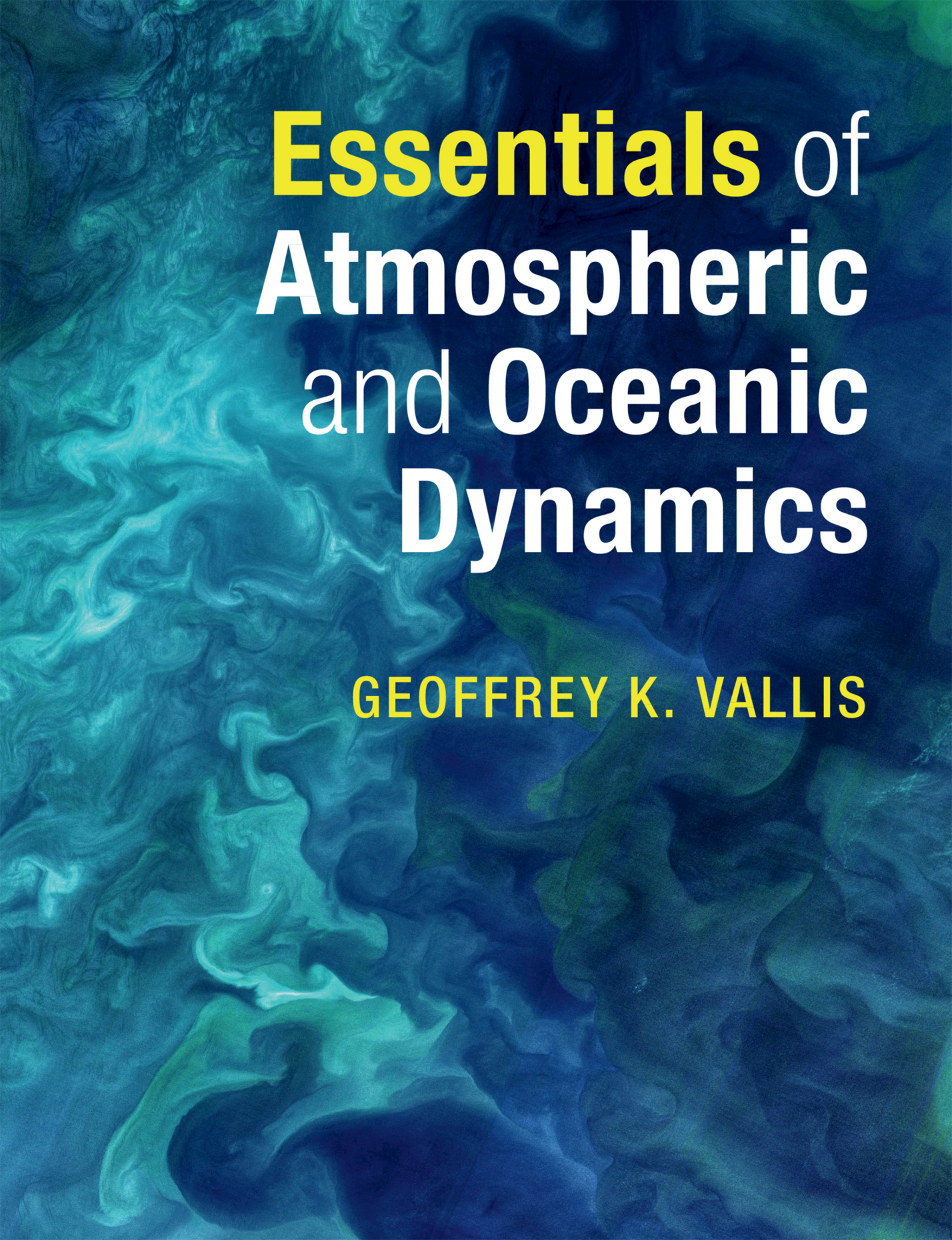




Essentials of **Atmospheric and Oceanic Dynamics**

GEOFFREY K. VALLIS

Essentials of Atmospheric and Oceanic Dynamics

This is a modern, introductory textbook on the dynamics of the atmosphere and ocean, with a healthy dose of geophysical fluid dynamics. It will be invaluable for intermediate to advanced undergraduate and graduate students in meteorology, oceanography, mathematics and physics. It is unique in taking the reader from very basic concepts to the forefront of research. It also forms an excellent refresher for researchers in atmospheric science and oceanography. It differs from other books at this level in both style and content: as well as very basic material, it includes some elementary introductions to more advanced topics. The advanced sections can easily be omitted for a more introductory course, as they are clearly marked in the text. Readers who wish to explore these topics in more detail can refer to this book's parent, *Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-Scale Circulation*, now in its second edition.

Geoffrey K. Vallis is a professor of applied mathematics at the University of Exeter, UK. Prior to taking up his position there, he taught for many years at Princeton University in the USA. He has carried out research in the atmospheric sciences, oceanography and the planetary sciences, and has published over 100 peer-reviewed journal articles. He is the recipient of various prizes and awards, including the Adrian Gill Prize (Royal Meteorological Society) and the Stanislaw M. Ulam Distinguished Scholar Award (Los Alamos National Laboratory). He is the author of *Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-Scale Circulation, Second Edition* (2017, Cambridge University Press).

‘Vallis’ insights into the fundamentals and applications go a long way towards making otherwise complex topics readily grasped by those willing to study. He does not shy away from mathematics where needed, nor does he smother the reader with mathematics where pedagogically unnecessary. Those making it through this book will be ready to tackle a huge suite of research questions related to atmosphere and ocean fluid mechanics. Hence, this book serves an incredibly important role to the academic community. In a nutshell, we need more smart researchers who are adept at atmosphere and ocean dynamics to help understand how those dynamics are increasingly being affected by humanity’s choices.

Essentials of Atmospheric and Oceanic Dynamics (EAOD) fills an important niche by offering an articulate and authoritative textbook to be worked through by advanced undergraduates and/or entering graduate students taking courses. The inclusion of exercises in *EAOD* is incredibly valuable for both students and teachers clamouring for more problem sets to test understanding. Whereas Vallis’ previous book *Atmospheric and Oceanic Fluid Dynamics (AOFD)* is the mother reference, *EAOD* offers a pedagogical entrée for those wishing to test the waters, including some deep waters. I will happily keep both books on my shelf and make use of them for personal study and to support the teaching of geophysical fluid dynamics.

Vallis has a clear writing style that brings the reader into the subject in an authoritative and friendly manner. He is a wise guru and gentle tutor. The subject of ocean and atmosphere fluid mechanics has matured greatly through his efforts at writing *AOFD*. *EAOD* furthers that maturation by allowing for a broader readership to tap into his brain. Well done Geoff!

- Stephen M. Griffies,
Geophysical Fluid Dynamics Laboratory,
National Oceanic and Atmospheric Administration

‘The “big book” [*AOFD*] by Vallis is a treasure, but I suspect that this new *Essentials* is destined to be used much more widely in classrooms. Vallis does a superb job of communicating the peculiar tensions between deductive reasoning and physical intuition that underlie this science. The new book is more approachable but no less rigorous. I especially appreciate how the various equation sets are derived in succinct but meaningful ways in the first few chapters, and then used as tools to explore the dynamics in the chapters that follow. It’s almost the perfect introductory textbook on this subject, and I plan to use it in my own courses.’

- Brian E. J. Rose,
University at Albany

‘He’s done it again. In *Essentials*, Geoff Vallis has produced a text that is useful to the student and the experienced scientist alike. While the content is simplified and shortened compared to its parent text, Vallis now provides even more descriptive explanations to support readers in their quest to navigate the physics of fluid flows. These explanations pair well with the theory, serving as an accessible introduction to students while also supporting the more experienced scientist as they put all of the pieces together. This will certainly be a future favourite for reading groups. Even readers with dog-eared versions of the parent book will want a copy of *Essentials*, for in it Vallis has added an entirely new chapter on planetary atmospheres, allowing the interested reader to venture into outer space to apply their newly honed GFD expertise.’

- Elizabeth A. Barnes,
Colorado State University

‘For the past decade, Geoff Vallis’ book *Atmospheric and Oceanic Fluid Dynamics* has been the “go to” encyclopaedic resource, but it is too lengthy and comprehensive to use as a course textbook. With this superb new shorter volume, Geoff Vallis provides us with the definitive graduate-level textbook, with just the right balance of essential topics alongside glimpses of more advanced topics at the cutting edge of research. The extensive use of margin notes, diamonds to indicate advanced topics, and a comprehensive set of problems will ensure that *Essentials of Atmospheric and Oceanic Dynamics* has much to offer students and researchers at all levels. The book opens with the quote: “Seek simplicity, accept complexity. Exploit simplification, avoid complication.” On all counts, this book succeeds magnificently!’

- David Marshall,
University of Oxford

‘As its parent book became the bible of the field, but also grew in size and the number of topics it covered in its latest edition, this new book provides a perfect balance and introduction to the essential topics, giving a quick reference without going into all the details. In the Vallis tradition, it is presented clearly, perfectly packaged, and is well organized for both atmospheric and oceanic fluid dynamics. Its simplicity will make it majestically appealing both for people outside the discipline looking for an accessible, yet complete, introduction, and for students within the field at all levels. The inclusion of planetary atmospheres broadens the scope and makes it appealing to a wider and growing audience. Anyone with a background in physics can get the essentials using this book.’

- Yohai Kaspi,
Weizmann Institute of Science

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GEOFFREY K. VALLIS

University of Exeter



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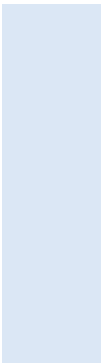
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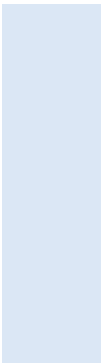
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Note: In the text itself more advanced sections are marked with a diamond, ♦, and may be omitted on a first reading. If a section is so marked then the marking applies to all the subsections within it.



Preface

*Seek simplicity, accept complexity.
Exploit simplification, avoid complication.*

This is an introductory book on the dynamics of atmospheres and oceans, with a healthy dose of geophysical fluid dynamics. It is written roughly at the level of advanced or upper-division undergraduates and beginning graduate students, but parts of it will be accessible to first- or second-year undergraduates and I hope that practising scientists will also find it useful. The book is designed for students and scientists who want an introduction to the subject but who may not want all the detail, at least not yet, and its prerequisites are just familiarity with some vector calculus and basic classical physics. Thus, it is meant to be accessible to non-specialists and students who will not necessarily go on to become professional dynamists. However, as well as very basic material the book does include some elementary introductions to a few ‘advanced’ topics, such as the residual circulation and turbulence theory, as well as material on the general circulation of the atmosphere and ocean. The more advanced parts could easily be omitted for a first course and, like difficult ski slopes, are marked with a diamond, ♦. Readers may explore these topics more in the references provided, or in this book’s parent, *Atmospheric and Oceanic Fluid Dynamics*. Nearly all the topics in this book, except those in the chapter on planetary atmospheres, are dealt with in greater detail there.

What is in the book

The book is divided into three Parts. The first, and longest, provides the foundation for the study of the dynamics of the atmosphere and ocean. It does not assume any prior knowledge of fluid dynamics or thermodynamics, although readers who have such knowledge may be able to skim Chapter 1. The rest of Part I provides an introduction to ‘geophysical fluid dynamics’, the subject that remains at the heart of atmospheric and oceanic dynamics and without which the subject would be largely qualitative and/or computational. Here we discuss the effects of rota-

tion and stratification, leading into shallow water theory and the quasi-geostrophic and planetary-geostrophic equations. Rossby waves, gravity waves, baroclinic instability and elementary treatments of wave–mean-flow interaction and turbulence round out Part I.

Parts II and III focus on the large scale dynamics and circulation of the atmosphere and ocean, respectively. Our main focus in both Parts is what is sometimes called ‘the general circulation’, meaning the large-scale quasi-steady and/or time-averaged circulation, but this circulation depends on the effects of time-dependent eddies — the atmosphere’s Ferrel Cell may be considered to be ‘driven’ by the effects of baroclinic instability and Rossby waves. And the El Niño phenomenon, described in the final chapter, is explicitly time dependent. One feature of this book that is not in the parent book is a chapter discussing some of the general principles of planetary atmospheres, a topic of increasing interest because of the new, sometimes quite spectacular, observations of the planets in our Solar System and beyond.

How to use the book

The contents of the book are about enough for a two-term course in atmosphere–ocean dynamics. A term-long, first course in geophysical fluid dynamics could, for example, be based on Part I, omitting some of the earlier or later chapters depending on the students’ backgrounds and interests. A term-long course in atmospheric and/or oceanic circulation could be based on Part II and/or Part III, supplementing the material with review articles or research papers as needed, perhaps using data sets to look at the real world (and other planets, if Chapter 13 is to be studied). Alternatively, one could combine aspects of Parts I and II, or Parts I and III, to construct an ‘Atmospheric Dynamics’ or ‘Oceanic Dynamics’ course.

If the book is to be used for self-study it could simply be read from beginning to end, although many other pathways are possible and may be preferable. Parts II and III depend on the material in Part I, but the material is reasonably self-contained, and readers who already have some knowledge of geophysical fluid dynamics should feel free to start at a later chapter, or with Part II or Part III. A few problems are collected at the end of some chapters; these are designed to test understanding as well as to fill in gaps and extend the material in the book itself. Many other problems at varying levels of difficulty can be found on the web site of this book, which can easily be found with a search engine. The reader will also see a number of margin notes throughout the book, rather like the ones to the left. The book itself was typeset using $\text{\LaTeX}$ with Crimson fonts for text, Cronos Pro for sans serif and Minion Math for equations.

I would like to thank Matt Lloyd, Zoë Pruce and Richard Smith at Cambridge University Press for their expert guidance through the writing and production process, as well as many colleagues and students — too many to list, but they know who they are — for their many comments, corrections and criticisms. If you, the reader, have other comments, major or minor, do please contact me.

Margin notes that are set in a roman (i.e., upright) font emphasize or expand on something that is in the main text.

Margin notes set in italics are asides or historical anecdotes.

Part I

GEOPHYSICAL FLUIDS

Fluid Fundamentals

FLUIDS, LIKE SOLIDS, move if they are pushed and they warm if they are heated. But, unlike solids, they flow and deform. In this chapter we establish the governing equations of motion for a fluid, with particular attention to air and seawater — the fluids of the atmosphere and ocean, respectively. Readers who already have knowledge of fluid dynamics may skim this chapter and begin reading more seriously at Chapter 2, where we begin to look at the effects of rotation and stratification.

1.1 TIME DERIVATIVES FOR FLUIDS

1.1.1 Field and Material Viewpoints

In solid-body mechanics one is normally concerned with the position and momentum of an identifiable object, such as a football or a planet, as it moves through space. In principle we could treat fluids the same way and try to follow the properties of individual fluid parcels as they flow along, perhaps getting hotter or colder as they move. This perspective is known as the *material* or *Lagrangian* viewpoint. However, in fluid dynamical problems we generally would like to know what the values of velocity, density and so on are at *fixed points* in space as time passes. A weather forecast we care about tells us how warm it will be where we live and, if we are given that, we may not care where a particular fluid parcel comes from or where it subsequently goes. Since the fluid is a continuum, this knowledge is equivalent to knowing how the fields of the dynamical variables evolve in space and time. This viewpoint is known as the *field* or *Eulerian* viewpoint.

Although the field viewpoint will often turn out to be the most practically useful, the material description is invaluable both in deriving the equations and in the subsequent insight it frequently provides. This is because the important quantities from a fundamental point of view are

The fluid dynamical equations of motion determine the evolution of a fluid. The equations are based on Newton's laws of motion and the laws of thermodynamics, and embody the principles of conservation of momentum, energy and mass. Initial conditions and boundary conditions are needed to solve the equations.

The Lagrangian viewpoint is named for the Franco-Italian J. L. Lagrange (1736–1813), one of the most renowned mathematicians of his time. The Eulerian point of view is named for Leonhard Euler (1707–1783), the great Swiss mathematician. In fact, Euler is also largely responsible for the Lagrangian view, but the attribution became tangled over time.

often those which are associated with a given fluid element: it is these which directly enter Newton's laws of motion and the thermodynamic equations. It is thus important to have a relationship between the rate of change of quantities associated with a given fluid element and the local rate of change of a field. The material derivative (also called the advective derivative or Lagrangian derivative) provides this relationship.

1.1.2 The Material Derivative of a Fluid Property

A *fluid element* is an infinitesimal, indivisible, piece of fluid — effectively a very small fluid parcel of fixed mass. The *material derivative*, or the *Lagrangian derivative*, is the rate of change of a property (such as temperature or momentum) of a particular fluid element or finite mass of fluid; that is, it is the total time derivative of a property of a piece of fluid.

Let us suppose that a fluid is characterized by a given velocity field $\mathbf{v}(\mathbf{x}, t)$, which determines its velocity throughout. Let us also suppose that the fluid has another property φ , and let us seek an expression for the rate of change of φ of a fluid element. Since φ is changing in time and in space we use the chain rule,

$$\delta\varphi = \frac{\partial\varphi}{\partial t}\delta t + \frac{\partial\varphi}{\partial x}\delta x + \frac{\partial\varphi}{\partial y}\delta y + \frac{\partial\varphi}{\partial z}\delta z = \frac{\partial\varphi}{\partial t}\delta t + \delta\mathbf{x} \cdot \nabla\varphi. \quad (1.1)$$

This is true in general for any δt , δx , etc. The total time derivative is then

$$\frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial t} + \frac{d\mathbf{x}}{dt} \cdot \nabla\varphi. \quad (1.2)$$

If this equation is to provide a material derivative we must identify the time derivative in the second term on the right-hand side with the rate of change of position of a fluid element, namely its velocity. Hence, the material derivative of the property φ is

$$\frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial t} + \mathbf{v} \cdot \nabla\varphi. \quad (1.3)$$

The right-hand side expresses the material derivative in terms of the local rate of change of φ plus a contribution arising from the spatial variation of φ , experienced only as the fluid parcel moves. Because the material derivative is so common, and to distinguish it from other derivatives, we denote it by the operator D/Dt . Thus, the material derivative of the field φ is

$$\frac{D\varphi}{Dt} = \frac{\partial\varphi}{\partial t} + (\mathbf{v} \cdot \nabla)\varphi. \quad (1.4)$$

The brackets in the last term of this equation are helpful in reminding us that $(\mathbf{v} \cdot \nabla)$ is an operator acting on φ . The operator $\partial/\partial t + (\mathbf{v} \cdot \nabla)$ is the *Eulerian representation of the Lagrangian derivative as applied to a field*.

Material derivative of vector field

The material derivative may act on a vector field $\mathbf{b}$, in which case

$$\frac{D\mathbf{b}}{Dt} = \frac{\partial \mathbf{b}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{b}. \quad (1.5)$$

In Cartesian coordinates this is

$$\frac{D\mathbf{b}}{Dt} = \frac{\partial \mathbf{b}}{\partial t} + u \frac{\partial \mathbf{b}}{\partial x} + v \frac{\partial \mathbf{b}}{\partial y} + w \frac{\partial \mathbf{b}}{\partial z}, \quad (1.6)$$

and for a particular component of $\mathbf{b}$, b^x say,

$$\frac{Db^x}{Dt} = \frac{\partial b^x}{\partial t} + u \frac{\partial b^x}{\partial x} + v \frac{\partial b^x}{\partial y} + w \frac{\partial b^x}{\partial z}, \quad (1.7)$$

and similarly for b^y and b^z . In coordinate systems other than Cartesian the advective derivative of a vector is not simply the sum of the advective derivatives of its components, because the coordinate vectors themselves change direction with position; this will be important when we deal with spherical coordinates.

1.1.3 Material Derivative of a Volume

The volume that a given, unchanging, mass of fluid occupies is deformed and advected by the fluid motion, and there is no reason why it should remain constant. Rather, the volume will change as a result of the movement of each element of its bounding material surface, and in particular it will change if there is a non-zero normal component of the velocity at the fluid surface. That is, if the volume of some fluid is $\int dV$, then

$$\frac{D}{Dt} \int_V dV = \int_S \mathbf{v} \cdot d\mathbf{S}, \quad (1.8)$$

where the subscript V indicates that the integral is a definite integral over some finite volume V , and the limits of the integral are functions of time since the volume is changing. The integral on the right-hand side is over the closed surface, S , bounding the volume. Although intuitively apparent (to some), this expression may be derived more formally using Leibniz's formula for the rate of change of an integral whose limits are changing. Using the divergence theorem on the right-hand side, (1.8) becomes

$$\frac{D}{Dt} \int_V dV = \int_V \nabla \cdot \mathbf{v} dV. \quad (1.9)$$

The rate of change of the volume of an infinitesimal fluid element of volume ΔV is obtained by taking the limit of this expression as the volume tends to zero, giving

$$\lim_{\Delta V \rightarrow 0} \frac{1}{\Delta V} \frac{D\Delta V}{Dt} = \nabla \cdot \mathbf{v}. \quad (1.10)$$

The Eulerian derivative is the rate of change of a property at a fixed location in space. The material derivative is the rate of change of a property of a given piece of fluid, which may be moving and so changing its position.

We will often write such expressions informally as

$$\frac{D\Delta V}{Dt} = \Delta V \nabla \cdot \mathbf{v}, \quad (1.11)$$

with the limit implied.

Consider now the material derivative of some fluid property, ξ say, multiplied by the volume of a fluid element, ΔV . Such a derivative arises when ξ is the amount per unit volume of ξ -substance — the mass density or the amount of a dye per unit volume, for example. Then we have

$$\frac{D}{Dt}(\xi \Delta V) = \xi \frac{D\Delta V}{Dt} + \Delta V \frac{D\xi}{Dt}. \quad (1.12)$$

Using (1.11) this becomes

$$\frac{D}{Dt}(\xi \Delta V) = \Delta V \left(\xi \nabla \cdot \mathbf{v} + \frac{D\xi}{Dt} \right), \quad (1.13)$$

and the analogous result for a finite fluid volume is just

$$\frac{D}{Dt} \int_V \xi \, dV = \int_V \left(\xi \nabla \cdot \mathbf{v} + \frac{D\xi}{Dt} \right) dV. \quad (1.14)$$

This expression is to be contrasted with the Eulerian derivative for which the volume, and so the limits of integration, are fixed and we have

$$\frac{d}{dt} \int_V \xi \, dV = \int_V \frac{\partial \xi}{\partial t} \, dV. \quad (1.15)$$

Now consider the material derivative of a fluid property φ multiplied by the mass of a fluid element, $\rho \Delta V$, where ρ is the fluid density. Such a derivative arises when φ is the amount of φ -substance per unit mass (note, for example, that the momentum of a fluid element is $\rho \mathbf{v} \Delta V$). The material derivative of $\varphi \rho \Delta V$ is given by

$$\frac{D}{Dt}(\varphi \rho \Delta V) = \rho \Delta V \frac{D\varphi}{Dt} + \varphi \frac{D}{Dt}(\rho \Delta V). \quad (1.16)$$

But $\rho \Delta V$ is just the mass of the fluid element, and that is constant — that is how a fluid element is defined. Thus the second term on the right-hand side vanishes and

$$\frac{D}{Dt}(\varphi \rho \Delta V) = \rho \Delta V \frac{D\varphi}{Dt} \quad \text{and} \quad \frac{D}{Dt} \int_V \varphi \rho \, dV = \int_V \rho \frac{D\varphi}{Dt} \, dV, \quad (1.17a,b)$$

where (1.17b) applies to a finite volume. That expression may also be derived more formally using Leibniz's formula for the material derivative of an integral, and the result also holds when φ is a vector. The result is quite different from the corresponding Eulerian derivative, in which the volume is kept fixed; in that case we have:

$$\frac{d}{dt} \int_V \varphi \rho \, dV = \int_V \frac{\partial}{\partial t}(\varphi \rho) \, dV. \quad (1.18)$$

Various material and Eulerian derivatives are summarized in the shaded box on the facing page.

Material and Eulerian Derivatives

The material derivatives of a scalar (φ) and a vector ($\mathbf{b}$) field are given by:

$$\frac{D\varphi}{Dt} = \frac{\partial\varphi}{\partial t} + \mathbf{v} \cdot \nabla\varphi, \quad \frac{D\mathbf{b}}{Dt} = \frac{\partial\mathbf{b}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{b}. \quad (\text{D.1})$$

Various material derivatives of integrals are:

$$\frac{D}{Dt} \int_V \varphi dV = \int_V \left(\frac{D\varphi}{Dt} + \varphi \nabla \cdot \mathbf{v} \right) dV = \int_V \left(\frac{\partial\varphi}{\partial t} + \nabla \cdot (\varphi \mathbf{v}) \right) dV, \quad (\text{D.2})$$

$$\frac{D}{Dt} \int_V dV = \int_V \nabla \cdot \mathbf{v} dV, \quad (\text{D.3})$$

$$\frac{D}{Dt} \int_V \rho \varphi dV = \int_V \rho \frac{D\varphi}{Dt} dV. \quad (\text{D.4})$$

These formulae also hold if φ is a vector. The Eulerian derivative of an integral is:

$$\frac{d}{dt} \int_V \varphi dV = \int_V \frac{\partial\varphi}{\partial t} dV, \quad (\text{D.5})$$

so that

$$\frac{d}{dt} \int_V dV = 0 \quad \text{and} \quad \frac{d}{dt} \int_V \rho \varphi dV = \int_V \frac{\partial\rho\varphi}{\partial t} dV. \quad (\text{D.6})$$

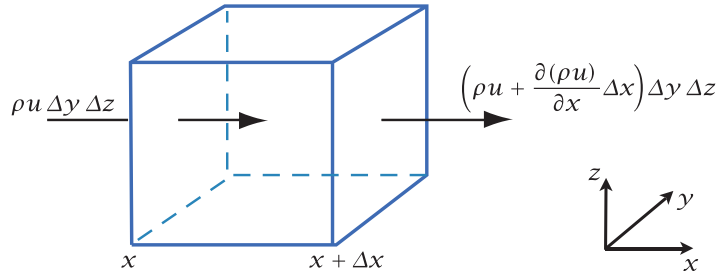
1.2 THE MASS CONTINUITY EQUATION

In classical mechanics mass is absolutely conserved and in solid-body mechanics we normally do not need an explicit equation of mass conservation. However, in fluid mechanics a fluid may flow into and away from a particular location, and fluid density may change, and we need an equation to describe that change.

1.2.1 An Eulerian Derivation

We first derive the mass conservation equation from an Eulerian point of view; that is, our reference frame is fixed in space and the fluid flows through it. Consider an infinitesimal, rectangular cuboid, control volume, $\Delta V = \Delta x \Delta y \Delta z$ that is fixed in space, as in Fig. 1.1. Fluid moves into or out of the volume through its surface, including through its faces in the y - z plane of area $\Delta A = \Delta y \Delta z$ at coordinates x and $x + \Delta x$. The accumulation of fluid within the control volume due to motion in the x -direction is

Fig. 1.1: Mass conservation in an Eulerian cuboid control volume. The mass convergence, $-\partial(\rho u)/\partial x$ (plus contributions from the y and z directions), must be balanced by a density increase equal to $\partial\rho/\partial t$.



evidently

$$\Delta y \Delta z [(\rho u)(x, y, z) - (\rho u)(x + \Delta x, y, z)] = -\left. \frac{\partial(\rho u)}{\partial x} \right|_{x,y,z} \Delta x \Delta y \Delta z. \quad (1.19)$$

To this must be added the effects of motion in the y - and z -directions, namely

$$- \left[\frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] \Delta x \Delta y \Delta z. \quad (1.20)$$

This net accumulation of fluid must be accompanied by a corresponding increase of fluid mass within the control volume. This is

$$\frac{\partial}{\partial t} (\text{density} \times \text{volume}) = \Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t}, \quad (1.21)$$

because the volume is constant. Thus, because mass is conserved, (1.19), (1.20) and (1.21) give

$$\Delta x \Delta y \Delta z \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right] = 0. \quad (1.22)$$

The quantity in square brackets must be zero and we therefore have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \quad (1.23)$$

This is called the *mass continuity equation* for it recognizes the continuous nature of the mass field in a fluid. There is no diffusion term in (1.23), no term like $\kappa \nabla^2 \rho$. This is because mass is transported by the macroscopic movement of molecules; even if this motion appears diffusion-like, any net macroscopic molecular motion constitutes, by definition, a velocity field.

Neither (1.23) nor the derivation that leads to it depends in any way on Cartesian geometry; a more general vector derivation using an arbitrary control volume is left as an easy exercise for the reader.

1.2.2 Mass Continuity via the Material Derivative

We now derive the mass continuity equation (1.23) from a material perspective. This is the most fundamental approach of all since the principle of mass conservation states simply that the mass of a given element

of fluid is, by definition of the element, constant. Thus, consider a small mass of fluid of density ρ and volume ΔV . Then conservation of mass may be represented by

$$\frac{D}{Dt}(\rho \Delta V) = 0. \quad (1.24)$$

Both the density and the volume of the parcel may change, so

$$\Delta V \frac{D\rho}{Dt} + \rho \frac{D\Delta V}{Dt} = \Delta V \left(\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} \right) = 0, \quad (1.25)$$

where the second expression follows using (1.11). Since the volume element is non-zero the term in brackets must vanish and

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0. \quad (1.26)$$

After expansion of the first term this becomes identical to (1.23). (A slightly more formal way to derive this result uses (1.14) with ξ replaced by ρ .) Summarizing, equivalent partial differential equations representing conservation of mass are

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0, \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \quad (1.27a,b)$$

1.2.3 Incompressible Fluids

A near-universal property of liquids is that their density is nearly constant; that is, they are essentially *incompressible*. If we write the density as

$$\rho(x, y, z, t) = \rho_0 + \delta\rho(x, y, z, t), \quad (1.28)$$

where ρ_0 is a constant, then a truly incompressible fluid has $\delta\rho = 0$. No fluid is incompressible in this strict sense so we relax the meaning slightly and simply require $|\delta\rho| \ll \rho_0$. When this is satisfied the mass continuity equation, (1.27a) takes on a different form. Equation (1.27a) may be written, without approximation, as

$$\frac{D\delta\rho}{Dt} + (\rho_0 + \delta\rho) \nabla \cdot \mathbf{v} = 0. \quad (1.29)$$

If the fluid is incompressible then the terms involving $\delta\rho$ are much smaller than those involving ρ_0 and hence may be neglected, giving

$$\nabla \cdot \mathbf{v} = 0. \quad (1.30)$$

This is the mass continuity equation for an incompressible fluid, and its satisfaction may be taken as the defining quality of an incompressible fluid. The *prognostic* equation, (1.27) has become a *diagnostic* equation.

An incompressible fluid is sometimes defined as one whose density is not affected by pressure. This definition may usefully be generalized to mean a fluid whose density is very nearly constant (and so also not affected by temperature or composition) such that the mass continuity equation takes the form (1.30). That equation is normally a very good approximation for seawater, less so for air.

The viscous form of the fluid dynamical equations of motion was established by Claude-Louis-Marie-Henri Navier (1785–1836) a French civil engineer, and George Stokes (1819–1903), an Anglo-Irish applied mathematician, who further elucidated viscous effects. Prior to their work the great Swiss mathematician Leonard Euler (1707–1783) had established the general form of the fluid equations for an inviscid incompressible flow, namely the Euler equations.

1.3 THE MOMENTUM EQUATION

The momentum equation is a partial differential equation that describes how the velocity or momentum of a fluid responds to internal and imposed forces. We derive it here using material methods, with a very heuristic treatment of the terms representing pressure and viscous forces.

1.3.1 Advection

Let $\mathbf{m}(x, y, z, t)$ be the momentum-density field (momentum per unit volume) of the fluid. Thus, $\mathbf{m} = \rho \mathbf{v}$ and the total momentum of a volume of fluid is given by the volume integral $\int_V \mathbf{m} dV$. Now, for a fluid the rate of change of momentum of an identifiable fluid mass is given by the material derivative, and by Newton's second law this is equal to the force acting on it. Thus,

$$\frac{D}{Dt} \int_V \rho \mathbf{v} dV = \int_V \mathbf{F} dV, \quad (1.31)$$

where $\mathbf{F}$ is the force per unit volume. Now, using (1.17b) (with φ replaced by $\mathbf{v}$) to transform the left-hand side of (1.31), we obtain

$$\int_V \left(\rho \frac{D\mathbf{v}}{Dt} - \mathbf{F} \right) dV = 0. \quad (1.32)$$

Because the volume is arbitrary the integrand itself must vanish and we obtain

$$\rho \frac{D\mathbf{v}}{Dt} = \mathbf{F} \quad \text{or} \quad \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \frac{\mathbf{F}}{\rho}, \quad (1.33a,b)$$

having used (1.5) to expand the material derivative. We have thus obtained an expression for how a fluid accelerates if subject to known forces. As well as external forces (like gravity), a stress arises from the direct contact between one fluid parcel and another, giving rise to pressure and viscous forces, sometimes referred to as *contact* forces.

1.3.2 Pressure and Viscous Forces

Pressure

Within or at the boundary of a fluid the pressure is the normal force per unit area due to the collective action of molecular motion. Thus

$$d\hat{\mathbf{F}}_p = -p d\mathbf{S}, \quad (1.34)$$

where p is the pressure, $\hat{\mathbf{F}}_p$ is the pressure force and $d\mathbf{S}$ an infinitesimal surface element. If we grant ourselves this intuitive notion, it is a simple matter to assess the influence of pressure on a fluid, for the pressure force on a volume of fluid is the integral of the pressure over the its boundary and so

$$\hat{\mathbf{F}}_p = - \int_S p d\mathbf{S}. \quad (1.35)$$

The forces due to pressure and viscosity are 'contact forces' arising because of the inter-molecular forces and/or collisions in a fluid. The net pressure force on a fluid element is proportional to the gradient of pressure.

The minus sign arises because the pressure force is directed inwards, whereas $\mathbf{S}$ is a vector normal to the surface and directed outwards. Applying a form of the divergence theorem to the right-hand side gives

$$\hat{\mathbf{F}}_p = - \int_V \nabla p \, dV, \quad (1.36)$$

where the volume V is bounded by the surface S . The pressure gradient force per unit volume, $\mathbf{F}_p$, is therefore just $-\nabla p$.

Viscosity

The effects of viscosity are apparent in many situations — the flow of treacle or volcanic lava are obvious examples. The viscous force per unit volume is approximately equal to $\mu \nabla^2 \mathbf{v}$, where μ is the coefficient of viscosity. With the pressure and viscous terms the momentum equation becomes,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v} + \mathbf{F}_b, \quad (1.37)$$

where $\nu = \mu/\rho$ is the *kinematic viscosity* and $\mathbf{F}_b$ represents body forces (per unit mass) such as gravity, $\mathbf{g}$. For most large-scale flows in the atmosphere and ocean the viscous term is, in fact, negligibly small.

1.3.3 The Hydrostatic Approximation

Neglecting viscosity, the vertical component (the component parallel to the gravitational force, $\mathbf{g}$) of the momentum equation is

$$\frac{Dw}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g, \quad (1.38)$$

where w is the vertical component of the velocity and $\mathbf{g} = -g\hat{\mathbf{k}}$. If the fluid is static the gravitational term is balanced by the pressure term and we have

$$\frac{\partial p}{\partial z} = -\rho g, \quad (1.39)$$

and this relation is known as *hydrostatic balance*, or *hydrostasy*. It is clear in this case that the pressure at a point is given by the weight of the fluid above it, provided that $p = 0$ at the top of the fluid. The flow need not be static for hydrostasy to hold — equation (1.39) is a good *approximation* to (1.38) provided that the vertical acceleration, Dw/Dt , is sufficiently small compared to gravity, which is nearly always the case in both atmosphere and ocean except in intense storms. However, because the pressure also appears in the horizontal momentum equations, hydrostatic balance must be *very* well satisfied to ensure that (1.39) provides an accurate enough pressure to determine the horizontal pressure gradients, a point we return to in Section 3.2.

Equation (1.37) is sometimes called the Navier–Stokes equation. If viscosity is absent the equation is the Euler equation. Sometimes these names are taken as applying to the complete set of equations of motion.

Hydrostatic balance is an approximation to the vertical momentum equation, valid for large-scale motion in both atmosphere and ocean and normally very well satisfied for flows of horizontal scales greater than a few tens of kilometres. It is one of the most fundamental and useful approximations in atmospheric and oceanic dynamics.

Table 1.1: Various thermodynamic parameters used in ideal gas theory, with the specific values being those for dry air.

Symbol	Description	Value
k_B	Boltzmann constant	$1.38 \times 10^{-23} \text{ J K}^{-1}$
N_A	Avogadro constant	$6.02214076 \times 10^{23} \text{ mol}^{-1}$
R^*	universal gas constant ($= k_B N_A$)	$8.31 \text{ J K}^{-1} \text{ mol}^{-1}$
μ	molar mass of dry air	$29 \times 10^{-3} \text{ kg mol}^{-1}$
R	specific gas constant ($= R^* / \mu$)	$287 \text{ J kg}^{-1} \text{ K}^{-1}$
c_v	specific heat capacity at const. volume	$717 \text{ J kg}^{-1} \text{ K}^{-1}$
c_p	specific heat capacity at const. pressure	$1004 \text{ J kg}^{-1} \text{ K}^{-1}$
c_s	sound speed at $T = 273 \text{ K}$	331 m s^{-1}

1.4 THE EQUATION OF STATE

In three dimensions the momentum and continuity equations provide four equations, but contain five unknowns — three components of velocity, density and pressure. Obviously other equations are needed, and an *equation of state* is an expression that diagnostically relates the various thermodynamic variables to each other. Most commonly the equation of state is written in a form that relates temperature, density, pressure and composition, and such an equation is known as the *thermal equation of state*, and it differs from fluid to fluid. In this book we will mainly be dealing with an ideal gas (for the atmosphere) or with seawater (for the ocean). The composition of air varies slightly with water vapour content and the composition of seawater varies slightly with salinity.

1.4.1 Ideal Gas

For an ideal gas of constant composition the equation of state is commonly written as

$$pV = Nk_B T = nR^* T, \quad (1.40)$$

where k_B is Boltzmann's constant, N is the total number of molecules in the volume V , R^* is the universal gas constant and n is the number of moles in that volume, where a mole is the amount of substance that contains Avogadro's number of elementary units. The two expressions on the right-hand side of (1.40) are equivalent because $N = nN_A$ and $R^* = k_B N_A$, where N_A is the Avogadro constant (see Table 1.1 and margin note). For fluid dynamical purposes we divide (1.40) by the total mass, $M = n\mu$, where μ is the molar mass (the mass per mole, often referred to as the molecular weight) of the gas, and obtain

$$p = \rho R T, \quad (1.41)$$

where $R = R^* / \mu$ is the *specific gas constant*, which varies from substance to substance. For dry air $R = 287 \text{ J kg}^{-1} \text{ K}^{-1}$. Air has virtually constant composition except for variations in water vapour; these variations make R a weak function of the water vapour content but we regard R as a constant. Finally, it is common in fluid dynamics to work with the inverse of

A *mole* is the amount of substance that contains exactly Avogadro's number, N_A , of elementary entities (usually atoms or molecules), and $N_A \equiv 6.02214076 \times 10^{23}$, by definition. The (quasi-dimensional) Avogadro constant, N_A , is the number of elementary units per mole, that number being Avogadro's number. A mole is almost the same as the atomic or molecular weight, or molar mass, in grams. Thus, a mole of molecular oxygen (O_2) has a mass of very nearly 32 grams.

Symbol	Description	Value
ρ_0	reference density	$1.027 \times 10^3 \text{ kg m}^{-3}$
α_0	reference specific volume	$9.738 \times 10^{-4} \text{ m}^3 \text{ kg}^{-1}$
T_0	reference temperature	283 K
S_0	reference salinity	35 ppt = 35 g kg ⁻¹
c_{s0}	reference sound speed	1490 m s ⁻¹
β_T	thermal expansion coefficient	$1.67 \times 10^{-4} \text{ K}^{-1}$
β_S	haline contraction coefficient	$0.78 \times 10^{-3} \text{ ppt}^{-1}$
β_p	compressibility coefficient ($= \alpha_0/c_{s0}^2$)	$4.39 \times 10^{-10} \text{ m s}^2 \text{ kg}^{-1}$
c_{p0}	specific heat capacity at const. pressure	$3986 \text{ J kg}^{-1} \text{ K}^{-1}$

Table 1.2: Various thermodynamic and equation-of-state parameters appropriate for seawater, as used in the equation of state (1.42) and elsewhere. The unit ppt is parts per thousand by weight, or g/kg.

density, or specific volume (i.e., volume per unit mass), $\alpha = 1/\rho$, whence the equation of state becomes $p\alpha = RT$.

1.4.2 Seawater

Water is nearly incompressible: its density changes very little with temperature, salinity, or pressure. However, these variations, small as they are, *are* important in oceanography for they allow the ocean currents to transport large quantities of heat in the great ocean gyres and in deep abyssal currents. There is no accurate, simple equation of state but for many purposes we can approximate it as

$$\rho = \rho_0 [1 - \beta_T(T - T_0) + \beta_S(S - S_0) + \beta_p(p - p_0)], \quad (1.42)$$

where β_T , β_S and β_p are empirical parameters and S_0 , T_0 and p_0 are constants, and usually we take $p_0 = 0$. (A still more accurate equation is required for quantitative oceanography.) Typical values of these parameters are given in Table 1.2. The parameter β_p is related to the speed of sound, c_s , given by $c_s^2 = (\partial p / \partial \rho)$. Using this result and (1.42) gives, to a good approximation, $\beta_p = 1/\rho_0 c_s^2$, and $c_s \approx 1500 \text{ m s}^{-1}$. *None* of the terms in (1.42) give rise to large variations in density in the ocean.

Unfortunately (perhaps) the equation of state introduces another unknown, temperature, into our equation set. We thus have to introduce another physical principle — one coming from thermodynamics — to obtain a complete set of equations, as we now explore.

Density variations *are* important for generating ocean currents but they are *not* important in the mass continuity equation for seawater, and to a very good approximation that equation may be written as $\nabla \cdot \mathbf{v} = 0$.

1.5 THERMODYNAMICS

1.5.1 A Few Fundamentals

The *first law of thermodynamics* states that the internal energy, I , of a body may change because of work done by or on it, or because of a heat input, or because of a change in its chemical composition. We will neglect the last effect so that

$$dI = dQ + dW, \quad (1.43)$$

where dW is the work done on the body, dQ is the heat input to the body and dI is the change in internal energy, and we take all these quantities to be per unit mass. (Heating arises from such things as radiation and conduction, and work done occurs when a body is compressed.) The quantities on the right-hand side (with a d) are ‘imperfect’ differentials or infinitesimals: Q and W are not functions of the state of a body, and the internal energy cannot be regarded as the sum of a ‘heat’ and a ‘work’. That is, we should think of heat and work as having meaning only as fluxes of energy, or rates of energy input, and not as amounts of energy; their sum changes the internal energy of a body, which is a function of its state. However, both the heat input and the work done are related to state variables, as follows:

A state variable is, by definition, a function of the state of a fluid, or more generally the state of any body. The internal energy and the entropy are both state variables. Heat and work done are *not* state variables; they have meaning only as inputs or fluxes of energy.

Heat input: Although heat is not itself a state function, there is a state function that responds directly to heating and this is the *entropy*. Specifically, in an infinitesimal quasi-static or reversible process, if an amount of heat dQ (per unit mass) is externally supplied then the specific entropy η will change according to

$$T d\eta = dQ. \quad (1.44)$$

That is to say, the entropy changes by an amount equal to the heat input divided by the temperature.

Work done: The work done on a body during a reversible process is equal to the pressure times its change in volume, and if the work is positive then the volume change is negative. Thus if an infinitesimal amount of work dW (per unit mass) is applied to a body then its thermodynamic state will change according to

$$-p d\alpha = dW, \quad (1.45)$$

where, we recall, $\alpha = 1/\rho$ is the specific volume of the fluid and p is the pressure.

Putting equations (1.43)–(1.45) together we have

$$dI = T d\eta - p d\alpha. \quad (1.46)$$

This expression is called the *fundamental thermodynamic relation*. Let’s see how it applies to a fluid.

1.5.2 Thermodynamic Equation for a Fluid

Let us suppose that the heating of a fluid arises from external agents, such as radiation from the sun, and that the changes in density and internal energy are consequences of this heating. It then makes sense to write (1.46) as

$$dI + p d\alpha = dQ. \quad (1.47)$$

Now, (1.47) applies to a particular fluid element, not to a particular location. The rate of change of the internal energy and volume are thus obtained by taking the material derivative giving

$$\frac{DI}{Dt} + p \frac{D\alpha}{Dt} = \dot{Q} \quad \text{or} \quad \frac{DI}{Dt} - \frac{p}{\rho^2} \frac{D\rho}{Dt} = \dot{Q}, \quad (1.48)$$

where $\dot{Q}$ is the rate of heat input. We can now use the mass conservation equation, (1.27a), to rewrite the time derivative of density to give

$$\frac{DI}{Dt} + \frac{p}{\rho} \nabla \cdot \mathbf{v} = \dot{Q}. \quad (1.49)$$

This is the ‘thermodynamic equation’ of a fluid, and it tells us how the internal energy responds to heating.

An ideal gas

In an ideal gas the internal energy is a function of temperature alone, and is given by

$$I = c_v T, \quad (1.50)$$

where c_v is the heat capacity at constant volume. For the gases in Earth’s atmosphere, c_v itself is, to a very good approximation, a constant and (1.49) becomes

$$c_v \frac{DT}{Dt} + \frac{p}{\rho} \nabla \cdot \mathbf{v} = \dot{Q}. \quad (1.51)$$

Equation (1.51) is perhaps the most commonly used form of the thermodynamic equation in the atmospheric sciences.

We can rewrite (1.47) as

$$dI + d(\alpha p) - \alpha dp = dQ \quad \text{or} \quad d(c_v T) + d(RT) - \alpha dp = dQ, \quad (1.52a,b)$$

where the second expression holds for an ideal gas. We let $c_p = c_v + R$ and (1.52b) then becomes, for constant c_p ,

$$c_p dT - \alpha dp = dQ \quad \text{implying} \quad c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} = \dot{Q}, \quad (1.53a,b)$$

From (1.53a), we can see that c_p is just the heat capacity at constant pressure. Equation (1.53b) is equivalent to (1.51), although perhaps not as widely used. The complete set of equations of motion for a fluid are summarized on the next page.

In an ideal gas, c_p , c_v and R are functions only of temperature and they are related by $c_p - c_v = R$. In Earth’s atmosphere they are all in fact very nearly constant.

1.6 POTENTIAL TEMPERATURE AND ENTROPY

When a fluid is heated its entropy increases, obeying $T d\eta = dQ$ as in (1.44). Taking the material derivative gives

$$T \frac{D\eta}{Dt} = \dot{Q}. \quad (1.54)$$

The Equations of Motion for a Fluid

The momentum equation

The momentum equation is an evolution equation for velocity and embodies Newton's second law, namely that force equals mass times acceleration. It is

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} + \mathbf{g}, \quad (\text{F.1})$$

where $\mathbf{g}$ is the gravitational force per unit mass.

The mass continuity equation

The mass continuity equation embodies the principle of conservation of mass. Two equivalent forms are

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{v} = 0, \quad \frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0. \quad (\text{F.2})$$

If the fluid is incompressible the above equations become

$$\nabla \cdot \mathbf{v} = 0. \quad (\text{F.3})$$

The thermodynamic equation

The thermodynamic equation is a statement of the first law of thermodynamics and is

$$\frac{DI}{Dt} + \frac{p}{\rho}\nabla \cdot \mathbf{v} = \dot{Q}, \quad (\text{F.4})$$

where $\dot{Q}$ is the total heating. In an ideal gas $I = c_v T$ and $p = \rho RT$ and the equation becomes

$$\frac{DT}{Dt} + \frac{RT}{c_v}\nabla \cdot \mathbf{v} = \frac{\dot{Q}}{c_v} = \kappa\nabla^2 T + J, \quad (\text{F.5})$$

and here we split the heating into a diffusion of temperature and an external source, J .

Equation of state

The equation of state is a diagnostic equation that connects pressure, density, temperature and, if the constituents of the fluid vary, composition. For a single-component ideal gas the equation of state is

$$p = \rho RT. \quad (\text{F.6})$$

For seawater an approximate equation of state that gives density as a function of pressure, temperature and salinity is

$$\rho = \rho_0 \left[1 - \beta_T(T - T_0) + \beta_S(S - S_0) + \beta_p(p - p_0) \right]. \quad (\text{F.7})$$

For quantitative oceanography we need a more accurate equation of state and that necessitates the inclusion of nonlinear terms.

This is a form of the thermodynamic equation that could be used *instead* of the internal energy equation (1.49), but to do so involves relating entropy to the other thermodynamic variables with an equation of state of the form

$$\eta = \eta(T, p),$$

or, in general, any two of pressure, density and temperature. (Entropy is also a function of composition but we treat that as constant.) We can obtain such an expression for an ideal gas, as follows.

Using the ideal gas equation of state we may write the first law of thermodynamics in two equivalent ways, namely

$$c_v dT + p d\alpha = dQ \quad \text{or} \quad c_p dT - \alpha dp = dQ. \quad (1.56a,b)$$

Then, since the entropy increase obeys $T d\eta = dQ$, we have

$$c_v dT + p d\alpha = T d\eta \quad \text{or} \quad c_p dT - \alpha dp = T d\eta. \quad (1.57a,b)$$

These are both forms of the fundamental thermodynamic relation, applied to an ideal gas. Using the equation of state in the form $\alpha = RT/p$ the above equations may be written

$$c_v \frac{dT}{T} - R \frac{dp}{p} = d\eta \quad \text{or} \quad c_p \frac{dT}{T} - R \frac{dp}{p} = d\eta. \quad (1.58a,b)$$

We can integrate (1.58) to give two equivalent explicit expressions for the entropy in an ideal gas, namely

$$\eta = c_v \log T - R \log p + \text{constant} = c_p \log T - R \log p + \text{constant}. \quad (1.59)$$

As always in classical mechanics, entropy contains an arbitrary constant.

For convenience we now define a quantity, θ , called *potential temperature* such that

$$\eta = c_p \ln \theta, \quad (1.60)$$

so that $d\eta = c_p (d\theta/\theta)$. Using this expression in (1.57b) and integrating gives

$$c_p \log T - R \log p = c_p \log \theta + \text{constant}. \quad (1.61)$$

If we fold the constant of integration into the pressure term this equation can be written as

$$\theta = T \left(\frac{p_R}{p} \right)^{R/c_p}, \quad (1.62)$$

where p_R is a constant. And given (1.60), we can write (1.54) as

$$c_p \frac{D\theta}{Dt} = \frac{\theta}{T} \dot{Q}. \quad (1.63)$$

This form of the thermodynamic equation is equivalent to (1.51) and in some ways is a simpler expression, although it is valid only for an ideal gas — obtaining an analogous expression for seawater is more difficult because of the nonlinearity of the true seawater equation of state and because c_p for seawater is not a constant.

Although entropy responds directly to the heating, it is *not* a measure of the heat content of a body, nor does any such measure properly exist. Note, for example, that the entropy increase depends on the temperature at which heat is added. Similarly, a body does not contain a certain amount of work. Rather, both work done and heating change the internal energy.

1.6.1 Meaning of Potential Temperature

We introduced potential temperature as a measure of the entropy of a fluid and that is how it is best regarded. However, it has a useful physical interpretation, as follows. Suppose a parcel of constant composition is moved adiabatically from one location to another at a different pressure, and by adiabatic we mean that no heat enters or leaves the parcel; that is, $\dot{Q} = 0$. In a reversible process (such as fluid flow) the entropy and potential temperature are then conserved. However, the temperature of the parcel *will* change, because the fluid may be compressed or expand, as (1.56) tells us. Consider a fluid parcel at some pressure p_1 with temperature T_1 and potential temperature θ_1 . Now move that parcel adiabatically to a pressure p_R . The temperature of the parcel changes but its potential temperature does not, so that the final potential temperature is just θ_1 . The final temperature of the parcel, T_2 say, is equal to θ_1 , because $T = \theta$ when $p = p_R$. Thus, the final temperature of the parcel is equal to its initial potential temperature. We may thus say that *the potential temperature of a parcel is equal to the temperature that a parcel will acquire if taken adiabatically to a standard pressure, p_R* . This statement is commonly taken as the *definition* of potential temperature, and if we begin here the connection with entropy is then made by realizing that in an adiabatic process at constant composition the entropy also does not change, and so is a function of potential temperature.

Potential temperature is the temperature that a fluid parcel will have if taken adiabatically and at constant composition to a reference pressure. Potential density is the analogous quantity for density. Both of these quantities are functions of the entropy of a parcel, and functions of the entropy alone if composition is fixed.

Potential temperature is sometimes a more convenient variable to use than entropy, and its temperature-like quality gives it an intuitive appeal; still, entropy is the more fundamental variable. Finally, although p_R may be chosen arbitrarily, in most atmospheric applications $p_R = 1000$ hPa, which is the approximate pressure at sea level. Thus, potential temperature is the temperature that a parcel achieves when adiabatically brought to the surface; the potential temperature is higher than the *in situ* temperature because the parcel is compressed as it descends, and the compression increases the internal energy, and hence increases the temperature, of the parcel.

1.6.2 Potential Density

By analogy with potential temperature, the *potential density*, ρ_θ , is the density that a fluid parcel would have if moved adiabatically and at constant composition to a reference pressure, p_R . It is a useful quantity because it turns out to be a measure of the stability of a fluid parcel with respect to convection, in both air and water, as we will discover later.

Ideal gas

Suppose a fluid with temperature T and at pressure p is moved to a pressure p_R , where its temperature becomes equal to θ . For an ideal gas its potential density at pressure p is equal to the density it has at p_R so that

$$\rho_\theta = \frac{p_R}{R\theta} = \frac{p_R}{RT} \left(\frac{p}{p_R} \right)^{R/c_p} = \rho \left(\frac{p_R}{p} \right)^{c_v/c_p}, \quad (1.64)$$

using $RT = p/\rho$ to obtain the last expression.

Seawater

There is no corresponding exact expression for potential density of seawater. However, we can obtain a good approximation by first noting that density is almost constant, and then Taylor-expanding the density around the density at the reference level, at which $T = \theta$ and $p = p_R$. At first order we have

$$\rho(p) \approx \rho(p_R) + (p - p_R) \frac{\partial \rho}{\partial p}. \quad (1.65)$$

The first term on the right-hand side is, by definition, the potential density and the derivative in the second term is the inverse of the square of speed of sound, which is nearly constant in seawater. Thus

$$\rho_\theta \approx \rho - \frac{1}{c_s^2} (p - p_R). \quad (1.66)$$

Because the speed of sound is a measurable quantity, (1.66) is a very useful practical expression for potential density, although because there are small variations in the speed of sound the expression is not especially accurate for problems involving large depth variations.

1.7 THE ENERGY BUDGET

The total energy of a fluid includes the kinetic, potential and internal energies. Both the fluid flow and pressure forces will move energy from place to place, but we nevertheless expect that total energy will be conserved. Let us therefore see what form energy conservation takes in a fluid.

1.7.1 An Energy Equation

We begin with the inviscid momentum equation with a time-independent potential Φ ,

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p - \rho \nabla \Phi. \quad (1.67)$$

In a uniform gravitational field $\Phi = gz$ but we can be a little more general in our derivation. Now take the dot product of (1.67) with $\mathbf{v}$ and obtain an equation for the evolution of kinetic energy,

$$\frac{1}{2} \rho \frac{D\mathbf{v}^2}{Dt} = -\mathbf{v} \cdot \nabla p - \rho \mathbf{v} \cdot \nabla \Phi = -\nabla \cdot (p\mathbf{v}) + p \nabla \cdot \mathbf{v} - \rho \mathbf{v} \cdot \nabla \Phi. \quad (1.68)$$

The internal energy equation for adiabatic flow is

$$\rho \frac{DI}{Dt} = -p \nabla \cdot \mathbf{v}. \quad (1.69)$$

Finally, and somewhat trivially, the potential satisfies

$$\rho \frac{D\Phi}{Dt} = \rho \mathbf{v} \cdot \nabla \Phi. \quad (1.70)$$

Energy is transported from place to place within a fluid both by advection and by the pressure force, which does work on a fluid.

Adding (1.68), (1.69) and (1.70) we obtain

$$\rho \frac{D}{Dt} \left(\frac{1}{2} \mathbf{v}^2 + I + \Phi \right) = -\nabla \cdot (p\mathbf{v}), \quad (1.71)$$

which, on expanding the material derivative and using the mass conservation equation, becomes

$$\frac{\partial}{\partial t} \left[\rho \left(\frac{1}{2} \mathbf{v}^2 + I + \Phi \right) \right] + \nabla \cdot \left[\rho \mathbf{v} \left(\frac{1}{2} \mathbf{v}^2 + I + \Phi + p/\rho \right) \right] = 0. \quad (1.72)$$

This may be written

$$\frac{\partial E}{\partial t} + \nabla \cdot [\mathbf{v}(E + p)] = 0, \quad (1.73)$$

where $E = \rho(\mathbf{v}^2/2 + I + \Phi)$ is the total energy per unit volume of the fluid, with contributions from the kinetic energy ($\rho\mathbf{v}^2/2$), the internal energy (ρI) and the potential energy ($\rho\Phi$). Equation (1.73) is the energy equation for an unforced, inviscid and adiabatic, compressible fluid. The energy flux term vanishes when integrated over a closed domain with rigid boundaries, implying that the total energy is conserved. However, there can be an exchange of energy between kinetic, potential and internal components. It is the divergent term, $\nabla \cdot \mathbf{v}$, that connects the kinetic energy equation, (1.68), and the internal energy equation, (1.69). In an incompressible fluid this term is absent, and the internal energy is divorced from the other components of energy. This consideration will be important when we consider the Boussinesq equations in Section 2.5. Note finally that the flux of energy, $F_E = \mathbf{v}(E + p)$ is not equal to the velocity times the energy; rather, energy is also transferred by pressure. We may write the energy flux as

$$F_E = \rho \mathbf{v} \left(\frac{\mathbf{v}^2}{2} + \Phi + h \right), \quad (1.74)$$

where $h = I + p/\rho$ is the enthalpy. That is, the local rate of change of energy is determined by the fluxes of kinetic energy, potential energy and *enthalpy*, not internal energy, because enthalpy can take into account the work done by the pressure.

Bernoulli's theorem

The quantity

$$B = \left(E + \frac{p}{\rho} \right) = \left(\frac{1}{2} \mathbf{v}^2 + I + \Phi + p\alpha \right) = \left(\frac{1}{2} \mathbf{v}^2 + h + \Phi \right), \quad (1.75)$$

is the general form of the Bernoulli function, equal to the sum of the kinetic energy, the potential energy and the enthalpy. Equation (1.73) may be written as

$$\frac{\partial E}{\partial t} + \nabla \cdot (\rho \mathbf{v} B) = 0. \quad (1.76)$$

The flux of energy is given by the velocity times the Bernoulli function, where the Bernoulli function is a combination of the kinetic, potential and internal energies and the pressure. It is equal to the sum of the kinetic energy, potential energy and enthalpy. The Bernoulli function is constant along streamlines in steady flow.

The Bernoulli function itself is not conserved, even for adiabatic flow. However, for steady flow $\nabla \cdot (\rho \mathbf{v}) = 0$, and the $\partial/\partial t$ terms vanish so that (1.76) may be written $\mathbf{v} \cdot \nabla B = 0$, or even $DB/Dt = 0$. The Bernoulli function is then a constant along streamlines, a result commonly known as Bernoulli's theorem. For adiabatic flow at constant composition we also have $D\theta/Dt = 0$. Thus, steady flow is both along surfaces of constant θ and along surfaces of constant B , and the vector

$$\mathbf{l} = \nabla \theta \times \nabla B \quad (1.77)$$

is parallel to streamlines.

1.7.2 Energy Conservation for Constant Density Fluids

If the density of a fluid is constant the derivation goes through just as above, but with an important simplification. The divergence of the velocity is zero, and so the $\nabla \cdot \mathbf{v}$ term does not appear on the right-hand side of the kinetic energy equation, (1.68). The right-hand side of the internal energy equation, (1.69), is then zero and the internal energy and the kinetic energy are then de-coupled. The kinetic energy equation straightforwardly becomes

$$\frac{\partial K}{\partial t} + \nabla \cdot (\mathbf{v}B) = 0, \quad (1.78)$$

where $K = \mathbf{v}^2/2$ and, here, $B = \phi + \Phi + \mathbf{v}^2/2$ where $\phi = p/\rho_0$ is the 'kinematic pressure'. The Bernoulli function does not contain the internal energy and the total *kinetic* energy is conserved.

1.7.3 Viscous Effects

We might expect that viscosity will always act to reduce the kinetic energy of a flow, and here we demonstrate this for a constant density fluid satisfying

$$\frac{D\mathbf{v}}{Dt} = -\nabla(\phi + \Phi) + \nu \nabla^2 \mathbf{v}. \quad (1.79)$$

The energy equation becomes

$$\frac{d\hat{E}}{dt} \equiv \frac{d}{dt} \int_V E dV = \mu \int_V \mathbf{v} \cdot \nabla^2 \mathbf{v} dV. \quad (1.80)$$

The right-hand side is negative definite. To see this we use the vector identity

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}, \quad (1.81)$$

and because $\nabla \cdot \mathbf{v} = 0$ we have $\nabla^2 \mathbf{v} = -\nabla \times \boldsymbol{\omega}$, where $\boldsymbol{\omega} \equiv \nabla \times \mathbf{v}$. Thus,

$$\frac{d\hat{E}}{dt} = -\mu \int_V \mathbf{v} \cdot (\nabla \times \boldsymbol{\omega}) dV = -\mu \int_V \boldsymbol{\omega} \cdot (\nabla \times \mathbf{v}) dV = -\mu \int_V \boldsymbol{\omega}^2 dV, \quad (1.82)$$

after integrating by parts, providing $\mathbf{v} \times \boldsymbol{\omega}$ vanishes at the boundary. Thus, viscosity acts to extract kinetic energy from the flow. The loss of kinetic energy reappears as an irreversible warming of the fluid and the total energy of the fluid is still conserved. The warming effect is small in Earth's atmosphere but it can be large in other planets.

Bernoulli's theorem was developed mainly by Daniel Bernoulli (1700–1782). The Bernoulli family produced several (at least eight) talented mathematicians over three generations in the seventeenth and eighteenth centuries, and is often regarded as the most mathematically distinguished family of all time.

Notes and References

Many books on fluid dynamics go into more detail about the equations of motion than we have, and Kundu *et al.* (2015) and Acheson (1990) both provide accessible introductions. The official definitions of the Avogadro constant and mole will change in May 2019 (as part of a larger redefinition of various SI units) and from then on a mole will bear no reference to carbon-12 (which it does in the old definition), although the change has no consequences for us. We use the new definitions in the text. The Avogadro constant is then defined as $6.02214076 \times 10^{23} \text{ mol}^{-1}$ and is to be regarded as a dimensional constant; Avogadro's number is the nondimensional number with the same numerical value, and a mole is the amount of substance containing Avogadro's number of elementary units.

Problems

- 1.1 Show that the derivative of an integral is given by

$$\frac{d}{dt} \int_{x_1(t)}^{x_2(t)} \varphi(x, t) dx = \int_{x_1}^{x_2} \frac{\partial \varphi}{\partial t} dx + \frac{dx_2}{dt} \varphi(x_2, t) - \frac{dx_1}{dt} \varphi(x_1, t). \quad (\text{P1.1})$$

By generalizing to three dimensions show that the material derivative of an integral of a fluid property is given by

$$\frac{D}{Dt} \int_V \varphi(\mathbf{x}, t) dV = \int_V \frac{\partial \varphi}{\partial t} dV + \int_S \varphi \mathbf{v} \cdot d\mathbf{S} = \int_V \left[\frac{\partial \varphi}{\partial t} + \nabla \cdot (\mathbf{v} \varphi) \right] dV, \quad (\text{P1.2})$$

where the surface integral ($\int_S$) is over the surface bounding the volume V . Hence deduce that

$$\frac{D}{Dt} \int_V \rho \varphi dV = \int_V \rho \frac{D\varphi}{Dt} dV. \quad (\text{P1.3})$$

- 1.2 (a) If molecules move quasi-randomly, why is there no diffusion term in the mass continuity equation?
 (b) Suppose that a fluid contains a binary mixture of dry air and water vapour. Show that the change in mass of a parcel of air due to the diffusion of water vapour is exactly balanced by the diffusion of dry air in the opposite direction.
- 1.3 If it is momentum, not velocity, that responds when a force is applied (according to Newton's second law), why is the (inviscid) momentum equation given by $\rho D\mathbf{v}/Dt = -\nabla p$ and not $D(\rho\mathbf{v})/Dt = -\nabla p$?
- 1.4 Using the observed value of molecular diffusion of heat in water, estimate how long it would take for a temperature anomaly to mix from the top of the ocean to the bottom, assuming that molecular diffusion alone is responsible. Comment on whether you think the real ocean has reached equilibrium after the last ice age (which ended about 12,000 years ago).
- 1.5 Show that viscosity will dissipate kinetic energy in a compressible fluid.
- 1.6 (a) Suppose that a sealed, insulated container consists of two compartments, and that one of them is filled with an ideal gas and the other is a vacuum. The partition separating the compartments is removed. How does the temperature of the gas change? (Answer: it stays the same. Explain.) Obtain an expression for the final potential temperature, in terms of the initial temperature of the gas and the volumes of the two compartments.

- (b) A dry parcel that is ascending adiabatically through the atmosphere will generally cool as it moves to lower pressure and expands, and its potential temperature stays the same. How can this be consistent with your answer to part (a)?

Reconcile your answers with the first law of thermodynamics for an ideal gas, namely that

$$\mathrm{d}Q = T \mathrm{d}\eta = c_p \frac{\mathrm{d}\theta}{\theta} = \mathrm{d}I + \mathrm{d}W = c_v \mathrm{d}T + p \mathrm{d}\alpha. \quad (\text{P1.4})$$

- 1.7 Beginning with the expression for potential temperature for a simple ideal gas, $\theta = T(p_R/p)^\kappa$, where $\kappa = R/c_p$, show that

$$\mathrm{d}\theta = (\theta/T)(\mathrm{d}T - (\alpha/c_p) \mathrm{d}p), \quad (\text{P1.5})$$

and that the first law of thermodynamics may be written as

$$\mathrm{d}Q = T \mathrm{d}\eta = c_p(T/\theta)\mathrm{d}\theta. \quad (\text{P1.6})$$

- 1.8 Show that adiabatic flow in an ideal gas satisfies $pp^{-\gamma} = \text{constant}$, where $\gamma = c_p/c_v$.
- 1.9 Show that for an ideal gas in hydrostatic balance, changes in dry static energy ($M = c_p T + gz$) and potential temperature (θ) are related by $\delta M = c_p(T/\theta)\delta\theta$. (The quantity $c_p T/\theta$ is known as the ‘Exner function’.)
- 1.10 Using the equation of state for seawater given in the text, or a more accurate one obtained from the literature, estimate the fractional change in density of the world’s oceans due to changes in pressure, salinity and temperature.
- 1.11 Using an accurate equation of state obtained from the literature (for example, from Vallis (2017) or IOC *et al.* (2010)) estimate by how much (1.42) is in error in the world’s oceans.

Equations for a Rotating Planet

PLANETS ARE ALMOST SPHERES. They also rotate. Here we consider how the equations of motion are affected by these facts, first by looking at how rotation affects the dynamics and then by expressing the equations in spherical coordinates.

2.1 EQUATIONS IN A ROTATING FRAME OF REFERENCE

Newton's second law of motion, that the rate of change of momentum of a body is proportional to the imposed force, applies in so-called inertial frames of reference that are either stationary or moving only with a constant rectilinear velocity relative to the distant galaxies. Now Earth spins around its axis once a day, so the surface of the Earth is *not* an inertial frame. Nevertheless, it is very convenient to describe the motion of the atmosphere or ocean relative to Earth's surface rather than in some inertial frame. How we do that is the subject of this section.

2.1.1 Rate of Change of a Vector

Consider first a vector $\mathbf{C}$ of constant length rotating relative to an inertial frame at a constant angular velocity $\boldsymbol{\Omega}$. Then, in a frame rotating with that same angular velocity it appears stationary and constant. If in a small interval of time δt the vector $\mathbf{C}$ rotates through a small angle $\delta\lambda$ then the change in $\mathbf{C}$, as perceived in the inertial frame, is given by (see Fig. 2.1)

$$\delta\mathbf{C} = |\mathbf{C}| \cos \vartheta \delta\lambda \mathbf{m}, \quad (2.1)$$

where the vector $\mathbf{m}$ is the unit vector in the direction of change of $\mathbf{C}$, which is perpendicular to both $\mathbf{C}$ and $\boldsymbol{\Omega}$. But the rate of change of the angle λ is just, by definition, the angular velocity so that $\delta\lambda = |\boldsymbol{\Omega}|\delta t$ and

$$\delta\mathbf{C} = |\mathbf{C}||\boldsymbol{\Omega}| \sin \hat{\vartheta} \mathbf{m} \delta t = \boldsymbol{\Omega} \times \mathbf{C} \delta t, \quad (2.2)$$

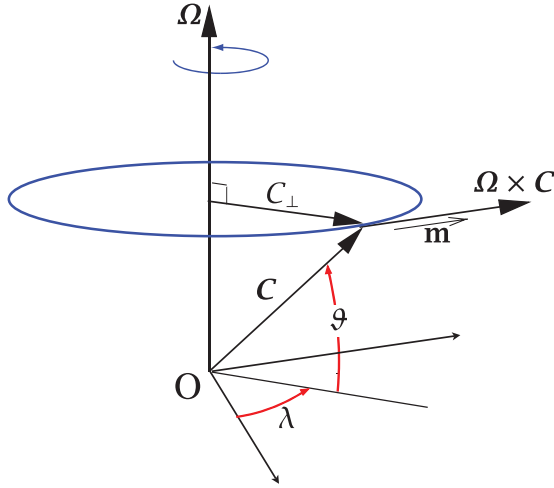


Fig. 2.1: A vector $\mathbf{C}$ rotating at an angular velocity Ω . It appears to be a constant vector in the rotating frame, whereas in the inertial frame it evolves according to $(d\mathbf{C}/dt)_I = \Omega \times \mathbf{C}$.

using the definition of the vector cross-product, where $\hat{\theta} = (\pi/2 - \theta)$ is the angle between Ω and $\mathbf{C}$. Thus

$$\left(\frac{d\mathbf{C}}{dt} \right)_I = \Omega \times \mathbf{C}, \quad (2.3)$$

where the left-hand side is the rate of change of $\mathbf{C}$ as perceived in the inertial frame.

Now consider a vector $\mathbf{B}$ that changes in the inertial frame. In a small time δt the change in $\mathbf{B}$ as seen in the rotating frame is related to the change seen in the inertial frame by

$$(\delta \mathbf{B})_I = (\delta \mathbf{B})_R + (\delta \mathbf{B})_{\text{rot}}, \quad (2.4)$$

where the terms are, respectively, the change seen in the inertial frame, the change due to the vector itself changing as measured in the rotating frame, and the change due to the rotation. Using (2.2) $(\delta \mathbf{B})_{\text{rot}} = \Omega \times \mathbf{B} \delta t$, and so the rates of change of the vector $\mathbf{B}$ in the inertial and rotating frames are related by

$$\left(\frac{d\mathbf{B}}{dt} \right)_I = \left(\frac{d\mathbf{B}}{dt} \right)_R + \Omega \times \mathbf{B}. \quad (2.5)$$

This relation applies to a vector $\mathbf{B}$ that, as measured at any one time, is the same in both inertial and rotating frames.

2.1.2 Velocity and Acceleration in a Rotating Frame

The velocity of a body is not measured to be the same in the inertial and rotating frames, so care must be taken when applying (2.5) to velocity. First apply (2.5) to $\mathbf{r}$, the position of a particle, to obtain

$$\left(\frac{d\mathbf{r}}{dt} \right)_I = \left(\frac{d\mathbf{r}}{dt} \right)_R + \Omega \times \mathbf{r} \quad (2.6)$$