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Ido Cohen and Guy Gilboa

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A Koopman Theory Analysis

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Abstract: Extracting the latent underlying structures of complex nonlinear local and nonlocal flows is essential for their analysis and modeling. In this Element the authors attempt to provide a consistent framework through Koopman theory and its related popular discrete approximation – dynamic mode decomposition (DMD). They investigate the conditions to perform appropriate linearization, dimensionality reduction, and representation of flows in a highly general setting. The essential elements of this framework are Koopman eigenfunctions (KEFs) for which existence conditions are formulated. This is done by viewing the dynamic as a curve in state-space. These conditions lay the foundations for system reconstruction, global controllability, and observability for nonlinear dynamics. They examine the limitations of DMD through the analysis of Koopman theory and propose a new mode decomposition technique based on the typical time profile of the dynamics.

Keywords: nonlinear decomposition, dynamic mode decomposition, homogeneous operators, gradient flows, nonlinear spectral theory, Koopman eigenfunctions, Koopman mode decomposition

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1 Introduction

Knowing the latent space of certain data allows one to represent it concisely and to differentiate between signal and clutter parts. Recovering this space in a data-driven manner is a long-standing research problem. Data resulting from dynamical systems is represented commonly as spatial structures (modes) that are attenuated or enhanced with time. A common technique in linear flows is *separation of variables*. It is assumed that a solution $u(x,t)$ of a linear flow can be expressed as

$$u(x,t) = X(x)T(t). \tag{1.1}$$

That is, the solution is a multiplication of a function of the spatial variable x and a function of the temporal variable t. In this study we examine, from various angles, the following paradigm: a *nonlinear flow* can be well approximated (or even exactly expressed) by a linear combination of variable separated functions,

$$u(x,t) \approx \sum_{i=1}^{m} X_i(x)T_i(t). \tag{1.2}$$

In this context, the spatial structures X_i are referred to as *modes* and T_i are *time-profiles*. For such an approximation, if the error is negligible and m is small, we obtain a significant simplification of the system. This enables better understanding and modeling, allowing accurate interpolation and prediction of the dynamics.

The theory of Koopman argues that for many nonlinear systems data measurements evolve as if the dynamical system is linear (in some infinite-dimensional space). A well-known algorithm to approximate these measurements is *Dynamic Mode Decomposition* (DMD) of Schmid (2010). In this work, we formulate sufficient and necessary conditions for the existence of these measurements. These findings highlight certain flaws of DMD. Finally, we suggest a new mode decomposition to overcome some of these problems, originated in an algorithm for general spectral decomposition of Gilboa (2018).

In many dynamical processes, there are measurements of the observations that evolve linearly, or approximately so; see Otto and Rowley (2021). A theoretical justification for that can be traced back to the seminal work of Koopman (1931). These measurements are referred to as *Koopman Eigenfunctions* (KEFs). An algorithm was proposed by Mezić (2005), *Koopman Mode Decomposition* (KMD), to reconstruct the dynamics using spatial structures, termed as modes, which are the coefficients of Koopman eigenfunctions. Since KEFs evolve as if they were observations in a linear dynamical system, KMD can interpret the original dynamics as a linear one.