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Classgroups of Group Rings

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To Sharon

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PREFACE

It gives me great pleasure to thank Ali Fröhlich for all the help and encouragement he has given me, and in particular for the many suggestions and helpful remarks he has made concerning the writing of this book.

I should like to express my thanks to Steve Ullom for numerous conversations and valuable insights, and I also wish to thank Joan Bunn for the typing.

This book first appeared in the form of an essay submitted for the Adams prize at Cambridge University.

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INTRODUCTION

For the most part this book is concerned with modules, which are locally free over an integral group ring, and the consequent problem of determining whether or not the module is globally free. Such questions arise naturally in both algebraic number theory and in algebraic topology. In the former, the standard such question is that of the existence of a normal integral basis. That is to say, given a Galois extension of number fields N/K, we wish to know whether $\mathcal{O}_N$, the ring of integers of N, has a basis over the ring of integers of a subfield $F \subset K$ which has the form $\{a_i^\gamma\}$, for γ running through Gal(N/K). An alternative way of considering this is to ask whether or not $\mathcal{O}_N$ is free over the group algebra $\mathcal{O}_F$ Gal(N/K). In the second area of application, that of algebraic topology, C.T.C. Wall has introduced a locally free module whose deviation from being globally free represents an obstruction to finding a finite complex in the homotopy type of a given space.

For a group ring R, $C\ell(R)$ is defined in a manner which closely resembles the way in which the ideal classgroup of a Dedekind domain is defined. In place of taking ideals modulo principal ideals, in essence we take locally free modules modulo free modules. This construction will be made precise in chapter 1.

The principal aim of this book is to instruct the reader in sufficient techniques to enable him, when given a locally free R module M, to calculate the class of M in $C\ell(R)$ and thereby, in many cases, determine whether or not M is globally free.

In the 1960's Swan and Jacobinski gave abstract descriptions of locally free (or projective) classgroups. However, explicit calculations of these classgroups were possible in only a few cases. Subsequently Reiner and Ullom introduced the elegant technique of Mayer-Vietoris sequences to the subject. This proved to be quite a powerful tool and it substantially increased our knowledge of classgroups of group rings. There is an excellent account of the level of knowledge obtained by such methods in S. Ullom's survey article [U3].

There then came a very important turning point in the theory of such classgroups when A. Fröhlich, motivated by the normal integral basis problem, introduced a completely new description of such classgroups. He was able to show that such classgroups are naturally isomorphic to the quotient of two groups of homomorphisms from the virtual characters of the group in question, taking idelic values. With this new viewpoint even the most basic properties were immediately better understood. As an example we consider certain elementary functorial properties. Previous descriptions of such classgroups had nearly always been in terms of families of ideals, one for each absolutely irreducible character of the underlying group. If we now change group, by means of induction resp. restriction of module, this then corresponds to restriction resp. induction on the characters of the underlying group. Of course, in general, induction and restriction do not preserve irreducibility of characters. They do, however, induce natural maps on homomorphisms from virtual characters. To underline the advantage of this change in viewpoint, we mention one further development. Presently we shall see that the Adams operations of character theory play a fundamental role in the theory of classgroups. In general, however, an Adams operation takes an irreducible character to a virtual character (and not an actual character, let alone an irreducible character).

Whilst Fröhlich's description of classgroups represents a fundamental change in view point, it does not, however, solve the basic problems. The point being that while the numerator in his description of the classgroup as the quotient of two groups was very clearly understood, the denominator was exceedingly difficult to handle. This necessitated a further development called the group logarithm which was first introduced by the author in [T1]. Essentially the group logarithm is the usual p-adic logarithm twisted by means of an Adams operation. Very often, adroit use of this logarithm, together with Fröhlich's description, allows us to decide whether or not the class of a module is trivial or not. The power of this technique is well illustrated by the normal integral basis problem. Here, the above work, when allied to Fröhlich's Gauss sum resolvent relation, enables one to describe the class of a ring of integers over a group ring whose coefficients are $\mathbb{Z}$, whenever the ring