate system, it is often convenient to express vector quantities like position, velocity, acceleration, or force using **unit vectors**. Each unit vector has unit length and points in one of the orthogonal directions corresponding to the coordinates in the system. It follows that the dot product of any unit vector with itself is unity, while the dot product of any unit vector with any other unit vector in the same system is zero.

For Cartesian coordinates in two dimensions the unit vectors are $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, where

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = 1, \;\; \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = 1, \;\; \hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{x}} = 0. \tag{1.53}$$

The position vector of a particle is then

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} \tag{1.54}$$

and the particle's velocity and acceleration vectors are

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{x}\hat{\mathbf{x}} + \dot{y}\hat{\mathbf{y}} \quad \text{and} \quad \mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{x}\hat{\mathbf{x}} + \ddot{y}\hat{\mathbf{y}}. \tag{1.55}$$

In differentiating $\mathbf{r}$ and $\mathbf{v}$ we differentiated their components, but did not have to differentiate the unit vectors, because $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are constants: neither the length of these unit vectors nor their directions in space change with time. If plane polar coordinates r, θ are chosen instead, the unit vectors are $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$, where

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1, \;\; \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} = 1, \;\; \hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} \cdot \hat{\mathbf{r}} = 0. \tag{1.56}$$

Now whereas Cartesian unit vectors do not change with time, the plane polar unit vectors generally *do* change as the particle moves, because their directions may change. For example, if the particle moves in a circle around the origin, both unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ change direction in space. In fact, their time derivatives are

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\hat{\boldsymbol{\theta}} \quad \text{and} \quad \frac{d\hat{\boldsymbol{\theta}}}{dt} = -\dot{\theta}\hat{\mathbf{r}}. \tag{1.57}$$

In plane polar coordinates the position vector of a particle is simply $\mathbf{r} = r\hat{\mathbf{r}}$. Therefore the velocity is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}, \tag{1.58}$$

and the acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}. \tag{1.59}$$

This equation contains within it the well-known results that a particle circling the origin at constant radius r and constant angular velocity $\dot{\theta} \equiv \omega$ will have an inward ("centripetal") acceleration $-r\omega^2\hat{r} = -(v^2/r)\,\hat{r}$, and a person walking outward $\dot{r} > 0$ on a steadily rotating carousel with angular velocity $\dot{\theta} > 0$ will be accelerating *sideways*, in the $\hat{\boldsymbol{\theta}}$ direction. Much more on all of this in Chapter 9.

Of course, motion in all three dimensions requires three unit vectors, typically for Cartesian, spherical, or cylindrical coordinates. These unit vectors are given in Appendix A.

Example 1.3 **A Slingshot on the Moon**

Someday we may want to construct spacecraft or space colonies not on the earth or the moon but in space itself, using mined metals and other materials lifted off the moon. The moon has the advantage of a much smaller escape velocity than that of the earth, and no atmosphere to retard motion. Instead of using expensive rockets and fuel, could it be possible to achieve the escape velocity from the airless moon by *slinging* containers of material from its surface using a rapidly rotating boom? A sturdy boom of length R might swing around in a horizontal plane on the moon's surface about a central vertical axis at constant angular velocity ω. A payload container starting near the rotation axis of the boom might then slide with increasing speed out along the length of the boom and then project outward at a very high velocity when it leaves the end of the boom.

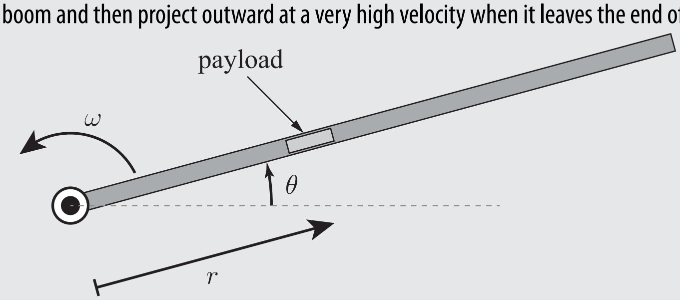


Fig. 1.10 A boom with payload on the moon's surface, rotating in a horizontal plane.

Plane polar coordinates are the obvious choice here, with r measured outward from the rotation axis and θ the angle of the boom from some initial angle $\theta = 0$ when the payload is released on the rotating boom at a small initial radius r_0, with $\dot{r}_0 = 0$. The boom keeps swinging around at constant angular velocity, so the angle of the payload is $\theta = \omega t$ until it finally flies off the end of the boom (see Figure 1.10).

We assume the payload slides frictionlessly along the boom, so the radial force $F_r = 0$. The tangential force is $F_\theta \neq 0$, which is the normal force of the boom on the payload, keeping it moving with constant angular velocity ω as it slides outward. Newton's second law is then

$$\mathbf{F} = F_\theta\hat{\boldsymbol{\theta}} = m\mathbf{a} = m[(\ddot{r} - r\omega^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}], \tag{1.60}$$

so

$$\ddot{r} - \omega^2 r = 0 \quad \text{and} \quad F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}). \tag{1.61}$$

The first equation is a linear, second-order differential equation with solution $r = Ae^{\omega t} + Be^{-\omega t}$, where A and B are arbitrary constants. We can find A and B from the given initial conditions at $t = 0$, which are $r = r_0$ and $\dot{r} = 0$. This gives $A = B = r_0/2$, so

$$r = (r_0/2)(e^{\omega t} + e^{-\omega t}) \equiv r_0 \cosh \omega t \tag{1.62}$$

in terms of the hyperbolic cosine function. Then the velocity of the payload as a function of time, including both the radial and tangential components, is

$$\mathbf{v} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} = r_0\omega \sinh \omega t\,\hat{\mathbf{r}} + r_0\omega \cosh \omega t\,\hat{\boldsymbol{\theta}}. \tag{1.63}$$

We can find the payload velocity when it reaches the end of the boom. At that point $R = r_0 \cosh \omega t_f$, where t_f is the time when this happens. Then $\cosh \omega t_f = R/r_0$ and $\sinh \omega t_f = \sqrt{\cosh^2 \omega t_f - 1} = \sqrt{(R/r_0)^2 - 1)}$, where we have used the identity $1 + \sinh^2 = \cosh^2$. Substituting these results into the expression for $\mathbf{v}$:

$$\mathbf{v} = \omega \left[\sqrt{R^2 - r_0^2}\,\hat{\mathbf{r}} + R\hat{\boldsymbol{\theta}} \right] \tag{1.64}$$

and from this we can find the speed of the payload as it flies off the end:

$$v_f = \sqrt{v_r^2 + v_\theta^2} = \omega\sqrt{2R^2 - r_0^2}, \tag{1.65}$$

which must equal or exceed the moon's escape velocity. Finally, we can calculate the tangential force the boom must exert upon the payload to keep $\theta = \omega\, t$, as a function of time and as a function of r:

$$F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = m(0 + 2\omega^2 r_0 \sinh \omega t) = 2m\omega^2\sqrt{r^2 - r_0^2}, \tag{1.66}$$

which is greatest when $r = R$, at the tip of the boom. There will be an equal but opposite reaction force back on the boom due to the payload, so the boom must be strong enough to withstand this tangential force at its tip.

Putting in some numbers, the escape velocity on the moon is approximately 2.4 km/s, and we can choose $\omega = 2\pi\ \mathrm{s}^{-1}$ and $r_0 = 1$ m. The radius of the boom must then be $R \simeq 270$ m. ■

1.7 Systems of Particles

Up to now we have concentrated on the dynamics of *single* particles. We will now expand our horizons to encompass systems of an arbitrary number of particles. A system of particles might be an entire solid object like a bowling ball, in which tiny parts of the ball can be viewed as individual infinitesimal particles. Or we might have a liquid in a glass, or the air in a room, or a planetary system, or a galaxy of stars, all made of constituents we treat as "particles."

The location of the ith particle of a system can be identified by a position vector $\mathbf{r}_i$ extending from the origin of coordinates to that particle, as illustrated in Figure 1.11. Using the laws of classical mechanics for each particle in the system, we can find the laws that govern the system as a whole.

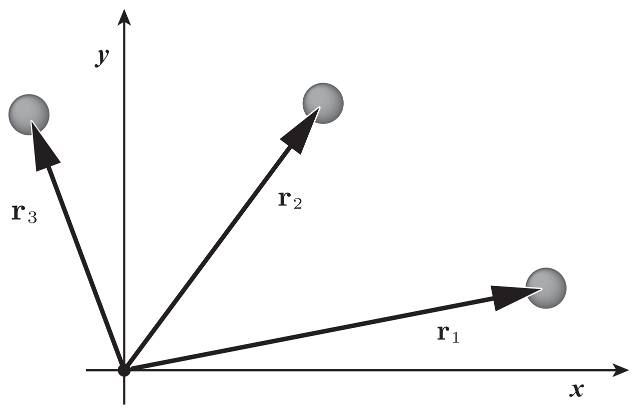


Fig. 1.11 A system of particles, with each particle identified by a position vector $\mathbf{r}_i$ with $i = 1, 2, 3$.

Define the total momentum $\mathbf{P}$ of the system as the sum of the momenta of the individual particles:

$$\mathbf{P} = \sum_i \mathbf{p}_i. \tag{1.67}$$

Similarly, define the total force $\mathbf{F}_{\mathrm{T}}$ on the system as the sum of all the forces on all the particles:

$$\mathbf{F}_{\mathrm{T}} = \sum_i \mathbf{F}_i. \tag{1.68}$$

It then follows that $\mathbf{F}_{\mathrm{T}} = d\mathbf{P}/dt$, just by adding up the individual $\mathbf{F}_i = d\mathbf{p}_i/dt$ equations for all the particles. If we further split up the total force $\mathbf{F}_{\mathrm{T}}$ into $\mathbf{F}_{\mathrm{ext}}$ (the sum of the forces exerted by external agents, like earth's gravity or air resistance on the system of particles that form a golfball) and $\mathbf{F}_{\mathrm{int}}$ (the sum of the internal forces between members of the system themselves, like the mutual forces between particles within the golfball), then

$$\mathbf{F}_{\mathrm{T}} = \mathbf{F}_{\mathrm{int}} + \mathbf{F}_{\mathrm{ext}} = \mathbf{F}_{\mathrm{ext}}, \tag{1.69}$$

because all the internal forces cancel out by Newton's third law. That is, for any two particles i and j, the force of i on j is equal but opposite to the force of j on i. Finally, we can write a grand second law for the system as a whole:

$$\mathbf{F}_{\mathrm{ext}} = \frac{d\mathbf{P}}{dt}, \tag{1.70}$$

showing how the system as a whole moves in response to external forces.

Now the importance of momentum is clear. For if no external forces act on the collection of particles $\mathbf{F}_{\text{ext}} = 0$, their *total* momentum cannot depend upon time, so $\mathbf{P}$ is conserved. Individual particles in the collection may move in complicated ways, but they always move in such a way as to keep the total momentum constant.

Another useful quantity characterizing a system of particles is their **center of mass** position $\mathbf{R}_{\text{CM}}$. Let the ith particle have mass m_i, and define the center of mass of the collection of particles as

$$\mathbf{R}_{\text{CM}} = \frac{\sum_i m_i \mathbf{r}_i}{M}, \tag{1.71}$$

where $M = \sum_i m_i$ is the total mass of the system. We can write the position vector of a particle as the sum $\mathbf{r}_i = \mathbf{R}_{\text{CM}} + \mathbf{r}'_i$, where $\mathbf{r}'_i$ is the position vector of the particle measured from the center of mass, as illustrated in Figure 1.12.

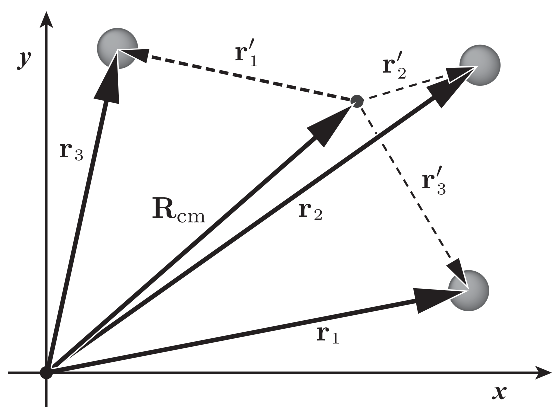


Fig. 1.12 A collection of particles, each with a position vector $\mathbf{r}_i$ from a fixed origin. The center of mass $\mathbf{R}_{\text{CM}}$ is shown, and also the position vector $\mathbf{r}'_i$ of the ith particle measured from the center of mass.

The velocity of the center of mass is

$$\mathbf{V}_{\text{CM}} = \frac{d\mathbf{R}_{\text{CM}}}{dt} = \frac{\sum_i m_i \mathbf{v}_i}{M} = \frac{\mathbf{P}}{M}, \tag{1.72}$$

differentiating term by term, and using the fact that the particle masses are constant. Again $\mathbf{P}$ is the total momentum of the particles, so we have proven that the center of mass moves at constant velocity whenever $\mathbf{P}$ is conserved – that is, whenever there is no net external force. In particular, if there is no external force on the particles, their center of mass stays at rest if it starts at rest.

This result is also very important because it shows that a real object composed of many smaller "particles" can be considered a particle itself: it obeys all of Newton's laws with a position vector given by $\mathbf{R}_{\text{CM}}$, a momentum given by $\mathbf{P}$, and the only relevant forces being the external ones. It relieves us of having to draw a distinct line between *particles* and *systems* of particles. For some purposes we think of a star as composed of many smaller particles, and for other purposes the star as a whole could be considered a single particle in the system of stars called a galaxy.

1.8 Conservation Laws

Using Newton's laws we can show that under the right circumstances there are as many as three dynamical properties of a particle that remain constant in time, *i.e.*, that are *conserved*. These properties are **momentum, angular momentum,** and **energy**. They are conserved under different circumstances, so in any particular case all of them, none of them, or only one or two of them may be conserved. As we will see, a conservation law typically leads to a *first-order* differential equation, which is generally much easier to tackle than the usual second-order equations we get from Newton's second law. This makes identifying conservation laws in a system a powerful tool for problem solving and characterizing the motion. We will see later in Chapter 6 that there are deep connections between conservation laws and symmetries in Nature.

Momentum

From Newton's second law in the form $\mathbf{F} = d\mathbf{p}/dt$ it follows that if there is no net force on a particle, its momentum $\mathbf{p} = m\mathbf{v}$ is conserved, so its velocity $\mathbf{v}$ is also constant. Conservation of momentum for a single particle simply means that a free particle (a particle with no force on it) moves in a straight line at constant speed. For a single particle, conservation of momentum is equivalent to Newton's first law.

For a *system* of particles, however, momentum conservation becomes nontrivial, because it requires the conservation of only *total* momentum $\mathbf{P}$. When there are no *external* forces acting on a system of particles, the total momentum of the individual constituents remains constant, even though the momentum of each single particle may change:

$$\mathbf{P} = \sum_i \mathbf{p}_i = \text{constant}. \tag{1.73}$$

As we saw earlier, this is the momentum of the center of mass of the system if we were to imagine the sum of all the constituent masses added up and placed at the center of mass. This relation can be very handy when dealing with several particles.

Example 1.4 **A Wrench in Space**

We are sitting within a spaceship watching a colleague astronaut outside holding a wrench. The astronaut-plus-wrench system is initially at rest from our point of view. The astronaut (of mass M) suddenly throws the wrench (of mass m), with some unknown force. We then see the astronaut moving with velocity $\mathbf{V}$. Without knowing anything about the force with which she threw the wrench, we can compute the velocity of the wrench. No external forces act on the system consisting of wrench plus astronaut, so its total momentum is conserved:

$$\mathbf{P} = M\mathbf{V} + m\mathbf{v} = \text{constant}, \tag{1.74}$$

where $\mathbf{v}$ is the unknown velocity of the wrench. Since the system was initially at rest, we know that $\mathbf{P} = 0$ for all time. We then deduce

$$\mathbf{v} = -\frac{M\mathbf{V}}{m} \tag{1.75}$$

without needing to use Newton's second law or any other differential equation. ■

Example 1.5

Rockets

In the preceding example the astronaut gains velocity in a direction opposite to the direction in which she throws the wrench, thereby conserving overall momentum. A rocket behaves exactly the same way, for exactly the same reason, except the single throw of a wrench is replaced by the continuous exhaust of burned fuel streaming out from the combustion chamber at the rear of the rocket. Figure 1.13 shows the rocket moving to the right in gravity-free empty space; there are no external forces, so the total momentum of the rocket plus expelled combustion gases must be conserved. At time t, shown in Figure 1.13(a), the rocket (including onboard fuel) has mass m and velocity v. Slightly later, at time $t + \Delta t$, as shown in Figure 1.13(b), the rocket has mass $m + \Delta m$ (where Δm is *negative*, since the rocket has expelled some fuel in the exhaust) and velocity $v + \Delta v$. In addition, there is now an exhaust mass $-\Delta m = |\Delta m|$, where $-\Delta m$ is positive. Note that our system of rocket plus exhaust has constant mass, which is essential here, because it only makes sense to conserve momentum for a system in which the mass stays the same.

What is the velocity of the bit of exhaust $|\Delta m|$ in the second figure? We will suppose that its velocity is u relative to the rocket, called the *exhaust velocity*, directed in the backwards direction, and so in the inertial frame in which we are viewing the rocket the rocket has velocity v (or $v + \Delta v$) to the right – it will make no difference which we choose – so the bit of exhaust has velocity $u - v$ to the *left* from our point of view. (Note that if at some instant the rocket happens to be moving to the right at speed u relative to us, then the bit of exhaust will be at rest in our frame; if the rocket is moving faster than u, the bit of exhaust will actually be moving to the right, since $u - v$ will be negative.)

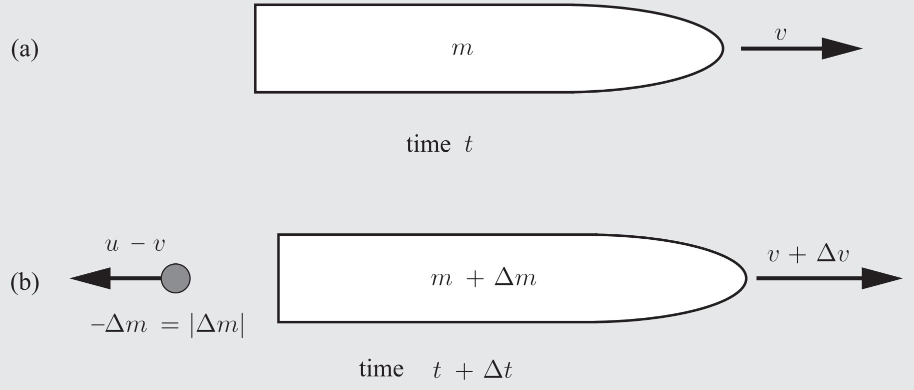


Fig. 1.13 A rocket and expelled exhaust (a) at time t and (b) at time $t + \Delta t$.

We can now conserve momentum between times t and $t + \Delta t$. That is:

$$(m + \Delta m)(v + \Delta v) - (-\Delta m)(u - v) = mv. \tag{1.76}$$

So

$$m\,\Delta v + \Delta m\,\Delta v + \Delta m\,u = 0. \tag{1.77}$$

Dividing by the brief time interval Δt and taking the limit $\Delta t \rightarrow 0$, the doubly small term $\Delta m \Delta v$ goes away in the limit, so we find that the equation of motion is

$$m(t)\frac{dv}{dt} = -u\frac{dm}{dt}. \tag{1.78}$$

This looks very similar to Newton's second law in the form $mdv/dt = F$, except that here the mass of the rocket changes with time. The "force" term on the right is called the *thrust* of the rocket:

$$\text{Thrust} \equiv -u\frac{dm}{dt}, \tag{1.79}$$

which is *positive* because the rocket mass is decreasing with time as its fuel is burned. The equation makes intuitive sense: the thrust is proportional to both the exhaust velocity and the rate at which the fuel is burned.

We can now integrate the rocket's equation of motion if we assume that the exhaust velocity u is constant. First, multiply Eq. (1.78) by dt and divide by m: this removes t as a variable, and we are left with $dv = -udm/m$. The remaining variables v and m have been separated, so we can integrate both sides:

$$\int_{v_0}^{v} dv = -u\int_{m_0}^{m}\frac{dm}{m}, \tag{1.80}$$

giving

$$v = v_0 + u\ln(m_0/m), \tag{1.81}$$

which is often called the *rocket equation*. If, for example, 90% of the initial mass of the rocket consists of fuel, while only 10% is "payload," then when all the fuel has burned the rocket has only 10% of its original mass, so its velocity has increased by

$$v - v_0 = u\ln\left(\frac{m_0}{m_{\text{payload}}}\right) = u\ln\left(\frac{m_0}{0.1m_0}\right) \simeq 2.30\,u. \tag{1.82}$$

By the end, the rocket is traveling faster than the fuel speed relative to the rocket. ∎

Finding the motion of a rocket is an example of a "variable mass" problem, called that because the mass of the object of interest (the rocket in this case) changes mass as time goes on. There are dozens of analogous problems, including for example (i) a hailstone that gains mass with time, freezing and accreting water molecules in the air as it falls; (ii) a jet aircraft whose mass increases as its wings ice up while its mass *decreases* as fuel is burned; (iii) a railroad boxcar moving along a horizontal track, open at the top and gaining mass as rain falls in, while losing mass due to a hole in the bottom of the boxcar through which water is leaking. Note that the total mass of the system does not change; it simply moves from one part of the system to another.

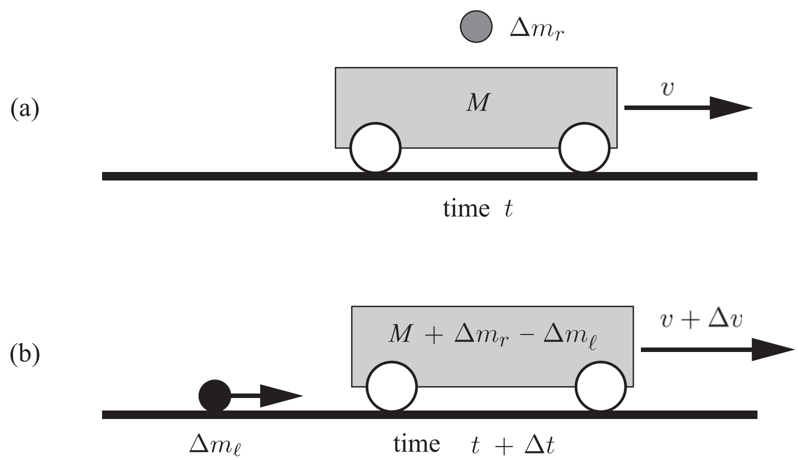


Fig. 1.14 A leaky open boxcar in a rainstorm. (a) At time t the boxcar is moving at velocity v and some raindrops of mass Δm_r are about to fall in, with no horizontal component of velocity. (b) At time $t + \Delta t$ the boxcar is moving at velocity $v + \Delta v$. The raindrops Δm_r have fallen in, and a quantity of water Δm_ℓ has leaked out, still moving with horizontal velocity v.

The technique for solving such problems is to use Newton's second law $F = dp/dt$ in the form $\Delta p = F\Delta t$ over the short time interval Δt *for a system whose mass is the same at time $t + \Delta t$ as it was at time t.* That is, we can only be confident that $F = dp/dt$ is valid if the system has fixed mass. So in the case of the boxcar, for example, we draw two pictures (see Figure 1.14). The first at time t shows a boxcar of mass M moving to the right at speed v plus a small quantity of rain of mass Δm_r falling with no horizontal velocity (its vertical velocity is irrelevant here). Thus, the horizontal momentum of the system at time t is simply $p_0 = Mv$. The second picture is at time $t + \Delta t$, and shows a boxcar of mass $M + \Delta m_r - \Delta m_\ell$, indicating that the boxcar has gained mass Δm_r due to the falling rain, while losing mass Δm_ℓ due to the leak. In this picture there is also a mass Δm_ℓ, the leaked mass, moving to the right at speed v, because it "remembers" the speed it had just before it leaked out by the law of inertia, Newton's first law. The momentum of the entire system at $t + \Delta t$ is $p_1 = (M + \Delta m_r - \Delta m_\ell)(v + \Delta v) + \Delta m_\ell v$. Now if we pretend there is no horizontal force on the system due to air resistance or friction with the tracks, the total momentum of the fixed-mass system is the same at $t + \Delta t$ as it was at time t. Therefore, setting $p_1 = p_0$:

$$(M + \Delta m_r - \Delta m_\ell)(v + \Delta v) + \Delta m_\ell v = Mv. \tag{1.83}$$

Now cancel the Mv terms, divide by Δt, and take the limit $\Delta t \to 0$. The result is the differential equation of motion of the boxcar:

$$M\frac{dv}{dt} = -\lambda_r v, \quad \text{where} \quad \lambda_r = \frac{dm_r}{dt}. \tag{1.84}$$

Here, λ_r is the rate at which rain is falling in. Note that this equation *looks* just like the equation for the bacterium subject to a linear drag force. The cause of the "drag" here is that the boxcar has to speed up the horizontal velocity of the raindrops that fall in, and the rain reacts back upon the boxcar tending to slow it down. Appearances may be deceiving, however, because in the boxcar problem M changes with time unless the rate of rainfall happens to be exactly the same as the rate of leakage. Nevertheless, we can solve the problem completely for $v(t)$ and then $x(t)$ if we assume the rates of rainfall and leaking are both constants, λ_r and λ_ℓ (see the Problems section at the end of this chapter). We can also find the differential equation of motion if there is air resistance or friction by adding nonzero forces to $\Delta p = F\Delta t$, and perhaps solve the equation exactly if F has a sufficiently simple form.

Angular Momentum

Let a position vector $\mathbf{r}$ extend from an origin of coordinates to a particle, as shown in Figure 1.15. The angular momentum of the particle is defined to be

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p}, \tag{1.85}$$

the vector cross product of $\mathbf{r}$ with the particle's momentum $\mathbf{p}$. Note that in a given inertial frame the angular momentum of the particle depends not only on properties of the particle itself, namely its mass and velocity, *but also upon our choice of origin*.

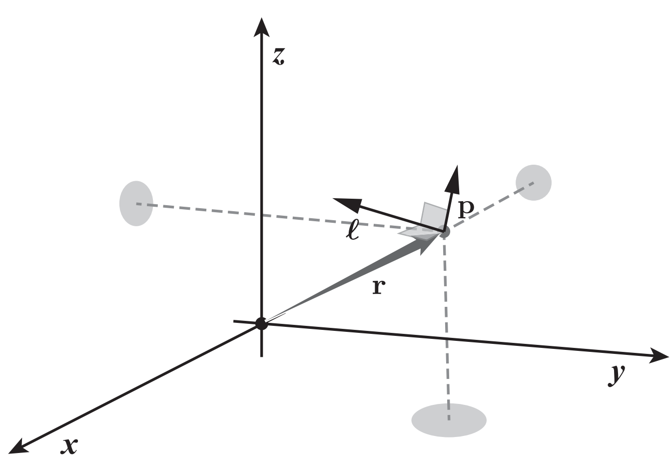


Fig. 1.15 The position vector for a particle. Angular momentum is always defined with respect to a chosen point from where the position vector originates.

Using the product rule, the time derivative of ℓ is

$$\frac{d\boldsymbol{\ell}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt}. \tag{1.86}$$

The first term on the right is $\mathbf{v} \times m\mathbf{v}$, which vanishes because the cross product of two parallel vectors is zero. In the second term, we have $d\mathbf{p}/dt = \mathbf{F}$ using Newton's second law, where $\mathbf{F}$ is the net force acting on the particle. It is therefore convenient to define the **torque** $\mathbf{N}$ on the particle due to $\mathbf{F}$ as

$$\mathbf{N} = \mathbf{r} \times \mathbf{F}, \tag{1.87}$$

so that

$$\mathbf{N} = \frac{d\boldsymbol{\ell}}{dt}. \tag{1.88}$$

That is, the net *torque* on a particle is responsible for any change in its angular momentum, just as the net *force* on the particle is responsible for any change in its momentum. The angular momentum of a particle is conserved if there is no net torque on it.

Sometimes the momentum $\mathbf{p}$ is called the "linear momentum" to distinguish it from the angular momentum $\boldsymbol{\ell}$. They have different units and are conserved under different circumstances. The momentum of a particle is conserved if there is no net external *force* and the angular momentum of the particle is conserved if there is no net external *torque*. It is easy to arrange forces on an object so that it experiences a net force but no net torque, and equally easy to arrange them so there is a net torque but no net force. For example, if the force $\mathbf{F}$ is parallel to $\mathbf{r}$, we have $\mathbf{N} = 0$; yet there is a nonzero force.

There is another striking difference between momentum and angular momentum. In a given inertial frame, the value of a particle's momentum $\mathbf{p}$ is independent of where we choose to place the origin of coordinates. But because the angular momentum $\boldsymbol{\ell}$ of the particle involves the position vector $\mathbf{r}$, the value of $\boldsymbol{\ell}$ does depend on the choice of origin. This makes angular momentum more abstract than momentum, in that in the exact same problem different people at rest in the same inertial frame may assign it different values depending on where they choose to place the origin of their coordinate system.

The angular momentum of *systems* of particles is sufficiently complex and sufficiently interesting to devote much of Chapter 12 to it. For now, we can simply say that as with linear momentum, angular momentum can be exchanged between particles in the system. The total angular momentum of a system of particles is conserved if there is no net *external* torque on the system.

Example 1.6 **A Particle in Two Dimensions Attached to a Spring**

A block of mass m is free to move on a frictionless tabletop under the influence of an attractive Hooke's-law spring force $\mathbf{F} = -k\mathbf{r}$, where the vector $\mathbf{r}$ is the position vector of the particle measured from the origin. We will find the motion $x(t), y(t)$ of the ball and show that the angular momentum of the ball about the origin is conserved.

The vector $\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}$, where x and y are the Cartesian coordinates of the ball and $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are unit vectors pointing in the positive x and positive y directions, respectively. Newton's second law $-k\mathbf{r} = m\ddot{\mathbf{r}}$ becomes

$$-k(x\hat{\mathbf{x}} + y\hat{\mathbf{y}}) = m(\ddot{x}\hat{\mathbf{x}} + \ddot{y}\hat{\mathbf{y}}), \tag{1.89}$$

which separates into the two simple harmonic oscillator equations

$$\ddot{x} + \omega_0^2 x = 0 \quad \text{and} \quad \ddot{y} + \omega_0^2 y = 0, \tag{1.90}$$

where $\omega_0 = \sqrt{k/m}$. It is interesting that the x and y motions are completely independent of one another in this case; the two coordinates have been decoupled, so we can solve the equations separately. The solutions are

$$x = A_1 \cos(\omega_0 t + \varphi_1) \quad \text{and} \quad y = A_2 \cos(\omega_0 t + \varphi_2), \tag{1.91}$$

showing that the ball oscillates simple harmonically in both directions.

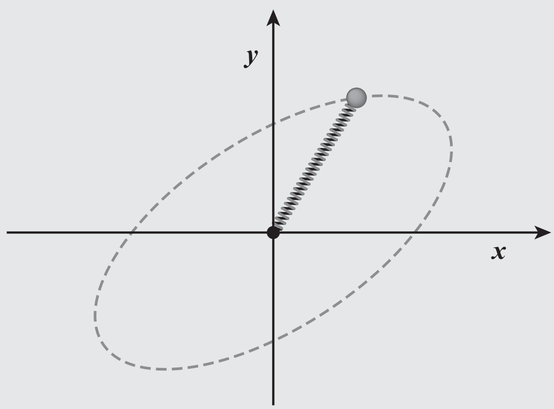


Fig. 1.16 A two-dimensional elliptical orbit of a ball subject to a Hooke's-law spring force, with one end of the spring fixed at the origin. The spring's rest length is zero.

The four constants $A_1, A_2, \varphi_1, \varphi_2$ can be evaluated in terms of the four initial conditions $x_0, y_0, v_{x_0}, v_{y_0}$. The oscillation frequencies are the same in each direction, so orbits of the ball are all closed. In fact, the orbit shapes are ellipses centered at the origin, as shown in Figure 1.16.[a] Note that in this two-dimensional problem, the motion of the ball is determined by *four* initial conditions (the two components of the position vector and the two components of the velocity vector), together with the known force throughout the motion. This is what is expected for two second-order differential equations.

The spring exerts no torque on the ball about the origin, since the cross product of any vector with itself vanishes, so $\mathbf{N} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times -k\mathbf{r} = 0$. Therefore the angular momentum of the ball is conserved about the origin. In this case, this angular momentum is given by

$$\ell = (x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}) \times (m\,\dot{x}\,\hat{\mathbf{x}} + m\,\dot{y}\,\hat{\mathbf{y}}) = (m\,x\,\dot{y} - m\,y\,\dot{x})\hat{\mathbf{z}}, \tag{1.92}$$

so the special combination $m\,x\,\dot{y} - m\,y\,\dot{x}$ remains constant for all time. That is certainly a highly nontrivial statement.

The angular momentum is *not* conserved about any other point in the plane, because then the position vector and the force vector would be neither parallel nor antiparallel. The angular momentum of a particle is *always* conserved if the force is purely central, *i.e.*, if it is always directly toward or away from a fixed point, as long as that same point is chosen as origin of the coordinate system.

We still have not used the conservation of angular momentum in this problem to our advantage, because we solved the full second-order differential equation. To see how we can tackle this problem without ever needing to invoke Newton's second law or *any* second-order differential equation, we need to first look at another very useful conservation law, the conservation of *energy.* ■

[a] Remember that the equation of an ellipse in the x–y plane can be written as

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1, \tag{1.93}$$

where (x_0, y_0) is the center of the ellipse, and a and b are the minor and major radii. One can show that Eq. (1.91) indeed satisfies this equation for appropriate relations between $\varphi_1, \varphi_2, A_1, A_2$ and x_0, y_0, a, b.

Energy

Energy is the third quantity that is sometimes conserved. Of momentum, angular momentum, and energy, energy is the most subtle and most abstract, yet it is often the most useful.

We begin by writing Newton's law for a particle in the form $\mathbf{F}_\mathrm{T} = m d\mathbf{v}/dt$, where $\mathbf{F}_\mathrm{T}$ is the total force on the particle. Dotting this equation with the particle's velocity $\mathbf{v}$:

$$\mathbf{F}_\mathrm{T} \cdot \mathbf{v} = m\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) \equiv \frac{dT}{dt}, \tag{1.94}$$

where we have defined

$$T = \frac{1}{2}mv^2 \tag{1.95}$$

as the *kinetic energy* of the particle.[4] If $\mathbf{F}$ is the force of gravity, for example, then if the particle is falling vertically its velocity is parallel to $\mathbf{F}$, so $\mathbf{F} \cdot \mathbf{v}$ is positive, causing the kinetic energy of the particle to *increase*; and if the particle is rising, its velocity is antiparallel to $\mathbf{F}$, so $\mathbf{F} \cdot \mathbf{v}$ is negative, causing the kinetic energy of the particle to *decrease*. If $\mathbf{F}_\mathbf{T}$ is the total force acting on the particle, the time rate of change

[4] In deriving Eq. (1.94), we have used the identity

$$\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = v_x\frac{dv_x}{dt} + v_y\frac{dv_y}{dt} + v_z\frac{dv_z}{dt} = \frac{1}{2}\frac{d}{dt}(v_x^2 + v_y^2 + v_z^2) = \frac{1}{2}\frac{d(v^2)}{dt}. \tag{1.96}$$

$$\frac{dT}{dt} = \mathbf{F_T} \cdot \mathbf{v} \tag{1.97}$$

is called the *net power input* to the particle.

Example 1.7

Charged Particle in a Magnetic Field

The force exerted by a magnetic field $\mathbf{B}$ on a particle of electric charge q moving with velocity $\mathbf{v}$ is given by

$$\mathbf{F}_B = q\mathbf{v} \times \mathbf{B}. \tag{1.98}$$

What is the change in a particle's kinetic energy if this is the only force acting on it?

Using the fact that the cross product of *any* two vectors is perpendicular to both vectors, it follows that $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}) = 0$. Therefore, the kinetic energy of a particle moving in a magnetic field is constant in time. Seen another way, the particle generally accelerates, but its acceleration $\mathbf{a} = q(\mathbf{v} \times \mathbf{B})/m$ is always perpendicular to $\mathbf{v}$, so the *magnitude* of $\mathbf{v}$ remains constant, and therefore the kinetic energy $T = (1/2)mv^2$ remains constant as well. The particle may move along very complicated paths, but its kinetic energy never changes. ■

We can integrate Eq. (1.94) over time to find the change in a particle's kinetic energy as it moves from some point a to another point b. The result is

$$\Delta T \equiv T_b - T_a = \int_a^b \mathbf{F}_\mathrm{T} \cdot \mathbf{v}\, dt = \int_a^b \mathbf{F}_\mathrm{T} \cdot d\mathbf{s}, \tag{1.99}$$

since $\mathbf{v} \equiv d\mathbf{s}/dt$, where $d\mathbf{s}$ is the instantaneous displacement vector. At each point on the path the vector $d\mathbf{s}$ is directed along the path, and its magnitude is an infinitesimal distance along the path.

Now define the **work** W done by any one of the forces $\mathbf{F}$ acting on the particle, as it moves from a to b, as the line integral (or *path* integral)

$$W = \int_a^b \mathbf{F} \cdot d\mathbf{s}. \tag{1.100}$$

Note from the dot product that it is only the component of $\mathbf{F}$ *parallel* to the path at some point that does work on the particle. Figure 1.17 illustrates the setup.

We can then define the total work done on the particle by all of the forces $\mathbf{F}_1$, $\mathbf{F}_2, \ldots$ to be

$$W_T = W_1 + W_2 + \cdots = \int_a^b \mathbf{F}_1 \cdots d\mathbf{s} + \int_a^b \mathbf{F}_2 \cdot d\mathbf{s} + \cdots \tag{1.101}$$

so it follows from Eq. (1.99) that

$$W_T = T_b - T_a, \tag{1.102}$$

which is known as the **work–energy theorem**: *the change in kinetic energy of a particle is equal to the total work done upon it.* If we observe that the kinetic energy of a particle has changed, there must have been a net amount of work done upon it.

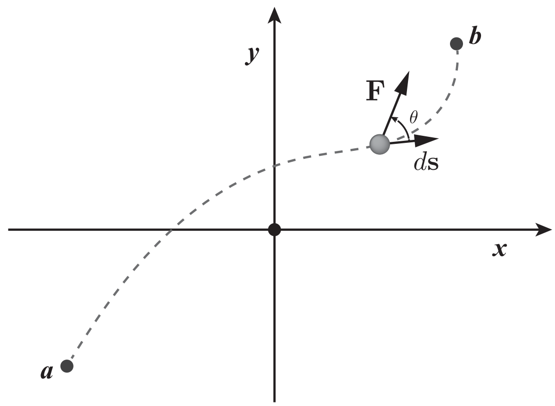


Fig. 1.17 The work done by a force on a particle is the line integral $\int_a^b \mathbf{F} \cdot d\mathbf{s}$ along the path traced by the particle.

Often the work done by a particular force **F** depends upon which path the particle takes as it moves from a to b. The frictional work done by air resistance on a ball as it flies from the bat to an outfielder depends upon how high it goes, that is, whether its total path length is short or long. There are other forces, however, like the static force of gravity, for which the work done is independent of the particle's path. For example, the work done by earth's gravity on the ball is the same no matter how it gets to the outfielder. For such forces the work depends only upon the endpoints a and b. That implies that the work can be written as the difference[5]

$$W_{\mathrm{a}\to\mathrm{b}} = -U_b + U_a \tag{1.103}$$

between a **potential energy** function U evaluated at the final point b and the initial point a.

A force **F** for which the work $W = \int_a^b \mathbf{F} \cdot d\mathbf{s}$ between any two points a and b is independent of the path is said to be **conservative**. There are several tests for conservative forces that are mathematically equivalent, in that if any one of them is true the others are true as well. The conditions are:

1 $W = \int_a^b \mathbf{F} \cdot d\mathbf{s}$ is path independent.
2 The work done around *any* closed path is $\oint \mathbf{F} \cdot d\mathbf{s} = 0$.
3 The curl of the force function vanishes: $\nabla \times \mathbf{F} = 0$.
4 The force function can always be written as the negative gradient of some scalar function U: $\mathbf{F} = -\nabla U$.

Often the third of these conditions makes the easiest test. For example, the curl of the uniform gravitational force $\mathbf{F} = -mg\,\hat{\mathbf{z}}$ is, using the determinant expression for the curl:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} = 0, \tag{1.104}$$

[5] The reason for this choice of signs will soon become clear.

since each component of $\mathbf{F}$ is zero or a constant. Therefore this force is conservative. That means it must have a potential energy given by the indefinite integral

$$U = -\int \mathbf{F} \cdot d\mathbf{s} \tag{1.105}$$
$$= -\int (-mg\hat{\mathbf{z}}) \cdot d\mathbf{s} = mg \int dz = mgz.$$

The work done by a conservative force is equal to the *difference* between two potential energies, so it follows that the physics is exactly the same for a particle with potential energy $U(\mathbf{r})$ as it is for a potential energy $U(\mathbf{r}) + C$, where C is any constant. For example, the potential energy of a particle of mass m in a uniform gravitational field g is $U_{\text{grav}} = mgh$, where h is the altitude of the particle. The fact that any constant can be added to U in this case is equivalent to the fact that it doesn't matter from what point the altitude is measured, as long as this is done consistently throughout a problem. The motion of a particle is the same whether we measure altitude from the ground or from the top of a building.

Not all forces are conservative: for example, the curl of the hypothetical force $\mathbf{F} = \alpha xy\hat{\mathbf{z}}$, where α is a constant, is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & \alpha xy \end{vmatrix}$$
$$= \hat{\mathbf{x}}\frac{\partial}{\partial y}(\alpha xy) - \hat{\mathbf{y}}\frac{\partial}{\partial x}(\alpha xy) = \alpha(x\hat{\mathbf{x}} - y\hat{\mathbf{y}}) \neq 0, \tag{1.106}$$

so this force is not conservative, and does not possess a potential energy function.

Typically both conservative ($\mathbf{F}_{\text{C}}$) and nonconservative forces ($\mathbf{F}_{\text{NC}}$) act on a particle, so the total work done on it is

$$W_{\text{T}} = W_{\text{C}} + W_{\text{NC}} = -U_{\text{b}} + U_{\text{a}} + W_{\text{NC}} = T(b) - T(a) \tag{1.107}$$

from the work–energy theorem equation (1.102), where now the potential energies U_{a} and U_{b} are the total potential energies due to all of the conservative forces. Rewriting this equation in the form

$$[T_{\text{b}} + U_{\text{b}}] - [T_{\text{a}} + U_{\text{a}}] = W_{\text{NC}}, \tag{1.108}$$

we can finally define the **energy** E of the particle as the sum of the kinetic and potential energies:

$$E \equiv T + U. \tag{1.109}$$

The change in a particle's energy as it travels from a to b is therefore

$$\Delta E = E_{\text{b}} - E_{\text{a}} = W_{\text{NC}}, \tag{1.110}$$

the total work done by nonconservative forces. The energy is *conserved*, with $E_b = E_a$, if only conservative forces act on the particle (that is, if $W_{\rm NC} = 0$).[6]

Example 1.8

A Child on a Swing

A child of mass m is being pushed on a swing. Suppose there are just four forces acting on her: (i) the normal force of the seat; (ii) the hands of the pusher; (iii) air resistance; and (iv) gravity. What is the work done by each?

(i) As long as the normal force of the swing seat is perpendicular to the instantaneous displacement, the work it does must be *zero* at all times, $\mathbf{F}_N \cdot d\mathbf{s} = 0$.

(ii) While the pusher is pushing, the force is in the direction of the displacement and $\mathbf{F} \cdot d\mathbf{s} > 0$, so the work it does is *positive*. The net work done over a complete cycle is also positive, $\oint \mathbf{F} \cdot d\mathbf{s} > 0$.

(iii) The work done by air resistance is *negative*, because air resistance is opposite to the direction of motion, and hence $\mathbf{F} \cdot d\mathbf{s} < 0$. The net work done by air resistance is therefore negative, $\oint \mathbf{F} \cdot d\mathbf{s} < 0$.

(iv) The work done by gravity is positive while she is descending, and negative while she is ascending; they exactly cancel out over a complete cycle. That is, gravity is a conservative force, or $\oint \mathbf{F} \cdot d\mathbf{s} = 0$.

The only two forces that do a net amount of work on her over a complete cycle are the hands pushing (positive) and air resistance (negative). Neither force is conservative, so $\Delta E = E_b - E_a = W_{NC} = W_{\rm hands} + W_{\rm air}$. If the right-hand side is positive (the net work done by the pusher exceeds the magnitude of the (negative) net work done by air resistance), her energy increases; but if $W_{\rm hands} < |W_{\rm air}|$, her energy decreases. If the pusher stops pushing, and if we could remove air resistance, then her energy would be conserved, continually oscillating between kinetic energy (maximum at her lowest point) and gravitational potential energy (maximum at her highest points). ∎

It is useful to expand the concept of energy beyond kinetic and potential energies by regarding the work done by nonconservative forces as external sources or sinks of the total energy. For example, in the case of the friction force, a decrease in the "mechanical energy" $T + U$ shows up in some other external form, such as heat. That is, conservation of energy is more general than one might expect from classical mechanics alone; in addition to kinetic and potential energies, there is thermal energy, the energy of deformation, energy in the electromagnetic field, and many other forms as well. Energy is a useful concept across many disparate physical systems.

Example 1.9

A Particle Attached to a Spring Revisited

We want to demonstrate the power of conservation laws in solving the previous problem of a particle of mass m confined to a two-dimensional plane and attached to a spring of force constant k (see Figure 1.16). The only force law is Hooke's law $\mathbf{F} = -k\mathbf{r}$. We can check that $\nabla \times \mathbf{F} = 0$, and then find that the potential energy for this conservative force is

[6] That, of course, is responsible for the term "conservative forces."

$$U_b - U_a = -\int_a^b \mathbf{F} \cdot d\mathbf{r} = -k \int_a^b \mathbf{r} \cdot d\mathbf{r} \Rightarrow U = \frac{1}{2} k r^2. \tag{1.111}$$

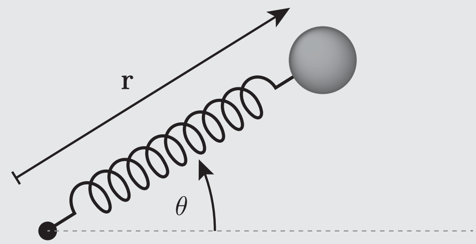


Fig. 1.18 A ball free to move in two dimensions subject to the spring force $\mathbf{F} = -k\,\mathbf{r}$. We assume the spring has natural length $r = 0$.

The total energy is therefore

$$E = \frac{1}{2} m v^2 + \frac{1}{2} k r^2. \tag{1.112}$$

The problem has rotational symmetry, so it is helpful to use polar coordinates. The velocity of the particle is

$$\mathbf{v} = \dot{r}\,\hat{\mathbf{r}} + r\,\dot{\theta}\,\hat{\boldsymbol{\theta}}, \tag{1.113}$$

where r and θ are the polar coordinates (see Appendix A for a review of coordinate systems). We then have

$$E = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2\right) + \frac{1}{2} k r^2. \tag{1.114}$$

Since E is a constant, this would be a very nice *first-order* differential equation for $r(t)$ if we could get rid of the pesky $\dot{\theta}$ term. Angular momentum conservation comes to the rescue. We know that

$$\boldsymbol{\ell} = \mathbf{r} \times (m\,\mathbf{v}) = m\,r\,\hat{\mathbf{r}} \times (\dot{r}\,\hat{\mathbf{r}} + r\,\dot{\theta}\,\hat{\boldsymbol{\theta}}) = m\,r^2\,\dot{\theta}\,\hat{\mathbf{z}} = \text{constant}. \tag{1.115}$$

We can then write

$$m\,r^2\,\dot{\theta} = \ell \Rightarrow \dot{\theta} = \frac{\ell}{m\,r^2} \tag{1.116}$$

with ℓ a constant. Putting this back into Eq. (1.114):

$$E = \frac{1}{2} m\,\dot{r}^2 + \frac{\ell^2}{2\,m\,r^2} + \frac{1}{2} k\,r^2, \tag{1.117}$$

which is a first-order differential equation from which $r(t)$ can be determined; after that we can find $\theta(t)$ using Eq. (1.116). We have thus solved the problem without ever dealing with the second-order differential equation arising from Newton's second law. This is not particularly advantageous here, given that the original second-order differential equations corresponded to harmonic oscillators. In general, however, tackling only first-order differential equations is likely to be a huge advantage.

It is instructive to analyze the boundary conditions and conservation laws of this system. Newton's second law provides two second-order differential equations in two dimensions. Each differential equation requires two boundary conditions to yield a unique solution, for a total of *four* required constants. If we use conservation laws instead, we know that both energy and angular momentum are conserved. Energy conservation provides us with a single first-order differential equation requiring a single boundary condition. But the value of energy E is another constant to be specified, so there are altogether two constants to fix using energy conservation. Angular momentum conservation gives us another first-order differential equation, with a single boundary condition plus the value ℓ of the angular momentum itself, so there are another two constants. The energy and angular momentum conservation equations together thus again require a total of *four* constants to yield a unique solution. The four boundary conditions of Newton's second law are directly related to the four constants required to solve the problem using conservation equations. ■

Example 1.10

Newtonian Central Gravity and its Potential Energy

Newton's law of gravity for the force on a "probe" particle of mass m due to a "source" particle of mass M is $\mathbf{F} = -(GMm/r^2)\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a unit vector pointing *from* the source particle *to* the probe in spherical coordinates. The minus sign means that the force is attractive, in the negative $\hat{\mathbf{r}}$ direction. We can check to see whether this force is conservative by taking its curl.

In spherical coordinates, the curl of a vector $\mathbf{F}$ in terms of unit vectors in the r, θ, and ϕ directions is

$$\nabla \times \mathbf{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}, \tag{1.118}$$

so the curl of $\mathbf{F}$ is

$$\nabla \times \left(-\frac{GMm}{r^2}\hat{\mathbf{r}}\right) = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ -GMm/r^2 & 0 & 0 \end{vmatrix} = 0. \tag{1.119}$$

Therefore, Newton's inverse-square gravitational force is conservative, and must have a corresponding potential energy function

$$U(r) = -\int \mathbf{F}\cdot d\mathbf{r} = GMm\int \frac{dr}{r^2} = -\frac{G\,M\,m}{r} + \text{constant}, \tag{1.120}$$

where by convention we ignore the constant of integration, which in effect makes $U \to 0$ as $r \to \infty$. ■

Example 1.11

Dropping a Particle in Spherical Gravity

Armed with the potential energy expression due to a spherical gravitating body of mass M, we write the total energy of a probe particle of mass m as

$$E = T + U(r) = \frac{1}{2}mv^2 - \frac{G\,M\,m}{r}, \tag{1.121}$$

which is conserved. Suppose that the probe particle is dropped from rest some distance r_0 from the center of M, which we assume is so large, $M \gg m$, that it does not move appreciably as the small mass m falls toward it. The particle has no initial tangential velocity, so it will fall radially with $v^2 = \dot{r}^2$. Energy conservation gives

$$E = \frac{1}{2}m\dot{r}^2 - \frac{G\,M\,m}{r}. \tag{1.122}$$

The initial conditions are $r = r_0$ and $\dot{r} = 0$, so it follows that

$$E = -GMm/r_0. \tag{1.123}$$

Equation (1.122) is a first-order differential equation in $r(t)$. It is said to be a "first integral" of the second-order differential equation $\mathbf{F} = m\mathbf{a}$, which in this case is

$$-\frac{GMm}{r^2} = m\ddot{r}. \tag{1.124}$$

That is, if we want to find the motion $r(t)$ it is a great advantage to begin with energy conservation, because that equation already represents one of the necessary two integrations of $\mathbf{F} = m\mathbf{a}$. Solving Eq. (1.122) for $\dot{r}$, we get

$$\dot{r} = \pm\sqrt{\frac{2}{m}\left(E + \frac{GMm}{r}\right)} = \pm\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)}. \tag{1.125}$$

We have to choose the minus sign, because when the particle is released from rest it will subsequently fall toward the origin with $\dot{r} < 0$. Separating the variables r and t and integrating both sides:

$$\int_{r_0}^{r} \frac{dr\sqrt{r}}{\sqrt{1 - r/r_0}} = -\sqrt{2\,G\,M}\int_0^t dt = -\sqrt{2\,G\,M}\,t. \tag{1.126}$$

At this point we say that the problem has been **reduced to quadrature**, an old-fashioned phrase which simply means that all that remains to find $r(t)$ (or in this case $t(r)$) is to evaluate an indefinite integral, which in the problem at hand is the integral on the left. If we are lucky, the integral can be evaluated in terms of known functions, in which case we have an *analytic* solution. If we are not so lucky, the integral can at least be evaluated numerically to any level of accuracy we need. See Chapter 14 on techniques of numerical integration.

An analytic solution of the integral in Eq. (1.126), using the substitution $r = r_0 \sin^2\theta$, gives

$$t(r) = \sqrt{\frac{r_0^3}{2\,G\,M}}\left[\frac{\pi}{2} - \sin^{-1}\sqrt{\frac{r}{r_0}} + \sqrt{\frac{r}{r_0}}\sqrt{1 - \frac{r}{r_0}}\right] \tag{1.127}$$

from which we can find the time it takes to fall to r given some initial value r_0. We cannot solve explicitly for $r(t)$ in this case, because the right-hand side is a transcendental function of r. Note that the constant r_0 in this equation is directly related to the energy E through Eq. (1.123).

The problem is much simplified if the particle falls from a great altitude to a much smaller altitude, so that $r \ll r_0$, in which case the first term in Eq. (1.127) is much bigger than the others. For example, the time it takes an astronaut to fall from rest at radius r_0 to the surface of an asteroid of radius R, where $r_0 \gg R$, is essentially

$$t = \frac{\pi}{2}\sqrt{\frac{r_0^3}{2GM}}, \tag{1.128}$$

which is independent of R! This insensitivity to the asteroid radius is due to the fact that nearly all of the travel time is spent at large radii, during which the astronaut is moving slowly. Changes in the asteroid radius R affect the overall travel time very little, because the astronaut is falling so fast near the end. On the contrary, the travel time is clearly quite sensitive to the initial position r_0. ■

Example 1.12

Potential Energies for Positive Power-Law Forces

A particle moves in one dimension subject to the power-law force $F = -kx^n$, where the coefficient k is positive, and n is a positive integer. Let us find the potential energy of the particle and also the maximum distance x_{max} it can reach from the origin, in terms of its maximum speed v_{max}. The maximum distance is the **turning point** of the particle, because as the particle approaches this position it slows down, stops at x_{max}, and turns around and heads in the opposite direction.

The potential energy of the particle is the indefinite integral

$$U = -\int F(x)dx = -\int(-kx^n)dx = \frac{k}{n+1}x^{n+1} \tag{1.129}$$

plus an arbitrary constant of integration, which we choose to be zero. Two of these potential energy functions, one with odd n and one with even n, illustrate the range of possibilities, as shown in Figure 1.19. The case $n = 1$, corresponding to a linear restoring force, corresponds to a Hooke's-law spring, where k is the spring constant and the potential energy is $U = (1/2)kx^2$. In this case the lowest possible energy is $E = 0$, when the particle is stuck at $x = 0$. There are two turning points for energies $E > 0$, one at the right and one at the left.

Energy is conserved for any value of n, where

$$E = \frac{1}{2}mv^2 + \left(\frac{k}{n+1}\right)x^{n+1}. \tag{1.130}$$

The potential energy increases with increasing positive x, so the maximum speed of the particle is at the origin, where $x = 0$ and $E = (1/2)mv_{max}^2$. The speed goes to zero at the maximum value of x attainable, *i.e.*, where $E = k\,x_{max}^{n+1}/(n+1)$. Eliminating E and solving for x_{max}, we find

$$x_{max} = \left[\frac{n+1}{2}\left(\frac{m}{k}\right)\right]^{1/(n+1)}(v_{max})^{2/(n+1)}. \tag{1.131}$$

For the spring force, which corresponds to $n = 1$, x_{max} is directly proportional to v_{max}, so if we double the particle's velocity at the origin we double the maximum x it can achieve.

Note that the conservation of energy equation (1.130) can also be solved for $v \equiv \dot{x}$ to give

$$\dot{x} = \pm\sqrt{\frac{2}{m}\left(E - \left(\frac{k}{n+1}\right)x^{n+1}\right)}, \tag{1.132}$$

which is a first-order differential equation. Dividing by the right-hand side and integrating over time yields

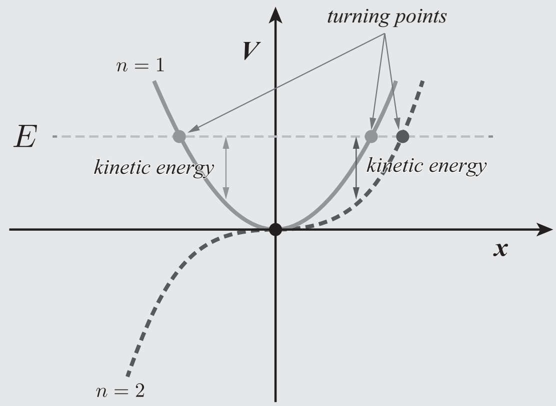


Fig. 1.19 Potential energy functions for selected positive powers n. A possible energy E is drawn as a horizontal line, since E is constant. The difference between E and $U(x)$ at any point is the value of the kinetic energy T. The kinetic energy is zero at the *turning points*, where the E line intersects $U(x)$. Note that for $n = 1$ there are two turning points for $E > 0$, but for $n = 2$ there is only a single turning point. The quadratic force with $n = 2$ has a cubic potential $U = (1/3)kx^3$ which is positive for $x > 0$ and negative for $x < 0$. Note that the slope of this potential is everywhere positive except at $x = 0$, so the force on any particle at $x \neq 0$ is toward the left, since $F = -dU/dx$ is then negative. So particles at positive x are pulled toward the origin, while particles at negative x are pushed away from the origin.

$$\int \frac{dx}{\sqrt{E - [k/(n+1)]x^{n+1}}} = \pm\sqrt{\frac{2}{m}} \int dt = \pm\sqrt{\frac{2}{m}}\, t + C, \tag{1.133}$$

where C is a constant of integration. The problem has been reduced to quadrature.

For some values of n, the integral on the left can be evaluated in terms of standard functions; this includes the cases $n = 0$ and $+1$, for example. For other values of n the integral can be evaluated numerically; that is, there are algorithms such as "Simpson's Rule" that can be implemented on a computer to provide a numerical value for the integral, given numerical values of E, k, n, and the limits of integration. Note that conservation of energy results in a first-order differential equation, so specifying the constant of integration C is equivalent to specifying a single initial condition.

Rather than integrating Eq. (1.130), which leads to Eq. (1.133), we can *differentiate* the equation instead. The time derivative of Eq. (1.130) is

$$0 = m\dot{x}\ddot{x} + \left(\frac{k}{n+1}\right)(n+1)x^n\dot{x} = 0, \tag{1.134}$$

since $dE/dt = 0$. The velocity $\dot{x}$ is not generally zero, so we can divide it out, leaving

$$m\ddot{x} = -kx^n, \tag{1.135}$$

which we recognize as $m\,a = F$ for the given force $F = -kx^n$. That is, the time derivative of the energy conservation first-order differential equation is simply $F = m\,a$, which is a second-order differential equation. Often, energy conservation serves as a **first integral of motion**, halfway toward a complete solution of the second-order equation $F = m\,a$. ■

1.9 Collisions

Collisions are commonplace: billiard balls on a billiard table, nitrogen molecules in the air, protons in a synchrotron, cars on the highway. Typically, colliding objects exert very strong equal but opposite forces on one another during a short time interval Δt, before and after which they hardly interact at all. It is true that there are usually also external forces acting on the objects during this brief time interval, such as gravity or the normal and frictional forces exerted by a pool table or road surface. However, during the brief collision times Δt such external forces are negligible compared with the internal smashing forces of one object on the other, so we can safely neglect them. Therefore, to an excellent approximation the total momentum of the colliding objects is conserved during the collision. And since their momentum is conserved, the center of mass (CM) of the colliding objects moves in a straight line at constant speed during the time just before, during, and after the collision. There is therefore an inertial frame in which the CM of the system stays at rest, called the center-of-mass (CM) frame. Analyzing the collision in the CM frame can be particularly useful.

The velocity of the CM frame in the original frame, which we will call the "lab frame," is

$$\mathbf{V}_{\mathrm{CM}} = \frac{m_0\mathbf{v}_0 + m_1\mathbf{v}_1}{m_0 + m_1} = \frac{\mathbf{P}}{M}, \tag{1.136}$$

where $\mathbf{P}$ is the total momentum and M is the total mass. Here m_0 and m_1 are the masses of the initial particles, and $\mathbf{v}_0$ and $\mathbf{v}_1$ are their velocities in the original lab frame. It is sometimes convenient to analyze the collision in the CM frame first, then transform results to the lab frame, or vice versa, using this relative velocity to transform between them.

In addition to momentum conservation, kinetic energy is sometimes also conserved in collisions, at least to a good approximation. Such kinetic-energy-conserving collisions are said to be *elastic*. Proton–proton collisions or ideal billiard-ball collisions may be nearly elastic, for example. We think of the billiard balls deforming slightly during such a collision, and then springing back to their original shape; that is, their initial kinetic energy is temporarily converted into a spring-like potential energy, and then returned to kinetic energy as soon as the balls separate. However, this is often just an approximation, although sometimes a pretty good one, because some of their initial energy is turned into oscillations within the balls themselves, which turns eventually into heat, robbing the balls of their macroscopic kinetic or potential energies. If a collision does not conserve macroscopic kinetic energy, it is said to be *inelastic*. And if the incident and target particles in a collision stick together during the collision, so two particles become one, that collision is said to be *totally inelastic.* A meteorite strikes the earth in a totally inelastic collision; the sum of their macroscopic kinetic energies decreases in the collision and the overall system becomes warmer to compensate.

There is an interesting special case, the elastic collision between two protons or two billiard balls of *equal mass*, where there is a "target" ball m_0 initially at rest, and an "incident" ball m_1 moving at velocity $\mathbf{v}_1$ toward its target in the "forward" direction, as shown in Figure 1.20. After they collide, and relative to the forward direction, ball m_1 bounces off at angle θ with velocity $\mathbf{v}'_1$, while ball m_0 moves off at angle φ with velocity $\mathbf{v}'_0$. Conservation of momentum tells us that

$$m\mathbf{v}_1 = m\mathbf{v}'_0 + m\mathbf{v}'_1 \quad \text{so} \quad \mathbf{v}_1 = \mathbf{v}'_0 + \mathbf{v}'_1, \tag{1.137}$$

while conservation of kinetic energy (for such an elastic collision) gives

$$\frac{1}{2}m(v_1)^2 = \frac{1}{2}m(v'_0)^2 + \frac{1}{2}m(v'_1)^2 \quad \text{so} \quad (v_1)^2 = (v'_0)^2 + (v'_1)^2. \tag{1.138}$$

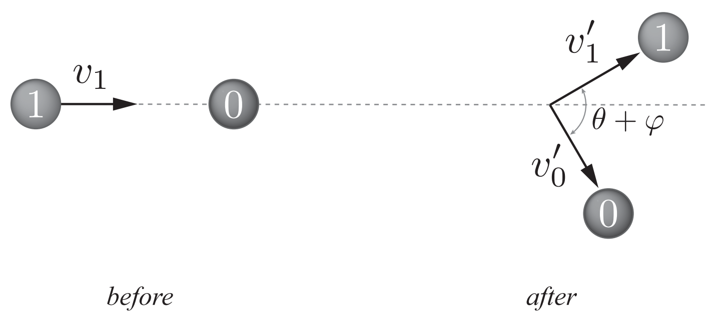


Fig. 1.20 A collision of equal-mass balls with ball 0 initially at rest. For an elastic collision, the two balls move at right angles to one another after the collision.

Squaring the conservation of momentum equation (i.e., dotting it with itself) gives

$$\mathbf{v}_1 \cdot \mathbf{v}_1 \equiv (v_1)^2 = (v'_0)^2 + 2\mathbf{v}'_0 \cdot \mathbf{v}'_1 + (v'_1)^2. \tag{1.139}$$

Comparing this last equation with the conservation of kinetic energy equation, clearly $\mathbf{v}'_0 \cdot \mathbf{v}'_1 = 0$, so the two balls must emerge from the collision in directions perpendicular to one another, with $\theta + \varphi = 90°$. The only exception occurs for an absolutely head-on collision in which the incident ball stops dead (with $\mathbf{v}'_1 = 0$) and all of its momentum and kinetic energy are transferred to ball m_0.

1.10 Forces of Nature

The hallmark of Newtonian mechanics – the relationship $\mathbf{F} = m\,\mathbf{a}$ – is only one part of a mechanics problem. To determine the dynamics of a particle, we also need to know the left-hand side of the equation. That is, we need to specify the forces. This is a separate requirement: we need to discover and learn about what forces are present through experimentation and additional theoretical considerations. We may then be tempted to ask the bold question: *what are all of the possible forces that can arise on the left-hand side of Newton's second law*? Surprisingly, this question has a complete answer at the fundamental level, an exhaustive and finite catalogue of possibilities.

To date, depending upon what one counts as a force, there are at most four *fundamental* forces in Nature, and only two of the four can be used in classical Newtonian mechanics. For the sake of completeness, let us list these four:

1 The **electromagnetic force** can be attractive or repulsive, and acts only on particles that carry a certain mysterious attribute we call "electric charge." This force is relevant from subatomic length scales to planetary length scales, and plays a role in virtually every physical setting.
2 The **gravitational force** is an omnipresent force in classical physics, which acts on anything that has mass or energy. Gravity is by far the weakest of the four forces, but at macroscopic length scales it is very noticeable nonetheless if objects are essentially electrically neutral – so that the much stronger electromagnetic force vanishes. To make things especially mysterious, our best and current theory of gravity is Einstein's theory of general relativity, and in this theory gravity is not a force at all, but an effect of the curvature of space and time. We will discuss this theory further in Chapter 10.
3 The **weak force** is subatomic in nature, acting only over very short distances, around 10^{-15} m – a regime where it is essential to use quantum mechanics. The weak force therefore plays no role in typical classical mechanics problems. The weak force is important for understanding radioactivity, neutrinos, and the Higgs boson particle. We have also learned that the weak force is closely related to electromagnetism. The electromagnetic and weak forces collectively are sometimes referred to as the **electroweak** force.
4 The **strong force**, which is also a force of subatomic relevance at around 10^{-18} m, binds quarks together and underlies nuclear energy. This is the strongest of all the forces, but in spite of its great importance it is not directly relevant to classical mechanics, since it arises in contexts requiring the use of quantum mechanics.

In summary, if we consider electromagnetism and the weak force to be two aspects of a single electroweak force, and if we take Einstein's point of view that gravity is not in fact a force at all, then we are left with only two truly fundamental forces, the electroweak and strong forces. If, however, we look at physics from the point of view of the large-scale, classical world, the forces that matter in our day-to-day experience can be taken to be gravity and electromagnetism. That is, in a setting where the strong and weak forces play a relevant dynamical role, the framework of classical mechanics itself is typically already faltering and a full extension to quantum mechanics is needed. And if we need to take account of gravitational effects more subtle or much more exotic than Newtonian gravity, the classical laws of motion have to be modified as well.

Hence, our classical mechanics world will deal primarily with Newtonian gravity and electromagnetic forces. But what can we say about the friction and spring forces encountered already in many examples, like the normal force, the tension force in a rope, and a myriad of other force laws that make prominent appearances on the left-hand side of Newton's second law? The answer is that these are all *macroscopic effective forces*, and are not fundamental. Microscopically, they originate entirely from the electromagnetic force law. For example, when two surfaces in contact rub against one another, the atoms at the interface interact microscopically through Coulomb's law of electrostatics. When we add a large number of these tiny forces, we have an effective macroscopic force that we call **friction**. The microscopic details can often, to a good approximation, be tucked into one single parameter, the coefficient of friction. Similarly, the effect of a large number of liquid molecules on a bacterium averages out into a simple force law, $F = -b\,v$, where b is the only parameter left over from the detailed microscopic interactions – which are once again electromagnetic in origin. **Contact forces**, as they are called, are again not fundamental; they originate with the electromagnetic force law.

The reader may rightfully be surprised that complicated microscopic dynamics can lead to rather simple effective force laws – often described by a few macroscopic parameters. This is a rather general feature of the natural laws. When microscopic complexity is averaged over a large number of particles and length scales, it is expected that the resulting macroscopic system is described through simpler laws with fewer parameters. This is not supposed to be obvious, although it may feel intuitive. Realization of its significance and implications in physics underlies several physics Nobel prizes in the late twentieth century.[1]

1.11 Summary

So much for our brief survey of Newtonian particle mechanics. Particles obey Newton's laws of motion, and depending upon the nature of the forces on a particle, one or another of momentum, angular momentum, and energy may be conserved.

[1] The Nobel prize for the development of the renormalization group was awarded to Kenneth G. Wilson in 1982. Wilson described most concisely and elegantly the idea that physics at large length scales can be sensitive to physics at small length scales only through a finite number of parameters. However, the idea pervades other major benchmarks of theoretical physics, such as the Nobel prizes of 1999 to Gerardus 't Hooft and Martinus J. G. Veltman and of 1965 to Sin-Itiro Tomonaga, Julian S. Schwinger, and Richard P. Feynman.

The momentum of a particle is conserved if there is no net force on it, while the angular momentum of the particle is conserved if there is no net torque on it. Energy is conserved if all the forces acting are conservative and time independent; *i.e.*, if the work done by each force is independent of the path of the particle. Similar laws apply to systems of particles.

Given the forces on a particle together with its initial position and velocity, a classical particle moves along a single, precise path. That is the vision of Isaac Newton: particles follow deterministic trajectories. When viewed from an inertial frame, a particle moves in a straight line at constant speed unless a net force is exerted on it, in which case it accelerates according to $\mathbf{a} = \mathbf{F}/m$.

We have required that the fundamental laws of mechanics obey what is called the principle of relativity, which means that if a fundamental law is valid in one inertial frame it is valid in all inertial frames. According to the principle, there is no preferred inertial frame: the fundamental laws can be used by observers at rest in any one of them. This physical statement can be translated into a mathematical statement that given a mathematical transformation of coordinates and other quantities from one frame to another, the fundamental equations should look the same in all inertial frames. We have assumed that the Galilean transformation is the correct transformation of coordinates, and have shown that Newton's laws are invariant under that transformation (provided that any particular force considered is the same in all inertial frames). It is therefore consistent to take Newton's laws as fundamental laws of mechanics.

Then what is left to do in classical mechanics? First of all, since the time of Newton extremely useful and elegant mathematical methods have been developed that give us deep insights into mechanics and may allow us to solve whole classes of problems more easily than with the methods discussed so far. These include Chapter 3 on variational methods culminating in Lagrange's approach to mechanics in Chapter 4; also the relation between symmetries and conservation laws as summarized by Noether's theorem in Chapter 6; Hamilton's equations as presented in Chapter 11; and the Hamilton–Jacobi equation in Chapter 15. Then there are a number of chapters on special cases and applications of classical mechanics, including motion in central-force gravity in Chapter 7 and in electromagnetic fields in Chapter 8; motion as viewed in non-inertial frames of reference in Chapter 9, rigid-body rotation in Chapter 12, motion of coupled oscillators in Chapter 13, and chaotic motion in Chapter 14. Finally, to illustrate how classical mechanics fits inside the larger world of physics, the path-integral approach to quantum mechanics is discussed in capstone Chapter 5, and how Newtonian physics emerges from quantum mechanics in a certain limit; also how Einstein's general theory of relativity describes the motion of particles subject to gravity in capstone Chapter 10; and then how Schrödinger discovered his famous equation of quantum mechanics using the Hamilton–Jacobi equation of classical mechanics as a guide, in final capstone Chapter 15. But before all of this, we first introduce special relativity in Chapter 2 and show how Einstein's very simple postulates have modified classical mechanics, especially for high-energy, fast-moving particles,