Geometry of String Theory Compactifications

Alessandro Tomasiello

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String theory is a leading candidate for the unification of universal forces and matter, and one of its most striking predictions is the existence of small additional dimensions that have escaped detection so far. This book focuses on the geometry of these dimensions, beginning with the basics of the theory, the mathematical properties of spinors, and differential geometry. It further explores advanced techniques at the core of current research, such as G-structures and generalized complex geometry. Many significant classes of solutions to the theory's equations are studied in detail, from special holonomy and Sasaki–Einstein manifolds to their more recent generalizations involving fluxes for form fields. Various explicit examples are discussed, of interest to graduates and researchers.

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Preface

There are already several excellent references on string theory [1–11]. This book focuses on one particular aspect: the geometry of the extra dimensions. Many interesting techniques have been developed over the years to find and classify string theory vacuum solutions, such as G-structures and pure spinors; I felt it would be useful to collect these ideas in a single place.

The intended audience is mostly advanced graduate students, but I tried to make the book interesting also to more experienced researchers who are not already working on this subject. I assume the reader has basic knowledge of general relativity, Lie groups, and algebras, and nodding acquaintance of the main ideas of supersymmetry. Proficiency in quantum field theory is very welcome but not heavily used. The basics of string theory and supergravity are recalled in Chapter 1, but in a presentation skewed toward the needs of the rest of the book, and not meant to give a complete picture of the field. Many details, such as the supersymmetry transformations, are postponed to a later stage, after a long mathematical detour in Chapters 2–7 allows us to present them with the appropriate level of sophistication.

Chapters 2 and 3 focus on the algebraic properties of spinors, and their deep relationship with forms. Here spacetime is taken to be flat. With respect to other introductions to these topics, I have emphasized the relation between a spinor and its bilinear tensors, and reviewed how forms can be considered as spinors for a doubled Clifford algebra; these are central to efforts in later chapters to rewrite supersymmetry in terms of exterior algebra. I have also considered a wide range of dimensions, both in Lorentzian and Euclidean signatures; this is perhaps a bit more than is really needed in later chapters, but it might be useful for readers who intend to go beyond the topics covered in the book. These chapters are also the most technically detailed; the aim was to teach how to carry out these computations as painlessly as possible – I have taken care to describe most steps, without resorting too often to the magical sentence “it can be shown that” to hide inaccessible derivations.

Chapters 4–7 are dedicated to geometry, but I have tried to keep them focused on physics needs. Chapter 4 is an introduction to differential geometry; these are standard topics, but sometimes they give an occasion to put the techniques of the earlier chapters to good use. In Chapter 5, we encounter G-structures, a well-known geometrical concept that has become very useful in supersymmetry. It is a very general framework, and I have discussed complex, Kähler, and Calabi–Yau geometry from this point of view. Kähler manifolds are those where computations are easiest, and so the entire Chapter 6 is devoted to them. Chapter 7 is devoted to manifolds with special holonomy where the Ricci tensor vanishes, such as Calabi–Yau’s. This includes a lengthy close-up on conical manifolds, which are important later for AdS compactifications.

We get back to physics with Chapter 8. This is an elementary introduction to compactifications with relatively little mathematics, in the simple settings of pure gravity and string theory without flux fields. I have also provided here a quick review of four-dimensional supergravity, for later use. Chapter 9 starts with Calabi–Yau compactifications; these are not too realistic but are still the field's gold standard for rigor and depth. We later modify them by including D-branes and fluxes, covering in particular the important F-theory and conformal Kähler classes of Minkowski vacua.

Chapter 10 is a more systematic investigation of vacuum solutions. Here we finally introduce the supersymmetry transformation in full generality, and rewrite them in terms of forms using G-structures, and more precisely their *doubled* variants using pure forms from Chapter 3. This chapter is again rather technical at times, but the result is a very general system of form equations, which we can then use to look for supersymmetric solutions without having to consider spinors any more. Here and elsewhere, parts marked by an asterisk are harder and can be skipped on first reading. At the end of the chapter, we give a geometrical interpretation to this system in terms of so-called generalized complex geometry. We then proceed in Chapter 11 to a more detailed review of AdS_d solutions in $d \geq 4$, focusing on supersymmetric ones but mentioning supersymmetry-breaking in various instances. In some cases, we can give a complete classification of explicit solutions. We end in Chapter 12 with a quicker review of efforts to obtain dS vacua and of the swampland program, and with some final thoughts.

The book is not meant to be comprehensive. Notably, I have mostly focused on vacuum solutions, perhaps not paying enough attention to the broader geometry of reductions. After Chapter 1, I have devoted most attention to type II supergravity, and perhaps not enough to M-theory and heterotic strings. In general, I almost always avoided $d < 4$ vacua; and I have covered holography only superficially, in Chapter 11. Several other important topics have not been given the space they deserved. I hope readers disappointed by such omissions will forgive me after checking the total page count, which is already testing my editors' patience; I was also wary of the danger of producing a soulless encyclopedia. In general, the number of pages dedicated to a subject should not be construed as a judgment of its importance. I have been lengthier on topics that I feel are less thoroughly covered in other books, and sketchier on those where lots of great material is available already, and which I am including for context and completeness. On controversial issues, I have tried to represent all sides as fairly as I could; I have not tried to hide my opinion, but I believe the proper place to articulate it is in research articles.

I have learned these topics from my teachers, my collaborators, and my students. I am especially grateful to Loriano Bonora, Michael Douglas, Davide Gaiotto, Mariana Graña, Shamit Kachru, Dario Martelli, Ruben Minasian, Michela Petrini, and Alberto Zaffaroni. During the writing phase, I was helped by Bruno De Luca, Suvendu Giri, Andrea Legramandi, Gabriele Lo Monaco, Luca Martucci, Achilleas Passias, Vivek Saxena, and Riccardo Villa, and by the great staff at CUP. Special thanks go to Francesca Baviera, Concetta Fratantonio, and Luciano Tomasiello, although I am sorry the latter could not wait to see this finished. In spite of all this help, I am of course aware that the final product will turn out to have lots of typos, imprecisions, and outright mistakes; I will maintain a list of corrections on my personal website.

Conventions

- Lorentzian signature is "mostly plus."
- The word "generic" means "for any choice except for a set of measure zero."
- The antisymmetrizer of k indices is denoted by square brackets and includes a $1/k!$; for example, $v_{[m}w_{n]} = \frac{1}{2}(v_m w_n - v_n w_m)$. The symmetrizer is denoted by round brackets, so $v_{(m}w_{n)} = \frac{1}{2}(v_m w_n + v_n w_m)$. A vertical slash | is used to exclude indices from these operations.
- The floor function $\lfloor x \rfloor$ is the integer part of x or, in other words, the largest integer n such that $n \leq x$.
- The chiral matrix is $\gamma = c\,\gamma^0 \ldots \gamma^{d-1}$ $(c\,\gamma^1 \ldots \gamma^d)$ for Lorentzian (Euclidean) signature. The constant c is constrained by (2.1.20) so that $\gamma^2 = 1$: notably, we take $c = \mathrm{i}$ for $d = 4$ Lorentzian, $c = -\mathrm{i}$ for $d = 6$ Euclidean, $c = 1$ for $d = 10$ Lorentzian.
- The identity matrix in d dimensions is denoted by 1_d or often simply by 1.
- d is most often the real dimension of a manifold; occasionally it denotes degree of a polynomial. Complex dimension is sometimes denoted by N.
- When working in an index-free notation, we use the same symbol v for a vector field with components v^μ and its associated one-form $g_{\mu\nu}v^\nu$.
- Indices $\mu, \nu \ldots$ are for Lorentzian signature; $m, n, \ldots$ for Euclidean signature; i, $j \ldots$ are holomorphic indices. An exception is $d = 10$ (and $d = 26$) Lorentzian, where we use $M, N \ldots$. Flat (vielbein) indices are $a, b \ldots$. Indices $\alpha, \beta \ldots$ are usually spinorial.
- The vielbein $e^a = e^a_m \mathrm{d}x^m$ is an orthonormal basis of one-forms, so the line element is $\mathrm{d}s^2 = g_{mn}\mathrm{d}x^m \mathrm{d}x^n = e^a e_a$. Its inverse is denoted by $E_a = E^m_a \partial_m$, an orthonormal basis of vector fields. We also often need a holomorphic vielbein, defined by (5.1.22), (5.1.35), and hence $\mathrm{d}s^2 = \sum_{a=1}^{d/2} h^a \bar{h}^{\bar{a}}$.
- The spinor covariant derivative is $D_m = \partial_m + \frac{1}{4}\omega_m^{ab}\gamma_{ab}$ (Section 4.3.3).
- The components of a k-form are defined by $\alpha_k = \frac{1}{k!}\alpha_{m_1\ldots m_k}\mathrm{d}x^{m_1} \wedge \cdots \wedge \mathrm{d}x^{m_k}$. The Clifford map associates to it a bispinor $\alpha\!\!\!/_k \equiv \frac{1}{k!}\alpha_{m_1\ldots m_k}\gamma^{m_1\ldots m_k}$, with $\gamma^{m_1\ldots m_k} \equiv \gamma^{[m_1} \ldots \gamma^{m_k]}$. For lengthy expressions, we also use the notation $(\alpha_k)_/ \equiv \alpha\!\!\!/_k$, but often we don't use any symbol at all and denote by α_k both a form and the associated bispinor, with an abuse of language.
- A vector field acts on a spinor as $v \cdot \eta \equiv v^\mu \gamma_\mu \eta = v_\mu \gamma^\mu \eta = v\!\!\!/\eta$, or also just $v\eta$ by the previous point.
- $\mathrm{vol} = e^1 \wedge \ldots \wedge e^d$ is the volume form, while the volume of a manifold M is denoted by $\mathrm{Vol}(M)$.
- A chiral spinor η_+ is said to be pure if it is annihilated by $d/2$ gamma matrices; in flat space, this defines a notion of a (anti-)holomorphic index, for which we use the convention $\gamma_{\bar{\imath}}\eta_+ = 0$

- The complex conjugate of a complex number z is denoted by z^* or $\bar{z}$. For a complex matrix M, $M^\dagger \equiv (M^*)^t$. The conjugate of a spinor is $\zeta^c \equiv B\zeta^*$ (Section 2.3.1). In Lorentzian signature, we also define $\overline{\zeta} = \zeta^\dagger \gamma_0$.
- We typically use the letter ζ for spinors in Lorentzian signature, and η for Euclidean signature.
- Span$\{v_1, \ldots, v_k\}$ is the vector subspace of linear combinations of the v_a.

Introduction

The idea that spacetime might have additional dimensions might seem preposterous at first. It has come to the fore of current research in theoretical physics by two strands of thought.

0.1 String theory

One comes from attempts at quantizing gravity. The problems one encounters in general relativity at high energies suggest that it is superseded in that regime by a different theory. A prominent candidate is *string theory* (to be reviewed in Chapter 1). It describes interacting strings, which at low energies are seen as particles, some of which behave as gravitons, thus reducing at low energies to a version of general relativity (GR). The other particles behave in ways that look complicated enough to accommodate the phenomena we see in particle physics. So string theory solves the high-energy problems of GR, and gives a possible strategy to unify not only all forces but also all matter. Remarkably, the theory is so constrained by various anomalies that it has no free parameters. This is perhaps what one should expect from a unified theory of all physics.

String theory does come, however, with a heavy conceptual framework. This includes supersymmetry, which plays an important role in the theory's internal consistency; it could be broken spontaneously at Planck energies and be hidden from observations for a long time. More importantly for us, string theory only works in more than four dimensions. In its best-understood phase, *six* additional dimensions are needed, with a seventh also sometimes emerging. To avoid conflict with observations, we need to postulate that the compact space M_6 they span is small enough that current experiments have not revealed it yet. A *compactification* is a spacetime that looks four-dimensional macroscopically, even if it actually has a larger number of dimensions.

0.2 Kaluza–Klein reduction

This idea is natural enough that it had been considered long before string theory [12, 13]. The reason is that it gives a simpler, independent way to unify gravity with other elementary forces. This was first noticed in GR with a single additional dimension, wound up on a circle S^1 with an extra coordinate x^4. The various components of the metric are viewed by a four-dimensional observer as fields of different spin: $g_{\mu\nu}$ as a four-dimensional metric, $g_{\mu 4}$ as a vector field, and g_{44} as a scalar. The first two can be interpreted as describing gravity and electromagnetism in four dimensions.

The field dependence on the extra dimension x^4 also gives rise to a "tower" of massive spin-two fields with masses

$$m_k = \frac{2\pi k}{L} \,, \tag{0.2.1}$$

where L is the size of the S^1. A similar phenomenon can be seen already with a free scalar σ on $\mathbb{R}^4 \times S^1$: if we expand σ in a Fourier series with respect to x^4 and plug the expansion in the Klein–Gordon equation $(\partial_\mu \partial^\mu + \partial_4^2)\sigma = 0$, the term ∂_4^2 gives a mass (0.2.1) to the kth Fourier mode. With gravity, $g_{\mu\nu}$, $g_{\mu 4}$, and g_{44} all undergo the same phenomenon: the massive spin-two fields then "eat" the massive modes of the other components, in a version of the Brout–Englert–Higgs (BEH) mechanism. The infinite sequence of masses (0.2.1) is called a *KK tower*, and the corresponding fields are called *KK modes*.

As expected by dimensional analysis, (0.2.1) are inversely proportional to L; so when L is small, these masses are large and might have avoided detection so far. Even $L \sim 10^{-19} m$ leads to $m_k \sim O(\text{TeV})$. We will review in more depth the physics of this five-dimensional model in Section 8.1.1.

With $d > 1$ additional dimensions, one can consider more complicated spaces M_d, which can now also realize Yang–Mills (YM) theories. The symmetry group of M_d becomes the YM gauge group. From this point of view, the idea of extra dimensions is an evolution of that of "internal symmetry" in the world of elementary particles. It is the postulate that those symmetries have a geometrical origin.

0.3 String compactifications

The topic of this book is the study of the "internal space" of string theory. While the theory itself has no free parameters, the choice of M_6 introduces a lot of freedom. As we will see in Section 4.2.5, the possible topologies for a six-dimensional compact space are classified by a few algebraic data (the dimensions of two vector spaces and two polynomial functions). But the space of possible metrics for each given topology is infinite dimensional.

Perhaps the simplest question we can ask is whether by compactifying on M_6 we can find at least a *vacuum solution*: namely, one where the macroscopic spacetime is empty. This means that the stress energy tensor is zero, or consists at most of a cosmological constant. Such a space should locally have as many symmetries as flat Minkowski space, and is said to be *maximally symmetric (MS)* (Section 4.5); the possibilities are Minkowski space itself, de Sitter (dS) space, and anti-de Sitter (AdS) space, with a positive, zero, or negative cosmological constant Λ. We will argue in Chapter 8 that for such a solution the line element for the ten-dimensional metric reads

$$ds_{10}^2 = e^{2A} ds_{\text{MS}_4}^2 + ds_{M_6}^2 \,. \tag{0.3.1}$$

$A = A(y)$ is called the *warping function* of the coordinates y^m on M_6, and this form of the metric is called *warped product*.

The question of which M_6 lead to vacuum solutions is already rather hard: it involves solving the equations of motion, which reduce to partial differential equations on M_6. An easy case is when all of string theory's fields have zero expectation value except the metric; the equations of motion then say that $R_{MN} = 0$. These imply that the maximally symmetric $\mathrm{MS}_4 = \mathrm{Mink}_4$, and that M_6 is Ricci-flat (Chapter 7). Many such spaces are the so-called *Calabi–Yau manifolds*. When the other fields are also present, finding and classifying solutions is much harder. This is one of the main topics of this book.

After finding a vacuum solution, one would like a description of the physics one would observe in it. Here one faces a choice between precision and broadness. At one extreme, the *KK spectrum* is the information about all the particle masses and spins for a single given vacuum, but without any information about their interactions.

At the other extreme, one focuses on a small subset of fields, but with complete information about their interactions, and in particular about the potential for the scalars. Such an action S_4 might describe many vacua at once. As usual, we call it an effective theory if the scalars we kept span a "valley" with a relatively mild potential V_{eff}, much smaller than the potential for the scalars we discarded. Sometimes in string theory we are forced to work without such a scale separation; one calls this a *nonlinear reduction*, and we then want at least that its vacua correspond to vacuum solutions of fully fledged string theory. We will give a longer introduction to these ideas in Section 8.3, and then see several examples in later parts of the book.

Sometimes one reverses this procedure, and one uses an effective theory or nonlinear reduction to find new vacuum solutions rather than to describe the physics of one that was previously found. Some of the theory's most celebrated solutions have been found this way. However, our focus will be on techniques to find vacua directly in ten (or eleven) dimensions.

0.4 Supersymmetric vacua and geometry

To find vacuum solutions, it proves easier to start from those where supersymmetry is partially preserved and to break it later, rather than trying to solve string theory's equations of motion in general. Technically, preserved supersymmetry gives a first-order system of equations, which partially implies the second-order equations of motion. More importantly, supersymmetry helps finding solutions because it naturally invokes several deep geometrical ideas.

We cannot cover the internal M_6 with a single coordinate system; we have to use several, related by coordinate changes called transition functions, in general valued in the group $GL(6, \mathbb{R})$ of invertible matrices. A *G-structure* on M_6 is a choice of transition functions valued in a smaller group G (Chapter 5). The infinitesimal parameters for supersymmetry are spinors η; they naturally define a G-structure, with $G = \mathrm{Stab}(\eta)$ their little group (or *stabilizer*), the group of rotations that keep η invariant. This helps trading η with other, nonspinorial geometrical objects on M_6; often antisymmetric tensors, or *forms*.

For example, a single η on M_6 defines an SU(3)-structure, which can also be defined by the metric and a complex three-form (an antisymmetric tensor with three indices) Ω. It is further possible to trade even the metric for a real two-form J. The G-structure techniques allow one to recast the supersymmetry equations, originally involving η, directly in terms of J and Ω. Often one then recognizes a well-known mathematical concept, and this helps finding solutions. This procedure is also natural because most of the string theory fields beyond the metric are themselves forms, analogues with many indices of the electromagnetic field-strength $F_{\mu\nu}$.

Some of the most interesting vacua are in type II, where there are two η^a. In this case, it proves more fruitful to work with a doubled, or *generalized*, version of the rotation group. Again, the upshot is that we may trade the data of the η^a and of the metric with forms, this time a pair $\Phi_\pm$ of them with an algebraic property called *purity* (Chapters 2 and 3). In this language, the supersymmetry equations become particularly elegant, making contact with *generalized complex geometry* (Chapter 10).

The fact that the metric is included in this trade-off with forms is particularly intriguing. It is reminiscent of previous attempts to reformulate GR such as [14, 15].

0.5 The cosmological constant

The observed cosmological constant is positive, so one would like to focus on the case $\mathrm{MS}_4 = \mathrm{dS}_4$. These are actually the hardest solutions to obtain. To see why, consider a general gravitational theory with an Einstein–Hilbert (EH) kinetic term [16–18]. The equations of motion read $R_{MN} - \frac{1}{2}g_{MN}R = 8\pi G_\mathrm{N} T_{MN}$, where as usual G_N is Newton's constant, R_{MN} is the Ricci tensor, R its trace, and T_{MN} the stress–energy tensor of various matter fields; the indices $M, N = 0, \ldots, 3 + d$. The "trace-reversed" Einstein equations are

$$R_{MN} = 8\pi G_\mathrm{N} \left(T_{MN} - \frac{1}{2+d} g_{MN} T_P{}^P \right) . \tag{0.5.1}$$

For a warped product metric as in (0.3.1) (generalizing $M_6 \to M_d$),

$$R_{\mu\nu} = \left(\Lambda - \frac{1}{4} \mathrm{e}^{-2A} \nabla^2 \mathrm{e}^{4A} \right) g^4_{\mu\nu} , \tag{0.5.2}$$

where $g^4_{\mu\nu}$ are the components of the metric of the external MS_4 space, Λ its cosmological constant (normalized as $R^4_{\mu\nu} = \Lambda g^4_{\mu\nu}$), and ∇ the internal covariant derivative. (We will derive (0.5.2) in an exercise in Chapter 4.) In coordinates where $g^4_{00} = -1$, the time components of (0.5.1) give

$$-\mathrm{e}^{2A}\Lambda + \frac{1}{4}\nabla^2 \mathrm{e}^{4A} = 8\pi G_\mathrm{N} \mathrm{e}^{2A} \left(T_{00} - \frac{1}{2+d} g_{00} T_P{}^P \right) . \tag{0.5.3}$$

The parenthesis on the right-hand side is nonnegative if the higher-dimensional theory obeys the *strong energy condition*. This is an assumption often made in general relativity, for example in proving singularity theorems for black holes and cosmology; see, for example, the discussions in [19, 4.3;8.2] and [20, chap. 9].

Supposing it holds, we integrate (0.5.3) on the compact M_6; the second term on the left-hand side is a total derivative and gives no contribution, so

$$\Lambda \leq 0\,. \tag{0.5.4}$$

We conclude that a theory with an EH kinetic term obeying the strong energy condition has *no de Sitter compactifications*. Even Minkowski compactifications are only marginally allowed, requiring the parenthesis in (0.5.3) to vanish.

Does this apply to string theory? We mentioned that it reduces at low energies to a version of GR, which due to supersymmetry is called *supergravity*. In this regime, the graviton has an EH kinetic term, coupled to various other fields and to certain localized sources. All the fields satisfy the strong energy condition, except for a term called *Romans mass*. But this possible loophole in the argument was closed in [18] (Section 10.3.1).

For localized sources, violating the strong energy condition requires negative tension, leading to repulsive gravity, and usually to instabilities for dynamical objects. The sources in string theory are of two types, called *D-branes* and *O-planes* (Sections 1.3 and 1.4.4). The first are defined as spacetime defects on which strings can end; this makes them dynamical. The second arise after quotienting string theory by a parity-like symmetry, and arise at its fixed loci; so they are not dynamical. Some of them have indeed negative tension, and so they do invalidate (0.5.3).

The conclusion is that, in the regime where string theory is described by supergravity, de Sitter compactifications require O-planes. Minkowski compactifications also need them, unless all the fields except gravity are turned off.

0.6 Beyond supergravity

Let us consider now a $d = 10$ effective field theory S_{10}. This is useful in a regime of energies high enough to see the extra dimensions, but low enough to see strings as particles; it is not to be confused with the $d = 4$ effective action S_4, relevant at lower energies where we cannot resolve the extra dimensions. In S_{10}, supergravity is the collection of the most relevant operators at energies well below the Planck scale, which in turn is related to a new fundamental length scale l_s, the "typical length" of strings. But supergravity is not renormalizable; this manifests itself in the presence of higher-derivative corrections, which become relevant at high energies, or when the curvature gets large. These are not known completely, but there is no reason to expect that the result (0.5.4) still holds when they are introduced. For example, one famous leading correction has the form $\int \mathrm{d}^{10}x(\text{Riemann})^4$. So not even the metric appears simply through an EH kinetic term, as in supergravity. Unfortunately, only the first few terms have been computed.

This introduces other challenges. These corrections will contribute to the potential of the $d = 4$ effective action S_4 (Chapter 8), whose vacua should approximate string theory's vacuum solutions as defined earlier. Very naively, suppressing the dependence on other fields, we will see that the EH term and the $(\text{Riemann})^4$ term give two contributions to the potential:

$$V_4 \sim ar^{-2} + br^{-8}\,, \tag{0.6.1}$$

where $r \equiv R/l_s$ is the length scale of the internal space M_6, in units of the string length. If a and b have opposite sign, this has an extremum at $r = (-4b/a)^{1/6}$; but if $a, b \sim O(1)$, also $r \sim O(1)$, and $R \sim l_s$. But in this regime, other (Riemann)^k, $k > 4$ corrections would also be relevant, contributing further terms r^{-2k} to (0.6.1); so we cannot trust our extremum. This illustrates a general issue: if we find a solution by using one string correction to supergravity, we can expect it to be in a regime where *all* string corrections are relevant, where we in fact cannot compute anything. This is the *Dine–Seiberg problem* [21].

Fortunately, not all the terms in the $d = 4$ effective potential are $\sim O(1)$. The aforementioned form fields have to satisfy a certain Dirac quantization; this introduces integers, which can be taken to be large, introducing a hierarchy that eventually makes $R \gg l_s$. From the point of view of S_4, this is behind the existence of most vacua, but usually the terms that compete originate from the leading supergravity approximation. (For the Calabi–Yau vacua, the leading supergravity contribution vanishes; we will see in Chapter 9 a more delicate argument to show that the corrections don't destroy the vacuum.)

0.7 Overview of vacua

The argument (0.5.4) indicates that finding vacua is easiest when the cosmological constant is negative, which as we mentioned is contrary to observations. These AdS vacua have found applications in *holography*, which relates them to quantum field theory models with conformal invariance, or conformal field theories (CFTs). Another reason not to discard them is that the supersymmetry-breaking procedure sometimes also changes Λ. For these reasons, we dedicate Chapter 11 to a survey of such vacua. Several classification results are available here, and several more are likely to emerge in the near future, as techniques improve. For example, a list of all supersymmetric AdS_6 and AdS_7 solutions has been achieved relatively recently.

Minkowski vacua are more tightly constrained. Relative to AdS, this is expected, if nothing else because $\Lambda = 0$ is an equality, not an inequality. A priori the general equations seem to allow for M_6 of any curvature; but so far the vast majority of known supersymmetric vacua are related to Calabi–Yau manifolds, one way or another. For example, in a famous class in Chapter 9 the Calabi–Yau metric is only modified by an overall function. Until recently, one might have thought this to be an artifact of technical limitations in including O-planes, which as we have already argued are necessary. However, many AdS vacua with O-planes have now been constructed and seem to allow for a far greater variety of internal spaces.

Finally, for the de Sitter case, the situation gets even less clear (Chapter 12). One additional complication is that supersymmetry is necessarily broken and cannot guide us any more. Most models evade (0.5.4) by involving both O-planes and quantum effects, which are harder to control. As a result, all of them have attracted some objections.

The first and most successful proposal, the KKLT model [22], again obtained by modifying a Calabi–Yau metric on M_6, generates a large quantity of dS vacua, with numbers such as 10^{hundreds} or even $10^{\text{thousands}}$ often quoted. The resulting picture has been dubbed the *string theory landscape* [23], borrowing a metaphor from protein-folding research. Leaving aside for now any criticism of this and other models, the possibility that a seemingly formidable obstacle such as the cosmological constant could be overcome so easily has suggested that the number of vacua reproducing *all* other observed features of our Universe could still be very large. This has created much confusion in causal observers of the field. Perhaps string theory has no predictive power?

This question appears misguided. First of all, the large numbers 10^N arise from discretizing an N-dimensional continuous space; in other words, from allowing several discrete possibilities to a set of N free parameters. Conceptually, this is not that different from the 19 (or more) free parameters in the standard model, which prior to experiment has an ∞^{19} of possibilities. Rightly, no one complains about the latter large number because, given enough experiments to fix the parameters within a certain range, the standard model makes testable predictions about other experiments. This would be true as well for string theory; the fact that such experiments are beyond human capabilities for the foreseeable future is of course unfortunate, but is in the nature of the problem of quantum gravity, whose characteristic scale is after all m_{Planck}.

Even more importantly, string *theory* is a framework, within which there are *models* with free parameters, such as the KKLT model. It would of course be senseless to criticize quantum field theory because it cannot predict the standard model of particle physics from first principles, or criticize quantum mechanics because it does not predict the potential in the Schrödinger equation. Of course, quantum field theory is a scientific *theory*, in the appropriate sense: given enough experimental data, it can provide a *model*, such as the Standard Model, which makes new experimental predictions.

An alternative point of view is to focus on the vacua that *cannot* be found in string theory. The *swampland* program [24] looks for models that look consistent in field theory but cannot be coupled to quantum gravity. While the inspiration often comes from string theory, the aim is to find universal properties that are valid beyond it. In recent years, this program has started to clash with many of the predictions of the effective field theory approach, including the existence of dS vacua. Some of this debate is covered in the final Chapter 12, with the unfortunate result of ending with more questions than answers.

1 String theory and supergravity

As stated in the Preface, this book assumes some rudimentary knowledge of string theory, but it is a good idea to recall the basics. The field is notoriously vast and complex, so this chapter should not be understood as a replacement for serious study on one of the many great introductions [1–11]. In most of the book, we will approximate string theory by *supergravity*, an effective theory of gravitons and other fields; the presentation will be biased toward that.

In this chapter, we also assume knowledge of general relativity (GR) and some acquaintance with spinors, but we will try to keep mathematical sophistication at a minimum. We will develop some ideas, such as spinors and differential geometry, in much greater detail in the next few chapters before we return to physics. Still, already in this chapter we will pepper our presentation with occasional forward references to those mathematically more advanced treatments, to whet the reader's appetite.

1.1 Perturbative strings

A quantum field collects creation (and annihilation) operators for a representation of the Poincaré group. Once one fixes the value of the momentum p of the created state, the remaining degrees of freedom are a representation of the *little group*, or *stabilizer*, of p, namely the subgroup $\mathrm{Stab}(p) \subset \mathrm{SO}(d)$ of elements that leave p invariant. This is

$$\mathrm{Stab}(p) = \mathrm{SO}(d-1)\ (p^2 < 0)\,, \qquad \mathrm{Stab}(p) = \mathrm{SO}(d-2) \ltimes \mathbb{R}^{d-2}\ (p^2 = 0)\,, \quad (1.1.1)$$

for the massive or massless case. (We will review the $p^2 = 0$ case in Section 3.3.6.) In the massless case, we would also have the possibility of selecting an infinite-dimensional representation, but this is usually regarded as exotic; so we select a finite-dimensional representation, ignoring the $\mathbb{R}^{d-2}$ factor. Ordinary fields then represent objects with finitely many degrees of freedom, which we call spin and helicity for $m^2 > 0$ and $= 0$, respectively. Moreover, we usually take these objects to interact via terms of the type $\int \mathrm{d}^d x \phi_1(x) \dots \phi_2(x)$: these allow the value of a field to influence directly that of another only at the same point.

All these reasons make us think of the quanta of a field as point particles. To describe a quantum theory of interacting extended objects, we need to change this picture somehow. First of all, a string can have infinitely many vibration modes, so a field that creates a string must be somehow a collection of infinitely many ordinary fields. Second, extended objects can interact when their centers of mass are not superimposed. So the interaction terms should be nonlocal.

Such a *string field theory* (SFT) is fascinating but also just as complicated as our description suggests. So in fact most studies of interacting strings focus on an approach that is *first-quantized*: one first decides the Feynman diagram one wants to consider, and then computes the amplitude associated with it. (A similar approach is used sometimes in quantum field theory too, under the name of *world-line formalism*.)

In this section, we will review quickly some aspects of this perturbative treatment of string interactions. There are five possible consistent string models:

- Type IIA
- Type IIB
- Heterotic with gauge group $E_8 \times E_8$
- Heterotic with gauge group SO(32)
- Type I

All these select $d = 10$ as spacetime dimension, in a sense we will clarify later in this chapter. The last case, type I, can be viewed as a certain quotient procedure from IIB strings, which we will introduce in Section 1.4.4. So in this section we will discuss the other four. We will actually start our discussion from a model that has a *tachyon*, namely a scalar with a negative mass, but whose discussion is simpler: the *bosonic* string.

1.1.1 Bosonic strings

The action for a particle moving in a curved background is proportional to its "length in spacetime," namely, to the proper time measured along its world-line (its trajectory γ in spacetime):

$$S_{\text{part}} = -m \int_\gamma d\sigma_0 \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}, \tag{1.1.2}$$

where $x^\mu(\sigma_0)$ are the coordinates of the point in spacetime as a function of the world-line coordinate σ_0, and $\dot{x}^\mu \equiv \partial_0 x^\mu$. In flat space, this is indeed minimized on straight lines in spacetime, which maximize proper time. For curved $g_{\mu\nu}$, (1.1.2) is minimized on geodesics. If we also have a Maxwell field and our particle is charged, we have to add a term

$$S_{\text{part,EM}} = q \int_\gamma d\sigma_0 A_\mu \partial_0 x^\mu, \tag{1.1.3}$$

where q is the charge, and A_μ is the vector potential. In Section 4.1.4, we will see that the integrand is an example of a natural operation called *pull-back*.

String action

By analogy with (1.1.2), the natural action for a string would seem to be the volume of its two-dimensional world-sheet in spacetime. However, it is classically equivalent to the *Polyakov action*, which is easier to quantize:

$$S_{\text{F1},g} = -\frac{1}{2} T_{\text{F1}} \int_\Sigma d^2\sigma h^{\alpha\beta}\sqrt{-h} g_{MN} \partial_\alpha x^M \partial_\beta x^N . \tag{1.1.4}$$

This type of action is also called a *sigma model*, for reasons going back to four-dimensional models of mesons, or sometimes *nonlinear* sigma model when g_{MN} is not flat. The $x^M(\sigma_0, \sigma_1)$, $M = 0, \ldots, d-1$, describe the embedding of Σ in physical spacetime (often called *target space*), and h is a metric on Σ. The mass m in (1.1.2) has been replaced by the mass/length ratio, or *tension*:

$$T_{\mathrm{F1}} = \frac{1}{2\pi l_s^2}\,. \tag{1.1.5}$$

"F" stands for *fundamental*, to distinguish this string from other extended objects that will appear later; 1 denotes the space extension of the string. The constant l_s is called *string length*. (We will always keep it explicit in this chapter, but later we will often work in string units and set $l_s = 1$.)

In this section, we are going to focus on strings that are *closed* or, in other words, that have no boundary. A generic[1] time slice is then a collection of several copies of the circle S^1. The time evolution of each of these for a finite time will be a cylinder; then σ^1 is a periodic coordinate, $\sigma^1 \sim \sigma^1 + \pi$. These cylinders are then glued together at some values of σ^0 to obtain a general Σ.

Spectrum in flat space

Quantizing (1.1.4) is challenging for general g_{MN} but relatively easy in Minkowski space $g_{MN} = \eta_{MN}$: superficially (1.1.4) then becomes a collection of free bosons, with equations of motion $\partial^2 x^M = 0$. For a closed string, the slice at $\sigma^0 =$ constant is an S^1; there are then discrete Fourier modes for each x^M. Since the equation of motion is of second order, the states are in correspondence to the values of these Fourier modes and their derivatives. Alternatively, we can write a solution of the world-sheet equations of motion as $x^M = x^M(\sigma^+) + x^M(\sigma^-)$, where $\sigma^\pm = \sigma^1 \pm \sigma^0$, and introduce Fourier modes α_i^M, $\tilde{\alpha}_i^M$ for the left- and right-movers $x^M(\sigma^\pm)$. The only subtlety is that the world-sheet metric $h_{\alpha\beta}$ is a Lagrange multiplier, which gives a constraint. This can be taken care of in many ways: by solving the constraint, or by introducing Faddeev–Popov ghosts and the Becchi–Rouet–Stora–Tyutin (BRST) method (the so-called covariant quantization). Skipping many interesting details, here we will just give the results.

Even for a fixed momentum, the spectrum has infinitely many states, of the form

$$\alpha_{-i_1}^{N_1} \ldots \alpha_{-i_n}^{N_n} \tilde{\alpha}_{-\tilde{\imath}_1}^{\tilde{N}_1} \ldots \tilde{\alpha}_{-\tilde{\imath}_n}^{\tilde{N}_n} |0\rangle, \tag{1.1.6}$$

where $|0\rangle$ is the world-sheet vacuum, and i_k, $j_k \geq 0$ (possibly repeated). As we mentioned, these correspond to the vibration modes of the string, and in a spacetime picture they would require infinitely many ordinary quantum fields to create them. Their masses are

$$m^2 = \frac{4}{l_s^2}\left(\frac{2-d}{24} + N\right), \tag{1.1.7}$$

where $N = \sum i_k = \sum \tilde{\imath}_k$ is a nonnegative integer. The identity between these two expressions is called *level matching* and is the link between the left- and right-moving sectors, which otherwise proceed on parallel tracks. If $d > 2$, we see that the lowest

[1] The mathematical meaning of the word *generically*, which we will use in this book, is "for any choice except for a set of measure zero."

value of m^2, for $N = 0$, is actually *negative*. Such a mode is usually called a *tachyon* and signals an instability. For this reason, the bosonic string we are discussing in this section is usually only considered a toy model.

Nevertheless, it already displays a very interesting feature. For the *critical dimension* $d = 26$, the modes with $n = 1$ in (1.1.6) and (1.1.7) are massless. They read

$$\alpha^M_{-1}\tilde{\alpha}^N_{-1}|0\rangle\,, \tag{1.1.8}$$

and so they correspond to fields with two indices. Among these we thus find a *massless spin-two field* $h_{MN} = \delta g_{MN}$. The action (1.1.4) can then be thought of as a string moving in a condensate of such a field. This is a bit similar to expanding a quantum field theory (QFT) around a vacuum where a field has acquired a nonzero expectation value.

So we have found that in $d = 26$, the string modes include those that would normally be associated with a graviton. Remarkably, the scattering amplitudes one obtains with this formalism are *finite*. The string tension acts as a regulator: Taking the limit $l_s \to 0$, the scattering amplitudes become divergent again. In this limit, the theory becomes a local QFT model again, and a local theory of gravity has divergent amplitudes.

Coupling to condensates of other fields

Among (1.1.8), we find other massless modes. Following (1.1.1), we need to consider only the components of (1.1.8) in the $d - 2 = 24$ dimensions transverse to the momentum p, which are 24^2. The physical components h_{MN} of a graviton are represented by a traceless 24×24 matrix; this is the generalization of the transverse traceless (TT) gauge familiar from the treatment of gravitational waves in four dimensions. The remaining modes are thus the antisymmetric part of (1.1.8) and its trace. The fields that create these states are an antisymmetric *Kalb–Ramond* field $B_{MN} = -B_{NM}$, and a scalar field ϕ called *dilaton*. So in total the massless fields of the bosonic string are

$$g_{MN}\,, \qquad B_{MN}\,, \qquad \phi\,. \tag{1.1.9}$$

We can consider condensates of B_{MN} and ϕ, too; this leads to the extra terms in the action:

$$S_{\mathrm{F1},B,\phi} = -\frac{1}{2}T_{\mathrm{F1}}\int_\Sigma \mathrm{d}^2\sigma\left[\epsilon^{\alpha\beta}B_{MN}\partial_\alpha x^M\partial_\beta x^N + l_s^2\sqrt{-h}R_{(2)}\phi\right]. \tag{1.1.10}$$

Here $R_{(2)}$ is the scalar curvature of the world-sheet metric $h_{\alpha\beta}$, and $\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The coupling with B is the natural generalization of the coupling (1.1.3). The coupling with the dilaton is peculiar in that

$$\frac{1}{4\pi}\int_\Sigma \sqrt{-h}R_{(2)} = 2 - 2g\,, \tag{1.1.11}$$

where g is the *genus* of the world-sheet Σ. This is the stringy analogue of the number of loops, and can be intuitively described (when Σ has no boundary) as the *number* *of handles*; a more formal definition will be given in Section 4.1.10. Because of this, the computation of all scattering amplitudes is organized in powers:

$$g_s^{2g-2}\,, \qquad g_s \equiv e^{\phi}\,. \tag{1.1.12}$$

We can think of g_s as a *string coupling constant*: when it is small, the powers (1.1.12) are smaller for Riemann surfaces Σ of increasing g, which can be thought of as the stringy analogue of Feynman diagrams of increasing complexity.

The action

$$S_{\text{bos}} = S_{\text{F1},g} + S_{\text{F1},B,\phi} \tag{1.1.13}$$

is classically invariant under general coordinate transformation $\sigma_\alpha \to \sigma'_\alpha(\sigma_0, \sigma_1)$, if we also take care to transform the world-sheet metric $h_{\alpha\beta}$. This is a gauge invariance, in that it doesn't affect the physical configuration, the image of the world-sheet embedding $x^\mu(\sigma)$, but only how we parameterize it. Equation (1.1.13) is also invariant under Weyl rescaling $h_{\alpha\beta} \to e^f h_{\alpha\beta}$. In two dimensions, one can fix the coordinate-change freedom by taking, for example, $h_{\alpha\beta}$ to have constant scalar curvature. Even so, a residual invariance remains: coordinate transformations that leave the metric invariant up to a Weyl transformation. These are called *conformal transformations*.

Conformal invariance and effective action

It is crucial that this residual gauge invariance remains at the quantum level. It decouples potentially harmful negative-norm states that would come from the fact that x^0 in (1.1.4) has a wrong-sign kinetic term. This is similar to what happens in the quantization of the electromagnetic field, for example. Conformal invariance is also behind the absence of high-energy divergences. Usually scattering amplitudes become problematic when two particles collide at a small impact parameter. The world-sheet of a string scattering is a non-compact Riemann surface with several spikes s_i corresponding to the incoming and outgoing strings. Conformal invariance means that the distance between two points on the world-sheet has no intrinsic meaning: only ratios of distances do. So a small impact parameter might seem to correspond to two such spikes s_1 and s_2 getting close, but that only means that they are close relative to their distance from other external strings s_i. This corresponds to a Riemann surface that develops a long neck, where the two s_i are both attached, far from the others.

The Noether current associated to dilatations in a field theory is $T^{\mu\nu}x_\nu$, where $T_{\mu\nu}$ is the stress–energy tensor. This is conserved if $0 = \partial_\mu(T^{\mu\nu}x_\nu) = T^{\mu\nu}g_{\mu\nu} = T^\mu_\mu$. Evaluating the expectation value $\langle T^\mu_\mu \rangle$ of this trace is thus a way to check if there is a *Weyl anomaly*.

From the point of view of the world-sheet, the spacetime fields (1.1.9) are really couplings for the action of the fields $x^M(\sigma)$. So a Weyl anomaly can also be detected by computing the beta functions of the action (1.1.13) for the couplings (1.1.9). This can be obtained by the usual perturbative methods; the coupling for this computation is given by l_s^2, or rather the dimensionless combination $l_s^2\times$ (spacetime curvature). This results in the following three conditions:

$$R_{MN} + 2\nabla_M \partial_N \phi - \frac{1}{4} H_{MPQ} H_N{}^{PQ} + O(l_s^2) = 0\,, \tag{1.1.14a}$$
$$\nabla_M (\mathrm{e}^{-2\phi} H^M{}_{NP}) + O(l_s^2) = 0\,, \tag{1.1.14b}$$
$$\frac{2}{3l_s^2}(26-d) + R - \frac{1}{2}|H|^2 - 4\mathrm{e}^{\phi}\nabla^2 \mathrm{e}^{-\phi} + O(l_s^2) = 0\,. \tag{1.1.14c}$$

We have introduced

$$H_{MNP} = \partial_M B_{NP} + \partial_N B_{PM} + \partial_P B_{MN}\,, \qquad |H|^2 \equiv \frac{1}{6} H_{MNP} H^{MNP}\,. \tag{1.1.15}$$

This can be considered as a *field-strength* for the *potential* B_{MN}, similar to the relation between $F_{MN} = \partial_M A_N - \partial_N A_M$ and A_M in electromagnetism. Indeed, there is also a gauge transformation

$$B_{MN} \to B_{MN} + \partial_M \hat{\lambda}_N - \partial_N \hat{\lambda}_M\,, \tag{1.1.16}$$

under which (1.1.15) is invariant. The world-sheet action (1.1.10) is invariant too under this, because the transformation adds a total derivative term.

From spacetime point of view, where (1.1.9) are fields, (1.1.14) are to be interpreted as equations of motion. They can be obtained by extremizing[2]

$$S_{\mathrm{bos}} = \frac{1}{2\kappa_{\mathrm{b}}^2} \int \mathrm{d}^d x \sqrt{-g}\, \mathrm{e}^{-2\phi} \left(\frac{2}{3l_s^2}(26-d) + R + 4\partial_M \phi \partial^M \phi - \frac{1}{2}|H|^2 + O(l_s^2) \right) \tag{1.1.17}$$

with respect to (1.1.9). By dimensional reasons, κ_{b} has dimension l_s^{12}. (The metric coefficients have no dimension, while R contains two derivatives and has mass dimension two.) In general, the *Planck mass* m_{P} is defined as the mass scale entering the Einstein–Hilbert action; the *Planck length* l_{P} is its inverse, and (1.1.17) tells us that it is proportional to l_s.

As a consistency check, we see that flat space is a solution of (1.1.14) only if we set $d = 26$, which is the value where we found the massless fields (1.1.9) in the first place. More generally, to trust (1.1.14) we have to make sure that the expansion parameter $l_s^2\times$ (curvature) is small, so we better solve those equations of motion separately at every order. This leads again to taking

$$d = 26\,. \tag{1.1.18}$$

It is conceptually possible to consider solutions where $d \neq 26$, and the first term in (1.1.14c) competes with the others, but in that case we have to worry that the other terms in the l_s expansion become relevant too, and we have not given them in (1.1.14). If, on the other hand, one is able to prove that a certain world-sheet model is conformal exactly, without using the l_s expansion at all, then $d = 26$ is not necessary. There are not many such cases: one is the *linear dilaton* background, where ϕ is linear in one of the coordinates. This leads to *noncritical string theories*, which historically have been important toy models.

Another point of view on the critical dimension is this. We observed that (1.1.13) has conformal invariance. Conformal transformations form a group; for flat space it is $\mathrm{SO}(d-2, 2)$ for $d > 2$, but for $d = 2$ it becomes infinite dimensional. Indeed,

[2] This variation is a little more involved than the usual Einstein–Hilbert action variation because of the prefactor $\mathrm{e}^{-2\phi}$. More details will be given in Section 10.1.2.

any transformation $x^\pm \to x^{\pm\prime}(x^\pm)$ is conformal for any metric of the type $\mathrm{d}s^2 = \mathrm{e}^f \mathrm{d}x^+ \mathrm{d}x^-$. The generators L_m, $m \in \mathbb{Z}$, of such transformations on the x^+ obey the Lie algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\, \delta_{n+m,0}\,, \tag{1.1.19}$$

called *Virasoro algebra*. The L_0, $L_{\pm 1}$ form an SO(1, 2) subalgebra where c does not appear. As usual, spacetime transformations are generated by the stress–energy tensor, so these L_m are related to it. After a Wick rotation, $x_+ \to z = \sigma^1 + \mathrm{i}\sigma^0$, and we can collect all the generators in

$$T_{zz}(z) = \sum_n L_n z^{-n-2}\,. \tag{1.1.20}$$

In a Lie algebra, the commutation relations should always be linear, so we need to think of the second term in (1.1.19) as containing a new generator c, which commutes with all the others, and thus lies in the center of the algebra; so c in (1.1.19) is called *central charge*. The $\tilde{L}_m$ on the x_- variable generate a second copy of the same algebra (1.1.19), and they are collected in $T_{\bar{z}\bar{z}}$.

This c is also a measure of the Weyl anomaly: for any QFT model that is conformal on a flat (world-sheet) metric $h_{\alpha\beta} = \eta_{\alpha\beta}$, a nonzero c tells us that conformal invariance is broken for more general $h_{\alpha\beta} \neq \eta_{\alpha\beta}$. A free boson contributes $c = 1$, while the ghosts give -26. Thus if we quantize *around flat space*, where the $x^M(\sigma)$ bosons are free, for quantum conformal invariance we need to take $d = 26$.

The fact that the action (1.1.17) exists at all is nontrivial from the point of view of the world-sheet derivation we described. We can think of it as being an approximation to the string field theory action S_{SFT}, which would also contain the massive fields creating all the states (1.1.6). We can call it an effective action, in the usual quantum field theory sense: It reproduces the results one would obtain from S_{SFT}, at energies that are *low*, namely much smaller than l_s^{-1}. Indeed, another way to compute (1.1.17) is to compute string scattering amplitudes using the world-sheet approach, and then guessing what spacetime action would reproduce them.

The diagrams leading to (1.1.17) have $g = 0$ in (1.1.12), leading to $g_s^{-2} = \mathrm{e}^{-2\phi}$, thus explaining the presence of that exponential. The higher powers of l_s hidden in (1.1.17) also receive contributions from higher values of g (and thus from more complicated Feynman diagrams). So the effective action will have a double expansion in powers of both:

$$S = \sum_{j,k} S_{j,k}\, l_s^j \mathrm{e}^{k\phi}\,. \tag{1.1.21}$$

These higher-order corrections can in principle be computed; we will see some examples for superstrings. When we first discover that GR is non-renormalizable and needs (curvature)^2 counterterms [25], we might perhaps hope that by adding more and more such counterterms, with arbitrary powers (curvature)^k, we might eventually find a theory that has no divergence. Finding such a renormalizable theory of gravity would be very hard without some sort of guidance: not only would we have to find a fixed point of the renormalization group (RG) flow by going backward in energy, but we would also have to worry about modes with wrong kinetic energy, which in such theories generically abound. (Adding operators with higher numbers of derivatives to a Lagrangian also adds propagating modes, each of which might be a ghost.) String theory is renormalizable, and in principle we can reexpress it precisely as such a sum of infinitely many corrections to (1.1.17).

This discussion seems, however, to assume that the effective action is analytic in the parameters l_s, e^ϕ, or, in other words, that it coincides with its Taylor expansion (1.1.21). In mathematics, we know many functions that are not analytic, and they might also appear here. This is the reason we have put the word "perturbative" in the title of this section; we will make amends in Section 1.4.

Some critics of string theory complain that the theory has not been proven to be *background independent*. What they mean is that in the world-sheet approach based on (1.1.13), we first have to fix a background configuration for the spacetime fields (1.1.9), and then we can compute an action for the small fluctuations around it. A priori, it might even be unclear if this procedure is describing a single theory or a collection of theories that have nothing to do with each other. The emergence of (1.1.17) should be reassuring in this respect: that effective action can be expanded around any background, and matches the result of the world-sheet method around it. A more satisfactory rebuttal is the proof at the level of string field theory in [26].

Torus compactification

Finally, let us have a first taste of string compactifications, by supposing that the theory lives on $\mathbb{R}^{25} \times S^1$. Thus we declare one of the coordinates to be periodically identified, say $x^{25} \equiv x^{25} + 2\pi R$. Now $x^{25}(\sigma^0, \sigma^1)$ is no longer necessarily periodic as a function of σ^1, even for a closed string: rather, if we take $\sigma^1 \sim \sigma^1 + \pi$, we demand

$$x^{25}(\sigma^0, \sigma^1 + \pi) = x^{25}(\sigma^0, \sigma^1) + 2\pi w R\,. \tag{1.1.22}$$

This represents a string that winds $w \in \mathbb{Z}$ times around the S^1. Another new effect is familiar from quantum mechanics: the overall momentum of the string in the S^1 direction is now not continuous but quantized: $p^{25} = \frac{q}{R}$, $q \in \mathbb{Z}$.

The mass spectrum in $\mathbb{R}^{25}$ is now modified from (1.1.7) to

$$m^2 = \frac{4}{l_s^2}(-1 + N) + \left(\frac{q}{R} - w\frac{R}{l_s^2}\right)^2 = \frac{4}{l_s^2}(-1 + \tilde{N}) + \left(\frac{q}{R} + w\frac{R}{l_s^2}\right)^2 , \tag{1.1.23}$$

where now $N = \sum i_k$ and $\tilde{N} = \sum_k \tilde{i}_k$ are no longer necessarily equal (as they were in (1.1.7)); comparing the two expressions, we have $N - \tilde{N} = wq$.

For a generic value of R, the massless spectrum is still (1.1.8) and (1.1.9); but now it should be reinterpreted. The components

$$g_{M\,25}\,, \qquad B_{M\,25} \tag{1.1.24}$$

are now two vector fields in $\mathbb{R}^{25}$; $g_{25\,25}$ is a scalar. The remaining components of (1.1.9) then give a metric, a Kalb–Ramond field, and a scalar in $\mathbb{R}^{25}$.

From (1.1.23), however, we also see another option: if $\frac{q}{R} - w\frac{R}{l_s^2} = \pm\frac{2}{l_s}$, then we have a new massless state for $N = 0$. This is possible for

$$R = l_s\,, \tag{1.1.25}$$

taking $q = -w = \pm 1$; then $\tilde{N} = \sum_k \tilde{i}_k = 1$. This state $\tilde{\alpha}^M_{-1}|0\rangle$ has a single index, and so it is created by a vector field. At this value of R, we also have the possibility of using the same trick with the other expression in (1.1.23), this time leading to

$q = w = \pm 1$, $\tilde{N} = 0$, $N = 1$. So we have a total of four more vector fields in $\mathbb{R}^{25}$. It turns out that these combine with the previous two (1.1.24) to give a nonabelian gauge group

$$\mathrm{SU}(2) \times \mathrm{SU}(2)\,. \tag{1.1.26}$$

This compactification was rather nice in that the string could be quantized exactly, at least perturbatively in g_s. In more complicated cases, we won't be so lucky, and we will have to limit ourselves to the less powerful effective field theory methods, potentially missing phenomena such as this non-abelian gauge group enhancement.

1.1.2 Type II superstrings

Supersymmetric world-sheet action

The world-sheet action (1.1.13) can be made supersymmetric. At the most basic level, this means that we promote the $x^M(\sigma)$ to a function of σ and of new formal coordinates $\theta^\pm$ that anticommute: $\theta^+\theta^- = -\theta^-\theta^+$, $(\theta^\pm)^2 = 0$. The Taylor expansion in the new coordinates truncates:

$$X^M = x^M + \theta^+\psi_+^M + \theta^-\psi_-^M + \theta^+\theta^- F^M\,. \tag{1.1.27}$$

We can also introduce the derivative operators

$$D_\pm = \partial_{\theta^\pm} + \mathrm{i}\theta^\pm\partial_\pm\,, \qquad \partial_\pm \equiv \partial_{\sigma^\pm}\,. \tag{1.1.28}$$

Then, (1.1.4), for example, is replaced by

$$S^{1,1}_{\mathrm{F1},g} = -\frac{1}{2}T_{\mathrm{F1}}\int_\Sigma \mathrm{d}^2\sigma\mathrm{d}^2\theta(g+B)_{MN}(X)D_+X^MD_-X^N\,, \tag{1.1.29}$$

with the integration rule $\int \mathrm{d}\theta^\pm\theta^\pm = 1$, $\int \mathrm{d}\theta^\pm 1 = \int \mathrm{d}\theta^\pm\theta^\mp = 0$. We also added the contribution from B. The terms (1.1.10) can also be supersymmetrized in this way. The final result is quite messy for a general background where g_{MN} and B_{MN} are arbitrary; it can be found, for example, in [27, sec. 6.3.1]. For example, it contains a kinetic term

$$g_{MN}(\psi_+^M\partial_-\psi_+^N + \psi_-^M\partial_+\psi_-^N) \tag{1.1.30}$$

for the world-sheet fermions $\psi_\pm^M$.[3] The F^M in (1.1.27) are auxiliary fields: they have no kinetic term, and can be replaced with the solutions of their equations of motion.

Since we have introduced a single θ^+ and a single θ^-, the resulting model is said to have $\mathcal{N} = (1,1)$ supersymmetry. Any two-dimensional bosonic model can be promoted to such a model. In the context of compactifications, one often needs to separate external and internal dimensions, and the supersymmetrization of the world-sheet model in the latter has more supercharges; a common case one needs is $\mathcal{N} = (2,2)$. This is more challenging to achieve, because such *extended* supersymmetry requires that one combine the x^M with each other in pairs. Such a pairing is reminiscent of the idea of complex coordinates, and is at the root of why differential geometry is useful for compactifications. This idea will return in Chapter 9.

[3] The bosonic world-sheet indices $\pm$ are conceptually not the same as the $\pm$ on the fermions, denoting chirality. To emphasize the difference, some authors change the world-sheet indices to $+ \to \,+\!\!\!+$ and $- \to =$. This also has the benefit that every term in a world-sheet will then have an equal number of pluses and minuses; see for example [28].

Equation (1.1.29) is called the *Neveu–Schwarz–Ramond* (NSR) model. While we introduced it by supersymmetrizing the world-sheet action, we will see later that the resulting spacetime theory also has the much more nontrivial property of *spacetime* supersymmetry.

Spectrum

Even around flat space, the spectrum of (1.1.29) is now more complicated because it depends on what we impose on the fermionic $\psi^M_\pm$. Since a fermion should only get back to itself after a 4π rotation, under 2π we can impose either periodic or antiperiodic boundary conditions, called Neveu–Schwarz (NS) and Ramond (R) respectively. These can be imposed independently on the $\psi^M_\pm$, leading to four sectors: NSNS, NSR, RNS, and RR. The spectrum has to be analyzed in each sector separately, because the Fourier modes for the $\psi^M_\pm$ behave differently in each.

In the NS sector, the fermionic Fourier modes are $b^M_{-i-1/2}$, $i \geq 0$. The two lowest-lying states are

$$|0\rangle_{\rm NS}\,, \qquad \mathbf{1}\,, \qquad m^2 = \frac{1}{8l_s^2}(2-d)\,; \tag{1.1.31a}$$

$$b^M_{-1/2}|0\rangle_{\rm NS}\,, \qquad \mathbf{8}_{\rm V}\,, \qquad m^2 = \frac{1}{l_s^2}\left(\frac{(2-d)}{8}+1\right). \tag{1.1.31b}$$

We have also indicated what representation these states form under the compact part $\mathrm{SO}(d-2) = \mathrm{SO}(8)$ of the massless little group (1.1.1). For (1.1.31b), the subscript "V" is because there are two more dimension-eight representations of SO(8), which will soon play a role too.

In the R sector, the fermionic Fourier modes are d^M_{-i}, $i \geq 0$. In this case, the vacuum has already $m^2 = 0$, but in fact it is not unique: the modes d^M_0 now don't raise the energy, and they act on the space of vacua. These d^M_0 satisfy a *Clifford algebra* $\{d^M_0, d^N_0\} = 2g^{MN}1$, and as a consequence the space of R vacua transforms as a spinor under spacetime symmetries. In Section 2.1, we will attack Clifford algebras and spinors systematically in every dimension; for now, we only state the main features we need, which are quite similar to the properties of gamma matrices in four dimensions.

- Gamma matrices Γ^M can be defined in every dimension as matrices that satisfy $\{\Gamma^M, \Gamma^N\} = 2g^{MN}1$.
- In $d = 10$ dimensions, they are 32×32 matrices; in $d = 8$, they are 16×16.
- The space of spinors on which the Γ^M act is a representation for the Lorentz group; in $d =$ even, it decomposes in two chiralities, for which we introduce indices α, $\dot\alpha$. Multiplication by a single Γ^M changes chirality, so the nonzero blocks are $\Gamma^M_{\alpha\dot\beta}$ and $\Gamma^M_{\dot\alpha\beta}$.
- In both $d = 10$ with Lorentzian signature and $d = 8$ with Euclidean signature, there is a choice of Γ^M that are all real. (This aspect will be treated more specifically in Sections 2.2.3 and 2.3.)

As a representation of the transverse SO(8) in the little group (1.1.1), the R states then form a reducible representation of dimension 16, which further splits in two representation of dimension eight, traditionally called $\mathbf{8}_\mathrm{S}$ and $\mathbf{8}_\mathrm{C}$. We summarize all this by writing

$$|0,\alpha\rangle_\mathrm{R}\,, \qquad \mathbf{8}_\mathrm{S}\,, \qquad m^2 = 0\,; \tag{1.1.32a}$$
$$|0,\dot\alpha\rangle_\mathrm{R}\,, \qquad \mathbf{8}_\mathrm{C}\,, \qquad m^2 = 0\,. \tag{1.1.32b}$$

The closed-string spectrum is obtained by taking tensor products of these two sectors. For example, if we impose NS conditions for both $\psi^M_\pm$, we have the NSNS sector, whose low-lying states are obtained by taking tensor products of two copies of (1.1.31). Here we see once again the presence of a tachyon, $|0\rangle_\mathrm{NS}\otimes|0\rangle_\mathrm{NS}$. In the critical dimension

$$d = 10\,, \tag{1.1.33}$$

this sector contains massless states $b^M_{-1/2}\tilde b^M_{-1/2}|0\rangle_\mathrm{NS}\otimes|0\rangle_\mathrm{NS}$, which would be created by the fields (1.1.9) we encountered for the bosonic string, so we are going to stick to $d = 10$ from now on.

Quite nontrivially, just like in the bosonic string, this is also the dimension where the *Weyl anomaly vanishes*. This can be checked in any of the methods we saw for the bosonic string. In terms of the central charge of the Virasoro algebra, for example, there are now ghosts and superghosts, which give a total contribution -15. Quantizing around flat space, each boson gives a contribution $+1$, but now each fermion gives an extra $+1/2$. Since (1.1.29) has an equal number of bosons of fermions, we get $(1+1/2)d = 15$, which leads again to $d = 10$.

A less welcome similarity with the bosonic string is the presence of a tachyon. Fortunately, this can be eliminated by the so-called *Gliozzi–Scherk–Olive (GSO) projection* on the spectrum. Initially an ad hoc prescription, it later emerged to be required by consistency when one takes the world-sheet Σ to be a torus, an invariance under coordinate changes called *modular invariance*. It projects out $|0\rangle_\mathrm{NS}$ in the NS sector, and also eliminates one of the two sets in (1.1.32). For the left-moving ψ^M_+, it is immaterial which one we choose; we keep $\mathbf{8}_\mathrm{S}$. However, now the choice for the right-moving ψ^M_- does matter; this leads to two different theories.

If we keep $\mathbf{8}_\mathrm{C}$ for the ψ^M_-, we obtain the theory called *IIA string theory*; its spectrum is given in Table 1.1. We use SO(8) group theory to decompose the tensor products of representations as direct sums. In the last column, we have named the corresponding spacetime fields.

- In the NSNS sector, the decomposition is that of an 8×8 matrix in its symmetric traceless, antisymmetric and trace, with the same logic that led us to the fields (1.1.9) for the bosonic string.

Table 1.1. Massless IIA spectrum.

NSNS	$\mathbf{8}_\mathrm{V}\otimes\mathbf{8}_\mathrm{V} = \mathbf{35}_\mathrm{V}\oplus\mathbf{28}_\mathrm{V}\oplus\mathbf{1}_\mathrm{V}$	g_{MN}, B_{MN}, ϕ
RNS	$\mathbf{8}_\mathrm{S}\otimes\mathbf{8}_\mathrm{V} = \mathbf{56}_\mathrm{S}\oplus\mathbf{8}_\mathrm{C}$	$\psi^1_{M\alpha}, \lambda^1_{\dot\alpha}$
NSR	$\mathbf{8}_\mathrm{V}\otimes\mathbf{8}_\mathrm{C} = \mathbf{56}_\mathrm{C}\oplus\mathbf{8}_\mathrm{S}$	$\psi^2_{M\dot\alpha}, \lambda^2_{\alpha}$
RR	$\mathbf{8}_\mathrm{S}\otimes\mathbf{8}_\mathrm{C} = \mathbf{56}_\mathrm{V}\oplus\mathbf{8}_\mathrm{V}$	C_{MNP}, C_M

- In the RNS sector, the $\mathbf{8}_\mathrm{S} \otimes \mathbf{8}_\mathrm{V}$ representation has a vector and a spinor index, leading to an object $\Psi_{M\alpha}$. This is not irreducible: we can use an eight-dimensional gamma matrix to extract a sort of trace $\lambda_{\dot\alpha} \equiv (\Gamma^M)_\alpha{}^{\dot\beta}\Psi_{M\dot\beta}$, transforming as $\mathbf{8}_\mathrm{C}$; the remaining traceless part of $\Psi_{M\alpha}$ is the $\mathbf{56}_\mathrm{S}$. Both have one spinor index, and thus are spacetime fermions. Since they only have an index of one kind (either α or $\dot\alpha$), they are chiral, or *Weyl*. In the aforementioned real basis for the Γ^M, these fermions are also all real. The basis-independent notion is called the *Majorana* property (Section 2.3.1).
- The NSR sector is similar, but with $\mathbf{8}_\mathrm{S} \leftrightarrow \mathbf{8}_\mathrm{C}$. These are also spacetime fermions. The $\psi^1_{\alpha M}$, $\psi^2_{\dot\alpha M}$ are related by supersymmetry to the metric field, and hence are called *gravitinos*. The $\lambda^1_{\dot\alpha}$, ψ^2_α are called *dilatinos*.
- In the RR sector, we have an object with two spinorial indices:

$$C_{\alpha\dot\beta}\,. \tag{1.1.34}$$

  These have two spinor indices, and thus are sometimes called *bispinors*; they are spacetime *bosons*.

A more familiar description for the RR (1.1.34) is obtained by expanding them in a basis for bispinors. This is familiar from QFT in $d = 4$: it consists of antisymmetrized products

$$\Gamma^{MN} \equiv \frac{1}{2}(\Gamma^M\Gamma^N - \Gamma^N\Gamma^M)\,, \qquad \Gamma^{MNP} = \frac{1}{6}(\Gamma^M\Gamma^N\Gamma^P \pm \text{perm.})\,, \tag{1.1.35}$$

and so on. The expansion on this basis is called a *Fierz identity* and will be analyzed systematically in Section 3.4; for now, we sketch the result. If we contract (1.1.34) with a single eight-dimensional gamma matrix, we obtain the vector

$$C_M = (\Gamma_M)^{\alpha\dot\beta}C_{\alpha\dot\beta}\,, \tag{1.1.36}$$

which is the $\mathbf{8}_\mathrm{V}$ there. One can further contract with products of gamma matrices; with two of them, we obtain $\Gamma^{MN}_{\alpha\beta}$ or $\Gamma^{MN}_{\dot\alpha\dot\beta}$, which cannot be contracted with (1.1.34), but with three we do obtain $C_{MNP} = (\Gamma_{MNP})^{\alpha\dot\beta}C_{\alpha\dot\beta}$. This explains the entries in the bottom-right corner of Table 1.1. By construction, this is completely antisymmetric:

$$C_{MNP} = -C_{NMP} = -C_{PNM} = -C_{MPN}\,. \tag{1.1.37}$$

The number of independent components of a completely antisymmetric tensor with k indices in d (transverse) dimensions is $\binom{d}{k}$; for (1.1.37) this gives $\binom{8}{3} = 56$, confirming the last row of Table 1.1. One might think that we also need to consider $\Gamma^{M_1\ldots M_k}$ with higher odd k, but these are actually related to the ones already present in Table 1.1, as shown by the matching dimension count in the representation. Sometimes it is convenient to work with a redundant set of RR potentials; this *democratic* formalism will be presented in Section 10.1.

The second choice is to keep $\mathbf{8}_\mathrm{S}$ for the ψ^M_-. The resulting theory is now called *IIB string*, and its spectrum is shown in Table 1.2. Most differences with Table 1.1 are straightforward; the one worthy of most attention is that now the RR sector consists of a bispinor

$$C_{\alpha\beta}\,. \tag{1.1.38}$$

Table 1.2. Massless IIB spectrum.

NSNS	$\mathbf{8}_V \otimes \mathbf{8}_V = \mathbf{35}_V \oplus \mathbf{28}_V \oplus \mathbf{1}_V$	g_{MN}, B_{MN}, ϕ
RNS	$\mathbf{8}_S \otimes \mathbf{8}_V = \mathbf{56}_S \oplus \mathbf{8}_C$	$\psi^1_{M\alpha}, \lambda^1_{\dot\alpha}$
NSR	$\mathbf{8}_V \otimes \mathbf{8}_S = \mathbf{56}_S \oplus \mathbf{8}_C$	$\psi^2_{M\alpha}, \lambda^2_{\dot\alpha}$
RR	$\mathbf{8}_S \otimes \mathbf{8}_S = \mathbf{35}_C \oplus \mathbf{28}_C \oplus \mathbf{1}_C$	C^+_{MNPQ}, C_{MN}, C_0

In the Fierz expansion, now the Γ^{MN} has the correct index structure, leading to $C_{MN} = (\Gamma_{MN})^{\alpha\beta} C_{\alpha\beta}$; a similar projection can be defined for all the products $\Gamma^{M_1 \ldots M_k}$ with even k, but in fact $k > 6$ are redundant, and even the one for $k = 4$ has a "self-duality property" that halves its degrees of freedom, and whose consequences we will see soon. This is the reason of the superscript $+$ on C^+_{MNPQ}. Just like for (1.1.37), by construction these tensors are all completely antisymmetric.

Spacetime supersymmetry

In both IIA and IIB, the spacetime bosons arise from the NSNS and RR sectors, while the fermions arise from the NSR and RNS sectors. They have the same total number (128) of degrees of freedom. This is a symptom of the aforementioned *spacetime* supersymmetry. These mix the bosonic fields (NSNS, RR) with the fermionic ones (NSR, RNS). They are rather complicated, and at this stage their expression would not look very informative. We will have a first look at them in Section 8.2 for the case where only the metric is present, and in Section 10.1 in full. For now, we just comment about their infinitesimal parameters, which are *two* Majorana–Weyl spinors:

$$\epsilon^1 , \qquad \epsilon^2 . \tag{1.1.39}$$

(This is the original reason these theories are called "type II.") They have the same chirality as the gravitino: thus in IIA ϵ^1_α has positive chirality, and $\epsilon^2_{\dot\alpha}$ has negative chirality, while in IIB both ϵ^a_α have positive chirality. Altogether this gives 32 supercharges, which is the highest number for any supersymmetric theory.

This property is made manifest in the alternative *Green–Schwarz* model. A more recent formulation is the *Berkovits*, or *pure spinor*, model [29, 30]. (Pure spinors will play an important role in this book, but for somewhat different reasons.) These alternative formulations are more complicated, but are better at describing strings in condensates of RR fields, while with the NSR model (1.1.29) this is difficult.

1.1.3 Heterotic strings

Definition

In both the bosonic string and the superstring, quantization of the left- and right-movers seems to proceed almost independently. This is because they are almost free as two-dimensional field theories, apart from the constraint associated to the Lagrange multiplier $h_{\alpha\beta}$. For example, in the bosonic string spectrum (1.1.6), the constraint is only visible in the level matching condition mentioned after (1.1.7).

This creates an opportunity: we can try to define a hybrid, or *heterotic* theory which looks like the bosonic string for the left-movers, and like the superstring for the right-movers [31]. This might look impossible: is the spacetime dimension going to be 26 or 10? The dimension of spacetime, however, is a macroscopic concept, irrelevant at length scales below l_P, where quantum gravity sets in. So a solution to the puzzling mismatch in dimension is to compactify 16 of the 26 bosonic coordinates. Provided we manage to satisfy the level matching condition relating the left- and right-moving sectors, all should be well.

The easiest way to compactify the 16 left-moving bosons x^I_+ is to take them to belong to a *torus*. In Section 1.1.1, we saw a circle compactification of the bosonic string: we took one of the coordinates $x^{25} \sim x^{25} + 2\pi R$. The simplest torus would be obtained as $T^{16} \equiv (S^1)^{16}$, with the same identification for all the x^I_+. A straightforward generalization is to take a basis $\{R^I_a\}$, $a = 1, \ldots, 16$ of $\mathbb{R}^{16}$, and to introduce equivalence relations

$$x^I = x^I + \pi R^I_a \quad \forall a\,. \tag{1.1.40}$$

The set of integer multiples of the $\{R^I_a\}$ is called a *lattice* Γ. The space defined by (1.1.40) is topologically still T^{16}; the choice of Γ affects its size and shape. The momenta are quantized according to

$$p^I R^I_a \in \mathbb{Z}\,. \tag{1.1.41}$$

The solutions to this equation are the elements of a *colattice*: there is a dual set of vectors p^I_a of which all solutions to (1.1.41) are integer multiples.

Spectrum

As usual, now the spectrum is computed independently for the left-movers and for the right-movers, imposing level matching. For the left-movers, $(\partial_0 - \partial_1)x^I = 0$ relates p^I to the R^I_a: this generalizes setting $q/R = wR/l_s^2$ in (1.1.23). So in fact the lattice and colattice coincide,[4] and Γ is said to be *self-dual*. Moreover, the analogue of (1.1.23) now reads

$$m^2 = \frac{4}{l_s^2}\left(-1 + N + \frac{1}{2}\hat{p}^2\right) = \frac{4}{l_s^2}\tilde{N}\,. \tag{1.1.42}$$

From this we see that $\frac{1}{2}\hat{p}^2 \in \mathbb{Z}$: the lattice is said to be *even*.

The massless spectrum is again obtained as a tensor product of left- and right-movers. The latter have both an NS and an R sector, just as in the superstring (recall (1.1.31) and (1.1.32)). For the left-movers, (1.1.42) gives us two possibilities. We can take $\hat{p}^2 = 0$ and $N = 1$: a single Fourier mode acting on the vacuum (compare with the left-moving part of (1.1.8)). This can be α^M_{-1}, with $M = 0, \ldots, 9$, or α^I_{-1}, with $I = 1, \ldots, 16$ one of the extra 16 bosonic directions. So far, this gives us the massless states

$$\alpha^M_{-1}|0\rangle_{\rm bos} \otimes b^M_{-1/2}|0\rangle_{\rm NS}\,, \qquad \alpha^M_{-1}|0\rangle_{\rm bos} \otimes |0,\alpha\rangle_{\rm R}\,; \tag{1.1.43a}$$

$$\alpha^I_{-1}|0\rangle_{\rm bos} \otimes b^M_{-1/2}|0\rangle_{\rm NS}\,, \qquad \alpha^I_{-1}|0\rangle_{\rm bos} \otimes |0,\alpha\rangle_{\rm R}\,. \tag{1.1.43b}$$

The first in (1.1.43a) gives the familiar (g_{MN}, B_{MN}, ϕ) in the NSNS sector of the superstring; the second is identical to the RNS sector of the superstring. These are

[4] At this point, one might also take more generally the lattice to be a sublattice of the colattice, but modular invariance (already mentioned for the superstring) eliminates this possibility.

Table 1.3. Massless heterotic spectrum.

NS	$\mathbf{8}_V \otimes \mathbf{8}_V = \mathbf{35}_V \oplus \mathbf{28}_V \oplus \mathbf{1}_V$	g_{MN}, B_{MN}, ϕ
R	$\mathbf{8}_V \otimes \mathbf{8}_S = \mathbf{56}_S \oplus \mathbf{8}_C$	$\psi_{M\alpha}, \lambda_{\dot\alpha}$
NS	$\mathbf{8}_V$	A^a_M
R	$\mathbf{8}_S$	χ^a_α

the first two lines in Table 1.3. The (1.1.43b) are 16 abelian vector fields A^I_M and spinors χ^I_α; they are the Cartan subalgebra part of the second two lines in Table 1.3.

Indeed, we are not done yet, because (1.1.42) allows us a second possibility: we can take $N = 0$ and look for $\hat{p}^2 = 2$, namely vectors in Γ of length two. There exist only *two* lattices in $\mathbb{R}^{16}$ that are both self-dual and even; both are the root lattices $\Gamma_{\mathfrak{g}}$ for a Lie algebra $\mathfrak{g}$ of a Lie group that can be either

$$\mathrm{SO}(32) \quad \text{or} \quad E_8 \times E_8 \,. \tag{1.1.44}$$

The elements of Γ of length two are the nonzero roots of $\mathfrak{g}$. A tensor product of all these states with $b^M_{-1/2}|0\rangle_{\rm NS}$ and $|0,\alpha\rangle_{\rm R}$, as in (1.1.43b), produces one vector field and one spinor for each nonzero root of $\mathfrak{g}$. Both (1.1.44) have rank 16, which is just the number of the A^I_M we found previously. So in total we have A^a_M, χ^a_α with $a = 1,\ldots,\dim(\mathfrak{g})$, and we reproduce the missing part of the last two lines in Table 1.3. The A^a_M are the gauge vectors of a nonabelian gauge algebra $\mathfrak{g}$. This enhancement is similar to (1.1.26) for the bosonic string.

The heterotic string has the advantage of having a built-in nonabelian gauge symmetry. We will see in Section 1.3 that D-branes give an alternative way of obtaining nonabelian gauge groups G in type II, but only of the type $G = \mathrm{U}(N)$; later in Section 1.4.4 we will see $G = \mathrm{SO}(N)$ and $\mathrm{Sp}(N)$. Obtaining a more interesting gauge group such as E_8 is in fact possible in IIB, but requires more sophisticated techniques that we will only study much later, in Section 9.4.[5]

1.2 Supergravity

An effective spacetime action for superstrings can be found using the same methods that led us to (1.1.17). Because of its spacetime supersymmetry, it is called 10-*dimensional supergravity*.

1.2.1 RR fields

In this book, we will actually only see the action for the *bosonic* fields. The superstring NSNS fields are the same as those (1.1.9) of the bosonic string. The new

[5] It might seem that a unitary gauge group is quite enough to accommodate the Standard Model's $\mathrm{SU}(3)\times \mathrm{SU}(2) \times \mathrm{U}(1)$. Even if these three factors unify at higher energies, the gauge groups that seem most promising are $G = \mathrm{SU}(5)$ and $\mathrm{SO}(10)$. However, the particular representations that one needs for those *grand unified theories (GUT)* are easier to obtain by starting from a group such as E_8 or its subgroup E_6.

bosonic fields are those in the RR sector. In this subsection, we focus on them before writing the effective actions for IIA and IIB in the next subsections.

Completely antisymmetric tensors

The RR fields are all completely antisymmetric tensors with k-indices. Such tensors are also called k-forms; we give here a minimal introduction to their properties, leaving a deeper treatment to Sections 3.1 and 4.1.

The *antisymmetrized derivative* or *exterior differential*

$$(\mathrm{d}A)_{M_1\ldots M_k} \equiv k\partial_{[M_1}C_{M_2\ldots M_k]} \equiv \partial_{M_1}C_{M_2\ldots M_k} - \partial_{M_2}C_{M_1\ldots M_k} \pm \ldots\,, \tag{1.2.1}$$

takes a $(k-1)$-form to a k-form. The index antisymmetrizer $[M_1\ldots M_k]$ sums over all permutations with a ± 1 equal to the sign $\sigma(i_1\ldots i_k)$ of the permutation taking $1\ldots k$ to $i_1\ldots i_k$, and divides by a $k!$. So, for example, $(\mathrm{d}A)_{MN} = \partial_M C_N - \partial_N C_M$ is the Maxwell field-strength of C_M. As another example, (1.1.15) can be written as $H_{MNP} = (\mathrm{d}B)_{MNP} = 3\partial_{[M}B_{NP]}$, with only three terms instead of six because of antisymmetry of B. To avoid index proliferation, often we will write $C_{M_1\ldots M_k}$ symbolically simply as C_k, and denote (1.2.1) by $\mathrm{d}C_k$. An important property of (1.2.1) is

$$\mathrm{d}\,\mathrm{d}\,\alpha_k = 0\,. \tag{1.2.2}$$

Explicitly,

$$(\mathrm{d}\mathrm{d}\alpha)_{M_1\ldots M_k} = k(k-1)\partial_{[M_1}\partial_{M_2}\alpha_{M_3\ldots M_k]} = 0\,, \tag{1.2.3}$$

because $\partial_{[M_1}\partial_{M_2]} = 0$ on smooth functions. For an electromagnetic potential A_M, this gives $\partial_{[M}F_{NP]} = \partial_{[M}\partial_N A_{P]} = 0$, one of the Maxwell equations. As another example, from (1.1.15) it now follows $\partial_{[M}H_{NPQ]} = 0$, or in other words,

$$\mathrm{d}H = 0\,. \tag{1.2.4}$$

The antisymmetrized, or *wedge*, product of two forms α_k, $\alpha'_{k'}$ is defined as

$$(\alpha\wedge\alpha')_{M_1\ldots M_k N_1\ldots N_{k'}} \equiv \frac{(k+k')!}{k!\,k'!}\alpha_{[M_1\ldots M_k}\alpha'_{N_1\ldots N_{k'}]}\,. \tag{1.2.5}$$

This satisfies $\alpha_k\wedge\alpha_{k'} = (-1)^{kk'}\alpha_{k'}\wedge\alpha_k$. In particular,

$$H\wedge H = 0\,. \tag{1.2.6}$$

We now also have the Leibniz identity:

$$\mathrm{d}(\alpha_k\wedge\alpha_{k'}) = \mathrm{d}\alpha_k\wedge\alpha_{k'} + (-1)^k\alpha_k\wedge\mathrm{d}\alpha_{k'}\,. \tag{1.2.7}$$

In $d=10$, the largest possible number of indices of a k-form is $k=10$. Moreover, a ten-form α_{10} is unique up to rescaling: it must be proportional to the Levi-Civita tensor $\epsilon^{(0)}_{M_1\ldots M_{10}} \equiv \sigma(M_1\ldots M_{10})$, the sign of the permutation taking $M_1\ldots M_{10}$ to $0\ldots 9$. (The (0) label is used here because a more mathematically sophisticated version of this tensor will enter the scene later.) Writing then $\alpha_{0\ldots 9} = f$, we define

$$\int \alpha_{10} \equiv \int \mathrm{d}^{10}x\, f\,. \tag{1.2.8}$$

A similar definition holds in any dimension d.

Twisted field-strengths

RR forms usually appear in the action through a *twisted* field-strength[6]

$$F_k \equiv \mathrm{d}C_{k-1} - H \wedge C_{k-3}\,. \tag{1.2.9}$$

For example, we will soon encounter $F_{MNPQ} = 4\partial_{[M}C_{NPQ]} - 4H_{[MNP}C_{Q]}$. (The factors come from (1.2.1) and (1.2.5).) For $k < 3$, the second term is absent; for example, $F_2 \equiv \mathrm{d}C_1$. The F_k satisfy a *Bianchi identity*:

$$\mathrm{d}F_k = H \wedge F_{k-2}, \tag{1.2.10}$$

as one sees using (1.2.2), (1.2.4), and (1.2.6). In a sense, this is more fundamental than (1.2.9): sometimes we will modify (1.2.9), with (1.2.10) remaining true. Later in this chapter, we will consider sources, which will violate (1.2.10) on some spacetime defects. A form field H or F_k satisfying (1.2.10) with no source is often called a *flux*. By an abuse of language, one sometimes calls by this name any form field, sourced or not.

Gauge transformations

In electromagnetism, the field-strength $F_{MN} = \partial_M A_N - \partial_N A_M$ is invariant under the gauge transformation $A_M \to \partial_M \lambda_0$. We already observed after (1.1.15) that the Kalb–Ramond field has a gauge transformation $B \to B + \mathrm{d}\hat{\lambda}_1$, which leaves invariant its field-strength $H = \mathrm{d}B$. For the RR fields, (1.2.9) are invariant under the gauge transformations:

$$\delta C_k = \mathrm{d}\lambda_{k-1} - H \wedge \lambda_{k-3}\,. \tag{1.2.11}$$

So, for example, under λ_1 we have $\delta C_2 = \mathrm{d}\lambda_1$, $\delta C_4 = -H \wedge \lambda_1$, while under λ_3 we have $\delta C_4 = \mathrm{d}\lambda_3$, $\delta C_2 = 0$.

In electromagnetism, λ_0 need not be a function: it can be multivalued, as long as

$$\mathrm{e}^{\mathrm{i}q_\mathrm{e}\lambda_0} \tag{1.2.12}$$

is single-valued, where q_e is the elementary electric charge. Multivalued λ_0 are called a *large* gauge transformation. They exist because the gauge group is compact, which in turn comes from the requirement that the wave function should be single-valued. This is relevant when there is a nontrivial loop in spacetime, either because of an excluded region (as in the Aharanov–Bohm effect) or because spacetime has a nontrivial topology. Consider for example an S^1 with a periodic coordinate $x \sim x + L$. Single-valuedness of (1.2.12) requires $\lambda_0 \to \lambda_0 + 2\pi N/q_\mathrm{e}$, $N \in \mathbb{Z}$. The gauge transformation $A_M \to A_M + \partial_M\lambda_0$ is more correctly rewritten as

$$A_M \to A_M + \Lambda_M\,, \tag{1.2.13}$$

where Λ_M is required to have $\int \mathrm{d}x\Lambda_x = 2\pi N/q_\mathrm{e}$.

Suppose we want to give a *constant* expectation value to a component, $A_x = A_x^0$. This has no physical meaning if the direction x is noncompact, because we can gauge it away with a gauge transformation $\lambda = xA_x^0$. But if $x \sim x + L$ is a coordinate on a

[6] Many authors use the symbol F_k for $\mathrm{d}C_{k-1}$ alone, and then put a tilde on our (1.2.9).

circle, then with a periodic λ we cannot gauge away A^0_x any more; with a large gauge transformation, (1.2.13) gives us the identification

$$A^0_x \cong A^0_x + \frac{2\pi}{L q_{\rm e}} \,. \tag{1.2.14}$$

In the limit $L \to \infty$ we see again that all constant values can be gauged away. Similar large generalizations of (1.2.11) exist; we will discuss them after we introduce the analogue of the elementary electric charge for the C_k in Section 1.3.

1.2.2 IIA supergravity

We now come to the effective action for IIA string theory. As we anticipated, we will only show the action for the bosonic fields.

Bosonic action of IIA supergravity

The leading bosonic action for type IIA superstrings is called type IIA supergravity. It reads

$$S_{\rm IIA} = S_{\rm kNS} + \frac{1}{4\kappa^2}\left[-\int d^{10}x\sqrt{-g}(|F_2|^2 + |F_4|^2) + \int B \wedge dC_3 \wedge dC_3\right] , \tag{1.2.15}$$

where $|F_k|^2 \equiv \frac{1}{k!} F_{M_1 \ldots M_k} F^{M_1 \ldots M_k}$, extending the definition in (1.1.15);

$$2\kappa^2 = (2\pi)^7 l_s^8 \, ; \tag{1.2.16}$$

and the kinetic NSNS term

$$S_{\rm kNS} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\,e^{-2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H|^2\right) \tag{1.2.17}$$

is the same Lagrangian as in (1.1.17), only now integrated in $d = 10$ dimensions rather than 26. This term will also appear in all the supergravity effective actions we will see later. The $e^{-2\phi}$ prefactor signals its origin from $g = 0$ string diagrams, as we remarked following (1.1.17). In a region where the dilaton is constant, the Planck length is now

$$l_{\rm P} = g_s^{1/4} l_s \, . \tag{1.2.18}$$

The string scattering amplitudes originally give a prefactor $e^{-2\phi}$ for the rest of (1.2.15) as well, but the action happens to look nicer if one rescales it away by redefining the C_k.

RR terms

The first parenthesis in (1.2.15) can be regarded as the kinetic term for $F_2 = dC_1$, and for $F_4 = dC_3 - H \wedge C_1$ from (1.2.9). The last term is more peculiar in (1.2.15). Unpacking the form notation (1.2.1), (1.2.5), and (1.2.8), one obtains:

$$\int B \wedge dC_3 \wedge dC_3 = \frac{10!}{2 \cdot (3!)^2}\int d^{10}x B_{[01}\partial_2 C_{345}\partial_6 C_{789]} \, . \tag{1.2.19}$$

It does not involve the metric, and it contains a potential B_2 without an exterior derivative. For this reason, it is called the *Chern–Simons* term, after the Chern–Simons action in three dimensions $S_{\rm CS} = \int_{M_3} {\rm CS}_A$, where

$$ {\rm CS}_A \equiv {\rm Tr}\left(A \wedge {\rm d}A + \frac{2}{3} A \wedge A \wedge A\right) \tag{1.2.20} $$

is a three-form with the property ${\rm dCS}_A = {\rm Tr}(F \wedge F)$; the two-form $F = {\rm d}A + A \wedge A$ is the nonabelian field-strength in the form notation of Section 1.2.1. Alternative expressions for (1.2.19), such as $-\int H \wedge C_3 \wedge {\rm d}C_3$, can be obtained by integration by parts. Note that if we worked in $d = 11$ rather than $d = 10$, we could compute[7]

$$ {\rm d}(B \wedge {\rm d}C_3 \wedge {\rm d}C_3) = H \wedge {\rm d}C_3 \wedge {\rm d}C_3 \overset{(1.2.9),(1.2.6)}{=} H \wedge F_4 \wedge F_4 \,. \tag{1.2.21} $$

Frame change

Superficially, the sign of the kinetic term for ϕ in (1.2.17) looks wrong: it is not of the usual form "kinetic energy minus potential energy," since $\dot{\phi}^2$ appears with a minus sign. (This issue of course was already present in (1.1.17).) But ϕ also appears multiplying the Einstein–Hilbert term R; so its dynamics is less simple than it looks. To make it more transparent, one can define an alternative metric

$$ g^{\rm E}_{MN} \equiv {\rm e}^{-\phi/2} g_{MN} \,. \tag{1.2.22} $$

This is called *Einstein frame* metric, because in terms of it (1.2.17) becomes

$$ S_{\rm E,\,kNS} = \frac{1}{2\kappa^2} \int {\rm d}^{10}x \sqrt{-g_{\rm E}} \left(R_{\rm E} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}{\rm e}^{-\phi}|H|^2\right) , \tag{1.2.23} $$

with the Einstein–Hilbert term now appearing without the dilaton prefactor; indices are also now contracted with the Einstein frame metric. Here the dilaton's kinetic term has the conventional sign. In most of the book, we will use the original *string frame* metric g_{MN}.

Supersymmetry

As we mentioned in Section 1.1.2, type II superstrings are symmetric under the 32 supercharges ϵ^a; so the fermionic completion of the supergravity action (1.2.15) will enjoy such symmetry too. Theories with 32 supercharges are relatively rare: besides IIB, an important example we will see later in this chapter is eleven-dimensional supergravity, of which IIA is a *dimensional reduction*.[8] This large amount of supersymmetry has consequences on the structure of the l_s and g_s corrections; for example, it fixes the two-derivative action completely.

This is a symmetry of the *theory*: not all supercharges will leave invariant a particular field configuration. The generic field configuration breaks supersymmetry

7 Equation (1.2.21), together with a generalization of Stokes's theorem that we will study in Section 4.1.10, allows us to reinterpret the CS term as an integral over an eleven-dimensional space of which the ten-dimensional one is a boundary.

8 Most other theories with 32 supercharges are further dimensional reductions of IIB or eleven-dimensional supergravity, some of which we'll see in Chapter 11; but there now exist examples which are not thought to arise this way [32].

completely, just like the generic metric g_{MN} has no Killing vectors. One of the topics of interest for this book will be the study of how many supercharges leave invariant given field configurations, or vice versa of which field configurations are invariant under a certain number of supercharges.

String corrections

Corrections to the supergravity approximation first occur at the eight-derivative level. The curvature terms read

$$S_{\mathrm{IIA},R^4} = \frac{1}{2\kappa^2}\frac{l_s^6\zeta(3)}{3\cdot 2^{11}}\int \mathrm{d}^{10}x\sqrt{-g}\mathrm{e}^{-2\phi}\left(t_{M_1\ldots M_8}t^{N_1\ldots N_8} + \frac{1}{8}\epsilon_{PQM_1\ldots M_8}\epsilon^{PQN_1\ldots N_8}\right) \cdot R^{M_1M_2}{}_{N_1N_2}R^{M_3M_4}{}_{N_3N_4}R^{M_5M_6}{}_{N_5N_6}R^{M_7M_8}{}_{N_7N_8} \tag{1.2.24}$$

at the leading order in g_s. Here ζ is Riemann's zeta function, and the tensor t is defined by

$$t_{M_1\ldots M_8}M^{M_1M_2}M^{M_3M_4}M^{M_4M_5}M^{M_7M_8} = 24\mathrm{Tr}M^4 - 6(\mathrm{Tr}M^2)^2\,. \tag{1.2.25}$$

Just like for supergravity, (1.2.24) can be obtained either by computing string amplitudes [33], or by computing the world-sheet beta functions beyond the leading approximation [34]. At $g = 1$, or in other words $\mathrm{e}^{0\cdot\phi}$ according to (1.1.12), there is a similar term, where the combination $t_8t_8 - \epsilon\epsilon/8$ appears.

Even at this l_s^8 level, the complete structure of the action is not completely established beyond (1.2.24). For example, one expects couplings to the form field strengths. There is a famous coupling of the type $\int B_2R^4$, related to anomalies [35, 36]. One can try to infer the remaining terms by using dualities [37] or supersymmetry [38]. A complementary approach is via string amplitudes in the four-field approximation (which gives a contribution to (1.2.24), when one linearizes it in the metric fluctuation $g_{MN} \sim \eta_{MN} + h_{MN}$); with this restriction, one can go to arbitrary precision in l_s in the aforementioned Berkovits formalism [39].

Massive IIA

IIA supergravity has a deformation by a parameter F_0 called *Romans mass* [40], still preserving 32 supercharges.[9] This theory was originally obtained by realizing a *Stückelberg mechanism*, by which the Kalb–Ramond field B acquires a mass, at the price of absorbing (or "eating") the degrees of freedom of C_1.

The original Stückelberg mechanism [43] is a variant of (and predates) the more familiar Brout–Englert–Higgs (BEH) mechanism. In both cases, a vector field A_μ acquires a mass by eating the degrees of freedom of a scalar a. In BEH, we gauge a rotation in field space, leading to a covariant derivative schematically of the form $\partial_\mu a + A_\mu a$. In the Stückelberg case, we instead introduce

$$D_\mu a \equiv \partial_\mu a + mA_\mu\,, \tag{1.2.26}$$

where m is a mass. This is invariant under a translation: $a \to a - m\lambda$, if also $A_\mu \to A_\mu + \partial_\mu\lambda$. This transformation can of course be used to set $a = 0$; in this sense, a

[9] There is in fact also a second deformation of IIA with 32 supercharges [41]; this arises as a dimensional reduction of eleven-dimensional supergravity where one identifies spacetime with itself up to an overall rescaling. These are the only maximally supersymmetric deformations [42].

is "eaten" by A_μ. The covariant kinetic term $D_\mu a D^\mu a$, when expanded, is now seen to contain a mass term $m^2 A_\mu A^\mu$ for the vector field. Equation (1.2.26) appears often in string compactifications with fluxes. We will comment further on the difference between BEH and Stückelberg, and generalize both, in Section 4.2.2.

The variant of (1.2.26) we need in IIA is obtained by adding an index to both participant fields, thus introducing

$$F_2 \equiv \mathrm{d}C_1 + F_0 B. \tag{1.2.27}$$

This modifies (1.2.9), which would have given $F_2 = \mathrm{d}C_1$. Now C_1 and B plays the role of a and A_μ in (1.2.26), respectively. We have now used the symbol F_0 for the mass parameter. The reason for this name becomes apparent when we notice that

$$\mathrm{d}F_2 = HF_0\,. \tag{1.2.28}$$

This is of the form (1.2.10), suggesting that F_0 should be regarded as a new, non-dynamical RR "field-strength" with zero indices. We will see later more cogent reasons for this interpretation.

We also have to change F_4 with respect to (1.2.9):

$$F_4 = \mathrm{d}C_3 - H \wedge C_1 + \frac{1}{2}F_0 B \wedge B\,; \tag{1.2.29}$$

(1.2.10) still holds. The gauge transformations (1.2.11) still leave these invariant, but that of B does not, unless we modify it:

$$\delta B = \mathrm{d}\hat{\lambda}_1\,, \qquad \delta C_1 = -F_0\hat{\lambda}_1\,, \qquad \delta C_3 = -F_0\hat{\lambda}_1 \wedge B\,. \tag{1.2.30}$$

With these new definitions for the F_k, the IIA action is now modified as

$$S_{\mathrm{IIA}} = S_{\mathrm{kNS}} + \frac{1}{4\kappa^2}\left[-\int \mathrm{d}^{10}x\sqrt{-g}(F_0^2 + |F_2|^2 + |F_4|^2) + \int \mathrm{mCS}_{10}\right], \tag{1.2.31}$$

where mCS_{10} is a ten-form that again has the formal property

$$\mathrm{d}(\mathrm{mCS}_{10}) = H \wedge F_4 \wedge F_4 \tag{1.2.32}$$

as in (1.2.21). Viewed in this way the modification with respect to (1.2.15) is minimal, although an explicit expression is less nice:

$$\mathrm{mCS}_{10} = B \wedge \left(\mathrm{d}C_3^2 + \frac{1}{3}F_0\mathrm{d}C_3 \wedge B^2 + \frac{1}{20}F_0^2 B^4\right)\,, \tag{1.2.33}$$

where $B^k \equiv B \wedge \ldots \wedge B$.

We will see a more elegant way of understanding both (1.2.15) and (1.2.31) in Section 10.1, at the cost of a slightly more sophisticated mathematical apparatus. On that occasion, we will also see the supersymmetry transformations.

1.2.3 IIB supergravity

IIB supergravity is very similar to IIA [44–46]. The most important difference is that it is a *chiral* theory: in Table 1.2, we see that the two ψ^a and the two λ^a have the same chirality. Usually the presence of chiral spinors threatens an anomaly.

While there are no gauge fields, in a gravitational theory we have to worry about potential anomalies for *diffeomorphisms*, which are just as lethal as their gauge counterparts. The chiral ψ^a and λ_a indeed give a nonvanishing contribution to such a diffeomorphism anomaly. Fortunately, however, the RR field C^+_{MNPQ} also gives a contribution, which exactly cancels the fermionic ones [47], avoiding a potential catastrophe in a nontrivial way.

C^+_{MNPQ} gives a contribution to an anomaly in spite of being bosonic because of its self-duality property:

$$F_{M_1\dots M_5} = \frac{1}{5!}\sqrt{-g}\,\epsilon^{(0)}_{M_1\dots M_5}{}^{M_6\dots M_{10}} F_{M_6\dots M_{10}}\,, \tag{1.2.34a}$$

where again $\epsilon^{(0)}$ is the completely antisymmetric tensor defined later in (1.2.6). In the condensed notation of the previous subsection, (1.2.34a) is written as

$$F_5 = *F_5\,. \tag{1.2.34b}$$

Just like in (1.2.9), we can take $F_5 = \mathrm{d}C_4^+ - H \wedge C_2$. There are alternatives, such as $F_5' = \mathrm{d}C_4^+ + \frac{1}{2}(B \wedge \mathrm{d}C_2 - H \wedge C_2)$. Both F_5 and F_5' satisfy the Bianchi identities (1.2.10); one can bring one into the other by redefining C_4^+ (with no ill effect on the action (1.2.36)).

Pseudoaction

While self-duality of F_5 saves the theory, it also makes it hard to write down an action. A self-duality property similar to (1.2.34) can be introduced for any k-form potential a_k with self-dual field-strength $\mathrm{d}a_k = *\mathrm{d}a_k$ in $d = 2(k+1)$ dimensions. The simplest example is for $k = 0$: a_0 is then a scalar in $d = 2$ dimensions, and self-duality implies

$$\partial_M a_0\, \partial^M a_0 = \sqrt{-g}\,\epsilon^{MN}_{(0)}\,\partial_M a_0\, \partial_N a_0 = 0\,, \tag{1.2.35}$$

so the usual Lagrangian density vanishes. More generally, this happens for any even k, and in particular for our (1.2.34): the naive Lagrangian density $|F_5|^2 = 0$. There are strategies to cope with this issue; see [48, 49] for a recent proposal. In the following, we will simply write a *pseudoaction*: all equations of motion can be derived by varying it, *except* for the constraint (1.2.34). From now on, we will drop the superscript and call $C_4^+ \to C_4$, for uniformity with the other potentials in both type II theories.

A pseudoaction for IIB is then

$$S_{\mathrm{IIB}} = S_{\mathrm{kNS}} + \frac{1}{4\kappa^2}\left[-\int \mathrm{d}^{10}x\sqrt{-g}\left(|F_1|^2 + |F_3|^2 + \frac{1}{2}|F_5|^2\right) + \int B\wedge \mathrm{d}C_2 \wedge \mathrm{d}C_4\right], \tag{1.2.36}$$

where S_{kNS} was given in (1.2.17). As in IIA, the last term is of Chern–Simons type, and has the formal property that in eleven dimensions $\mathrm{d}(B\wedge \mathrm{d}C_2\wedge \mathrm{d}C_4) = H\wedge F_3\wedge F_5$, similar to (1.2.21). Alternative expressions can be obtained by integration by parts, such as $-\int H\wedge C_2\wedge \mathrm{d}C_4$. The fermionic completion of (1.2.36) is supersymmetric with 32 supercharges, just like IIA.

The leading l_s corrections are the same as in IIA, (1.2.24). At the $g = 1$ level, the same combination $t_8 t_8 - \epsilon\epsilon/8$ is present. Knowledge of the full eight-derivative corrections is as incomplete as in IIA, but in this case dualities are more powerful and determine the dependence on ϕ of the terms in (1.2.24) beyond the $g = 0$ and $g = 1$ terms [50]. This is based on an important additional symmetry of IIB that is already present at the level of supergravity, to which we now turn.

$\mathrm{SL}(2,\mathbb{R})$ symmetry

Given a 2×2 matrix with unit determinant,

$$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}), \tag{1.2.37}$$

we define the transformation law

$$\begin{gathered} \tau \to m \cdot \tau \equiv \frac{a\tau + b}{c\tau + d}, \qquad g_{MN} \to |c\tau + d|\, g_{MN} \\ F_5 \to F_5, \qquad \begin{pmatrix} C_2 \\ B \end{pmatrix} \to m \begin{pmatrix} C_2 \\ B \end{pmatrix}, \end{gathered} \tag{1.2.38}$$

where

$$\tau \equiv C_0 + \mathrm{i} e^{-\phi} \tag{1.2.39}$$

is called the *axiodilaton*.

Equation (1.2.38) is a symmetry of the action (1.2.36). This type of nonlinear action on τ is called a *Möbius transformation*; it will appear in several other contexts. The string-frame metric transforms, but the Einstein frame metric (1.2.22) does not:

$$g^{\mathrm{E}}_{MN} \to g^{\mathrm{E}}_{MN}. \tag{1.2.40}$$

There is also an alternative expression for the transformation law of the two-form potentials, in terms of

$$G_3 \equiv \mathrm{d}C_2 - \tau H = \mathrm{d}C_2 - HC_0 - \mathrm{i}e^{-\phi} H = F_3 - \mathrm{i}e^{-\phi} H. \tag{1.2.41}$$

From (1.2.38):

$$G_3 \to (a\,\mathrm{d}C_2 + bH) - \frac{a\tau + b}{c\tau + d}(c\,\mathrm{d}C_2 + dH) \overset{(1.2.37)}{=} \frac{1}{c\tau + d} G_3. \tag{1.2.42}$$

To see why this symmetry is remarkable, consider for example $s = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Starting from a configuration where $C_0 = 0$, we see that

$$e^{\phi} \to e^{-\phi}. \tag{1.2.43}$$

The string coupling $g_s \to 1/g_s$: strong coupling is mapped to weak coupling.

We hasten to add that this is a symmetry of the *supergravity* approximation; it does not fully survive in string theory. One reason is the general expectation that theories of quantum gravity should have no continuous symmetries, which we will review in Chapter 12. A perhaps more concrete argument is that the fundamental string couples to B; an arbitrary transformation in (1.2.38) transforms $B \to cC_2 + dB$, and if this were a symmetry for arbitrary c and d there would have to exist a continuum of string-like objects, coupling to any such linear combination. We will see in Section 1.4.3 that generalizations of the fundamental string do exist, but only a *discrete* infinity of them. This argument suggests that $\mathrm{SL}(2,\mathbb{R})$ gets discretized; we will see this in Section 1.4.3.

1.2.4 Heterotic supergravity

Action

The bosonic part of the action for heterotic supergravity is

$$S_{\rm het} = S_{\rm kNS} + \frac{l_s^2}{8\kappa^2}\int_{M_{10}} \mathrm{d}^{10}x\sqrt{-g_{10}}e^{-2\phi}\mathrm{Tr}|F^2|\,. \qquad (1.2.44)$$

(The positive sign is because $\mathrm{Tr}(T^aT^b)$ is negative-definite; we do not include i's in the gauge fields.) The term $S_{\rm kNS}$ is the usual (1.2.17), but now

$$H \equiv \mathrm{d}B - \frac{l_s^2}{4}\mathrm{CS}_A\,, \qquad (1.2.45)$$

where CS_A was given in (1.2.20). As in type II supergravity, there are l_s corrections; some of these will play a role in anomaly cancellation, to which we now turn.

Anomaly cancellation

Since we only supersymmetrized the right-movers, (1.2.44) only has half the supersymmetry of the type II theories: 16 supercharges. Each line marked R in Table 1.3 is related by supersymmetry to the line marked NS above it. As in IIB, the spectrum is chiral: the gravitino $\psi_{M\alpha}$ and the gaugino χ_α have positive chirality, while the $\lambda_{\dot\alpha}$ have negative chirality. Again, this creates the danger of an anomaly, this time not only for diffeomorphisms but also for gauge transformations. Superficially, this does not appear to vanish, but exactly for the two gauge groups (1.1.44) it takes the factorized form

$$\delta\Gamma = \int \omega_2^1 \wedge Y_8\,. \qquad (1.2.46)$$

Here the symbol δ denotes variation under both diffeomorphisms and gauge transformations; as usual in QFT, Γ is the quantum effective action (the action whose tree-level amplitudes equal the full quantum amplitudes of S). Y_8 is an eight-form, quartic in curvature and in the Yang–Mills field-strength F^a. The two-form ω_2^1 is such that

$$\mathrm{d}\omega_2^1 = \delta(\mathrm{CS}_g - \mathrm{CS}_A)\,, \qquad (1.2.47)$$

where now CS_g is a three-form with the property $(\mathrm{dCS}_g)_{MNPQ} = 6R^{AB}{}_{[MN}R_{PQ]BA}$, similar to (1.2.20).

Equation (1.2.46) is nonzero, but it is not the full story; there are contributions from the higher-derivative corrections. First of all, (1.2.45) should be modified to

$$H \equiv \mathrm{d}B + \frac{l_s^2}{4}(\mathrm{CS}_g - \mathrm{CS}_A)\,. \qquad (1.2.48)$$

CS_g contains up to three derivatives, since its derivative contains by definition two Riemann tensors. Now (1.2.47) implies $\delta H \neq 0$, unless we also make B transform as $\delta B = -\frac{l_s^2}{4}\omega_2^1$. Another l_s correction is the term

$$\int B \wedge Y_8 \, ; \tag{1.2.49}$$

its transformation now cancels the anomaly (1.2.46). This only works for the two choices (1.1.44) of the gauge group, which allowed us to write (1.2.46) in the first place. For more general gauge groups, $\delta\Gamma$ does not factorize.

Exercise 1.2.1 Compute the equations of motion for the RR fluxes in IIA and IIB by varying the actions (1.2.15), (1.2.31), and (1.2.36) with respect to the C_p. Check that they are formally identical to the Bianchi identities (1.2.10), if we extend them to $k \geq 6$ by defining

$$F_6 = *F_4 \, , \qquad F_7 = - * F_3 \, , \qquad F_8 = - * F_2 \, , \qquad F_9 = *F_1 \, , \qquad F_{10} = *F_0 \, , \tag{1.2.50}$$

in the notation of (1.2.34b).

Exercise 1.2.2 Use (1.2.38) and (1.2.39) to show

$$\mathrm{e}^{\phi} \to |c\tau + d|^2 \mathrm{e}^{\phi} \, . \tag{1.2.51}$$

1.3 D-branes

We saw that string theory has two expansion parameters: l_s (or rather the dimensionless l_s/r, with r a curvature radius), and e^{ϕ}, which when constant is also called g_s. In (1.1.21), we wrote the expansion as a power series, but we wondered whether this really captures the whole dependence. Recall that a function is called (real) analytic if it coincides with its Taylor series around every point. A famous nonanalytic function is

$$f(g) = \mathrm{e}^{-1/g^2} \, . \tag{1.3.1}$$

Both f and all its derivatives vanish at $g = 0$, so its Taylor series vanishes identically, but $f(g) \neq 0$. In physics, we call an effect with such a dependence on the coupling *nonperturbative*. In attempts at describing nonperturbative aspects of field theory, two types of objects often come up: solitons, which are large and stable field configurations, localized in space; and instantons, which occur in the Euclidean path integral and are localized in time as well. In string theory, the most prominent objects that play these roles are *D-branes*; this section is devoted to them.

1.3.1 Solitons and instantons

We begin with a quick reminder of solitons and instantons in field theory.

Solitons

In perturbative field theory, we often deal with quanta of small field perturbations around the vacuum, which we call elementary particles. However, many field theories also contain other objects. A *soliton* is a "localized" solution of the equations of motion that does not dissipate. This is in contrast with plane waves, which are delocalized. Already in a linear system we find both types of solutions. For example, for a free scalar ϕ in $d = 2$, the equation of motion $(-\partial_t^2 + \partial_x^2)\phi = 0$ is solved by $\phi = \phi_0(t - x)$ for any ϕ_0; this includes the plane wave solutions $\phi = e^{i(t-x)}$, but we can also take ϕ_0 to be any localized function, such as $\phi = (\cosh(x - t))^{-2}$, which is very small everywhere except for a narrow band of order one.

More notably, solitons exist in nonlinear systems also: the venerable Korteweg–De Vries (KdV) equation

$$\partial_t \phi + \partial_x^3 \phi + 6\phi\partial_x\phi = 0\,, \tag{1.3.2}$$

describing water waves in shallow channels, has many solitonic solutions, such as

$$\phi = \frac{c}{2}\left(\cosh\left(\frac{\sqrt{c}}{2}(x - ct)\right)\right)^{-2} \tag{1.3.3}$$

for any c. Once again, this is localized in a narrow band at any t, and velocity c. There also exist *multisoliton* solutions, where many waves similar to (1.3.3) coexist and interact, just like particles. The dissipative and nonlinear effects from the second and third term of (1.3.2) compete in exactly such a way that these waves do not dissipate. (At a deeper level, this phenomenon is really due to the presence of infinitely many conserved quantities for (1.3.2); see, for example, [51].)

Going back to relativistic field theories, many nonlinear theories also have such stable solutions. Any $d = 2$ theory of a single scalar ϕ with a potential $V(\phi)$ with two or more vacua $\phi_\pm$ will have solitons: in this case, they are field configurations where $\phi \to \phi_\pm$ at $x \to \pm\infty$. This is protected from dissipation by topological reasons: one cannot take such a configuration to a vacuum without an infinite energy expenditure. Unlike (1.3.3), these solitons can be at rest. Solitons also exist for $d > 2$, and again they are usually stable for topological reasons. They are usually heavy when the theory is weakly coupled. Nevertheless, just like the soliton waves of the KdV equation, they behave almost as particles, and we should be able to compute their dynamics, for example, their scattering amplitudes.

Monopoles

A *magnetic monopole* is a soliton around which the flux integral of the magnetic field is nonzero. In $\mathbb{R}^3$, this means that

$$\int_{S_2} \mathrm{d}a_i B_i = q_\mathrm{m} \neq 0 \tag{1.3.4}$$

on a sphere S_2. $\mathrm{d}a_i$ is an outward-directed vector whose norm $\mathrm{d}a$ is the infinitesimal area. Usually in electromagnetism $q_\mathrm{m} = 0$, because of the Maxwell equation $\partial_i B_i = 0$, the space part of $\partial_{[\mu} F_{\nu\rho]} = 0$. So to introduce such objects, we should change this equation to $\partial_i B_i = \Pi_{i=1}^3 \delta^i(x^i)$. A striking feature is their *charge quantization*. Suppose we have a monopole localized at the origin. Consider a potential A_μ^N, and integrate it on the equator E of S_2. Since $B_i = \epsilon_{ijk}\partial_j A_k^\mathrm{N}$, we can apply Stokes's theorem to the semisphere $\mathrm{U_N}$ of S_2 bounded by E:

$$\oint_\mathrm{E} \mathrm{d}x_i A_i^\mathrm{N} = \int_{U_\mathrm{N}} \mathrm{d}a_i B_i\,. \tag{1.3.5}$$

However, we could do the same with the other portion of the sphere U_S; because of orientation, there is a minus sign, so this time we would have $\oint_E dx_i A_i^N \stackrel{?}{=} -\int_{U_S} da_i B_i$. Subtracting this from (1.3.5), we would find $\int_{S^2} da_i B_i \stackrel{?}{=} 0$, a contradiction. The way out is to have a second potential A_μ^S for the region U_S; then $\oint_E dx_i A_i^S = -\int_{US} da_i B_i$, and subtraction from (1.3.5) now just gives

$$\int_{S_2} da_i B_i = \oint_E dx_i (A^N - A^S)_i = q_m \,. \tag{1.3.6}$$

Two potentials are related by a gauge transformation, so $(A^N - A^S)_i = \partial_i \lambda_0$; now (1.3.6) tells us that λ_0 is not periodic, but undergoes a shift q_m after a turn around E. Now, a gauge transformation acts on the wave function of a particle with electric charge q_e by $\psi \to e^{-iq_e\lambda_0}\psi$. This should be periodic, so $e^{-iq_e q_m} = 1$, or in other words,

$$\frac{1}{2\pi} q_e q_m \in \mathbb{Z} \,. \tag{1.3.7}$$

This is called *Dirac quantization*, and will play many roles in this book.[10]

Monopole solutions can be found in the Yang–Mills–Higgs model

$$S_{\text{YMH}} = -\int d^4x \, \text{Tr}\left(\frac{1}{2g_{\text{YM}}^2}|F|^2 + D_\mu a D^\mu a + \lambda(a^2 - a_0^2)^2\right), \tag{1.3.8}$$

where F is an SU(2) gauge field, a an adjoint scalar, and $D_\mu a \equiv \partial_\mu a + [A_\mu, a]$ the gauge covariant derivative. For the *'t Hooft–Polyakov monopole* solutions, q_m is defined as the magnetic charge under $\text{Tr}(aF)$, and all fields are nonsingular. One can prove the *Bogomolnyi–Prasad–Sommerfield (BPS) bound* for the mass of any monopole solution:

$$m_{\text{mon}} \geq a_0 q_m = \frac{2\pi a_0}{g_{\text{YM}}} \,. \tag{1.3.9}$$

As anticipated, the mass is large at weak coupling. Conversely, at strong coupling they may become light.

These effects are under better control in supersymmetric theories. A famous example is the $\mathcal{N} = 2$-supersymmetric version of Yang–Mills (*super-YM*), which is a bit similar to (1.3.8). The $\mathcal{N} = 1$ super-YM involves the gauge field A_μ and a gaugino λ_α. The $\mathcal{N} = 2$ version involves *two* gauginos, and a scalar a (now complex), all in the adjoint representation of the gauge group. The bosonic Lagrangian is[11]

$$S_{\text{sYM}} = \int d^4x \text{Tr}\left(-\frac{1}{2g^2}|F|^2 + \frac{\theta}{64\pi^2}\epsilon_{(0)}^{\mu\nu\rho\sigma} F_{\mu\nu}F_{\rho\sigma} - D_\mu a^\dagger D^\mu a - \frac{1}{2}\text{Tr}([a, a^\dagger])^2\right) . \tag{1.3.10}$$

[10] Monopoles in electromagnetism have never been detected; but in quantum chromodynamics (QCD), the theory of strong interactions, their condensation is believed to play an important role in confinement. The mechanism is similar to the Meissner effect in superconductors, where the electric field effectively acquires a mass and the magnetic field is zero almost everywhere, except in thin tubes. The same should happen in QCD, but now with the electric field confined in thin tubes; since its flux lines no longer disperse, this leads to a potential that grows with distance.

[11] The θ term is a total derivative, but it does have physical effects on instantons and on monopoles. In QCD, it is allowed but observed to be $< 10^{-9}$ for unknown reasons (the *strong CP* problem). In supersymmetric theories, it appears naturally.

The anticommutators of the $\mathcal{N} = 2$ algebra read

$$\{Q^I, \bar{Q}_J\} = P_\mu \gamma^\mu \delta^I_J \,, \qquad \{Q^I, Q^J\} = \epsilon^{IJ} Z \,. \tag{1.3.11}$$

The generator Z commutes with all other generators in the algebra, and as such it is called *central* charge, just like c in (1.1.19). The BPS bound is now reinterpreted as $m_{\text{mon}} \geq |Z|$. The representation theory of (1.3.11) shows that BPS states form a special short representation; because of this, they are protected against time evolution and against deformations of the theory, so they cannot just disappear as the coupling is changed.

There is a low-energy effective description where we only keep the "abelian" part of the fields, in the Cartan subalgebra. For $G = \mathrm{SU}(2)$, this is $\mathrm{U}(1)$, which we can take along the σ^3 generator of su(2); so we keep A^3_μ and its supersymmetric partners. The bosonic part of the effective action is now an Abelian version of (1.3.10), but with both $g = g(a_3)$ and $\theta = \theta(a_3)$ depending on the vacuum expectation value for the scalar a_3. $\mathcal{N} = 2$ supersymmetry determines this *Seiberg–Witten (SW) effective theory* [52] exactly as $\tau(a_3) = \partial^3_{a_3} \mathcal{F}$, where the *prepotential* $\mathcal{F}$ is a holomorphic function of a_3, and we defined it:

$$\tau \equiv \frac{4\pi \mathrm{i}}{g^2_{\mathrm{YM}}} + \frac{\theta}{2\pi} \,. \tag{1.3.12}$$

In this solution, monopoles do become light at strong coupling, and there is a "dual" description of the theory, of which they are the elementary photons, described by a vector $\tilde{A}_\mu$.

In conclusion, solitons behave a lot like particles; at weak coupling, they are collective excitations, but at strong coupling they may become the fundamental degrees of freedom. So in quantum field theory it is not always clear which objects are made of which others.

Instantons

In cases where exact results are available, such as in the $\mathcal{N} = 2$ SW theory, the effective action is not analytic: it does not coincide with its perturbative expansion. The prepotential $\mathcal{F}$ depends on a_3, but after reexpressing it in terms of the high-energy g_{YM}, we find a sum of contributions of the type

$$\mathrm{e}^{-s/g^2_{\mathrm{YM}}} \,, \tag{1.3.13}$$

with $s = 8\pi^2 k$, for k an integer.

Such effects are ubiquitous: they appear even in quantum mechanics, in the dependence on $\hbar$ of the energy spectrum, or of the tunnel effect probability (see, for example, [53–55]). The easiest example is perhaps the energy of the vacuum in a double-well potential $V_{\mathrm{dw}} = \lambda(x^2 - x_0^2)^2$. Near each vacuum $x = \pm x_0$, it is approximately harmonic, $V_{\mathrm{dw}} \sim \frac{1}{2}\omega^2 (x \mp x_0)^2$, $\omega^2 = 8x_0^2 \lambda$; so we expect the lowest energies to be $\sim \frac{\hbar}{2}\omega$, with a small splitting due to tunneling between the two. To find this, one computes the probability $\langle x_0 | \mathrm{e}^{-\mathrm{i}HT/\hbar} | -x_0 \rangle$, which becomes an integral over histories:

$$\int Dx(t) \mathrm{e}^{-\mathrm{i}S/\hbar} \,. \tag{1.3.14}$$

After a Wick rotation $T \to -iT_E$, the integral is dominated by the Euclidean-time history $x(t_E)$ that solves the Euclidean equations of motion, which are obtained from the ordinary ones by an overall sign of the potential, $V_E = -V$. This classical solution x_{cl} needs to asymptote to $\pm x_0$ for $t_E \to \pm\infty$; it is called an *instanton* because most of its action comes from a particular time t_0, when x switches from one vacuum to the other. Evaluating the integrand on this solution gives already a good approximation to the integral; a better one is found by performing the integral over fluctuations $(x - x_{cl})(t)$ around it. For the double-well V_{dw}, both these steps can be carried out exactly, and the energy of the lowest eigenstate is

$$E_0 \sim \frac{\hbar}{2}\omega - \hbar\omega\sqrt{\frac{6s}{\pi\hbar}}e^{-s/\hbar}\,, \qquad s = \frac{2}{3}\omega x_0^2\,. \tag{1.3.15}$$

The splitting of the levels contains the nonanalytic $e^{-s/\hbar}$. The coefficient $s = S(x_{cl})$ is the action evaluated on the classical solution; the coefficient multiplying it comes from the integral over small fluctuations.

In field theory, one again Wick-rotates the path integral, looking for classical Euclidean solutions ϕ_{cl} with finite action $s = S(\phi_{cl})$, so that their contribution $e^{-s} \neq 0$. So a field theory instanton should be localized in space and time. A famous example occurs for YM theories in $d = 4$, where instantons are solutions of the *self-duality equation*

$$F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}\,, \tag{1.3.16}$$

which in the notation of (1.2.34b) reads

$$F = *F\,. \tag{1.3.17}$$

Computing these effects exactly is more challenging, but sometimes it can be done; again, supersymmetry helps (see, for example, [56] for a review). While the SW $\mathcal{N} = 2$ solution was found with other methods, one can in fact reproduce it exactly by counting gauge instantons [57].

World-sheet instantons in string theory

In string theory, we have two expansion parameters, l_s and $g_s = e^{\phi}$. For l_s, we can apply the preceding discussion to the two-dimensional QFT on the world-sheet. An instanton for this model is a finite-action solution to the Euclidean equations of motion. Equation (1.1.4) is not directly the world-sheet area, but it is classically equivalent to it, so we can look for maps $x(\sigma)\colon \Sigma \to M_{10}$, where the area $A(\Sigma)$ is minimized. In flat space, this would give the degenerate situation where Σ is shrunk to a point. But for spacetimes with nontrivial topology, such as compactifications, the internal space may contain a two-dimensional subspace S_2 that cannot be continuously shrunk to a point; so its minimal area $A(\Sigma) \neq 0$, and this embedding is a *world-sheet instanton*, which, recalling (1.1.4) and (1.1.5), contributes

$$e^{-A(\Sigma)/4\pi l_s^2} \tag{1.3.18}$$

to the path integral. Remarkably, these effects can sometimes be computed exactly, as we will see in Section 7.1.

Effects that are nonperturbative in g_s are a different matter altogether. The world-sheet approach is intrinsically perturbative in g_s; so we need a new ingredient.

1.3.2 Open string definition

A *D-brane* is an additional extended object in string theory; so far we have not encountered it because we limited ourselves to a perturbative approach.

In general relativity, no solitons exist, if defined as fully regular solutions with localized energy and flat asymptotics ([58]; for a recent account, see, for example, [59, IV.8]). In a general gravitational theory, we might be tempted to say that black holes are the analogue of solitons: they have localized energy, and they are stable. They do have a singularity, unlike the solitons we have studied in the previous section. It is natural to think, however, that this singularity signals the breakdown of general relativity, and that in a full theory of quantum gravity it disappears.

Besides black holes, string theory has solutions whose energy is localized along a subspace, and not just a point in space; so they are called *p-branes*, a generalization of the word "membrane," where p denotes the number of space dimensions they span. As solutions of the supergravity equations, they have a singularity at their core; but string theory does provide an alternative, smooth understanding, in terms of *open strings*.

An open string is one whose time slice is an interval rather than a circle. It requires choosing boundary conditions for the world-sheet fields; Dirichlet boundary conditions for some of the $x^M(\sigma)$ correspond to open strings that end on a subspace of spacetime, which is accordingly called a Dirichlet-brane, or more commonly a *D-brane*.

These D-branes are the fundamental definition of the p-brane gravity solutions [60]. Consider, for example, a D-brane in flat space. We have both closed and open strings, interacting with one another, both described by a world-sheet model with a flat background metric $g_{MN} = \eta_{MN}$. Suppose we now perform the path integral over the open string degrees of freedom – in QFT jargon, we "integrate them out." As in QFT, the action for the remaining degrees of freedom, those of closed strings, is now modified: the background metric is distorted to a different metric g_{MN}. So we obtain an effective description, with only closed strings in a curved background metric, which is nothing but the p-brane metric, or in other words the "back-reaction" of the D-brane on spacetime. Because of the difficulties with the world-sheet description of RR fields that we mentioned at the end of Section 1.1.2, this identification is a bit difficult to demonstrate explicitly, but several indirect arguments point toward it.

We will now review the definition of D-branes as loci where open strings end, and then their gravitational description.

To describe an open string, we take $\sigma^1 \in [0, \pi]$, rather than a periodic coordinate as in Section 1.1. When we vary the action, we now have to pay attention to boundary terms. For example, let us consider flat space $g_{MN} = \eta_{MN}$ and $h_{\alpha\beta} = \eta_{\alpha\beta}$, and focus on a single coordinate x. Then the variation gives

$$\begin{aligned} \frac{1}{2}\delta \int_\Sigma \mathrm{d}^2\sigma \partial^\alpha x \partial_\alpha x &= \int \mathrm{d}^2\sigma \partial^\alpha x \partial_\alpha \delta x = \int \mathrm{d}^2\sigma \left(\partial_\alpha(\delta x \partial^\alpha x) - \delta x \partial^2 x\right) \\ &= \int_{\partial\Sigma} \mathrm{d}\sigma^0 \delta x \partial_{\sigma^1} x - \int_\Sigma \mathrm{d}^2\sigma \delta x \partial^2 x\,. \end{aligned} \tag{1.3.19}$$

The second term gives us the free equation of motion $\partial^2 x = 0$, familiar from closed strings; the first is new. To set it to zero, we can either set $\partial_{\sigma^1} x = 0$ or $\delta x = 0$. The first is called *Neumann (N)* boundary condition; the second sets to zero the variation of this world-sheet scalar at the boundary, and is called *Dirichlet (D)* boundary condition.

All this was for a single x; still remaining in flat space, we can select N or D boundary conditions for each of the x^M independently. Each x^M for which we are choosing, a D boundary condition is fixed, $x^M = x_0^M$, at the boundary $\partial\Sigma$, or in other words at the endpoints of the open string. So if we choose N boundary conditions for $p+1$ fields x^M (including time), and D for the remaining $9-p$, we have a theory of open strings that end on a $p+1$-dimensional object; by definition, this is called a Dp-brane.

Clearly, the presence of a Dp breaks some of the symmetries of the background; in flat space, a flat Dp breaks the ten-dimensional Poincaré group ISO$(1,9)$ to a ISO$(1,p+1)\times$ SO$(9-p)$. For applications to compactifications, we will want Dp-branes that are completely extended along $\mathbb{R}^4$ and localized along some of the internal dimensions. This preserves the four-dimensional Poincaré group.

The analysis in (1.3.19) is enough for D-branes in bosonic string theory; for type II superstrings, we also need to give boundary conditions for world-sheet spinors. The procedure (1.3.19) applied to the spinorial action (1.1.30) shows that at the boundary $\partial\Sigma$ the combination $\psi_+\delta\psi_+ - \psi_-\delta\psi_-$ should vanish; this can be arranged by having

$$(\psi_+ = \pm\psi_-)|_{\partial\Sigma}\,. \tag{1.3.20}$$

Both for the x^M and ψ^M, the boundary conditions now relate left- and right-movers, and the open string spectrum has only one set of oscillators. Just like in the closed string sector, massless states come from the fermionic oscillators; the sign in (1.3.20) gives two different sectors, which again we call NS and R. After GSO projection, we have the massless spectrum

$$b^M_{-1/2}|0\rangle_{\rm NS}\,,\qquad \mathbf{8}_{\rm V}\,;\qquad |0,\alpha\rangle_{\rm R}\,,\qquad \mathbf{8}_{\rm S}\,. \tag{1.3.21}$$

If we are considering N boundary conditions, the endpoints are not fixed to lie anywhere, and we have a D9-brane; in this case (1.3.21) is interpreted as a vector field a_M and a gaugino λ_α. If we have D directions $i = p+1,\ldots,9$, then the components $b^i_{-1/2}|0\rangle_{\rm NS}$ are transverse to the Dp-brane, and behave as scalars under ISO$(1,p+1)$; the parallel components $b^a_{-1/2}|0\rangle_{\rm NS}$, $a = 0,\ldots,p$ still represent a vector field. As for the $|0,\alpha\rangle_{\rm R}$, they represent a spinor under both the parallel and transverse rotations. So in total we have

$$a^a\,,\qquad x^i\,,\qquad \lambda^\alpha\,. \tag{1.3.22}$$

Calling the scalars x^i might seem confusing, given that we also called x^M the world-sheet scalars. But we will see shortly that the scalars x^i on a D-brane parameterize its transverse fluctuations, and thus give an embedding of Dp into spacetime.

Just as closed (bosonic) strings can couple to condensates of their massless fields (1.1.9), we can couple open strings to condensates of (1.3.22). For example, the coupling to a^a reads

$$\int_{\partial\Sigma} d\sigma^0 a_a \partial_0 x^a\,; \tag{1.3.23}$$

recalling (1.1.3), this means that the endpoints behave as charged particles under a. Rewriting this as $-\int_\Sigma d^2\sigma\partial_1(a_a\partial_0 x^a)$ and adding the total derivative

$0 = \int \mathrm{d}^2\sigma\partial_0(a_a\partial_1 x^a)$, this becomes $\int_\Sigma \mathrm{d}^2\sigma f_{ab}\partial_0 x^a\partial_1 x^b$, where the two-form $f = \mathrm{d}a$ is the field-strength of the vector a. This has the same form as in (1.1.10); in the parallel directions, the two can be collected together, and the combination $(B + 2\pi l_s^2 f)_{ab}$ appears.

1.3.3 Effective action

So far, we considered a Dp-brane extended along a flat subspace $\mathbb{R}^{p+1} \subset \mathbb{R}^{10}$. We can try to generalize this to arbitrary subspaces in arbitrary background metrics, but we still need to impose conformal invariance. For closed strings, this led to the beta functions (1.1.14), which we reinterpreted as spacetime equations of motion for the effective action (1.1.17); for open strings, it leads to equations of motion for the brane, which again come from an effective action $S_{\mathrm{D}p}$. The fields appearing in this action are those of the closed string spectrum in Tables 1.1 and 1.2, plus the open string fields (1.3.22). Once again, we only give here the bosonic part:

$$S_{\mathrm{D}p} = \tau_{\mathrm{D}p}\left[-\int \mathrm{d}^{p+1}\sigma \mathrm{e}^{-\phi}\sqrt{-\det(g|_{\mathrm{D}p} + \mathcal{F})} \pm \sum_k (-1)^k \frac{1}{k!}\int_{\mathrm{D}p} C_{p+1-2k}\wedge \mathcal{F}^k\right] \tag{1.3.24}$$

where

$$\tau_{\mathrm{D}p} = \frac{1}{(2\pi)^p l_s^{p+1}}\,. \tag{1.3.25}$$

The coordinates σ^a, $a = 0, \ldots, p$ parameterize the world-volume[12] Dp. The restriction or pull-back operation $|_{\mathrm{D}p}$ consists in contracting each index M with $\partial_a x^M$, as in (1.1.3) and (1.1.10); the $x^M(\sigma^0, \ldots, \sigma^p)$ gives the embedding of Dp into spacetime. In the second term of (1.3.24), the spacetime forms are integrated on the Dp world-volume by first pulling them back; so, for example, we should really write $\int_{\mathrm{D}p} C_p \equiv \int_{\mathrm{D}p} C_p|_{\mathrm{D}p}$. The sign $\pm$ is related to the "orientation" on Dp, namely to the order of the coordinates in the world-volume measure; we will see this in more detail later, and fix this convention more precisely in Chapter 10.

Recall that $f^k = f\wedge \ldots \wedge f$; we also defined

$$\mathcal{F}_{ab} \equiv (2\pi l_s^2 f + B|_{\mathrm{D}p})_{ab} = 2\pi l_s^2 f_{ab} + B_{MN}\partial_a x^M \partial_b x^N\,, \tag{1.3.26}$$

in line with the remark that follows (1.3.23). The gauge transformation for B no longer leaves the action invariant, unless it also acts on a:

$$B \to B + \mathrm{d}\hat{\lambda}_1\,, \qquad a \to a - 2\pi l_s^2 \hat{\lambda}_1\,. \tag{1.3.27}$$

For open strings, (1.1.12) has to be modified as $g_s^{2g-2+\#b}$, where $\#b$ is the number of boundaries added to Σ; the prefactor $\mathrm{e}^{-\phi}$ in (1.3.29) indicates then that this action originates from a sphere to which we add a single boundary, which is topologically a disk. This is the simplest open string diagram: an open string that is created and later disappears. The term containing C_p does not contain this prefactor because of the

[12] We will call "world-volume" the subspace swept in spacetime by a D-brane, and keep using "world-sheet" for a fundamental string.

customary RR rescaling noted following (1.2.17). For constant $e^{\phi} = g_s$ effectively the tension of a D-brane is not really $\tau_{\mathrm{D}p}$, but rather

$$T_{\mathrm{D}p} = \frac{\tau_{\mathrm{D}p}}{g_s}\,. \tag{1.3.28}$$

So at small g_s, D-branes are heavy, similar to the bound on monopole mass in (1.3.9) in terms of g_{YM}.

The DBI term

If we set to zero all fields except g_{MN}, (1.3.24) reduces to

$$-\,\tau_{\mathrm{D}p}\int \mathrm{d}^{p+1}\sigma\, \mathrm{e}^{-\phi}\sqrt{-\det g_{MN}\partial_a x^M \partial_b x^N}\,. \tag{1.3.29}$$

This is the natural generalization of the particle action (1.1.2) to an extended object; it measures the volume of the $(p+1)$-dimensional object Dp in spacetime, relative to the background metric g_{MN}. In the flat background metric $g_{MN} = \eta_{MN}$, a subspace that is itself flat extremizes (1.3.29); a curved Dp will tend to relax to such a flat subspace, or shrink to a point. Without loss of generality, we can locally take the first $p+1$ coordinates of spacetime x^M to coincide with the σ^a; then the remaining $9-p$ can be identified with the transverse scalars x^i in (1.3.22). If we choose the latter so that $g_{aj} = 0$, then

$$(g|_{\mathrm{D}p})_{ab} = g_{ab} + g_{ij}\partial_a x^i \partial_b x^j\,. \tag{1.3.30}$$

For small x^i, (1.3.29) contains then the usual kinetic term, and terms with more than two derivatives.

The full first term $\sqrt{-\det(g|_{\mathrm{D}p} + \mathcal{F})}$ is called *(Dirac–)Born–Infeld (DBI)* because it is reminiscent of early proposals to improve the short-distance behavior of the electromagnetic field, inspired by the resemblance to the relativistic contraction factor $\gamma^{-1} = \sqrt{1-v^2}$. While this combination might seem odd, we will see that it has many natural properties (including a duality with Pythagoras' theorem in Section 1.4.2). We can expand it for small open string fields. Recall that $\log\det M = \mathrm{Tr}\log(M)$; then for a matrix $M \sim 1 + m$ near the identity, $m \ll 1$:

$$\begin{aligned}\det(1+m) &= \exp\left[\mathrm{Tr}\log(1+m)\right] = \exp\left[\mathrm{Tr}\, m - \frac{1}{2}\mathrm{Tr}(m^2) + \ldots\right] \\ &= 1 + \mathrm{Tr}\, m + \frac{1}{2}\left((\mathrm{Tr}\, m)^2 - \mathrm{Tr}(m^2)\right) + \ldots\,.\end{aligned} \tag{1.3.31}$$

Using this and (1.3.30), for $B = 0$:

$$\sqrt{-\det(g|_{\mathrm{D}p} + \mathcal{F})} \sim \sqrt{-\det_{ab} g_{ab}}\left(1 + \frac{1}{2}g_{ij}\partial_a x^i \partial^a x^j + \frac{1}{2}|2\pi l_s^2 f|^2\right) + \ldots\,, \tag{1.3.32}$$

where we kept only terms with two derivatives, and indices are raised and contracted with g_{ab}. Taking into account the prefactors in (1.3.24), the YM coupling is given by

$$g_{\mathrm{YM}}^2 = (2\pi)^{p-2} l_s^{p-3} g_s\,. \tag{1.3.33}$$

In particular, it is dimensionless for $p = 3$, as expected.

The Wess–Zumino term

We now turn to a discussion of the term involving C_{p+1-2k} in (1.3.24). This is variously called *Chern–Simons* or *Wess–Zumino* (WZ) term, inspired by the names of two famous actions that don't involve the metric. When $\mathcal{F} = 0$, the integral on Dp is defined as in (1.2.8), after the pull-back operation described following (1.3.24). Explicitly,

$$\int_{\mathrm{D}p} C_{p+1} = \int C_{M_0 \ldots M_p} \partial_0 x^{M_0} \ldots \partial_p x^{M_p} \,. \tag{1.3.34}$$

For $k = 1$ this has the form (1.1.3); for $k = 2$, it looks like the coupling (1.1.10) of the fundamental string to B. So the coefficient of (1.3.34) is a charge density; we see from (1.3.24) that it equals the brane tension $T_{\mathrm{D}p}$.

Recall from Section 1.1.2 that the C_{p+1} exist with p = even in IIA, and p = odd in IIB; so there are

$$\mathrm{D}p\text{-branes:} \quad p = \begin{cases} \text{even} & \text{IIA}\,, \\ \text{odd} & \text{IIB}\,. \end{cases} \tag{1.3.35}$$

In IIA, the smallest object is a point-like soliton in IIA, the D0-brane. In IIB we also have the possibility of a D(-1)-brane or *D-instanton*, with a D boundary condition for the time coordinate as well, so that it is localized also in time. In the context of compactifications, there are additional brane instanton that wrap the internal directions and are completely localized in the noncompact directions.

A subtlety mentioned following (1.3.24) is that the overall sign of (1.3.34) can change if we embed the world-volume differently: if we flip the sign of one of the σ^a,

$$x^M(\sigma^0, \sigma^1, \ldots, \sigma^p) \to x^M(\sigma^0, -\sigma^1, \ldots, \sigma^p)\,, \tag{1.3.36}$$

the image of the embedding remains the same, but (1.3.34) changes sign, because $\partial_{\sigma^1} x^M$ appears only a single time in it. A similar sign change happens upon exchanging two coordinates. Since the charge density changes sign, we call this an *anti-Dp-brane*. The difference between a brane and antibrane is conventional.

Effect of world-volume flux

When $\mathcal{F} \neq 0$, a Dp-brane also couples to RR fields C_k with $k < p$. For example, the WZ term for a D2-brane becomes

$$\int_{\mathrm{D}2} (C_3 - \mathcal{F} \wedge C_1)\,. \tag{1.3.37}$$

In (1.3.34), C_1 would couple to a D0; so we interpret the second term in (1.3.37) as the presence of a distribution of D0-branes on the D2. In other words, we have a D2 that also has D0 charge, or a *D2/D0 bound state*.

A final subtlety about the WZ term regards the Romans mass F_0 [61, 62]. Since this flux has no potential, it might seem that no brane couples to it. In fact, for $F_0 \neq 0$ the WZ term in (1.3.24) needs to be modified to

$$\int_{Dp} \mathrm{wz}_{p+1}\,, \qquad \mathrm{dwz}_{p+1} = \sum \frac{1}{k!}(-1)^k F_{p+2-2k} \wedge \mathcal{F}^k\,. \tag{1.3.38}$$

For $F_0 = 0$, $\mathrm{wz}_{p+1} = \sum_k \frac{1}{k!}(-1)^k C_{p+1-2k} \wedge \mathcal{F}^k$ as in (1.3.24); for $F_0 \neq 0$, we can use $\sum_k F_{p+2-2k} \mathrm{cs}_k$, where cs_k generalizes the Chern–Simons form (1.2.20). For example, for $p = 0$, taking (1.3.24) literally would give a term $\int C_1 - C_{-1} \wedge \mathcal{F}$; the correct coupling can be obtained by formally integrating the second term by part:

$$\int_{\mathrm{D}0} (C_1 - F_0 a)\,. \tag{1.3.39}$$

Magnetic dual potentials

In (1.3.34), RR potentials C_p appear for any p, while in Section 1.2 we only saw $p \le 4$. We need to extend the definition to higher p by introducing the *magnetic dual* potentials, defined through

$$\mathrm{d}C_p - H \wedge C_{p-2} \equiv F_{p+1} \equiv (-1)^{p(p-1)/2} * F_{9-p} \quad (p > 4) \tag{1.3.40}$$

in the notation of (1.2.34b). These new C_p, $p \ge 5$ are the higher-dimensional analogue of the magnetic dual gauge fields we mentioned for the SW solution that follows (1.3.12). For example, a D6-brane couples to C_7, which when $H = 0$ is defined by $\mathrm{d}C_7 = F_8$, or more explicitly

$$F_{M_1 \ldots M_8} = -\sqrt{-g}\,\epsilon^{(0)}_{M_1 \ldots M_8}{}^{M_9 M_{10}} \partial_{M_9} C_{M_{10}}\,. \tag{1.3.41}$$

In Exercise 1.2.1, it was noted that the RR equations of motion look like an extension of the Bianchi identities in terms of the magnetic duals. This allows to rewrite the action by swapping one or more of the original potentials with their magnetic duals. In presence of a Dp with $p \ge 4$, it is convenient to use this reformulation to vary with respect to C_{p+1}. The combined action, (1.3.24) plus the supergravity action in (1.2.31) or (1.2.36), can be rewritten as an integral over ten dimensions by using delta functions. In the language of forms, recalling our notation x^i for the directions transverse to the Dp, we can introduce a form

$$\delta_{\mathrm{D}p} \equiv \delta(x^1) \ldots \delta(x^{9-p}) \mathrm{d}x^1 \wedge \ldots \mathrm{d}x^{9-p}\,. \tag{1.3.42}$$

After the variation, we then obtain (for $\mathcal{F} = 0$)

$$\mathrm{d}F_p - H \wedge F_{p-2} = \mp 2\kappa^2 \tau_{\mathrm{D}(8-p)} \wedge \delta_{\mathrm{D}(8-p)}\,; \tag{1.3.43}$$

the source term on the right-hand side is the effect of D-branes on the RR field strengths. A similar source term appears in the equations of motion for the other closed string fields.

1.3.4 Flux quantization

The coupling (1.3.34) is the natural generalization of (1.1.3) to extended objects. In this sense, a Dp-brane is the elementary charge for the RR field C_{p+1}, much as the electron for the electric field. The generalization of the large gauge transformation (1.2.13) is

$$C_{p+1} \to C_{p+1} + \Lambda_{p+1}\,, \qquad \frac{\tau_{\mathrm{D}p}}{2\pi} \int_{S_{p+1}} \Lambda_{p+1} \in \mathbb{Z} \tag{1.3.44}$$

for any subspace S_{p+1}. This includes our old "small" RR gauge transformations (1.2.11): $\Lambda_{p+1} = \mathrm{d}\lambda_p$ is a total derivative, and the integral (1.3.44) is zero.

The non-single-valuedness of a gauge transformation λ_0 was used in the Dirac quantization argument (1.3.7). This suggests a generalization of that result to RR fields. Surround a Dp-brane with a sphere S^{8-p}, consider an "equator" E on it (itself a sphere S^{7-p}), and the two semispheres $U_{\rm N}$, $U_{\rm S}$ into which it divides S^{8-p}. The integral $\tau_{{\rm D}(6-p)} \int_{U_{\rm N}} ({\rm d}C)_{8-p}$ can be rewritten using an analogue of Stokes's theorem (which we will formalize in Section 4.1.10) as

$$\tau_{{\rm D}(6-p)} \int_{\rm E} C_{7-p} \,. \tag{1.3.45}$$

Repeating this over $U_{\rm S}$ yields a competing result for (1.3.45); the two results in general disagree, their difference being exactly the integral of F_{8-p} over all of S^{8-p}. But there is no inconsistency if we use two different C_{p+1} on $U_{\rm N}$ and $U_{\rm S}$, differing by a large gauge transformation (1.3.44). This leads to the condition

$$\tau_{{\rm D}(6-p)} \int_{S^k} ({\rm d}C)_{8-p} = 2\pi n \,, \tag{1.3.46}$$

where $n \in \mathbb{Z}$. Recalling (1.3.25),

$$\frac{1}{(2\pi l_s)^{k-1}} \int_{S^k} ({\rm d}C)_k \in \mathbb{Z} \,. \tag{1.3.47}$$

Although for concreteness we presented our argument with spheres, (1.3.47) holds for any k-dimensional subspace S_k. Often such a subspace can be continuously deformed to a point, and in that case the integral in (1.3.47) just vanishes. The integral can be nonzero in two types of situations:

- If S_k surrounds a brane, we cannot continuously deform it to zero without crossing it. The integral $\int F_{8-p}$ can be taken as the definition of the charge of a Dp; one could use this to rederive (1.3.25). In this case, flux quantization can also be derived by suitably integrating (1.3.43).
- For compactifications, some S_k in the internal space cannot be continuously deformed to a point for *topological* reasons (Section 4.1.10). In this case, (1.3.47) can be nonzero even if there are no D-branes.

The logic behind (1.3.47) can also be applied to the magnetic dual potentials (1.3.40); one concludes that (1.3.47) is valid for all $k \leq 10$, and not just for the F_k with $k \leq 5$ that appear in the supergravity actions. Notice that the $*$ in (1.3.40) contains the metric, as we see explicitly in (1.3.41). This seems to create a paradox: if we impose flux quantization both for F_3 and for its dual $*F_3$ (say), it seems we are imposing some quantization conditions on the metric itself. In practice, however, there are always some noncompact directions in spacetime, and only one of the two flux integrals makes sense. For example, for $M_{10} = \mathbb{R}^4 \times M_6$, if we impose flux quantization for F_3, the one for $*F_3$ would involve an integral over $\mathbb{R}^4$, which would diverge.

The case $k = 0$ in (1.3.47) deserves a separate discussion. There is no potential C_{-1}, and $S^0 = \{x_1^2 = 1\}$ is just the union of two points. But integrating (1.3.43) for $p = 0$ on a segment crossing a D8, we still obtain

$$2\pi l_s \Delta F_0 = N_{\rm D8} \in \mathbb{Z} \,. \tag{1.3.48}$$

Strictly speaking, in this case we only quantized the jumps of F_0 rather than F_0 itself. It would now be possible to entertain the notion that $2\pi l_s F_0 = n_0 + \theta$, with $n_0 \in \mathbb{Z}$ and $\theta \in [0, 1)$, fixed for any background. This looks unlikely; we will see that T-duality relates the RR fluxes to one another, and if $\theta \neq 0$ were allowed it would eventually violate one of the (1.3.47) for $k > 0$. So we will take

$$2\pi l_s F_0 \in \mathbb{Z} \,. \tag{1.3.49}$$

Our discussion can also be easily applied to B, since its coupling to the fundamental string is identical to the one of C_1 to the D1. Looking at (1.3.44) and (1.3.47), we conclude

$$B \to B + \tilde{\Lambda}_2 \,, \qquad \frac{1}{(2\pi l_s)^2} \int_{S^2} \tilde{\Lambda}_2 \in \mathbb{Z} \,; \tag{1.3.50a}$$

$$\frac{1}{(2\pi l_s)^2} \int H \in \mathbb{Z} \,. \tag{1.3.50b}$$

1.3.5 Supersymmetry

D-branes break some of the bosonic symmetries of a background; perhaps more importantly, D-branes also partially break some of its supersymmetry. Recall that both type II string theories are invariant under 32 supercharges, whose infinitesimal parameters are two spacetime fermions ϵ^a. A Dp extended along a flat $\mathbb{R}^{p+1} \subset \mathbb{R}^{10}$ is invariant only if the two ϵ^a are related by

$$\epsilon^1 = \Gamma_{\|} \epsilon^2 \,, \tag{1.3.51}$$

where

$$\Gamma_{\|} = \frac{1}{p! \sqrt{-\det(g|_{\mathrm{D}p})}} \epsilon_{(0)}^{a_0 \ldots a_p} \Gamma_{a_0 \ldots a_p} \,, \qquad \Gamma_a \equiv \Gamma_M \partial_a x^M \qquad (\mathcal{F} = 0) \,. \tag{1.3.52}$$

The flat space background is invariant under all 32 supercharges. If we have a Dp-brane extended along a $\mathbb{R}^{p+1} \subset \mathbb{R}^{10}$ subspace, without loss of generality we can take it to be $\{x^{p+1} = \ldots = x^9 = 0\}$; (1.3.52) then becomes

$$\Gamma_{\|} = \Gamma_0 \ldots \Gamma_p \,. \tag{1.3.53}$$

Equation (1.3.51) gives a relation among the two ϵ^a: only ϵ^1 is now independent, and thus we are left with 16 supercharges. This preserved supersymmetry makes these flat D-branes the analogue of the field-theory BPS states in Section 1.3.1. In particular, they are stable against time evolution and deformations of the theory.[13] This is the reason they are useful: if we make the string coupling g_s large, we lose perturbative control, but we know that D-branes have to remain in the spectrum. Since their tension (1.3.28) is the inverse of the string coupling, they become light at strong coupling and might become new fundamental objects, similar to the SW solution of $\mathcal{N} = 2$-supersymmetric Yang–Mills (Section 1.3.1).

We saw in Section 1.1.2 that the ϵ^a have opposite chiralities in IIA, and equal in IIB, and that multiplication by a single Γ^M changes chirality. It follows that

[13] A D-brane with a more general shape is not guaranteed to preserve supersymmetry, and hence to be stable under time evolution. In Section 9.2, we will consider this further, and also generalize (1.3.52) to $\mathcal{F} \neq 0$.

(1.3.51) can only be solved for p = even in IIA, and for p = odd in IIB; this confirms (1.3.35).[14]

The origin of the constraint (1.3.51) is roughly the following. The fermionic completion of the effective action (1.3.24) includes two world-sheet fermions θ^a. Closed string supersymmetry acts on these as a spinorial translation:

$$\delta\theta^a = \epsilon^a \,. \tag{1.3.54}$$

Imposing supersymmetry would then set both $\epsilon^a = 0$. But there is also a gauge equivalence among the θ^a, called *κ-symmetry*, acting as

$$\delta\theta^1 = \Gamma_{\|}\kappa \,, \qquad \delta\theta^2 = \kappa \,. \tag{1.3.55}$$

Imposing invariance under a combined supersymmetry and κ-symmetry gives (1.3.51).

1.3.6 Multiple D-branes

D-brane stacks and nonabelian gauge groups

To each endpoint of an open string, it is possible to add an extra discrete quantum number $I = 1, \ldots, N$, called the *Chan–Paton (CP) label*. It can be interpreted as the presence of $N > 1$ superimposed D-branes, or in other words a stack of N D-branes.

Let us see why. Since open strings have two endpoints, the states (1.3.21) and the fields (1.3.22) acquire two extra labels IJ, and so they are now promoted to Hermitian matrices. Now a becomes a nonabelian $\mathrm{U}(N)$ gauge field, and x a scalar in the adjoint representation. In the action (1.3.24), we should turn all fields into matrices with an overall trace, but there are potential ordering ambiguities.

The situation is much clearer for the two-derivative approximation (1.3.32), whose nonabelian extension is dictated by supersymmetry. We know from (1.3.51) that a brane extended along a flat $\mathbb{R}^{p+1} \subset \mathbb{R}^{10}$ subspace preserves 16 supercharges; we expect its effective action to have this invariance. This is the maximum number of supercharges for a QFT model (not including gravity), and models with this property are quite constrained. For example, for $p = 9$, when we have a D9 extended along all of spacetime, there are no scalars x^i, and in (1.3.32) only the $|f|^2$ term remains. Its nonabelian version is the YM Lagrangian density $\mathrm{Tr}|f|^2$; supersymmetry involves also the gaugino λ^α of (1.3.22), and the requirement of 16 supercharges fixes the action uniquely.

For lower p, some components of the gauge field now become x^i, because of their common origin in (1.3.57). The resulting supersymmetric YM theory with 16 supercharges is again uniquely fixed: it is the dimensional reduction of the action for $p = 9$ along the directions to the Dp. The components of the gauge field along the transverse directions become the transverse scalars x_i; the world-volume field strengths then become

$$F_{ai} = D_a x_i = \partial_a x_i + [A_a, x_i] \,, \qquad F_{ij} = [x_i, x_j] \,. \tag{1.3.56}$$

[14] There also exist non-BPS branes that violate (1.3.35) and are of course not described by the effective action (1.3.24). These branes are unstable, but they become stable after some quotients, including the orientifolds we will introduce in Section 1.4.4 [63].

In particular, the YM term generates a potential for the scalars

$$V \propto \mathrm{Tr}\left([x_i, x_j][x_i, x_j]\right), \tag{1.3.57}$$

which vanishes in the Abelian case. The x_i are diagonalizable because Hermitian; the vacua of (1.3.57) are given by configurations where $[x_i, x_j] = 0$, which means that the x_i are simultaneously diagonalizable. In the generic vacuum where they are all different, the BEH mechanism gives a mass to all fields except the diagonal ones; the action of these massless modes is again the Abelian (1.3.32). The D-brane positions are the eigenvalues λ^I_i. This picture motivates the proposed interpretation of the CP label as N superimposed D-branes.

An additional check of this conclusion is that the off-diagonal modes receive a mass that is proportional to the $\sum_i (\lambda^I - \lambda^J)^2_i$; these can be interpreted as the lengths of the strings going from the Ith to the Jth D-brane. Another is supersymmetry: the condition for (1.3.51) is the same for parallel Dp, since $\partial_a x^M$ is insensitive to translations $x^M \to x^M + x^M_0$. So having parallel Dp still preserves 16 supercharges, which is what we assumed for the nonabelian super-YM theory in presence of the CP label.

Brane–antibrane system

Recall from (1.3.36) that parity in one of the embedding coordinates changes the WZ coupling (1.3.34) by a sign; we called this an anti-Dp. Now in the condition for supersymmetry (1.3.51) the matrix $\Gamma_\parallel$ changes sign, too. If we have a Dp and an anti-Dp together, we have to solve

$$\epsilon^1 = \Gamma_\parallel \epsilon^2 \,, \qquad \epsilon^1 = -\Gamma_\parallel \epsilon^2, \tag{1.3.58}$$

which is impossible. So a brane–antibrane system breaks supersymmetry.

Moreover, an analysis of the open string modes reveals the presence of a tachyon. It has been shown [64, 65] that in fact this tachyon has a nontrivial potential, with a stable vacuum, which represents the closed-string vacuum without any branes; in other words, the tachyon can condense (get a nonzero expectation value) and become stable. The spacetime interpretation is that the Dp–anti-Dp pair annihilates, much as a particle–antiparticle pair. The story becomes even more interesting when one of the two has a nontrivial flux of the world-volume field-strength f; in this case, after the annihilation we are left with a lower-dimensional brane instead of the vacuum. This motivates the *K-theory* interpretation of D-branes [66], also suggested by the higher-derivative corrections to (1.3.24) [67]. (This will reappear in Section 9.2.3.)

1.3.7 Gravity solutions

We anticipated at the beginning of this section that a D-brane back-reacts on closed string fields, distorting the flat space metric to a so-called p-brane solution. These can be found by solving the closed string equations of motion with the symmetries we expect the brane to preserve, namely $\mathrm{ISO}(p+1)\times\mathrm{SO}(d-p)$ (parallel Poincaré and transverse rotations), and 16 supercharges. Here we give the result; we will check in later chapters that they are supersymmetric solutions, in various ways.

The solutions for N superimposed Dp-branes, extended along a flat $\{x^{p+1} = \ldots = x^9 = 0\} = \mathbb{R}^{p+1} \subset \mathbb{R}^{10}$, is

$$ds^2_{\mathrm{D}p} = h^{-1/2} ds^2_{\parallel} + h^{1/2} ds^2_{\perp}\,, \qquad e^{\phi} = g_s h^{\frac{3-p}{4}}\,, \tag{1.3.59a}$$

$$F_{i_1 \ldots i_{8-p}} = \frac{f_{8-p}}{r^{9-p}} x^i \epsilon^{\perp}_{i i_1 \ldots i_{8-p}}\,, \qquad f_{8-p} \equiv \frac{(2\pi l_s)^{7-p} N}{v_{8-p}}\,. \tag{1.3.59b}$$

$ds^2_{\parallel} = -(dx^0)^2 + \sum_{a=1}^{p} (dx^a)^2$, and $ds^2_{\perp} = \sum_{i=p+1}^{9} (dx^i)^2$ are the metrics respectively of the $1+p$ parallel and $9-p$ transverse dimensions prior to introducing the D-branes; recall that a and i are our names in this section for the parallel and transverse indices respectively. $\epsilon^{\perp}$ is the completely antisymmetric tensor in the transverse directions (such that $\epsilon^{\perp}_{p+1\ldots 9} = 1$); $v_d \equiv \mathrm{Vol}(S^d) = 2\frac{\pi^{(d+1)/2}}{\Gamma((d+1)/2)}$ is the volume of the unit-radius sphere S^d; in form notation, as we will see in Exercise 4.1.9, $F = f_p \mathrm{vol}_{S^d}$, the volume form of S^d.

h is a function of the radial direction $r = (\sum_{i=p+1}^{9} (x^i)^2)^{1/2}$, which satisfies

$$\sum_{i=p+1}^{9} \partial_i^2 h = 0 \tag{1.3.60}$$

away from $r = 0$. For $p < 7$, we can take

$$h = 1 + N \frac{r_0^{7-p}}{r^{7-p}}\,, \qquad r_0^{7-p} \equiv \frac{g_s (2\pi l_s)^{7-p}}{(7-p) v_{8-p}}\,. \tag{1.3.61}$$

The fields not mentioned in (1.3.59) are zero, and in particular so is $H = dB$.

For the sake of a unified description, in (1.3.59) we have given the transverse RR field-strength F_{8-p}. For $p \geq 4$, this is one of the RR forms that appear in the type II supergravity actions of Section 1.2. For $p \leq 2$, this is one of the magnetic duals defined in (1.3.40); using that definition backward on (1.3.59) gives the original RR form F_{p+2}, which has components $F_{a_0 \ldots a_p r}$. For $p = 3$, the flux F_5 is self-dual, and both the transverse and parallel directions appear.

The $x^i \epsilon^{\perp}_{i i_1 \ldots i_{8-p}}$ in (1.3.59b) can be interpreted as the angular part of the transverse directions, as we will see better in Section 4.1.3; its integral is simply v_{8-p}, and the prefactor is fixed so that (1.3.46) and (1.3.47) holds with $n = 1$. The equations of motion then determine the coefficient of r^{7-p} in (1.3.61), but only up to a sign; the correct choice can be decided by examining the long-distance behavior of the gravitational field, which should be attractive given that we want positive tension as in (1.3.28).

Generalizations

The solutions (1.3.59) are part of a more general family, where there are two horizons; (1.3.59) is the extremal limit where the two horizons have coalesced, and have been set to $r = 0$ by a coordinate change.[15] This extremality is related to the equality of the tension and the charge density of a D-brane, or in other words the coefficients of the DBI and WZ terms in (1.3.24). It is quite common for the extremal limit to saturate the BPS bound, and vice versa.

[15] See, for example, [68, sec. 1] or [69, chap. 19] for more details; the change of variables is similar to the one we will see in Section 11.1 for charged black holes.

Another generalization, which will be more important for us, consists in considering parallel branes. This can be achieved by solving (1.3.60) with a harmonic function, which has several point-like sources in the transverse space rather than just one. For $p < 7$, (1.3.61) is replaced by

$$h = 1 + r_0^{7-p} \sum_{\alpha=1} \frac{1}{|x - x_I|^{7-p}}\,, \tag{1.3.62}$$

where x_I^i is the position of the Ith D-brane in transverse space. The fact that this solution is static signals that *parallel Dp-branes don't exert any force on each other*. We could also have expected this from the analysis in Section 1.3.6, where we concluded that parallel Dp preserve the same 16 supercharges as a single one.

Asymptotics and singularities

Returning to a single stack (1.3.61), at large r we would expect the solution to be asymptotic to flat space. The radius r_0 is interpreted as the region where the gravitation field is strong. For $g_s \to 0$, it shrinks to zero; this might seem strange, since the D-brane's tension (1.3.28) gets large in this limit. The reason is the prefactor $\mathrm{e}^{-2\phi}$ in the NSNS sector action (1.2.17), so that the effective Newton constant is $2\kappa^2 g_s^2$; this overcomes the g_s^{-1} in the D-brane tension, so that their product goes like g_s, which is the power observed in r_0^{7-p}. (Indeed, $(2\pi l_s)^{7-p} = 2\kappa^2 \tau_{\mathrm{D}p}$.)

In fact, as we will now see, $r = 0$ is in most cases a singularity; either curvature, or dilaton, diverge there. As we mentioned, this singularity is expected to be resolved in fully fledged string theory. For example, in Section 7.2.3 we are going to see how this happens for the D6-brane solution.

The behavior of this solution as $r \to 0$ depends on p:

- If $p < 3$, the curvature invariants remain finite as $r \to 0$; in particular, the Ricci scalar $R \to 0$. But the string coupling diverges. $r = 0$ is at infinite distance.
- If $p = 3$, again the curvature invariants are finite, and moreover ϕ is constant. In fact, an analytic continuation beyond $r = 0$ exists [70], but once again $r = 0$ is at infinite distance.
- If $3 < p < 7$, the curvature diverges as $r \to 0$, but the string coupling $\mathrm{e}^{\phi} \to 0$.
- For $p = 7$, there are two transverse directions, and a function satisfying (1.3.60) is a logarithm:

  $$h = -\frac{g_s N}{2\pi} \log(r/r_0)\,. \tag{1.3.63}$$

  $r = 0$ is again a singularity. The metric is no longer asymptotic to flat space at large distance: for $r > r_0$ the function h becomes negative, and the solution loses meaning. In practice, this is seldom a problem, because in the context of compactifications the transverse directions to a D7 are compact anyway.
- For $p = 8$, there is only one transverse direction x^9, and a function satisfying (1.3.60) is piecewise-linear:

  $$h = h_0 + \frac{g_s N}{2\pi l_s} |x_9|\,. \tag{1.3.64}$$

  Again, the solution is no longer asymptotic to flat space, and has a critical distance, dependent on the integration constant h_0, which is related to the value of the dilaton at $x_9 = 0$.

NS5-branes

Similar solutions to (1.3.59) exist that are supersymmetric and charged under the Kalb–Ramond field B.

One represents the back-reaction of a stack of superimposed fundamental strings F1; we will see it later in an exercise. Another describes a new object, extended along five space dimensions and hence called *NS5-brane*. The solution is rather similar to (1.3.59) for $p = 5$:

$$\mathrm{d}s^2_{\mathrm{NS5}} = \mathrm{d}s^2_{\parallel} + h\mathrm{d}s^2_{\perp}\,, \qquad e^{\phi} = g_s h^{1/2}\,, \tag{1.3.65a}$$

$$H_{ijk} = 2Nx^l \epsilon^{\perp}_{lijk}\,. \tag{1.3.65b}$$

$\mathrm{d}s^2_{\parallel}$, $\mathrm{d}s^2_{\perp}$ describe the six parallel and four transverse directions; $h = 1 + Nl_s^2/r^2$. (In form notation, (1.3.65b) reads $H = 2N\mathrm{vol}_{S^3}$.) A definition of this solitonic object from open strings is not available, but we will see later that it plays an important role in string dualities, which give indirect information. One can infer from this an effective action in the style of (1.3.24). In particular, one obtains that it couples to the dual potential B_6, defined similar to (1.3.40) as

$$* H = -\mathrm{d}B_6\,. \tag{1.3.66}$$

The tension is

$$T_{\mathrm{NS5}} \equiv \frac{\tau_{\mathrm{NS5}}}{g_s^2} \equiv \frac{1}{(2\pi)^5 l_s^6 g_s^2}\,, \tag{1.3.67}$$

which differs from $T_{\mathrm{D5}} = 1/((2\pi)^5 l_s^6 g_s)$ from (1.3.28) in the power of g_s, signaling that the NS5 is not a D-brane. Its effect on the Bianchi identity is

$$\mathrm{d}H = -2\kappa^2 \tau_{\mathrm{NS5}} \delta_{\mathrm{NS5}}\,, \tag{1.3.68}$$

in the spirit of (1.3.43).

Exercise 1.3.1 In flat ten-dimensional space $\mathbb{R}^{10}$, consider a D-brane on $S^p \times \mathbb{R}$, where $S^p = \{\sum_{m=1}^{p}(x^m)^2 = R^2\}$. Does it satisfy (1.3.51)?

Exercise 1.3.2 Obtain (1.3.47) from (1.3.43).

Exercise 1.3.3 Show that the metric of the $D8$ solution can be put in the *conformally flat* form

$$\mathrm{d}s^2_{10} = f\mathrm{d}s^2_{\mathrm{Mink}_{10}}\,, \tag{1.3.69}$$

where f is a function and Mink_{10} denotes as usual *flat* Minkowski space.

Exercise 1.3.4 Check the sign of the coefficient of r^{7-p} in (1.3.61) by considering the potential of a p-brane with DBI action (1.3.29) in the solution (1.3.59): the force should be attractive. Next, consider a Dp-brane probe, with its full action (1.3.24); check that the total force in this case is zero.

1.4 Dualities

We have learned that in flat space D-branes extended along flat subspaces are BPS solitons, the analogue of monopoles in supersymmetric YM theories; in this section, we will use them to find information on the strong-coupling behavior of string theory.