Representation Theorems in Hardy Spaces

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Representation Theorems in Hardy Spaces

JAVAD MASHREGHI
Université Laval

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To My Parents: **Masoumeh Farzaneh**
and
Ahmad Mashreghi

Contents

Preface **xi**

1 Fourier series **1**
- 1.1 The Laplacian . . . 1
- 1.2 Some function spaces and sequence spaces . . . 5
- 1.3 Fourier coefficients . . . 8
- 1.4 Convolution on $\mathbb{T}$ . . . 13
- 1.5 Young's inequality . . . 16

2 Abel–Poisson means **21**
- 2.1 Abel–Poisson means of Fourier series . . . 21
- 2.2 Approximate identities on $\mathbb{T}$ . . . 25
- 2.3 Uniform convergence and pointwise convergence . . . 32
- 2.4 Weak* convergence of measures . . . 39
- 2.5 Convergence in norm . . . 43
- 2.6 Weak* convergence of bounded functions . . . 47
- 2.7 Parseval's identity . . . 49

3 Harmonic functions in the unit disc **55**
- 3.1 Series representation of harmonic functions . . . 55
- 3.2 Hardy spaces on $\mathbb{D}$ . . . 59
- 3.3 Poisson representation of $h^\infty(\mathbb{D})$ functions . . . 60
- 3.4 Poisson representation of $h^p(\mathbb{D})$ functions $(1 < p < \infty)$ . . . 65
- 3.5 Poisson representation of $h^1(\mathbb{D})$ functions . . . 66
- 3.6 Radial limits of $h^p(\mathbb{D})$ functions $(1 \le p \le \infty)$ . . . 70
- 3.7 Series representation of the harmonic conjugate . . . 77

4 Logarithmic convexity **81**
- 4.1 Subharmonic functions . . . 81
- 4.2 The maximum principle . . . 84
- 4.3 A characterization of subharmonic functions . . . 88
- 4.4 Various means of subharmonic functions . . . 90
- 4.5 Radial subharmonic functions . . . 95
- 4.6 Hardy's convexity theorem . . . 97

4.7 A complete characterization of $h^p(\mathbb{D})$ spaces . . . 99

5 Analytic functions in the unit disc 103

5.1 Representation of $H^p(\mathbb{D})$ functions $(1 < p \leq \infty)$ . . . 103
5.2 The Hilbert transform on $\mathbb{T}$ . . . 106
5.3 Radial limits of the conjugate function . . . 110
5.4 The Hilbert transform of $\mathcal{C}^1(\mathbb{T})$ functions . . . 113
5.5 Analytic measures on $\mathbb{T}$ . . . 116
5.6 Representations of $H^1(\mathbb{D})$ functions . . . 120
5.7 The uniqueness theorem and its applications . . . 123

6 Norm inequalities for the conjugate function 131

6.1 Kolmogorov's theorems . . . 131
6.2 Harmonic conjugate of $h^2(\mathbb{D})$ functions . . . 135
6.3 M. Riesz's theorem . . . 136
6.4 The Hilbert transform of bounded functions . . . 142
6.5 The Hilbert transform of Dini continuous functions . . . 144
6.6 Zygmund's $L \log L$ theorem . . . 149
6.7 M. Riesz's $L \log L$ theorem . . . 153

7 Blaschke products and their applications 155

7.1 Automorphisms of the open unit disc . . . 155
7.2 Blaschke products for the open unit disc . . . 158
7.3 Jensen's formula . . . 162
7.4 Riesz's decomposition theorem . . . 166
7.5 Representation of $H^p(\mathbb{D})$ functions $(0 < p < 1)$ . . . 168
7.6 The canonical factorization in $H^p(\mathbb{D})$ $(0 < p \leq \infty)$ . . . 172
7.7 The Nevanlinna class . . . 175
7.8 The Hardy and Fejér–Riesz inequalities . . . 181

8 Interpolating linear operators 187

8.1 Operators on Lebesgue spaces . . . 187
8.2 Hadamard's three-line theorem . . . 189
8.3 The Riesz–Thorin interpolation theorem . . . 191
8.4 The Hausdorff–Young theorem . . . 197
8.5 An interpolation theorem for Hardy spaces . . . 200
8.6 The Hardy–Littlewood inequality . . . 205

9 The Fourier transform 207

9.1 Lebesgue spaces on the real line . . . 207
9.2 The Fourier transform on $L^1(\mathbb{R})$ . . . 209
9.3 The multiplication formula on $L^1(\mathbb{R})$ . . . 218
9.4 Convolution on $\mathbb{R}$ . . . 219
9.5 Young's inequality . . . 221

10 Poisson integrals **225**
10.1 An application of the multiplication formula on $L^1(\mathbb{R})$ 225
10.2 The conjugate Poisson kernel 227
10.3 Approximate identities on $\mathbb{R}$ 229
10.4 Uniform convergence and pointwise convergence 232
10.5 Weak* convergence of measures 238
10.6 Convergence in norm 241
10.7 Weak* convergence of bounded functions 243

11 Harmonic functions in the upper half plane **247**
11.1 Hardy spaces on $\mathbb{C}_+$ 247
11.2 Poisson representation for semidiscs 248
11.3 Poisson representation of $h(\overline{\mathbb{C}}_+)$ functions 250
11.4 Poisson representation of $h^p(\mathbb{C}_+)$ functions $(1 \le p \le \infty)$ 252
11.5 A correspondence between $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{D}}$ 253
11.6 Poisson representation of positive harmonic functions 255
11.7 Vertical limits of $h^p(\mathbb{C}_+)$ functions $(1 \le p \le \infty)$ 258

12 The Plancherel transform **263**
12.1 The inversion formula 263
12.2 The Fourier–Plancherel transform 266
12.3 The multiplication formula on $L^p(\mathbb{R})$ $(1 \le p \le 2)$ 271
12.4 The Fourier transform on $L^p(\mathbb{R})$ $(1 \le p \le 2)$ 273
12.5 An application of the multiplication formula on $L^p(\mathbb{R})$ $(1 \le p \le 2)$ 274
12.6 A complete characterization of $h^p(\mathbb{C}_+)$ spaces 276

13 Analytic functions in the upper half plane **279**
13.1 Representation of $H^p(\mathbb{C}_+)$ functions $(1 < p \le \infty)$ 279
13.2 Analytic measures on $\mathbb{R}$ 284
13.3 Representation of $H^1(\mathbb{C}_+)$ functions 286
13.4 Spectral analysis of $H^p(\mathbb{R})$ $(1 \le p \le 2)$ 287
13.5 A contraction from $H^p(\mathbb{C}_+)$ into $H^p(\mathbb{D})$ 289
13.6 Blaschke products for the upper half plane 293
13.7 The canonical factorization in $H^p(\mathbb{C}_+)$ $(0 < p \le \infty)$ 294
13.8 A correspondence between $H^p(\mathbb{C}_+)$ and $H^p(\mathbb{D})$ 298

14 The Hilbert transform on $\mathbb{R}$ **301**
14.1 Various definitions of the Hilbert transform 301
14.2 The Hilbert transform of $\mathcal{C}_c^1(\mathbb{R})$ functions 303
14.3 Almost everywhere existence of the Hilbert transform 305
14.4 Kolmogorov's theorem 308
14.5 M. Riesz's theorem 311
14.6 The Hilbert transform of $\mathrm{Lip}_{\alpha(t)}$ functions 321
14.7 Maximal functions 329
14.8 The maximal Hilbert transform 336

A Topics from real analysis **339**
A.1 A very concise treatment of measure theory . . . 339
A.2 Riesz representation theorems . . . 344
A.3 Weak* convergence of measures . . . 345
A.4 $\mathcal{C}(\mathbb{T})$ is dense in $L^p(\mathbb{T})$ $(0 < p < \infty)$ . . . 346
A.5 The distribution function . . . 347
A.6 Minkowski's inequality . . . 348
A.7 Jensen's inequality . . . 349

B A panoramic view of the representation theorems **351**
B.1 $h^p(\mathbb{D})$ . . . 352
B.1.1 $h^1(\mathbb{D})$ . . . 352
B.1.2 $h^p(\mathbb{D})$ $(1 < p < \infty)$ . . . 354
B.1.3 $h^\infty(\mathbb{D})$ . . . 355
B.2 $H^p(\mathbb{D})$ . . . 356
B.2.1 $H^p(\mathbb{D})$ $(1 \le p < \infty)$ . . . 356
B.2.2 $H^\infty(\mathbb{D})$ . . . 358
B.3 $h^p(\mathbb{C}_+)$ . . . 359
B.3.1 $h^1(\mathbb{C}_+)$ . . . 359
B.3.2 $h^p(\mathbb{C}_+)$ $(1 < p \le 2)$ . . . 361
B.3.3 $h^p(\mathbb{C}_+)$ $(2 < p < \infty)$ . . . 362
B.3.4 $h^\infty(\mathbb{C}_+)$ . . . 363
B.3.5 $h^+(\mathbb{C}_+)$ . . . 363
B.4 $H^p(\mathbb{C}_+)$ . . . 364
B.4.1 $H^p(\mathbb{C}_+)$ $(1 \le p \le 2)$ . . . 364
B.4.2 $H^p(\mathbb{C}_+)$ $(2 < p < \infty)$ . . . 365
B.4.3 $H^\infty(\mathbb{C}_+)$ . . . 366

Bibliography **367**

Index **369**

Preface

In 1915 Godfrey Harold Hardy, in a famous paper published in the *Proceedings of the London Mathematical Society*, answered in the affirmative a question of Landau [7]. In this paper, not only did Hardy generalize Hadamard's three-circle theorem, but he also put in place the first brick of a new branch of mathematics which bears his name: the theory of Hardy spaces. For three decades afterwards Hardy, alone or with others, wrote many more research articles on this subject.

The theory of Hardy spaces has close connections to many branches of mathematics, including Fourier analysis, harmonic analysis, singular integrals, probability theory and operator theory, and has found essential applications in robust control engineering. I have had the opportunity to give several courses on Hardy spaces and some related topics. A part of these lectures concerned the various representations of harmonic or analytic functions in the open unit disc or in the upper half plane. This topic naturally leads to the representation theorems in Hardy spaces.

There are excellent books [5, 10, 13] and numerous research articles on Hardy spaces. Our main concern here is only to treat the representation theorems. Other subjects are not discussed and the reader should consult the classical textbooks. A rather complete description of representation theorems of $H^p(\mathbb{D})$, the family of Hardy spaces of the open unit disc, is usually given in all books. To study the corresponding theorems for $H^p(\mathbb{C}_+)$, the family of Hardy spaces of the upper half plane, a good amount of Fourier analysis is required. As a consequence, representation theorems for the upper half plane are not discussed thoroughly in textbooks mainly devoted to Hardy spaces. Moreover, quite often it is mentioned that they can be derived by a conformal mapping from the corresponding theorems on the open unit disc. This is a useful technique in certain cases and we will also apply it at least on one occasion. However, in the present text, our main goal is to give a complete description of the representation theorems with *direct proofs* for both classes of Hardy spaces. Hence, certain topics from Fourier analysis have also been discussed. But this is not a book about Fourier analysis, and we have been content with the minimum required to obtain the representation theorems. For further studies on Fourier analysis many interesting references are available, e.g. [1, 8, 9, 15, 21].

I express my appreciation to the many colleagues and students who made valuable comments and improved the quality of this book. I deeply thank Colin Graham and Mostafa Nasri, who read the entire manuscript and offered several suggestions and Masood Jahanmir, who drew all the figures. I am also grateful to Roger Astley of the Cambridge University Press for his great management and kind help during the publishing procedure.

I have benefited from various lectures by Arsalan Chademan, Galia Dafni, Paul Gauthier, Kohur GowriSankaran, Victor Havin, Ivo Klemes and Paul Koosis on harmonic analysis, potential theory and the theory of Hardy spaces. As a matter of fact, the first draft of this manuscript dates back to 1991, when I attended Dr Chademan's lectures on the theory of H^p spaces. I take this opportunity to thank them with all my heart.

Thanks to the generous support of Kristian Seip, I visited the Norwegian University of Science and Technology (NTNU) in the fall semester of 2007–2008. During this period, I was able to concentrate fully on the manuscript and prepare it for final submission to the Cambridge University Press. I am sincerely grateful to Kristian, Yurii Lyubarski and Eugenia Malinnikova for their warm hospitality in Trondheim.

I owe profound thanks to my friends at McGill University, Université Laval and Université Claude Bernard Lyon 1 for their constant support and encouragement. In particular, Niky Kamran and Kohur GowriSankaran have played a major role in establishing my mathematical life, Thomas Ransford helped me enormously in the early stages of my career, and Emmanuel Fricain sends me his precious emails on a daily basis. The trace of their efforts is visible in every single page of this book.

Québec
August 2008

Chapter 1

Fourier series

1.1 The Laplacian

An open connected subset of the complex plane $\mathbb{C}$ is called a *domain*. In particular, $\mathbb{C}$ itself is a domain. But, for our discussion, we are interested in two special domains: the open *unit disc*

$$\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$$

whose boundary is the *unit circle*

$$\mathbb{T} = \{ \zeta \in \mathbb{C} : |\zeta| = 1 \}$$

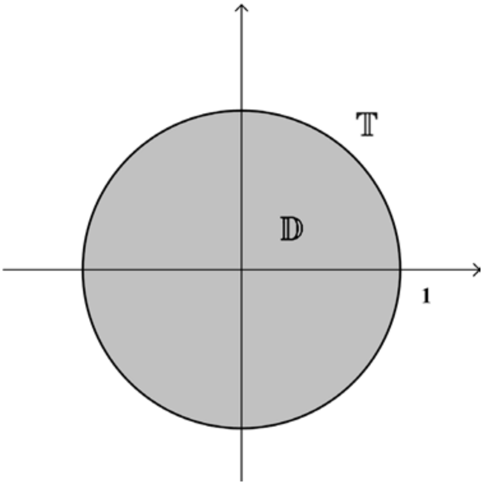

Fig. 1.1. The open unit disc $\mathbb{D}$ and its boundary $\mathbb{T}$.

and the *upper half plane*

$$\mathbb{C}_+ = \{\, z \in \mathbb{C} \,:\, \Im z > 0 \,\}$$

whose boundary is the *real line* $\mathbb{R}$ (see Figures 1.1 and 1.2). They are essential domains in studying the theory of Hardy spaces.

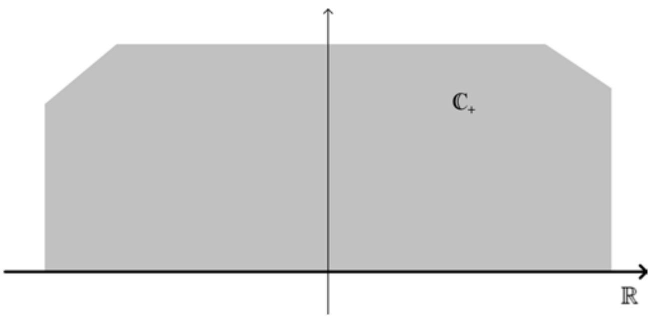

Fig. 1.2. The upper half plane $\mathbb{C}_+$ and its boundary $\mathbb{R}$.

The notations

$$\begin{aligned} D(a,r) &= \{\, z \in \mathbb{C} \,:\, |z-a| < r \,\}, \\ \overline{D(a,r)} &= \{\, z \in \mathbb{C} \,:\, |z-a| \le r \,\}, \\ \partial D(a,r) &= \{\, z \in \mathbb{C} \,:\, |z-a| = r \,\} \end{aligned}$$

for the open or closed discs and their boundaries will be used frequently too. We will also use D_r for a disc whose center is the origin, with radius r.

The *Laplacian* of a twice continuously differentiable function $U : \Omega \longrightarrow \mathbb{C}$ is defined by

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}.$$

If $0 \notin \Omega$ and we use polar coordinates, then the Laplacian becomes

$$\nabla^2 U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}.$$

We say that U is *harmonic* on Ω if it satisfies the *Laplace equation*

$$\nabla^2 U = 0 \tag{1.1}$$

at every point of Ω. By direct verification, we see that

$$U(re^{i\theta}) = r^n \cos(n\theta), \qquad (n \ge 0),$$

and

$$U(re^{i\theta}) = r^n \sin(n\theta), \qquad (n \geq 1),$$

are real harmonic functions on $\mathbb{C}$. Since (1.1) is a linear equation, a complex-valued function is harmonic if and only if its real and imaginary parts are real harmonic functions. The complex version of the preceding family of real harmonic functions is

$$U(re^{i\theta}) = r^{|n|}\, e^{in\theta}, \qquad (n \in \mathbb{Z}).$$

A special role is played by the constant function $U \equiv 1$ since it is the only member of the family whose integral means

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} U(re^{i\theta})\, d\theta$$

are not zero. This fact is a direct consequence of the elementary identity

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\theta}\, d\theta = \begin{cases} 1 & \text{if} \quad n = 0, \\ \\ 0 & \text{if} \quad n \neq 0, \end{cases} \tag{1.2}$$

which will be used frequently throughout the text.

Let F be analytic on a domain Ω and let U and V represent respectively its real and imaginary parts. Then U and V are infinitely continuously differentiable and satisfy the *Cauchy–Riemann equations*

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \qquad \text{and} \qquad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}. \tag{1.3}$$

Using these equations, it is straightforward to see that U and V are real harmonic functions. Hence an analytic function is a complex harmonic function. In the following we will define certain classes of harmonic functions, and in each class there is a subclass containing only analytic elements. Therefore, any representation formula for members of the larger class will automatically be valid for the corresponding subclass of analytic functions.

Exercises

Exercise 1.1.1 Let F be analytic on Ω and let $U = \Re F$ and $V = \Im F$. Show that U and V are real harmonic functions on Ω.
Hint: Use (1.3).

Exercise 1.1.2 Let $F : \Omega \longmapsto \mathbb{C}$ be analytic on Ω, and suppose that $F(z) \neq 0$ for all $z \in \Omega$. Show that $\log |F|$ is harmonic on Ω.

Exercise 1.1.3 Let U be harmonic on the annular domain

$$A(R_1, R_2) = \{\, z \in \mathbb{C} \,:\, 0 \le R_1 < |z| < R_2 \le \infty \,\}.$$

Show that $U(\frac{1}{r}\, e^{i\theta})$ is harmonic on the annular domain $A(\frac{1}{R_2}, \frac{1}{R_1})$.

Exercise 1.1.4 Let $F : \Omega_1 \longmapsto \Omega_2$ be analytic on Ω_1, and let $U : \Omega_2 \longmapsto \mathbb{C}$ be harmonic on Ω_2. Show that $U \circ F : \Omega_1 \longmapsto \mathbb{C}$ is harmonic on Ω_1.

Exercise 1.1.5 Define the differential operators

$$\begin{aligned} \partial &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right), \\ \bar{\partial} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right). \end{aligned}$$

Show that

$$\nabla^2 = 4\partial\bar{\partial}.$$

Exercise 1.1.6 Let $F = U + iV$ be analytic. Show that the Cauchy–Riemann equations (1.3) are equivalent to the equation

$$\bar{\partial} F = 0.$$

Remark: We also have $F' = \partial F$.

Exercise 1.1.7 Let U_1 and U_2 be real harmonic functions on a domain Ω. Under what conditions is $U_1\, U_2$ also harmonic on Ω?
Remark: We emphasize that U_1 and U_2 are *real* harmonic functions. The answer to this question changes dramatically if we consider complex harmonic functions. For example, if F_1 and F_2 are analytic functions, then, under no extra condition, $F_1\, F_2$ is analytic. Note that an analytic function is certainly harmonic.

Exercise 1.1.8 Let U be a real harmonic function on a domain Ω. Suppose that U^2 is also harmonic on Ω. Show that U is constant.
Hint: Use Exercise 1.1.7.

Exercise 1.1.9 Let F be analytic on a domain Ω, and let Φ be a twice continuously differentiable function on the range of F. Show that

$$\nabla^2 (\Phi \circ F) = \big(\, (\nabla^2 \Phi) \circ F \,\big)\, |F'|^2.$$

Exercise 1.1.10 Let F be analytic on a domain Ω, and let $\alpha \in \mathbb{R}$. Suppose that F has no zeros on Ω. Show that

$$\nabla^2(|F|^\alpha) = \alpha^2\, |F|^{\alpha-2}\, |F'|^2.$$

Hint: Apply Exercise 1.1.9 with $\Phi(z) = |z|^\alpha$.

Exercise 1.1.11 Let F be analytic on a domain Ω. Under what conditions is $|F|^2$ harmonic on Ω?
Hint: Apply Exercise 1.1.10.

Exercise 1.1.12 Let F be a complex function on a domain Ω such that F and F^2 are both harmonic on Ω. Show that either F or $\bar{F}$ is analytic on Ω.
Hint: Use Exercise 1.1.7.

1.2 Some function spaces and sequence spaces

Let f be a measurable function on $\mathbb{T}$, and let

$$\|f\|_p = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{it})|^p \, dt \right)^{\frac{1}{p}}, \qquad (0 < p < \infty),$$

and

$$\|f\|_\infty = \inf_{M>0} \big\{\, M \,:\, |\{\, e^{it} \,:\, |f(e^{it})| > M \,\}| = 0 \,\big\},$$

where $|E|$ denotes the Lebesgue measure of the set E. Then *Lebesgue spaces* $L^p(\mathbb{T})$, $0 < p \le \infty$, are defined by

$$L^p(\mathbb{T}) = \{\, f \,:\, \|f\|_p < \infty \,\}.$$

If $1 \le p \le \infty$, then $L^p(\mathbb{T})$ is a Banach space. In particular, $L^2(\mathbb{T})$, equipped with the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it})\, \overline{g(e^{it})} \, dt,$$

is a Hilbert space. It is easy to see that

$$L^\infty(\mathbb{T}) \subset L^p(\mathbb{T}) \subset L^1(\mathbb{T})$$

for each $p \in (1, \infty)$. In the following, we will mostly study $L^p(\mathbb{T})$, $1 \le p \le \infty$, and their subclasses and thus $L^1(\mathbb{T})$ is the largest function space that enters our discussion. This simple fact has important consequences. For example, we will define the Fourier coefficients of functions in $L^1(\mathbb{T})$, and thus the Fourier coefficients of elements of $L^p(\mathbb{T})$, for all $1 \le p \le \infty$, are automatically defined too. We will appreciate this fact when we study the Fourier transform on the real line. Spaces $L^p(\mathbb{R})$ do not form a chain, as is the case on the unit circle, and thus after defining the Fourier transform on $L^1(\mathbb{R})$, we need to take further steps in order to define the Fourier transform for some other $L^p(\mathbb{R})$ spaces.

A continuous function on $\mathbb{T}$, a compact set, is necessarily bounded. The space of all continuous functions on the unit circle $\mathcal{C}(\mathbb{T})$ can be considered as a subspace of $L^\infty(\mathbb{T})$. As a matter of fact, in this case the maximum is attained and we have

$$\|f\|_\infty = \max_{e^{it}\in\mathbb{T}} |f(e^{it})|.$$

On some occasions we also need the smaller subspace $\mathcal{C}^n(\mathbb{T})$ consisting of all n times continuously differentiable functions, or even their intersection $\mathcal{C}^\infty(\mathbb{T})$ consisting of functions having derivatives of all orders. The space $\mathcal{C}^n(\mathbb{T})$ is equipped with the norm

$$\|f\|_{\mathcal{C}^n(\mathbb{T})} = \sum_{k=1}^{n} \frac{\|f^{(k)}\|_\infty}{k!}.$$

Lipschitz classes form another subfamily of $\mathcal{C}(\mathbb{T})$. Fix $\alpha \in (0,1]$. Then $\mathrm{Lip}_\alpha(\mathbb{T})$ consists of all $f \in \mathcal{C}(\mathbb{T})$ such that

$$\sup_{\substack{t,\tau\in\mathbb{R}\\ \tau\neq 0}} \frac{|f(e^{i(t+\tau)}) - f(e^{it})|}{|\tau|^\alpha} < \infty.$$

This space is equipped with the norm

$$\|f\|_{\mathrm{Lip}_\alpha(\mathbb{T})} = \|f\|_\infty + \sup_{\substack{t,\tau\in\mathbb{R}\\ \tau\neq 0}} \frac{|f(e^{i(t+\tau)}) - f(e^{it})|}{|\tau|^\alpha}.$$

The space of all complex *Borel measures* on $\mathbb{T}$ is denoted by $\mathcal{M}(\mathbb{T})$. This space equipped with the norm

$$\|\mu\| = |\mu|(\mathbb{T}),$$

where $|\mu|$ denotes the total variation of μ, is a Banach space. Remember that the total variation $|\mu|$ is the smallest positive Borel measure satisfying

$$|\mu(E)| \le |\mu|(E)$$

for all Borel sets $E \subset \mathbb{T}$. To each function $f \in L^1(\mathbb{T})$ there corresponds a Borel measure

$$d\mu(e^{it}) = \frac{1}{2\pi}\, f(e^{it})\, dt.$$

Clearly we have $\|\mu\| = \|f\|_1$, and thus the map

$$\begin{array}{ccc} L^1(\mathbb{T}) & \longrightarrow & \mathcal{M}(\mathbb{T}) \\ f & \longmapsto & f(e^{it})\, dt/2\pi \end{array}$$

is an embedding of $L^1(\mathbb{T})$ into $\mathcal{M}(\mathbb{T})$.

In our discussion, we also need some *sequence spaces*. For a sequence of complex numbers $\mathcal{Z} = (z_n)_{n\in\mathbb{Z}}$, let

$$\|\mathcal{Z}\|_p = \left(\sum_{n=-\infty}^{\infty} |z_n|^p \right)^{\frac{1}{p}}, \qquad (0 < p < \infty),$$

and

$$\|\mathcal{Z}\|_\infty = \sup_{n\in\mathbb{Z}} |z_n|.$$

Then, for $0 < p \le \infty$, we define

$$\ell^p(\mathbb{Z}) = \{\, \mathcal{Z} \,:\, \|\mathcal{Z}\|_p < \infty \,\}$$

and

$$c_0(\mathbb{Z}) = \{\, \mathcal{Z} \in \ell^\infty(\mathbb{Z}) \,:\, \lim_{|n|\to\infty} |z_n| = 0 \,\}.$$

If $1 \le p \le \infty$, then $\ell^p(\mathbb{Z})$ is a Banach space and $c_0(\mathbb{Z})$ is a closed subspace of $\ell^\infty(\mathbb{Z})$. The space $\ell^2(\mathbb{Z})$, equipped with the inner product

$$\langle \mathcal{Z}, \mathcal{W} \rangle = \sum_{n=-\infty}^{\infty} z_n \, \overline{w}_n,$$

is a Hilbert space. The subspaces

$$\ell^p(\mathbb{Z}^+) = \{\, \mathcal{Z} \in \ell^p(\mathbb{Z}) \,:\, z_{-n} = 0,\ n \ge 1 \,\}$$

and

$$c_0(\mathbb{Z}^+) = \{\, \mathcal{Z} \in c_0(\mathbb{Z}) \,:\, z_{-n} = 0,\ n \ge 1 \,\}$$

will also appear when we study the Fourier transform of certain subclasses of $L^p(\mathbb{T})$.

Exercises

Exercise 1.2.1 Let f be a measurable function on $\mathbb{T}$, and let

$$f_\tau(e^{it}) = f(e^{i(t-\tau)}).$$

Show that

$$\lim_{\tau\to 0} \|f_\tau - f\|_X = 0$$

if $X = L^p(\mathbb{T})$, $1 \le p < \infty$, or $X = \mathcal{C}^n(\mathbb{T})$. Provide examples to show that this property does not hold if $X = L^\infty(\mathbb{T})$ or $X = \mathrm{Lip}_\alpha(\mathbb{T})$, $0 < \alpha \le 1$. However, show that

$$\|f_\tau\|_X = \|f\|_X$$

in all function spaces mentioned above.

Exercise 1.2.2 Show that $\ell^p(\mathbb{Z}^+)$, $0 < p \le \infty$, is closed in $\ell^p(\mathbb{Z})$.

Exercise 1.2.3 Show that $c_0(\mathbb{Z})$ is closed in $\ell^\infty(\mathbb{Z})$.

Exercise 1.2.4 Show that $c_0(\mathbb{Z}^+)$ is closed in $c_0(\mathbb{Z})$.

Exercise 1.2.5 Let $c_c(\mathbb{Z})$ denote the family of sequences of compact support, i.e. for each $\mathcal{Z} = (z_n)_{n\in\mathbb{Z}} \in c_c(\mathbb{Z})$ there is $N = N(\mathcal{Z})$ such that $z_n = 0$ for all $|n| \ge N$. Show that $c_c(\mathbb{Z})$ is dense in $c_0(\mathbb{Z})$.

1.3 Fourier coefficients

Let $f \in L^1(\mathbb{T})$. Then the nth *Fourier coefficient* of f is defined by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\, e^{-int}\, dt, \qquad (n \in \mathbb{Z}).$$

The two-sided sequence $\hat{f} = (\,\hat{f}(n)\,)_{n\in\mathbb{Z}}$ is called the *Fourier transform* of f. We clearly have

$$|\hat{f}(n)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{it})|\, dt, \qquad (n \in \mathbb{Z}),$$

which can be rewritten as $\hat{f} \in \ell^\infty(\mathbb{Z})$ with

$$\|\hat{f}\|_\infty \le \|f\|_1.$$

Therefore, the Fourier transform

$$\begin{array}{ccc} L^1(\mathbb{T}) & \longrightarrow & \ell^\infty(\mathbb{Z}) \\ f & \longmapsto & \hat{f} \end{array}$$

is a linear map whose norm is at most one. The constant function shows that the norm is indeed equal to one. We will show that this map is one-to-one and its range is included in $c_0(\mathbb{Z})$. But first, we need to develop some techniques.

The *Fourier series* of f is formally written as

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{int}.$$

The central question in Fourier analysis is to determine *when, how* and *toward what* this series converges. We will partially address these questions in the following. Any formal series of the form

$$\sum_{n=-\infty}^{\infty} a_n\, e^{int}$$

is called a *trigonometric series*. Hence a Fourier series is a special type of trigonometric series. However, there are trigonometric series which are not Fourier series. In other words, the coefficients a_n are not the Fourier coefficients of any integrable function. Using Euler's identity

$$e^{int} = \cos(nt) + i\sin(nt),$$

a trigonometric series can be rewritten as

$$\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos(nt) + \beta_n \sin(nt).$$

It is easy to find the relation between α_n, β_n and a_n.

An important example which plays a central role in the theory of harmonic functions is the *Poisson kernel*

$$P_r(e^{it}) = \frac{1-r^2}{1+r^2-2r\cos t}, \qquad (0 \le r < 1). \tag{1.4}$$

(See Figure 1.3.)

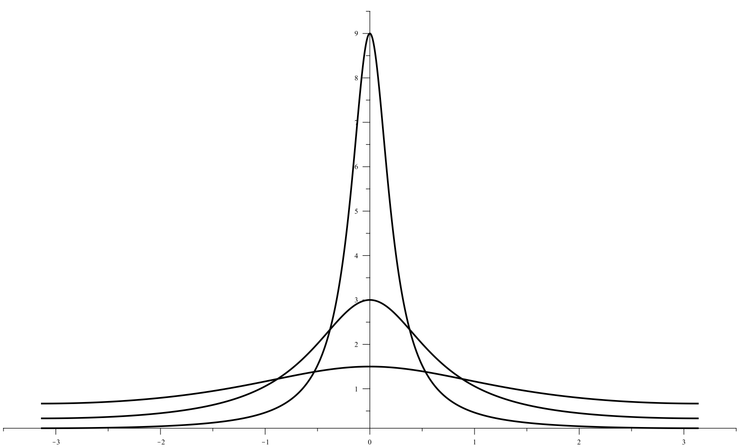

Fig. 1.3. The Poisson kernel $P_r(e^{it})$ for $r = 0.2,\ 0.5,\ 0.8$.

Clearly, for each fixed $0 \le r < 1$,

$$P_r \in \mathcal{C}^\infty(\mathbb{T}) \subset L^1(\mathbb{T}).$$

Direct computation of $\widehat{P}_r$ is somehow difficult. But, the following observation makes its calculation easier. We have

$$\begin{aligned}\frac{1-r^2}{1+r^2-2r\cos t} &= \frac{1-r^2}{1+r^2-r(e^{it}+e^{-it})}\\ &= \frac{1-r^2}{(1-re^{it})(1-re^{-it})}\\ &= \frac{1}{1-re^{it}}+\frac{1}{1-re^{-it}}-1\end{aligned}$$

and thus, using the geometric series

$$1+w+w^2+\cdots=\frac{1}{1-w}, \qquad (|w|<1),$$

we obtain

$$P_r(e^{it})=\sum_{n=-\infty}^{\infty} r^{|n|}\,e^{int}. \tag{1.5}$$

Moreover, for each fixed $r<1$, the partial sums are *uniformly* convergent to P_r. The uniform convergence is the key to this shortcut method. Therefore, for each $n\in\mathbb{Z}$,

$$\begin{aligned}\widehat{P_r}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(e^{it})\,e^{-int}\,dt\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{m=-\infty}^{\infty} r^{|m|}\,e^{imt}\right)e^{-int}\,dt\\ &= \sum_{m=-\infty}^{\infty} r^{|m|}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-n)t}\,dt\right)\\ &= r^{|n|}. \end{aligned}\tag{1.6}$$

(See Figure 1.4.)

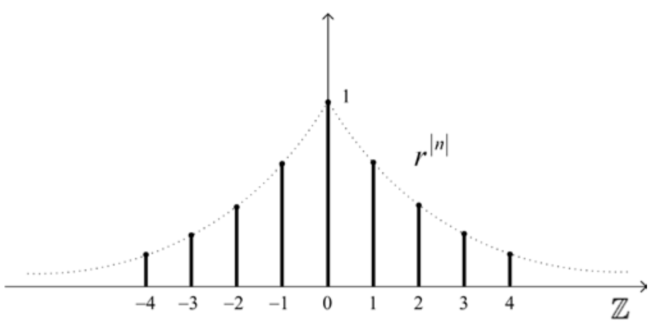

Fig. 1.4. The spectrum of P_r.

We emphasize that the uniform convergence of the series enables us to change the order of $\sum$ and $\int$ in the third equality above. This phenomenon will appear frequently in our discussion. The identity (1.5) also shows that P_r is equal to its Fourier series at all points of $\mathbb{T}$.

It is rather easy to extend the definition of Fourier transform for Borel measures on $\mathbb{T}$. Let $\mu \in \mathcal{M}(\mathbb{T})$. The nth Fourier coefficient of μ is defined by

$$\hat{\mu}(n) = \int_{\mathbb{T}} e^{-int} \, d\mu(e^{it}), \qquad (n \in \mathbb{Z}),$$

and the Fourier transform of μ is the two-sided sequence $\hat{\mu} = (\hat{\mu}(n))_{n\in\mathbb{Z}}$. Considering the embedding

$$\begin{array}{ccc} L^1(\mathbb{T}) & \longrightarrow & \mathcal{M}(\mathbb{T}) \\ f & \longmapsto & f(e^{it})\, dt/2\pi, \end{array}$$

if we think of $L^1(\mathbb{T})$ as a subspace of $\mathcal{M}(\mathbb{T})$, it is easy to see that the two definitions of Fourier coefficients are consistent. In other words, if

$$d\mu(e^{it}) = \frac{1}{2\pi} \, f(e^{it})\, dt,$$

where $f \in L^1(\mathbb{T})$, then we have

$$\hat{\mu}(n) = \hat{f}(n), \qquad (n \in \mathbb{Z}).$$

Lemma 1.1 *Let $\mu \in \mathcal{M}(\mathbb{T})$. Then $\hat{\mu} \in \ell^\infty(\mathbb{Z})$ and*

$$\| \hat{\mu} \|_\infty \le \| \mu \|.$$

In particular, for each $f \in L^1(\mathbb{T})$,

$$\| \hat{f} \|_\infty \le \| f \|_1.$$

Proof. For each $n \in \mathbb{Z}$, we have

$$\begin{aligned} |\hat{\mu}(n)| &= \left| \int_{\mathbb{T}} e^{-int} \, d\mu(e^{it}) \right| \\ &\le \int_{\mathbb{T}} |e^{-int}| \, d|\mu|(e^{it}) \\ &= \int_{\mathbb{T}} d|\mu|(e^{it}) = |\mu|(\mathbb{T}) = \|\mu\|. \end{aligned}$$

The second inequality is a special case of the first one with $d\mu(e^{it}) = f(e^{it})\, dt/2\pi$. In this case, $\hat{\mu}(n) = \hat{f}(n)$ and $\|\mu\| = \|f\|_1$. It was also proved directly at the beginning of this section. □

Based on the preceding lemma, the Fourier transform

$$\begin{array}{ccc} \mathcal{M}(\mathbb{T}) & \longrightarrow & \ell^\infty(\mathbb{Z}) \\ \mu & \longmapsto & \hat{\mu} \end{array}$$

is a linear map whose norm is at most one. The Dirac measure δ_1 shows that the norm is actually equal to one. We will show that this map is also one-to-one. However, since

$$\hat{\delta}_1(n) = 1, \qquad (n \in \mathbb{Z}),$$

its range is not included in $c_0(\mathbb{Z})$.

Exercises

Exercise 1.3.1 Let $f \in L^p(\mathbb{T})$, $1 \le p \le \infty$. Show that $\| \hat{f} \|_\infty \le \| f \|_p$.

Exercise 1.3.2 Let $z = re^{i\theta} \in \mathbb{D}$. Show that

$$P_r(e^{i(\theta - t)}) = \Re\left(\frac{e^{it} + z}{e^{it} - z} \right).$$

Exercise 1.3.3 Let $f \in L^1(\mathbb{T})$. Define

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \, f(e^{it}) \, dt, \qquad (z \in \mathbb{D}).$$

Show that

$$F(z) = \hat{f}(0) + 2 \sum_{n=1}^{\infty} \hat{f}(n) \, z^n, \qquad (z \in \mathbb{D}).$$

Hint: Note that

$$\frac{e^{it} + z}{e^{it} - z} = 1 + 2 \sum_{n=1}^{\infty} z^n \, e^{-int}.$$

Exercise 1.3.4 Let $f \in L^1(\mathbb{T})$ and define

$$g(e^{it}) = f(e^{i2t}).$$

Show that

$$\hat{g}(n) = \begin{cases} \hat{f}(\frac{n}{2}) & \text{if} \quad 2 | n, \\ 0 & \text{if} \quad 2 \not| n. \end{cases}$$

Consider a similar question if we define

$$g(e^{it}) = f(e^{ikt}),$$

where k is a fixed positive integer.

Exercise 1.3.5 Let $f \in \mathrm{Lip}_\alpha(\mathbb{T})$, $0 < \alpha \le 1$. Show that

$$\hat{f}(n) = O(1/n^\alpha),$$

as $|n| \to \infty$.
Hint: If $n \neq 0$, we have

$$\hat{f}(n) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \big(f(e^{it}) - f(e^{i(t+\pi/n)}) \big) \, e^{-int} \, dt.$$

1.4 Convolution on $\mathbb{T}$

Let $f, g \in L^1(\mathbb{T})$. Then we cannot conclude that $fg \in L^1(\mathbb{T})$. Indeed, it is easy to manufacture an example such that

$$\int_{-\pi}^{\pi} |f(e^{it}) \, g(e^{it})| \, dt = \infty.$$

Nevertheless, by Fubini's theorem,

$$\begin{aligned} &\int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} |f(e^{i\tau}) \; g(e^{i(t-\tau)})| \, d\tau \right) dt \\ = \quad &\int_{-\pi}^{\pi} |f(e^{i\tau})| \left(\int_{-\pi}^{\pi} |g(e^{i(t-\tau)})| \, dt \right) d\tau \\ = \quad &\left(\int_{-\pi}^{\pi} |f(e^{i\tau})| \; d\tau \right) \left(\int_{-\pi}^{\pi} |g(e^{is})| \, ds \right) < \infty. \end{aligned}$$

Therefore, we necessarily have

$$\int_{-\pi}^{\pi} |f(e^{i\tau}) \; g(e^{i(t-\tau)})| \, d\tau < \infty$$

for almost all $e^{it} \in \mathbb{T}$. This observation enables us to define

$$(f * g)(e^{it}) = \int_{-\pi}^{\pi} f(e^{i\tau}) \; g(e^{i(t-\tau)}) \; d\tau$$

for almost all $e^{it} \in \mathbb{T}$, and besides the previous calculation shows that

$$f * g \in L^1(\mathbb{T})$$

with

$$\|f * g\|_1 \le \|f\|_1 \; \|g\|_1. \tag{1.7}$$

The function $f * g$ is called the *convolution* of f and g. It is straightforward to see that the convolution is

(i) commutative: $f * g = g * f$,

(ii) associative: $f * (g * h) = (f * g) * h$,

(iii) distributive: $f * (g + h) = f * g + f * h$,

(iv) homogenous: $f * (\alpha g) = (\alpha f) * g = \alpha(f * g)$,

for all $f, g, h \in L^1(\mathbb{T})$ and $\alpha \in \mathbb{C}$. In technical terms, $L^1(\mathbb{T})$, equipped with the convolution as its product, is a *Banach algebra*. As a matter of fact, this concrete example inspired most of the abstract theory of Banach algebras.

Let $f, g \in L^1(\mathbb{T})$ and let $\varphi \in \mathcal{C}(\mathbb{T})$. Then, by Fubini's theorem,

$$\begin{aligned} \int_{\mathbb{T}} \varphi(e^{is})\, (f * g)(e^{is})\, \frac{ds}{2\pi} &= \int_{\mathbb{T}} \int_{\mathbb{T}} \varphi(e^{is}) \left(f(e^{i(s-\tau)})\, g(e^{i\tau})\, \frac{d\tau}{2\pi} \right) \frac{ds}{2\pi} \\ &= \int_{\mathbb{T}} \int_{\mathbb{T}} \varphi(e^{i(t+\tau)}) \left(f(e^{it})\, \frac{dt}{2\pi} \right) \left(g(e^{i\tau})\, \frac{d\tau}{2\pi} \right). \end{aligned}$$

This fact enables us to define the convolution of two Borel measures on $\mathbb{T}$ such that if we consider $L^1(\mathbb{T})$ as a subset of $\mathcal{M}(\mathbb{T})$, the two definitions are consistent. Let $\mu, \nu \in \mathcal{M}(\mathbb{T})$, and define $\Lambda : \mathcal{C}(\mathbb{T}) \longrightarrow \mathbb{C}$ by

$$\Lambda(\varphi) = \int_{\mathbb{T}} \int_{\mathbb{T}} \varphi(e^{i(t+\tau)})\, d\mu(e^{it})\, d\nu(e^{i\tau}), \qquad (\varphi \in \mathcal{C}(\mathbb{T})).$$

The functional Λ is clearly linear and satisfies

$$|\,\Lambda(\varphi)\,| \le \|\mu\|\; \|\nu\|\; \|\varphi\|_\infty, \qquad (\varphi \in \mathcal{C}(\mathbb{T})),$$

which implies

$$\|\,\Lambda\,\| \le \|\mu\|\; \|\nu\|.$$

Therefore, by the Riesz representation theorem for bounded linear functionals on $\mathcal{C}(\mathbb{T})$, there exists a unique Borel measure, which we denote by $\mu * \nu$ and call the convolution of μ and ν, such that

$$\Lambda(\varphi) = \int_{\mathbb{T}} \varphi(e^{it})\; d(\mu * \nu)(e^{it}), \qquad (\varphi \in \mathcal{C}(\mathbb{T})),$$

and moreover,

$$\|\Lambda\| = \|\mu * \nu\|.$$

Hence, $\mu * \nu$ is defined such that

$$\int_{\mathbb{T}} \varphi(e^{it})\; d(\mu * \nu)(e^{it}) = \int_{\mathbb{T}} \int_{\mathbb{T}} \varphi(e^{i(t+\tau)})\; d\mu(e^{it})\, d\nu(e^{i\tau}), \tag{1.8}$$

for all $\varphi \in \mathcal{C}(\mathbb{T})$, and it satisfies

$$\|\,\mu * \nu\,\| \le \|\mu\|\; \|\nu\|. \tag{1.9}$$

The following result is easy to prove. Nevertheless, it is the most fundamental connection between convolution and the Fourier transform. Roughly speaking, it says that the Fourier transform changes convolution to multiplication.

Theorem 1.2 *Let μ, $\nu \in \mathcal{M}(\mathbb{T})$. Then*

$$\widehat{\mu * \nu}(n) = \hat{\mu}(n)\,\hat{\nu}(n), \qquad (n \in \mathbb{Z}).$$

In particular, if f, $g \in L^1(\mathbb{T})$, then

$$\widehat{f * g}(n) = \hat{f}(n)\,\hat{g}(n), \qquad (n \in \mathbb{Z}).$$

Proof. Fix $n \in \mathbb{Z}$ and put

$$\varphi(e^{it}) = e^{-int}$$

in (1.8). Hence,

$$\begin{aligned}
\widehat{\mu * \nu}(n) &= \int_{\mathbb{T}} e^{-int}\, d(\mu * \nu)(e^{it}) \\
&= \int_{\mathbb{T}} \int_{\mathbb{T}} e^{-in(t+\tau)}\, d\mu(e^{it})\, d\nu(e^{i\tau}) \\
&= \int_{\mathbb{T}} e^{-int}\, d\mu(e^{it}) \times \int_{\mathbb{T}} e^{-in\tau}\, d\nu(e^{i\tau}) = \hat{\mu}(n)\,\hat{\nu}(n).
\end{aligned}$$

□

We saw that if we consider $L^1(\mathbb{T})$ as a subset of $\mathcal{M}(\mathbb{T})$, the two definitions of convolution are consistent. Therefore, $\mathcal{M}(\mathbb{T})$ contains $L^1(\mathbb{T})$ as a subalgebra. But, we can say more in this case. We show that $L^1(\mathbb{T})$ is actually an ideal in $\mathcal{M}(\mathbb{T})$.

Theorem 1.3 *Let $\mu \in \mathcal{M}(\mathbb{T})$ and let $f \in L^1(\mathbb{T})$. Let*

$$d\nu(e^{it}) = f(e^{it})\; dt.$$

Then $\mu * \nu$ is also absolutely continuous with respect to Lebesgue measure and we have

$$d(\mu * \nu)(e^{it}) = \left(\int_{\mathbb{T}} f(e^{i(t-\tau)})\, d\mu(e^{i\tau}) \right) dt.$$

Proof. According to (1.8), for each $\varphi \in \mathcal{C}(\mathbb{T})$ we have

$$\begin{aligned}
\int_{\mathbb{T}} \varphi(e^{it})\; d(\mu * \nu)(e^{it}) &= \int_{\mathbb{T}} \int_{\mathbb{T}} \varphi(e^{i(t+\tau)})\; d\mu(e^{it})\, d\nu(e^{i\tau}) \\
&= \int_{\mathbb{T}} \int_{\mathbb{T}} \varphi(e^{i(t+\tau)})\; d\mu(e^{it})\;\; f(e^{i\tau})\, d\tau \\
&= \int_{\mathbb{T}} \varphi(e^{is}) \left(\int_{\mathbb{T}} f(e^{i(s-t)})\, d\mu(e^{it}) \right) ds.
\end{aligned}$$

Therefore, by the uniqueness part of the Riesz representation theorem,

$$d(\mu * \nu)(e^{is}) = \left(\int_{\mathbb{T}} f(e^{i(s-t)})\, d\mu(e^{it}) \right) ds.$$

□

Let $\mu \in \mathcal{M}(\mathbb{T})$ and $f \in L^1(\mathbb{T})$. Considering f as a measure, by the preceding theorem, $\mu * f$ is absolutely continuous with respect to Lebesgue measure and we may write

$$(\mu * f)(e^{it}) = \int_{\mathbb{T}} f(e^{i(t-\tau)})\, d\mu(e^{i\tau}). \tag{1.10}$$

Theorem 1.3 ensures that $(\mu * f)(e^{it})$ is well-defined for almost all $e^{it} \in \mathbb{T}$, $\mu * f \in L^1(\mathbb{T})$ and, by (1.9),

$$\|\mu * f\|_1 \leq \|f\|_1 \, \|\mu\|. \tag{1.11}$$

We will need a very special case of (1.10) where $f \in \mathcal{C}(\mathbb{T})$. In this case, $(\mu * f)(e^{it})$ is defined for *all* $e^{it} \in \mathbb{T}$.

Exercises

Exercise 1.4.1 Let $\chi_n(e^{it}) = e^{int}$, $n \in \mathbb{Z}$. Show that

$$f * \chi_n = \hat{f}(n)\, \chi_n$$

for any $f \in L^1(\mathbb{T})$.

Exercise 1.4.2 Show that $\mathcal{M}(\mathbb{T})$, equipped with convolution as its product, is a commutative Banach algebra. What is its unit?

Exercise 1.4.3 Are you able to show that the Banach algebra $L^1(\mathbb{T})$ does not have a unit?
Hint: Use Theorem 1.2. Come back to this exercise after studying the Riemann–Lebesgue lemma in Section 2.5.

1.5 Young's inequality

Since $L^p(\mathbb{T}) \subset L^1(\mathbb{T})$, for $1 \leq p \leq \infty$, and since the convolution was defined on $L^1(\mathbb{T})$, then a priori $f * g$ is well-defined whenever $f \in L^r(\mathbb{T})$ and $g \in L^s(\mathbb{T})$ with $1 \leq r, s \leq \infty$. The following result gives more information about $f * g$, when we restrict f and g to some smaller subclasses of $L^1(\mathbb{T})$.

Theorem 1.4 (Young's inequality) *Let $f \in L^r(\mathbb{T})$, and let $g \in L^s(\mathbb{T})$, where $1 \leq r, s \leq \infty$ and*

$$\frac{1}{r} + \frac{1}{s} \geq 1.$$

Let

$$\frac{1}{p} = \frac{1}{r} + \frac{1}{s} - 1.$$

Then $f * g \in L^p(\mathbb{T})$ and

$$\|f * g\|_p \leq \|f\|_r \, \|g\|_s.$$

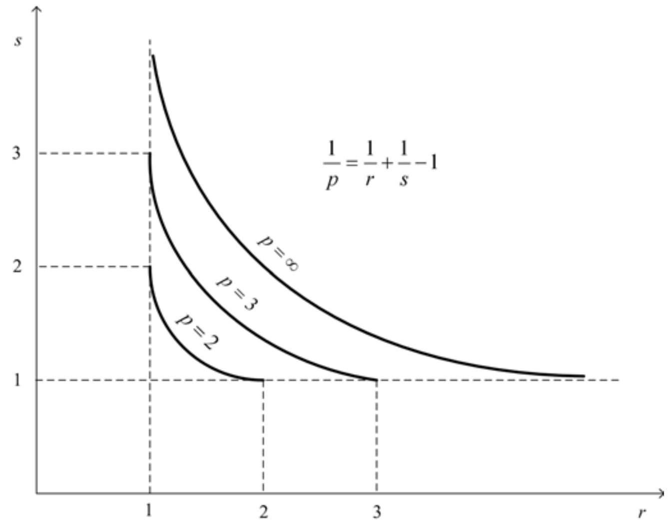

Fig. 1.5. The level curves of p.

Proof. (Figure 1.5 shows the level curves of p.) If $p = \infty$, or equivalently $1/r + 1/s = 1$, then $f * g$ is well-defined for *all* $e^{i\theta} \in \mathbb{T}$ and Young's inequality reduces to Hölder's inequality. Now, suppose that $1/r + 1/s > 1$. We need a generalized form of Hölder's inequality. Let $1 < p_1, \ldots, p_n < \infty$ such that

$$\frac{1}{p_1} + \cdots + \frac{1}{p_n} = 1,$$

and let $f_1, \ldots, f_n$ be measurable functions on a measure space $(X, \mathfrak{M}, \mu)$. Then

$$\int_X |f_1 \cdots f_n|\, d\mu \le \left(\int_X |f_1|^{p_1}\, d\mu \right)^{\frac{1}{p_1}} \cdots \left(\int_X |f_n|^{p_n}\, d\mu \right)^{\frac{1}{p_n}}.$$

This inequality can be proved by induction and the ordinary Hölder's inequality.

Let r' and s' be respectively the conjugate exponents of r and s, i.e.

$$\frac{1}{r} + \frac{1}{r'} = 1 \qquad \text{and} \qquad \frac{1}{s} + \frac{1}{s'} = 1.$$

Then, according to the definition of p, we have

$$\frac{1}{r'} + \frac{1}{s'} + \frac{1}{p} = 1.$$

Fix $e^{i\theta} \in \mathbb{T}$. To apply the generalized Hölder's inequality, we write the integrand $|f(e^{it})\ g(e^{i(\theta-t)})|$ as the product of three functions respectively in $L^{r'}(\mathbb{T})$, $L^{s'}(\mathbb{T})$

and $L^p(\mathbb{T})$. Write

$$\begin{aligned} |f(e^{i\tau})\ g(e^{i(t-\tau)})| &= \left(|g(e^{i(t-\tau)})|^{1-\frac{s}{p}} \right) \\ &\times \left(|f(e^{i\tau})|^{1-\frac{r}{p}} \right) \\ &\times \left(|f(e^{i\tau})|^{\frac{r}{p}}\ |g(e^{i(t-\tau)})|^{\frac{s}{p}} \right). \end{aligned}$$

Hence, by the generalized Hölder's inequality,

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\tau})\ g(e^{i(t-\tau)})|\ d\tau &\leq \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |g(e^{i(t-\tau)})|^{r'(1-\frac{s}{p})}\ d\tau \right)^{\frac{1}{r'}} \\ &\times \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\tau})|^{s'(1-\frac{r}{p})}\ d\tau \right)^{\frac{1}{s'}} \\ &\times \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\tau})|^{r}\ |g(e^{i(t-\tau)})|^{s}\ d\tau \right)^{\frac{1}{p}}. \end{aligned}$$

But $r'(1-s/p) = s$ and $s'(1-r/p) = r$. Thus

$$|(f*g)(e^{it})| \leq \|g\|_s^{s/r'}\ \|f\|_r^{r/s'} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\tau})|^{r}\ |g(e^{i(t-\tau)})|^{s}\ d\tau \right)^{\frac{1}{p}}$$

for almost all $e^{it} \in \mathbb{T}$. Finally, by Fubini's theorem,

$$\begin{aligned} \|f*g\|_p &= \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |(f*g)(e^{it})|^p\ dt \right)^{\frac{1}{p}} \\ &\leq \|g\|_s^{s/r'}\ \|f\|_r^{r/s'} \left(\frac{1}{(2\pi)^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(e^{i\tau})|^{r}\ |g(e^{i(t-\tau)})|^{s}\ dt\ d\tau \right)^{\frac{1}{p}} \\ &= \|g\|_s^{s/r'}\ \|f\|_r^{r/s'} \times \|g\|_s^{s/p}\ \|f\|_r^{r/p} = \|f\|_r\ \|g\|_s. \end{aligned}$$

□

Another proof of Young's inequality is based on the Riesz–Thorin interpolation theorem and will be discussed in Chapter 8. The following two special cases of Young's inequality are what we need later on.

Corollary 1.5 *Let $f \in L^p(\mathbb{T})$, $1 \leq p \leq \infty$, and let $g \in L^1(\mathbb{T})$. Then $f*g \in L^p(\mathbb{T})$, and*

$$\|f*g\|_p \leq \|f\|_p\ \|g\|_1.$$

Corollary 1.6 *Let $f \in L^p(\mathbb{T})$, and let $g \in L^q(\mathbb{T})$, where q is the conjugate exponent of p. Then $(f*g)(e^{it})$ is well-defined for all $e^{it} \in \mathbb{T}$, $f*g \in \mathcal{C}(\mathbb{T})$, and*

$$\|f*g\|_\infty \leq \|f\|_p\ \|g\|_q.$$

Proof. As we mentioned in the proof of Theorem 1.4, Hölder's inequality ensures that $(f * g)(e^{it})$ is well-defined for all $e^{it} \in \mathbb{T}$. The only new fact to prove is that $f * g$ is a continuous function on $\mathbb{T}$.

At least one of p or q is not infinity. Without loss of generality, assume that $p \neq \infty$. This assumption ensures that $\mathcal{C}(\mathbb{T})$ is dense in $L^p(\mathbb{T})$ (see Section A.4). Thus, given $\varepsilon > 0$, there is $\varphi \in \mathcal{C}(\mathbb{T})$ such that

$$\|f - \varphi\|_p < \varepsilon.$$

Hence

$$\begin{aligned} |(f * g)(e^{it}) - (f * g)(e^{is})| &\leq |\big((f-\varphi) * g\big)(e^{it})| + |\big((f-\varphi) * g\big)(e^{is})| \\ &+ |(\varphi * g)(e^{it}) - (\varphi * g)(e^{is})| \\ &\leq 2\,\|f-\varphi\|_p\,\|g\|_q + \omega_\varphi(|t-s|)\,\|g\|_q, \end{aligned}$$

where

$$\omega_\varphi(\delta) = \sup_{|t-s|\leq\delta} |\varphi(e^{it}) - \varphi(e^{is})|$$

is the *modulus of continuity* of φ. Since φ is uniformly continuous on $\mathbb{T}$,

$$\omega_\varphi(\delta) \longrightarrow 0$$

as $\delta \to 0$. Therefore, if $|t-s|$ is small enough, we have

$$|(f * g)(e^{it}) - (f * g)(e^{is})| \leq 3\varepsilon\,\|g\|_q.$$

□

Exercises

Exercise 1.5.1 What can we say about $f * g$ if $f \in L^r(\mathbb{T})$ and $g \in L^s(\mathbb{T})$ with $1 \leq r, s \leq \infty$ and

$$\frac{1}{r} + \frac{1}{s} \leq 1?$$

(See Figure 1.6.)

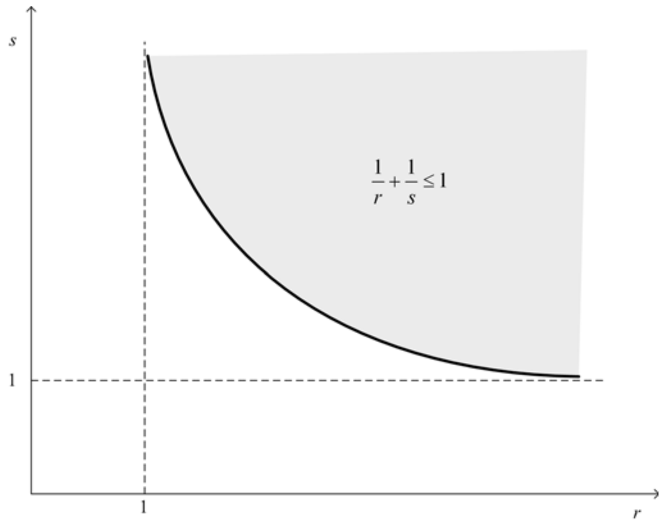

Fig. 1.6. The region $\frac{1}{r}+\frac{1}{s}\leq 1$.

Exercise 1.5.2 Show that Young's inequality is sharp in the following sense. Given r, s with $1 \leq r, s \leq \infty$ and

$$\frac{1}{r}+\frac{1}{s} \geq 1,$$

there are $f \in L^r(\mathbb{T})$ and $g \in L^s(\mathbb{T})$ such that $f * g \in L^p(\mathbb{T})$, where

$$\frac{1}{p}=\frac{1}{r}+\frac{1}{s}-1,$$

but $f * g \notin L^t(\mathbb{T})$ for any $t > p$.
Hint: Start with the case $r = s = 1$ and a function $\varphi \in L^1(\mathbb{T})$ such that $\varphi \notin L^t(\mathbb{T})$ for any $t > 1$.

Chapter 2

Abel–Poisson means

2.1 Abel–Poisson means of Fourier series

Let $\{F_r\}_{0\leq r<1}$ be a family of functions on the unit circle $\mathbb{T}$. Define

$$F(re^{it}) = F_r(e^{it}), \qquad (re^{it} \in \mathbb{D}).$$

Hence, instead of looking at the family as a collection of individual functions F_r which are defined on $\mathbb{T}$, we deal with *one* single function defined on the open unit disc $\mathbb{D}$. On the other hand, if $F(re^{it})$ is given first, for each fixed $r \in [0,1)$, we can define F_r by considering the values of F on the circle $\{|z| = r\}$. This dual interpretation will be encountered many times in what follows. An important example of this phenomenon is the Poisson kernel which was defined as a family of functions on the unit circle by (1.4). This kernel can also be considered as one function

$$P(re^{it}) = \frac{1-r^2}{1+r^2-2r\cos t}$$

on $\mathbb{D}$.

Let $\mu \in \mathcal{M}(\mathbb{T})$. Then, by (1.10), we have

$$(P_r * \mu)(e^{i\theta}) = \int_{\mathbb{T}} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)}\, d\mu(e^{it}) \tag{2.1}$$

which is called the *Poisson integral* of μ. Moreover, by (1.6) and Theorem 1.2, the Fourier coefficients of $P_r * \mu$ are given by

$$\widehat{P_r * \mu}\,(n) = r^{|n|}\;\hat{\mu}(n), \qquad (n \in \mathbb{Z}).$$

Thus the formal Fourier series of $P_r * \mu$ is

$$\sum_{n=-\infty}^{\infty} \hat{\mu}(n)\, r^{|n|}\, e^{in\theta}.$$

These sums are called the *Abel–Poisson means* of the Fourier series

$$\sum_{n=-\infty}^{\infty} \hat{\mu}(n)\, e^{in\theta}.$$

The Fourier series of μ is not necessarily pointwise convergent. However, we show that its Abel–Poisson means behave much better. The following theorem reveals the relation between the Abel–Poisson means of μ and its Poisson integral.

Theorem 2.1 *Let $\mu \in \mathcal{M}(\mathbb{T})$, and let*

$$U(re^{i\theta}) = (P_r * \mu)(e^{i\theta}) = \int_{\mathbb{T}} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)}\, d\mu(e^{it}), \qquad (re^{i\theta} \in \mathbb{D}).$$

Then

$$U(re^{i\theta}) = \sum_{n=-\infty}^{\infty} \hat{\mu}(n)\, r^{|n|}\, e^{in\theta}, \qquad (re^{i\theta} \in \mathbb{D}).$$

The series is absolutely and uniformly convergent on compact subsets of $\mathbb{D}$, and U is harmonic on $\mathbb{D}$.

Proof. Since

$$|\, \hat{\mu}(n)\, r^{|n|}\, e^{in\theta}\, | \leq \|\mu\|\, r^{|n|},$$

the series $\sum \hat{\mu}(n)\, r^{|n|}\, e^{in\theta}$ is absolutely and uniformly convergent on compact subsets of $\mathbb{D}$. Fix $0 \leq r < 1$ and θ. Then, by (1.5),

$$\begin{aligned} U(re^{i\theta}) &= \int_{\mathbb{T}} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)}\, d\mu(e^{it}) \\ &= \int_{\mathbb{T}} \left(\sum_{n=-\infty}^{\infty} r^{|n|}\, e^{in(\theta-t)} \right) d\mu(e^{it}). \end{aligned}$$

Since the series is uniformly convergent (as a function of e^{it}), and since $|\mu|$ is a finite positive Borel measure on $\mathbb{T}$, we can change the order of summation and integration. Hence,

$$U(re^{i\theta}) = \sum_{n=-\infty}^{\infty} \left(\int_{\mathbb{T}} e^{-int}\, d\mu(e^{it}) \right) r^{|n|}\, e^{in\theta} = \sum_{n=-\infty}^{\infty} \hat{\mu}(n)\, r^{|n|}\, e^{in\theta}.$$

There are several ways to verify that U is harmonic on $\mathbb{D}$. We give a direct proof. Fix $k \geq 0$. Then the absolute and uniform convergence of $\sum_{n=-\infty}^{\infty} n^k \hat{\mu}(n)\, r^{|n|}\, e^{in\theta}$ on compact subsets of $\mathbb{D}$ enables us to change the order of summation and any linear differential operator. In particular, let us apply the Laplace operator. Hence, remembering that each term $r^{|n|}\, e^{in\theta}$ is a harmonic function, we obtain

$$\nabla^2 U = \nabla^2 \left(\sum_{n=-\infty}^{\infty} \hat{\mu}(n)\, r^{|n|}\, e^{in\theta} \right) = \sum_{n=-\infty}^{\infty} \hat{\mu}(n)\, \nabla^2 (r^{|n|}\, e^{in\theta}) = 0.$$

$\square$

As a special case, if the measure μ in Theorem 2.1 is absolutely continuous with respect to the Lebesgue measure, i.e. $d\mu(e^{it}) = u(e^{it})\,dt/2\pi$ with $u \in L^1(\mathbb{T})$, then

$$
\begin{aligned}
U(re^{i\theta}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)}\; u(e^{it})\,dt \\
&= \sum_{n=-\infty}^{\infty} \hat{u}(n)\, r^{|n|}\, e^{in\theta}, \qquad (2.2)
\end{aligned}
$$

where the series is absolutely and uniformly convergent on compact subsets of $\mathbb{D}$, and U represents a harmonic function there.

Exercises

Exercise 2.1.1 Let $(a_n)_{n\geq 0}$ be a sequence of complex numbers. Suppose that the series

$$S = \sum_{n=0}^{\infty} a_n$$

is convergent. For each $0 \leq r < 1$, define

$$S(r) = \sum_{n=0}^{\infty} a_n\, r^n.$$

Show that $S(r)$ is absolutely convergent and moreover

$$\lim_{r\to 1} S(r) = S.$$

Hint: Let

$$S_m = \sum_{n=0}^{m} a_n, \qquad (m \geq 0).$$

Then

$$S(r) = S + (1-r)\sum_{n=0}^{\infty} (S_n - S)\, r^n.$$

Exercise 2.1.2 Let $(a_n)_{n\geq 0}$ be a bounded sequence of complex numbers and let

$$S(r) = \sum_{n=0}^{\infty} a_n\, r^n, \qquad (0 \leq r < 1).$$

Find $(a_n)_{n\geq 0}$ satisfying the following properties:

(i) the series $\sum_{n=0}^{\infty} a_n$ is divergent;

(ii) for each $0 \leq r < 1$, $S(r)$ is absolutely convergent;

(iii) $\lim_{r\to 1} S(r)$ exists.

Exercise 2.1.3 Let $(a_n)_{n\geq 0}$ be a sequence of complex numbers. Suppose that the series

$$S = \sum_{n=0}^{\infty} a_n$$

is convergent. Let

$$S_n = \sum_{k=0}^{n} a_k$$

and define

$$C_n = \frac{S_0 + S_1 + \cdots + S_n}{n+1} = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) a_k.$$

Show that

$$\lim_{n\to\infty} C_n = S.$$

Remark: The numbers C_n, $n \geq 0$, are called the *Cesàro means* of S_n.

Exercise 2.1.4 Let $(a_n)_{n\geq 0}$ be a sequence of complex numbers and define

$$C_n = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) a_k.$$

Find $(a_n)_{n\geq 0}$ such that

$$\lim_{n\to\infty} C_n$$

exists, but the sequence

$$\sum_{n=0}^{\infty} a_n$$

is divergent.

Exercise 2.1.5 Let $(a_n)_{n\geq 0}$ be a sequence of complex numbers and define

$$C_n = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) a_k.$$

Suppose that the series $\sum_{n=0}^{\infty} C_n$ is convergent and

$$\sum_{n=0}^{\infty} n|a_n|^2 < \infty.$$

Show that the series $\sum_{n=0}^{\infty} a_n$ is also convergent.

Remark: Compare with Exercises 2.1.3 and 2.1.4.

2.2 Approximate identities on $\mathbb{T}$

We saw that $L^1(\mathbb{T})$, equipped with convolution, is a commutative Banach algebra. This algebra does not have a unit element since such an element must satisfy

$$\hat{f}(n) = 1, \qquad (n \in \mathbb{Z}),$$

and we will see that the nth Fourier coefficient of any integrable function tends to zero as $|n| \to \infty$. To overcome this difficulty, we consider a family of integrable functions $\{\Phi_\iota\}$ satisfying

$$\lim_\iota \hat{\Phi}_\iota(n) = 1 \tag{2.3}$$

for each fixed $n \in \mathbb{Z}$. The condition (2.3) alone is not enough to obtain a family that somehow plays the role of a unit element. For example, the Dirichlet kernel satisfies this property but it is not a proper replacement for the unit element (see Exercise 2.2.2). We choose three other properties to define our family and then we show that (2.3) is fulfilled.

Let $\Phi_\iota \in L^1(\mathbb{T})$, where the index ι ranges over a directed set. In the examples given below, it ranges either over the set of integers $\{1, 2, 3, \dots\}$ or over the interval $[0, 1)$. Therefore, in the following, $\lim_\iota$ means either $\lim_{n\to\infty}$ or $\lim_{r\to 1^-}$. Similarly, $\iota \succ \iota_0$ means $n > n_0$ or $r > r_0$. The family $\{\Phi_\iota\}$ is called an *approximate identity* on $\mathbb{T}$ if it satisfies the following properties:

(a) for all ι,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_\iota(e^{it})\, dt = 1;$$

(b)

$$C_\Phi = \sup_\iota \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |\Phi_\iota(e^{it})|\, dt \right) < \infty;$$

(c) for each fixed δ, $0 < \delta < \pi$,

$$\lim_\iota \int_{\delta \le |t| \le \pi} |\Phi_\iota(e^{it})|\, dt = 0.$$

The condition (a) forces $C_\Phi \ge 1$. If $\Phi_\iota(e^{it}) \ge 0$, for all ι and for all $e^{it} \in \mathbb{T}$, then $\{\Phi_\iota\}$ is called a *positive approximate identity*. In this case, (b) follows from (a) with

$$C_\Phi = 1.$$

We give three examples of a positive approximate identity below. Further examples are provided in the exercises. Our main example of a positive approximate identity is the Poisson kernel

$$P_r(e^{it}) = \frac{1 - r^2}{1 + r^2 - 2r\cos\theta} = \sum_{n=-\infty}^{\infty} r^{|n|}\, e^{int}, \qquad (0 \le r < 1).$$