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VARIATIONAL PRINCIPLES IN MATHEMATICAL PHYSICS, GEOMETRY, AND ECONOMICS

Alexandru Kristály, Vicenţiu D. Rădulescu and Csaba György Varga

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VARIATIONAL PRINCIPLES IN MATHEMATICAL PHYSICS, GEOMETRY, AND ECONOMICS

This comprehensive introduction to the calculus of variations and its main principles also presents their real-life applications in various contexts: mathematical physics, differential geometry, and optimization in economics.

Based on the authors' original work, it provides an overview of the field, with examples and exercises suitable for graduate students entering research. The method of presentation will appeal to readers with diverse backgrounds in functional analysis, differential geometry, and partial differential equations. Each chapter includes detailed heuristic arguments, providing thorough motivation for the material developed later in the text.

Since much of the material has a strong geometric flavor, the authors have supplemented the text with figures to illustrate the abstract concepts. Its extensive reference list and index also make this a valuable resource for researchers working in a variety of fields who are interested in partial differential equations and functional analysis.

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ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Variational Principles in Mathematical Physics, Geometry, and Economics

Qualitative Analysis of Nonlinear Equations and Unilateral Problems

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A. Kristály dedicates this book to the memory of his father, Vilmos Kristály.
V. D. Rădulescu dedicates this book to the memory of his father, Dumitru Rădulescu.
Cs. Gy. Varga dedicates this book to the memory of his parents, Irma and György Varga.

Contents

Foreword

The use of variational principles has a long and fruitful history in mathematics and physics, both in solving problems and shaping theories, and it has been introduced recently in economics. The corresponding literature is enormous and several monographs are already classical. The present book *Variational Principles in Mathematical Physics, Geometry, and Economics*, by Kristály, Rădulescu and Varga, is original in several ways.

In Part I, devoted to variational principles in mathematical physics, unavoidable classical topics such as the Ekeland variational principle, the mountain pass lemma, and the Ljusternik–Schnirelmann category, are supplemented with more recent methods and results of Ricceri, Brezis–Nirenberg, Szulkin, and Pohozaev. The chosen applications cover variational inequalities on unbounded strips and for area-type functionals, nonlinear eigenvalue problems for quasilinear elliptic equations, and a substantial study of systems of elliptic partial differential equations. These are challenging topics of growing importance, with many applications in natural and human sciences, such as demography.

Part II demonstrates the importance of variational problems in geometry. Classical questions concerning geodesics or minimal surfaces are not considered, but instead the authors concentrate on a less standard problem, namely the transformation of classical questions related to the Emden–Fowler equation into problems defined on some four-dimensional sphere. The combination of the calculus of variations with group theory provides interesting results. The case of equations with critical exponents, which is of special importance in geometrical problems since Yamabe's work, is also treated.

Part III deals with variational principles in economics. Some choice is also necessary in this area, and the authors first study the minimization of cost-functions on manifolds, giving special attention to the Finslerian–Poincaré disc. They then consider best approximation problems on manifolds before approaching Nash equilibria through variational inequalities.

The high level of mathematical sophistication required in all three parts could be an obstacle for potential readers more interested in applications. However, several appendices recall in a precise way the basic concepts and results of convex analysis, functional analysis, topology, and set-valued analysis. Because the present in science depends upon its past and shapes its future, historical and bibliographical notes are complemented by perspectives. Some exercises are proposed as complements to the covered topics.

Among the wide recent literature on critical point theory and its applications, the authors have had to make a selection. Their choice has of course been influenced by their own tastes and contributions. It is a happy one, because of the interest and beauty of selected topics, because of their potential for applications, and because of the fact that most of them have not been covered in existing monographs. Hence I believe that the book by Kristály, Rădulescu, and Varga will be appreciated by all scientists interested in variational methods and in their applications.

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Preface

> For since the fabric of the universe is most perfect and the work of a most wise Creator, nothing at all takes place in the universe in which some rule of maximum or minimum does not appear.
>
> Leonhard Euler (1707–1783)

An understanding of nature is impossible without an understanding of the partial differential equations and variational principles that govern a large part of physics. That is why it is not surprising that nonlinear partial differential equations first arose from an interplay of physics and geometry. The roots of the calculus of variations go back to the seventeenth century. Indeed, Johann Bernoulli raised as a challenge the "brachistochrone problem" in 1696. The same year, Sir Isaac Newton heard of this problem and he found that he could not sleep until he had solved it. Having done so, he published the solution anonymously. Bernoulli, however, knew at once that the author of the solution was Newton and, in a famous remark asserted that he "recognized the Lion by the print of its paw" [224].

However, the modern calculus of variations appeared in the middle of the nineteenth century, as a basic tool in the qualitative analysis of models arising in physics. Indeed,

> *it was Riemann who aroused great interest in them [problems of the calculus of variations] by proving many interesting results in function theory by assuming Dirichlet's principle*
>
> (Charles B. Morrey Jr. [162])

The characterization of phenomena by means of variational principles has been a cornerstone in the transition from classical to contemporary physics. Since the middle part of the twentieth century, the use of variational principles has developed into a range of tools for the study of nonlinear partial differential equations and many problems arising in applications. As stated by Ioffe and Tikhomirov [103],

> *the term "variational principle" refers essentially to a group of results showing that a lower semi-continuous, lower bounded function on a complete metric space possesses arbitrarily small perturbations such that the perturbed function will have an absolute (and even strict) minimum.*

Very often, important equations and systems (Yang–Mills equations, Einstein equations, Ginzburg–Landau equations, etc.) describing phenomena in applied sciences arise from the minimization of energy functionals such as

$$E(u) = \int_{\Omega} f(x, u(x), \nabla u(x))\, dx\,.$$

The class $\mathcal{C}$ of admissible functions $u = (u^1, \dots, u^n) : \Omega \subset \mathbb{R}^N \to \mathbb{R}^n$ may be constrained, for instance, by boundary conditions, while the function $f = f(x, u, p) : \Omega \times \mathbb{R}^n \times \mathbb{R}^{Nn} \to \mathbb{R}$ is assumed to be sufficiently smooth and verifying natural growth conditions. Formally, the variational problem

$$\min_{u \in \mathcal{C}} E(u) \tag{0.1}$$

gives rise to the nonlinear elliptic system of partial differential equations

$$-\sum_{j=1}^{n} \frac{\partial}{\partial x^j} f_{p_j^i}(x, u(x), \nabla u(x)) + f_{u^i}(x, u(x), \nabla u(x)) = 0\,, \tag{0.2}$$

for all $1 \le i \le n$. The simplest example corresponding to

$$f(x, u, p) = \frac{1}{2}\, |p|^2$$

implies that problem (0.1) is associated to the minimization of the Dirichlet integral

$$E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2\, dx$$

and (0.2) reduces to the Laplace equation

$$\Delta u = 0\,.$$

A more sophisticated example corresponds to the function

$$f(x, u, p) = \sum_{i,j=1}^{n} \sum_{s,t=1}^{N} g_{ij}(u) \gamma^{st}(x) p_s^i p_t^j\,,$$

where $\gamma = (\gamma_{st})_{1 \le s,t \le N}$ is an invertible matrix with inverse $\gamma^{-1} = (\gamma^{st})_{1 \le s,t \le N}$, while $g = (g_{ij})_{1 \le i,j \le n}$ is a uniformly positive definite matrix. In this case, problem (0.1) yields a generalization of the Dirichlet integral on suitable manifolds and (0.2) becomes an equation of the type

$$-\Delta_{\mathcal{M}} u = \sum_{i,j,k=1}^{n} \sum_{s,t=1}^{N} \gamma^{st} \Gamma^i_{jk} u^j_{x^s} u^k_{x^t}\,.$$

The differential operator $\Delta_{\mathcal{M}}$ denotes the Laplace–Beltrami operator and Γ^i_{jk} are the Christoffel symbols.

This book is an original attempt to develop the modern theory of the calculus of variations from the points of view of several disciplines. This theory is one of the twin pillars on which nonlinear functional analysis is built. The authors of this volume are fully aware of the limited achievements of this volume as compared with the task of understanding the force of variational principles in the description of many processes arising in various applications. Even though necessarily limited, the results in this book benefit from many years of work by the authors and from interdisciplinary exchanges between them and other researchers in this field.

One of the main objectives of this book is to let physicists, geometers, engineers, and economists know about some basic mathematical tools from which they might benefit. We would also like to help mathematicians learn what applied calculus of variations is about, so that they can focus their research on problems of real interest to physics, economics, and engineering, as well as geometry or other fields of mathematics. We have tried to make the mathematical part accessible to the physicist and economist, and the physical part accessible to the mathematician, without sacrificing rigor in either case. The mathematical technicalities are kept to a minimum within the book, enabling the discussion to be understood by a broad audience. Each problem we develop in this book has its own difficulties. That is why we intend to develop some standard and appropriate methods that are useful and that can be extended to other problems. However, we do our best to restrict the prerequisites to the essential knowledge. We define as few concepts as possible and give only basic theorems that are useful for our topic. We use a first-principles approach, developing only the minimum background necessary to justify mathematical concepts and placing mathematical developments in context. The only prerequisites for this volume are standard graduate courses in partial differential equations and differential geometry, drawing especially from linear elliptic equations to elementary variational methods, with a special emphasis on the maximum principle (weak and strong variants). This volume may be used for self-study by advanced graduate students and as a valuable reference for researchers in pure and applied mathematics and related fields. Nevertheless, both the presentation style and the choice of the material make the present book accessible to all newcomers to this modern research field, which lies at the interface between pure and applied mathematics.

Each chapter gives full details of the mathematical proofs and subtleties. The book also contains many exercises, some included to clarify simple points of exposition, others to introduce new ideas and techniques, and a few containing relatively deep mathematical results. Each chapter concludes with historical notes. Five appendices illustrate some basic mathematical tools applied in this book: elements of convex analysis, function spaces, category and genus, Clarke and Degiovanni gradients, and elements of set-valued analysis. These auxiliary chapters deal with some analytical methods used in this volume, but also include some complements. This unique presentation should ensure a volume of interest to mathematicians, engineers, economists, and physicists. Although the text is geared toward graduate students at a variety of levels, many of the book's applications will be of interest even to experts in the field.

We are very grateful to Diana Gillooly, Editor for Mathematics, for her efficient and enthusiastic help, as well as for numerous suggestions related to previous versions of this book. Our special thanks go also to Clare Dennison, Assistant Editor for Mathematics and Computer Science, and to the other members of the editorial technical staff of Cambridge University Press for the excellent quality of their work.

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Our vision throughout this volume is closely inspired by the following prophetic words of Henri Poincaré [186] on the role of partial differential equations in the development of other fields of mathematics and in applications:

> *A wide variety of physically significant problems arising in very different areas (such as electricity, hydrodynamics, heat, magnetism, optics, elasticity, etc...) have a family resemblance and should be treated by common methods.*

Part I

Variational principles in mathematical physics

1

Variational principles

> A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator the smaller the fraction.
>
> Leo Tolstoy (1828–1910)

Variational principles are very powerful techniques that exist at the interface between nonlinear analysis, calculus of variations, and mathematical physics. They have been inspired by and have deep applications in modern research fields such as geometrical analysis, constructive quantum field theory, gauge theory, superconductivity, etc.

In this chapter we briefly recall the main variational principles which will be used in the rest of the book, such as Ekeland and Borwein–Preiss variational principles, minimax- and minimization-type principles (the mountain pass theorem, Ricceri-type multiplicity theorems, the Brezis–Nirenberg minimization technique), the principle of symmetric criticality for nonsmooth Szulkin-type functionals, as well as Pohozaev's fibering method.

1.1 Minimization techniques and Ekeland's variational principle

Many phenomena arising in applications such as geodesics or minimal surfaces can be understood in terms of the minimization of an energy functional over an appropriate class of objects. For the problems of mathematical physics, phase transitions, elastic instability, and diffraction of light are among the phenomena that can be studied from this point of view.

A central problem in many nonlinear phenomena is whether a bounded from below and lower semi-continuous functional f attains its infimum. A simple function for which the above statement clearly fails is $f : \mathbb{R} \to \mathbb{R}$ defined by $f(s) = e^{-s}$. Nevertheless, further assumptions either on f or on its domain may give a satisfactory answer. In the following chapters we present two useful forms of the well-known Weierstrass theorem.

Theorem 1.1. (Minimization; compact case) *Let X be a compact topological space and let $f : X \to]-\infty, \infty]$ be a lower semi-continuous functional. Then f is bounded from below and its infimum is attained on X.*

Proof. The set X can be covered by the open family $S_n := \{u \in X : f(u) > -n\}$, $n \in \mathbb{N}$. Since X is compact, there exists a finite number of sets $S_{n_0}, \ldots, S_{n_l}$ which also cover X. Consequently, $f(u) > -\max\{n_0, \ldots, n_l\}$ for all $u \in X$.

Let $s = \inf_X f > -\infty$. Arguing by contradiction, we assume that s is not achieved, which means in particular that $X = \cup_{n=1}^{\infty}\{u \in X : f(u) > s + 1/n\}$. Due to the compactness of X, there exists a number $n_0 \in \mathbb{N}$ such that $X = \cup_{n=1}^{n_0}\{u \in X : f(u) > s + 1/n\}$. In particular, $f(u) > s + 1/n_1$ for all $u \in X$, which is in contradiction with $s = \inf_X f > -\infty$. □

The following result is a very useful tool in the study of various partial differential equations where no compactness is assumed on the domain of the functional.

Theorem 1.2. (Minimization; noncompact case) *Let X be a reflexive Banach space, let M be a weakly closed, bounded subset of X, and let $f : M \to \mathbb{R}$ be a sequentially weakly lower semi-continuous function. Then f is bounded from below and its infimum is attained on M.*

Proof. We argue by contradiction, that is, we assume that f is not bounded from below on M. Then for every $n \in \mathbb{N}$ there exists $u_n \in M$ such that $f(u_n) < -n$. Since M is bounded, the sequence $\{u_n\} \subset M$ is also. Due to the reflexivity of X, one may subtract a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ which weakly converges to an element $\tilde{x} \in X$. Since M is weakly closed, $\tilde{x} \in M$. Since $f : M \to \mathbb{R}$ is sequentially weakly lower semi-continuous, we obtain that $f(\tilde{x}) \leq \liminf_{k\to\infty} f(u_{n_k}) = -\infty$, a contradiction. Therefore, f is bounded from below.

Let $\{u_n\} \subset M$ be a minimizing sequence of f over M, that is, $\lim_{n\to\infty} f(u_n) = \inf_M f > -\infty$. As before, there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ which weakly converges to an element $\bar{x} \in M$. Due to the sequentially weakly lower semi-continuity of f, we have that $f(\bar{x}) \leq \liminf_{k\to\infty} f(u_{n_k}) = \inf_M f$, which concludes the proof. □

For any bounded from below, lower semi-continuous functional f, Ekeland's variational principle provides a minimizing sequence whose elements minimize an appropriate sequence of perturbations of f which converges locally uniformly to f. Roughly speaking, Ekeland's variational principle states that there exist points which are almost points of minima and where the "gradient" is small. In particular, it is not always possible to minimize a nonnegative continuous function on a complete metric space. Ekeland's variational principle is a very basic tool, has effective in numerous situations, which has led to many new results and strengthened a series of known results in various fields of analysis, geometry, the Hamilton–Jacobi theory, extremal problems, the Ljusternik–Schnirelmann theory, etc.

Its precise statement is as follows.

Theorem 1.3. (Ekeland's variational principle) *Let (X, d) be a complete metric space and let $f : X \to]-\infty, \infty]$ be a lower semi-continuous, bounded from below functional with $D(f) = \{u \in X : f(u) < \infty\} \neq \emptyset$. Then for every $\varepsilon > 0$, $\lambda > 0$, and $u \in X$*

such that

$$f(u) \leq \inf_X f + \varepsilon,$$

there exists an element $v \in X$ *such that*

(a) $f(v) \leq f(u)$;
(b) $d(v, u) \leq 1/\lambda$;
(c) $f(w) > f(v) - \varepsilon\lambda d(w, v)$ *for each* $w \in X \setminus \{v\}$.

Proof. It is sufficient to prove our assertion for $\lambda = 1$. The general case is obtained by replacing d by an equivalent metric λd. We define the relation on X as follows:

$$w \leq v \Longleftrightarrow f(w) + \varepsilon d(v, w) \leq f(v).$$

It is easy to see that this relation defines a partial ordering on X. We now construct inductively a sequence $\{u_n\} \subset X$ as follows: $u_0 = u$, and assuming that u_n has been defined, we set

$$S_n = \{w \in X : w \leq u_n\}$$

and choose $u_{n+1} \in S_n$ so that

$$f(u_{n+1}) \leq \inf_{S_n} f + \frac{1}{n+1}.$$

Since $u_{n+1} \leq u_n$, then $S_{n+1} \subset S_n$ and, by the lower semi-continuity of f, S_n is closed. We now show that $\operatorname{diam} S_n \to 0$. Indeed, if $w \in S_{n+1}$, then $w \leq u_{n+1} \leq u_n$, and consequently

$$\varepsilon d(w, u_{n+1}) \leq f(u_{n+1}) - f(w) \leq \inf_{S_n} f + \frac{1}{n+1} - \inf_{S_n} f = \frac{1}{n+1}.$$

This estimate implies that

$$\operatorname{diam} S_{n+1} \leq \frac{2}{\varepsilon(n+1)},$$

and our claim follows. The fact that X is complete implies that $\bigcap_{n\geq 0} S_n = \{v\}$ for some $v \in X$. In particular, $v \in S_0$; that is, $v \leq u_0 = u$ and hence

$$f(v) \leq f(u) - \varepsilon d(u, v) \leq f(u),$$

and moreover

$$d(u, v) \leq \frac{1}{\varepsilon}(f(u) - f(v)) \leq \frac{1}{\varepsilon}(\inf_X f + \varepsilon - \inf_X f) = 1.$$

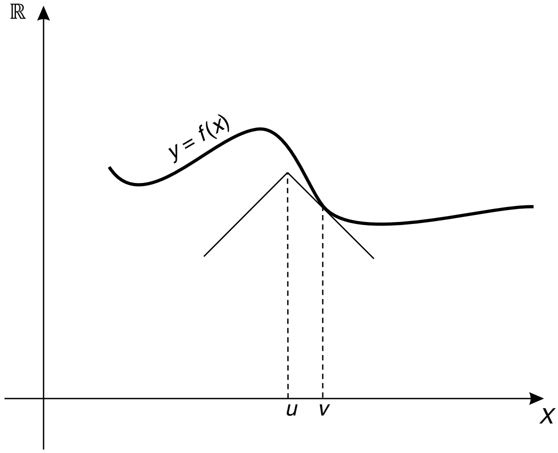

Figure 1.1. Geometric illustration of Ekeland's variational principle.

Now let $w \neq v$. To complete the proof we must show that $w \preceq v$ implies $w = v$. If $w \preceq v$, then $w \preceq u_n$ for each integer $n \geq 0$, that is $w \in \cap_{n\geq 0} S_n = \{v\}$. So, $w \not\preceq v$, which is actually (c). □

In $\mathbb{R}^N$ with the Euclidean metric, properties (a) and (c) in the statement of Ekeland's variational principle are completely intuitive as Figure 1.1 shows. Indeed, assuming that $\lambda = 1$, let us consider a cone lying below the graph of f, with slope $+1$ and vertex projecting onto u. We move up this cone until it first touches the graph of f at some point $(v, f(v))$. Then the point v satisfies both (a) and (c).

In the particular case $X = \mathbb{R}^N$, we can give the following simple alternative proof to Ekeland's variational principle, due to Hiriart-Urruty [100]. Indeed, consider the perturbed functional

$$g(w) := f(w) + \varepsilon\lambda \,\|w - u\|\,, \qquad w \in \mathbb{R}^N\,.$$

Since f is lower semi-continuous and bounded from below, then g is lower semi-continuous and $\lim_{\|w\|\to\infty} g(w) = \infty$. Therefore there exists $v \in \mathbb{R}^N$ minimizing g on $\mathbb{R}^N$ such that, for all $w \in \mathbb{R}^N$,

$$f(v) + \varepsilon\lambda \,\|v - u\| \leq f(w) + \varepsilon\lambda \,\|w - u\|\,. \tag{1.1}$$

By letting $w = u$ we obtain

$$f(v) + \varepsilon\lambda \,\|v - u\| \leq f(u)$$

and (a) follows. Now, since $f(u) \leq \inf_{\mathbb{R}^N} f + \varepsilon$, we also deduce that $\|v - u\| \leq 1/\lambda$.

We infer from relation (1.1) that, for any w,

$$f(v) \leq f(w) + \varepsilon\lambda \,[\|w - u\| - \|v - u\|] \leq f(w) + \varepsilon\lambda \,\|w - u\|\,,$$

which is the desired inequality (c).

Taking $\lambda = \frac{1}{\sqrt{\varepsilon}}$ in Theorem 1.3 we obtain the following property.

Corollary 1.4. *Let (X, d) be a complete metric space and let $f : X \to]-\infty, \infty]$ be lower semi-continuous, bounded from below, and let $D(f) = \{u \in X : f(u) < \infty\} \neq \emptyset$. Then for every $\varepsilon > 0$ and every $u \in X$ such that*

$$f(u) \leq \inf_X f + \varepsilon,$$

there exists an element $u_\varepsilon \in X$ such that

(a) $f(u_\varepsilon) \leq f(u)$;
(b) $d(u_\varepsilon, u) \leq \sqrt{\varepsilon}$;
(c) $f(w) > f(u_\varepsilon) - \sqrt{\varepsilon} d(w, u_\varepsilon)$ *for each* $w \in X \setminus \{u_\varepsilon\}$.

Let $(X, \|\cdot\|)$ be a real Banach space, and let $X^\star$ be its topological dual endowed with its natural norm, denoted for simplicity also by $\|\cdot\|$. We denote by $\langle\cdot,\cdot\rangle$ the duality mapping between X and $X^\star$; that is, $\langle x^\star, u\rangle = x^\star(u)$ for every $x^\star \in X^\star, u \in X$. Theorem 1.3 readily implies the following property, which asserts the existence of *almost critical points*. In other words, Ekeland's variational principle can be viewed as a generalization of the Fermat theorem which establishes that interior extrema points of a smooth functional are, necessarily, critical points of this functional.

Corollary 1.5. *Let X be a Banach space and let $f : X \to \mathbb{R}$ be a lower semi-continuous functional which is bounded from below. Assume that f is Gâteaux differentiable at every point of X. Then for every $\varepsilon > 0$ there exists an element $u_\varepsilon \in X$ such that*

(i) $f(u_\varepsilon) \leq \inf_X f + \varepsilon$;
(ii) $\|f'(u_\varepsilon)\| \leq \varepsilon$.

Letting $\varepsilon = 1/n$, $n \in \mathbb{N}$, Corollary 1.5 gives rise to a minimizing sequence for the infimum of a given function which is bounded from below. Note, however, that such a sequence need not converge to any point. Indeed, let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(s) = e^{-s}$. Then, $\inf_{\mathbb{R}} f = 0$, and any minimizing sequence fulfilling (a) and (b) from Corollary 1.5 tends to ∞. The following definition is dedicated to handle such situations.

Definition 1.6. (a) A function $f \in C^1(X, \mathbb{R})$ satisfies the Palais–Smale condition at level $c \in \mathbb{R}$ (abbreviated to $(PS)_c$-condition) if every sequence $\{u_n\} \subset X$, such that $\lim_{n\to\infty} f(u_n) = c$ and $\lim_{n\to\infty} \|f'(u_n)\| = 0$, possesses a convergent subsequence.
(b) A function $f \in C^1(X, \mathbb{R})$ satisfies the Palais–Smale condition (abbreviated to (PS)-condition) if it satisfies the Palais–Smale condition at every level $c \in \mathbb{R}$.

Combining this compactness condition with Corollary 1.5, we obtain the following result.

Theorem 1.7. *Let X be a Banach space and let f be a function $f \in C^1(X, \mathbb{R})$ which is bounded from below. If f satisfies the $(PS)_c$-condition at level $c = \inf_X f$, then c*

is a critical value of f; that is, there exists a point $u_0 \in X$ such that $f(u_0) = c$ and u_0 is a critical point of f, that is, $f'(u_0) = 0$.

1.2 Borwein–Preiss variational principle

The Borwein–Preiss variational principle [32] is an important tool in infinite dimensional nonsmooth analysis. This basic result is strongly related to Stegall's variational principle [212], *smooth bumps* on Banach spaces, Smulyan's test describing the relationship between Fréchet differentiability and the strong extremum, properties of continuous convex functions on separable Asplund spaces, variational characterizations of Banach spaces, the Bishop–Phelps theorem, or Phelps' lemma [180]. The generalized version we present here is due to Loewen and Wang [146] and enables us to deduce the standard form of the Borwein–Preiss variational principle, as well as other related results.

Let X be a Banach space and assume that $\rho : X \to [0, \infty[$ is a continuous function satisfying

$$\rho(0) = 0 \quad \text{and} \quad \rho_M := \sup\{\|x\|;\ \rho(x) < 1\} < \infty . \tag{1.2}$$

An example of function with these properties is $\rho(x) = \|x\|^p$ with $p > 0$.

Given the families of real numbers $\mu_n \in]0, 1[$ and vectors $e_n \in X$ $(n \geq 0)$, we associate to ρ the *penalty function* ρ_∞ defined for all $x \in X$:

$$\rho_\infty(x) = \sum_{n=0}^{\infty} \rho_n(x - e_n), \quad \text{where } \rho_n(x) := \mu_n \rho((n+1)x) . \tag{1.3}$$

Definition 1.8. For the function $f : X \to]-\infty, \infty]$, a point $x_0 \in X$ is a strong minimizer if $f(x_0) = \inf_X f$ and every minimizing sequence $\{z_n\}$ of f satisfies $\|z_n - x_0\| \to 0$ as $n \to \infty$.

We observe that any strong minimizer of f is, in fact, a strict minimizer, that is $f(x) > f(x_0)$ for all $x \in X \setminus \{x_0\}$. The converse is not true, as shown by $f(x) = x^2 e^x$, $x \in \mathbb{R}$, $x_0 = 0$.

The generalized version of the Borwein–Preiss variational principle due to Loewen and Wang is given the following.

Theorem 1.9. *Let $f : X \to]-\infty, \infty]$ be a lower semi-continuous function. Assume that $x_0 \in X$ and $\varepsilon > 0$ satisfy*

$$f(x_0) < \varepsilon + \inf_X f .$$

Let $\{\mu_n\}$ be a decreasing sequence in $]0, 1[$ such that the series $\sum_{n=0}^{\infty} \mu_n$ is convergent. Then for any continuous function ρ satisfying (1.2), there exists a sequence $\{e_n\}$ in X converging to e such that

(i) $\rho(x_0 - e) < 1$;
(ii) $f(e) + \varepsilon \rho_\infty(e) \leq f(x_0)$;

(iii) *e is a strong minimizer of $f + \varepsilon\rho_\infty$. In particular, e is a strict minimizer of $f + \varepsilon\rho_\infty$, that is*

$$f(e) + \varepsilon\rho_\infty(e) < f(x) + \varepsilon\rho_\infty(x) \qquad \text{for all } x \in X \setminus \{e\}.$$

Proof. Define the sequence $\{f_n\}$ such that $f_0 = f$ and, for any $n \geq 0$,

$$f_{n+1}(x) := f_n(x) + \varepsilon\rho_n(x - e_n)\,.$$

Then $f_n \leq f_{n+1}$ and f_n is lower semi-continuous.

Set $e_0 = x_0$. We observe that, for any $n \geq 0$,

$$\inf_X f_{n+1} \leq f_{n+1}(e_n) = f_n(e_n)\,. \tag{1.4}$$

If this inequality is strict, then there exists $e_{n+1} \in X$ such that

$$f_{n+1}(e_{n+1}) \leq \frac{\mu_{n+1}}{2} f_n(e_n) + \left(1 - \frac{\mu_{n+1}}{2}\right) \inf_X f_{n+1} \leq f_n(e_n)\,. \tag{1.5}$$

If equality holds in relation (1.4) then (1.5) also holds, but for e_{n+1} replaced with e_n. Consequently, there exists a sequence $\{e_n\}$ in X such that relation (1.5) holds true.

Set

$$D_n := \left\{x \in X;\, f_{n+1}(x) \leq f_{n+1}(e_{n+1}) + \frac{\varepsilon\mu_n}{2}\right\}\,.$$

Then D_n is not empty, since $e_{n+1} \in D_n$. By the lower semi-continuity of functions f_n we also deduce that D_n is a closed set. Since $\mu_{n+1} \in]0, 1[$, relation (1.5) implies

$$\begin{aligned} f_{n+1}(e_{n+1}) - \inf_X f_{n+1} &\leq \frac{\mu_{n+1}}{2}\,[f_n(e_n) - \inf_X f_{n+1}] \\ &\leq f_n(e_n) - \inf_X f_n\,. \end{aligned} \tag{1.6}$$

We also observe that

$$f_0(e_0) - \inf_X f_0 = f(x_0) - \inf_X f < \varepsilon\,.$$

Next, we prove that

$$\text{the sequence } \{D_n\} \text{ is decreasing} \tag{1.7}$$

and

$$\operatorname{diam}(D_n) \to 0 \quad \text{as } n \to \infty. \tag{1.8}$$

In order to prove (1.7), assume that $x \in D_n$, $n \geq 1$. Since the sequence $\{\mu_n\}$ is decreasing, relation (1.5) implies

$$f_n(x) \leq f_{n+1}(x) \leq f_{n+1}(e_{n+1}) + \frac{\varepsilon\mu_n}{2} \leq f_n(e_n) + \frac{\varepsilon\mu_{n-1}}{2}\,,$$

hence $x \in D_{n-1}$.

Since $f_n \geq f_{n-1}$, relations (1.5) and (1.6) imply

$$\begin{aligned} f_n(e_n) - \inf_X f_n &\leq \frac{\mu_n}{2}\,[f_{n-1}(e_{n-1}) - \inf_X f_n] \\ &\leq \frac{\mu_n}{2}\,[f_{n-1}(e_{n-1}) - \inf_X f_{n-1}] < \frac{\varepsilon\mu_n}{2}\,. \end{aligned} \tag{1.9}$$

For any $x \in D_n$, combining relation (1.9) and the definitions of f_{n+1} and D_n we obtain

$$\begin{aligned} \varepsilon\mu_n\rho((n+1)(x-e_n)) &\leq f_{n+1}(e_{n+1}) - f_n(x) + \frac{\varepsilon\mu_n}{2} \\ &\leq f_{n+1}(e_{n+1}) - \inf_X f_n + \frac{\varepsilon\mu_n}{2} \\ &\leq f_n(e_n) - \inf_X f_n + \frac{\varepsilon\mu_n}{2} < \varepsilon\mu_n\,. \end{aligned} \tag{1.10}$$

Therefore $\rho((n+1)(x-e_n)) < 1$. So, by (1.2),

$$(n+1)\|x - e_n\| \leq \rho_M\,,$$

which shows that diam $(D_n) \leq 2\rho_M/(n+1)$. This implies (1.8).

Since D_n is a closed set for any $n \geq 1$, then (1.7) and (1.8) imply that $\cap_{n=1}^{\infty} D_n$ contains a single point, denoted by e. Then $e_n \to e$ as $n \to \infty$. Thus, using $\rho((n+1)(x-e_n)) < 1$ for all $n \geq 0$ and $x \in X$, we deduce that $\rho(x_0 - e) < 1$.

Since the sequence $\{f_n(e_n)\}$ is nonincreasing and $f_0(e_0) = f(x_0)$, it follows that, in order to prove (ii), it is enough to deduce that

$$f(e) + \varepsilon\rho_\infty(e) \leq f_n(e_n)\,. \tag{1.11}$$

For this purpose we define the nonempty closed sets

$$C_n := \{x \in X;\ f_{n+1}(x) \leq f_{n+1}(e_{n+1})\}\,.$$

Since $f_n \leq f_{n+1}$ and $f_n(e_n) \geq f_{n+1}(e_{n+1})$ for all n, it follows that the sequence $(C_n)_{n\geq 0}$ is nested and $C_n \subset D_n$ for all n. Therefore $\cap_{n=0}^{\infty} C_n = \{e\}$ and

$$f_m(e) \leq f_m(e_m) \leq f_n(e_n) \leq f(x_0) \qquad \text{provided that } m > n. \tag{1.12}$$

Taking $m \to \infty$ we obtain (1.11).

It remains to argue that e is a strong minimizer of $f_\varepsilon := f + \varepsilon\rho_\infty$. Since

$$f_\varepsilon(x) \le \inf_X f_\varepsilon + \frac{\varepsilon\mu_n}{2},$$

relation (1.12) yields

$$f_{n+1}(x) \le f_\varepsilon(x) \le f_\varepsilon(e) + \frac{\varepsilon\mu_n}{2} \le f_{n+1}(e_{n+1}) + \frac{\varepsilon\mu_n}{2}.$$

Setting

$$A_n := \left\{x \in X;\, f_\varepsilon(x) \le \frac{\varepsilon\mu_n}{2} + \inf_X f\right\},$$

the above relation shows that $A_n \subset D_n$. So, by (1.8), we deduce that diam $(A_n) \to 0$ as $n \to \infty$, which shows that e is a strong minimizer of $f_\varepsilon := f + \varepsilon\rho_\infty$. $\square$

Assume that $p \ge 1$ and $\lambda > 0$. Taking

$$\rho(x) = \frac{\|x\|^p}{\lambda^p} \qquad \text{and} \qquad \mu_n = \frac{1}{2^{n+1}(n+1)}$$

we obtain the initial smooth version of the Borwein–Preiss variational principle. Roughly speaking, it asserts that the Lipschitz perturbations obtained in Ekeland's variational principle can be replaced by *superlinear perturbations* in a certain class of admissible functions.

Theorem 1.10. *Given that $f: X \to]-\infty, \infty]$ is a lower semi-continuous function, $x_0 \in X$, $\varepsilon > 0$, $\lambda > 0$, and $p \ge 1$, suppose*

$$f(x_0) < \varepsilon + \inf_X f.$$

Then there exists a sequence $\{\mu_n\}$ with $\mu_n \ge 0$, $\sum_{n=0}^\infty \mu_n = 1$, and a point e in X, expressible as the limit of some sequence $\{e_n\}$, such that for all $x \in X$,

$$f(x) + \frac{\varepsilon}{\lambda^p}\sum_{n=0}^\infty \mu_n \|x - e_n\|^p \ge f(e) + \frac{\varepsilon}{\lambda^p}\sum_{n=0}^\infty \mu_n \|e - e_n\|^p.$$

Moreover, $\|x_0 - e\| < \lambda$ and $f(e) \le \varepsilon + \inf_X f$.

We have seen in Corollary 1.5 of Ekeland's variational principle that any smooth bounded from below functional on a Banach space admits a sequence of "almost critical points." The next consequence of Borwein–Preiss' variational principle asserts that, in the framework of Hilbert spaces, such a functional admits a sequence of *stable* "almost critical points."

Corollary 1.11. *Let f be a real-valued C^2-functional that is bounded from below on a Hilbert space X. Assume that $\{u_n\}$ is a minimizing sequence of f. Then there exists* *a minimizing sequence* $\{v_n\}$ *of* f *such that the following properties hold true:*

(i) $\lim_{n\to\infty} \|u_n - v_n\| = 0$;
(ii) $\lim_{n\to\infty} \|f'(v_n)\| = 0$;
(iii) $\liminf_{n\to\infty} \langle f''(v_n)w, w\rangle \geq 0$ *for any* $w \in X$.

1.3 Minimax principles

In this section we are interested in some powerful techniques for finding solutions of some classes of stationary nonlinear boundary value problems. These solutions are viewed as critical points of a natural functional, often called the *energy* associated to the system. The critical points obtained in this section by means of topological techniques are generally *nonstable* critical points which are neither maxima nor minima of the energy functional.

1.3.1 Mountain pass type results

In many nonlinear problems we are interested in finding solutions as stationary points of some associated "energy" functionals. Often such a mapping is unbounded from above and below, so that it has no maximum or minimum. This forces us to look for saddle points, which are obtained by minimax arguments. In such a case one maximizes a functional f over a closed set A belonging to some family Γ of sets and then one minimizes with respect to the set A in the family. Thus we define

$$c = \inf_{A\in\Gamma} \sup_{u\in A} f(u) \tag{1.13}$$

and one tries to prove, under various hypotheses, that this number c is a critical value of f, hence there is a point u such that $f(u) = c$ and $f'(u) = 0$. Indeed, it seems intuitively obvious that c defined in (1.13) is a critical value of f. However, this is not true in general, as shown by the following example in the plane: let $f(x,y) = x^2 - (x-1)^3y^2$. Then $(0,0)$ is the only critical point of f but c is not a critical value. Indeed, looking for sets A lying in a small neighborhood of the origin, then $c > 0$. This example shows that the heart of the matter is to find appropriate conditions on f and on the family Γ.

One of the most important minimax results is the so-called *mountain pass theorem*, whose geometrical interpretation will be briefly described in the following. Denote by f the function which measures the altitude of a mountain terrain and assume that there are two points in the horizontal plane, e_0 and e_1, representing the coordinates of two locations such that $f(e_0)$ and $f(e_1)$ are the deepest points of two separated valleys. Roughly speaking, our aim is to walk along an optimal path on the mountain from the point $(e_0,f(e_0))$ to $(e_1,f(e_1))$, spending the smallest amount of energy by passing the mountain ridge between the two valleys. Walking on a path $(\gamma,f(\gamma))$ from $(e_0,f(e_0))$ to $(e_1,f(e_1))$ such that the maximal altitude along γ is the smallest among all such continuous paths connecting $(e_0,f(e_0))$ and $(e_1,f(e_1))$, we reach a point L on γ passing the ridge of the mountain which is called a *mountain pass point*.

In the following, we give a first formulation of the mountain pass theorem.

Theorem 1.12. (Mountain pass theorem; positive altitude) *Let X be a Banach space, and let $f : X \to \mathbb{R}$ be a function of class C^1 such that*

$$\inf_{\|u-e_0\|=\rho} f(u) \geq \alpha > \max\{f(e_0), f(e_1)\}$$

for some $\alpha \in \mathbb{R}$ and $e_0 \neq e_1 \in X$ with $0 < \rho < \|e_0 - e_1\|$. If f satisfies the $(PS)_c$-condition at level

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = e_0,\ \gamma(1) = e_1\},$$

then c is a critical value of f with $c \geq \alpha$.

There are two different ways to prove Theorem 1.12: via Ekeland's variational principle, or by using a sort of deformation lemma. We present its proof by means of the latter argument. We refer to Figure 1.2 for a geometric illustration of Theorem 1.12.

Proof of Theorem 1.12. We may assume that $e_0 = 0$ and let $e := e_1$. Since $f(u) \geq \alpha$ for every $u \in X$ with $\|u\| = \rho$ and $\rho < \|e\|$, the definition of the number c shows that $\alpha \leq c$.

It remains to prove that c is a critical value of f. Arguing by contradiction, assume $K_c = \emptyset$. Thus, on account of Remark D.9 (See Appendix D), we may choose $\mathcal{O} = \emptyset$

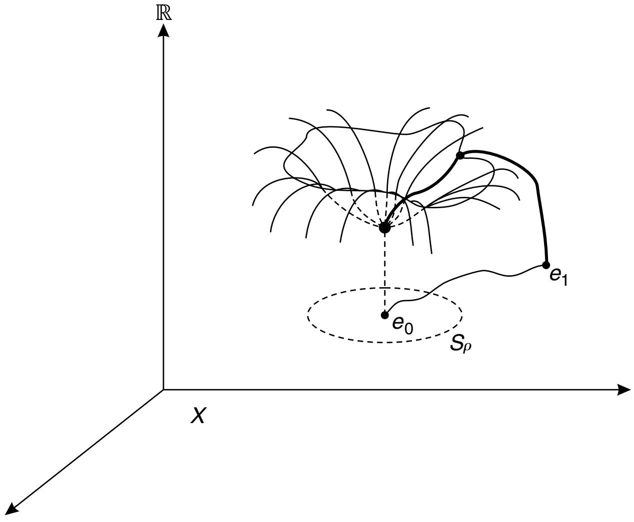

Figure 1.2. Mountain pass landscape between "villages" e_0 and e_1.

in Theorem D.10, and $\overline{\varepsilon} > 0$ such that $\overline{\varepsilon} < \min\{\alpha - f(0), \alpha - f(e)\}$. Consequently, there exist $\varepsilon > 0$ and a continuous map $\eta : X \times [0,1] \to X$ verifying the properties (i)–(iv) from Theorem D.10.

From the definition of the number c, there exists $\gamma \in \Gamma$ such that

$$\max_{t\in[0,1]} f(\gamma(t)) \le c + \varepsilon. \tag{1.14}$$

Let $\gamma_1 : [0,1] \to X$, defined by $\gamma_1(t) = \eta(\gamma(t), 1)$, $\forall t \in [0,1]$.

We prove that $\gamma_1 \in \Gamma$. The choice of $\overline{\varepsilon} > 0$ gives that $\max\{f(0), f(e)\} < \alpha - \overline{\varepsilon} \le c - \overline{\varepsilon}$, thus $0, e \notin f^{-1}(]c - \overline{\varepsilon}, c + \overline{\varepsilon}[)$. Consequently, due to Theorem D.10(iii), we have $\gamma_1(0) = \eta(\gamma(0), 1) = \eta(0,1) = 0$ and $\gamma_1(1) = \eta(\gamma(1), 1) = \eta(e,1) = e$. Therefore, $\gamma_1 \in \Gamma$.

Note that (1.14) means that $\gamma(t) \in f^{c+\varepsilon} = \{u \in X : f(u) \le c + \varepsilon\}$ for all $t \in [0,1]$. Then by Theorem D.10(iv) and the definition of γ_1 we have

$$c \le \max_{t\in[0,1]} f(\gamma_1(t)) = \max_{t\in[0,1]} f(\eta(\gamma(t), 1)) \le c - \varepsilon,$$

a contradiction. □

Remark 1.13. Using Theorem D.10, we can provide an alternative proof to Theorem 1.7. Indeed, if we suppose that $K_c = \emptyset$, with $c = \inf_X f > -\infty$, one may deform continuously the level set $f^{c+\varepsilon}$ (for $\varepsilon > 0$ small enough) into a subset of $f^{c-\varepsilon}$; see Theorem D.10(iv). But $f^{c-\varepsilon} = \emptyset$, a contradiction.

Remark 1.14. Note that the choice of $0 < \overline{\varepsilon} < \min\{\alpha - f(0), \alpha - f(e)\}$ with $\inf_{\|u\|=\rho} f(u) \ge \alpha$ is crucial in the proof of Theorem 1.12. Actually, it means that the ridge of the mountain between the two valleys has a positive altitude. However, a more involved proof allows us to handle the so-called "zero altitude" case. More precisely, the following result holds.

Theorem 1.15. (Mountain pass theorem; zero altitude) *Let X be a Banach space, and let $f : X \to \mathbb{R}$ be a function of class C^1 such that*

$$\inf_{\|u-e_0\|=\rho} f(u) \ge \max\{f(e_0), f(e_1)\}$$

for some $e_0 \ne e_1 \in X$ with $0 < \rho < \|e_0 - e_1\|$. If f satisfies the $(PS)_c$-condition at level

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = e_0,\ \gamma(1) = e_1\},$$

then c is a critical value of f with $c \ge \max\{f(e_0), f(e_1)\}$.

Remark 1.16. In both Theorems 1.12 and 1.15 the Palais–Smale condition may be replaced by a weaker one, called the *Cerami compactness condition*. More precisely, a function $f \in C^1(X, \mathbb{R})$ satisfies the Cerami condition at level $c \in \mathbb{R}$ (abbreviated to $(C)_c$-condition) if every sequence $\{u_n\} \subset X$, such that $\lim_{n\to\infty} f(u_n) = c$ and $\lim_{n\to\infty}(1 + \|u_n\|)\|f'(u_n)\| = 0$, possesses a convergent subsequence.

1.3.2 Minimax results via the Ljusternik–Schnirelmann category

A very useful method to find multiple critical points for a given functional is related to the notion of the *Ljusternik–Schnirelmann category*. In the case of variational problems on finite dimensional manifolds, this notion is useful to find a lower bound for the number of critical points in terms of topological invariants. In this subsection we present a general form of this approach. We first present some basic properties of Finsler–Banach manifolds.

Let M be a Banach manifold of class C^1; let TM be the usual tangent bundle; and let T_pM be the tangent space at the point $p \in M$. A Finsler structure on the Banach manifold M can be introduced as follows.

Definition 1.17. A Finsler structure on TM is a continuous function $\|\cdot\|: TM \to [0, \infty[$ such that

(a) for each $p \in M$ the restriction $\|\cdot\|_p = \|\cdot\||_{T_pM}$ is a norm on T_pM;
(b) for each $p_0 \in M$ and $k > 1$, there is a neighborhood $U \subset M$ of p_0 such that

$$\frac{1}{k}\|\cdot\|_p \leq \|\cdot\|_{p_0} \leq k\|\cdot\|_p$$

for all $p \in U$.

The Banach manifold M of class C^1 endowed with a Finsler structure is called the Banach–Finsler manifold M of class C^1.

Definition 1.18. Let M be a Banach–Finsler manifold of class C^1 and let $\sigma : [a, b] \to M$ be a C^1-path. The length of σ is defined by $l(\sigma) = \int_a^b \|\sigma'(t)\|dt$. If p and q are in the same component of M we define the distance from p to q as follows:

$$\rho(p, q) = \inf\{l(\sigma) : \sigma \text{ is a } C^1 - \text{path from } p \text{ to } q\}.$$

Definition 1.19. A Banach–Finsler manifold M of class C^1 is said to be complete if the metric space (M, ρ) is complete.

Theorem 1.20. *If M is a Banach–Finsler manifold of class C^1 then the function $\rho: M \times M \to \mathbb{R}$ (from Definition 1.18) is a metric function for each component of M. Moreover, this metric is consistent with the topology of M.*

Theorem 1.21. *If M is a Banach–Finsler manifold of class C^1 and N is a C^1 submanifold of M, then $\|\cdot\| |_{T(N)}$ is a Finsler structure for N (called the Finsler structure* *induced from M). If M is complete and N is a closed C^1 submanifold of M, then N is a complete Banach–Finsler manifold in the induced Finsler structure.*

On a Banach–Finsler manifold M the cotangent bundle $TM^\star$ has a dual Finsler structure defined by

$$\|y\| = \sup\{\langle y, v\rangle : v \in T_pM, \|v\|_p = 1\},$$

where $y \in T_pM^\star$ and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $T_pM^\star$ and T_pM. Consequently, for a functional $f : M \to \mathbb{R}$ of class C^1, the application $p \mapsto \|f'(p)\|$ is well defined and continuous. These facts allow us to introduce the following definition.

Definition 1.22. Let M be a Banach–Finsler manifold of class C^1.

(a) A function $f \in C^1(M, \mathbb{R})$ satisfies the Palais–Smale condition at level $c \in \mathbb{R}$ (abbreviated to $(PS)_c$-condition) if every sequence $\{u_n\} \subset M$, such that $\lim_{n\to\infty} f(u_n) = c$ and $\lim_{n\to\infty} \|f'(u_n)\| = 0$, possesses a convergent subsequence.
(b) A function $f \in C^1(M, \mathbb{R})$ satisfies the Palais–Smale condition (abbreviated to (PS)-condition) if it satisfies the Palais–Smale condition at every level $c \in \mathbb{R}$.

Let M be a Banach–Finsler manifold of class C^1. For $h \geq 1$, we introduce the set

$$\Gamma_h = \{A \subseteq M : \mathrm{cat}_M(A) \geq h,\ A \text{ compact}\}.$$

Here, $\mathrm{cat}_M(A)$ denotes the *Ljusternik–Schnirelmann category* of the set A relative to M; see Appendix C.

If $h \leq \mathrm{cat}_M(M) = \mathrm{cat}(M)$, then $\Gamma_h \neq \emptyset$. Moreover, if $\mathrm{cat}(M) = \infty$, we may take any positive integer h in the above definition. If $f : M \to \mathbb{R}$ is a functional of class C^1, we denote by K the set of all critical points of f on M; that is, $K = \{u \in M : f'(u) = 0\}$ and $K_c = K \cap f^{-1}(c)$. Finally, for every $h \leq \mathrm{cat}(M)$ we define the number

$$c_h = \inf_{A\in\Gamma_h} \max_{u\in A} f(u).$$

Proposition 1.23. *Let $f : X \to \mathbb{R}$ be a functional of class C^1. Using the above notations, we have*

(a) $c_1 = \inf_M f$;
(b) $c_h \leq c_{h+1}$;
(c) $c_h < \infty$ *for every* $h \leq \mathrm{cat}(M)$;
(d) *if $c := c_h = c_{h+m-1}$ for some $h, m \geq 1$, and f satisfies the $(PS)_c$-condition, then $\mathrm{cat}_M(K_c) \geq m$. In particular, $K_c \neq \emptyset$.*

Proof. Properties (a)–(c) are obvious. To prove (d), we argue by contradiction. Thus, we assume that $\mathrm{cat}_M(K_c) \leq m - 1$. Since M is an absolute neighborhood retract (ANR), there exists a neighborhood $\mathcal{O}$ of K_c such that $\mathrm{cat}_M(\overline{\mathcal{O}}) = \mathrm{cat}_M(K_c) \leq m-1$. By using Theorem D.10 in Appendix D, there exists a continuous map $\eta : X \times [0, 1] \to X$ and $\varepsilon > 0$ such that $\eta(f^{c+\varepsilon} \setminus \mathcal{O}, 1) \subset f^{c-\varepsilon}$ and $\eta(u, 0) = u$ for all $u \in X$. Let

$A_1 \in \Gamma_{h+m-1}$ such that $\max_{u \in A_1} f(u) \leq c + \varepsilon$. Considering the set $A_2 = \overline{A_1 \setminus \mathcal{O}}$, we obtain

$$\mathrm{cat}_M(A_2) \geq \mathrm{cat}_M(A_1) - \mathrm{cat}_M(\overline{\mathcal{O}}) \geq h + m - 1 - (m - 1) = h.$$

Therefore, $A_2 \in \Gamma_h$. Moreover, $\max_{u \in A_2} f(u) \leq \max_{u \in A_1} f(u) \leq c + \varepsilon$ and $A_2 \cap \mathcal{O} = \emptyset$. In conclusion, we have $A_2 \subset f^{c+\varepsilon} \setminus \mathcal{O}$. Due to Theorem D.10 (iv), $\eta(A_2, 1) \subset f^{c-\varepsilon}$. Moreover, $\mathrm{cat}_M(\eta(A_2, 1)) \geq \mathrm{cat}_M(A_2) \geq h$, thus $\eta(A_2, 1) \in \Gamma_h$. But $c = \inf_{A \in \Gamma_h} \max_{u \in A} f(u) \leq \max f(\eta(A_2, 1)) \leq c - \varepsilon$, a contradiction. □

The main result of the present subsection is the following theorem.

Theorem 1.24. *Let M be a complete Banach–Finsler manifold of class C^1 and let $f : M \to \mathbb{R}$ be a functional of class C^1 which is bounded from below on M. If f satisfies the (PS)-condition then f has at least* $\mathrm{cat}(M)$ *critical points.*

Proof. Since f is bounded from below, every c_h is finite, $h = 1, \ldots, \mathrm{cat}(M)$; see Proposition 1.23 (c). To prove the statement, it is enough to show that

$$\mathrm{card}(K \cap f^{c_h}) \geq h, \quad h = 1, \ldots, \mathrm{cat}(M). \tag{1.15}$$

We proceed by induction. For $h = 1$, relation (1.15) is obvious since the global minimum belongs to K. Now we assume that (1.15) holds for $h = 1, \ldots, k$. We will prove that (1.15) also holds for $k + 1$. There are two cases as follows.

(I) $c_k \neq c_{k+1}$. Due to Proposition 1.23(d), we have that $K_{c_{k+1}} \neq \emptyset$. Consequently, each element of the set $K_{c_{k+1}}$ clearly differs from those of the set $K \cap f^{c_k}$. Therefore, $K \cap f^{c_{k+1}}$ contains at least $k + 1$ points.

(II) $c_k = c_{k+1} := c$. Let m be the least positive integer such that $c_m = c_{k+1}$. Due to Proposition 1.23(d), we have $\mathrm{cat}_M(K_c) \geq k + 1 - m + 1 = k - m + 2$. In particular, $\mathrm{card}(K_c) = \mathrm{card}(K_{c_{k+1}}) \geq k - m + 2$. We distinguish again two subcases.

(IIa) $m = 1$; the claim easily follows.

(IIb) $m > 1$. Since $m \leq k$, then $\mathrm{card}(K \cap f^{c_{m-1}}) \geq m - 1$. Consequently,

$$\begin{aligned}\mathrm{card}(K \cap f^{c_{k+1}}) &\geq \mathrm{card}(K \cap f^{c_{m-1}}) + \mathrm{card}(K_{c_{k+1}}) \\ &\geq (m - 1) + (k - m + 2) \\ &= k + 1,\end{aligned}$$

which completes the proof of (1.15), and thus the theorem. □

Let X be an infinite dimensional, separable real Banach space and let $S = \{u \in X : \|u\| = 1\}$ be its unit sphere. Important questions appear when the study of some classes of nonlinear elliptic problems reduce to finding critical points of a given functional on S. Note that $\mathrm{cat}(S) = 1$; therefore, Theorem 1.24 does not give multiple critical points on S. However, exploiting the symmetric property of S by means of the *Krasnoselski genus*, see Appendix C, a multiplicity result may be given, similar to Theorem 1.24. In order to state this result, we construct a general framework.

Let X be a real Banach space; we denote the family of sets by

$$\mathcal{A} = \{A \subset X \setminus \{0\} : A = -A,\ A \text{ is closed}\}.$$

Assume that $M = G^{-1}(0) \in \mathcal{A}$ is a submanifold of X, where $G \colon X \to \mathbb{R}$ is of class C^1 with $G'(u) \neq 0$ for every $u \in M$.

For any $h \leq \gamma_0(M) := \sup\{\gamma(K) : K \subseteq M, K \in \mathcal{A},\ K \text{ compact}\}$, we introduce the set

$$\mathcal{A}_h = \{A \subseteq M : A \in \mathcal{A}, \gamma(A) \geq h,\ A \text{ compact}\}.$$

Here, $\gamma(A)$ denotes the Krasnoselski genus of the set A; see Appendix C. Finally, if $f \colon X \to \mathbb{R}$ is an even function, we denote

$$\tilde{c}_h = \inf_{A \in \mathcal{A}_h} \max_{u \in A} f(u).$$

Proposition 1.25. *Let $f : X \to \mathbb{R}$ be an even functional of class C^1. Using the above notations, we have*

(a) $\tilde{c}_1 > -\infty$ *whenever f is bounded from below on M.*
(b) $\tilde{c}_h \leq \tilde{c}_{h+1}$.
(c) $\tilde{c}_h < \infty$ *for every $h \leq \gamma_0(M)$.*
(d) *If $c := \tilde{c}_h = \tilde{c}_{h+m-1}$ for some $h, m \geq 1$, and f satisfies the $(PS)_c$-condition on M, then $\gamma(K_c) \geq m$. In particular, $K_c \neq \emptyset$.*

The following result may be stated.

Theorem 1.26. *Let $M = G^{-1}(0) \subset X$ be a submanifold of a real Banach space, where $G : X \to \mathbb{R}$ is of class C^1 with $G'(u) \neq 0$ for every $u \in M$. Assume that $M \in \mathcal{A}$ and let $f : X \to \mathbb{R}$ be a functional of class C^1 which is bounded from below on M and even. If $\gamma(M) = \infty$ and f satisfies the (PS)-condition on M, then f has infinitely many critical points $\{u_k\} \subset M$ with $\lim_{k\to\infty} f(u_k) = \sup_M f$.*

We conclude this section by stating the following version of the Lagrange multiplier rule.

Theorem 1.27. *Let $(X, \|\cdot\|)$ be a real Banach space, let $\Omega \subset X$ be an open subset, let $f \colon \Omega \to \mathbb{R}$ be a differentiable function, and let $G \colon \Omega \to \mathbb{R}^n$ be of class C^1. Also let $a \in \mathbb{R}^n$ be such that $M = G^{-1}(a)$ is not empty. If $x_0 \in M$ is a solution of the minimization problem*

$$f(x_0) = \min_{x \in M} f(x),$$

and if $DG(x_0)$ is surjective, then there exists $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ such that

$$Df(x_0) = \sum_{i=1}^{n} \lambda_i DG_i(x_0),$$

where $G = (G_1, \dots, G_n)$.

1.4 Ricceri's variational results

The main part of this section is dedicated to Ricceri's recent multiplicity results. First, we discuss three critical points-type results with one or two parameters. In Section 1.4.2 a general variational principle of Ricceri is presented. Finally, in Section 1.4.3 a new kind of multiplicity result of Ricceri is given which guarantees the existence of $k \geq 2$ distinct critical points for a fixed functional.

1.4.1 Three critical point results

It is a simple exercise to show that a function $f : \mathbb{R} \to \mathbb{R}$ of class C^1 having two local minima necessarily has a third critical point. However, once we are dealing with functions defined on a multi-dimensional space, the problem becomes much deeper. The well-known form of the three critical point theorem is due to Pucci and Serrin [187, 188], and it can be formulated as follows.

Theorem 1.28. *A function $f : X \to \mathbb{R}$ of class C^1 satisfies the (PS)-condition and it has two local minima. Then f has at least three distinct critical points.*

Proof. Without loss of generality we may assume that 0 and $e \in X \setminus \{0\}$ are the two local minima of f and $f(e) \leq f(0) = 0$. We have two cases, as follows.

(I) If there exist constants $\alpha, \rho > 0$ such that $\|e\| > \rho$ and $\inf_{\|u\|=\rho} f(u) \geq \alpha$, then the existence of a critical point of f at a minimax level c with $c \geq \alpha$ is guaranteed by Theorem 1.12. Consequently, this critical point certainly differs from 0 and e.

(II) Assume now that constants such as those in (I) do not exist. Since 0 is a local minima of f, we may choose $r < \|e\|$ such that $f(u) \geq 0$ for every $u \in X$ with $\|u\| \leq r$. We apply Ekeland's variational principle, see Theorem 1.3, with $\varepsilon = 1/n^2$, $\lambda = n$, and f restricted to the closed ball $B[0, r] = \{u \in X : \|u\| \leq r\}$. Let us fix $0 < \rho < r$ small enough. Since $\inf_{\|u\|=\rho} f(u) = 0$, there exist $z_n \in X$ with $\|z_n\| = \rho$ and $u_n \in B[0, r]$ such that

$$0 \leq f(u_n) \leq f(z_n) \leq \frac{1}{n^2}, \quad \|u_n - z_n\| \leq \frac{1}{n},$$

and

$$f(w) - f(u_n) \geq -\frac{1}{n}\|w - u_n\|,$$

for all $w \in B[0, r]$. Since $\rho < r$, for n large enough, we have $\|u_n\| < r$. Fix $v \in X$ arbitrarily and let $w = u_n + tv$. If $t > 0$ is small enough then $w \in B[0, r]$.

Therefore, with this choice, the last inequality gives that $\|f'(u_n)\| \leq \frac{1}{n}$. Since $0 \leq f(u_n) \leq \frac{1}{n^2}$ and f satisfies the (PS)-condition, we may assume that $u_n \to u \in X$, which is a critical point of f. Since $|\|u_n\| - \rho| = |\|u_n\| - \|z_n\|| \leq \frac{1}{n}$, we actually have that $\|u\| = \rho$, thus $0 \neq u \neq e$.

□

Let $f : X \times \Lambda \to \mathbb{R}$ be a function such that $f(\cdot, \lambda)$ is of class C^1 for every $\lambda \in \Lambda \subset \mathbb{R}$. In view of Theorem 1.28, the main problem Ricceri dealt with is the *stability* of the three critical points of $f(\cdot, \lambda)$ with respect to the parameters $\lambda \in \Lambda$. Ricceri's main three critical point theorem is the following.

Theorem 1.29. *Let X be a separable and reflexive real Banach space, let $\Lambda \subseteq \mathbb{R}$ be an interval, and let $f : X \times \Lambda \to \mathbb{R}$ be a function satisfying the following conditions:*

(α_1) *for each $u \in X$, the function $f(u, \cdot)$ is continuous and concave;*
(β_1) *for each $\lambda \in \Lambda$, the function $f(\cdot, \lambda)$ is sequentially weakly lower semi-continuous and Gâteaux differentiable, and $\lim_{\|u\| \to \infty} f(u, \lambda) = \infty$;*
(γ_1) *there exists a continuous concave function $h\colon \Lambda \to \mathbb{R}$ such that*

$$\sup_{\lambda \in \Lambda} \inf_{u \in X} (f(u, \lambda) + h(\lambda)) < \inf_{u \in X} \sup_{\lambda \in \Lambda} (f(u, \lambda) + h(\lambda)).$$

Then, there exist an open interval $J \subseteq \Lambda$ and a positive real number ρ, such that, for each $\lambda \in J$, the equation

$$f'_u(u, \lambda) = 0$$

has at least two solutions in X whose norms are less than ρ. If, in addition, the function f is (strongly) continuous in $X \times \Lambda$, and, for each $\lambda \in \Lambda$, the function $f(\cdot, \lambda)$ is of class C^1 and satisfies the (PS)-condition, then the above conclusion holds with "three" instead of "two."

The proof of Theorem 1.29 is quite involved, combining deep arguments from nonlinear analysis as a topological minimax result of Saint Raymond [199], a general selector result of Kuratowski and Ryll-Nardzewski [104, 135], and the mountain pass theorem. Instead of its proof, we give a useful consequence of Theorem 1.29.

Corollary 1.30. *Let X be a separable and reflexive real Banach space, and let $\Phi\colon X \to \mathbb{R}$ be a continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on $X^\star$. Also, let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, and let $\Lambda \subseteq \mathbb{R}$ be an interval. Assume that*

$$\lim_{\|u\| \to \infty} (\Phi(u) - \lambda \Psi(u)) = \infty,$$

for all $\lambda \in \Lambda$, and that there exists a continuous concave function $h : \Lambda \to \mathbb{R}$ such that

$$\sup_{\lambda \in \Lambda} \inf_{u \in X} (\Phi(u) - \lambda \Psi(u) + h(\lambda)) < \inf_{u \in X} \sup_{\lambda \in \Lambda} (\Phi(u) - \lambda \Psi(u) + h(\lambda)).$$

Then there exist an open interval $J \subseteq \Lambda$ and a positive real number ρ such that, for each $\lambda \in J$, the equation

$$\Phi'(u) - \lambda \Psi'(u) = 0$$

has at least three solutions in X whose norms are less than ρ.

Proof. We may apply Theorem 1.29 to the function $f : X \times \Lambda \to \mathbb{R}$ defined by $f(u, \lambda) = \Phi(u) - \lambda \Psi(u)$ for each $(u, \lambda) \in X \times \Lambda$. In particular, the fact that Ψ' is compact implies that Ψ is sequentially weakly continuous. Moreover, our assumptions ensure that, for each $\lambda \in \Lambda$, the function $f(\cdot, \lambda)$ satisfies the (PS)-condition. □

In the following we point out a useful result concerning the *location* of parameters in Corollary 1.30. Actually, this location result is an improved version of Theorem 1.29 given by Bonanno [30].

Theorem 1.31. *Let X be a separable and reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that there exists $u_0 \in X$ such that $\Phi(u_0) = \Psi(u_0) = 0$ and $\Phi(u) \geq 0$ for every $u \in X$ and that there exists $u_1 \in X$, $r > 0$ such that*

(i) $r < \Phi(u_1)$;

(ii) $\displaystyle\sup_{\Phi(u)<r} \Psi(u) < r\frac{\Psi(u_1)}{\Phi(u_1)}$.

Further, put

$$\overline{a} = \frac{\zeta r}{r\dfrac{\Psi(u_1)}{\Phi(u_1)} - \displaystyle\sup_{\Phi(u)<r} \Psi(u)},$$

with $\zeta > 1$, and assume that the functional $\Phi - \lambda\Psi$ is sequentially weakly lower semi-continuous, satisfies the (PS)-condition, and

(iii) $\displaystyle\lim_{\|u\|\to\infty} (\Phi(u) - \lambda\Psi(u)) = \infty$ *for every* $\lambda \in [0, \overline{a}]$.

Then there exists an open interval $\Lambda \subset [0, \overline{a}]$ and a positive real number ρ such that, for each $\lambda \in \Lambda$, the equation

$$\Phi'(u) - \lambda \Psi'(u) = 0$$

admits at least three solutions in X whose norms are less than ρ.

Proof. We prove the minimax inequality from Corollary 1.30 with $\Lambda_0 = [0, \overline{a}]$ and the function h suitably chosen. Fix $\zeta > 1$. Due to (ii), there exists $\sigma \in \mathbb{R}$ such that

$$\sup_{\Phi(u)<r} \Psi(u) + \frac{r\dfrac{\Psi(u_1)}{\Phi(u_1)} - \displaystyle\sup_{\Phi(u)<r} \Psi(u)}{\zeta} < \sigma < r\frac{\Psi(u_1)}{\Phi(u_1)}. \tag{1.16}$$