Engineering Mechanics

Problems and Solutions

Arshad Noor Siddiquee
Zahid Akhtar Khan
Pankul Goel

CAMBRIDGE

Engineering Mechanics

Engineering mechanics is a foundation subject. A sound understanding of the subject is required during the analysis of complex problems in several core engineering disciplines. This textbook, adapted to meet the syllabi requirements of most universities, begins with an introduction to systems of units and the representation, interaction and concatenation of forces in cartesian space. It shows ways to articulate forces and force systems, work out resultants, and equilibrium conditions through free body diagrams. Forces and moments are related to real applications by introducing beams and loading of beams. Besides applied and incident forces, friction and its related aspects are introduced. This is followed up by an analysis of trusses; the concepts of centroid and center of gravity, moment of inertia, bending moment etc. Moving beyond the mechanics of static bodies, the text takes up problems related to particles and bodies in motion with separate treatments for constant, uniform and variable accelerations. Kinematics and the concept of virtual work conclude this comprehensive introduction to engineering mechanics. The book's strength lies in its ability to relate abstract engineering concepts to real life situations.

Arshad Noor Siddiquee is a Professor at the Department of Mechanical Engineering, Jamia Millia Islamia, New Delhi. He served as Assistant Director at All India Council of Technical Education (AICTE) from 2005 to 2007. He has published more than 100 articles in national and international journals. His research interests include materials structure property correlation, welding engineering, machining, optimization of design and process parameters using fuzzy modelling.

Zahid A. Khan is a Professor at the Department of Mechanical Engineering, Jamia Millia Islamia, New Delhi. He has published more than 90 articles in national and international journals. His research interests include optimization of design and manufacturing processes parameters, artificial neural network (ANN), fuzzy modelling and environmental ergonomics.

Pankul Goel is Associate Professor at the Department of Mechanical Engineering, IMS Engineering College, Ghaziabad. He has more than 15 years experience in teaching and has also spent five years in the industry. He has published number of articles in national and international journals. His research areas include machining, friction stir welding, optimization of design and process parameters.

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Dedicated

to

Family members

Contents

Preface

Proper design and analysis of the elements of a structure, machine or installation is critical to its rigidity, safety, cost and reliability. With methods and materials of fabrication, construction and manufacturing changing rapidly, a lot of attention to design and analysis is required so that the system is robust yet cost effective. Further, 'principles of mechanics' is a foundation subject and its importance to several engineering disciplines cannot be overemphasized. A sound understanding of this subject is extremely useful during the analysis of complex problems in core engineering subjects. The subject deals with a variety of materials, in different geometries and loading configurations subjected to various types of loads. During this course, the learner needs to be presented cases similar to those encountered in real engineering problem-situations. This helps the student develop faculties to choose the right approach to analyze problems, situations and arrive at a correct solution.

Due to the importance of engineering mechanics and its applications in many engineering disciplines, it is a part of common engineering curricula. The contents of this volume have been developed to match the syllabi of major universities. This treatise is organized to provide the basic concepts in the initial chapters; advancing to subsequent application of these concepts to a variety of situations encountered in engineering problem solving. The architecture of the volume makes it a self-sufficient introduction to the subject. The book begins with an introduction to basic building blocks of this subject, such as units, system of units of force and force systems. Representation, interaction and concatenation of forces in the Cartesian space are also dealt with. Concept of planar forces is developed through concurrent and non-concurrent forces in chapters 2 and 3. These chapters demonstrate articulation of forces and force systems to resolve forces, to work out resultants, and equilibrium conditions through free body diagrams. The concept and articulation of moments and couples is developed through forces and force systems. Forces and moments are further related to real applications by introducing beams and loading of beams.

Apart from applied and incident forces as dealt with in the beginning, friction and related aspects are introduced in chapters 4 and 5. Forces induced due to friction, their estimation and articulation is presented in these chapters. Frictional problems in common cases such as in wedge, screw, belt friction, etc. are discussed in these chapters. Structural problems related to trusses are dealt with in chapter 6. Analysis of trusses subjected to various loading conditions is demonstrated through joint and section methods. Centroid and centre of gravity is introduced in chapter 7. Work out of centre and centroids of various structural sections and common geometric shapes have been articulated in this chapter, so that the learner can appreciate it during real problem solving. The centroid is further developed and related to area and mass moment of inertia. Moment of inertia is treated in chapter 8. Beginning with simple shapes and sections, the centroid is developed for more complex cases like composite sections and bodies. Analysis of beams for bending through shear force and bending moment is presented in depth, separately, in chapter 9.

After dealing with problems that are static in nature, the analysis of bodies in motion is covered in chapters 10 through 17. Problems relating to dynamics of particles and bodies in motions along rectilinear and curvilinear paths have been presented in chapters 10 and 11. Cases are further structured through treatment of constant, uniform and variable accelerations separately. Concepts are developed through treatment of situations analytically and graphically both. Kinetics of particles and bodies is elaborated through laws of motion and applicable theorems detailed in chapter 12. Chapter 13 covers, in depth, analysis on work and energy.

The abstract concepts of impulse and momentum are related to real application situations such as elastic collision. These aspects are covered in detail in chapter 14. Kinetics and kinematics is covered in chapters 15 and 16. Analysis of motions of bodies and particles in rotation is detailed in chapter 15. Motion and acceleration of bodies in rotation is elaborated through equations of motion in rotation. Analogy has also been drawn by relating linear motion with rotational motion. Chapter 16 deals with kinetics of rigid bodies and develops the concepts of moment of momentum, torque and angular momentum and relates them to the work and energy. The concluding chapter 17 details the concept of virtual work.

This treatise is rich in illustration: a large number of diagrams and worked out examples have been provided. End of chapter exercises have been included so that the reader can apply the knowledge gained in each chapter to solve un-encountered problems.

Chapter 1

Introduction

1.1 Introduction to Engineering Mechanics

Engineering Mechanics is basically a branch of mechanics, associated with the study of effect of forces acting on rigid bodies. Such forces may keep a body or bodies in rest or in motion. If a body remains in rest condition, it means the net effect of all forces acting is zero; this condition of the body is termed as static or equilibrium. However, if the body moves, it means an unbalanced force is acting and causing the motion; this condition of body is termed as dynamics. Thus, engineering mechanics is mainly concerned with the effect of forces on rigid body. However, mechanics is broadly classified into various categories depending upon type (nature) of bodies influenced by forces. The bodies may be solid (rigid or deformable) or fluid. A pictorial chart describing mechanics is given below:

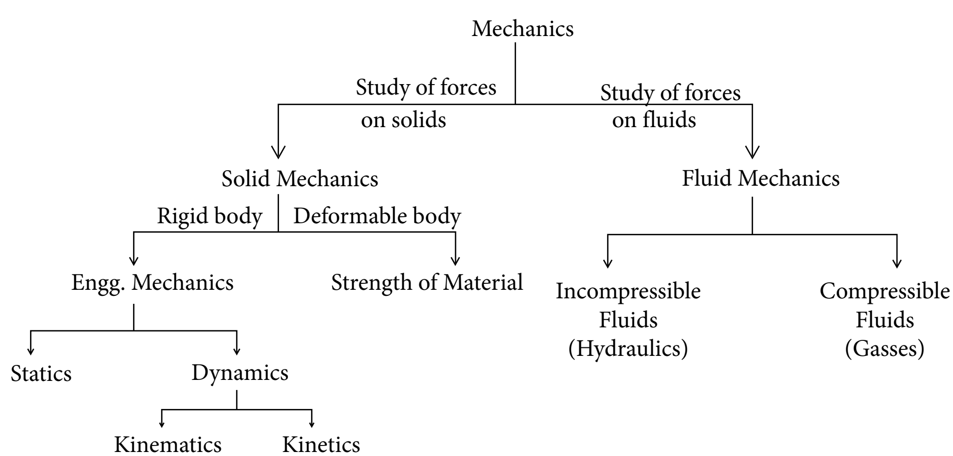


Fig. 1.1 Pictorial chart describing Mechanics

1.2 Basic Idealizations: Particle, Continuum and Rigid Body

A body has its distinct mass which is continuously distributed all over its entire volume. Sometimes the group of various components, i.e., a system; is termed as body. For example, a locomotive is modelled as a body for analysis of its motion but the locomotive is nothing but a group of various engineering components like piston, crank, connecting rod, chassis and wheels.

The different idealizations of bodies are:

Particle
A particle is defined as an individual body whose size does not affect the analysis of forces acting on it. In other words, it is nothing but a point of concentrated mass. For example a helicopter running at high altitude observed form ground, a football viewed by spectators, etc.

Continuum
Atoms and molecules are the main constituent of any matter. The analysis of forces on individual atom and molecule of a body is too complex, perhaps impossible. Thus matter is assumed to be continuously distributed and due to this average behaviour of a body can be measured. This assumption of matter distribution is termed as continuum. The continuum is valid for rigid body as well as for deformable body.

Rigid Body
In nature, there is no perfect rigid body. A rigid body is one that does not deform under the action of forces. Thus its change in volume is negligible in comparison to original volume of the body and each point remains at constant distance from other points in the body. For example, strings and belts are assumed to stretch less and treated as rigid bodies under application.

1.3 Units

Engineering mechanics and its activities involve considerable study of physical phenomena and experimental results that are quantified in terms of magnitude and a unit is attached to each physical quantity. Thus any measurement of a physical quantity is nothing but a comparison against a reference standard. For example if mass of a rod is 20 kg, it means that the mass is 20 times the magnitude of kg, as specified by the international standard unit of mass.

1.3.1 Types of units

Physical quantities are categorized into two types of units:

Fundamental or Base Units: Units which cannot be altered and have separate entities are called fundamental or base units. There are three physical quantities i.e., length, time and mass, whose units i.e., metre, sec and kg, respectively; are called fundamental or base units and extensively used in mechanics.

Derived Units: These are units which are derived or dependent on fundamental or base units. For example, the units of force, work, power, density, area, volume, velocity and acceleration; are derived units.

1.3.2 Systems of units

There are four systems of units which are as follows:

M.K.S. (Metre-Kilogram-Second) system

C.G.S. (Centimetre-Gram-Second) system

F.P.S. (Foot-Pound-Second) system

S.I. (International System) system

The units of length, mass and time in the first three systems are defined by individual name of the system. Different countries have adopted one of the first three systems based on their choice and convention followed by the scientific community. However, that has led to some confusion and inconvenience, particularly when conversion of one unit becomes necessary in order to apply in another country based on the system the latter follows. Thus to avoid confusion, a universal standard for units was framed, known as the **SI units** at the Eleventh General Conference of Weights and Measures held during 1960 in Paris. SI units in French and English is referred as Syste´me Internationale d’ Unite´s and International System of Units, respectively. This system is now widely used all over the world. In India SI units are being used since 1957 after a statutory decision.

Table 1.1 SI Units

Types of Units	Physical Quantity	Unit	Symbol
Base Units	amount of substance current length luminous intensity mass temperature time	mole ampere metre candela kilogram kelvin second	mol A m cd kg k s
Supplementary Units	plane angle solid angle	radian steradian	rad sr
Derived units with distinct name	force frequency power pressure, stress work, heat, energy	newton hertz watt pascal joule	N Hz W Pa J

Derived units in terms of Base units	acceleration activity(radioactive) area concentration luminance mass density specific volume velocity, speed volume	metre/sec 1/sec square metre mole/cubic metre candela/square metre kilogram/cubic metre cubic metre/kilogram metre/sec cubic metre	m/s^2 s^{-1} m^2 mol/m^3 cd/m^2 kg/m^3 m^3/kg m/s m^3

1.4 Scalar and Vector Quantities

All quantities involved in engineering mechanics are classified into scalar and vector quantities.

Scalar Quantities are those that can be described completely by its magnitude only, like mass, length, volume, power, temperature and time, etc. For example if a person asks a shopkeeper to pack 2 kg potatoes, it is providing sufficient information for the required task. Such quantities are called scalar quantities.

Vector Quantities are those which cannot be described completely by its magnitude only, like force, weight, moment, couple, displacement, velocity, acceleration and momentum, etc. For example, if a person is applying 20 Nm couple on cap of bottle, it does not specify whether person is tightening or loosing cap but when direction is stated clockwise or anticlockwise it serves the full information. Such quantities which are defined by both magnitude as well as direction are called vector quantities.

1.5 Force and its Characteristics

A force is a vector quantity that causes interaction between bodies. It changes or tends to change the position of a body whether the body is in rest or in motion. A force can cause a body to push, pull or twist. A force is specified by four characteristics, i.e., magnitude, point of application, line of action, and sense.

1.6 Force System

Force System is a collection or pattern or group of various forces acting on a rigid body. It is of two types and its classification depends upon number of forces acting in planes.

1.6.1 Classification of force system

The force system is classified into two categories

(a) Coplanar Force System (b) Non-Coplanar Force System

1.6.1.1 Co-planar force system

Coplanar force system is the one where a number of forces work in a single plane or common plane. It is further divided into:

(i) **Concurrent coplanar force system:** If a number of forces work through a common point in a common plane, then the force system is called concurrent coplanar force system.

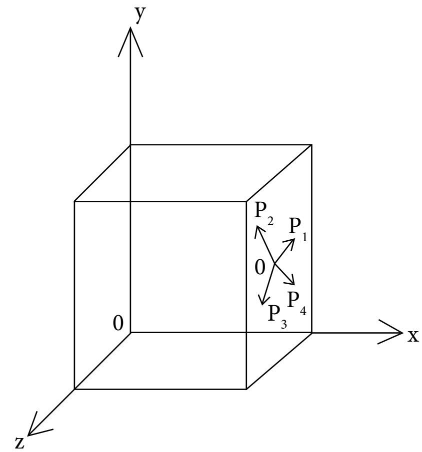


Fig. 1.2 Concurrent coplanar force system

(ii) **Collinear coplanar force system:** If a number of forces have single line of action in a common plane, then the force system is called collinear coplanar force system.

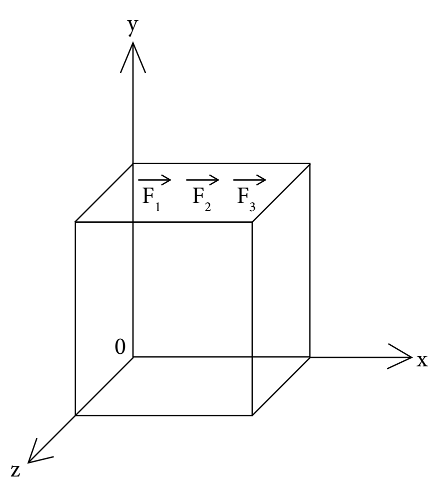


Fig. 1.3 Collinear coplanar force system

(iii) **Parallel coplanar force system:** If a number of forces have parallel line of action in a common plane, then the force system is called parallel coplanar force system.

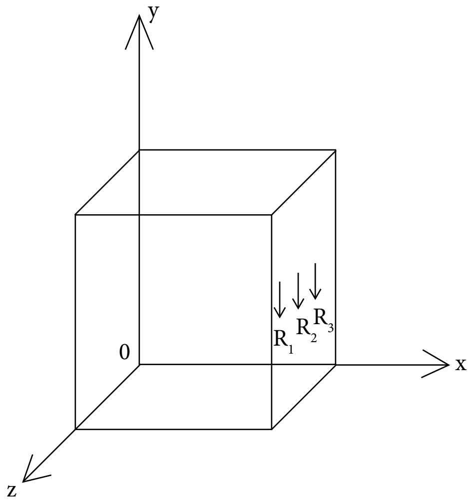


Fig. 1.4 Parallel coplanar force system

(iv) **Non-parallel coplanar force system:** If a number of forces do not have parallel line of action in a common plane, then the force system is known as non-parallel coplanar force system.

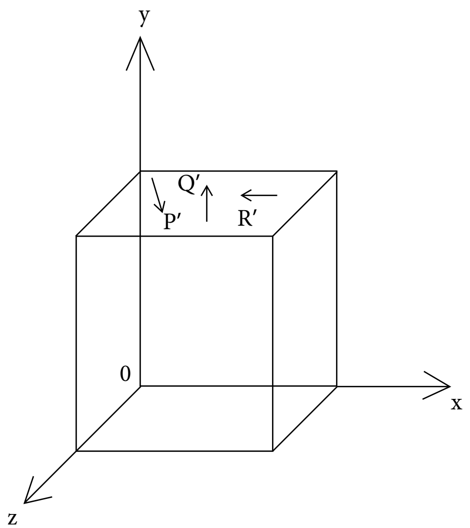


Fig. 1.5 Non parallel coplanar force system

1.6.1.2 Non-coplanar force system

It is the one where a number of forces work in different planes. It is further divided into:

(i) **Concurrent non-coplanar force system:** If a number of forces work through a common point in different planes, then the force system is called concurrent non-coplanar force system.

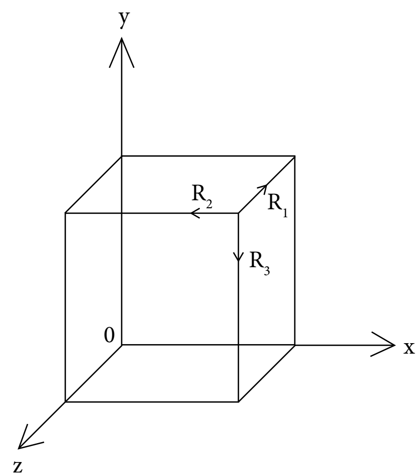


Fig. 1.6 Concurrent non-coplanar force system

(ii) **Non-parallel non-coplanar force system:** When a number of forces are having different line of action in three different planes, then the force system is called non-parallel non-coplanar force system.

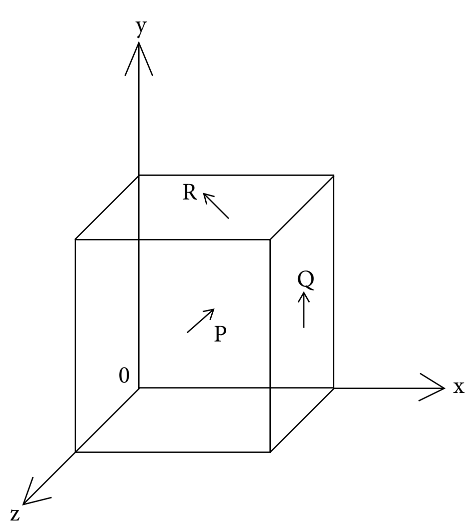


Fig. 1.7 Non-parallel non-coplanar force system

(iii) **Parallel non-coplanar force system:** When a number of forces are having parallel line of action in two different planes, then the force system is known as parallel non-coplanar force system. This force system can exist in two, two-dimensional planes only. For example: the compressive forces in four legs of a table represent this force system.

1.7 Laws of Mechanics

The whole mechanics relies on relatively few basic laws which lay down the foundation of mechanics. The laws are discussed briefly below:

- Laws of Motion
- The Gravitational Law of Attraction
- Laws of Forces

1.7.1 Laws of motion

Sir Isaac Newton was the first one who stated the three laws of motion in his treatise, ***Principia.*** These laws govern the motion of a particle and demonstrate their validity. These laws are as follows:

(i) **Newton's First law:** A particle always continues to remain at rest or in uniform motion in a straight line in the absence of applied force. This law is also known as the law of inertia.
(ii) **Newton's Second law:** If a particle is subjected to force, the magnitude of the acceleration will be directly proportional to the magnitude of the force and inversely proportional to the mass and lies in the direction of the force.
(iii) **Newton's Third law:** Every action is encountered with equal and opposite reaction, or two interacting bodies have forces of action and reaction collinearly equal in magnitude but opposite in direction.

1.7.2 The gravitational law of attraction

This law was formulated by Newton. The gravitational force on a body, computed by using this formula, reflects the weight of the body. The law is expressed by equation

$$F = G\frac{m_1 m_2}{r^2}$$

where G = universal gravitational constant
F = force of attraction between two particles
m_1, m_2 = masses of two particles
r = distance between the centres of two particles

1.7.3 Laws of forces

1.7.3.1 Parallelogram law

This law states that if two forces acting a point are represented as per magnitude and direction by two adjacent sides of a parallelogram, then the diagonal of such parallelogram will represent their resultant force in magnitude and direction. The parallelogram law can determine the resultant by two ways, one graphically and second analytically.

Consider a body where two forces P and Q are acting at a point O as shown in Fig. 1.8.

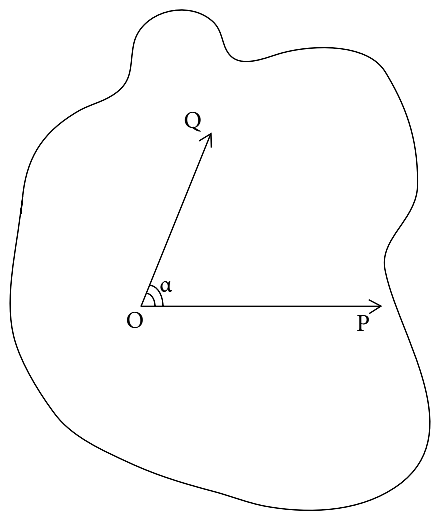


Fig. 1.8 Two concurrent forces acting on rigid body

In the graphical method, as shown in Fig. 1.9, if these forces are represented according to a suitable scale based on their magnitude and direction (α) by sides of parallelogram OA and OB respectively, then length of diagonal OC represents magnitude of the resultant and ∠COA (θ) represent its direction which can be measured by using protector.

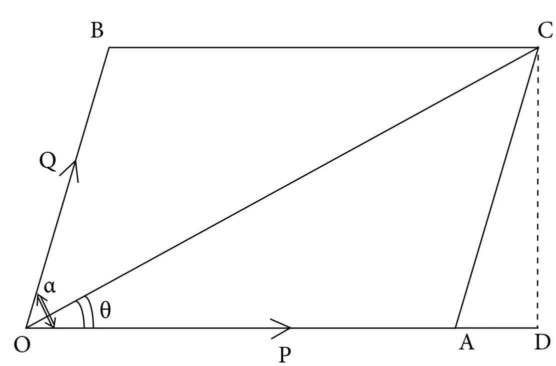


Fig. 1.9 Two concurrent forces represented by parallelogram

In analytical method it can be determined from Fig. 1.9 as follows:

$$\begin{aligned}
OC^2 &= OD^2 + CD^2 \\
&= (OA + AD)^2 + CD^2 \\
&= OA^2 + AD^2 + 2.OA.OD + CD^2 \\
&= P^2 + Q^2\cos^2\alpha + 2.P.Q.\cos\alpha + Q^2\sin^2\alpha \\
R^2 &= P^2 + Q^2 + 2.P.Q.\cos\alpha
\end{aligned}$$

Thus resultant, $R = \sqrt{P^2 + Q^2 + 2.P.Q.\cos\alpha}$

and direction, $\tan\theta = CD/OD$

$$\theta = \tan^{-1}\left(\frac{Q\sin\alpha}{P + Q\cos\alpha}\right)$$

1.7.3.2 Triangle law

This law states that if two forces acting at a point are represented as per magnitude and direction taken in order than the closing side of triangle taken from starting point to last point represents the resultant force.

Consider a body where two forces R_1 and R_2 are acting at a point O as shown in Fig. 1.10. The closing side AC of triangle as shown in Fig. 1.11 represents magnitude of the resultant force, R_3 and ∠CAB represents direction of resultant force.

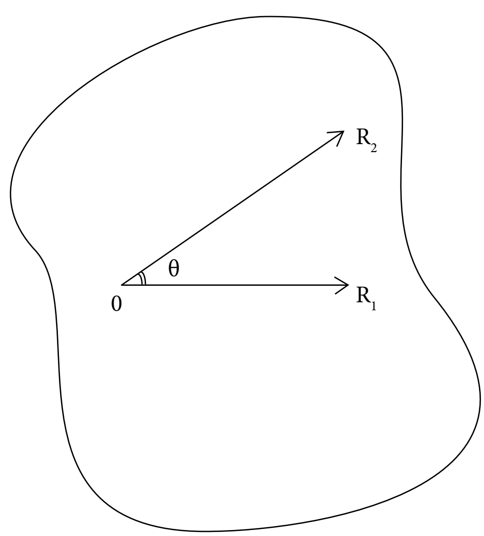


Fig. 1.10 Two concurrent forces acting on rigid body

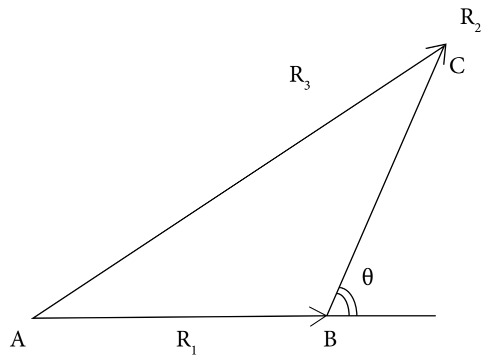


Fig. 1.11 Two concurrent forces representing triangle law

1.7.3.3 Polygon law

This law states that if a number of forces acting at a point are represented as per their magnitude and direction in the correct order, then the closing side from the starting point of first force to the last point of the last force represents the resultant force.

Consider a body where four forces P_1, P_2, P_3 and P_4 are acting at O as shown in Fig. 1.12.

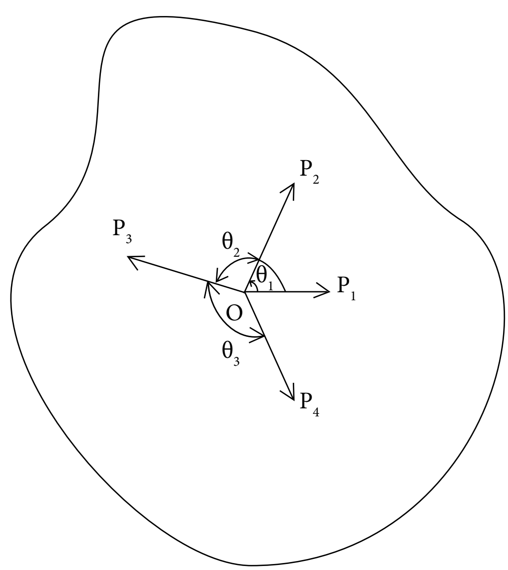


Fig. 1.12 Concurrent forces acting on rigid body

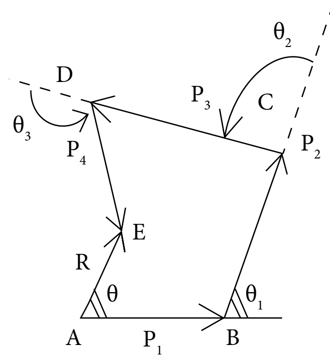


Fig. 1.13 Determination of resultant using polygon law

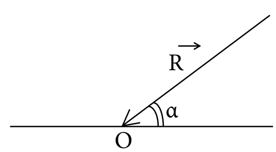
The length of the closing side AE of polygon as shown in Fig. 1.13 represents magnitude of the resultant force (R) and ∠EAB (θ) represents the direction of the resultant force.

1.8 Vector Algebra

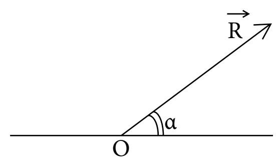
This section includes representation of vectors, classification of vectors and various mathematical operations like addition, subtraction and multiplication of vectors.

1.8.1 Vectors' representation

Graphically, a vector is drawn form a point of application and represented by a line segment whose length is proportional to its magnitude. The direction of vector is represented by its line of action made with reference line equal to angle of inclination. The arrowhead shows sensing of a vector i.e., vector is approaching towards or away to the point of application. Consider Figs. 1.14(a) and (b) as shown below:



(a)
Vector approaching towards point of application O.



(b)
Vector acting away from point of application O.

Fig. 1.14 Vector representation

However, a vector is either written by thick bold uppercase letter (**R**) or by an uppercase letter with an arrow over it ($\vec{R}$). Here an uppercase letter without arrow or bold feature represents the magnitude of vector

i.e., for vector $\vec{R} = |\vec{R}| = (\vec{R})$

and (R) represents the magnitude of vector.

1.8.2 Classification of vectors

Vectors are broadly classified under three categories, like sliding, fixed and free vectors. However depending upon their magnitude they are further classified as unit vector and null vector. The classification is described below:

Sliding Vector is a vector which can move along its line of action without altering its magnitude, direction and sensing. The point of application can lie anywhere along its line of action. For example, consider Fig. 1.15 where force P acting at point A of a rigid body can be transferred about new point of application B, C and so on, along its line of action without changing any effect.

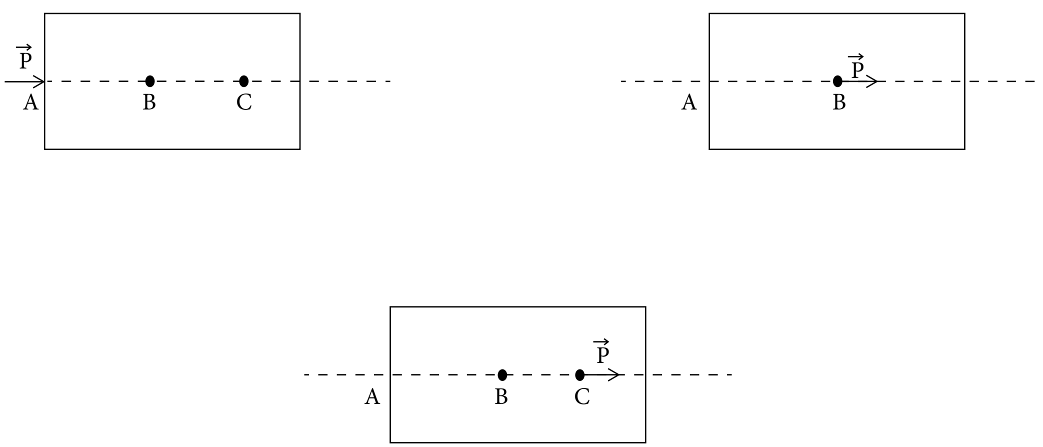


Fig. 1.15 Sliding Vector

Fixed Vector is vector whose point of application remains fixed, i.e., it cannot move without altering the conditions of the problem. Such vectors have specific magnitude, direction and line of action always passes through a particular point in space. The moment value of a force about a point represents fixed vector. For example, consider Fig. 1.16 where force P is acting at point A of a cantilever beam OA. The moment value of a force P about point O remains fixed vector. The force P cannot be transferred about new point of application B, C as its effect will change. Consider Fig. 1.16:

M_o when force P acts at A = P × 7
M_o when force P acts at B = P × 4
M_o when force P acts at C = P × 2

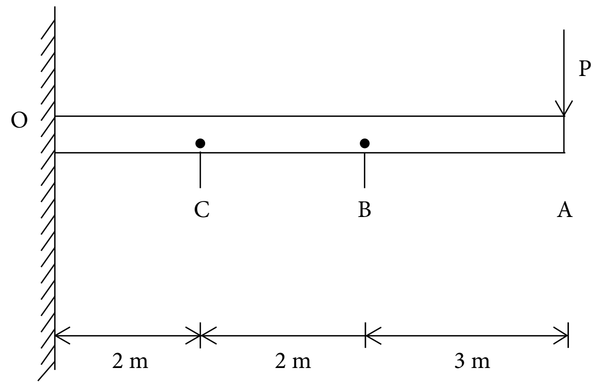


Fig. 1.16 Fixed Vector

Free Vector is a vector which can move freely in any plane or space. It has specific magnitude, direction and sensing but its line of action does not pass through a particular point in space. A couple is considered as free vector because it can move anywhere throughout in a plane without altering its effect. For example, consider Fig. 1.17 where Couple C acting at point O on a simple supported beam AB. The couple C can be transferred about any new point O_1 on the beam AB and its effect on beam will remain same.

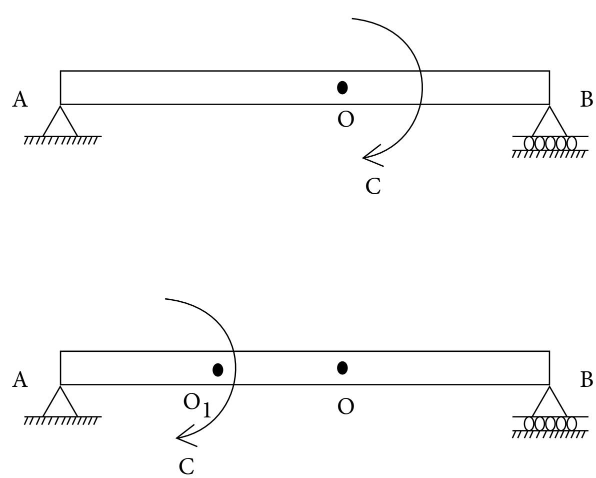


Fig. 1.17 Free Vector

Unit Vector is one whose magnitude is equal to one i.e., $\hat{n}=1$. It is a dimensionless quantity. It is shown by $\hat{n}$ where a hat (^) is placed over letter *n*. The other vectors can be shown in terms of unit vector by product of their magnitude with unit vector along their direction as shown below:

$$\vec{O}=|\vec{O}|\hat{n}=O\hat{n}$$

The unit vector along the direction of any vector may be obtained as

$$\hat{n}=\frac{\vec{O}}{|\vec{O}|}$$

The main importance of unit vector is to denote a direction in space. The unit vectors along the rectangular coordinate axis i.e., X, Y and Z are represented by $\hat{i},\hat{j}$ and $\hat{k}$ as shown in Fig. 1.18

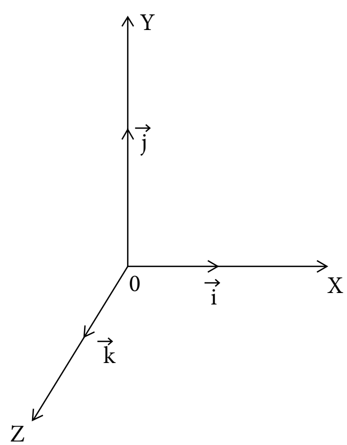


Fig. 1.18 Unit vector along rectangular coordinate axis

Null Vector is one whose magnitude is zero and it is similar to zero in scalar.

Vectors' equality is said when two vectors have same magnitude and direction. However, the point of application may be different like O_1 and O_2 as shown in Fig. 1.19.

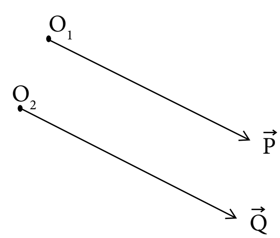


Fig. 1.19 Equal Vectors

Negative Vector is said about a vector when it is reversed from its direction as shown in Fig. 1.20.

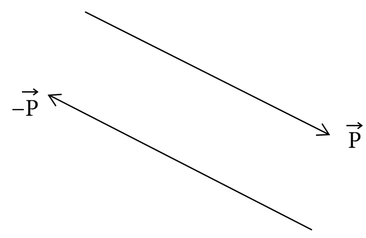


Fig. 1.20 Negative Vector

1.8.3 Vector operations

These include addition, subtraction and multiplication of vectors and are as follows:

1.8.3.1 Addition of vectors

The addition of vectors can be done graphically by using parallelogram law or triangle law.

According to the **parallelogram law,** if two concurrent vectors are represented by two adjacent sides of a parallelogram then its diagonal represents the addition of vectors in magnitude and direction. Consider Fig. 1.21 shows the addition of vectors where resultant is represented by diagonal AC which is equal to the summation of vectors AB and BC.

i.e.,

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AB} + \overrightarrow{BC}\\ &= \vec{P} + \vec{Q}\end{aligned}$$

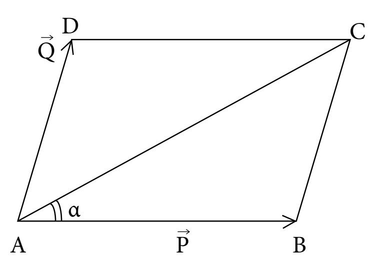


Fig. 1.21 Addition of vectors by parallelogram law

According to the **triangle law,** if two concurrent vectors are represented as per magnitude and direction in order than the closing side of triangle taken from starting point to last point represents the addition of vectors. Consider Fig. 1.22 shows the addition of vectors.

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AB} + \overrightarrow{BC}\\ &= \vec{P} + \vec{Q}\end{aligned}$$

Where α shows direction of addition of vectors

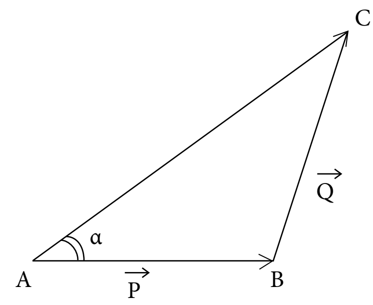


Fig. 1.22 Addition of vectors by triangle law

1.8.3.2 Subtraction of vectors

Vectors can be subtracted by addition of one vector with reversing the second vector (negative vector).This is illustrated as shown in Fig. 1.23.

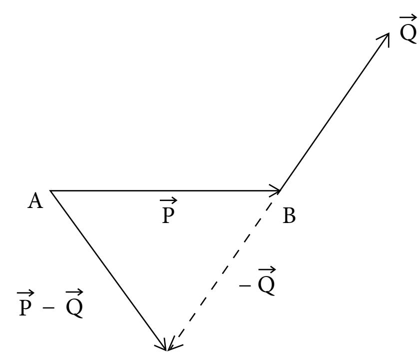


Fig. 1.23 Subtraction of vectors

1.8.3.3 Multiplication of vectors

It is of two types

(i) Dot or scalar product (ii) Cross or vector product

Dot or Scalar Product

The dot product of two vectors is equal to the product of their magnitudes and the cosine of the angle between them. If $\vec{p}$ *and* $\vec{q}$ are two vectors then dot product will be

$$\vec{p}.\vec{q} = pq\cos\alpha$$

which is called a scalar quantity because *p* and *q* are scalar quantities and *cos α* is a pure numeric value.

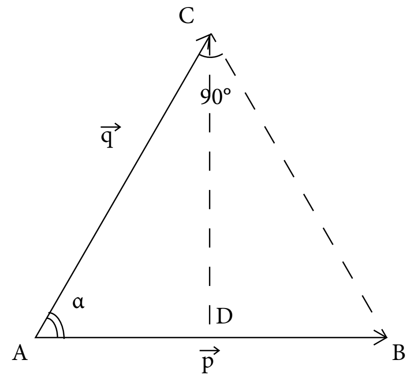


Fig. 1.24 Projection of Vectors in dot product

It can be observed from Fig. 1.24,

AC = projection of vector p on vector q = AB$cos\alpha = pcos\alpha$

Similarly,

AD = projection of vector q on vector p = AC$cos\alpha = qcos\alpha$

Consider Fig. 1.24 where the value of dot product is shown graphically:

$$
\begin{aligned}
\vec{p}.\vec{q} &= pq\cos\alpha \\
&= q(p\cos\alpha) \\
&= q(AC \text{ i.e projection of } \vec{p} \text{ on } \vec{q}) \\
\text{or } &= p(AD \text{ i.e projection of } \vec{p} \text{ on } \vec{q})
\end{aligned}
$$

Thus it can be concluded that the dot product of two vectors is the product of magnitude of one vector and the projection of another vector over the initial vector.

Note:

1. If α = 0° then both vectors $\vec{p}$ *and* $\vec{q}$ are collinear and their dot product will be equal to product of their magnitude, i.e.

$$\vec{p}.\vec{q} = pq\cos\alpha = pq\cos 0 = pq$$

Thus *if* $\hat{i}.\hat{j}$ and $\hat{k}$ are unit vectors along the rectangular coordinate axis X, Y and Z, then dot product between identical vector will be

$$\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1 \; as \; \alpha = 0^\circ$$

2. If α = 90° then both vectors $\vec{p}$ *and* $\vec{q}$ are perpendicular to each other and their dot product will be equal to zero i.e.

$$\vec{p}.\vec{q} = pq\cos\alpha = pq\cos 90^\circ = 0$$

Thus for unit vectors $\hat{i}.\hat{j}$ and $\hat{k}$, the dot product between different vectors will be

$$\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0 \; as \; \alpha = 90^\circ$$

3. The dot product is used to define **work** in dynamics Section.
4. If two vectors are given as

$$\vec{p} = p_x\check{i} + p_y\check{j} + p_z\check{k} \text{ and } \vec{q} = q_x\check{i} + q_y\check{j} + q_z\check{k}$$

$$\vec{p}.\vec{q} = \left(p_x\check{i} + p_y\check{j} + p_z\check{k}\right).\left(q_x\check{i} + q_y\check{j} + q_z\check{k}\right)$$

$$= \left(p_xq_x + p_yq_y + p_zq_z\right)$$

Cross or Vector Product

The cross product of two vectors is equal to the product of their magnitudes and the sine of the angle between them; however its direction is perpendicular to the plane containing vectors and can be determine with right hand screw rule. If $\vec{p}$ and $\vec{q}$ are two vectors then the cross product will be

$$\vec{p} \times \vec{q} = pq\sin\alpha\,\hat{n}$$

Where $\hat{n}$ is a unit vector and α (≤ 180°) is the angle between two vectors.
The magnitude of cross product is given by

$$|\vec{p} \times \vec{q}| = pq\sin\alpha$$

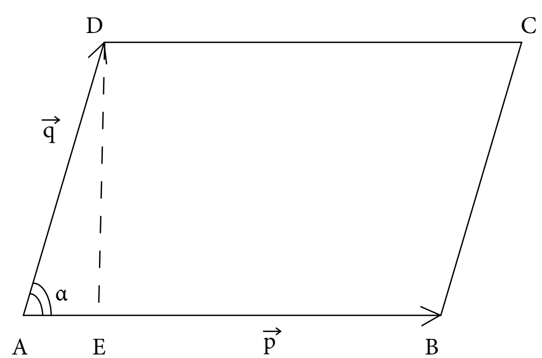


Fig. 1.25 Projection of Vectors in cross product

Consider Fig. 1.25 where the magnitude of cross product is shown graphically:

$$|\vec{p} \times \vec{q}| = pq\sin\alpha$$

where p and q are magnitude (of vectors $\vec{p}$ and $\vec{q}$) represented by AB and AD respectively and $DE = AD\sin\alpha$

$$|\vec{p} \times \vec{q}| = (AB)(q\sin\alpha)$$
$$|\vec{p} \times \vec{q}| = (AB)(AD\sin\alpha)$$
$$|\vec{p} \times \vec{q}| = (AB)(DE)$$
$$|\vec{p} \times \vec{q}| = \text{area of parallelogram ABCD}$$

Thus it can be concluded that the magnitude of cross product of two vectors is equal to the area of parallelogram whose adjacent sides are formed by two vectors.

Note:

1. If α = 0° then both vector $\vec{p}$ and $\vec{q}$ are collinear and their cross product will be equal to zero

$$\vec{p}\times\vec{q} = pq\sin\alpha\hat{n}$$
$$\vec{p}\times\vec{q} = pq\sin 0\hat{n} = 0$$

Thus *if* $\hat{i}.\hat{j}$ and $\hat{k}$ are units vectors along the rectangular coordinate axis X, Y and Z then cross product between identical vector will be

$$\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = 0 \text{ as } \alpha = 0^\circ$$

2. If α = 90° then both vectors $\vec{p}$ *and* $\vec{q}$ are perpendicular to each other and their cross product will be equal to product of their magnitude, i.e.

$$\vec{p}\times\vec{q} = pq\sin\alpha\hat{n}$$
$$\vec{p}\times\vec{q} = pq\sin 90^\circ\hat{n} = p.q\hat{n}$$

Thus for unit vectors $\hat{i}.\hat{j}$ and $\hat{k}$, the cross product between different vectors will be

$$\hat{i}\times\hat{j} = \hat{k};\ \hat{j}\times\hat{k} = \hat{i};\ \hat{k}\times\hat{i} = \hat{j} \text{ as } \alpha = 90^\circ$$

3. The cross product is used to define **moment** in statics.

4. If $\vec{p} = p_x\hat{i} + p_y\hat{j} + p_z\hat{k}$ and $\vec{q} = q_x\hat{i} + q_y\hat{j} + q_z\hat{k}$ the cross product in terms of rectangular components is defined as

$$\vec{p}\times\vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p_x & p_y & p_z \\ q_x & q_y & q_z \end{vmatrix}$$
$$\vec{p}\times\vec{q} = \left(p_y q_z - p_z q_y\right)\hat{i} - \left(p_x q_z - p_z q_x\right)\hat{j} + \left(p_x q_y - p_y q_x\right)\hat{k}$$

1.8.4 Vectorial representation of component of force

Force is a vector quantity as it is defined by both magnitude and direction. Consider a force R is shown by vector OA. If vector OA is extended in three-dimensional space, than it can be expressed in term of its components as R_x, R_y and R_z as shown in the Fig. 1.26.

In term of unit vectors force R is expressed as

$$\vec{R} = R_x\hat{i} + R_y\hat{j} + R_z\hat{k}$$

and the magnitude is given by

$$\left|\vec{R}\right| = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

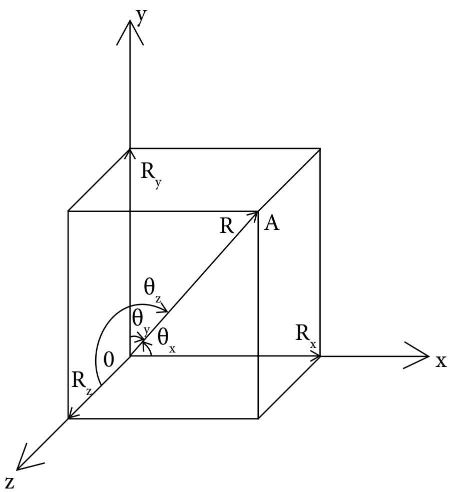


Fig. 1.26 Component of a force in three dimensions

If force R make angles θ_x, θ_y and θ_z with the rectangular coordinate axis X, Y and Z then it is expressed as

$$R = R\cos\theta_x \hat{i} + R\cos\theta_y \hat{j} + R\cos\theta_z \hat{k}$$

Force R can be expressed in terms of unit vector as

$$\vec{R} = R\hat{n}$$

Thus comparing two equations of R, unit vector is given by

$$\hat{n} = \cos\theta_x \hat{i} + \cos\theta_y \hat{j} + \cos\theta_z \hat{k}$$

As the magnitude of unit vector is unity thus

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$$

Where $\cos\theta_x$, $\cos\theta_y$ and $\cos\theta_z$ are known as direction of cosines of force R.

The direction of cosines can be further determined as

$$\cos\theta_x = \frac{R_x}{R},\ \cos\theta_y = \frac{R_y}{R} \text{ and } \cos\theta_z = \frac{R_z}{R}$$

1.8.5 Vectorial representation of force passing through two points in space

The earlier sections has dealt with where force vector acting at origin in space but generally force passes through any two points in space instead through origin. Consider Fig. 1.27 where force R passes through two points P and Q which are having coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively in space. Thus the position vectors of both points P and Q from origin will be

$\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

Similarly,

$\overrightarrow{OQ} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

Force $\vec{R}$ is given by $\overrightarrow{PQ}$

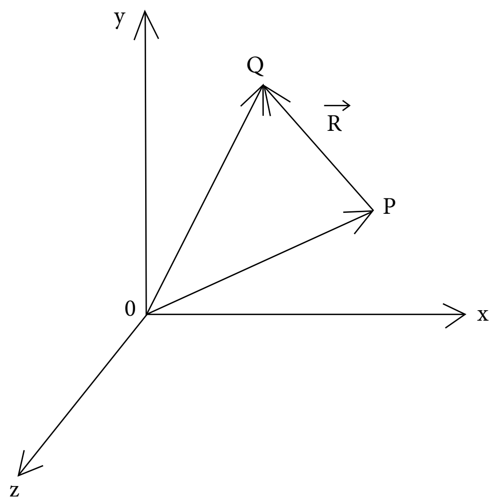


Fig. 1.27 Force passing through two points in space

The unit vector along PQ is given by $\hat{n}_{PQ} = \dfrac{\overrightarrow{PQ}}{\left|\overrightarrow{PQ}\right|}$

Thus Force $\vec{R} = R\hat{n}_{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

Example: 1.1

Two vectors acting parallel are given by

$\vec{p} = 3\vec{i} + 5\vec{j} - 2\vec{k}$ and $\vec{q} = 2\vec{i} + \vec{j} + 4\vec{k}$

Determine:

(i) Magnitude of vector $\vec{p}$ and $\vec{q}$

(ii) Unit vectors along their directions

Solution:

(i) Magnitude of vector, $\vec{p} = |\vec{p}| = p = \sqrt{(3)^2 + (5)^2 + (-2)^2}$

$$p = \sqrt{9 + 25 + 4}$$

$$p = \sqrt{38} \text{ units}$$

Similarly,

$$\text{Magnitude of vector } \vec{q} = |\vec{q}| = q = \sqrt{(2)^2 + (1)^2 + (4)^2}$$

$$q = \sqrt{4+1+16}$$

$$q = \sqrt{21} \text{ units}$$

(ii) Unit vector along the direction of vector $\vec{p}$ is given as

$$\hat{n} = \frac{\vec{p}}{|\vec{p}|} = \frac{3\vec{i} + 5\vec{j} - 2\vec{k}}{\sqrt{38}}$$

Similarly,
Unit vector along the direction of vector $\vec{q}$ is given by

$$\hat{n} = \frac{\vec{q}}{|\vec{q}|} = \frac{2\vec{i} + \vec{j} + 4\vec{k}}{\sqrt{21}}$$

Example: 1.2

In the previous question, if a vector $\vec{R} = 2\vec{p} - \vec{q}$ then determine magnitude and unit vector along its direction.

Solution:

$$\vec{R} = 2\vec{p} - \vec{q}$$

$$\vec{R} = 2\left(3\vec{i} + 5\vec{j} - 2\vec{k}\right) - \left(2\vec{i} + \vec{j} + 4\vec{k}\right)$$

$$= 6\vec{i} + 10\vec{j} - 4\vec{k} - 2\vec{i} - \vec{j} - 4\vec{k}$$

$$\vec{R} = \left(4\vec{i} + 9\vec{j} - 8\vec{k}\right)$$

$$\text{The magnitude of } \vec{R} = \left|\vec{R}\right| = R = \sqrt{(4)^2 + (9)^2 + (-8)^2}$$

$$= \sqrt{16+81+64}$$

$$= \sqrt{161} \text{ units}$$

Unit vector along its direction is given by

$$\hat{n} = \frac{\vec{R}}{\left|\vec{R}\right|}$$

$$\hat{n} = \frac{\left(4\vec{i} + 9\vec{j} - 8\vec{k}\right)}{\sqrt{161}}$$

Example: 1.3

If $\vec{p}=\vec{i}+2\vec{j}+3\vec{k}$ and $\vec{q}=2\vec{i}-3\vec{j}+4\vec{k}$ then find $\vec{p}.\vec{q}$ and the angle between them. Also determine the projection of $\vec{q}$ on $\vec{p}$.

Solution:

The dot product of two vectors is given by

$$\vec{p}.\vec{q}=\left(\vec{i}+2\vec{j}+3\vec{k}\right).\left(2\vec{i}-3\vec{j}+4\vec{k}\right)$$
$$=2-6+12$$
$$\vec{p}.\vec{q}=8$$

However, magnitude of vector $\vec{p}=\left|\vec{p}\right|=\sqrt{(1)^2+(2)^2+(3)^2}$

$$p=\sqrt{14}$$

Similarly, $q=\sqrt{(2)^2+(-3)^2+(4)^2}$

$$=\sqrt{29}$$

We know that the dot product is given by

$$\vec{p}.\vec{q}=|\vec{p}||\vec{q}|\cos\theta$$
$$=pq\cos\theta$$
$$\text{or }\cos\theta=\frac{\vec{p}.\vec{q}}{pq}=\frac{8}{\sqrt{14}\sqrt{29}}$$
$$\cos\theta=0.397 \text{ or } \theta=66.61°$$

$$\text{projection of } \vec{q} \text{ on } \vec{p}=\frac{\vec{p}.\vec{q}}{|\vec{p}|}=\frac{8}{\sqrt{14}}=2.14$$

Example: 1.4

Determine the angle between two vectors

$\left(\vec{p}+\vec{q}\right)$ and $\left(\vec{p}-\vec{q}\right)$

if $\vec{p}=\vec{i}-2\vec{j}+3\vec{k}$ and $\vec{q}=2\vec{i}+\vec{j}+\vec{k}$

Solution:

For given vectors $\vec{p}$ & $\vec{q}$,

$$\vec{p}+\vec{q}=\left(\vec{i}-2\vec{j}+3\vec{k}\right)+\left(2\vec{i}+\vec{j}+\vec{k}\right)$$
$$=\left(3\vec{i}-\vec{j}+4\vec{k}\right)$$

and $\vec{p}-\vec{q}=\left(\vec{i}-2\vec{j}+3\vec{k}\right)-\left(2\vec{i}+\vec{j}+\vec{k}\right)$

$$=\left(-\vec{i}-3\vec{j}+2\vec{k}\right)$$

We know that

$$(\vec{p}+\vec{q}).(\vec{p}-\vec{q})=|\vec{p}+\vec{q}||\vec{p}-\vec{q}|\cos\theta$$

$$\cos\theta=\frac{(\vec{p}+\vec{q}).(\vec{p}-\vec{q})}{|\vec{p}+\vec{q}||\vec{p}-\vec{q}|} \qquad \text{..... (1)}$$

Thus, $|\vec{p}+\vec{q}|=\sqrt{(3)^2+(-1)^2+(4)^2}=\sqrt{26}$

and $|\vec{p}-\vec{q}|=\sqrt{(-1)^2+(-3)^2+(2)^2}=\sqrt{14}$

from equation (1),

$$\cos\theta=\frac{(-3+3+8)}{\sqrt{26}\sqrt{14}}=0.42$$

$$\theta=65.21^\circ$$

Example: 1.5

Determine the cross product of two vectors and the angle between them if vectors are given by

$\vec{p}=\vec{i}+2\vec{j}+5\vec{k}$ and $\vec{q}=2\vec{i}-3\vec{j}+2\vec{k}$

Solution:

The cross product is given by

$$\vec{p}\times\vec{q}=pq\sin\theta\hat{n}$$

or $|\vec{p}\times\vec{q}|=pq\sin\theta$ ..... (1)

$$\text{thus, } \vec{p}\times\vec{q}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 2 & 5\\ 2 & -3 & 2\end{vmatrix}$$

$$=\vec{i}(4+15)-\vec{j}(2-10)+\vec{k}(-3-4)$$

$$=(19\vec{i}+8\vec{j}-7\vec{k})$$

thus, $|\vec{p}\times\vec{q}|=\sqrt{(19)^2+(8)^2+(-7)^2}$

$$=21.77$$

and $p=|\vec{p}|=\sqrt{1^2+2^2+5^2}=\sqrt{30}$

$q=|\vec{q}|=\sqrt{2^2+(-3)^2+2^2}=\sqrt{17}$

From equation (1),

$$\sin\theta=\frac{|\vec{p}\times\vec{q}|}{pq}$$

$$=\frac{21.77}{(\sqrt{30})(\sqrt{17})}$$

$$\sin\theta=0.96$$

$$\theta=74.58^\circ$$

Example: 1.6

Determine the area of a parallelogram whose adjacent sides are given by $\vec{i} - 4\vec{j}$ and $2\vec{i} - 3\vec{k}$.

Solution: Let

$\vec{p} = \vec{i} - 4\vec{j}$ and $\vec{q} = 2\vec{i} - 3\vec{k}$

Thus, area of parallelogram is given by the magnitude of cross product of two vectors i.e., $|\vec{p} \times \vec{q}|$,

$$\vec{p} \times \vec{q} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -4 & 0 \\ 2 & 0 & -3 \end{vmatrix}$$

$$= \vec{i}(12 - 0) - \vec{j}(-3 - 0) + \vec{k}(0 + 8)$$

$$= (12\vec{i} + 3\vec{j} + 8\vec{k})$$

and $|\vec{p} \times \vec{q}| = \sqrt{(12)^2 + (3)^2 + (8)^2}$

$$= \sqrt{217}$$

$$= 14.73$$

Thus area of parallelogram = 14.73 units.

Example: 1.7

A force, R, is given as $3\vec{i} - 2\vec{j} + 5\vec{k}$ is acting at a point. Find out its magnitude and the angles it makes with x, y and z axis.

Solution:

The magnitude of the force, R is given by

$$|\vec{R}| = \sqrt{(3)^2 + (-2)^2 + (5)^2}$$

$$R = \sqrt{38}$$

$$= 6.16 \text{ units}$$

If θ_x, θ_y and θ_z are the angles made by force, R with x, y and z axis, respectively

then $R_x = R \cdot \cos\theta_x$

i.e. $\cos\theta_x = \dfrac{R_x}{R} = \dfrac{3}{6.16} = 0.487$

$$\theta_x = 60.86°$$

Simillarly $\cos\theta_y = \dfrac{R_y}{R} = \dfrac{(-2)}{6.16} = 0.325$

$$\theta_y = 71.03°$$

and $\cos\theta_z = \dfrac{R_z}{R} = \dfrac{5}{6.16} = 0.811$

$$\theta_y = 35.81°$$

Example: 1.8

Three concurrent forces are given by $\left(2\vec{i}-3\vec{j}+2\vec{k}\right),\left(3\vec{i}+2\vec{j}-4\vec{k}\right)and\left(\vec{i}+3\vec{j}+3\vec{k}\right)$.

Determine their resultant force, magnitude and its direction cosines.

Solution:

The resultant force, $\vec{R}$ is given by

$$=\left(2\vec{i}-3\vec{j}+2\vec{k}\right)+\left(3\vec{i}+2\vec{j}-4\vec{k}\right)+\left(\vec{i}+3\vec{j}+3\vec{k}\right)$$
$$=\left(6\vec{i}+2\vec{j}+\vec{k}\right)$$

Thus magnitude of resultant force, R

$$=\left|\vec{R}\right|=\sqrt{(6)^2+(2)^2+(1)^2}$$
$$=\sqrt{36+4+1}$$
$$=\sqrt{41}$$
$$=6.40$$

Thus direction of cosines will be

$$\cos\theta_x=\frac{R_x}{\left|\vec{R}\right|}=\frac{R_x}{R}=\frac{6}{6.40}=0.94$$
$$\theta_x=19.95°$$

Similarly, $\cos\theta_y=\frac{R_y}{\left|\vec{R}\right|}=\frac{R_y}{R}=\frac{2}{6.4}=0.31$

$$\theta_y=71.94°$$

and $\cos\theta_y=\frac{R_z}{\left|\vec{R}\right|}=\frac{R_z}{R}=\frac{1}{6.4}=0.16$

$$\theta_z=80.79°$$

Example: 1.9

Prove that the two vectors are parallel to each other

$$\vec{p}=2\vec{i}-2\vec{j}-4\vec{k}\text{ and }\vec{q}=-\vec{i}+\vec{j}+2\vec{k}$$

Solution:

We know that

$$\vec{p}\times\vec{q}=pq\sin\theta\hat{n}$$

or, $\left|\vec{p}\times\vec{q}\right|=pq\sin\theta$

or, $\sin\theta=\frac{\left|\vec{p}\times\vec{q}\right|}{pq}$ ..... (1)

If $\theta = 0^\circ$ them $\vec{p}$ and $\vec{q}$ will be parallel to each other

thus, $\vec{p}\times\vec{q} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & -4 \\ -1 & 1 & 2 \end{vmatrix}$

$$= \vec{i}(-4+4) - \vec{j}(4-4) + \vec{k}(2-2)$$
$$= 0$$

thus from equation (1),

$$\sin\theta = 0^\circ$$
$$\theta = 0^\circ$$

Hence, both vectors are parallel to each other.

Alternative Method:

we know that $\vec{p}.\vec{q} = p.q\cos\theta$

$$i.e.\ \cos\theta = \frac{\vec{p}.\vec{q}}{pq} \qquad ..... (2)$$

$$\vec{p}.\vec{q} = (2\vec{i} - 2\vec{j} - 4\vec{k}).(-\vec{i} + \vec{j} + 2\vec{k})$$
$$= (-2-2-8)$$
$$= -12$$
$$p = |\vec{p}| = \sqrt{2^2 + (-2)^2 + (-4)^2} = \sqrt{24}$$
$$q = |\vec{q}| = \sqrt{(-1)^2 + (1)^2 + (2)^2} = \sqrt{6}$$

From equation (2),

$$\cos\theta = \frac{-12}{\sqrt{24}\sqrt{6}} = \frac{-12}{\sqrt{4\times 6}\sqrt{6}}$$
$$\cos\theta = -1$$
$$\theta = 180^\circ \text{ or } \pi$$

i.e., two vectors are parallel to each other but unlike in nature.

Example: 1.10

Prove that the given vectors are perpendicular to each other.

$\vec{p} = 8\vec{i} - 6\vec{j} + 5\vec{k}$ and $\vec{q} = 5\vec{i} + 10\vec{j} + 4\vec{k}$

Solution:

We know that $\vec{p}.\vec{q} = pq\cos\theta$

$$\text{or, } \cos\theta = \frac{\vec{p}.\vec{q}}{pq} \qquad ..... (1)$$

If $\theta = 90°$ then the vector will be perpendicular to each other.

$$\begin{aligned}\vec{p}.\vec{q} &= \left(8\vec{i} - 6\vec{j} + 5\vec{k}\right).\left(5\vec{i} + 10\vec{j} + 4\vec{k}\right) \\ &= (8\times 5) + (-6\times 10) + (5\times 4) \\ &= 40 - 60 + 20 \\ &= 0\end{aligned}$$

Thus from equation (1),

$\cos\theta = 0$ i.e $\theta = 90°$

Thus, both vectors are perpendicular to each other.

Example: 1.11

A force of 48 kN passes through two points A (2, –4, 3) and B (–5, 2, 1) in space as shown in Fig. 1.28. Represent the force in terms of unit vectors $\vec{i}, \vec{j}$ and $\vec{k}$.

Solution:

Vector between two points is given by

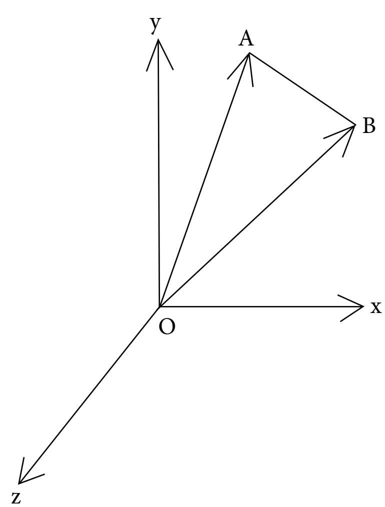


Fig. 1.28

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

where O is the origin in space

and $\overrightarrow{OA} = 2i - 4j + 3k$, $\overrightarrow{OB} = -5\vec{i} + 2\vec{j} + \vec{k}$

$$\begin{aligned}\text{thus } \overrightarrow{AB} &= (-5-2)\vec{i} + (2+4)\vec{j} + (1-3)\vec{k} \\ &= \left(-7\vec{i} + 6\vec{j} - 2\vec{k}\right)\end{aligned}$$

Unit vector along $\overrightarrow{AB} = \dfrac{\left(-7\vec{i} + 6\vec{j} - 2\vec{k}\right)}{\left|\overrightarrow{AB}\right|}$

$$= \frac{\left(-7\vec{i} + 6\vec{j} - 2\vec{k}\right)}{\sqrt{(-7)^2 + (6)^2 + (-2)^2}}$$

$$= \left(\frac{-7\vec{i} + 6\vec{j} - 2\vec{k}}{9.43}\right)$$

$$= \left(-0.742\vec{i} + 0.636\vec{j} - 0.212\vec{k}\right)$$

Thus force 48 kN in terms of unit vector will be

$\vec{R} = \text{Magnitude of force} \times \text{Unit vector}$

$= 48\left(-0.742\vec{i} + 0.636\vec{j} - 0.212\vec{k}\right)$

$\vec{R} = \left(-35.62\,\vec{i} + 30.53\,\vec{j} - 10.18\,\vec{k}\right)$

Example: 1.12

A force $\vec{R} = 4\vec{i} + 5\vec{j} - 2\vec{k}$ moves a body from point A (2, 4, -2) to point B (1, -2, 3). Determine work done by force on body.

Solution:

We know that work done is given by

W = $\vec{R}$. Displacement ..... (1)

The displacement will be given by

= position vector of A point – position vector of B point

$= \left(2\vec{i} + 4\vec{j} - 2\vec{k}\right) - \left(\vec{i} - 2\vec{j} + 3\vec{k}\right)$

$= \left(\vec{i} + 6\vec{j} - 5\vec{k}\right)$

Thus from equation (1),

$$\begin{aligned} W &= \left(4\vec{i} + 5\vec{j} - 2\vec{k}\right).\left(\vec{i} + 6\vec{j} - 5\vec{k}\right) \\ &= (4\times1) + (5\times6) + (-2)(-5) \\ &= 4 + 30 + 10 \\ &= 44 \text{ units} \end{aligned}$$

Theoretical Problems

T 1.1 Define rigid body.
T 1.2 Discuss briefly engineering mechanics, statics and dynamics.
T 1.3 Differentiate between fundamental unit and derived unit.
T 1.4 Define scalar and vector quantities with examples.
T 1.5 Define and classify the force system with examples.
T 1.6 What do you mean by Non concurrent coplanar force system?
T 1.7 Discuss collinear coplanar force system with example.
T 1.8 Define concurrent coplanar force system.
T 1.9 Illustrate the Gravitational law of attraction.
T 1.10 Define Parallelogram law, Triangle law and Polygon law of forces.
T 1.11 Define Sliding, Fixed and Free vectors with suitable examples.
T 1.12 Briefly explain unit vector and null vector
T 1.13 Discuss the dot product and cross product in multiplication of vectors.

Numerical Problems

N 1.1 Two parallel vectors are given as

$\vec{p} = \vec{i} - 2\vec{j} + 3\vec{k}$ and $\vec{q} = 3\vec{i} + 2\vec{j} - \vec{k}$

Determine: (i) Magnitude of both vectors

(ii) Unit vectors of each vector along their directions

N 1.2 Determine $\vec{p}.\vec{q}$ in the previous question. Also determine the angle between them.

N 1.3 Determine the angle between two vectors $(\vec{p} + \vec{q})$ and $(\vec{p} - \vec{q})$

If $\vec{p} = 2\vec{i} - 3\vec{j} + 2\vec{k}$ and $\vec{q} = 4\vec{i} - \vec{j} - 2\vec{k}$

N 1.4 Determine the cross product of vectors $\vec{p}$ and $\vec{q}$

$\vec{p} = \vec{i} + \vec{j} + 2\vec{k}$ and $\vec{q} = 3\vec{i} - 3\vec{j} + 2\vec{k}$

Also determine the angle between them.

N 1.5 The adjacent sides of a Parallelogram are given by

$2\vec{i} - 3\vec{j}$ and $\vec{i} + 2\vec{k}$

Determine area of such Parallelogram.

N 1.6 If three concurrent forces are given by

$(\vec{i} - \vec{j} - \vec{k}), (2\vec{i} + 2\vec{j} - 3\vec{k})$ and $(3\vec{i} - \vec{j} + \vec{k})$

Determine their resultant force, magnitude and its direction cosines.

N 1.7 Prove that following two vectors are perpendicular to each other,

$\vec{p} = 3\vec{i} + 4\vec{j} + 5\vec{k}$ and $\vec{q} = 7\vec{i} - 9\vec{j} + 3\vec{k}$

N 1.8 Prove that the following two vectors are parallel to each other.

$\vec{p} = 2\vec{i} - 6\vec{j} + 4\vec{k}$ and $\vec{q} = -\vec{i} + 3\vec{j} - 2\vec{i}$

N 1.9 Determine force 30kN in terms of unit vectors $\vec{i}, \vec{j}$ and $\vec{k}$ if passing through two points A (3, -2, 2) and B (-4, 3, 1) in space.

N 1.10 If a body is moved from point (1, 3, -2) to point B (2, -1, 4) by a force $\vec{p} = 3\vec{i} - 4\vec{j} + 5\vec{k}$ then determine work done by force.

Multiple Choice Questions

1. A body is called as rigid body if
   a. change in volume is significant
   b. change in volume is negligible
   c. distance between points remains variable
   d. all of these
2. Engineering mechanics deals with forces acting on
   a. elastic body
   b. plastic body
   c. rigid body
   d. all of these
3. Determine the vector quantity
   a. weight
   b. mass
   c. time
   d. none of these
4. A force is characterized by
   a. magnitude
   b. point of application
   c. line of action and sense
   d. all of these
5. If number of forces are acting in a common plane along a line, the force system will be called as
   a. collinear coplanar force system
   b. concurrent coplanar force system
   c. parallel coplanar force system
   d. none of these
6. If number of forces are acting at a point but in different planes, the force system will be called as
   a. collinear non coplanar force system
   b. concurrent coplanar force system
   c. concurrent non coplanar force system
   d. none of these
7. If number of forces are parallel in common plane, the force system will be called as
   a. collinear coplanar force system
   b. parallel non coplanar force system
   c. parallel coplanar force system
   d. none of these
8. Laws of Forces include following laws
   a. parallelogram law,
   b. triangle law
   c. polygon law
   d. all of these
9. Vectors are classified as
   a. Sliding vector
   b. Fixed vector
   c. Free vector
   d. all of these

10. Which is a free vector quantity
    - a. moment about a point
    - b. couple
    - c. force
    - d. none of these
11. The moment value of a force about a point represents
    - a. Sliding vector
    - b. Fixed vector
    - c. Free vector
    - d. none of these
12. In engineering mechanics, a force acting on a rigid body represents
    - a. Sliding vector
    - b. Fixed vector
    - c. Free vector
    - d. none of these

Answers

1. b 2. c 3. a 4. d 5. a 6. c 7. c 8. d 9. d 10. b 11. b 12. a

Chapter 2

Two Dimensional Concurrent Force Systems

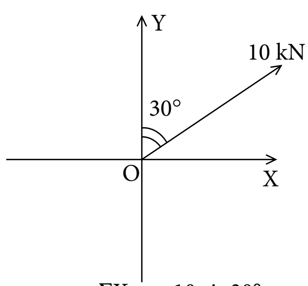
2.1 Resolution of Force and Force Systems

Resolution is a method of resolving the forces acting on a rigid body; in two perpendicular directions. The algebraic sum of all forces is determined in two perpendicular directions i.e., X and Y directions which are designated as ΣX and ΣY. Subsequently, the resultant force acting on the body can be determined as explained in section 2.2.

Points to be noted:

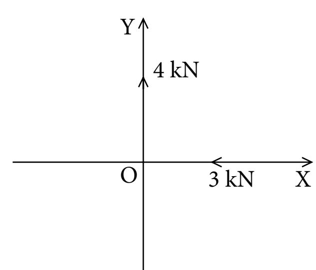
i. For ΣX take (+) sign if force is along X-direction and (–) sign if force is opposite to X-direction.
ii. For ΣY take (+) sign if force is along Y-direction and (–) sign if force is opposite to Y-direction.
iii. Angle must lie between line of action of force and any side of quadrant.
iv. If the force is resolved along angle side then use function "cosine" however use "sine" if it is not resolved along angle side.

Some of the examples are shown in Fig. 2.1 (a, b, c, d, e and f) to illustrate the resolution method.



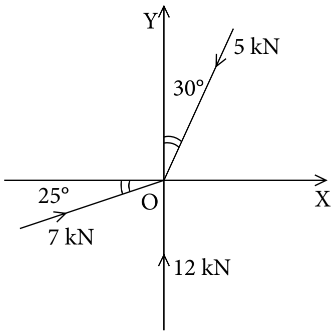
$\Sigma X = +\ 10.\sin 30°$
$\Sigma Y = +\ 10.\cos 30°$

Fig. 2.1 (a)



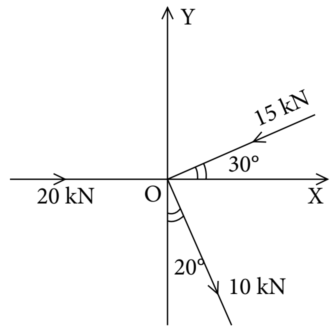
$\Sigma X = -\ 3$ kN
$\Sigma Y = +\ 4$ kN

Fig. 2.1 (b)



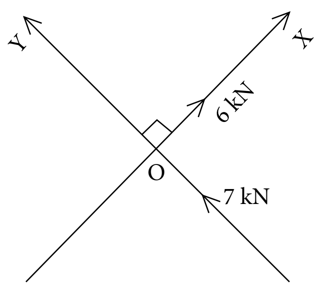
$\Sigma X = -5\sin 30° + 7.\cos 25°$
$\Sigma Y = -5\cos 30° + 7.\sin 25° + 12$

Fig. 2.1 (c)



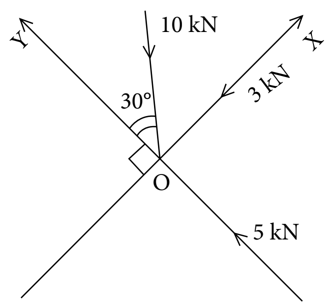
$\Sigma X = -15\cos 30° + 20 + 10\sin 20°$
$\Sigma Y = -15\sin 30° - 10\cos 20°$

Fig. 2.1 (d)



$\Sigma X = +6$ kN
$\Sigma Y = +7$ kN

Fig. 2.1 (e)

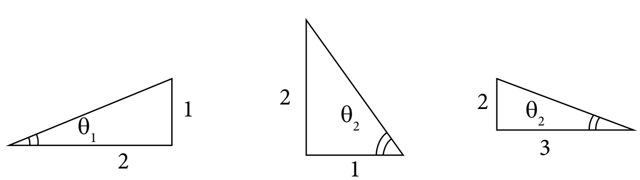


$\Sigma X = -3 - 10\sin 30°$
$\Sigma Y = +5 - 10\cos 30°$

Fig. 2.1 (f)

Example 2.1

Determine the resultant of the concurrent coplanar force system acting at point 'O' as shown in Fig. 2.2.



$\tan\theta_1 = 1/2$ $\quad$ $\tan\theta_2 = 2/1$ $\quad$ $\tan\theta_3 = 2/3$

$\theta_1 = 26.57°$ $\quad$ $\theta_2 = 63.43°$ $\quad$ $\theta_3 = 33.69°$

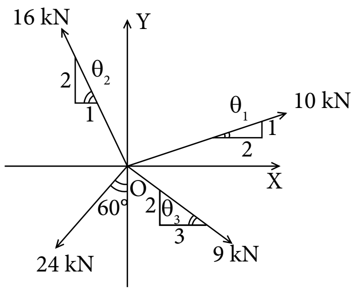


Fig. 2.2 (a)

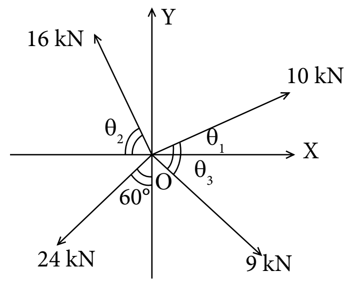


Fig. 2.2 (b)

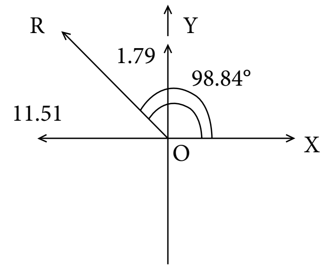


Fig. 2.2 (c)

$$\sum X = +10.\cos\theta_1 - 16\cos\theta_2 - 24\sin 60° + 9\cos\theta_3$$

$$= 10.\cos 26.57° - 16\cos 63.43° - 24\sin 60° + 9.\cos 33.69°$$

$$= -11.51 \text{ kN}$$

$$\sum Y = +10.\sin\theta_1 + 16.\sin\theta_2 - 24\cos 60° - 9\sin\theta_3$$

$$= 10.\sin 26.57° + 16\sin 63.43° - 24\cos 60° - 9.\sin 33.69°$$

$$= +1.79 \text{ kN}$$

$$R = \sqrt{\sum X^2 + \sum Y^2} = \sqrt{(-11.51)^2 + (+1.79)^2} = 11.65 \text{ kN}$$

$$\tan\alpha = \frac{\sum Y}{\sum X} = \frac{+1.79}{-11.51}$$

$$\alpha = 8.84° \text{ or } 98.84°$$

Example 2.2

A wheel has five equally spaced radial spokes. If the three consecutive spokes are in tension 800 N, 500 N and 300 N respectively, find the tensions in other two spokes.

Solution:

Each spoke is at an angle

$$= \frac{360°}{5} = 72°$$

The concurrent force system about point 'O' can considered as shown in Fig. 2.3. (a) and (b)

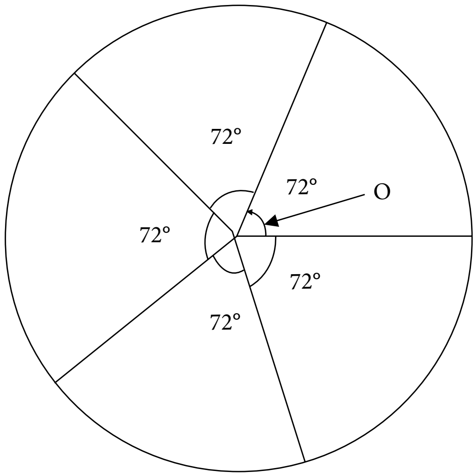


Fig. 2.3 (a)

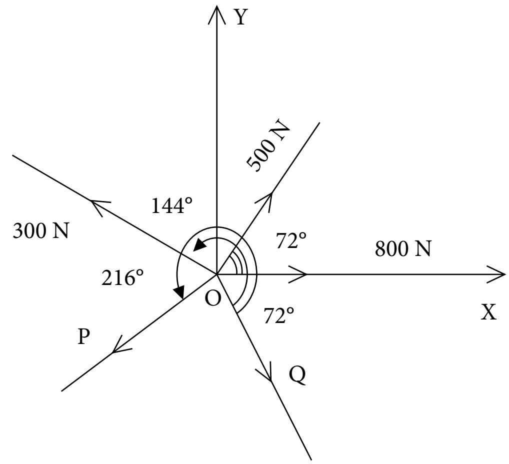


Fig. 2.3 (b)

$$\sum X = 800 + 500.\cos 72° + 300.\cos 144° + P.\cos 216° + Q.\cos 72°$$

$$0 \quad = 711.8 - 0.81P + 0.31Q \qquad .....(1)$$

$$\sum Y = 500.\sin 72° + 300 \ \sin 144° + P.\sin 216° - Q.\sin 72°$$

$$0 \quad = 651.86 - 0.59P - Q \times 0.95 \qquad .....(2)$$

Equation (1) and (2) can be further simplified as

$$-P + 0.38Q = -878.77 \qquad .....(3)$$

$$P + 1.61Q = 1104.85 \qquad .....(4)$$

Add equation (3) and (4)

$$1.99\ Q = 226.08$$

$Q = 113.61\,N$ substituting value of Q in equation (4)

$$P + 1.61(113.61) = 1104.85$$

$$P = 921.94\,N$$

2.2 Resultant of Two Dimensional Concurrent Forces

The resultant of concurrent forces acting in two dimensional plane can be determined by following ways:

a. Graphical methods- parallelogram law, triangle law and polygon law (stated in section 1.7.3 laws of forces)
b. Analytical methods- vector approach, resolution method, Lami's theorem, sine law and cosine law.

Example 2.3

Determine the resultant force acting on an eye bolt as shown in Fig. 2.4.

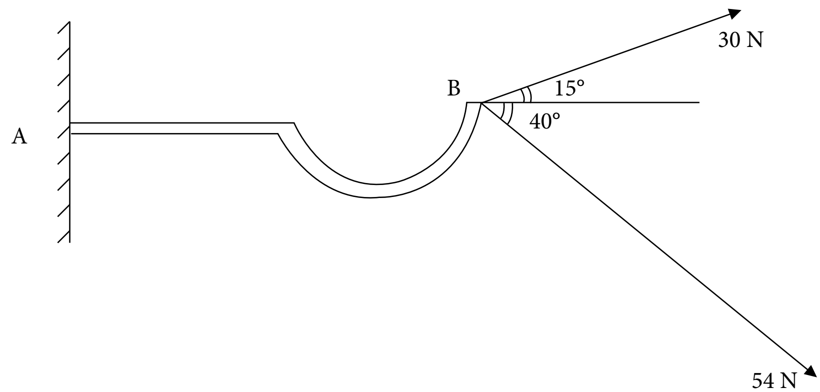


Fig. 2.4 Eye bolt

Solution:

(i) By Resolution Method:

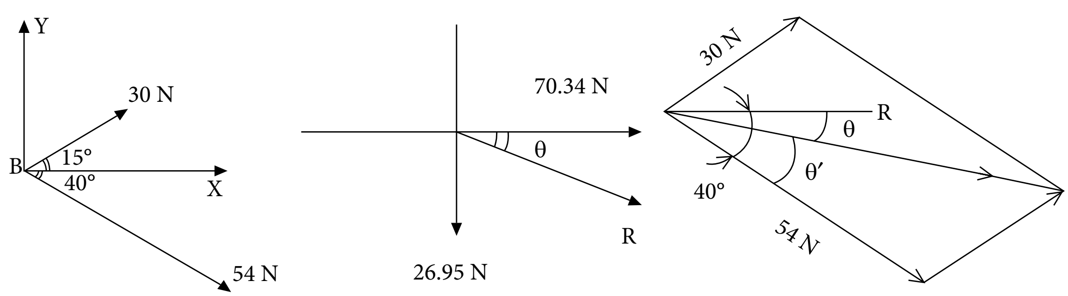


Fig. 2.4 (a) **Fig. 2.4 (b)** **Fig. 2.4 (c)**

$$\sum X = 30\ \cos 15° + 54\ \cos 40°$$
$$= 70.34\ \text{N}$$
$$\sum Y = +30 \sin 15° - 54 \sin 40°$$
$$= -26.95\ \text{N}$$
$$R = \sqrt{\sum X^2 + \sum Y^2} = \sqrt{(70.34)^2 + (-26.95)^2}$$
$$= 75.32\ \text{N}$$
$$\tan\theta = \frac{\sum Y}{\sum X} = \frac{-26.95}{70.34}$$
$$\theta = -20.96° \text{ (clockwise) or } 339.04° \text{ (anticlockwise)}$$

Note: As ΣX is +ve while ΣY is –ve. this shows that resultant is in fourth quadrant.

(ii) By using parallelogram law

$$P = 54 \text{ N},\ Q = 30 \text{ N},\ \alpha = 55°$$

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$$

$$= \sqrt{54^2 + 30^2 + 2 \times 54 \times 30 \cos 55°}$$

$$= 75.33 \text{ N}$$

$$\tan\theta' = \frac{Q\sin\alpha}{P + Q\cos\alpha} = \frac{30\sin 55°}{54 + 30\cos 55°}$$

$$\theta' = 19.04°$$

$$\theta' = 19.04°, \text{ from 54 N force or}$$

$$\theta = 40° - 19.04° = 20.96°, \text{ from horizontal line BX.}$$

Example 2.4

Two bars *AC and CB* are hinged together at *C* as shown in the Fig. 2.5. Find the forces induced in the bar. Assume weight of bars as negligible.

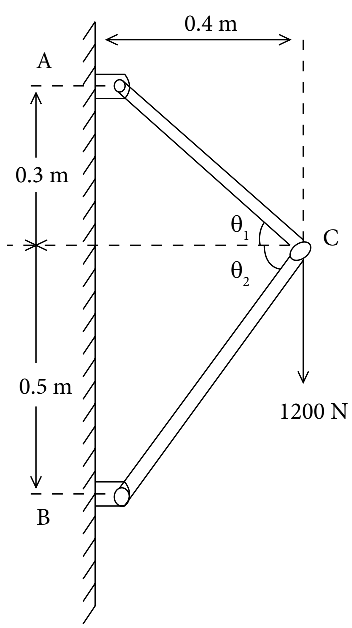


Fig. 2.5

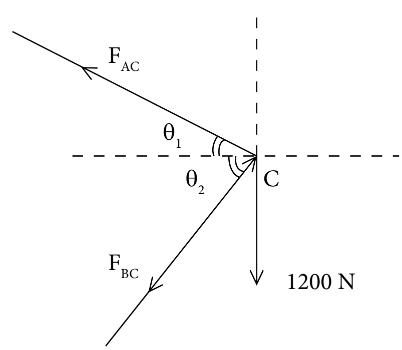


Fig. 2.5 (a)

Solution:

Let both bar *AC* and *BC* are in tensions.

Then considering equilibrium of force system about *C*, (Fig. 2.5 (*a*)).

As this is a concurrent coplanar force system,

$$\sum X = 0,$$
$$F_{AC} \cos\theta_1 + F_{BC} \cos\theta_2 = 0 \qquad .....(1)$$
$$\sum Y = 0,$$
$$F_{AC} \sin\theta_1 = 1200 + F_{BC} \sin\theta_2 \qquad .....(2)$$

From Fig. 2.5,

$$\tan\theta_1 = \frac{0.3}{0.4} \text{ and } \tan\theta_2 = \frac{0.5}{0.4}$$
$$\theta = 36.87° \text{ and } \theta_2 = 51.34°$$

Substituting values of θ_1 and θ_2 in equations (1) and (2)

$$F_{AC} \cos 36.87° + F_{BC} \cos 51.34° = 0$$
$$0.8\ F_{AC} + 0.62\ F_{BC} = 0$$
$$F_{BC} = -1.29\ F_{AC} \qquad .....(3)$$
$$F_{AC} \sin 36.87° = 1200 + F_{BC} \sin 51.34°$$
$$0.6\ F_{AC} = 1200 + 0.78\ F_{BC}$$
$$0.6\ F_{AC} = 1200 + 0.78\ (-1.29\ F_{AC})$$
$$0.6\ F_{AC} = 1200 - F_{AC}$$
$$1.6\ F_{AC} = 1200$$
$$F_{AC} = 750\,\text{N}$$

From equation (3)

$$F_{BC} = -1.29 \times 750$$
$$F_{BC} = -967.5\,\text{N}$$

–ve values shows that force in the member *BC* is compressive

Example 2.5

Determine the resultant of a force system consisting forces 3 *N*, $3\sqrt{3}\,N$, 5 *N*, 6 *N* and 4 *N* from an angular point of a regular hexagon acting towards the other angular points in a regular order.

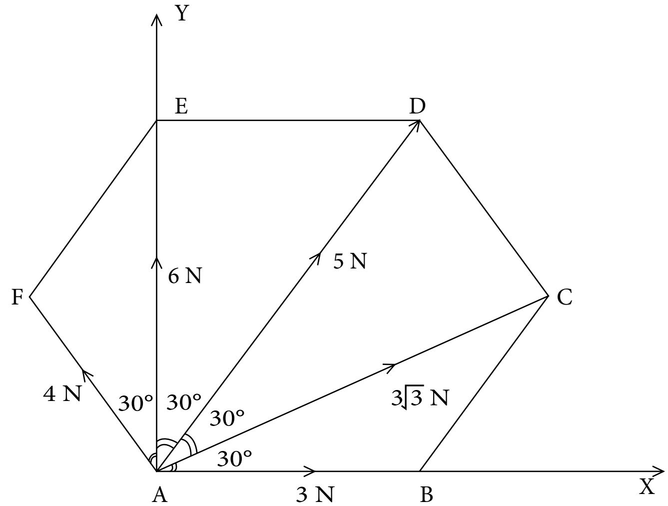


Fig. 2.6

Solution:

Consider Fig. 2.6 as shown. As the resultant is required to determine, this shows that force system about '*A*' is not in equilibrium thus resolved all force in two parts.

$$\begin{aligned}\Sigma X &= 3\cos 0° + 3\sqrt{3}\cos 30° + 5\cos 60° + 6\cos 90° + 4\cos 120° \\ &= 3 + 3\sqrt{3}\cdot\frac{\sqrt{3}}{2} + 5\cdot\frac{1}{2} + 0 + 4\left(-\frac{1}{2}\right) \\ &= 3 + 4.5 + 2.5 - 2 = +8 \\ \Sigma Y &= 3\sin 0° + 3\sqrt{3}\sin 30° + 5\sin 60° + 6\sin 90° + 4\sin 120° \\ &= 0 + 3\sqrt{3}\cdot\frac{1}{2} + \frac{5\sqrt{3}}{2} + 6.1 + \frac{4\sqrt{3}}{2} \\ &= 6\sqrt{3} + 6 = 16.39\ N \\ R &= \sqrt{\Sigma x^2 + \Sigma y^2} = \sqrt{(8)^2 + (16.39)^2} = 18.24\ \text{N} \\ \tan\theta &= \frac{\Sigma Y}{\Sigma X} = \frac{16.39}{8} \\ \theta &= 83.04°\end{aligned}$$

2.3 Principle of Transmissibility of Forces

This law states that force acting at a point on a body can be transferred to any new point but along its line of action only and its effect on the body remains unchanged. Using this law, forces are transferred to a suitable point and then force system is analyzed.

Consider a box lying on the ground as shown in Fig. 2.7. Where force R is acting at point A. Suppose we want to transfer it to new point B lying on line of action of force R as shown in Fig. 2.7(a). Let us introduce equal and opposite force R at point B as shown in Fig. 2.7(b). Now force at point A and B is collinear and opposite thus cancel out each other. Finally, the force remains at point B as shown in Fig. 2.7(c) confirms the transmission of force from A to B.

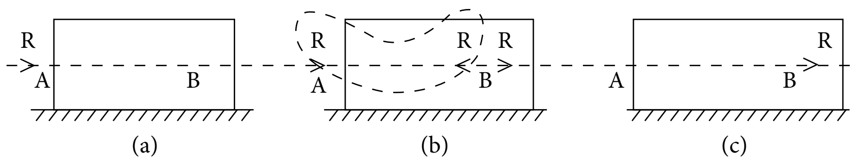


Fig. 2.7

2.4 Free Body Diagrams

Free Body Diagram is very crucial for all numerical problems as it represents the skeleton of the force system which can be analyzed easily by using law of forces. Following are the steps required to draw the free body diagram of a body:

i. Draw the figure of the body separately, i.e., draw the isolated body.
ii. Mention the weight of the body, if given.
iii. Mark the external forces as applied on the body, if given.
iv. Remove the contact surfaces one by one and mark their reactions on the body accordingly.

Some of the examples are shown in Fig. 2.8 (a, b and c) to illustrate the free body diagram:

i. Figure 2.8 (a) shows a cylinder of weight W rests in between two inclined planes which are inclined at angles α and β respectively from horizontal plane. The free body diagram of the cylinder is drawn by reactions R_A and R_B at both contact points and weight W as shown in Fig. 2.8 (b).
ii. Figure 2.8 (c) shows a cylinder of weight W is kept in equilibrium on inclined plane with the help of a string OA. The inclined plane is inclined at angle α from horizontal plane. The string OA remains under tension. The normal reaction of inclined plane on cylinder acts as R_B. The free body diagram of the cylinder is shown in Fig. 2.8 (d).
iii. Figure 2.8 (e) shows a beam supported on hinge and roller support bearing. The weight of beam is W and an external force P is acting at an angle α with the vertical as shown in the Fig. 2.8 (e). The hinge always restricts motion along both horizontal and vertical axis i.e., X and Y directions, thus offers reactions along both axis as R_{AH} and R_{AV}. However, the roller support bearing only restricts motion along vertical Y axis, thus offers reactions in Y axis as R_B. The free body diagram of the beam is shown in Fig. 2.8 (f).

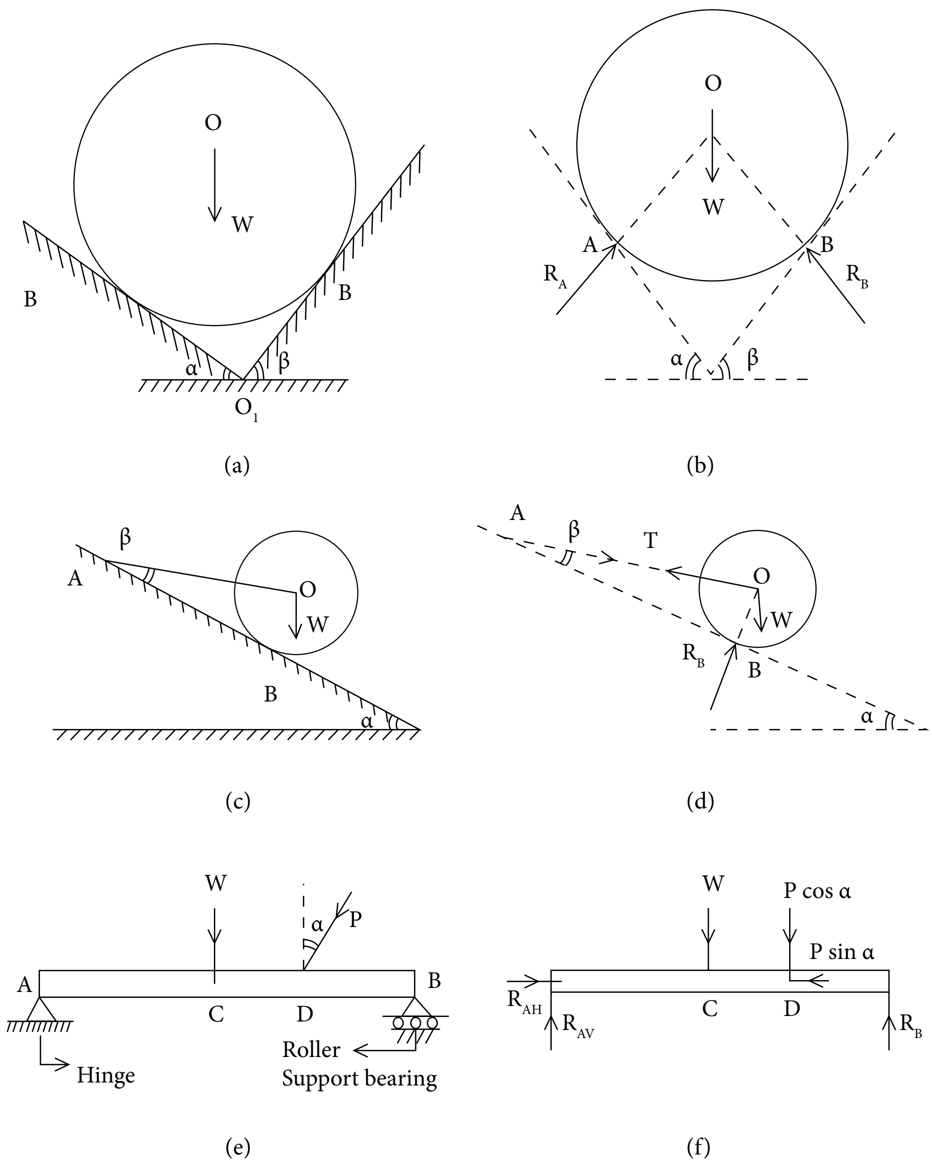


Fig. 2.8

2.5 Equations of Equilibrium Conditions

A body is said to be in equilibrium if the net resultant force (R) acting on the body is zero.

Thus, ΣX the algebraic sum of all forces in horizontal direction (horizontal component of resultant force) is equal to zero.

Similarly, ΣY the algebraic sum of all forces in vertical direction (vertical component of resultant force) is equal to zero.

Finally, the equations of equilibrium conditions for a two dimensional concurrent force system are:

$\Sigma X = 0$ and $\Sigma Y = 0$

However in two dimensional non-concurrent force system, the two conditions ($\Sigma X = 0$ and $\Sigma Y = 0$) are not enough to define the equilibrium of body; as non-concurrent forces causes rotation of body in their plane. Finally, the equations of equilibrium conditions for a two dimensional non-concurrent force system are:

$\Sigma X = 0$, $\Sigma Y = 0 =$ and $\Sigma M = 0$

2.6 Lami's Theorem

This theorem states that if a body is held in equilibrium under influence of three concurrent coplanar forces then each force is directly proportional to the sine of the angle between other two forces. We can determine maximum two unknown forces using this theorem as it is applicable only when a body is held in equilibrium just by three forces. For example, if three forces R_1, R_2 and R_3 are keeping a body in equilibrium as shown in Fig. 2.9 (a).

$R_1 \propto \sin\theta_1$ *i.e.*, $R_1 = k.\sin\theta_1$

$R_2 \propto \sin\theta_2$ *i.e.*, $R_2 = k.\sin\theta_2$

$R_3 \propto \sin\theta_3$ *i.e.*, $R_3 = k.\sin\theta_3$

$$\text{then } k = \frac{R_1}{\sin\theta_1} = \frac{R_2}{\sin\theta_2} = \frac{R_3}{\sin\theta_3}$$

Proof:

If three forces are drawn on the basis of magnitude and direction as per triangle law, it closes as shown in Fig. 2.9 (b).

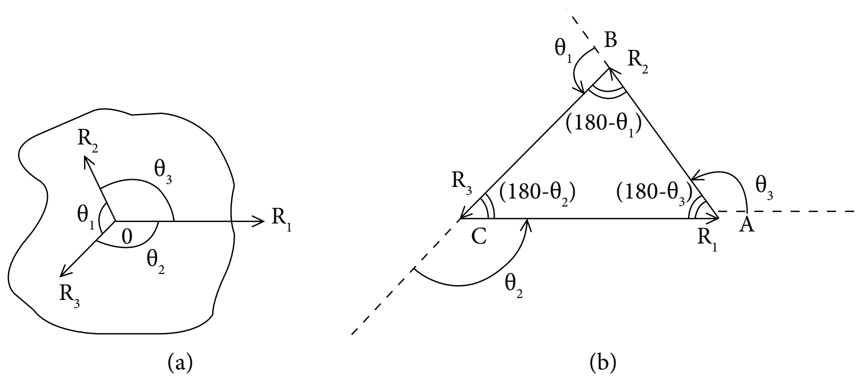


Fig. 2.9

Applying Law of Sines Fig. 2.9 (b).

$$\frac{AC}{\sin(180-\theta_1)} = \frac{AB}{\sin(180-\theta_2)} = \frac{BC}{\sin(180-\theta_3)}$$

$$\frac{AC}{\sin\theta_1} = \frac{AB}{\sin\theta_2} = \frac{BC}{\sin\theta_3}$$

thus, $\frac{R_1}{\sin\theta_1} = \frac{R_2}{\sin\theta_2} = \frac{R_3}{\sin\theta_3}$.

Example 2.6

A horizontal cylinder of weight 240 kN is held against a wall with the help of string as shown in Fig. 2.10. The string makes an angle 20° with the wall. Determine the reaction of the wall and tension induced in the string.

Solution:

Following steps are taken out:

(i) Draw free body diagram of cylinder as shown in Fig. 2.10 (*a*)
(ii) Apply Principle of Transmissibility so that Reaction R_B transmit to point '*O*' as shown in Fig. 2.10 (*b*)
(iii) Draw skeleton of forces as shown in Fig. 2.10 (*c*)
(iv) Now Fig. 2.10 (*c*) represents concurrent coplanar force system which can be analyzed by two methods:

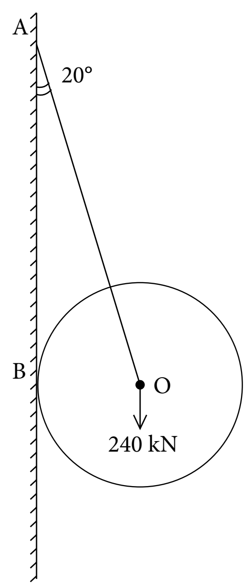


Fig. 2.10

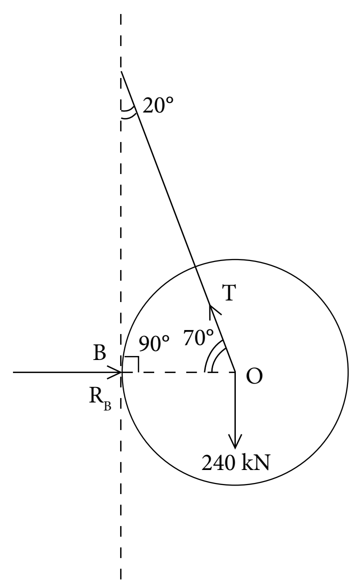


Fig. 2.10 (a)

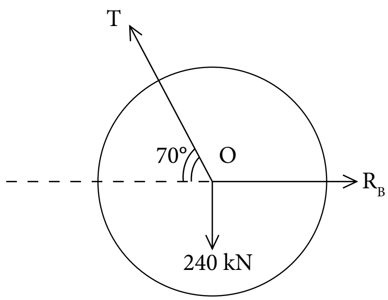


Fig. 2.10 (b) F.B.D. of Cylinder

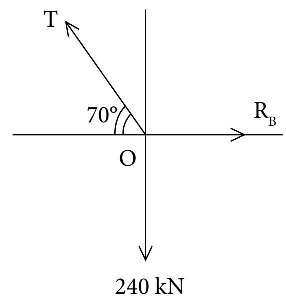


Fig. 2.10 (c) Concurrent coplanar force system

(a) By Resolution Method

As cylinder is in equilibrium thus $\Sigma X = 0$,

$$T \cos 70° = R_B \quad .....(1)$$

and $\sum Y = 0$,

$$T \sin 70° = 240 \quad .....(2)$$

$$T = \frac{240}{\sin 70°}$$

$$T = 255.40 \text{ kN}$$

Substituting value of T in equation (1),

$255.40 \cos 70° = R_B$

$$R_B = 87.35 \text{ kN}$$

(b) By Lami's Theorem

$$\frac{T}{\sin(90°)} = \frac{R_B}{\sin(90°+70°)} = \frac{240}{\sin(180°-70°)}$$

$$T = \frac{240 \cdot \sin 90°}{\sin 70°}$$

$$T = 255.40 \text{ kN}$$

and $R_B = 240.\cos 70° / \sin 70°$

$$R_B = 87.35 \text{ kN}$$

Example 2.7

Determine the force 'R' as shown in Fig. 2.11 causes tension in each part of the string 120 N.

Solution:

Given, $T_{OA} = T_{OB} = 120$ N.
Considering force system equilibrium about '*O*'
Figure 2.11 (*a*) can further simplified about '*O*' and can be easily resolved. As the force system is in equilibrium,

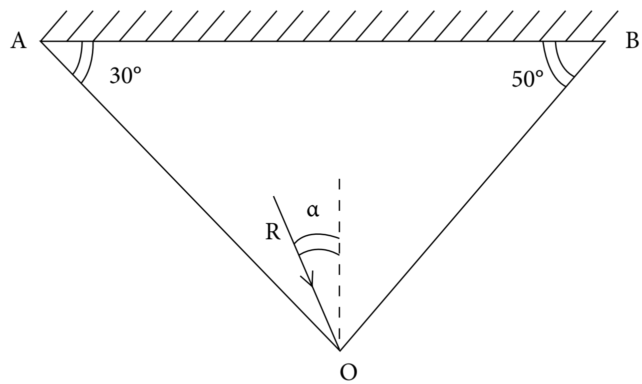


Fig. 2.11

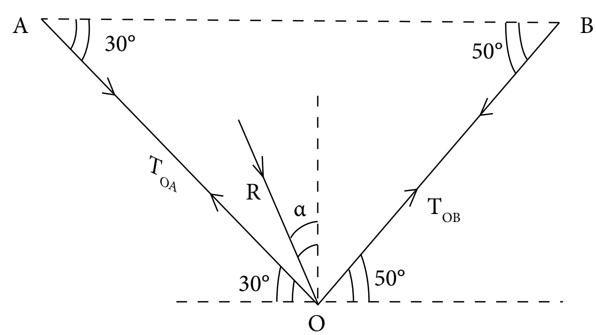


Fig. 2.11 (a)

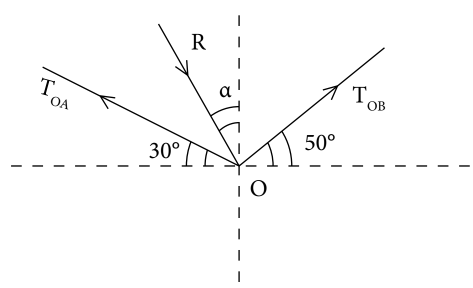


Fig. 2.11 (b)

$$120\cos 50° + R\sin\alpha - 120\cos 30° = 0$$

$$R\sin\alpha = +26.79 \qquad .....(1)$$

$$\sum Y = T_{OB}\sin 50° + T_{OA}\sin 30° - R\cos\alpha = 0$$

$$120\sin 50° + 120\sin 30° - R\cos\alpha = 0$$

$$R\cos\alpha = +151.93\text{ N} \qquad .....(2)$$

from equation (1) and (2)

$$R^2\sin^2\alpha + R^2\cos^2\alpha = (26.79)^2 + (151.93)^2$$

$$R^2 = (26.79)^2 + (151.93)^2$$

$$R^2 = 23800.43$$

$$R = 154.27\text{ N}$$

$$\tan\alpha = \frac{26.79}{151.93}$$

$$\alpha = 10°$$

Alternative Method

The above problem can be solved by using Lami's theorem as shown below Fig. 2.11(*a*) is simplified as in Fig. 2.11 (*c*),

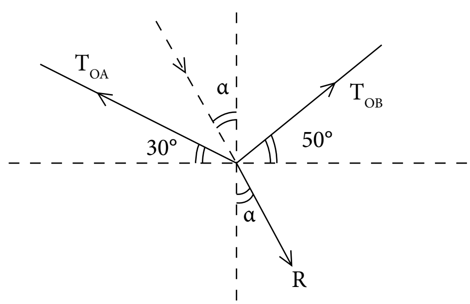


Fig. 2.11 (c)

$$\frac{T_{OA}}{\sin(90° - \alpha + 50°)} = \frac{T_{OB}}{\sin(90° + 30° + \alpha)} = \frac{R}{\sin(180° - 30° - 50°)}$$

$$\frac{120}{\sin(140° - \alpha)} = \frac{120}{\sin(120° + \alpha)} = \frac{R}{\sin(100°)}$$

i.e., $\sin(120° + \alpha) = \sin(140° - \alpha)$

or, $(120° + \alpha) = (140° - \alpha)$

$$2\alpha = 20°$$

$$\alpha = 10°$$

thus,
$$\frac{120}{\sin(120°+\alpha)} = \frac{R}{\sin(100°)}$$
$$R = \frac{120.\sin 100}{\sin(120°+10°)}$$
$$R = 154.269$$
$$R = 154.27\ N$$

Example 2.8

Determine the tension in the string *OA* and reaction of inclined plane when a roller of weight 600 N is supported as shown in Fig. 2.12.

Solution:

Considering Free body diagram of roller as shown in Fig. 2.12 (*a*).

The force system acting at point '*O*' can be further simplified as shown in Fig. 2.12 (*b*)

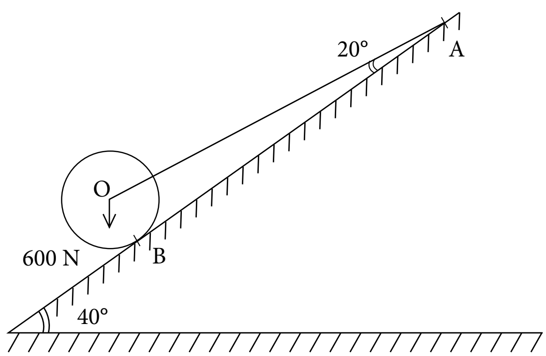


Fig. 2.12

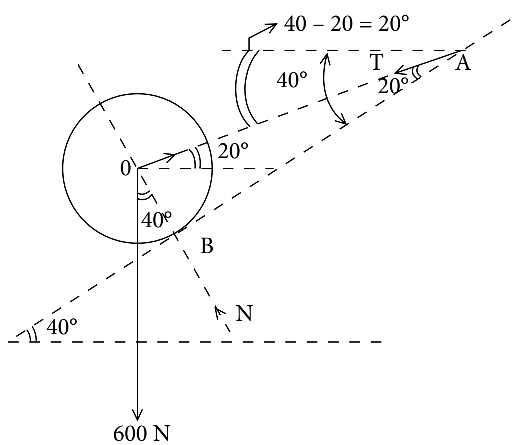


Fig. 2.12 (a) F.B.D. of Cylinder

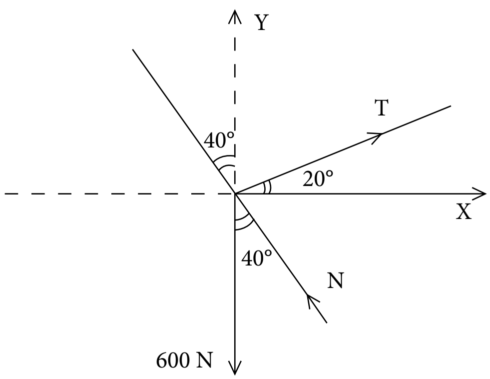


Fig. 2.12 (b)

As the roller is in equilibrium,

$$\sum X = 0 \text{ i.e. } T\cos 20° - N\sin 40° = 0$$

$$T = 0.684 \times N \qquad \text{.....(1)}$$

$$\sum Y = 0, \text{ i.e. } T\sin 20° + N\cos 40° - 600 = 0$$

$$T = 410.40 N$$

$$N = 600 N$$

Alternative Method

The above free body diagram can be resolved along and perpendicular to the inclined plane as shown in Fig. 2.12 (*c*).
The Fig. 2.12 (*c*) further can be simplified for force system about '*O*' as shown in Fig. 2.12 (*d*).
Resolving the forces along the inclined plane i.e., $\Sigma X = 0$
$T \cos 20° - 600 \sin 40° = 0$

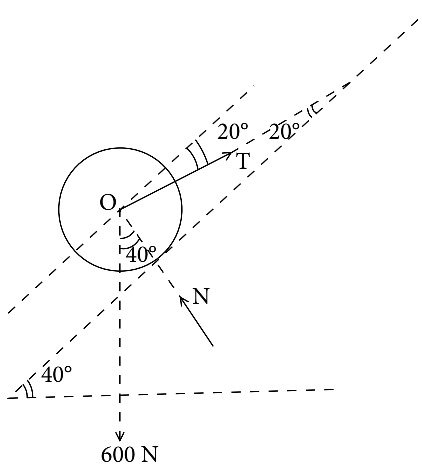


Fig. 2.12 (c) F.B.D. of Cylinder

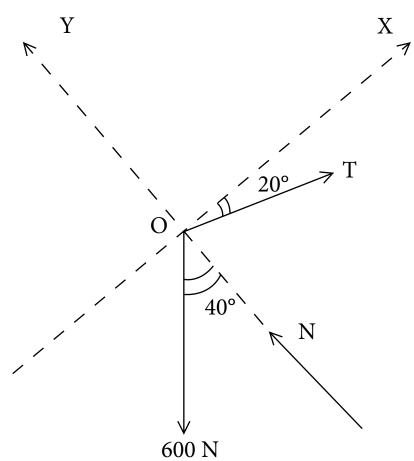


Fig. 2.12 (d)

or, $T \cos 20° = 600 \sin 40°$

$T = 410.42\ N$

Resolving the forces perpendicular to the inclined plane,

i.e., $\sum Y = 0$,

$$N = T \sin 20° + 600 \cos 40°$$
$$= 410.42 \sin 20° + 600 \cos 40°$$
$$N = 600\ N$$

Example 2.9

A roller of weight '*W*' rests over two inclined planes as shown in the Fig. 2.13. Determine support reactions at points *A* and *B*.

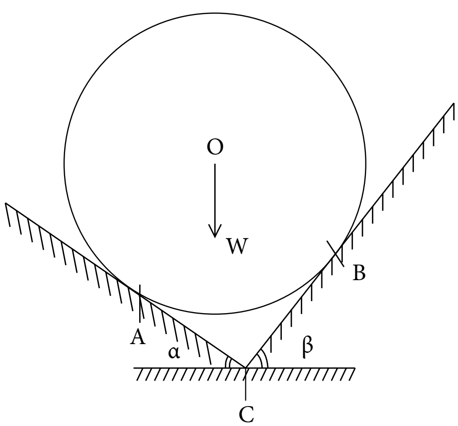


Fig. 2.13

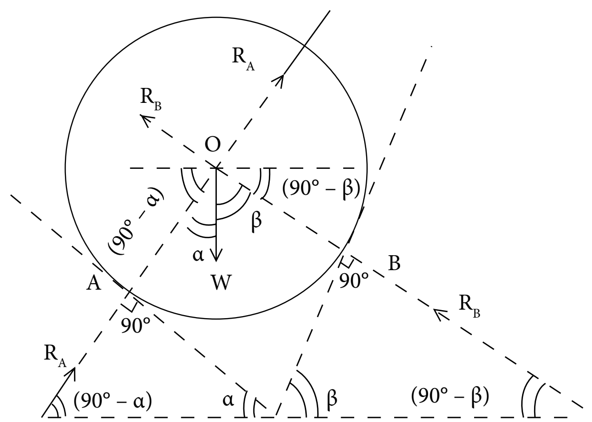


Fig. 2.13 (a) F.B.D. of Cylinder

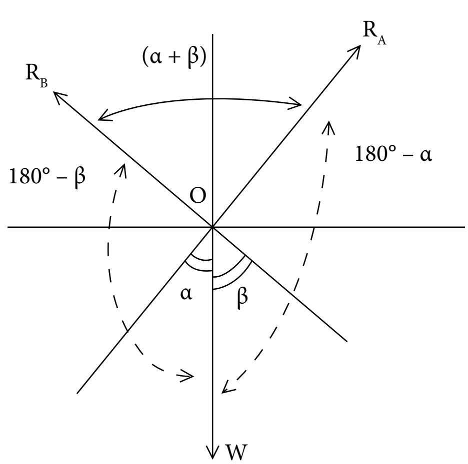


Fig. 2.13 (b)

Solution:

With application of Principle of Transmissibility the reactions R_A and R_B will be transferred through centre '*O*'. The force system acting at centre '*O*' can be simplified as shown in Fig. 2.13 (*a*) and (*b*) Applying Lami's theorem,

$$\frac{R_A}{\sin(180° - \beta)} = \frac{R_B}{\sin(180° - \alpha)} = \frac{W}{\sin(\alpha + \beta)}$$

$$R_A = \frac{W.\sin\beta}{\sin(\alpha + \beta)}$$

and $$R_B = \frac{W.\sin\alpha}{\sin(\alpha + \beta)}$$

Example 2.10

Figure 2.14 shows a sphere resting in a smooth V-shaped groove and subjected to a spring force. The spring is compressed to a spring force. The spring is compressed to a length of 100 mm from its free length of 150 mm. It the stiffness of spring is 2 N/mm, determine the contact reactions at *A* and *B*.

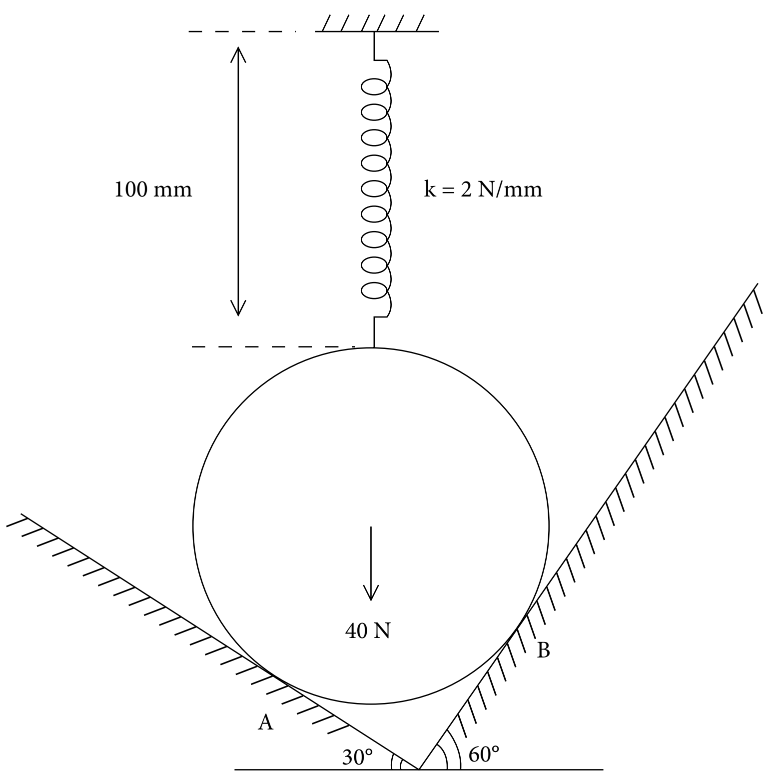


Fig. 2.14

Solution:

The spring is compressed by = 150 – 100 = 50 mm.

Thus compression force in the spring = 50 × stiffness of spring = 50 × 2 = 100 N

The free body diagram of both spring and sphere are as shown in Fig. 2.14 (a) and (b), respectively

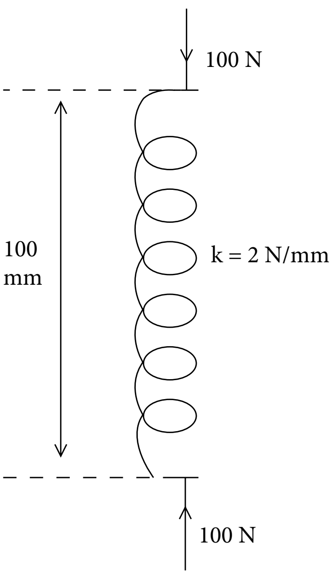


Fig. 2.14 (a) F.B.D. of Spring

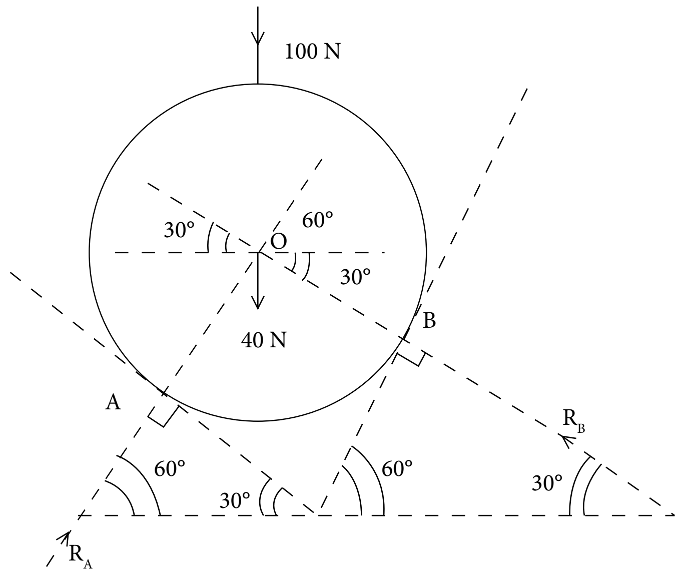


Fig. 2.14 (b) F.B.D. of Cylinder

Figure 2.14 *(b)* further simplified as shown in Fig. 2.14 (*c*) where force 100 N will shift to center O by using principle of transmissibility.

Figure 2.14 (*c*) can be evaluated further by using Lami's theorem or resolution.

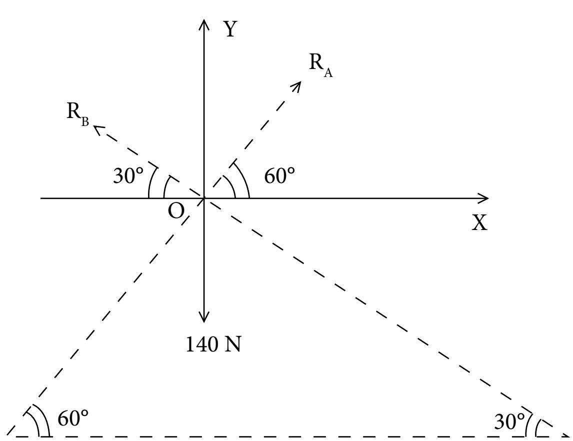


Fig. 2.14 (c)

By Lami's theorem,

$$\frac{R_A}{\sin(90°+30°)} = \frac{R_B}{\sin(90°+60°)} = \frac{140}{\sin(180°-30°-60°)}$$

$$\frac{R_A}{\cos 30°} = \frac{R_B}{\cos 60°} = \frac{140}{\sin 90°}$$

$$R_A = 121.24\ N$$

$$R_B = 70\ N$$

By Resolution method,

$\sum X = 0,$

$$R_B \cos 30° = R_A \cos 60°$$

$$R_A = \sqrt{3}.R_B \qquad \text{.....(1)}$$

$\sum Y = 0,$

$$R_A \sin 60° + R_B \sin 30° = 140$$

$$\frac{R_A \cdot \sqrt{3}}{2} + \frac{R_B}{2} = 140$$

$$\frac{\sqrt{3} \cdot R_B \cdot \sqrt{3}}{2} + \frac{R_B}{2} = 140$$

$$R_B = 70\ N$$

Substituting value of $R_B = 70\ N$ in equation (1), $R_A = 121.24\ N$.

Example 2.11

A 1500 kN cylinder is supported by the frame *ABC* as shown in Fig. 2.15. The frame is hinged to the wall at *A*. Determine the reactions at *A, B, C* and *D.* Take weight of the frame as negligible.

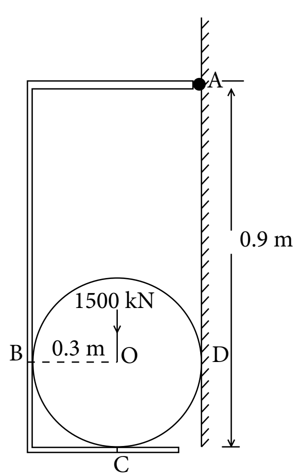


Fig. 2.15

Solution:

Considering free body diagram of cylinder as shown in Fig. 2.15 (*a*)

$$\sum X = 0,$$

$$R_B = R_D \qquad .....(1)$$

$$\sum Y = 0,$$

$$R_C = 1500\,\text{kN}$$

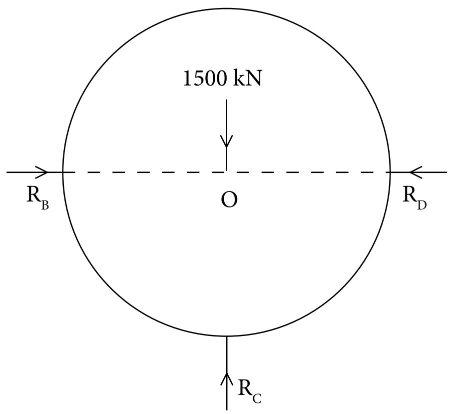


Fig. 2.15 (a) F.B.D. of Cylinder

Considering *F.B.D.* of channel, Fig. 2.15 (*b*) the force at *B* and *C* will be balanced by reaction of hinge *A* and this is possible only when three forces are concurrent.

$$\tan\theta = \frac{0.6}{0.3}$$

$$\theta = 63.43°$$

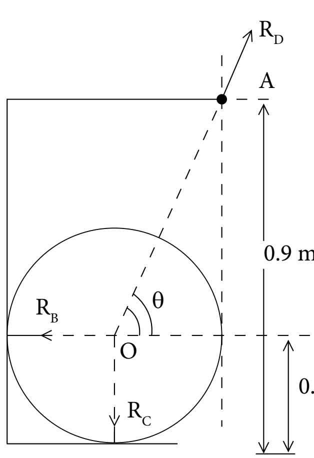


Fig. 2.15 (b) F.B.D. of Channel

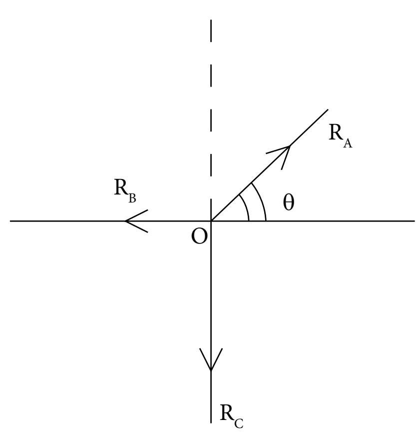


Fig. 2.15 (c)

$$\sum X = 0,$$

$$R_B = R_A \cos\theta$$

$$\sum Y = 0,$$

$$R_C = R_A \sin\theta$$

$$1500 = R_A \sin\theta$$

$$R_A = \frac{1500}{\sin 63.43°}$$

$$R_A = 1677.05 \text{ kN}$$

$$R_B = 1677.05 \times \cos 63.43°$$

$$R_B = 750.13 \text{ kN}$$

from equation (1),

$$R_D = 750.13 \text{ kN}.$$

Example 2.12

Two smooth spheres each of weight W and each of radius 'r' are in equilibrium in a horizontal channel of width 'b' ($b < 4r$) and vertical sides as shown in Fig. 2.16.
Find the three reactions from the sides of channel which are all smooth. Also find the force exerted by each sphere on the other.

(UPTU I sem, 2005–06)

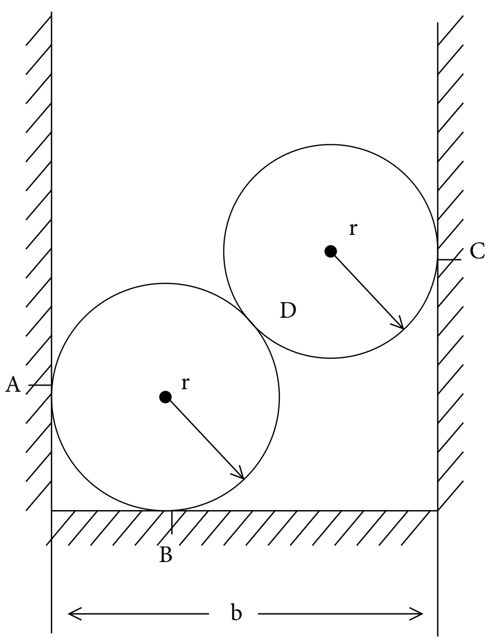


Fig. 2.16

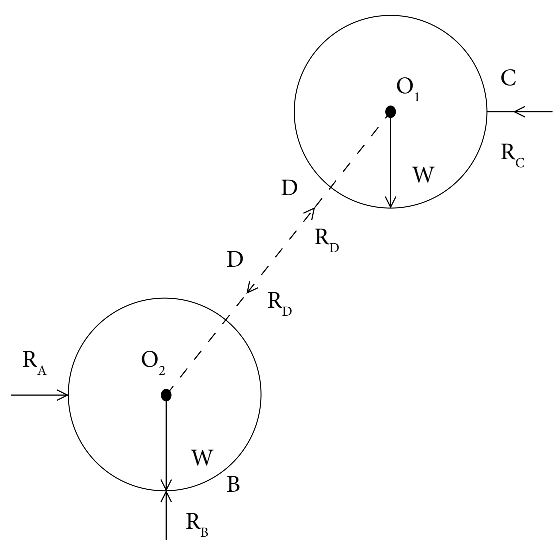


Fig. 2.16 (a) F.B.D. of Cylinder

Solution:

The following are the steps:

(i) Draw the *F.B.D.* of both spheres as shown in Fig. 2.16 (*a*)
(ii) Apply principle of transmissibility and transfer R_A, R_B, R_C, and R_D to the respective centres of the sphere as shown in Fig. 2.16 (*b*).
(iii) Draw the skeleton of force system as shown in Fig. 2.16 (*c*)

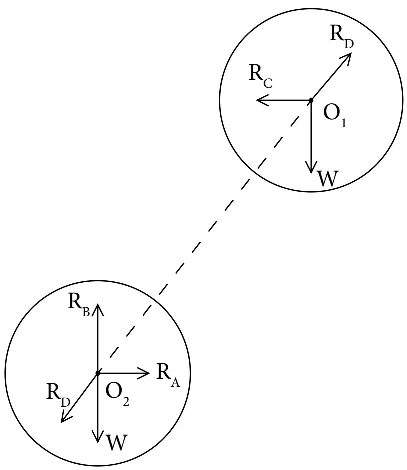


Fig. 2.16 (b)

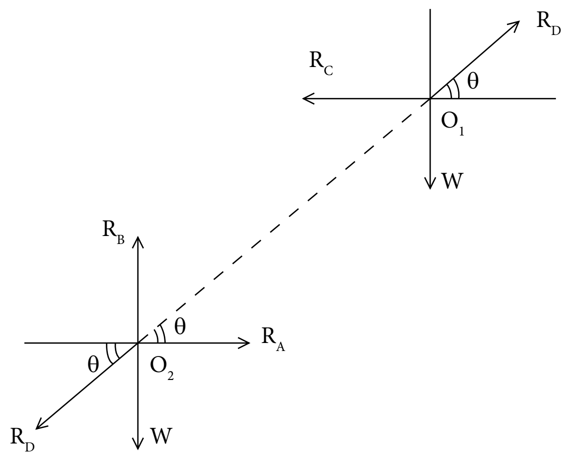


Fig. 2.16 (c)

Considering equilibrium of upper sphere.

$\Sigma X = 0, \qquad R_C = R_D \cdot \cos\theta$ .....(1)

$\Sigma Y = 0,$

$$R_D \sin\theta = W$$

i.e., $\qquad R_D = \dfrac{W}{\sin\theta}$ substituting in equation (1)

or, $\qquad R_C = \dfrac{W}{\sin\theta} \cdot \cos\theta$

$$R_C = W \cot\theta \text{ and } R_D = W / \sin\theta$$

Considering equilibrium of bottom sphere

$\Sigma X = 0,$

$R_D \cos\theta = R_A$ .....(2)

$\Sigma Y = 0,$

$R_B = W + R_D \sin\theta$ .....(3)

Substituting R_D value in equation (2),

$$\frac{W}{\sin\theta} \cdot \cos\theta = R_A$$

$$R_A = W.\cot\theta$$

Substituting R_D value in equation (3),

$$R_B = W + \frac{W}{\sin\theta}\cdot\sin\theta$$
$$R_B = 2W$$

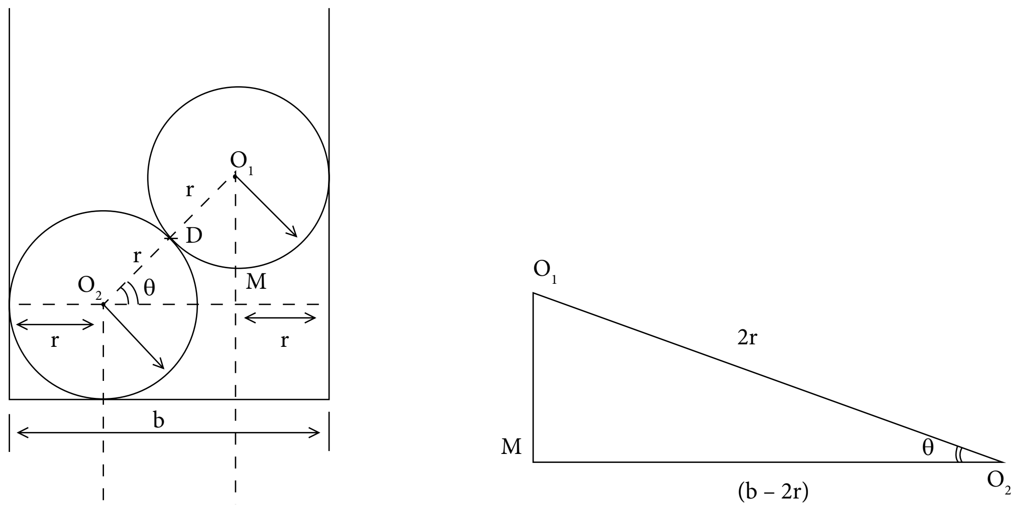


Fig. 2.16 (d)

From Fig. 2.16 (*d*)

$$O_2M = b - r - r$$
$$= (b - 2r)$$
$$O_1O_2 = r + r = 2r$$

Thus $\cos\theta = \dfrac{O_2M}{O_1O_2}$

$$\cos\theta = \frac{(b-2r)}{2r}$$
$$O_1M = \sqrt{(2r)^2 - (b-2r)^2}$$
$$= \sqrt{4r^2 - (b^2 + 4r^2 - 4br)}$$
$$O_1M = \sqrt{4br - b^2}$$

Thus, $\cot\theta = \left(\dfrac{(b-2r)}{\sqrt{4br - b^2}}\right)$ and

$$\sin\theta = \left(\frac{\sqrt{4br - b^2}}{2r}\right)$$

Finally, $R_A = R_C = W\cot\theta$

$$R_A = R_C = \left(\frac{W.(b-2r)}{O_1M}\right) \quad R_D = \frac{W}{\sin\theta},\ R_D = \frac{2rW}{\sqrt{4br - b^2}}.$$

Example 2.13

Two cylinders *A* and *B* of weight 1000 *N* and 500 *N* rest on smooth inclined planes as shown in the figure. A bar of negligible weight is hinged to each cylinder at its geometric centre by smooth pins. Determine the force *'P'* as applied can hold the system in equilibrium for given position as shown in Fig. 2.17.

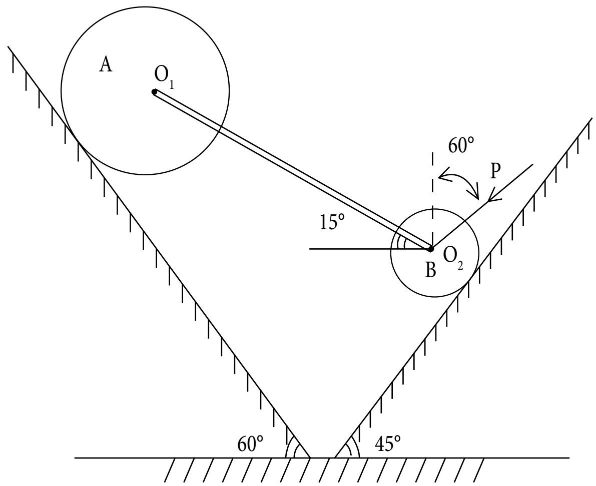


Fig. 2.17

Solution:

(i) Draw the *F.B.D.* of both cylinders and apply the Principle of Transmissibility.

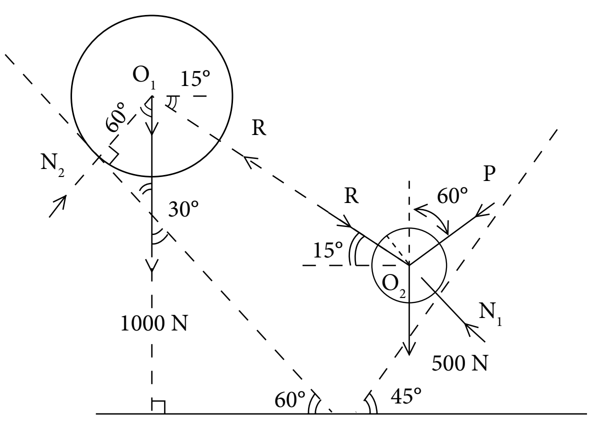


Fig. 2.17 (a) F.B.D. of Cylinder

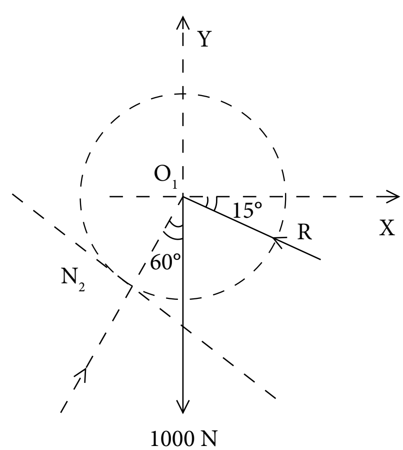


Fig. 2.17 (b) F.B.D. of Cylinder A

$\Sigma X = 0,$

$$N_2 \sin 60° = R \cos 15° \qquad .....(1)$$

$\Sigma Y = 0,$

$$N_2 \cos 60° + R \sin 15° = 1000\ N$$

$$N_2 \cos 60° = 1000 - R \sin 15° \qquad .....(2)$$

Divide equation (1) by equation (2),

$$\frac{N_2 \sin 60°}{N_2 \cos 60°} = \frac{R \cos 15°}{1000 - R \sin 15°}$$

$$\sqrt{3}(1000 - R \sin 15°) = R \cos 15°$$

$$1000\sqrt{3} = R(\sqrt{3} \sin 15° + \cos 15°)$$

$$R = \frac{1000\sqrt{3}}{(\sqrt{3} \sin 15° + \cos 15°)}$$

$$R = 1224.74\ N$$

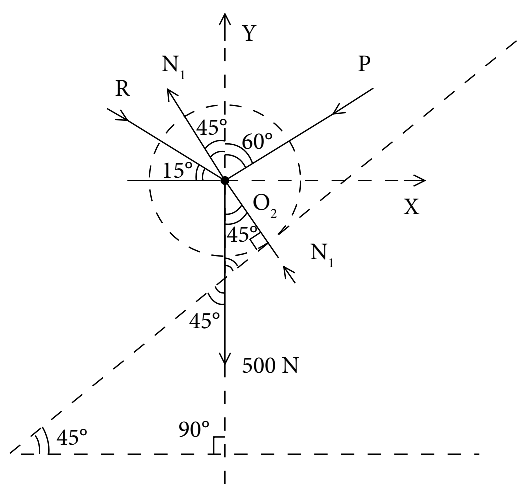


Fig. 2.17 (c) F.B.D. of Cylinder B

$\Sigma X = 0,$

$$R \cos 15° = P \sin 60° + N_1 \sin 45°$$

$$1224.74 \cos 45° = P \sin 60° + N_1 \sin 45°$$

$$P.\sin 60° + N_1 \sin 45° = 1183 \qquad .....(3)$$

$\Sigma Y = 0,$

$$N_1 \cos 45° = R \sin 15° + P \cos 60° + 500$$

$$N_1 \cos 45° = 1224.74 \sin 15° + P \cos 60° + 500$$

$$N_1 = (816.99 + P \cos 60°)\sqrt{2}$$

Substituting N_1 in equation (3),

$$P \sin 60° + (816.99 + P \cos 60°)\sqrt{2}.\sin 45° = 1183$$

$$P = 267.94\ N.$$

Example 2.14

A roller of weight 2400 N, radius 180 cm is required to be pulled over a brick of height 90 cm as shown in the Fig. 2.18 by a horizontal pull applied at the end of a string wound round the circumference of the roller. Determine the value of *P* which will just tend to turn the roller over the brick.

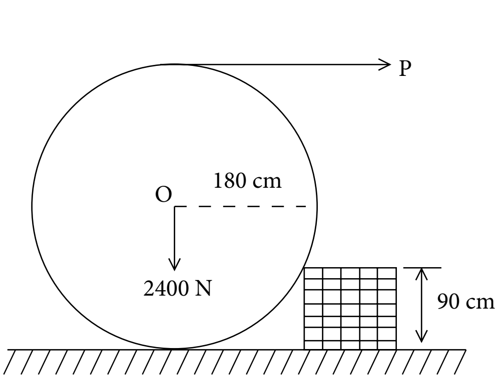


Fig. 2.18

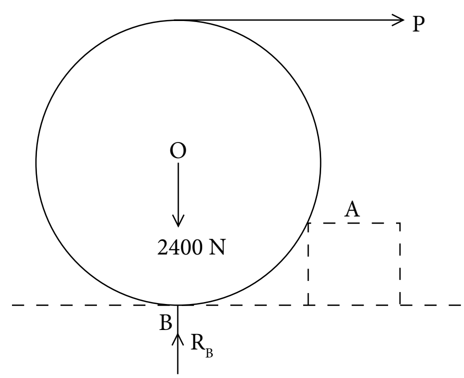


Fig. 2.18 (a)

Solution:

The steps will be taken as follows:

(i) Draw F.B.D. of roller, when roller just about to turn over the brick at that instant, the contact between roller and ground will break-off. thus $R_B = 0$

(ii) The roller will be in equilibrium under three forces *P*, 2400 N and R_A.

Thus three forces can hold a body in equilibrium only when they pass through a common point i.e., represent a concurrent system. Which can be possible only when weight. of roller is transferred to *O*' by principle of transmissibility thus reaction R_A passes through *O*' and this way roller will be held in equilibrium.

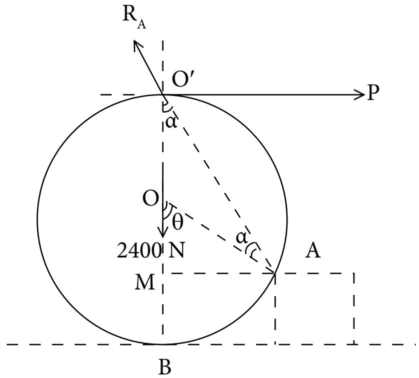


Fig. 2.18 (b) F.B.D. of roller

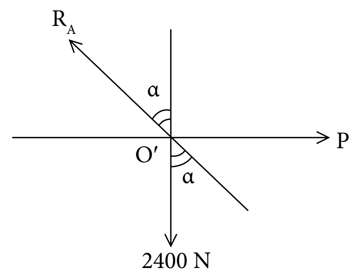


Fig. 2.18 (c)

From Fig. 2.18 (b),

$$OM = 180 - 90$$
$$OM = 90 \text{ cm}$$
$$OA = 180 \text{ cm}$$

Let $\angle MOA = \theta$, $\cos\theta = \dfrac{OM}{OA} = \dfrac{90}{180}$

$$\theta = 60°$$
$$\angle O'OA = 180 - \theta = 180 - 60° = 120°$$

As $OO' = OA = 90$ cm

$$\angle OO'A = \angle O'AO = \alpha, \text{ say}$$

thus for $\Delta OO'A$,

$$\alpha + \alpha + 120° = 180°$$
$$\alpha = 30°.$$

(iii) Draw the skeleton of forces as shown in Fig. 2.18 (c)
Consider equilibrium of skeleton of forces as shown in Fig. 2.18 (c)

$\sum X = 0$,

$$R_A.\cos\alpha = 2400 \qquad .....(1)$$

$\sum Y = 0$,

$$R_A.\sin\alpha = P \qquad .....(2)$$

from equation (1),

$$R_A = \frac{2400}{\cos 30°}$$
$$R_A = 2771.28\ N$$

Substituting value of R_A in equation (2),

$$P = R_A \sin\alpha = 2771.28 \sin 30°$$
$$P = 1385.64\ N$$

Example 2.15

In the previous question, if pull, P is applied at centre of the roller then determine the minimum pull P, which will cause the roller to just turn over the brick.

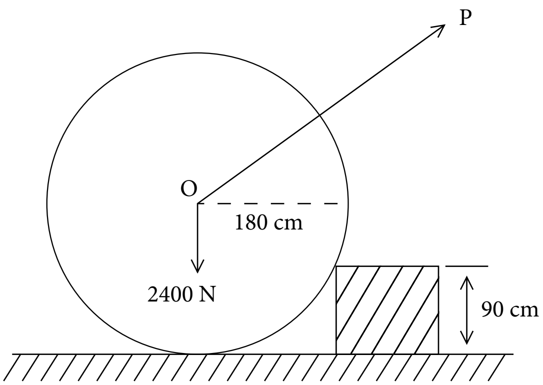


Fig. 2.19

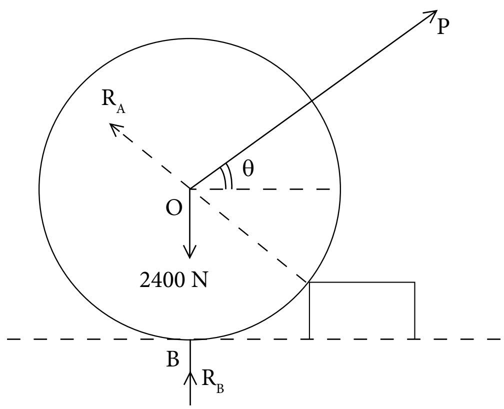


Fig. 2.19 (a)

Solution:

(i) Draw *F.B.D.* of the roller,

Let the minimum pull takes place at an angle θ from the horizontal line as shown in Fig. 2.19 (*a*)

$R_B = 0$ As stated in earlier question.

Roller will be in equilibrium under forces P, 2400 N and R_A. As two forces are acting at centre thus R_A will also go through O to formulate concurrent force system in equilibrium. Simplified *F.B.D.* as shown in Fig. 2.19 (*b*).

(ii) Draw the skeleton of force system as shown in Fig. 2.19 (*c*)

Let $\angle AOM = \alpha$,

$$OM = 180 - 90 = 90 \text{ cm}$$

$$\cos\alpha = \frac{90}{180} = \frac{1}{2}$$

$$\alpha = 60°$$

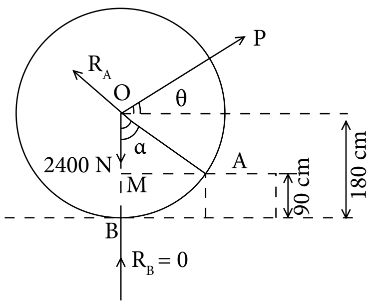


Fig. 2.19 (b)

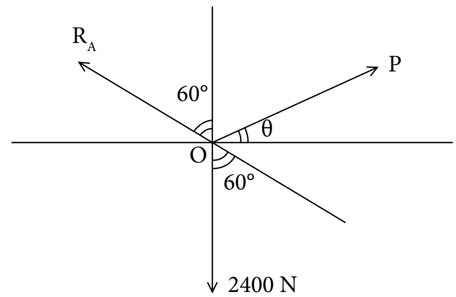


Fig. 2.19 (c)

(iii) By Lami's Theorem,

$$\frac{R_A}{\sin(90° + \theta)} = \frac{P}{\sin(180° - 60°)} = \frac{2400}{\sin(60 + 90 - \theta)}$$

$$P = \frac{2400 \cdot \sin 60°}{\sin(150° - \theta)} \quad .....(1)$$

P will be minimum when denominator is maximum.

thus, $\sin(150° - \theta) = 1$

$$\sin(150° - \theta) = \sin 90°$$

$$150° - \theta = 90°$$

$$\theta = 60°$$

Substituting 'θ' in equation (1),

$$P = 2078.46 \text{ N}$$

By Resolution method,

$\sum X = 0$, $R_A \sin 60° = P\cos\theta$ .....(2)

$\sum Y = 0$, $R_A \cos 60° + P\sin\theta = 2400$ .....(3)

Substituting R_A from equation (2) to equation (3)

$$\frac{P.\cos\theta}{\sin 60°} \cdot \cos 60° + P\sin\theta = 2400$$

$$P[\cos\theta \cos 60° + \sin\theta \sin 60°] = 2400 \sin 60°$$

$$P = \frac{2400\sqrt{3}}{(\cos\theta + \sqrt{3}\sin\theta)} \quad .....(4)$$

P will be minimum, if ($\cos\theta + \sqrt{3}\sin\theta$) is maximum, for maxima and minima condition,

$$i.e., \qquad \frac{d}{d\theta}(\cos\theta + \sqrt{3}\sin\theta) = 0$$

$$-\sin\theta + \sqrt{3}\cos\theta = 0$$

$$\tan\theta = \sqrt{3}$$

$$i.e., \quad \theta = 60°$$

if P is minimum at $\theta = 60°$ then second differentiation should be negative

$$i.e., \frac{d^2}{d\theta^2}(\cos\theta + \sqrt{3}\sin\theta) \text{ should be} - \text{ve}$$

$$= \frac{d}{d\theta}(-\sin\theta + \sqrt{3}\cos\theta)$$

$$= (-\cos\theta - \sqrt{3}\sin\theta)$$

$$= (-\cos 60° - \sqrt{3}\sin 60°)$$

$$= -\frac{1}{2} - \frac{\sqrt{3}.\sqrt{3}}{2} = -2$$

thus P is minimum when $\theta = 60°$
from equation (4)

$$P_{\min} = \frac{2400.\sqrt{3}}{(\cos 60° + \sqrt{3}\sin 60°)}$$

$$P_{\min} = 2078.46 \text{ N}$$

Example 2.16

Two identical spheres of weight 1000 N are held in equilibrium on inclined plane and against wall as shown in Fig. 2.20. Determine reactions at *A, B, C* and D.

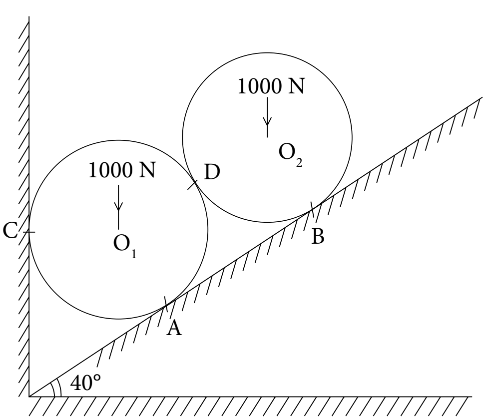


Fig. 2.20