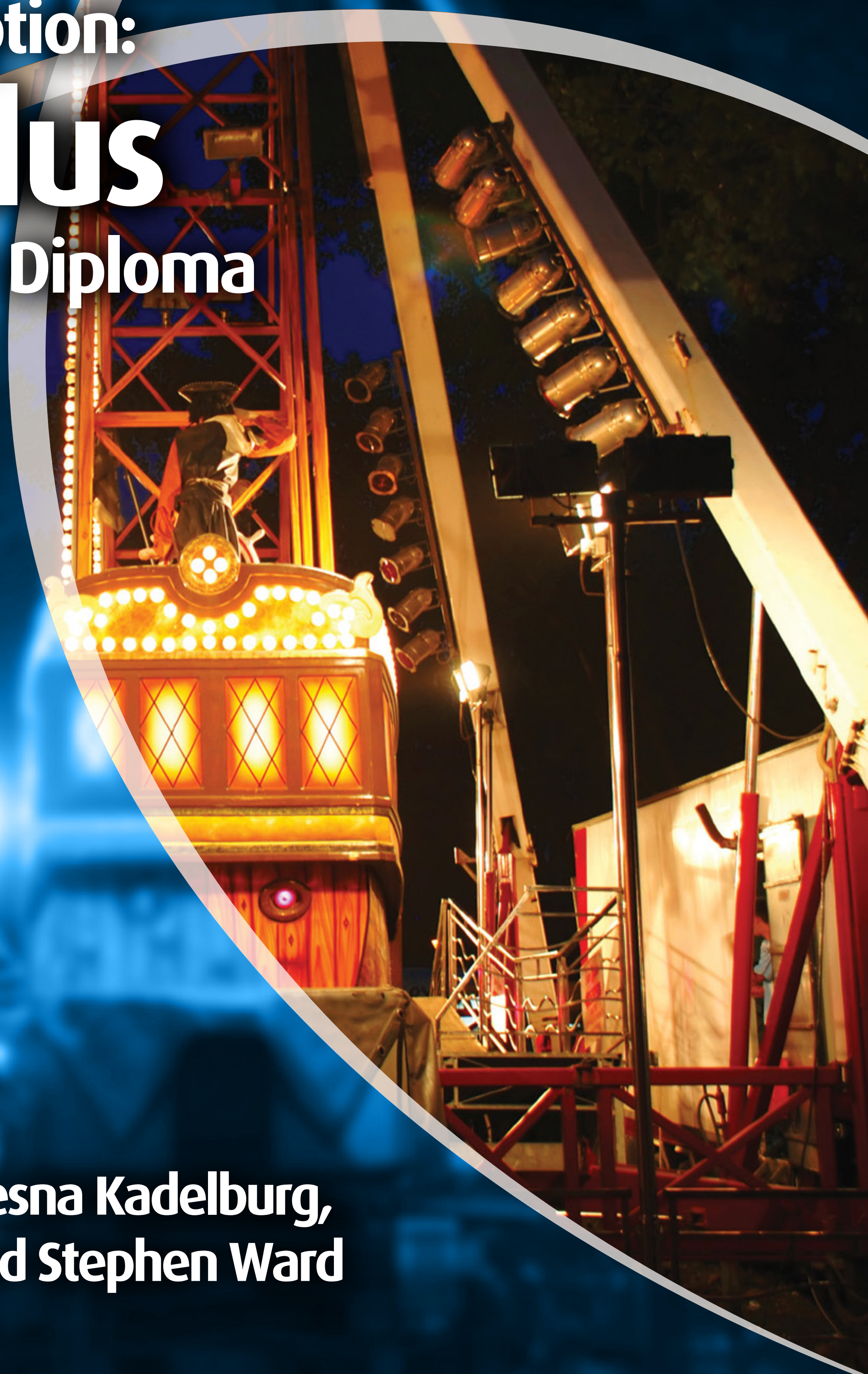
CAMBRIDGE

Mathematics Higher Level Topic 9 – Option: Calculus for the IB Diploma



Paul Fannon, Vesna Kadelburg, Ben Woolley and Stephen Ward

Mathematics Higher Level Topic 9 – Option:

Calculus

for the IB Diploma

Paul Fannon, Vesna Kadelburg,
Ben Woolley and Stephen Ward

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town,
Singapore, São Paulo, Delhi, Mexico City

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

www.cambridge.org
Information on this title: www.cambridge.org/9781107632899

First published 2013

Printed in Poland by Opolgraf

A catalogue record for this publication is available from the British Library

ISBN 978-1-107-63289-9 Paperback

Cover image: Thinkstock

Contents

How to use this book

Structure of the book

This book covers all the material for Topic 9 (Calculus Option) of the Higher Level Mathematics syllabus for the International Baccalaureate course. It assumes familiarity with the core material (syllabus topics 1 to 6), particularly Calculus (syllabus topic 6) and Sequences and Series (syllabus topic 1.1). We have tried to include in the main text only the material that will be examinable. There are many interesting applications and ideas that go beyond the syllabus and we have tried to highlight some of these in the 'From another perspective' and 'Research explorer' boxes.

The five main chapters are probably best covered in the order presented, although chapter 5 (except for the last section) only requires knowledge of the core calculus material. Chapter 6 contains a summary of all the topics and further examination practice, with many of the questions mixing several topics – a favourite trick in IB examinations.

Each chapter starts with a list of learning objectives to give you an idea about what the chapter contains. There is an introductory problem at the start of the topic that illustrates what you should be able to do after you have completed the topic. You should not expect to be able to solve the problem at the start, but you may want to think about possible strategies and what sort of new facts and methods would help you. The solution to the introductory problem is provided at the end of chapter 6.

Key point boxes

The most important ideas and formulae are emphasised in the 'KEY POINT' boxes. When the formulae are given in the Formula booklet, there will be an icon: ; if this icon is not present, then the formulae are ***not*** in the Formula booklet and you may need to learn them or at least know how to derive them.

Worked examples

Each worked example is split into two columns. On the right is what you should write down. Sometimes the example might include more detail then you strictly need, but it is designed to give you an idea of what is required to score full method marks in examinations. However, mathematics is about much more than examinations and remembering methods. So, on the left of the worked examples are notes that describe the thought processes and suggest which route you should use to tackle the question. We hope that these will help you with any exercise questions that differ from the worked examples. It is very deliberate that some of the questions require you to do more than repeat the methods in the worked examples. Mathematics is about thinking!

Signposts

There are several boxes that appear throughout the book.

Theory of knowledge issues

Every lesson is a Theory of knowledge lesson, but sometimes the links may not be obvious. Mathematics is frequently used as an example of certainty and truth, but this is often not the case. In these boxes we will try to highlight some of the weaknesses and ambiguities in mathematics as well as showing how mathematics links to other areas of knowledge.

From another perspective

The International Baccalaureate® encourages looking at things in different ways. As well as highlighting some international differences between mathematicians these boxes also look at other perspectives on the mathematics we are covering: historical, pragmatic and cultural.

Research explorer

As part of your course, you will be asked to write a report on a mathematical topic of your choice. It is sometimes difficult to know which topics are suitable as a basis for such reports, and so we have tried to show where a topic can act as a jumping-off point for further work. This can also give you ideas for an Extended essay. There is a lot of great mathematics out there!

Exam hint

Although we would encourage you to think of mathematics as more than just learning in order to pass an examination, there are some common errors it is useful for you to be aware of. If there is a common pitfall we will try to highlight it in these boxes.

EXAM HINT

Fast forward / rewind

Mathematics is all about making links. You might be interested to see how something you have just learned will be used elsewhere in the course, or you may need to go back and remind yourself of a previous topic. These boxes indicate connections with other sections of the book to help you find your way around.

How to use the questions

The colour-coding

The questions are colour-coded to distinguish between the levels.

Black questions are drill questions. They help you practise the methods described in the book, but they are usually not structured like the questions in the examination. This does not mean they are easy, some of them are quite tough.

Each differently numbered drill question tests a different skill. Lettered subparts of a question are of increasing difficulty. Within each lettered part there may be multiple roman-numeral parts ((i), (ii),...) , all of which are of a similar difficulty. Unless you want to do lots of practice we would recommend that you only do one roman-numeral part and then check your answer. If you have made a mistake then you may want to think about what went wrong before you try any more. Otherwise move on to the next lettered part.

Green questions are examination-style questions which should be accessible to students on the path to getting a grade 3 or 4.

Blue questions are harder examination-style questions. If you are aiming for a grade 5 or 6 you should be able to make significant progress through most of these.

Red questions are at the very top end of difficulty in the examinations. If you can do these then you are likely to be on course for a grade 7.

Gold questions are a type that are *not* set in the examination, but are designed to provoke thinking and discussion in order to help you to a better understanding of a particular concept.

At the end of each chapter you will see longer questions typical of the second section of International Baccalaureate® examinations. These follow the same colour-coding scheme.

Of course, these are just **guidelines**. If you are aiming for a grade 6, do not be surprised if you find a green question you cannot do. People are never equally good at all areas of the syllabus. Equally, if you can do all the red questions that does not guarantee you will get a grade 7; after all, in the examination you have to deal with time pressure and examination stress!

These questions are graded relative to our experience of the final examination, so when you first start the course you will find all the questions relatively hard, but by the end of the course they should seem more straightforward. Do not get intimidated!

We hope you find the Calculus Option an interesting and enriching course. You might also find it quite challenging, but do not get intimidated, frequently topics only make sense after lots of revision and practice. Persevere and you will succeed.

The author team.

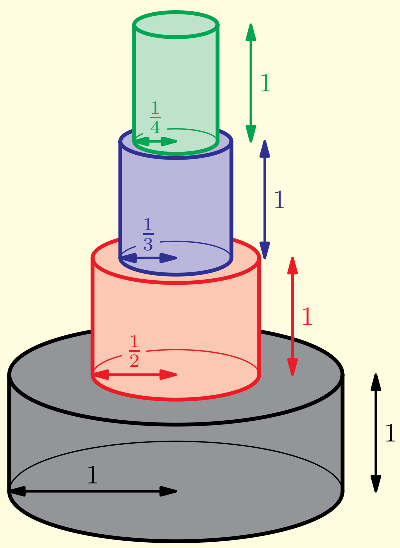
Introduction

In this Option you will learn:

- how to find a limit of a sequence (the value that the terms of the sequence approach)
- about various methods for finding limits of functions, including L'Hôpital's Rule
- formal definitions of continuity and differentiability of functions, and some useful properties of differentiable functions
- how we can extend definite integration to allow one of the limits to tend to infinity
- how to decide whether infinite series have a limit
- how to use series to approximate various functions (Maclaurin and Taylor series)
- how to solve certain types of differential equations, both exactly and approximately.

Introductory problem

Consider a wedding cake with four layers:



Each layer has a thickness of 1. The first layer has a radius of 1, the second a radius of $\frac{1}{2}$, the third a radius of $\frac{1}{3}$ and the fourth a radius of $\frac{1}{4}$.

Find the volume of the cake and the surface area (excluding the bottom of the first layer) that needs covering with icing.

Now imagine there are infinitely many layers to the cake. What can you say about the volume of the cake and the surface area that needs icing now?

The calculus option builds on many different areas of the course, using the theory of functions, sequences and series, differentiation and integration. Some of the ideas you will study here have interested many of the greatest minds in mathematics: Euler, Newton, Leibniz, Cauchy, Riemann and others; and have far reaching applications both within mathematics and other subjects such as physics and engineering.

The option is split into five chapters:

1 Limits of sequences and functions looks first at the behaviour of sequences as we take more and more terms. How can we determine which sequences head towards (converge to) a finite value when we take infinitely many terms, and which just become infinitely large? We then examine the values of functions as we get closer and closer to a particular value in their domain. The focus is particularly on quotients where both the numerator and denominator are tending to 0 or where they are both tending to ∞. Building on the idea of a limit of a function, we then look at where functions are continuous and where they can be differentiated, before considering two important theorems for differentiable functions.

2 Improper integrals looks at what happens when we take the upper limit on an integral to be ∞. Will this always give an infinite value for the integral (and hence an infinite area under the curve), or are there some circumstances in which that value (or area) is finite? And again how do we know which is the case for a given integral? This chapter starts with a look at the Fundamental Theorem of Calculus.

3 Infinite series builds on both the previous chapters and examines the sums of infinitely many terms. We have already seen with some geometric series that it is possible to add together infinitely many terms and yet get a finite value for the sum to infinity; here we look at a wide range of different series and develop ways to test if the series converges (has a finite value) or not. We also link improper integrals and infinite sums and look at ways of placing bounds on the value of an infinite sum.

4 Maclaurin and Taylor series establishes a way of representing many familiar functions as infinite series in increasing integer powers of x. We examine polynomial approximations for functions, and their accuracy, and use the series representation of functions for integration and to calculate limits, again particularly those where both numerator and denominator are tending to 0 or ∞.

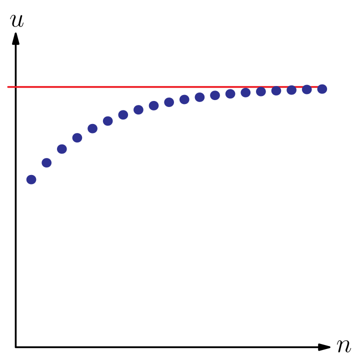
5 Differential equations looks at setting up and solving differential equations, which are used to describe the behaviour of many processes in nature and engineering. We look at three methods for solving different types of these equations and then consider some methods to find approximate solutions to differential equations that are difficult (if not impossible) to solve in the standard way.

1 Limits of sequences and functions

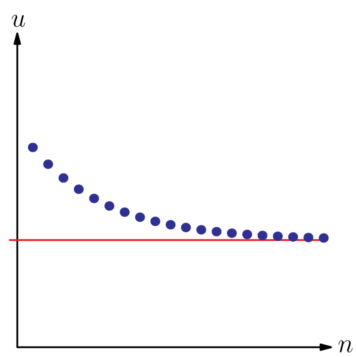
In this chapter you will learn:

- to use algebraic rules to calculate limits of some sequences
- to find the limit of a sequence by squeezing it between two sequences that both converge to the same limit
- to apply similar principles to find limits of functions
- to use l'Hôpital's Rule to find limits of the form '$\frac{0}{0}$' and '$\frac{\infty}{\infty}$'
- to determine where functions are continuous and where they are differentiable
- to apply Rolle's Theorem and the Mean Value Theorem to differentiable functions.

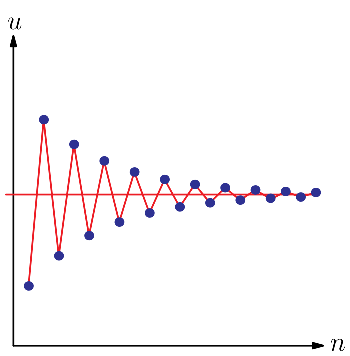
You should have met sequences in the core course, and will be familiar with using a general term u_n to define them. Our main focus now is to see what happens as we take more and more terms of a sequence: does there seem to be some finite value (a limit) which the sequence approaches, and if so does the sequence do so by increasing, decreasing, or perhaps by oscillating either side of the value?



terms increase to a limit

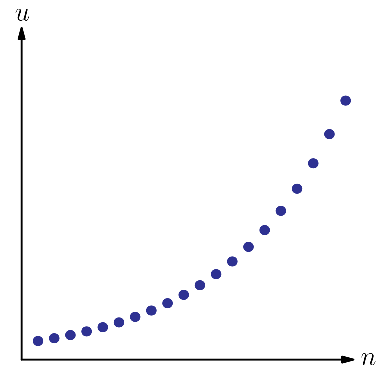


terms decrease to a limit

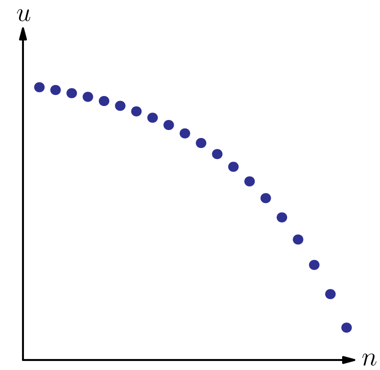


terms approach limit with oscillation

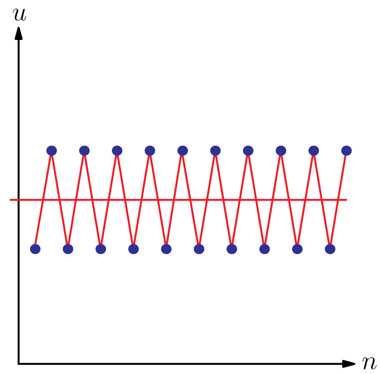
Alternatively, does the sequence not have a (finite) limit, due to just getting larger in magnitude or perhaps oscillating with the same magnitude either side of some value?



terms get progressively greater



terms get progressively smaller



terms oscillate

Sometimes it is immediately obvious which of these is taking place.

For example, the sequence with general term $u_n = 2^{n-1}$
$1, 2, 4, 8, 16, \ldots$
is clearly just getting larger term by term and will never approach a limit.

The sequence defined inductively by $u_1 = 3,\ u_{n+1} = \dfrac{1}{4-u_n}$

$$3,\ 1, \frac{1}{3}, \frac{3}{11}, \frac{11}{41}, \frac{41}{153} \ldots$$

seems to be decreasing, but it is not clear whether it will have a limit or if it does, what this limit might be.

In this chapter we start by looking at ways to answer these questions about sequences and then widen our focus to look at the limits of functions. We then use the idea of limits of functions to consider where a function is continuous and where it can be differentiated and finally apply these ideas to develop two well known and useful theorems.

1A The limit of a sequence

If the terms of a sequence u_k head towards a value L, we say that the sequence **converges** to a limit L, as $n \to \infty$ ('n tends to ∞') and we write:

$$\lim_{n\to\infty} u_n = L$$

This does not necessarily mean that any term of the sequence actually reaches the value L, but that, by taking more and more terms, the sequence becomes arbitrarily close to L.

For example, the sequence $\{u_n\} = \dfrac{1}{n}$

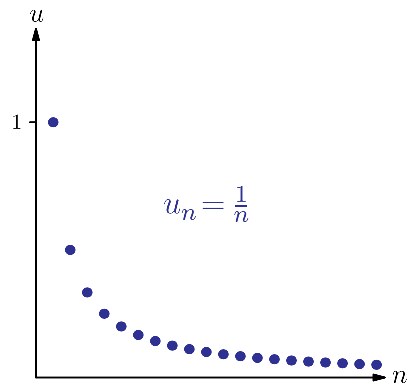
$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7} \ldots$$

appears to be converging to 0, that is:

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

although no term actually is 0.

Graphically we have (see alongside):



If the sequence does not converge then it is said to **diverge**.

Often, the behaviour of a sequence is not so apparent and we may want to find a possible limit without evaluating lots of terms or drawing a graph.

For a sequence such as $\{u_n\} = \frac{6n+72}{2n+97}$, we might consider trying to take the limits of both the numerator and denominator separately and then dividing one by the other. The problem here is that both numerator and denominator diverge and '$\frac{\infty}{\infty}$' is not defined.

We can, however, take this approach if the limit of both sequences is finite, as then the limits do behave much as you might expect (or at least hope!).

KEY POINT 1.1

Algebra of limits

If the sequence $\{a_n\}$ converges to a limit a and the sequence $\{b_n\}$ to a limit b, then:

$$\lim_{n\to\infty}(pa_n + qb_n) = p\lim_{n\to\infty} a_n + q\lim_{n\to\infty} b_n = pa + qb \qquad (p, q \in \mathbb{R})$$

$$\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = ab$$

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{a}{b} \quad (b \neq 0)$$

The algebra of limits only applies when the limits exist (they are finite) but we will often make use of the following two supplements to the final point.

KEY POINT 1.2

If the sequence $\{a_n\}$ diverges, then for any constant $c \in \mathbb{R}$ $(c \neq 0)$:

$$\lim_{n\to\infty}\left(\frac{c}{a_n}\right) = 0$$

$$\lim_{n\to\infty}\left(\frac{a_n}{c}\right) = \infty$$

While this might not seem helpful for $\{u_n\} = \frac{6n+72}{2n+97}$ where both numerator and denominator diverge, the following example illustrates a common way of dealing with this problem, allowing us to apply the algebra of limits result.

Worked example 1.1

Find the following:

(a) $\lim_{n\to\infty}\frac{6n+72}{2n+97}$ (b) $\lim_{n\to\infty}\frac{n^2+5}{2n^2-3n+8}$

Neither $6n+72$ nor $2n+97$ converges so we must rearrange the expression to give a numerator and denominator that both converge, by dividing through by n

(a)

$$\frac{6n+72}{2n+97}=\frac{6+\frac{72}{n}}{2+\frac{97}{n}}$$

Now apply the algebra of limits results

$$\therefore \lim_{n\to\infty}\frac{6n+72}{2n+97}=\lim_{n\to\infty}\frac{6+\frac{72}{n}}{2+\frac{97}{n}}$$

$\lim_{n\to\infty}\frac{72}{n}=0$ and $\lim_{n\to\infty}\frac{97}{n}=0$ by Key point 1.2

$$=\frac{\lim_{n\to\infty}\left(6+\frac{72}{n}\right)}{\lim_{n\to\infty}\left(2+\frac{97}{n}\right)}$$

$$=\frac{6+0}{2+0}$$

$$=3$$

Neither n^2+5 nor $2n^2-3n+8$ converges so rearrange the expression to give a numerator and denominator that both converge, by dividing through by n^2

(b)

$$\frac{n^2+5}{2n^2-3n+8}=\frac{1+\frac{5}{n^2}}{2-\frac{3}{n}+\frac{8}{n^2}}$$

Now apply the algebra of limits results

$$\therefore \lim_{n\to\infty}\frac{n^2+5}{2n^2-3n+8}=\lim_{n\to\infty}\frac{1+\frac{5}{n^2}}{2-\frac{3}{n}+\frac{8}{n^2}}$$

$\lim_{n\to\infty}\frac{5}{n^2}=0$, $\lim_{n\to\infty}\frac{3}{n}=0$ and $\lim_{n\to\infty}\frac{8}{n^2}=0$, all by Key point 1.2

$$=\frac{\lim_{n\to\infty}\left(1+\frac{5}{n^2}\right)}{\lim_{n\to\infty}\left(2-\frac{3}{n}+\frac{8}{n^2}\right)}$$

$$=\frac{1+0}{2-0+0}$$

$$=\frac{1}{2}$$

EXAM HINT

To apply the algebra of limits result to a fraction, first divide through by the highest power of n.

It is tempting to think that the limit of the difference of two sequences that diverge to infinity is 0, i.e. that '$\infty-\infty=0$', but we must not apply the algebra of limits to non-convergent sequences. (In fact '$\infty-\infty$' might be 0, but it might be some other finite value or it might even be ∞!) In these cases it is necessary to manipulate the expression into a convenient form for applying the algebra of limits and/or Key point 1.2.

Worked example 1.2

Find $\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right)$.

Neither $\sqrt{n^2+n}$ nor n converges so we can't apply the algebra of limits to the difference. Neither do we have the familiar situation of a quotient where we could divide through as in Worked example 1.1

So create a quotient by multiplying top and bottom by the same expression. The presence of $\sqrt{a}-b$ here suggests multiplying by $\sqrt{a}+b$

$$\sqrt{n^2+n}-n=\left(\sqrt{n^2+n}-n\right)\left(\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}\right)$$

$$=\frac{\left(\sqrt{n^2+n}\right)^2-n^2}{\sqrt{n^2+n}+n}$$

$$=\frac{\left(n^2+n\right)-n^2}{\sqrt{n^2+n}+n}$$

$$=\frac{n}{\sqrt{n^2+n}+n}$$

Divide through by n (this will mean dividing through inside the square root by n^2)

$$=\frac{1}{\sqrt{\frac{n^2}{n^2}+\frac{n}{n^2}}+1}$$

$$=\frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

Now apply the algebra of limits

$$\therefore \lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right)=\frac{1}{\sqrt{1+0}+1}=\frac{1}{2}$$

Exercise 1A

1. Find the following limits:

(a) (i) $\lim_{n\to\infty}\frac{3n-7}{2n+1}$

(ii) $\lim_{n\to\infty}\frac{4n+11}{3-2n}$

(b) (i) $\lim_{n\to\infty}\frac{n^2+3n-1}{3n^2+5n-7}$

(ii) $\lim_{n\to\infty}\frac{4n^4+3n^3-10}{n^4+5}$

(c) (i) $\lim_{n\to\infty}\frac{5-3n}{2n^2+4n+9}$

(ii) $\lim_{n\to\infty}\frac{n^2-6}{n^3+5n-3}$

(d) (i) $\lim_{n\to\infty}\left(\frac{5-3n}{7-4n}\right)^2$

(ii) $\lim_{n\to\infty}\sqrt{\frac{9n^2+2}{4n^2+n+1}}$

2. (a) Show that $\lim_{n\to\infty}\frac{3n-2}{5n+7}=\frac{3}{5}$

(b) Hence find $\lim_{n\to\infty}\left(\frac{3n-2}{5n+7}\right)^3$ *[3 marks]*

3. (a) Show that $\frac{n(n+4)}{n+2}-\frac{n^3}{n^2+3}=\frac{2n^3+3n^2+12n}{n^3+2n^2+3n+6}$

(b) Hence find $\lim_{n\to\infty}\left(\frac{n(n+4)}{n+2}-\frac{n^3}{n^2+3}\right)$ *[6 marks]*

4. For the sequence $\{u_n\}$, whose general term is given by

$$u_n=\sqrt{n+1}-\sqrt{n-1}$$

find $\lim_{n\to\infty}u_n$ *[6 marks]*

5. Use the algebra of limits to find $\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n-3}+2}$ *[7 marks]*

6. Use the algebra of limits to prove that

$$\lim_{n\to\infty}\frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n}}=2$$ *[4 marks]*

1B The Squeeze Theorem

In addition to the methods already described for finding limits of sequences, there is another result which we need to have at our disposal.

KEY POINT 1.3

Squeeze Theorem

If we have sequences $\{a_n\},\{b_n\}$ and $\{c_n\}$ such that

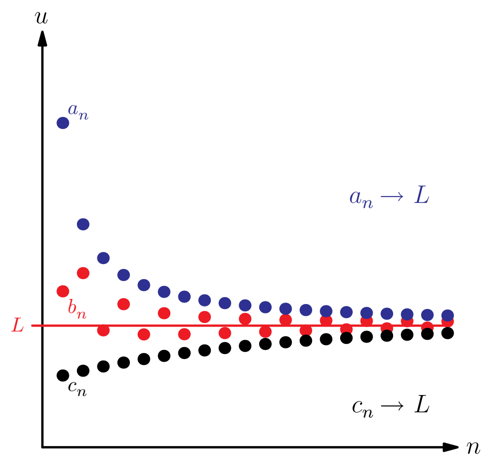
$a_n\le b_n\le c_n$ for all $n\in\mathbb{Z}^+$

and

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}c_n=L<\infty$$

then $\lim_{n\to\infty}b_n=L$.

The Squeeze Theorem says that if we can find two sequences that converge to the same limit and squeeze another sequence between them, then that sequence must also converge to the same limit.



To use this result we often need known inequalities for standard functions.

Worked example 1.3

Use the Squeeze Theorem to find $\lim_{n\to\infty} \frac{\sin n}{n}$.

The obvious place to start is by bounding $\sin n$

$-1 \le \sin n \le 1 \quad \text{for all } n \in \mathbb{Z}^+$

$\Rightarrow \frac{-1}{n} \le \frac{\sin n}{n} \le \frac{1}{n} \quad \text{for all } n \in \mathbb{Z}^+$

As both $\left\{\frac{-1}{n}\right\}$ and $\left\{\frac{1}{n}\right\}$ converge to 0 we can apply the Squeeze Theorem

Since

$\lim_{n\to\infty}\left(-\frac{1}{n}\right) = \lim_{n\to\infty}\frac{1}{n} = 0$

by the Squeeze Theorem

$\lim_{n\to\infty}\frac{\sin n}{n} = 0$

When you use the Squeeze Theorem it can be difficult to choose one, or both, of the sequences a_n and c_n. The following illustrates a common way of doing this.

Worked example 1.4

Show that $\lim_{n\to\infty}\frac{n!}{n^n} = 0$.

We can see that $\frac{n!}{n^n} > 0$ so we are looking for a sequence that is always at least as large as $\frac{n!}{n^n}$ but which tends to 0 as $n \to \infty$ so that we can squeeze $\frac{n!}{n^n}$

$\frac{n!}{n^n} > 0 \quad \text{for all } n \in \mathbb{Z}^+$

continued . . .

To find such a sequence start by writing out the general term of our sequence long-hand

Next,
$$\frac{n!}{n^n}=\frac{n}{n}\frac{(n-1)}{n}\frac{(n-2)}{n}\frac{(n-3)}{n}\cdots\frac{3}{n}\frac{2}{n}\frac{1}{n}$$

Each numerator is less than n so use that to introduce the inequality.
We leave the final term unaltered so that not everything cancels out

$$<\frac{n}{n}\frac{n}{n}\frac{n}{n}\frac{n}{n}\cdots\frac{n}{n}\frac{n}{n}\frac{1}{n}=\frac{1}{n}$$

$$0<\frac{n!}{n^n}<\frac{1}{n} \quad \text{for all } n\in\mathbb{Z}^+$$

Apply the Squeeze Theorem

Since $\lim\limits_{n\to\infty}\frac{1}{n}=0$

we can conclude that $\lim\limits_{n\to\infty}\frac{n!}{n^n}=0$

by the Squeeze Theorem

Exercise 1B

1. Use the Squeeze Theorem to find the following:

(a) $\lim\limits_{n\to\infty}\frac{\cos n}{2n}$

(b) $\lim\limits_{n\to\infty}\frac{4n^2-\sin 3n}{n^2+8}$

(c) $\lim\limits_{n\to\infty}\frac{n^n}{(2n)!}$

(d) $\lim\limits_{n\to\infty}\frac{\arctan(n^2)}{\sqrt{n}}$

EXAM HINT

It is a common trick to use part of the binomial expansion to introduce an inequality such as in question 3. Look back at the core course for a reminder of the binomial expansion.

2. (a) Show that $\frac{6^n}{n!}\le\frac{6^5}{5!}\times\frac{6}{n}$ for $n\ge 6$.

(b) Hence use the Squeeze Theorem to show that $\lim\limits_{n\to\infty}\frac{6^n}{n!}=0$.

[6 marks]

3. (a) Show that $(1+x)^n>\frac{n(n-1)}{2}x^2$ for all $x>0, n\in\mathbb{Z}^+$.

(b) By taking $x=\sqrt{\frac{2}{n-1}}$ show that $\left(\sqrt[n]{n}-1\right)^2<\frac{2}{n-1}$ $n\neq 1$.

(c) Hence find $\lim\limits_{n\to\infty}\sqrt[n]{n}$.

[8 marks]

4. (a) Find $\lim\limits_{n\to\infty}\frac{n}{\sqrt{n^2+n}}$

(b) Hence use the Squeeze Theorem to find $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{\sqrt{n^2+k}}$

[7 marks]

5. (a) By considering $(1+x)^n$ for a suitable value of x, show that

$$2^n>\frac{n(n-1)(n-2)}{3!} \text{ for all } n\in\mathbb{Z}^+.$$

(b) Hence find $\lim\limits_{n\to\infty}\frac{n^2}{2^n}$.

[8 marks]

6. (a) Find the value a at which the tangent to $y = \ln x$ passing through the origin touches the curve.

(b) Hence form an inequality involving x and $\ln x$ for $x > 0$.

(c) Find $\lim\limits_{n\to\infty} \dfrac{3n^3 + \ln n^3}{n^3}$.

7. (a) Show that $\dfrac{1}{n} < \ln n$ for all $n > 1$.

(b) Show that $\ln(1+n) < n$ for all $n \geq 1$.

(c) Hence show that $\dfrac{n}{1+n} < \ln\left(1+\dfrac{1}{n}\right)^n < 1$ for all $n \geq 1$.

(d) (i) Find $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$

(ii) What assumptions have you made? *[15 marks]*

The limit established in Question 7, part (d)(i) is one of the most important in maths. It is often introduced informally in the context of percentage increase, but has many wide-ranging applications. We will meet it again in chapter 3 of this option.

1C The limit of a function

The idea of the limit of a function $f(x)$ at a point a, $\lim\limits_{x\to a} f(x)$, is like that of the limit of a sequence: it is either the value that the function attains at a or approaches as $x \to a$. The difference is that the domain will not just be positive integers and that the limit can be taken as x tends to any value, not just as $x \to \infty$.

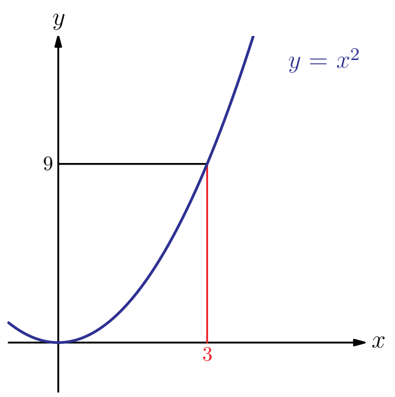
For example:

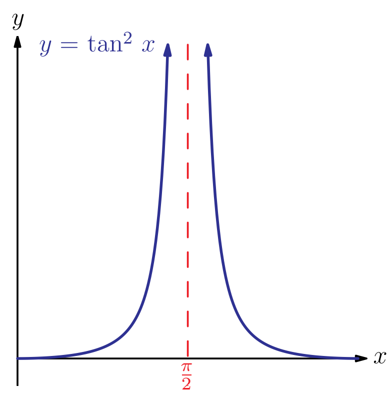
$$\lim_{x\to 3} x^2 = 3^2 = 9$$

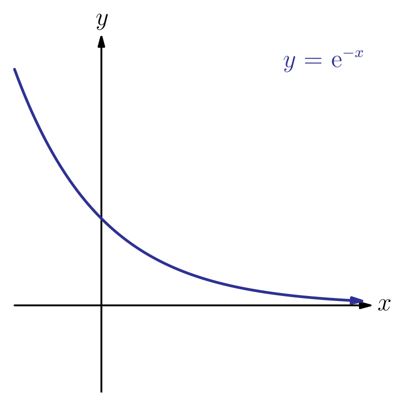
$$\lim_{x\to\frac{\pi}{2}} \tan^2 x = \infty$$

$$\lim_{x\to\infty} \mathrm{e}^{-x} = 0$$

While we cannot input $x = \dfrac{\pi}{2}$ into $\tan^2 x$ to get the limit in the second example above (as it is not in the domain) or $x = \infty$ into e^{-x} to get the limits of ∞ and 0 respectively, we can see that the functions tend to these limits by referring to their graphs.







Using a combination of two of these limits, we could also say

$$\lim_{x\to\frac{\pi}{2}} \mathrm{e}^{-\tan^2 x} = \mathrm{e}^{-\lim\limits_{x\to\frac{\pi}{2}} \tan^2 x} = 0$$