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Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras

F. F. Bonsall and J. Duncan

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Introduction

The numerical range of a linear operator on a normed linear space is a subset of the scalar field constructed in such a way that it is related both to the algebraic and the norm structures of the operator. In this it differs from the spectrum, which is related to the algebraic structure but independent of the norm (up to equivalence). For an operator on a Hilbert (or pre-Hilbert) space the numerical range has a very natural definition which was introduced, for finite dimensional spaces, by Toeplitz in 1918 [70], as follows. Let $\mathscr{H}$ denote a pre-Hilbert space with scalar product $\langle\,,\rangle$ and norm $\|\cdot\|$, and let $S(\mathscr{H})$ denote the unit sphere in $\mathscr{H}$, $S(\mathscr{H}) = \{x \in \mathscr{H} : \|x\| = 1\}$. Then the numerical range of a linear operator $T:\mathscr{H} \to \mathscr{H}$ is the set $W(T)$ of scalars defined by

$$W(T) = \{\langle Tx, x\rangle : x \in S(\mathscr{H})\} . \tag{1}$$

The numerical range $W(T)$ owes part of its motivation to the classical theory of quadratic forms, but modern developments are naturally given in terms of the theory of bounded linear operators.

We have not included an account of the numerical range of an operator on a Hilbert space in these notes. There are two reasons for this. First and most important is the ready availability of an excellent account of the subject in Halmos's recent book [30].

The second reason is that in the context of Hilbert spaces, with the varied and powerful methods available there, the numerical range has remained a relatively unimportant tool. By comparison, the theory of operators on Banach spaces and the general theory of Banach algebras are lacking in effective methods, and therefore such methods as we have are to be correspondingly treasured. Moreover the numerical range can be applied to problems, such as the metric characterization of B*-algebras, which become meaningful only in a wider context, even though, as in this instance, they may concern a Hilbert structure.

It is, however, appropriate to give a brief review of the main results for Hilbert spaces, since these have motivated much of the general theory. It is obvious that $W(T)$ contains all eigenvalues of T; for if λ is an eigenvalue, there exists $u \in S(\mathscr{H})$ with $Tu = \lambda u$, and then

$$\lambda = \lambda < u, u > = < Tu, u > \in W(T) .$$

Thus when $\mathscr{H}$ has finite dimension, $W(T)$ contains $Sp(T)$, the spectrum of T, and Toeplitz proved that the unbounded component of the complement of $W(T)$ has a convex curve for its boundary. The truth of this latter result became obvious when it was proved by Hausdorff [31], still for finite dimensional $\mathscr{H}$, that $W(T)$ is convex. This last result of Hausdorff was extended by Stone [68] to operators on pre-Hilbert spaces of arbitrary dimension. Let $\mathscr{H}$ denote a complex Hilbert space, let T be a bounded linear operator on $\mathscr{H}$ with operator norm $|T|$, and let

$$w(T) = \sup \{ |\zeta| : \zeta \in W(T) \} .$$

Further developments in the theory included the results that $Sp(T) \subset W(T)^{-}$ (the closure of $W(T)$), that, for a normal operator T, $W(T)^{-}$ is the convex hull of $Sp(T)$, and that for all T

$$|T| \leq 2w(T) . \tag{2}$$

These elementary results already give examples of the relationship of the numerical range to both the algebraic and the norm properties of the operator.

The study of the numerical range of an operator on a Hilbert space continued in an unspectacular fashion until quite recently. One very interesting recent result is the theorem of Berger [4] that

$$w(T^n) \leq (w(T))^n \quad (n = 1, 2, \ldots) , \tag{3}$$

which is remarkable because the inequality $w(ST) \leq w(S)w(T)$ is false, as is also the special case $w(T^{n+m}) \leq w(T^n)w(T^m)$. An elegant and simple proof of the inequality (3) is given by Pearcy [54]. For other references see Halmos [30] and Putnam [56].

By contrast to the long history of the Hilbert space numerical range, the birth of the general theory was long delayed and its growth has been spectacular. No concept of numerical range appropriate to general normed linear spaces appeared until 1961 and 1962, when distinct, though related, concepts were introduced independently by Lumer [40] and Bauer [3]. Lumer defined the concept of a semi-inner-product on a linear space, and showed that every normed linear space $(X, \|.\|)$ has at least one semi-inner-product $[,]$ such that

$$[x, x] = \|x\|^2 \quad (x \in X) . \tag{4}$$

In terms of a semi-inner-product satisfying (4), the definition (1) used for Hilbert spaces at once generalizes to give the definition of the numerical range $W(T)$ for a linear operator on X,

$$W(T) = \{[Tx, x] : x \in S(X)\} ,$$

where $S(X)$ denotes the unit sphere of X. On the face of it this definition has the serious defect that it is not an invariant of the normed space $(X, \|\cdot\|)$, since, except when the unit ball of X is smooth, there are infinitely many semi-inner-products on X satisfying (4). However, this defect is more apparent than real, for Lumer proved that

$$\sup\{\operatorname{Re}\lambda : \lambda \in W(T)\} = \lim_{\alpha \to 0+} \frac{1}{\alpha} \{|I + \alpha T| - 1\}, \tag{5}$$

from which it follows that $\overline{\text{co}}\, W(T)$, the closed convex hull of $W(T)$, is independent of the choice of semi-inner-product satisfying (4). In fact, (5) shows that $\overline{\text{co}}\, W(T)$ depends only on the norms of the operators in the two dimensional linear subspace spanned by I (the identity operator) and T.

Lumer's paper [40] was undoubtedly the most important in the development of the subject. Besides introducing a suitable general concept and obtaining its fundamental properties, including a generalization of the crucial inequality (2), this paper showed the power of the concept in applications both to linear operators and to Banach algebras. Given an element a of a normed algebra A, let T_a denote the left regular representation operator:

$$T_a x = ax \quad (x \in A) . \tag{6}$$

Then the numerical range $W(a)$ may be defined by $W(a) = W(T_a)$ in terms of some semi-inner-product on A related to the norm of A. Lumer showed that the numerical range is an effective tool for relating algebraic and geometric properties of a Banach algebra, giving, in particular, a simple proof of the theorem of Bohnenblust and Karlin [10] that the unit element is a vertex of a complex unital Banach algebra, and throwing fresh light on Vidav's metric characterization of B*-algebras [72].

Bauer's paper [3] was concerned only with finite dimensional normed linear spaces, but the concept of numerical range that he introduced is available without restriction of the dimension. Let $(X, \|.\|)$ be a normed linear space, $S(X)$ its unit sphere, X' its dual space, and let

$$\Pi = \{(x,f) \in S(X) \times S(X') : f(x) = 1\} .$$

Then, for an operator T on X, the numerical range $V(T)$ is defined by

$$V(T) = \{f(Tx) : (x,f) \in \Pi\} . \tag{7}$$

When X is a Hilbert space, $(x,f) \in \Pi$ if and only if $x \in S(X)$ and f is the functional given by $f(y) = \langle y, x \rangle$ $(y \in X)$. Thus $V(T)$ in this case coincides with the classical $W(T)$ given by (1) above. If X is a normed linear space with a smooth unit ball, then $V(T)$ coincides with the numerical range $W(T)$ corresponding to the unique semi-inner-product satisfying (4). For a general normed linear space X, $V(T)$ is the union of all the numerical ranges $W(T)$ corresponding to all choices of semi-inner-product satisfying (4), and for each choice of such a semi-inner-product

$$\overline{\text{co}}\, V(T) = \overline{\text{co}}\, W(T) .$$

Thus Lumer's results are immediately transferable to $V(T)$.

Since $V(T)$ is intrinsically defined, without intervention of a choice of semi-inner-product, it is to be expected that it will exhibit greater regularity than $W(T)$; and two recent theorems exemplify this. Williams [74] has proved that for any bounded linear operator T on a complex Banach space

$$\mathrm{Sp}(T) \subset V(T)^{-} ,$$

whereas we only know that $W(T)^{-}$ contains the approximate point spectrum of T; and Bonsall, Cain, and Schneider [12] have proved that for every bounded linear operator T on a normed linear space $V(T)$ is connected, whereas $W(T)$ may be disconnected. With one trivial exception, the connectedness of $V(T)$ holds for any continuous mapping T of $S(X)$ into X, and this suggests that $V(T)$ may be useful for the study of non-linear mappings. It should be remarked that $V(T)$ is not in general convex, and perhaps the failure of the best known property of the Hilbert space numerical range contributed to the long delay in the introduction of $V(T)$, which after all is a very simple generalization of the Hilbert space numerical range.

That the numerical range can give interesting information even about an $n \times n$ matrix is exemplified by the following theorem of Nirschl and Schneider [47]. Let λ be an eigenvalue of an operator T which belongs to the frontier of co $V(T)$; then λ has index (ascent) one, i. e.

$$(\lambda I - T)^2 x = 0 \implies (\lambda I - T)x = 0 .$$

This result is immediately applicable to the eigenvalues of modulus one of a stochastic matrix.

Let $v(T) = \sup\{ |\lambda| : \lambda \in V(T) \}$. Since $\overline{\text{co}}\, V(T) = \overline{\text{co}}\, W(T)$, we have $v(T) = \sup\{ |\lambda| : \lambda \in W(T) \}$ for any choice of semi-inner-product satisfying (4). For complex normed linear spaces, we have

$$|T| \leq e\, v(T) ,$$

an inequality essentially due to Bohnenblust and Karlin [10], in which the constant e ($= \exp 1$) has been shown by Glickfeld [28] to be best possible for the class of all complex normed linear spaces. For special classes of spaces better constants have been established. The reciprocal of the least constant valid for a given normed linear space is called the numerical index of the space. It has recently been proved by Duncan, McGregor, Pryce and White [21] that for every real number $\nu \in [e^{-1}, 1]$ there exists a complex normed linear space with numerical index ν.

The inequality (3) has not been established† for operators on a general Banach space (perhaps for the natural reason), but M. J. Crabb [17] has recently proved that if T is a bounded linear operator on a complex Banach space normalized so that $v(T) \leq 1$, then

$$\|T^n\| \leq e\, n^{\frac{1}{2}} \quad (n = 1, 2, \dots) .$$

He has further proved, using the theorem of Nirschl and Schneider mentioned earlier, that if X is finite dimensional (or more

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† See Remark (10) of § 14.

generally if T is a meromorphic operator), then with the same normalization the sequence $\{\|T^n\|\}$ is bounded.

Suppose now that A is a normed algebra, and is unital, i. e. has a unit element 1 with $\|1\| = 1$. Given $a \in A$, the numerical range $V(a)$ is defined by $V(a) = V(T_a)$, where T_a is the left regular representation operator on A, as in (6) above. In this case, however, a remarkably simple expression is available for $V(a)$. Let $D(1)$ denote the set of all normalized states on A, i. e. continuous linear functionals f on A such that

$$f(1) = \|f\| = 1 .$$

Then

$$V(a) = \{f(a): f \in D(1)\} .$$

It follows at once that $V(a)$ is a compact convex set; and a number of other fundamental properties of $V(a)$ are very easily established.

Given a bounded linear operator T on a normed linear space X, we can regard T as an element of the unital normed algebra $B(X)$ of all bounded linear operators on X, and we then have two intrinsic numerical ranges available for T. It turns out, as would be expected, that the numerical range of T as an element of the normed algebra $B(X)$ is the closed convex hull of the spatial numerical range $V(T)$ given by (7) above.

Of special interest are the Hermitian elements of a complex unital Banach algebra A, i. e. elements h of A such that $V(h) \subset \mathbb{R}$. By the lemma of Lumer mentioned above (5), h is Hermitian if and only if

$$\lim_{\alpha \to 0} \alpha^{-1} \{ \|1 + i\alpha h\| - 1 \} = 0 ,$$

i. e. if and only if h is Hermitian in the sense of Vidav [72]. Thus a fundamental lemma of Vidav shows that, for all Hermitian elements h of A, we have

$$\mathrm{co}\, \mathrm{Sp}(h) = V(h) . \tag{8}$$

This is a crucial lemma in the Vidav characterization of B*-algebras, already mentioned above, which, as recently improved by Palmer [49], states that A is a B*-algebra if and only if every element of A is of the form h + ik with h and k Hermitian. As is well known, B*-algebras are isometrically star isomorphic to C*-algebras, i. e. uniformly closed self-adjoint algebras of operators on Hilbert spaces. Thus an important theorem about algebras of operators on Hilbert spaces involves the concept of numerical range for general normed linear spaces and not just the classical Hilbert space concept. To show the effectiveness of the Vidav-Palmer theorem, we give a number of applications of the theorem. In particular we use it to give new and transparent proofs of the theorem of Glimm and Kadison on a Banach star algebra satisfying the condition

$$\|a^*a\| = \|a^*\| \ \|a\| ,$$

and of the well known theorems of Kaplansky and of Sherman that the quotient of a B*-algebra by a closed two-sided ideal and the second dual of a B*-algebra with the Arens multiplication are both B*-algebras.