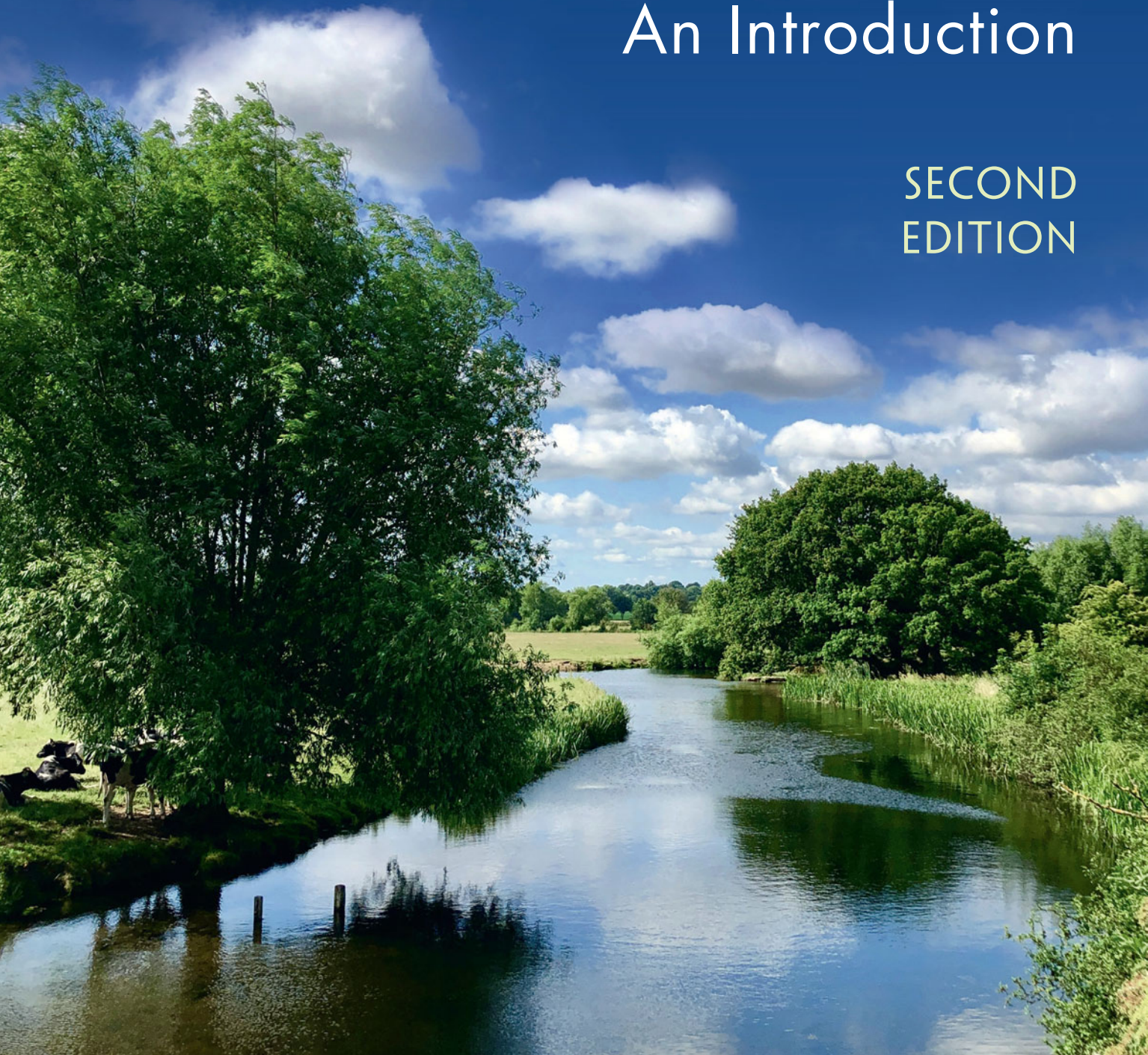


WILFRIED BRUTSAERT

# HYDROLOGY

An Introduction

SECOND  
EDITION





# Hydrology

*Second Edition*

Wilfried Brutsaert has revised and updated his classic textbook to take into account recent developments, while retaining the rigor and structure of the previous edition to introduce the fundamental principles of hydrology. New topics include the response of the global water cycle to climate change, the land-surface energy-budget closure, snow melt, groundwater trends and statistical surface variability with disturbed atmospheric boundary layers. Hydrologic phenomena are dealt with at the spatial and temporal scales at which they occur in nature. The physics and mathematics necessary to describe these phenomena are introduced and developed: readers will require a working knowledge of calculus and basic fluid mechanics. This classroom-tested textbook – based on the author’s long-running course at Cornell – is invaluable for entry-level courses in hydrology directed at advanced undergraduate and graduate students in physical science and engineering. In addition, it is also a great reference text for practising scientists and engineers.

**Professor Wilfried Brutsaert** has been active in education and research for over fifty years at Cornell University, where he has taught courses in hydrology, fluid mechanics, groundwater, and atmospheric boundary layer physics. He has been widely recognized for his research and has received many awards, including the Bowie Medal and the Horton Medal from the American Geophysical Union; the Charney Medal and Honorary Membership from the American Meteorological Society; and the International Award and Honorary Membership from the Japan Society of Hydrology and Water Resources. He was awarded a “Doctor Honoris Causa” title at Ghent University, and he was the recipient of the Prince Sultan Bin Abdulaziz International Prize for Water. Most notably, in 2022 he was awarded the Stockholm Water Prize (sometimes described as the Nobel Prize for water). He is a member of the U.S. National Academy of Engineering. Professor Brutsaert was President of the Hydrology Section of the American Geophysical Union, served as a Council member of the American Meteorological Society, and as Section Chair of the National Academy of Engineering. He is also the author of the book *Evaporation into the Atmosphere* (1982, Springer).



# Hydrology

## An Introduction

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***Second Edition***

**Wilfried Brutsaert**

Cornell University

# CAMBRIDGE

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# Preface to Second Edition

The first edition of this book was well received in the hydrologic community. It was even translated in several languages, namely Japanese, Albanian, Macedonian and most recently Chinese. In essence, this new edition is an updated version of the original, with clarifications and corrections and with inclusion of several additional sections. These new or greatly modified sections address physical phenomena in hydrology that have received continued or even increased attention during the past two decades. Briefly, these additions are about aspects of the global water cycle response to climate change ([Section 1.3](#)), land-surface energy-budget closure ([Sections 2.6](#) and [4.2.1](#)), statistical surface variability with disturbed atmospheric boundary layers ([Section 2.7](#)), atmospheric rivers and precipitation ([Section 3.2.6](#)), energetics of snow melt ([Section 3.5](#)), complementarity between actual evaporation and atmospheric evaporative demand ([Section 4.3.3](#)), the Schreiber-Oldekop hypothesis, aka the Budyko framework ([Section 4.4.2](#)), drying of soil ([Section 9.6.2](#)), base-flow decline rate ([Section 10.6.3](#)), groundwater storage changes ([Section 10.6.4](#)), simulation of subsurface storm flow to identify old and new water ([Section 11.2.2](#)), and long-term relationships between runoff and precipitation ([Section 12.5](#)).

Like in the first edition, the material is presented in two formats. The more basic material, which consists of topics fit for a first reading and covered in regular courses, is shown in standard type. More advanced topics and elaborations for further reading or research are presented in a slightly smaller type against a grey background to indicate them.

To facilitate the preparation of courses with this book, prospective teachers can request from the publisher an example syllabus outlining sections that may be covered in a first course in hydrology during a 42-lecture semester. Similarly, a solutions manual for the problems listed at the end of each chapter can also be provided to prospective teachers.

**Wilfried Brutsaert, 2023**



# Preface to First Edition

Water in its different forms has always been a source of wonder, curiosity and practical concern for humans everywhere. The goal of this book is to present a coherent introduction to some of the concepts and relationships needed to describe the distribution and transport of water in the natural environment. Thus it is an attempt to provide a more thorough understanding, and to connect the major paradigms that bear upon the hydrologic cycle, that is the never-ending circulation of water over the continents of the Earth.

Continental water transport processes take place above, on and below the Earth's land surfaces. Accordingly, in [Part I](#), water is considered as it passes through the lower atmosphere; this part consists of a general description of atmospheric transport in [Chapter 2](#), followed by the application of these concepts to precipitation and evaporation in [Chapters 3 and 4](#), respectively. In [Part II](#), water transport on the Earth's surface is dealt with; this part consists of a general description of the hydraulics of free surface flow in [Chapter 5](#), which is then applied to overland runoff and streamflow routing in rivers in [Chapters 6 and 7](#), respectively. Water below the surface is the subject of [Part III](#); again, a general introduction to flow in porous materials in [Chapter 8](#) is followed by applications to phenomena involving infiltration and capillary rise in [Chapter 9](#), and groundwater drainage and baseflow in [Chapter 10](#). [Part IV](#) is devoted to flow phenomena, mostly fluvial runoff, in response to precipitation at the catchment and river basin scales, which result from the combination of flows both above and below the Earth's surface, already treated at smaller scales separately in Parts II and III. Various interactions of these flow phenomena and the major paradigms regarding the subscale mechanisms are described in [Chapter 11](#). This is followed by a treatment of the available parameterizations in [Chapter 12](#). In [Chapter 13](#) the fourth part of the book concludes with a brief description of some of the more common statistical concepts that are useful in the analysis of hydrologic data. Finally, as an afterword, [Chapter 14](#) closes the book with a brief history of the ideas on the water cycle, which over the centuries evolved to our present understanding; Santayana's dictum may be a bit worn by now, but several recent reinventions of the hydrologic wheel could have been avoided, if the past had been better remembered.

These transport phenomena in the hydrologic cycle on land are treated at spatial and temporal scales, at which they are commonly encountered in everyday life and at which they are tractable with presently available data. Hydrology is a physical science, and the language of physics is mathematics. Accordingly, plausible assumptions are introduced and the mathematical formulations and parameterizations are derived, which describe the more relevant mechanisms involved in the different phases of the continental hydrologic cycle. The resulting equations are then examined and, if possible, solved for certain prototype situations and boundary conditions. The motivation for this is, first, to gain a better understanding of their structure and underlying assumptions, and of the physics they are intended to represent; and second, to provide the basis and background for more complex modeling exercises, simulations and predictions in practical applications.

The subject material covered in this book grew out of the lecture notes for my courses in hydrology and related topics in the School of Civil and Environmental Engineering, at Cornell University. I have not tried to cover all possible angles and points of view of the subject matter. Rather, I have followed a line of thought, which over the years I have come to find effective in conveying a broad understanding of the more important phenomena, and in stimulating further inquiry in the subject. Similarly, no attempt has been made to compile

a complete bibliography. But the references that are listed refer to other works, so that it should be possible to trace back the more important developments.

As its subtitle indicates, this book is intended as an introduction; as such, it should be suitable as a textbook for an entry-level course in hydrology directed at advanced seniors and beginning graduate students in engineering and physical science, who have a working knowledge of calculus and basic fluid mechanics. The book contains much more material than can reasonably be covered in a first course. Thus it will depend on the objectives of the course, and on the orientation and level of the students, which specific topics should be selected for coverage. Naturally, the instructor should be the ultimate judge in this. However, to facilitate this selection, the text is printed in two different type formats. The main subject matter, which in the experience of the author can be suggested for inclusion in a first course, is presented in regular type. An effort has been made to lay out this part of the text in such a way that the student should be able to grasp the material with little or no reliance on the more advanced sections. For certain topics, clarification by an experienced instructor in the lectures will undoubtedly be helpful. Subject matter of a more advanced or specialized nature is printed as indented text in a slightly smaller type and with a grey rule on the left-hand side of the page. This material is intended either as optional or explanatory reading for the first course, or as subject matter to be covered in a second and more advanced course. Sections of this second type of material have also been used as major portions in more specialized courses, namely in Groundwater Hydrology ([Chapters 8, 9 and 10](#)) and in Boundary Layer Meteorology ([Chapters 2, 3 and 4](#)) at Cornell.

The book is intended mainly for students of hydrology; it should, however, also be more broadly of interest to professional scientists and engineers, who are active in environmental matters, meteorology, agronomy, geology, climatology, oceanology, glaciology and other Earth sciences, and who wish to study some of the underlying concepts of hydrology, relevant to their discipline. In addition, it is hoped that the book will be of use to workers in fluid dynamics, who want to become acquainted with applications to some intriguing and fascinating phenomena in nature.

**Wilfried Brutsaert, 2005**

Ter nagedachtenis van mijn ouders Godelieve S. G. Bostijn (-B.) en Daniel P. C. Brutsaert

妻 トヨに捧 げる

And to the life of Siska, Hendrik, Erika and Karl

# 1

## Introduction

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### 1.1 Definition and Scope

Hydrology is literally the science of water. Etymologically, the word has its roots in ancient Greek, and is a composite, made up of  $\tilde{\upsilon}\delta\omega\rho$ , water, and  $\lambda\acute{o}\gamma\omicron\varsigma$ , word. Obviously, defined this way, the term is much too broad to be very useful, as it would have ramifications in all scientific disciplines.

Actually, the word hydrology has not always been well defined and even as recently as the 1960s it was not very clear exactly what hydrology was supposed to cover and encompass. Price and Heindl (1968), in a survey of many of the definitions that had appeared in the literature over the previous 100 years, were compelled to conclude that the question “What is hydrology?” had not been resolved by their review. Still, they felt that, in general, there seemed to be a consensus that hydrology is a physical science, which is concerned mainly with the water cycle of land and near-shore areas; moreover, there had been a tendency to broaden the term rather than to narrow it, even to the point of including socio-economic aspects.

Over the past few decades, however, with the growing activity level and the increasing maturity of this field of endeavor, a more precise definition has emerged. Hydrology is now widely (see, for example, Eagleson, 1991) accepted to be the science that deals with those aspects of the cycling of water in the natural environment that relate specifically to:

- *the continental water processes*, namely the physical and chemical processes along the various pathways of continental water (solid, liquid, and vapor) at all scales, including those biological processes that influence this water cycle directly; and to
- *the global water balance*, namely the spatial and temporal features of the water transfers (solid, liquid, and vapor) between all compartments of the global system, i.e. atmosphere, oceans, and continents, in addition to stored water quantities and residence times in these compartments.

Because it is defined as being concerned specifically with continental water processes, hydrology is a discipline distinct from meteorology, climatology, oceanology, glaciology, and others that also deal with the water cycle in their own specific domains, namely the atmosphere, the oceans, the ice masses, etc., of the Earth; at the same time, however, hydrology integrates and links these other geosciences, in that, through the global water balance, it is also concerned with the exchanges of water between all these separate compartments.

With this definition it is now also possible to delineate the practical scope of hydrologic analysis in engineering and in other applied disciplines. It consists of the determination of the amount and/or flow rate of water that will be found at a given location and at a given time under natural conditions, without direct human control or intervention. The latter specification, that no human control be involved, is necessary to distinguish hydrology from the related discipline of hydraulics. Hydraulics is concerned with the study of



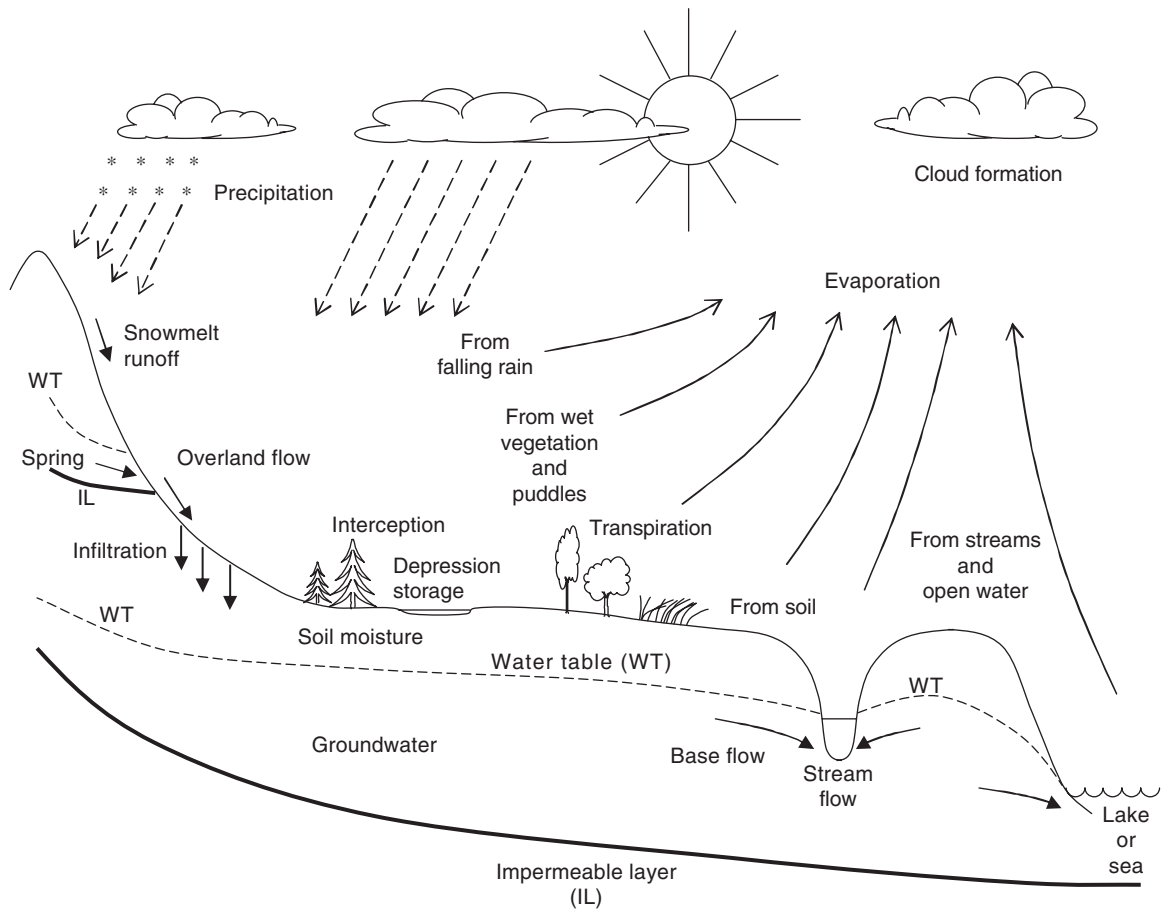
controlled fluid motion in well-defined and often in human-made environments. For instance, problems involving pipe flow, irrigation water distribution, or pumping of groundwater are not hydrologic in nature, but are more properly assigned to the realm of hydraulics.

## 1.2 The Hydrologic Cycle

The water cycle, also called the hydrologic cycle, refers to the pathway of water in nature, as it moves in its different phases through the atmosphere, down over and through the land, to the ocean, and back up to the atmosphere. When atmospheric water vapor condenses and precipitates over land, initially it moistens the surface and some amount of it is stored as *interception*, which later evaporates. As *precipitation* (and in a similar way *snowmelt*) continues, part of it may flow over the surface in the form of *overland flow* or *surface runoff*, and part of it may enter into the soil as *infiltration*. This surface runoff soon tends to collect locally, either in puddles or small ponds as *depression storage*, or in gullies or larger channels where it continues as *streamflow*, which ultimately ends up in a larger water body, such as a lake or the ocean. Streamflow is normally described by a *hydrograph*, that is the rate of flow at a gaging station as a function of time. The infiltrated water may flow rapidly through the near-surface soil layers to exit into springs or adjacent streams, or it may percolate more slowly through the profile to join the *groundwater*, which sooner or later seeps out into the natural river system, lakes and other open-water bodies; part of the infiltrated water is retained in the soil profile by capillarity and other factors, where it is available for uptake by the roots of vegetation.

Soil layers and other geologic formations, whose pores and interstices can transmit water, are called *aquifers*. When an aquifer is in direct contact with the land surface, it is referred to as *unconfined*. The locus of points in an unconfined aquifer, where the water pressure is atmospheric, is called the *water table*. Although the water table is not a true free surface separating a saturated zone from a dry zone, to simplify the analysis it is sometimes assumed to be the upper boundary of the groundwater in an unconfined aquifer. The partly saturated zone in an unconfined aquifer, between the water table and the ground surface, is sometimes referred to as the *vadose zone*. In an unconfined aquifer, the term *groundwater* refers usually to the water found below the water table; *soil water* or *soil moisture* refers to the water above the water table. A water-bearing geologic formation, that is separated from the surface by an impermeable layer, is referred to as a *confined aquifer*. Streamflow is fed both by surface runoff and by subsurface flow from riparian (i.e. located along the banks) aquifers. The streamflow, resulting from groundwater outflow is often called *base flow*; in the absence of *storm flow* or *storm runoff* caused by precipitation, base flow is also referred to as *drought flow* or *fair-weather flow*.

Finally, the hydrologic cycle is closed by *evaporation*, which returns the water, while in transit in the different flow paths and stages of storage along the way, back into the atmosphere. When evaporation takes place through the stomates of vegetation, it can be referred to as *transpiration*. Direct evaporation from open water or soil surfaces and transpiration of biological water from plants are not easy to separate; therefore the combined process is sometimes called *evapotranspiration*. Although this term may on occasion be helpful and explanatory locally during the growing season in agriculture and forestry, it can be misleading for general usage as it fails to include such important vapor flux components as evaporation from intercepted water due to ample rainfall, from small open-water bodies scattered in the landscape, and from snow and ice during winter in many regions of the world. Evaporation of ice is referred to as *sublimation*. While these distinctions can be useful at times, the term *evaporation* is normally not only adequate but also



**Figure 1.1** Sketch of some of the main processes in the land phase of the water cycle.

preferable to describe all processes of vaporization (Miralles et al., 2020). Some of the main processes are drawn schematically in Figure 1.1.

## 1.3 Some Estimates of the Global Water Balance

Numerous studies have been carried out to estimate the magnitude of the most important components of the water-budget equations on a global scale. Because the available database required for this purpose is still far from adequate, several of the methods used in these estimates may be open to criticism. Nevertheless, there is a fair agreement among some of the calculated values and, within certain limits, they provide a useful idea of the long-term average balance in different climatic regions of the world.

As shown in Table 1.1, for the entire Earth, the average annual precipitation and evaporation have the same magnitude, i.e.  $P = E$ , and are of the order of 1 m. However, their average values of  $P$  and  $E$  over the Earth's oceans and land surfaces are markedly different. Water is continuously removed from the ocean into the atmosphere by the small surplus of  $E$  over  $P$ , and this is counterbalanced by the larger surplus of  $P$  over  $E$  on land. Under steady conditions, that is for long time periods, the latter surplus, or

**Table 1.1 Estimates of world water balance ( $\text{m a}^{-1}$ )**

| Reference                      | Land<br>( $1.49 \times 10^8 \text{ km}^2$ ) |       |       | Oceans<br>( $3.61 \times 10^8 \text{ km}^2$ ) |      | Global $P = E$ |
|--------------------------------|---------------------------------------------|-------|-------|-----------------------------------------------|------|----------------|
|                                | $P$                                         | $R$   | $E$   | $P$                                           | $E$  |                |
| Budyko (1970, 1974)            | 0.73                                        | 0.31  | 0.42  | 1.14                                          | 1.26 | 1.02           |
| Lvovitch (1970)                | 0.73                                        | 0.26  | 0.47  | 1.14                                          | 1.24 | 1.02           |
| Lvovitch (1973)                | 0.83                                        | 0.29  | 0.54  | –                                             | –    | –              |
| Baumgartner and Reichel (1975) | 0.75                                        | 0.27  | 0.48  | 1.07                                          | 1.18 | 0.97           |
| Korzun et al. (1978)           | 0.80                                        | 0.315 | 0.485 | 1.27                                          | 1.40 | 1.13           |
| Rodell et al. (2015)           | 0.80                                        | 0.31  | 0.48  | 1.11                                          | 1.23 | 1.02           |

**Table 1.2 Some estimates of the mean precipitation (and river runoff) for the continents (in  $\text{m a}^{-1}$ )(\*)**

|                                | Europe           | Asia             | Africa           | North America    | South America    | Australia and Oceania | Antarctica       |
|--------------------------------|------------------|------------------|------------------|------------------|------------------|-----------------------|------------------|
| Percent of land area           | 6.7              | 29.6             | 20.0             | 16.2             | 12.0             | 6.0                   | 9.5              |
| Reference                      |                  |                  |                  |                  |                  |                       |                  |
| Lvovitch (1973)                | 0.734<br>(0.319) | 0.726<br>(0.293) | 0.686<br>(0.139) | 0.670<br>(0.287) | 1.648<br>(0.583) | 0.736<br>(0.226)      | –<br>–           |
| Baumgartner and Reichel (1975) | 0.657<br>(0.282) | 0.696<br>(0.276) | 0.696<br>(0.114) | 0.645<br>(0.242) | 1.564<br>(0.618) | 0.803<br>(0.269)      | 0.169<br>(0.141) |
| Korzoun et al. (1977)          | 0.790<br>(0.283) | 0.740<br>(0.324) | 0.740<br>(0.153) | 0.756<br>(0.339) | 1.600<br>(0.685) | 0.791<br>(0.280)      | 0.165<br>(0.165) |

\*The corresponding evaporation values can be determined with [Equation \(1.1\)](#).

remainder, can be considered to be runoff from the Earth's rivers into the oceans,  $R$  (expressed as height of water column per unit of time). Thus, over the Earth's land surfaces the average annual water balance can be described by

$$R = P - E \quad (1.1)$$

Over the land surfaces the average precipitation intensity,  $P$ , is about  $0.80 \text{ m a}^{-1}$ , whereas the corresponding average evaporation,  $E$ , is around  $0.50 \text{ m a}^{-1}$  or about 60 to 65% of the precipitation. Averaged over all continents and over long time periods, the annual runoff,  $R$ , is therefore around 35 to 40% of the precipitation. Except for South America and Antarctica (see [Table 1.2](#)), the values for the individual continents are not very different from the global values. Precipitation and streamflow runoff measurements have been and are being made routinely in many places on Earth. In contrast, evaporation has not received as much attention and only in recent years have attempts been made to obtain systematic measurements ([Pastorello et al., 2017](#)).

Estimates of the average distribution of water in different forms expressed as depth of water covering the globe, assumed to be a perfect sphere, are given in [Table 1.3](#). These indicate that the 1 m of average annual precipitation (shown in [Table 1.1](#)) is relatively large as compared to the active fresh water on Earth, that is the water which is not stored in permanent ice and deep groundwater. This means that the turnover

**Table 1.3 Estimates of different forms of global water storage (as depth in m over entire Earth's surface)**

| Source of data        | Lvovitch (1970) | Baumgartner and Reichel (1975)  | Korzun et al. (1978) |
|-----------------------|-----------------|---------------------------------|----------------------|
| Oceans                | 2686            | 2643                            | 2624                 |
| Ice caps and glaciers | 47.1            | 54.7                            | 47.2                 |
| Total groundwater     | 117.6           | 15.73<br>(excluding Antarctica) | 45.9                 |
| (Active groundwater)  | (7.84)          | (6.98)                          | —                    |
| Soil water            | 0.161           | 0.120                           | 0.0323               |
| Lakes                 | 0.451           | 0.248                           | 0.346                |
| Rivers                | 0.002 35        | 0.002 12                        | 0.004 16             |
| Atmosphere            | 0.0274          | 0.0255                          | 0.0253               |

of the active part of the hydrologic cycle is rather fast, and that the residence times in some of the major compartments of the water cycle are relatively short; the mean residence time can be taken as the ratio of the storage and the flux in or out of storage. For example, a continental runoff rate of  $0.30 \text{ m a}^{-1}$  (Table 1.1) and a storage in the rivers of  $(0.003/0.29) \text{ m}$  of water on the 29% of the world occupied by land, gives a mean residence time of the order of 13 days for the rivers of the world. Similarly, a global evaporation rate of  $1 \text{ m a}^{-1}$ , with  $0.025 \text{ m}$  of storage in the atmosphere, leads to a mean residence time of the order of 9 days for the atmosphere. These are very short residence times. Moreover, as the oceans occupy about 71% of the Earth's surface, the active fresh water in the hydrologic cycle is continually being distilled anew through ocean evaporation.

Maps depicting the approximate distribution of components of the water balance in different parts of the world have been presented by, among others, Lvovitch (1973), Budyko (1974), Baumgartner and Reichel (1975), and Korzoun et al. (1977). The relative and absolute magnitudes of the main components of the hydrologic cycle, namely  $P$ ,  $R$ , and  $E$ , can vary over a wide range from one location to another. Obviously, the long-term mean values of all three are negligible in desert locations. At the other extreme, maximal annual precipitation values of up to  $26.5 \text{ m}$  have been recorded in a mountainous monsoon environment (Cherrapunji, Meghalaya). Maximal mean evaporation values of up to  $3.73 \text{ m a}^{-1}$  have been inferred for the Gulf Stream in the western Atlantic (Bunker and Worthington, 1976) and up to 4 or even  $5 \text{ m a}^{-1}$  for the Gulf of Aqaba (Assaf and Kessler, 1976).

Much research has been directed in recent years into studying the evolution of today's climate in response to increasing greenhouse gases in the atmosphere. For some time now, there have been numerous indications of an accelerating hydrologic cycle in certain regions (Brutsaert and Parlange, 1998; Karl and Knight, 1998; Lins and Slack, 1999). But also globally, the fluxes shown in Table 1.1 are now known to be evolving in tandem with the gradual average rise in temperature of the atmosphere. For instance, an analysis of satellite measurements, mainly over the oceans supplemented with land data, by Wentz et al. (2007) has indicated that during the period 1987–2006 the average relative rate of increase of the total atmospheric water vapor  $W'/W$ , of the global precipitation  $P'/P$ , and of the global evaporation  $E'/E$  have been roughly the same, namely about  $0.0013 \text{ a}^{-1}$ . (Here the prime ( ' ) denotes the average time derivative  $d( )/dt$ .) During these two decades around the turn of the century, the global lower tropospheric warming rate was about  $T' = 0.02 \text{ }^{\circ}\text{C a}^{-1}$ , which means that the average trend of  $0.13\%$  per year amounts to about 6 to 7% per kelvin of warming, say  $0.065 \text{ K}^{-1}$ ; this is nearly the same as the rate produced by the Clausius–Clapeyron (or C-C) equation (see Equation (2.13)), with constant relative humidity at a temperature of  $14 \text{ }^{\circ}\text{C}$ ,

**Table 1.4** Estimates of mean global heat budget at the Earth's surface in  $\text{W m}^{-2}$ 

| Reference                      | Land  |         |     | Oceans |         |     | Global |         |     |
|--------------------------------|-------|---------|-----|--------|---------|-----|--------|---------|-----|
|                                | $R_n$ | $L_e E$ | $H$ | $R_n$  | $L_e E$ | $H$ | $R_n$  | $L_e E$ | $H$ |
| Budyko (1974)                  | 65    | 33      | 32  | 109    | 98      | 11  | 96     | 80      | 16  |
| Baumgartner and Reichel (1975) | 66    | 37      | 29  | 108    | 92      | 16  | 96     | 76      | 20  |
| Korzun et al. (1978)           | 65    | 36      | 29  | 121    | 109     | 12  | 105    | 89      | 16  |
| Ohmura (2005)                  | 62    | 36      | 26  | 125    | 110     | 15  | 104    | 85      | 19  |
| Trenberth et al. (2009)        | 65.5  | 38.5    | 27  | 110    | 97      | 12  | 98     | 80      | 17  |
| Wild et al. (2015)             | 70    | 38      | 32  | 117    | 100     | 16  | 103    | 82      | 21  |

the commonly accepted global average temperature in the lower troposphere. Application of this C-C similarity in Brutsaert (2017) showed that the global rate of increase of the hydrologic cycle during the second half of the twentieth century was of the order of  $P' = E' = 1.0 \text{ mm a}^{-2}$ ; the evaporation trend from the global land surfaces was found to be about half this value, namely of the order of  $0.4$  to  $0.5 \text{ mm a}^{-2}$ . Very similar values were obtained in other studies by different methods (Brutsaert, 2006; Miralles et al., 2014; Anabalón and Sharma, 2017). In response to the global precipitation trend, the overall global runoff has been observed to increase as well. From the monthly discharges of the largest rivers worldwide, Labat et al. (2004) estimated this increase to be about 4% per kelvin; interestingly, this is roughly of the same order as the Clausius–Clapeyron value of  $0.065 \text{ K}^{-1}$  mentioned above for precipitation and evaporation. However, the runoff changes also showed large variability over the different continents, including negative trends in some regions (Zhang et al., 2015).

The strong link between the water cycle and climate is further illustrated by the estimates of the mean global surface energy budget in Table 1.4. Over large areas and over sufficiently long periods, when effects of unsteadiness, melt and thaw, photosynthesis and burning, and lateral advection can be neglected, this surface energy balance can be written as

$$R_n = L_e E + H \quad (1.2)$$

where  $R_n$  is the specific flux of net incoming radiation,  $L_e$  is the latent heat of vaporization,  $E$  is the rate of evaporation, and  $H$  is the specific flux of sensible heat into the atmosphere. The major portion of the incoming radiation is absorbed near the surface of the Earth, and is transformed into internal energy. The subsequent partition of this internal energy into longwave back radiation, upward conduction and convection of sensible heat,  $H$ , and latent heat,  $L_e E$ , is one of the main processes driving the atmosphere. Table 1.4 (see also Figure 2.28) indicates that the net radiative energy is mainly disposed of as evaporation. Over the oceans, the latent heat flux  $L_e E$  is on average larger than 90% of the net radiation. But even over the land surfaces of the Earth,  $L_e E$  is on average still larger than half of  $R_n$ .

Because the global patterns of heating force the circulation of the planetary atmosphere, the implications of this large latent heat flux are clear. As a result of the relatively large latent heat of vaporization,  $L_e$ , evaporation of water involves the transfer and redistribution of large amounts of energy under nearly isothermal conditions. Because, even at saturation, air can contain only relatively small amounts of water vapor, which can easily be condensed at higher levels, the air can readily be dried out; this release of energy through condensation and subsequent precipitation is the largest single heat source for the atmosphere. Thus processes in the water cycle play a central role in governing weather and climate.

## 1.4 Methodologies and Procedures

This book aims primarily to describe the occurrence and transport of water in its continuous circulation over the land surfaces of the Earth. Before starting this task, it is worthwhile to review briefly the different strategies that are available and that can be used for this purpose.

### 1.4.1 Statistical Analysis and Data Transformation

As observed in [Section 1.2](#), one of the main practical objectives of hydrologic analysis is the determination of the quantity of water, in storage or in transit, to be found at a given time and place, free of any direct human control. When a reliable record of observed hydrologic data is available, a great deal can be learned simply by a statistical analysis of this record. Although such an approach is proper for stationary systems in the prediction of long-term behavior for general planning purposes, it cannot be used for short-term and emergency forecasting, for example, during floods, or for day-to-day resource-management decisions. Furthermore, reliable records are available for only a few locations over a limited period of time, and practically never where needed. Therefore, in hydrology the problem is often such that a method must be devised to transform some available data, which are of no direct interest, to the required hydrologic information. For instance, the problem may consist of determining the rate of flow in a river at a given location either from a known flow rate at some other point upstream or downstream, or from a known rainfall distribution over the upstream river basin. In other cases, the problem may consist of deducing the basin evaporation from soil and vegetation on the basis of available meteorological data.

### 1.4.2 The “Physical” versus the “Systems” Approach

The hydrologic literature is replete with attempts at classifying the methodologies and paradigms that have been used to transform hydrologic input into hydrologic output information. Until a few years ago it had become customary to consider two contrasting approaches, namely the “physical” approach and the “systems” approach. In the physical approach, the input–output relationship is sought by the solution of the known conservation equations of fluid mechanics and thermodynamics with appropriate boundary conditions to describe the flow and transport of water throughout the hydrologic cycle. This approach has obvious limitations; the physiographic and geomorphic characteristics of most hydrologic systems are so complicated and variable, and the degree of uncertainty in the boundary conditions so large, that solutions are feasible only for certain highly simplified situations. In other words, the properties of natural catchments can never be measured accurately enough, and solutions, based on internal descriptions starting from first principles of fluid mechanics, can be obtained only for grossly idealized conditions, which are coarse approximations of any real situation.

The hydrologic “systems” (also “operational” or “empirical”) approach is presumably based on a diametrically opposite philosophy. In this approach the physical structure of the various components of the hydrologic cycle and their inner mechanisms are not considered; instead, each component, however it may be defined, is thought of as a “black box,” and the analysis focuses on discovering a mathematical relationship between the external input (e.g. rainfall, air temperature, etc.) and the output (e.g. river flow, soil moisture, evaporation, etc.). The structure of this mathematical relationship is mostly quite remote from the physical structure of the prototype phenomena in nature. This lack of correspondence between the inner physical mechanisms and the postulated functional formalisms makes this approach quite general operationally, because it permits the use of well-known algorithms and objective criteria in identification and prediction. However, this also underlies the main limitations of this approach. First, in assigning cause and

effect, the definition of input and output variables is mostly based on intuition guided by past experience, and the danger exists that some important phenomena are overlooked. Second, the best that can ever be expected with a black-box approach is a satisfactory reproduction of a previously obtained input–output record; even when such data are available, it is difficult to accommodate fully the nonstationary effects in the system, and it is impossible to anticipate subsequent hydrologic changes, such as those resulting from urbanization, deforestation, reclamation, or climate change.

Because many hydrologic methods do not really fit in this physical-versus-empirical classification, a third possible approach was taken to be an intermediate one. In this view the performance of a hydrologic unit, say a catchment, is represented in terms of some idealized components or “gray boxes,” which correspond to recognizable elements in the prototype, whose input–output response functions are structured after solutions of some tractable or suitably simplified situations of the physical processes perceived to be relevant. This third way was often called the “conceptual model” approach.

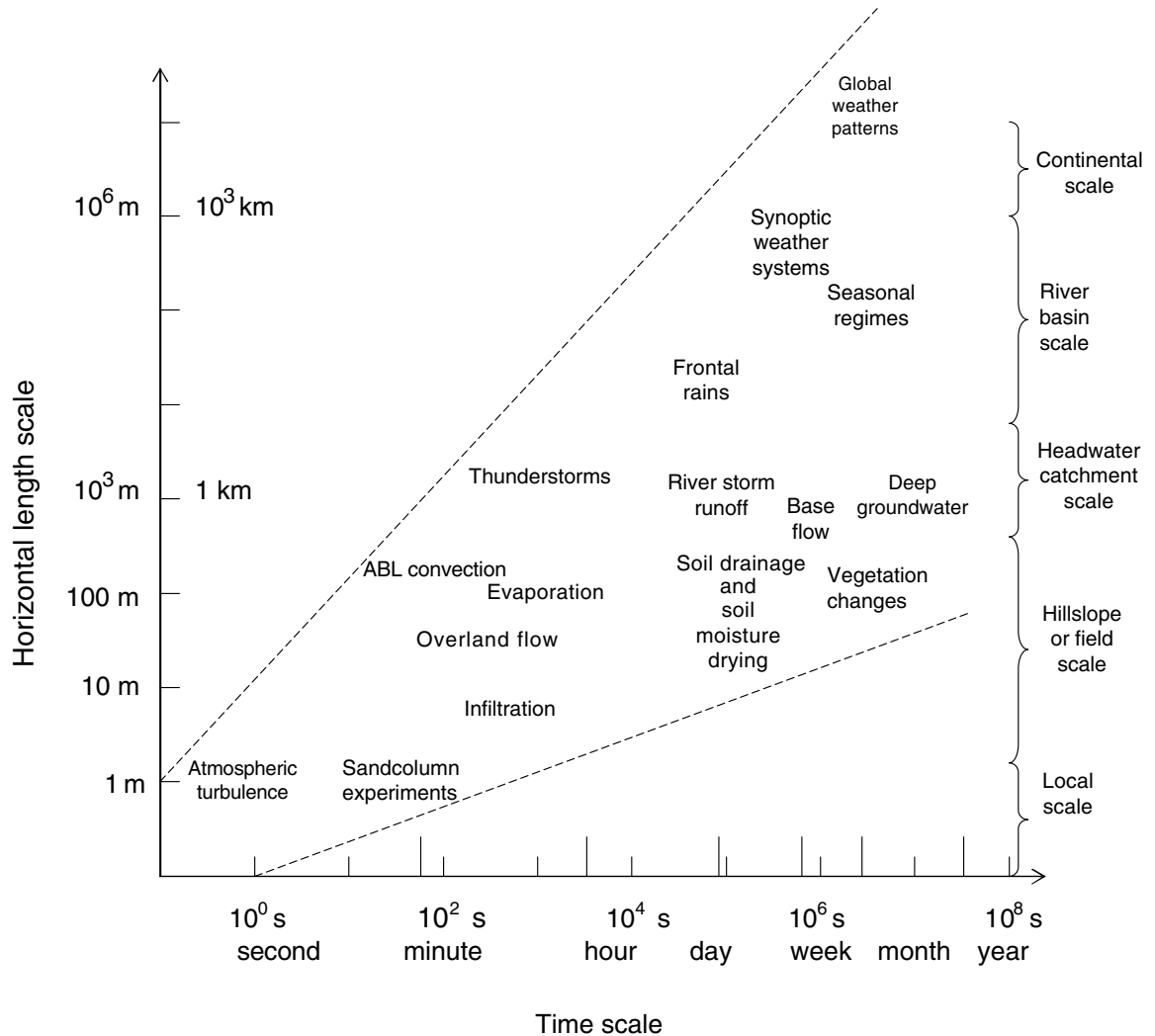
At first sight, a classification based on three distinct approaches, namely physical, empirical, and conceptual, may appear reasonable. However, it is less than obvious how this classification can be applied to specific cases. Indeed, one might ask what the difference is between physical and empirical. After all, the essence of physical science is experimentation and conceptualization. Moreover, the physical approach of one discipline is usually the empiricism or the conceptual model of another. For example, Newton’s “law” of viscous shear constitutes the physical basis of a wide area of fluid mechanics, whereas it represents a mere black-box simplification in molecular physics. Darcy’s law is the physical basis of much of groundwater hydrology, but in fluid mechanics it can be considered an operational approach, to avoid the complexity of flow analysis in an irregular and ill-defined pore network. The same dilemma is inherent in most other special concepts used in hydrology. This ambiguous difference between physical, empirical, and conceptual shows that the classification of the methodologies should be based on other criteria.

### 1.4.3 Spatial Scale and Parameterization

#### General Approach

All natural flow phenomena are governed by the principles of conservation of mass, momentum, and energy, which can be expressed by a number of equations to provide a mathematical description of what goes on. However, because there are normally more dependent variables than available conservation equations, in order to close the system, additional relationships must be introduced. These closure relationships, also called *parameterizations*, relate some of the variables to each other to describe certain specific physical mechanisms; the mathematical form of these relationships, and the values of the material constants or *parameters* are usually based on experimentation.

A second point is that any physical phenomenon must be considered at a given scale; this scale is the available (depending on the data) or chosen (depending on the objectives of the study) resolution. It will become clear later on in this book that, while the fundamental conservation equations remain unaffected by the scale at which the phenomenon is being considered, most closure relationships within them are quite sensitive to scale. Indeed, a parameterization can be considered as a mathematical means of describing the subresolution (or microscale) processes of the phenomenon, in terms of resolvable scale (or macroscale) variables; these macroscale variables are the ones that can be treated explicitly in the analysis or for which measured records are obtainable. Thus, the details of the microscale mechanisms are not considered explicitly, but their statistical effect is formulated mathematically by a parameterization in terms of macroscale variables.



**Figure 1.2** Approximate ranges of spatial and temporal scales of some common physical processes that are relevant in hydrology.

All this suggests that a sound criterion to distinguish, in principle at least, one approach from another may be the spatial scale at which the internal mechanisms are parameterized. For example, Newton's equation for viscous shear stress (see [Equation \(1.12\)](#) below) is a parameterization in terms of variables typically at the millimeter to centimeter scale; however, it reflects momentum exchanges at molecular scales, which are orders of magnitude smaller. The hydraulic conductivity is a parameter at the so-called Darcy scale (see [Chapter 8](#)), namely a scale somewhere intermediate between the Newtonian viscosity (or Navier–Stokes) scale for water and air inside the soil pores, on the one hand, and the field scales for infiltration and drainage, on the other. Several spatial scales are illustrated, with the corresponding characteristic temporal scales, in [Figure 1.2](#) for some general types of water transport processes, as they have been considered in hydrologic studies.



**Table 1.5 A common scale classification in the atmosphere (after Orlanski, 1975)**

| Nomenclature   | Scale range    |
|----------------|----------------|
| Micro $\gamma$ | <20 m          |
| Micro $\beta$  | 20–200 m       |
| Micro $\alpha$ | 200 m–2 km     |
| Meso $\gamma$  | 2 km–20 km     |
| Meso $\beta$   | 20 km–200 km   |
| Meso $\alpha$  | 200 km–2000 km |
| Macro $\gamma$ | >2000 km       |

On the land surfaces of the Earth, the catchment or river basin sizes appear as scales of central importance. The terms *basin*, *catchment*, *watershed*, and *drainage area* are roughly synonymous and are often used interchangeably. A basin can be defined as all of the upstream area, which contributes to the open channel flow at a given point along a river. The size of the basin depends on the selection of the point in the river system under consideration. Usually this point is taken where the river flows into a large water body, such as a lake or the ocean, or where it changes its name as a tributary into a larger river. However, a basin or catchment can also be defined by any point along the river where the river flow is being measured. Basins are delineated naturally by the land-surface topography, and topographic ridges are usually taken as their boundaries; they can be considered as the natural conveyance systems for mass and energy on the land surfaces of the Earth. In meteorology, the concern is more on atmospheric motions and weather systems, and this has led to a somewhat different scale classification; an example of a commonly used classification is shown in [Table 1.5](#).

To summarize, these observations indicate that, in deciding on a strategy to describe a hydrologic phenomenon, the relevant question is probably not so much whether a physical, a black box, or a conceptual approach should be used. Rather, it is more useful to determine what scales are appropriate for the available and measurable data, and for the problem at hand. In other words, what is the appropriate level of parameterization?

### Spatial Variability and Effective Parameters

As mentioned above, a parameterization can be defined as a functional relationship between the variables describing the phenomenon in question. This relationship invariably contains one or more constant terms, reflecting material and fluid properties, and vegetational, geomorphic, geologic, and other physiographic features; these are called parameters and they are normally determined by experiment. Most hydrologic parameters tend to be highly variable in space. It stands to reason, therefore, that the experimental determination of any such space-dependent parameter must be carried out at the scale at which it is to be applied to describe the flow.

A second important issue is that any given parameterization is usually valid only over a certain finite range of spatial scales, and that the computational scale, that is the integration domain or the discretization of the equations, must lie within that range. Because the necessary data may be available only at a coarser resolution, in practical application, a parameterization may have to be applied at scales for which it was not intended originally and which are larger than permissible. This means that in such a case the spatial variability of the

parameters at the finer scales, which is normally present in the natural environment, cannot be accounted for with the available data. This difficulty is often resolved by assuming that the parameterization is still valid at the larger scale, and that it can be implemented with averaged or *effective* values of the parameters. This approach is not always satisfactory, and it is still the subject of intense research. Some aspects of this issue related to land-surface–atmosphere interactions are discussed in [Section 2.7](#) and in [Brutsaert \(1998\)](#).

## Requirements

To be useful, a parameterization must satisfy several requirements. First, a parameterization must be *valid*, that is, it must be able to give a faithful description of the phenomenon in question. The word comes from the Latin “*validus*,” which means healthy or strong, and thus reliable. Validation is the term that refers to the testing of a parameterization, and it usually consists of the application of some goodness of fit measure to results of calculations with the parameterization relative to observations. Beside being valid, a useful parameterization must satisfy the dual requirements of parsimony and robustness.<sup>1</sup> A given parameterization is said to be more *parsimonious* than another one, when it needs fewer variables and parameters to describe the phenomenon.

A parameterization can be called *robust* when the outcome is relatively insensitive to its structure, and to errors and uncertainties in the input variables and parameters. In hydrology, a *model* usually refers to a combination of several parameterizations to simulate more complicated phenomena and their interactions.

## 1.5 Conservation Laws: The Equations of Motion

### 1.5.1 Rate of Change of Fluid Properties

Consider a fluid in motion with a velocity field  $\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ , in which  $(u, v, w)$  are the velocity components and  $(\mathbf{i}, \mathbf{j}, \mathbf{k})$  are the unit vectors in the  $(x, y, z)$  directions, respectively, and let  $C(x, y, z, t)$  denote some property of this fluid. The rate at which this property changes for a given particle of the fluid located at  $(x, y, z)$  at time  $t$ , can be determined by tracking the particle to its new position  $(x + u\delta t, y + v\delta t, z + w\delta t)$ , a small distance away at time  $t + \delta t$ . The fluid property has then become

$$C(x + u\delta t, y + v\delta t, z + w\delta t) = C + \frac{\partial C}{\partial x}u\delta t + \frac{\partial C}{\partial y}v\delta t + \frac{\partial C}{\partial z}w\delta t + \frac{\partial C}{\partial t}\delta t$$

Thus, after the small displacement, the property of the fluid assumes the new value  $C + (DC / Dt)\delta t$ . This shows that the rate of change of the property  $C$  of the moving fluid particle is given by

$$\frac{DC}{Dt} = \frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} \quad (1.3)$$

$DC / Dt$  is commonly referred to as the *substantial* time derivative, and is also variously called the *fluid mechanical* time derivative, the time derivative *following the motion*, or the *material* or *particle* derivative.

<sup>1</sup> The law of parsimony, also known as Ockham’s razor, comes to mind here. Actually, the principle was already promulgated by Aristotle, and the razor is essentially Ockham’s (1989; pp. 17, 20, 128) paraphrase of it. More than 2300 years ago [Aristotle \(1929\)](#) wrote, for instance, in *The Physics* (I, 6, 189a, 15) “inasmuch as it can be done from the limited, the limited is better,” and in (VIII, 6, 259a, 10) “for when the outcome is the same, the limited is always to be preferred, and indeed in matters of nature, the limited, being the better, occurs more when possible.”

Physically, Equation (1.3) is the total rate of change in the property, as seen by an observer moving with the fluid. The first term on the right describes the changes taking place locally at  $(x, y, z)$ . The last three terms describe the changes observed while moving between locations with different values of  $C$ ; the rate of change depends on the speed of the motion,  $(u, v, w)$ .

### 1.5.2 Conservation of Mass: The Continuity Equation

Because hydrology is concerned with amounts of water observed at different times and locations, conservation of mass is the main governing principle. There are several ways of deriving a formulation that embodies this principle.

#### At a Point

One way, after Euler's 1755 derivation (Lamb, 1932), is to consider an element of fluid mass, which occupies a small volume  $\delta\forall = (\delta x \delta y \delta z)$  at time  $t$ , shown as ABCD in two dimensions in Figure 1.3, and whose center moves at a velocity  $\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ . If the mass of fluid per unit volume, that is its density, is  $\rho$ , the mass of the element is given by  $(\rho\delta\forall)$ . In the absence of chemical reactions, or sources and sinks, the mass of this element does not change and must remain the same. Therefore, if the fluid property is taken as the mass of the fluid, or  $C = (\rho\delta\forall)$ , one has with Equation (1.3) that

$$\frac{D(\rho\delta\forall)}{Dt} = 0$$

or

$$\rho \frac{D(\delta\forall)}{Dt} + \delta\forall \frac{D\rho}{Dt} = 0 \quad (1.4)$$

The rate of change of the fluid element volume  $D(\delta\forall)/Dt$  can be derived by tracking the fluid element, shown in Figure 1.3, as it moves from ABCD to A'B'C'D' during the small time interval  $\delta t$ . The point H is then located at

$$x = x_0 - \frac{\delta x}{2} + \left( u - \frac{\partial u}{\partial x} \frac{\delta x}{2} + \frac{\partial^2 u}{\partial x^2} \left( \frac{\delta x}{2} \right)^2 \frac{1}{2} - \dots \right) \delta t$$

and at

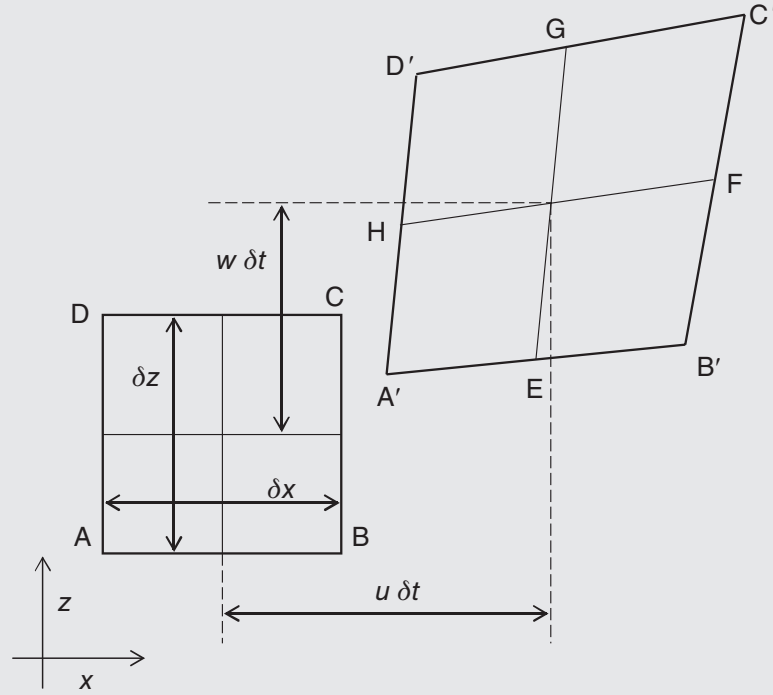
$$z = z_0 + \left( w - \frac{\partial w}{\partial x} \frac{\delta x}{2} + \frac{\partial^2 w}{\partial x^2} \left( \frac{\delta x}{2} \right)^2 \frac{1}{2} - \dots \right) \delta t$$

The point F is at

$$x = x_0 + \frac{\delta x}{2} + \left( u + \frac{\partial u}{\partial x} \frac{\delta x}{2} + \frac{\partial^2 u}{\partial x^2} \left( \frac{\delta x}{2} \right)^2 \frac{1}{2} + \dots \right) \delta t$$

and

$$z = z_0 + \left( w + \frac{\partial w}{\partial x} \frac{\delta x}{2} + \frac{\partial^2 w}{\partial x^2} \left( \frac{\delta x}{2} \right)^2 \frac{1}{2} + \dots \right) \delta t$$



**Figure 1.3** At time  $t$  the mass  $\rho \delta \forall$  occupies the volume  $ABCD$ , and at time  $t + \delta t$  this same fluid mass has moved to  $A'B'C'D'$ . The center of the volume has moved from  $(x_0, y_0, z_0)$  to  $(x_0 + u \delta t, y_0 + v \delta t, z_0 + w \delta t)$ . The figure is shown in two dimensions for clarity; the third coordinate  $y$  can be imagined as pointing into the plane of the drawing.

Therefore, the length of the segment  $HF$ , as projected on the  $x$ -axis, is given by  $[\delta x + (\partial u / \partial x) \delta x \delta t]$ ; in a similar way, one obtains for the projection of the length of the segment  $EG$  on the  $z$ -axis the value  $[\delta z + (\partial w / \partial z) \delta z \delta t]$ , and for the length of the segment in the  $y$ -direction (not shown in Figure 1.3)  $[\delta y + (\partial v / \partial y) \delta y \delta t]$ . If  $\delta x, \delta y, \delta z$ , and  $\delta t$  are all small enough, higher-order terms can be neglected, and the volume occupied by the mass at time  $t + \delta t$  can be taken as the product of these three segments. Thus the change in volume during  $\delta t$  becomes

$$\frac{D(\delta x \delta y \delta z)}{Dt} \delta t = \left(1 + \frac{\partial u}{\partial x} \delta t\right) \delta x \left(1 + \frac{\partial v}{\partial y} \delta t\right) \delta y \left(1 + \frac{\partial w}{\partial z} \delta t\right) \delta z - \delta x \delta y \delta z$$

so that

$$\frac{D(\delta \forall)}{Dt} = \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \delta \forall \quad (1.5)$$

In more concise vector notation this can also be written as

$$\frac{D(\delta \forall)}{Dt} = \nabla \cdot \mathbf{v} \delta \forall \quad (1.6)$$

where  $\nabla$  is the operator  $\nabla = (\partial / \partial x)\mathbf{i} + (\partial / \partial y)\mathbf{j} + (\partial / \partial z)\mathbf{k}$ . Equations (1.5) and (1.6) show that the divergence  $\nabla \cdot \mathbf{v}$  is indeed, as its name suggests, the fractional rate of change of the fluid element volume. With this result, Equation (1.4) can be written as

$$\frac{D\rho}{Dt} + \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0 \quad (1.7)$$

or again, in vector notation, as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (1.8)$$

Equations (1.7) and (1.8) are forms of the classical continuity equation. The form of (1.8) is applicable to describe the conservation of mass of any substance at a given point  $(x, y, z)$ , provided  $(\rho \mathbf{v})$  is made to represent the specific mass flux  $\mathbf{F}$ , that is the transport of mass of that substance per unit cross-sectional area and per unit time. Whenever the density of the substance in question can be considered constant, the continuity equation assumes its well-known form

$$\nabla \cdot \mathbf{v} = 0 \quad (1.9)$$

Note that the continuity equation is not usually derived this way; the present derivation is used to maintain unity and consistency of treatment with the conservation of momentum in Section 1.5.3; moreover, it is relevant to the study of deforming porous media later on in Chapter 8. A more common way to derive the continuity equation consists of setting up a mass balance for a certain volume fixed in space, also called a *control volume*. The mass balance states that the sum of all the inflow rates into the control volume minus the sum of all the outflow rates is equal to the time rate of change of mass stored in the control volume. For an infinitesimally small control volume this procedure also yields Equation (1.8). Regardless of the derivation, however, it should be remembered that Equations (1.8) and (1.9) describe the flow at a point; therefore, in principle, the integration of (1.8) or (1.9) should allow the determination of the distribution of the amount and transport of water in space and in time.

## Finite Control Volume

In a second but equally valid approach, the control volume is assumed to occupy the entire flow domain by integrating out the spatial dependence of the flux terms. Thus all flux terms are located on the boundaries of the flow domain and they can be grouped into bulk inflow rates  $Q_i$  and outflow rates  $Q_e$ . As a result, the continuity equation takes on the *lumped* form of the *storage equation*, as follows

$$Q_i - Q_e = \frac{dS}{dt} \quad (1.10)$$

where  $S$  is the amount of water stored in the control volume, and the ordinary derivative indicates that the time  $t$  is the only remaining independent variable. When Equation (1.10) describes the flow of liquid water with an assumed constant density, these variables can have the dimensions  $[Q] = [L^3/T]$  and  $[S] = [L^3]$ , where  $L$  and  $T$  represent the basic dimensions of length and time, respectively; if the  $Q$ -terms include precipitation and evaporation, it is often convenient to take these dimensions as  $[Q] = [L/T]$  and  $[S] = [L]$ . In the lumped formulation of Equation (1.10), all interior variables and parameters represent spatial averages over the entire control volume. Equation (1.1) is a straightforward implementation of Equation (1.10), for a control volume consisting of the geologic formations of all the land surfaces of the Earth over a sufficiently long time period, so that the storage at the beginning is the same as at the end.

### 1.5.3 Conservation of Momentum: Euler and Navier–Stokes Equations

The flow of a fluid is also subject to the principle of conservation of momentum. Again, there are several ways of obtaining a mathematical formulation of this principle.

#### At a Point

The simplest method is probably to consider, as before, a small element of an ideal fluid with a mass  $(\rho\delta\forall)$ , as illustrated in Figure 1.4, and to apply Newton's second law to it. This states that the rate of change of momentum is equal to the sum of the impressed forces. The pressure and velocity at the center  $(x, y, z)$  of this element are  $p(x, y, z, t)$  and  $\mathbf{v}(x, y, z)$ , respectively. Accordingly, the property of the fluid element in this case is its momentum, or  $\mathbf{C} = (\rho\delta\forall\mathbf{v})$ , and the rate of change is  $D(\rho\delta\forall\mathbf{v})Dt$ , as given by Equation (1.3). Because the fluid is assumed to be ideal, the only relevant forces are those owing to pressure and to the acceleration of gravity. The latter is a vector,  $\mathbf{g} = \mathbf{i}g_x + \mathbf{j}g_y + \mathbf{k}g_z$ , whose direction defines the local vertical, and whose absolute value is commonly denoted by  $g$ ; the coordinates shown in Figure 1.4 are oriented in such a way that  $g_x = -g \sin \theta$ ,  $g_y = 0$ , and  $g_z = -g \cos \theta$ . As illustrated in Figure 1.4, the  $x$ -component of the net force acting on the fluid element is the sum of the forces acting on AD and BC plus the force due the Earth's gravity; this sum equals  $-(\partial p / \partial x) + \rho g \sin \theta \delta x \delta y \delta z$ . Similarly, the sum of the impressed forces in the  $z$ -direction is  $-(\partial p / \partial z) + \rho g \cos \theta \delta x \delta y \delta z$ . Adding to these an analogous  $y$ -component, one has, in vector notation,

$$\frac{D}{Dt}(\rho\delta\forall\mathbf{v}) = -(\nabla p - \rho\mathbf{g})\delta\forall$$

Making use of Equation (1.4), one obtains immediately

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g} \quad (1.11)$$

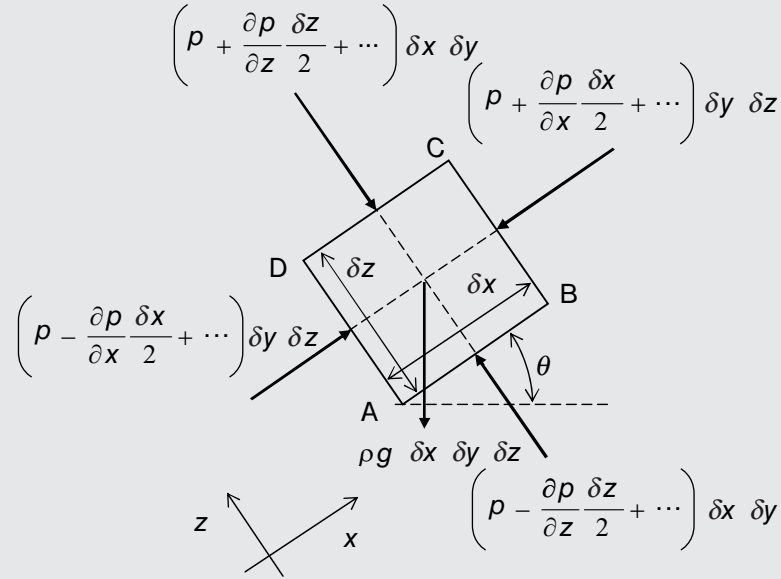
which is a form of Euler's equation. Inclusion of the effect of viscosity into Euler's equation produces the Navier–Stokes equation; expanding the substantial derivative according to its definition (Equation (1.3)), one can write it as follows:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{g} + \mathbf{f} \quad (1.12)$$

where  $\mathbf{f}$  denotes the frictional force (per unit mass); for an incompressible Newtonian fluid it can be shown that this is given by  $\mathbf{f} = \nu \nabla^2 \mathbf{v}$ , where  $\nu$  is the kinematic viscosity. To repeat briefly, the first term on the left represents the change in momentum (per unit mass) of the fluid due to local acceleration, i.e. changes in velocity at the point  $(x, y, z)$  under consideration. The second term represents the momentum changes resulting from acceleration (or deceleration) experienced by the fluid as it moves between points with different velocities. The first term on the right represents the force resulting from the pressure gradient, and the second the force resulting from the gravity field of the Earth. If the  $z$ -axis represents the vertical and points upward (or  $\theta = 0$  in Figure 1.4), Equation (1.12) can be written as

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho}\nabla p - g\mathbf{k} + \mathbf{f} \quad (1.13)$$

in which, it should be recalled,  $\mathbf{k}$  is the unit vector in the  $z$ -direction.



**Figure 1.4** Definition sketch for the conservation of momentum of a fluid element occupying the volume  $\delta\forall = (\delta x \delta y \delta z)$ , with its center at  $(x, y, z)$ . The element is subject to pressure forces and the acceleration of gravity. The  $y$ -coordinate, which is not shown, points into the plane of the drawing.

The viscosity of a liquid at standard pressure depends on temperature. For most practical purposes, in the range  $0 \leq T \leq 40^\circ\text{C}$  the kinematic viscosity of water (in units of  $\text{m}^2\text{s}^{-1}$ ) as a function of temperature (in  $^\circ\text{C}$ ) can be calculated with sufficient accuracy by means of  $\nu = 10^{-6} (1.785 - 0.05789T + 0.001128T^2 - 0.9671 \times 10^{-5}T^3)$ ; this yields roughly  $1.00 \times 10^{-6} \text{ m}^2\text{s}^{-1}$  at  $20^\circ\text{C}$ . This expression is based on measurements at the National Bureau of Standards, made available by J. F. Swindells. Similarly, for most applications in hydrology the density of water (in units of  $\text{kg m}^{-3}$ ) at 1 atmosphere can be calculated in the same temperature range by means of  $\rho = (999.8505 + 0.06001T - 0.007917T^2 + 4.1256 \times 10^{-5}T^3)$ .

## Finite Control Volume

Like [Equations \(1.8\) and \(1.9\)](#), also here [Equations \(1.11\), \(1.12\), and \(1.13\)](#) describe the flow phenomenon at a point. Again, they can be extended to a larger control volume by integrating out the spatial dependence of the terms. This can be accomplished by multiplication of each term in [Equation \(1.13\)](#) by the differential volume  $ds \cdot d\mathbf{A}$  (in which  $ds$  and  $d\mathbf{A}$  represent the differential flow path and cross-sectional area vectors, the latter pointing in the direction of flow) and by subsequent integration along all flow paths inside the control volume and across all areas of entry and exit of the control volume. For example, in the case of a conduit fixed in space occupied by a fluid volume  $S$  of constant density  $\rho$ , this yields for, say, the  $x$ -direction, approximately,

$$\rho \frac{d(S\bar{V}_x)}{dt} + \rho(Q_e V_{xe} - Q_i V_{xi}) = F_x \quad (1.14)$$

where  $F$  is the sum of all forces acting on the fluid in the control volume,  $Q_i$  and  $Q_e$  are the inflow and outflow rates of the control volume,  $\bar{V}$  is the average fluid velocity inside the control volume,  $V_i$  and  $V_e$  are the average fluid velocity over the entry and exit section, respectively, of the control volume, and the subscript  $x$  denotes the component direction of the momentum and of the forces.

### 1.5.4 The Kinematic Approach

In principle, the description of fluid flow phenomena should involve conservation of mass, conservation of momentum, and conservation of energy. However, whenever the relevant phenomena are isothermal, most of the energy is mechanical, and the energy conservation equation becomes redundant, so that it is often not included in the formulation. In this book, the energy conservation equation will be used only in relation to atmospheric phenomena, where it will be discussed further. In hydrologic applications, whenever both mass and momentum conservation principles are made use of, the mathematical description of the flow phenomena is called a *dynamic* formulation. However, in some situations, momentum changes, both temporal and spatial, are so small that they can be neglected. In such cases, the terms on the left-hand side of Equation (1.12) can be omitted and this greatly simplifies the formulation. In practice, the right-hand side of Equation (1.12) can then often be parameterized by an explicit functional relationship between the flow velocities in the system and some other variables, such as pressure, water depth, or water level height. Whenever only the continuity equation is required, and the momentum equation can be replaced by this type of relationship, the mathematical description is referred to as a *kinematic* formulation. The same idea can also be applied to larger control volumes. In this case, the combination of the storage equation (1.10) with a simple functional relationship between  $S$  and  $Q_e$  and/or  $Q_i$  is called a *lumped kinematic* formulation.

## Problems

**1.1** Assuming that the average volume of water storage in the soil equals 0.05 m (if spread over the entire Earth's surface; see Table 1.3), and that the average precipitation on the land surfaces of the Earth is equal to  $0.8 \text{ m a}^{-1}$  (Table 1.1), give an estimate (expressed in days) for the mean residence time of soil water. Consider conditions to be steady, so that precipitation is in balance with runoff and evaporation, and assume that all the precipitation infiltrates into the soil. What would this residence time be, if it is assumed that only one half of the precipitation infiltrates directly into the soil, and that the remaining half immediately evaporates as interception or runs off on the surface?

**1.2** Recent estimates of the average surface energy fluxes at the global scale (Table 1.4) are as follows; net radiation  $R_n = 103 \text{ W m}^{-2}$ , latent heat flux  $L_e E = 82 \text{ W m}^{-2}$ , and turbulent sensible heat flux  $H = 21 \text{ W m}^{-2}$ . Express these fluxes as equivalent quantities of liquid water evaporated in units of  $\text{mm a}^{-1}$ . Assume that the latent heat of vaporization of water is roughly  $L_e = 2.5 \times 10^6 \text{ J kg}^{-1}$  and the density  $\rho_w = 10^3 \text{ kg m}^{-3}$ .

**1.3** Using the data in Table 1.1, show that the estimates of the average evaporation excess ( $E - P$ ) over the world's oceans are indeed (roughly) equal to the estimates of the combined outflow rate from all the world's rivers. Which estimates satisfy Equation (1.1) best?

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**Part I**

# **Water in the Atmosphere**



# 2

## Water Aloft: Fluid Mechanics of the Lower Atmosphere

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On account of the short residence time and great mobility of water vapor in air, the lower atmosphere is one of the critical pathways in the global hydrologic cycle; it transports water and energy around the globe without regard to continental boundaries, and thus links the continents, the upper atmosphere, and the oceans. The transport and distribution of water vapor in the lower atmosphere, where it is most abundantly present, are among the main factors controlling precipitation and evaporation from the surface; these processes, in turn, determine soil and groundwater storage and the different runoff phenomena.

### 2.1 Water Vapor in Air

#### 2.1.1 Global Features

The global amount of water vapor contained in the air (see [Table 1.3](#)) is roughly equivalent to a layer of liquid water covering the Earth, with a thickness of around 25 mm, on average. The thickness of this layer, which is the total liquid equivalent of water vapor in the atmospheric column at a given location, is also called the *precipitable water*  $W_p$ . However, this quantity of water vapor is not distributed uniformly and it can vary greatly over a wide range of scales in space and in time. For instance, the water vapor content of the atmosphere, just like the temperature, generally tends to decrease with increasing latitude. Available data (Randel et al., 1996) show that the precipitable water is more likely to be well below 5 mm near the poles, and close to 50 mm near the equator. But this is not always the case; even at similar latitudes there can be huge regional variations, the most extreme example being the warm, dry deserts of the world. Most of the atmospheric water vapor is found relatively close to the ground, and at any given location water vapor decreases sharply with height; typically, about half the total water vapor in the atmospheric column can be found below a height of 1 or 2 km.

Because the global annual evaporation is around  $E = 1$  m, the average atmospheric residence time of water vapor,  $W_p/E$ , is only about 9 days. This time scale governs the hydrologic interactions and water transfers between the atmosphere and the other two compartments of the global system, the oceans and the continents. It is especially fundamental to the transport of atmospheric water vapor from its source regions – mainly evaporation from the oceans – to sinks in precipitating weather systems. Indeed, the excess precipitation on the continents, which does not evaporate, ultimately runs off to the seas and oceans of the world. A balance is maintained in the global system by the fact that over the oceans the situation is reversed and that evaporation is generally larger than precipitation, allowing the excess oceanic water vapor to be transported back to the continents. This transport of water from the oceans to the land areas, also called advection, takes place mostly in the form of water vapor and not as clouds; actually, in the

atmosphere the total amount of water in the liquid and ice phases is less than 0.5% of the water in the vapor phase.

But beside its central role in the hydrologic cycle, water vapor strongly affects other aspects of the Earth's weather and climate as well. It is one of the main agents in the overall energy budget of the atmosphere in a number of ways. Globally, as seen in [Table 1.4](#), the phase changes from liquid and solid to vapor are the main energy transfer mechanisms from the Earth's surface to the atmosphere; the subsequent condensation of this vapor in the air furnishes a large portion of the energy needed for the circulation of the atmosphere. Thus, the large-scale transport of water vapor is one of the main latent heat redistribution mechanisms for the uneven radiative input from the Sun. In addition, the concentration and spatial distribution of atmospheric water vapor are major factors controlling the amount and type of cloud, which in turn determine the solar radiation reaching the Earth's surface. Finally, as the most abundant greenhouse gas, water vapor absorbs and thus “traps” terrestrial infrared radiative energy, and then re-emits it at a lower temperature.

### 2.1.2 Some Physical Properties

For many practical purposes, the air of the lower atmosphere can be considered as a mixture of perfect gases; in the present context these may conveniently be assumed to be dry air of constant composition and water vapor. The water vapor content of the air can be expressed in terms of the mixing ratio, defined as the mass of water vapor per unit mass of dry air,

$$m = \rho_v / \rho_d \quad (2.1)$$

where  $\rho_v$  is the density of the water vapor and  $\rho_d$  the density of the air without the water vapor. The specific humidity is defined as the mass of water vapor per unit mass of moist air,

$$q = \rho_v / \rho \quad (2.2)$$

where  $\rho = \rho_v + \rho_d$ . The relative humidity is the ratio of the actual mixing ratio and the mixing ratio in water-vapor-saturated air at the same temperature and pressure,

$$r = m / m^* \quad (2.3)$$

This is nearly equal to  $(e/e^*)$ , the ratio of the actual vapor pressure and the saturation vapor pressure; the latter is the pressure of the vapor when it is in equilibrium with a plane surface of water or ice at the same temperature and pressure.

According to Dalton's law, the total pressure in a mixture of perfect gases equals the sum of the partial pressures, and each of the component gases obeys its own equation of state. Thus, the density of the dry air component is

$$\rho_d = \frac{p - e}{R_d T} \quad (2.4)$$

where  $p$  is the total pressure in the air,  $e$  the partial pressure of the water vapor,  $T$  the (“absolute”) temperature, and  $R_d$ , which is given in [Table 2.1](#), is the specific gas constant for dry air. Similarly, the density of water vapor is

$$\rho_v = \frac{0.622e}{R_d T} \quad (2.5)$$

where  $0.622 = (18.016/28.966)$  is the ratio of the molecular weights of water and dry air.

**Table 2.1 Some physical constants****Dry Air**Molecular weight:  $28.966 \text{ g mol}^{-1}$ Gas constant:  $R_d = 287.04 \text{ J kg}^{-1} \text{ K}^{-1}$ Specific heat:  $c_{pd} = 1005 \text{ J K}^{-1} \text{ kg}^{-1}$  $c_{vd} = 716 \text{ J K}^{-1} \text{ kg}^{-1}$ Density:  $\rho = 1.2923 \text{ kg m}^{-3}$  $(p = 1013.25 \text{ mb}, T = 273.16 \text{ K})$ **Water Vapor**Molecular weight:  $18.016 \text{ g mol}^{-1}$ Gas constant:  $R_w = 461.5 \text{ J kg}^{-1} \text{ K}^{-1}$ Specific heat:  $c_{pw} = 1846 \text{ J kg}^{-1} \text{ K}^{-1}$  $c_{vw} = 1386 \text{ J kg}^{-1} \text{ K}^{-1}$ 

*Note:* The values listed in [Tables 2.1, 2.4, and 2.5](#) are adapted from the Smithsonian Meteorological Tables ([List, 1971](#)), where the original references are cited.

The density of moist air is, from (2.4) and (2.5)

$$\rho = \frac{p}{R_d T} \left( 1 - \frac{0.378e}{p} \right) \quad (2.6)$$

showing that it is smaller than that of dry air at pressure  $p$ . This means that water vapor stratification plays a role in determining the stability of the atmosphere. The equation of state of moist air can be obtained by eliminating  $e$  from (2.4) and (2.5):

$$p = \rho T R_d (1 + 0.61 q) \quad (2.7)$$

This indicates that the air mixture behaves as a perfect gas provided it has a specific gas constant

$$R_m = R_d (1 + 0.61 q) \quad (2.8)$$

that is a function of the water vapor content. Therefore, (2.7) is often expressed as

$$p = R_d \rho T_V \quad (2.9)$$

where  $T_V$  is the virtual temperature defined by

$$T_V = (1 + 0.61 q) T \quad (2.10)$$

The virtual temperature is the temperature that dry air should have in order to have the same density as moist air with given  $q$ ,  $T$ , and  $\rho$ .

The precipitable water is the total mass of water vapor in a vertical atmospheric column; if it is assumed that the pressure is negligible at the top of the atmosphere, it can be written as

$$W = \int_0^{p_0} q \, dp / g \quad (2.11)$$



Table 2.2 Some useful units

|          | SI (mks)                                | cgs                                              |
|----------|-----------------------------------------|--------------------------------------------------|
| Length   | meter<br>m                              | centimeter<br>cm                                 |
| Mass     | kilogram<br>kg                          | gram<br>g                                        |
| Time     | second<br>s                             | second<br>s                                      |
| Force    | newton<br>$N = \text{kg m s}^{-2}$      | dyne<br>$\text{dyn} = \text{g cm s}^{-2}$        |
| Pressure | pascal<br>$\text{Pa} = \text{N m}^{-2}$ | microbar<br>$\mu\text{bar} = \text{dyn cm}^{-2}$ |
| Energy   | joule<br>$J = \text{N m}$               | erg<br>$\text{erg} = \text{dyn cm}$              |
| Power    | watt<br>$W = \text{J s}^{-1}$           | $\text{erg s}^{-1}$                              |

Table 2.3 Conversion factors

|                         |                                              |                                                                                                                                                                                    |
|-------------------------|----------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Pressure                | millibar<br>millimeter mercury<br>atmosphere | $1 \text{ mb} = 1 \text{ hPa} = 10^3 \mu\text{bar} = 10^3 \text{ dyn cm}^{-2}$<br>$1 \text{ mm Hg} = 1.333\,224 \text{ hPa}$<br>$1 \text{ atm} = 1.013\,25 \times 10^5 \text{ Pa}$ |
| Energy<br>(Energy/area) | calorie (IT)<br>(langley)                    | $1 \text{ cal} = 4.1868 \text{ J} = 4.1868 \times 10^7 \text{ erg}$<br>$(1 \text{ ly} = 1 \text{ cal cm}^{-2})$                                                                    |

where  $p_0$  is the surface pressure. Recall that the basic dimensions of these variables are  $[q] = [M_w / M_a]$ ,  $[p] = [M_a L^{-1} T^{-2}]$ , and  $[g] = [L T^{-2}]$ , in which it is convenient to distinguish between the mass of air  $M_a$  and the mass of water substance  $M_w$ . Therefore the basic dimensions of precipitable water are  $[W] = [M_w L^{-2}]$ , i.e. water mass per unit area. In SI units this can be expressed in  $\text{kg m}^{-2}$ , which is roughly equivalent to mm of vertical liquid water column, because the density of liquid water is around  $1000 \text{ kg m}^{-3}$ .

For convenient reference, some common units and conversion factors are listed in [Tables 2.2 and 2.3](#).

### 2.1.3 Saturation Vapor Pressure

The saturation vapor pressure depends only on the temperature, or  $e^* = e^*(T)$ . Some values are presented in [Tables 2.4 and 2.5](#). These values were obtained from the Goff–Gratch formulation (see [List, 1971](#)), which has been used as the international standard for some time. On the basis of this formulation, a simpler function as a close approximation, which should be suitable for most applications in the lower atmosphere, was proposed by [Bolton \(1980\)](#); it is accurate to 0.1% in the range  $-35^\circ\text{C} \leq T \leq 35^\circ\text{C}$  and can be written as

$$e^* = 6.112 \exp\left(\frac{17.67T}{T + 243.5}\right) \quad (2.12)$$

in which  $e^*$  is in hPa and here  $T$  is the temperature in degrees celsius. The derivative of this function is best expressed in the form of the original Clausius–Clapeyron equation (see [Iribarne and Godson, 1973](#)) as follows

**Table 2.4** Some properties of water

| Temperature<br>(°C) | $c_w$<br>(J kg <sup>-1</sup> K <sup>-1</sup> ) | $L_e$<br>(10 <sup>6</sup> J kg <sup>-1</sup> ) | $e^*$<br>(hPa) | $de^*/dT$<br>(hPa K <sup>-1</sup> ) |
|---------------------|------------------------------------------------|------------------------------------------------|----------------|-------------------------------------|
| -20                 | 4354                                           | 2.549                                          | 1.2540         | 0.1081                              |
| -10                 | 4271                                           | 2.525                                          | 2.8627         | 0.2262                              |
| 0                   | 4218                                           | 2.501                                          | 6.1078         | 0.4438                              |
| 5                   | 4202                                           | 2.489                                          | 8.7192         | 0.6082                              |
| 10                  | 4192                                           | 2.477                                          | 12.272         | 0.8222                              |
| 15                  | 4186                                           | 2.466                                          | 17.044         | 1.098                               |
| 20                  | 4182                                           | 2.453                                          | 23.373         | 1.448                               |
| 25                  | 4180                                           | 2.442                                          | 31.671         | 1.888                               |
| 30                  | 4178                                           | 2.430                                          | 42.430         | 2.435                               |
| 35                  | 4178                                           | 2.418                                          | 56.236         | 3.110                               |
| 40                  | 4178                                           | 2.406                                          | 73.777         | 3.933                               |

$c_w$ : specific heat;  $L_e$ : latent heat of vaporization;  $e^*$ : saturation vapor pressure.

**Table 2.5** Some properties of ice

| Temperature<br>(°C) | $c_i$<br>(J kg <sup>-1</sup> K <sup>-1</sup> ) | $L_{fu}$<br>(10 <sup>6</sup> J kg <sup>-1</sup> ) | $L_s$<br>(10 <sup>6</sup> J kg <sup>-1</sup> ) | $e_i^*$<br>(hPa) | $de_i^*/dT$<br>(hPa K <sup>-1</sup> ) |
|---------------------|------------------------------------------------|---------------------------------------------------|------------------------------------------------|------------------|---------------------------------------|
| -20                 | 1959                                           | 0.2889                                            | 2.838                                          | 1.032            | 0.09905                               |
| -15                 | —                                              | —                                                 | —                                              | 1.652            | 0.1524                                |
| -10                 | 2031                                           | 0.3119                                            | 2.837                                          | 2.597            | 0.2306                                |
| -5                  | —                                              | —                                                 | —                                              | 4.015            | 0.3432                                |
| 0                   | 2106                                           | 0.3337                                            | 2.834                                          | 6.107            | 0.5029                                |

$c_i$ : specific heat;  $L_{fu}$ : latent heat of fusion;  $L_s (= L_e + L_{fu})$ : latent heat of sublimation;  $e_i^*$ : saturation vapor pressure over ice.

$$\frac{de^*}{dT} = \frac{e^* L_e}{R_w T^2} \quad (2.13)$$

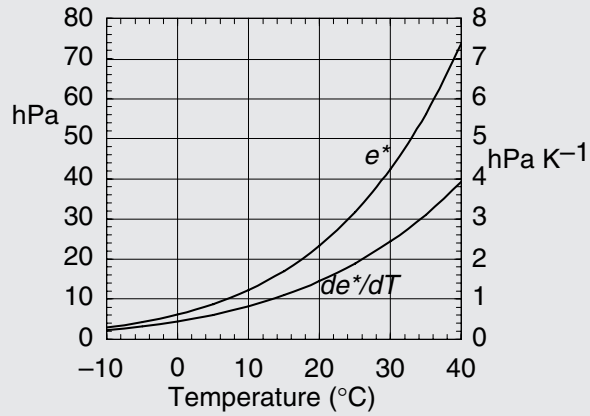
in which  $R_w = 461.5 \text{ J kg}^{-1} \text{ K}^{-1}$  is the gas constant for water vapor (see Table 2.1),  $T$  is in kelvin as usual, and the latent heat of vaporization can be given to a good approximation by  $L_e = [2.5011 - 0.002374(T - 273.15)] \times 10^6$  in units of  $\text{J kg}^{-1}$ . Both  $e^*$  and  $de^*/dT$  are sketched in Figure 2.1. Equation (2.13) can be adapted to vapor above ice by replacing  $L_e$  by the sum of the latent heat values of fusion and of vaporization of liquid water, namely  $L_s = L_{fu} + L_e$ . Lowe (1977), who has also compared other currently used expressions for the saturation vapor pressure has presented polynomials for  $e^*$ ,  $de^*/dT$ ,  $e_i^*$ , and  $de_i^*/dT$ , which are quite accurate and suitable for rapid computation. For computational speed these polynomials should be used in nested form; for  $e^*$  the representation takes the form

$$e^* = a_0 + T(a_1 + T(a_2 + T(a_3 + T(a_4 + T(a_5 + Ta_6)))))) \quad (2.14)$$

where the polynomial coefficients are as follows when  $T$  is in K:

$$a_0 = 6984.505\,294, \quad a_1 = -188.903\,931\,0, \quad a_2 = 2.133\,357\,675, \quad a_3 = -1.288\,580\,973 \times 10^{-2}, \\ a_4 = 4.393\,587\,233 \times 10^{-5}, \quad a_5 = -8.023\,923\,082 \times 10^{-8}, \quad \text{and} \quad a_6 = 6.136\,820\,929 \times 10^{-11}.$$

**Figure 2.1** The saturation vapor pressure  $e^*$  of water (in hPa, left-hand ordinate scale), and its derivative  $de^*/dT$  (in hPa K<sup>-1</sup>, right-hand ordinate scale), both as functions of temperature.



## 2.2 Hydrostatics and Atmospheric Stability

The first law of thermodynamics states that the heat added to a system equals the sum of the change in internal energy and the work done by the system on its surroundings. If these quantities are taken per unit mass and in differential form, this is, for partly saturated air

$$dh = du + p d\alpha \quad (2.15)$$

where  $\alpha = \rho^{-1}$  is the specific volume,  $\rho$  is the density of the air, and (in this Section 2.2 only)  $u$  represents the internal energy. The equation of state (2.7), which on account of (2.8) can also be written as

$$p = R_m T / \alpha \quad (2.16)$$

relates the three variables,  $\alpha$ , the temperature,  $T$ , and the pressure,  $p$ ; thus only two of the three are needed to define the state. If  $\alpha$  and  $T$  are chosen as these independent variables, (2.15) becomes

$$dh = \left( \frac{\partial u}{\partial T} \right)_\alpha dT + \left[ \left( \frac{\partial u}{\partial \alpha} \right)_T + p \right] d\alpha \quad (2.17)$$

Since, by definition, the specific heat capacity for constant volume is  $c_v = (\partial u / \partial T)_\alpha$  and since it can be shown that  $(\partial u / \partial \alpha)_T = 0$ , combination of the differential form of Equation (2.16) with (2.17) produces

$$dh = (c_v + R_m) dT - \alpha dp \quad (2.18)$$

or

$$dh = c_p dT + \alpha dp \quad (2.19)$$

where, by definition,  $c_p = (\partial h / \partial T)_p$  is the specific heat for constant pressure. With the hydrostatic law giving the pressure change with height in a fluid at rest, i.e.

$$dp = - \rho g dz \quad (2.20)$$

one obtains finally from (2.19)

$$dh = c_p dT + g dz \quad (2.21)$$

Equation (2.21) is derived here by a combination of the principle of conservation of energy with the equation of state and the hydrostatic equation. This result was obtained for air containing water vapor; however, the moisture-content-dependency of the specific heat at constant pressure, namely  $c_p = q c_{pw} + (1-q)c_{pd}$ , is very weak and therefore (2.21) is usually applied with the specific heat for dry air, i.e.  $c_{pd}$ .

The criterion for the stability of an atmosphere at rest can be obtained by the following thought experiment. Consider a small parcel of air with a temperature  $T_1$  that undergoes a small vertical displacement without mixing with the surrounding body of air; this displacement is sufficiently small and fast that the pressure of the particle adjusts to its new environment in an adiabatic fashion, that is as a reversible process without heat exchange with its surroundings. Two cases are of interest, depending on the degree of saturation of the air.

### 2.2.1 Stability of a Partly Saturated Atmosphere

#### Dry Adiabatic Lapse Rate

If the atmosphere can be assumed to remain partly saturated during this process, there is no vaporization or condensation, and the change in heat of the parcel is  $dh = 0$ . With (2.21) this yields, for the temperature change of the parcel as it moves up or down,

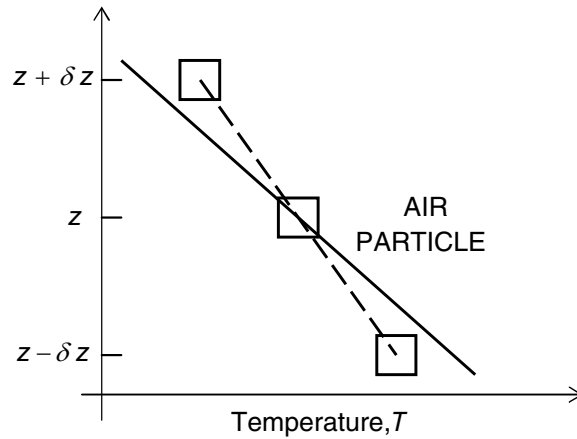
$$dT_1 / dz = -g / c_p \quad (2.22)$$

which is of the order of  $9.8 \text{ K km}^{-1}$ . The vertical rate of decrease in temperature of the atmosphere,  $-dT / dz$ , is the lapse rate of the air, denoted here by  $\Gamma$ . A lapse rate of the atmosphere, that is equal to  $g / c_p$ , is called a dry adiabatic lapse rate,  $\Gamma_d$ .

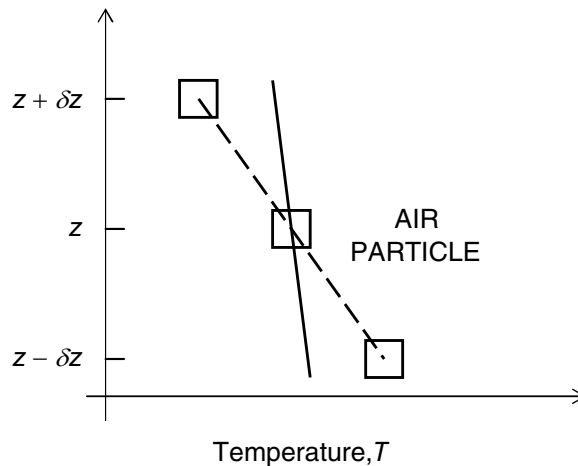
Whenever the actual lapse rate in the atmosphere  $\Gamma$  is larger than  $\Gamma_d$ , a particle undergoing a small upward displacement  $\delta z$  and changing its temperature according to (2.22), will be warmer and therefore lighter than its surroundings; this means that it will have a tendency to continue its upward motion (see Figure 2.2). By the same token, a particle undergoing a small downward displacement  $\delta z$  under the same lapse rate conditions will be colder and thus heavier than the surrounding air; again, it will have a tendency to continue its downward movement. In both situations, once displaced, however slightly, air parcels tend to continue their motion and amplify their displacements both upward and downward; under such lapse rate conditions the atmosphere is unstable. Conversely, in an atmosphere with  $\Gamma < \Gamma_d$ , the upward moving parcel, whose temperature change is given by (2.22), becomes surrounded by air that is relatively warmer (see Figure 2.3); thus it is heavier and it will tend to return to its original position, where it was in equilibrium with its surroundings. In this case, the parcel resists being moved away from its original position and vertical displacements are suppressed; the atmosphere is said to be stable. Under conditions when the actual lapse rate in the air is dry adiabatic, the atmospheric stability is neutral. In summary, one has for unsaturated air the following criteria:

$$\begin{aligned} \Gamma &> \Gamma_d && \text{unstable} \\ \Gamma &= \Gamma_d && \text{neutral} \\ \Gamma &< \Gamma_d && \text{stable} \end{aligned}$$

**Figure 2.2** Unstable atmosphere in which the temperature decrease with elevation (solid line) is larger than the dry adiabatic lapse rate (dashed line), so that  $\Gamma > \Gamma_d$ .



**Figure 2.3** Stable atmosphere in which the temperature decrease with elevation (solid line) is smaller than the dry adiabatic lapse rate (dashed line), so that  $\Gamma < \Gamma_d$ .



Unstable conditions typically occur whenever the atmosphere is being heated by the surface below, for example, as a result of solar radiation on days with clear skies, or as cold air moves over a relatively warmer surface, such as a lake or the ocean in fall and early winter. An unstable atmosphere is subject to more intense mixing and turbulence than a neutral one; this also results in larger turbulent transport. Under certain conditions an unstable atmosphere will even give rise to various types of organized motion, with scales ranging from mere local updrafts and dust devils to large tropical storms. The atmosphere is often stable when the air is being cooled from below. This typically occurs at night under clear skies, when the surface is cooled by outgoing long-wave radiation or when relatively warm air flows over a relatively colder lake or ocean surface in spring. In addition, stable conditions, also called inversions, may result from larger-scale weather patterns, when relatively warmer air masses move over colder layers. Not surprisingly, stable conditions have the opposite effect of unstable conditions. Thus mixing and turbulence are suppressed, and atmospheric transport is normally smaller. Under extreme conditions, the turbulence may be altogether eliminated and the flow of the air may be laminar. Such conditions are sometimes visible in the evening of a calm sunny day, around sunset, when the air near the ground becomes chilly and the smoke from a chimney can be seen moving slowly through the tree tops of a forest. On a somewhat larger scale, as a result of the reduced turbulence and dispersion, inversion conditions also tend to aggravate pollution problems in populated areas.

## Potential Temperature

To repeat briefly, during small displacements, parcels of air undergo adiabatic temperature changes in accordance with (2.22). In a perfectly neutral atmosphere the lapse rate of the atmosphere is also  $-g/c_p$ ; therefore under such conditions a displaced parcel will, on average at least, always be surrounded by air at the same temperature and, as a result, there is no net exchange of heat. This means that even though there is a vertical temperature difference under neutral conditions, the heat flux is zero; consequently, the temperature used in heat transfer formulations should be corrected to account for that fact. This can be done by using the potential temperature  $\theta$  instead of the actual temperature  $T$ . The potential temperature is the temperature which would result if air were brought adiabatically to a standard pressure level  $p_0 = 1000$  hPa; for such a process  $dh = 0$ , and (2.19), after substitution of  $\alpha$  with (2.16), can be integrated to yield Poisson's equation

$$\theta = T (p_0 / p)^{R_d/c_p} \quad (2.23)$$

which can serve to define the potential temperature  $\theta$  and also to calculate it for a given pressure  $p$  and temperature  $T$ ; note that in (2.23)  $R_m$  is replaced by  $R_d$  for convenience. During an adiabatic process the potential temperature is conserved and therefore under perfectly neutral conditions in a dry atmosphere, or when the specific humidity is constant with height, the potential temperature should be a constant. A dry atmosphere is unstable when  $\theta$  decreases, and stable when it increases with height. Nevertheless, the difference between  $T$  and  $\theta$  is usually rather small, especially in the lower layers of the air near the ground surface, where most measurements are made. Therefore, in many situations, when the height difference of the temperature measurements is only a few meters, the use of the actual temperature  $T$  is allowed; otherwise  $\theta$  must be used in heat transfer formulations.

## Density Stratification Due to Water Vapor

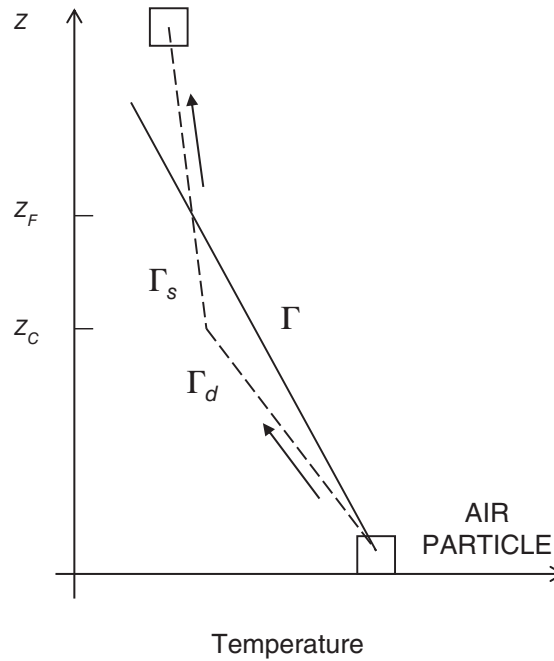
In the above considerations of atmospheric stability, the density stratification due to vertical humidity gradient  $\partial q / \partial z$  was not taken into account. Under some conditions this can be an important factor, but it can be readily shown (e.g. Brutsaert, 1982) that this may be incorporated into the analysis by means of the virtual potential temperature, defined as  $\theta_v = (1 + 0.61q)\theta$ ; the virtual potential temperature is related to the potential temperature, in the same way the virtual temperature is related to the actual temperature, as indicated in (2.10). Thus, strictly speaking, in the presence of humidity stratification, the atmosphere can be considered statically neutral, not when  $\theta$  is constant, but only when  $\theta_v$  is constant; it is unstable when  $\theta_v$  decreases, and stable when it increases with height. Put differently, under such conditions the stability criterion for an atmosphere is not the lapse rate of the temperature, but the lapse rate of the virtual temperature; in practice, however, this difference is often ignored.

### 2.2.2 Stability of a Saturated Atmosphere

When the air is saturated, any increase in heat content  $dh$  of a parcel of air during an adiabatic process can only be the result of condensation, that is a decrease in the water vapor content of the air; this can be written as  $dh = -L_e dq$ , in which  $L_e$  is the latent heat of vaporization and  $q$ , defined in (2.2), is the specific humidity. With (2.21) one obtains now

$$-\frac{dT_1}{dz} = \Gamma_d + \frac{L_e}{c_p} \frac{dq}{dz} \quad (2.24)$$

The quantity on the right-hand side of (2.24) is called the saturated adiabatic lapse rate,  $\Gamma_s$ . Since normally  $dq/dz < 0$ , the saturated adiabatic lapse rate must be smaller than the dry adiabatic lapse rate. Moreover,  $(L_e/c_p)(dq/dz)$  depends on the temperature. At high temperatures, near the equator, it is of the order of  $\Gamma_s \approx 0.35 \Gamma_d$ , whereas at lower temperatures, for example around  $-30^\circ\text{C}$ , it has approximately the same value as  $\Gamma_d$ , i.e.  $9.8^\circ\text{C km}^{-1}$ . In the lower layers of the atmosphere it is, on average, of the order of  $5.5^\circ\text{C km}^{-1}$ .



**Figure 2.4** Conditional instability in the atmosphere. A partly saturated air particle, which is raised, initially undergoes a rate of temperature decrease  $\Gamma_d$  (dashed line) which is larger than that of the environment,  $\Gamma$ . As it becomes saturated at  $z_C$ , the rate of temperature decrease is reduced to the saturated adiabatic lapse rate  $\Gamma_s$  (dashed line). Above  $z_F$  conditions become unstable.

If in a rising air mass the condensed moisture is being removed (e.g. through precipitation), the rate of temperature decrease is called the pseudo-adiabatic lapse rate. However, under most conditions, the heat loss by the removal of this condensed water is fairly negligible, and the saturated adiabatic lapse rate is a satisfactory approximation. Thus, for saturated air, one has the following stability criteria:

$$\begin{aligned}\Gamma &> \Gamma_s && \text{unstable} \\ \Gamma &= \Gamma_s && \text{neutral} \\ \Gamma &< \Gamma_s && \text{stable}\end{aligned}$$

### 2.2.3 Conditional Instability

It often happens that the actual lapse rate in the atmosphere is intermediate between the dry and the saturated adiabatic lapse rate, that is  $\Gamma_s < \Gamma < \Gamma_d$ ; this case is referred to as conditional instability. When a partly saturated parcel of air is raised in such an atmosphere, it will initially change its temperature at the dry adiabatic rate in accordance with Equation (2.22), and thus remain colder than the surroundings (see Figure 2.4). This situation is still stable. However, if the particle is made to rise further, and continues to cool down, it may reach the level  $z_C$ , where condensation takes place; above the condensation level  $z_C$  it will change its temperature at the saturated adiabatic lapse rate. If the rise continues, eventually at the level  $z_F$  the temperature of the particle will exceed that of the surroundings; the rising air is now lighter than its surroundings, and an unstable situation has been established. The air, which was originally forced to rise, will now take off by free

convection and continue to rise without any outside agent. The height  $z_F$  is the free convection level. Thus whether or not a vertical displacement results in instability depends largely on the moisture content of the atmosphere. In a moist atmosphere the condensation level is low, and relatively small vertical displacements readily produce unstable conditions. In a dry atmosphere, the level  $z_C$  is higher, and the atmosphere is more likely to remain stable, even with relatively large vertical displacements.

## 2.3 Turbulent Transport of Water Vapor

The flow of the atmosphere is almost invariably turbulent. In a turbulent flow, molecular diffusion can usually be neglected, and water vapor is moved from one place to another by advective transport, that is by being linked to the motions of the air that contains it. One exception, when molecular diffusion may be of some consequence, occurs near a wall where the no-slip condition reduces the velocity of the moving air to zero and the turbulence is largely suppressed. Thus in turbulent air flow, the specific mass flux of water vapor is given by

$$\mathbf{F}_v = \rho_v \mathbf{v} = \rho q \mathbf{v} \quad (2.25)$$

where  $\mathbf{v}$  is the velocity of the air,  $\rho_v$  the water vapor density, and  $q$  the specific humidity. The variables  $\mathbf{F}_v = \mathbf{i} F_{vx} + \mathbf{j} F_{vy} + \mathbf{k} F_{vz}$  and  $\mathbf{v} = \mathbf{i} u + \mathbf{j} v + \mathbf{k} w$  are both vectors, with  $x$  denoting the direction of the mean wind near the ground and  $z$  the vertical.

Note that the transport described by Equation (2.25) can also be referred to as *convection* in fluid mechanics. However, this usage may lead to some confusion because, especially in the atmospheric sciences, the term convection is commonly reserved to describe transport involving gravity effects, resulting from unstable density stratification. To avoid such confusion in this regard, in this book any transport which is linked to the motion of the fluid is called *advection*.

### Turbulent Flux of Water Vapor

In turbulent flow the detailed description of the velocity field and also the temperature, the content of water vapor, or other admixtures of the air, at any given point in time and space, is practically impossible and it can only be accomplished in a statistical sense. The simplest and probably most important statistic is the mean. Accordingly, ever since Reynolds introduced the idea, it has been common practice in the analysis of turbulent-flow phenomena to decompose the relevant variables into a mean and a turbulent fluctuation, namely  $F_{vx} = \overline{F_{vx}} + F'_{vx}, \dots, u = \bar{u} + u', \dots, q = \bar{q} + q'$ , etc. After applying the customary time-averaging over a suitable time period, one obtains from Equation (2.25) for the mean flux components of water vapor

$$\begin{aligned} \overline{F_{vx}} &= \rho (\bar{u} \bar{q} + \overline{u'q'}) \\ \overline{F_{vy}} &= \rho (\bar{v} \bar{q} + \overline{v'q'}) \\ \overline{F_{vz}} &= \rho (\bar{w} \bar{q} + \overline{w'q'}) \end{aligned} \quad (2.26)$$

The first terms on the right of these three equations represent the advective transport of water vapor by the mean motion of the air. The second terms are the components of the advective vapor transport by the turbulence; they are also often called the Reynolds fluxes, and statistically speaking, they are covariances. The estimation and parameterization of these flux components is one of the core problems of hydrology.



### Conservation Equation of Water Vapor

The standard procedure for a more thorough analysis of the water-vapor transport consists of combining the expressions for the fluxes, (2.26), with the principle of mass conservation (1.8) applied to water vapor. This is accomplished by substituting  $\rho_v$  for  $\rho$  and  $\mathbf{F}_v$  for  $(\rho \mathbf{v})$  in Equation (1.8); since in this derivation, the bulk air itself is of less concern, it can be assumed to have a constant density, which allows use of Equation (1.9) for the mean velocity  $\bar{\mathbf{v}}$ . Thus, one obtains the conservation equation for the mean specific humidity,  $\bar{q}$  (e.g. Brutsaert, 1982), as follows:

$$\frac{\partial \bar{q}}{\partial t} + \bar{u} \frac{\partial \bar{q}}{\partial x} + \bar{v} \frac{\partial \bar{q}}{\partial y} + \bar{w} \frac{\partial \bar{q}}{\partial z} = - \left[ \frac{\partial}{\partial x} (\overline{u'q'}) + \frac{\partial}{\partial y} (\overline{v'q'}) + \frac{\partial}{\partial z} (\overline{w'q'}) \right] \quad (2.27)$$

in which, it should be noted again, the molecular diffusion term is neglected. In principle, it should be possible to solve Equation (2.27) with appropriate boundary conditions to describe water-vapor transport in the atmosphere. However, this equation presents several difficulties, which make its solution extremely difficult. First, since the fluxes (2.26) are intrinsically dependent on the velocity of the air and the turbulence, it is necessary to consider the dynamics of the flow and to include the conservation equations for momentum and temperature in the solution process as well. A second and more fundamental difficulty is that this conservation equation for the mean specific humidity contains not only  $\bar{q}$ , as dependent variable, which is the first moment, but also the covariances of  $q'$  with the velocity fluctuations  $u'$ ,  $v'$ , and  $w'$ , which are second moments. This means that (2.27) has more than one unknown; this fact is an instance of the notorious closure problem of turbulence and it indicates that, without additional relationships, this equation cannot be solved mathematically.

Fortunately, it is possible to simplify the general problem, as formulated with the above fluxes, considerably and still obtain meaningful results. This is accomplished, first, by assuming that the atmosphere nearest the surface can be considered as a steady boundary layer above a quasi-uniform surface (Section 2.4), and second, by the application of similarity assumptions to alleviate the turbulence closure problem by appropriate parameterization (Section 2.5).

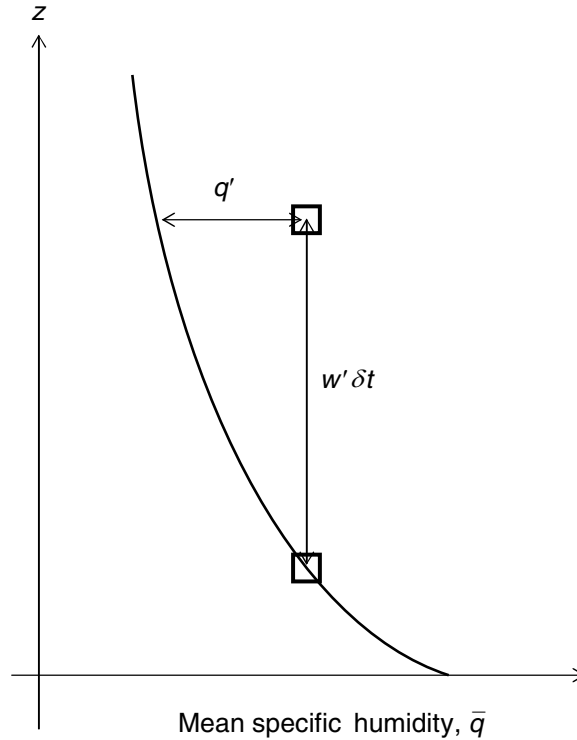
## 2.4 The Atmospheric Boundary Layer

### 2.4.1 Quasi-Uniform Conditions

In the atmosphere, the largest changes in wind velocity, temperature, and humidity are usually found in the vertical direction and in a distinct region near the surface. In contrast, horizontal changes are often relatively mild, and tend to occur over distances ranging from hundreds of meters to tens of kilometers. For this reason, the air near the surface may be regarded as a boundary layer, a concept set forth by Prandtl (1904) for momentum transport in the neighborhood of a solid wall. The atmospheric boundary layer (or ABL) can be defined as the lower part of the atmosphere, where the nature and properties of the surface affect the turbulence directly. Accordingly, the horizontal scales of most atmospheric flow phenomena of interest in hydrology are much larger than the vertical, so that the horizontal gradients are very small as compared to the vertical gradients, and the vertical velocities are very small relative to the horizontal velocities. Thus, many problems can be solved by simply assuming that

$$\left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = 0 \text{ and } \bar{w} = 0 \quad (2.28)$$

In addition, since  $x$  is the direction of the mean wind velocity near the ground, the mean velocity in the lateral  $y$ -direction can also be discarded, or  $\bar{v} = 0$ . Strictly speaking, (2.28) is valid only when the surface is



**Figure 2.5** Sketch illustrating a small fluid parcel, which rises a small distance  $w'\delta t$ , to a level, where the mean humidity  $\bar{q}$  is lower by an amount  $q'$  than at its original position.

perfectly homogeneous or uniform. Such conditions are rare, and the properties of most natural land surfaces are spatially quite variable; fortunately, in many situations of interest they can be considered to be at least statistically homogeneous, and the assumptions of (2.28) can still be used to describe the flow. This issue is further discussed in Section 2.7 at the end of this chapter.

More generally, (2.28) is tantamount to assuming that, as the air moves parallel to a uniform surface, on average (in the turbulence sense) the concentration of any property or admixture advected by the air changes only in the vertical and remains constant in the horizontal direction. The fact that the mean concentrations change only vertically is evidence that there is a source or a sink of the admixture at the surface, and thus the only turbulent fluxes of consequence are the vertical components. In the case of humidity with mean concentration (per unit mass of bulk air),  $\bar{q}$ , Equations (2.26) are thus effectively reduced to

$$F_{vz} = \rho \overline{w'q'} \quad (2.29)$$

in which henceforth the overbar on  $F_{vz}$  is omitted for convenience of notation.

While mathematically (2.26) and (2.29) are unambiguous, a more intuitive sense of their physical significance can be obtained by considering the mechanism sketched in Figure 2.5. A particle of air, which undergoes a vertical velocity fluctuation  $w'$ , travels a distance  $w'\delta t$  during a time interval  $\delta t$ . After that air parcel has risen a small distance  $w'\delta t$  from a level, where the mean specific humidity is  $\bar{q}$ , it has a specific humidity which is  $q'$  larger than the mean of its new environment; thus the rate (distance per unit time) at which this particular parcel transports absolute humidity upward is  $(\rho q'w')$  times its volume. There are many such parcels – or eddies – in turbulent flow moving in all directions, and the transport rate by all of them, i.e. the vertical transport of water vapor mass per unit horizontal area and per unit time, is on average as indicated in (2.29).

Similar expressions can be written for the fluxes of other properties or admixtures of the flow. The vertical flux component of horizontal momentum, with mean concentration  $\bar{u}$ , is

$$F_{mz} = \rho \overline{w'u'} \quad (2.30)$$

and that of sensible heat, with mean concentration  $c_p \bar{\theta}$ , can be written as

$$F_{hz} = \rho c_p \overline{w'\theta'} \quad (2.31)$$

Under steady conditions, on account of continuity, the inflow rate equals the outflow rate, which means that these vertical fluxes must be constant with elevation. Hence the water vapor flux in (2.29) is in fact equal to the rate of evaporation,  $E$ , from the surface, or  $F_{vz} = \rho \overline{w'q'}_0 \equiv E$ , in which the 0 subscript denotes the value near the surface. In the case of momentum, there is a sink at the surface in the form of a shear stress, so that  $F_{mz} \equiv -\tau = -\tau_0$ , in which  $\tau_0$  is the shear stress at the surface. Similarly the flux in (2.31) equals the sensible heat flux  $H$  at the surface, or  $F_{hz} = \rho c_p \overline{w'\theta'}_0 \equiv H$ . For convenience of notation, the surface shear stress, which is introduced here, is often also expressed as the friction velocity, defined as

$$u_* \equiv (\tau_0 / \rho)^{1/2} \quad (2.32)$$

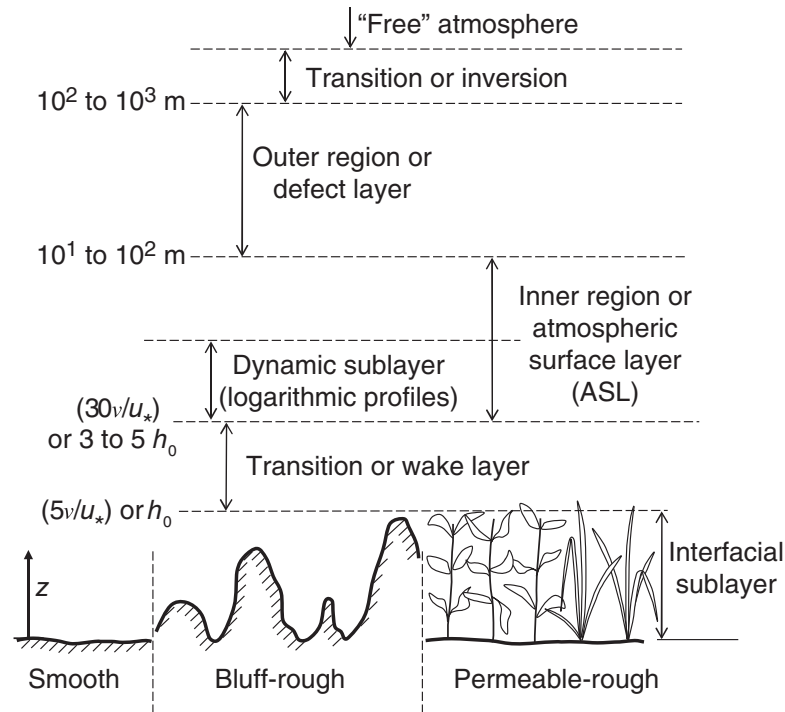
This shows that, in light of (2.30), under steady or nearly steady conditions, one has to a good approximation near the ground that  $u_*^2 = -\overline{w'u'}$ .

## 2.4.2 General Structure of the ABL

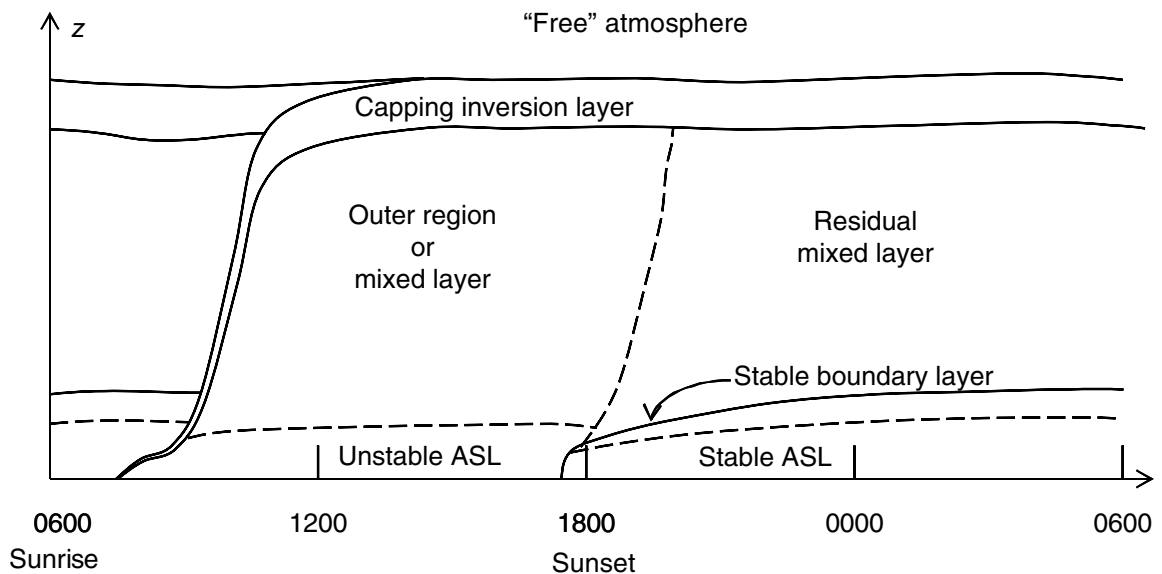
In the analysis, it is convenient to assume that the atmospheric boundary layer consists of a number of sub-layers, in which different sets of variables are important to different degrees in governing turbulent transport. The main subdivision is into an inner and an outer region. In the *outer region* or *defect layer* the flow is strongly dependent on the free stream velocity outside the boundary layer, whereas in the *inner region*, also called variously *atmospheric surface layer* (or ASL), *Prandtl layer* or *wall layer*, the flow is more strongly affected by the nature of the surface (see Figure 2.6).

In the atmosphere, under conditions not very different from neutral, the outer region is affected both by the pressure gradients, reflecting larger-scale weather patterns, and by the Coriolis forces, reflecting the effect of the rotation of the Earth. Under unstable conditions, the effects of the pressure and Coriolis forces are relatively small, and the outer region is more characterized by thermal convective turbulence; the outer region may then be referred to as the *mixed layer* or the *convection layer*. The upper limit of the unstable boundary layer is typically indicated by a sharp inversion, that is, a layer of stable air. Over land, the thickness of the boundary layer tends to vary in the course of the day. Consider, for instance, a typical evolution at mid-latitudes, in the absence of rapidly changing weather with passing fronts or precipitation activity. As stable conditions develop during the night, the boundary layer may range from a few tens of meters in the evening to about 500 m by early morning; then, after sunrise, a new unstable boundary layer develops which may eventually reach a thickness of one to two kilometers at full maturity by the middle of the day. This evolution is sketched in Figure 2.7. Figure 2.8 shows an example of the evolution of the temperature profile in the course of a sunny day. As a rule of thumb, the thickness of a typical boundary layer can be assumed to be of the order of 1 km; it is usually larger under unstable than under neutral conditions.

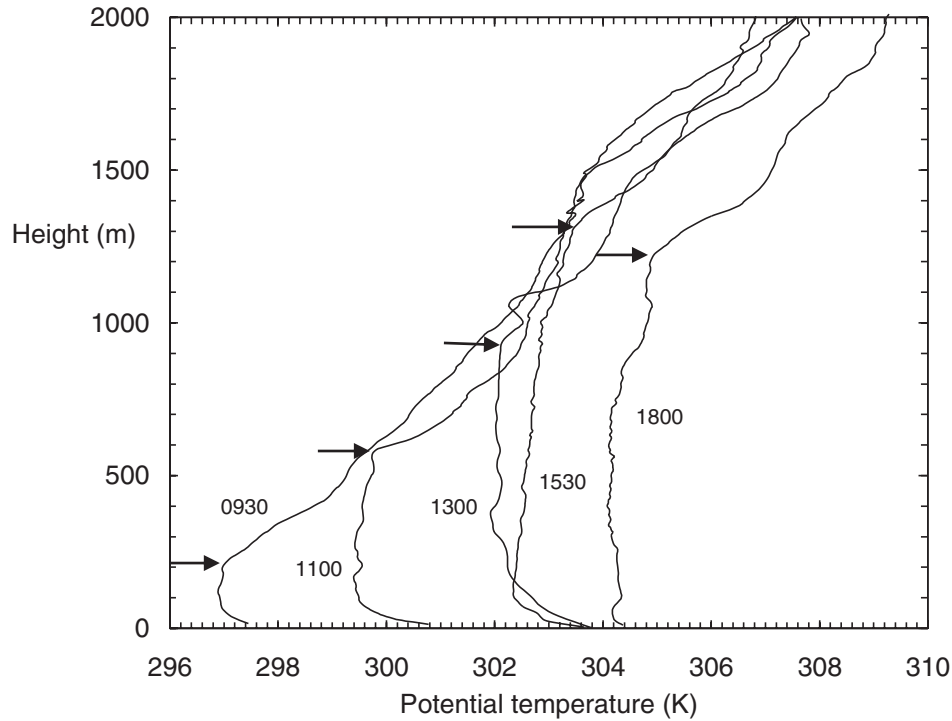
The thickness of the atmospheric surface layer (or ASL) is usually taken as the lower one-tenth of the boundary layer. While there are several ways of defining this thickness (see below), it coincides approximately



**Figure 2.6** Sketch of the typical structure of the atmospheric boundary layer (ABL) above three different types of uniform surfaces. The atmospheric surface layer (ASL) is the region where Monin–Obukhov similarity (MOS) is usually valid;  $h_0$  is a typical height of the roughness obstacles. Under unstable conditions the outer region is called the mixed layer, and it is capped by an inversion layer. (The vertical axis scale is distorted.)



**Figure 2.7** Sketch of the typical diurnal evolution of a fair-weather atmospheric boundary layer (ABL) over land under clear sky around the time of the equinox. The inner region or atmospheric surface layer (ASL) is unstable during the daytime as a result of solar heating at the surface; the ASL is stable at night as a result of radiative cooling. During the day the outer region is characterized by convective turbulence, fed by heating through the surface layer; after sunset this outer layer becomes virtually uncoupled from the surface, by the development of the stable nocturnal boundary layer.



**Figure 2.8** Example of profiles of potential temperature in and above the atmospheric boundary layer. The approximate heights of the inversion overlying the boundary layer are indicated by arrows. The measurements were made over gently rolling terrain ( $z_0 = 0.45$  m,  $d_0 = 8.9$  m, see [Asanuma et al., 2000](#)) by means of radiosoundings at the indicated times (Central Daylight Savings Time) on June 13, 1992, in the Washita River Basin in Oklahoma.

with the region where the direction of the wind remains constant with height; this absence of “turning” confirms that it is indeed the region where the effect of the rotation of the Earth is of little consequence. The surface sublayer is also sometimes assumed to be the layer where the vertical turbulent fluxes do not change appreciably from their value at the surface, say less than 10%. Although the ASL occupies the lower part of the turbulent boundary layer, it does not extend all the way down to the surface. As illustrated in [Figure 2.6](#), the height of the lower limit of the ASL can be assumed to be of the order of  $30\nu/u_*$  in the case of smooth flow, and of the order of  $3$  to  $5h_0$  in the case of rough flow; here  $\nu$  is the viscosity of the air and  $h_0$  is the characteristic height of the roughness obstacles.

In general, under non-neutral conditions, the air flow and the momentum transport are greatly affected by the transport of sensible heat and, to a lesser extent, water vapor, and vice versa. However, in the lower part of the atmospheric surface layer it is found that sensible heat and water vapor may be considered as merely passive admixtures, and that the effects of the density stratification resulting from temperature and humidity gradients are negligible. This lower region of the atmospheric surface layer is referred to as the *dynamic sublayer*. Under neutral conditions, the whole surface layer behaves as a dynamic layer.

Finally, in the immediate vicinity of the surface, the turbulence is strongly affected by the structure of the roughness elements, or it is greatly damped by the viscous effects; in most cases it is subjected to both effects. The region nearest to the surface where these effects are most important is sometimes referred to as the *interfacial (transfer) sublayer*. In the case of smooth flow, as may occur e.g. over snow, water, or salt flats, it is referred to as the *viscous sublayer*. Experiments have shown that its thickness is of the order of  $5\nu/u_*$ , in which again  $\nu$  is the viscosity of the air; the flow may be considered smooth when  $(u_*h_0/\nu) < 1$ , approximately, in which again  $h_0$  is the average height of the surface roughness elements. Experiments have also shown that a surface can be considered rough, when  $(u_*h_0/\nu) > 15$ , approximately; in this case the interfacial sublayer may be referred to as a *roughness sublayer*, and its thickness is of the order of the mean height of the roughness obstacles. When the roughness obstacles consist of vegetation, which is more or less porous or permeable for the air stream, the interfacial sublayer is commonly referred to as *canopy sublayer*.

## 2.5 Turbulence Similarity

Over the past century or so, various turbulence closure schemes have been proposed, essentially by invoking similarity on the basis of dimensional analysis. In this type of approach, after the relevant physical quantities are identified, either from the governing equations or simply by inspection, they are organized into a reduced number of dimensionless quantities. Dimensional analysis only establishes the possible existence of a functional relationship between these dimensionless quantities, and it is incapable of providing the specific form of the functional relationship; the form of that function must usually be determined by experiment or on the basis of some conceptual transport model or other theoretical considerations. This section does not present an exhaustive review but only a few ideas that will be useful in the determination of evaporation in [Chapter 4](#).

### 2.5.1 Parameterization of the Turbulent Transport

Most similarity formulations of turbulent flux have the common feature that the mean of the product of temporal fluctuations in expressions such as (2.29), (2.30), and (2.31), i.e. the second moment, is replaced simply by the product of the spatial changes of the corresponding mean quantities, i.e. of the first moments. In the case of specific humidity this is in general

$$\overline{w'q'} = -\text{Ce} (\bar{u}_2 - \bar{u}_1)(\bar{q}_4 - \bar{q}_3) \quad (2.33)$$

where the subscripts 1 through 4 refer to the measurement heights above the surface and Ce is a dimensionless parameter, also called the water-vapor transfer coefficient, or the Dalton number; Ce depends on the heights of the reference levels 1 through 4, beside a number of other (dimensionless) factors, as will be shown below; the minus sign indicates that the flux points in the direction of negative increments of  $\bar{q}$ . Note that the four heights in (2.33) need not all be different; thus levels 4 and 3 could be the same as 2 and 1, respectively. In the case of the momentum flux, one obtains in the same way

$$\overline{w'u'} = -\text{Cd} (\bar{u}_2 - \bar{u}_1)^2 \quad (2.34)$$

where  $C_d$  is the transfer coefficient for momentum, also called the drag coefficient; in the case of the sensible heat flux, one has similarly

$$\overline{w'\theta'} = -Ch(\bar{u}_2 - \bar{u}_1)(\bar{\theta}_4 - \bar{\theta}_3) \quad (2.35)$$

where  $Ch$  is the heat transfer coefficient, also called the Stanton number.

In many applications the lowest reference level of the wind speed is taken at the surface where  $\bar{u} = 0$ . When in addition the water-vapor flux refers to that at the ground surface, namely  $E$ , (2.33) assumes the common form

$$E = -Ce \rho \bar{u} \Delta \bar{q} \quad (2.36)$$

where  $\bar{u}$  is the wind speed at a certain reference height above the ground and  $\Delta \bar{q}$  the difference between the mean specific humidity at two other reference heights (one of which may also be at the water or ground surface level), whose values will, again, affect the magnitude of  $Ce$ . In the same way, (2.34) becomes for the surface shear stress

$$\tau_0 = Cd \rho \bar{u}^2 \quad (2.37)$$

and (2.35) for the surface sensible heat flux

$$H = -Ch \rho c_p \bar{u} \Delta \bar{\theta} \quad (2.38)$$

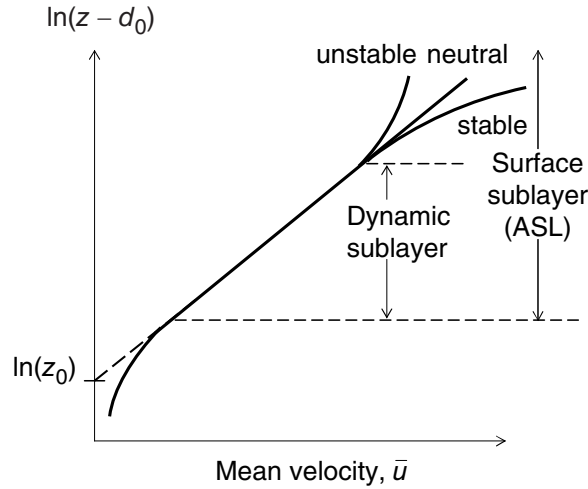
Recall that the difference between  $T$  and  $\theta$  is often small in the lower layers of the surface layer, where most measurements are made. Therefore in many situations, when the height difference of the temperature measurements is only a few meters, in expressions like (2.35) and (2.38) the use of  $\bar{T}$  is allowed instead of  $\bar{\theta}$ .

## 2.5.2 Some Specific Implementations: Flux-Profile Functions

The dimensionless transfer coefficients  $Ce$ ,  $Cd$ , and  $Ch$ , and their dependence on other dimensionless variables, have been the subject of much research. Major progress was made in the 1930s by means of mixing length theory, as a result of contributions by Prandtl, von Karman, and Taylor in the framework of the turbulent diffusion approach; this led initially to the formulation of the logarithmic profile equations for the mean wind speed, the potential temperature, the specific humidity, and other admixtures of the flow (e.g. Monin and Yaglom, 1971; Brutsaert, 1982; 1993) and subsequently to further developments by Monin and Obukhov and others. In this section a few similarity approaches are reviewed that have been useful in the practical estimation of surface fluxes.

### Neutral Atmospheric Surface Layer

It is now generally agreed, and almost accepted by definition, that in the dynamic sublayer, and under neutral conditions in the whole atmospheric surface layer, the concentration of any admixture of the flow is a logarithmic function of height above the ground. Many different derivations of this relationship have appeared in the literature, but the simplest is no doubt that given by Landau and Lifshitz (1959) in the 1944 edition of their book (see also Monin and Yaglom, 1971). The derivation is based strictly on dimensional analysis and on the observation that in plan-parallel flow an increase in velocity in the  $z$ -direction,  $(d\bar{u}/dz)$ , is evidence of a downward momentum flux and a sink at the surface. Thus, the mean velocity gradient in



**Figure 2.9** Schematic illustration of the mean wind profile  $\bar{u} = \bar{u}(z)$  in the dynamic and in the atmospheric surface layer (ASL, also called the surface sublayer).

a fluid of density,  $\rho$ , is determined by the shear stress at the wall,  $\tau_0$ , and the distance from the wall,  $(z - d_0)$ ; in the last variable the (zero-plane) displacement height,  $d_0$ , is introduced to account for the uncertainty of the position of the wall in the case of irregular and uneven surfaces. These variables can be combined into a single dimensionless quantity as follows:

$$\frac{u_*}{(z - d_0)(d\bar{u}/dz)} = k \quad (2.39)$$

where  $u_*$  is defined as in (2.32). Experimentally, this combination  $k$  has been found to be nearly invariant and close to 0.4 under many different conditions; it is referred to commonly as von Karman's constant. The logarithmic profile follows upon integration of (2.39).

In general, this logarithmic profile can be written as

$$\bar{u}_2 - \bar{u}_1 = \frac{u_*}{k} \ln \left( \frac{z_2 - d_0}{z_1 - d_0} \right) \quad (2.40)$$

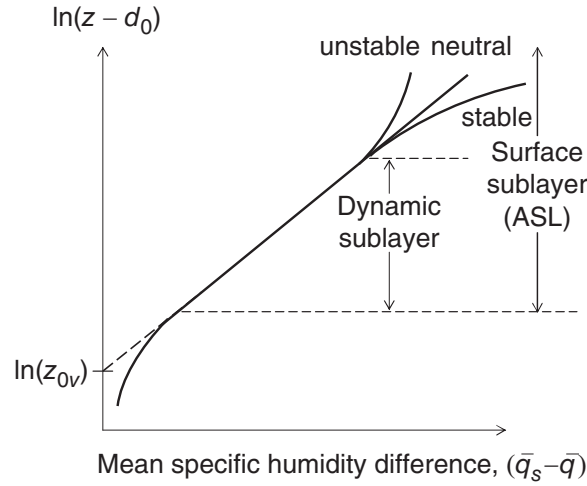
where the subscripts refer to two levels within the neutral surface layer. This result produces immediately the drag coefficient, as it appears in (2.34), namely  $C_d = \{k / \ln[(z_2 - d_0)/(z_1 - d_0)]\}^2$ . Equation (2.39) can also be integrated as follows:

$$\bar{u} = \frac{u_*}{k} \ln \left( \frac{z - d_0}{z_0} \right) \quad (2.41)$$

where  $z_0$  is an integration constant, whose dimensions are length; it is usually referred to as the momentum roughness parameter or the roughness length. Its value depends on the conditions at the lower boundary of the region of validity of (2.39). Graphically, it may be visualized as the zero-velocity intercept of the straight line resulting from a semi-logarithmic plot of mean velocity data versus height in a neutral surface layer (see Figure 2.9). Equation (2.41) leads to the drag coefficient, as it appears in (2.37), namely  $C_d = \{k / \ln[(z - d_0)/z_0]\}^2$ .



**Figure 2.10** Schematic illustration of the mean specific humidity profile,  $\bar{q} = \bar{q}(z)$ , in the dynamic and in the atmospheric surface layer;  $\bar{q}_s$  is its value at the surface.



Dimensional arguments, similar to those leading to the profiles of the mean wind speed, produce for the mean specific humidity gradient

$$\frac{E / \rho}{u_* (z - d_0) (d\bar{q} / dz)} = -k \quad (2.42)$$

Once again, integration yields a logarithmic profile as follows:

$$\bar{q}_1 - \bar{q}_2 = \frac{E}{k u_* \rho} \ln \left( \frac{z_2 - d_0}{z_1 - d_0} \right) \quad (2.43)$$

This result, combined with (2.33) and (2.37) produces a mass-transfer coefficient for water vapor; in the case where wind speed and specific humidity are measured at the same two levels,  $z_1$  and  $z_2$  one obtains  $C_e = \{k / \ln[(z_2 - d_0)/(z_1 - d_0)]\}^2$ ; it is remarkable that this transfer coefficient has the same form as that for momentum, i.e.  $C_e = C_d$ , as derived above. The fact that, under certain conditions, transfer coefficients of different admixtures in turbulent flow are the same, is also referred to as the *Reynolds analogy*. The alternative form of (2.43), when one of the specific humidity values is taken at the surface,  $z = 0$ , is

$$q_s - \bar{q} = \frac{E}{k u_* \rho} \ln \left( \frac{z - d_0}{z_{0v}} \right) \quad (2.44)$$

where  $q_s$  is the value of  $\bar{q}$  at the surface and  $z_{0v}$  is the (scalar) roughness for water vapor (see Figure 2.10). In this case, the transfer coefficient can be written as  $C_e = k^2 / \{\ln[(z_2 - d_0)/z_0] \ln[(z_1 - d_0)/z_{0v}]\}$ , in which the subscript 2 refers to the height of the wind measurement and 1 to that of the specific humidity. In this formulation  $C_e$  would be equal to  $C_d$  only if the two roughness parameters  $z_0$  and  $z_{0v}$  have the same value, which is rarely the case above land.

It would be possible to define a similar logarithmic relationship between the temperature and the surface sensible heat flux  $H$ ; however, since under neutral conditions the temperature differences and the sensible heat flux are relatively small, this is not very meaningful. In what follows, under non-neutral conditions the scalar roughness for sensible heat in the temperature profile will be denoted by  $z_{0h}$ .

In practical applications, the roughness parameters  $z_0$ ,  $z_{0h}$ ,  $z_{0v}$ , and  $d_0$  are best determined experimentally for each specific surface. However, in the absence of measurements, it may be necessary to estimate them

**Table 2.6 Typical roughness values for various surfaces**

| Surface description                                          | $z_0$ (m)     |
|--------------------------------------------------------------|---------------|
| Large water surfaces (“average”)                             | 0.0001–0.0005 |
| Fresh snow, mud flats                                        |               |
| Smooth runways                                               |               |
| Old snow, glaciers                                           | 0.0005–0.005  |
| Short grass                                                  | 0.008–0.02    |
| Long grass, prairie                                          | 0.02–0.06     |
| Short agricultural crops                                     | 0.05–0.10     |
| Tall agricultural crops                                      | 0.10–0.20     |
| Prairie or short crops with scattered bushes and tree clumps | 0.20–0.40     |
| Continuous bushland                                          |               |
| Bushland in rugged and hilly (50–100 m) terrain              | 1.0–2.0       |
| Mature pine forest                                           | 0.80–1.5      |
| Tropical forest                                              | 1.5–2.5       |
| Fore-Alpine terrain (200–300 m) with scattered tree stands   | 3.0–4.0       |

from simple geometric characteristics of the surface; numerous such relationships have appeared in the literature (e.g. Brutsaert, 1975a; 1982). Wieringa (1993) has presented a review of available experimental determinations of  $z_0$  over homogeneous terrain. A few typical values of  $z_0$  taken from the literature for various surfaces are given in Table 2.6. As a useful first approximation for surfaces with densely placed obstacles such as natural vegetation with average height  $h_0$ , the momentum roughness,  $z_0$ , can be assumed to be of the order of  $h_0/10$ ,  $d_0$  of the order of  $h_0/2$  to  $2h_0/3$ , and  $z_{0h}$  and  $z_{0v}$  of the order of  $h_0/100$  or smaller. The scalar roughness parameters  $z_{0h}$  and  $z_{0v}$  continue to be the subject of research (e.g. Brutsaert and Sugita, 1996; Qualls and Brutsaert, 1996; Sugita and Brutsaert, 1996; Cahill et al., 1997; Voogt and Grimmond, 2000; Kotani and Sugita, 2005; Demuzere et al., 2008; Li et al., 2017; Young et al., 2021).

### Monin–Obukhov Similarity (MOS) in the Surface Layer

Neutral conditions occur only seldom in the atmospheric boundary layer. Therefore, it is practically always necessary to include the effect of the stability, i.e. the density stratification, of the atmosphere in the formulation of the profile equations and of the corresponding transfer coefficients. One of the more common ways of doing this is based on the Monin–Obukhov (1954) approach, which assumes that the effect of the density stratification of the flow can be represented by the production rate of turbulent kinetic energy, resulting from the work of the buoyancy forces; it can be shown (e.g. Monin and Yaglom, 1971; Brutsaert, 1982) that near the ground this rate is given by  $(g/T_a)[(H/c_p\rho) + 0.61 T_a E/\rho]$ . The dimensionless variables in (2.39) and (2.42) have the variables  $(z - d_0)$  and  $u_*$  in common. Accordingly one can hypothesize that in a stratified turbulent flow, any dimensionless characteristic of the turbulence depends only on the following: the height above the virtual surface level,  $(z - d_0)$ ; the shear stress at the surface,  $\tau_0$ ; the density,  $\rho$  and the turbulent energy production rate by the buoyancy. These four quantities, which can be expressed in terms of three basic dimensions, viz. time, length, and air mass, can be combined into one dimensionless variable. This variable, which was proposed by Monin and Obukhov (1954) (originally for  $d_0 = 0$ ), is

$$\zeta = \frac{z - d_0}{L} \quad (2.45)$$

where  $L$  is known as the Obukhov stability length, defined by

$$L = \frac{-u_*^3}{k(g/T_a)(w'\theta'_0 + 0.61 T_a w'q'_0)} \quad (2.46)$$

in which  $T_a$  is a mean reference temperature (in K) of the air near the ground and the subscript 0 refers to near-surface values of the fluxes, so that, by definition, these fluxes represent  $(H/c_p \rho)$  and  $(E/\rho)$ , respectively. In the original formulation of  $L$  the turbulent water-vapor flux term did not appear; although in many cases the effect of the water vapor on the density stratification can be neglected, it is still advisable to include it whenever possible.

With this MOS, the dimensionless gradients of the mean wind, of the temperature, and of the humidity, can be written as

$$\frac{k(z-d_0)}{u_*} \frac{d\bar{u}}{dz} = \phi_m(\zeta) \quad (2.47)$$

$$-\frac{k u_*(z-d_0)}{w'\theta'_0} \frac{d\bar{\theta}}{dz} = \phi_h(\zeta) \quad (2.48)$$

$$-\frac{k u_*(z-d_0)}{w'q'_0} \frac{d\bar{q}}{dz} = \phi_v(\zeta) \quad (2.49)$$

in which the subscripts  $m$ ,  $h$ , and  $v$  refer to momentum, sensible heat, and water vapor, respectively. To be consistent with (2.39) and (2.42), in the dynamic sublayer or under neutral conditions, when  $\zeta \ll 1$  (but  $z-d_0 \gg z_0$ ) these  $\phi$ -functions become equal to unity. It is usually assumed that  $\phi_v = \phi_h$ , and thus that the Reynolds analogy is valid for scalar admixtures of the flow.

Equations (2.47) through (2.49) are formulated in terms of the gradients; these are not easy to determine from field measurements, which more often than not tend to be noisy. To avoid this problem, the MOS equations (2.47) through (2.49) can be expressed in integral form as follows:

$$\bar{u}_2 - \bar{u}_1 = \frac{u_*}{k} [\ln(\zeta_2 / \zeta_1) - \Psi_m(\zeta_2) + \Psi_m(\zeta_1)] \quad (2.50)$$

$$\bar{\theta}_1 - \bar{\theta}_2 = \frac{w'\theta'_0}{k u_*} [\ln(\zeta_2 / \zeta_1) - \Psi_h(\zeta_2) + \Psi_h(\zeta_1)] \quad (2.51)$$

$$\bar{q}_1 - \bar{q}_2 = \frac{w'q'_0}{k u_*} [\ln(\zeta_2 / \zeta_1) - \Psi_v(\zeta_2) + \Psi_v(\zeta_1)] \quad (2.52)$$

in which each of the  $\Psi$ -functions, with its respective subscript, is defined by

$$\Psi(\zeta) = \int_0^\zeta [1 - \phi(x)] dx / x \quad (2.53)$$

and  $x$  is the dummy integration variable. Under neutral conditions, when  $|L| \rightarrow \infty$  and  $\zeta \rightarrow 0$ , the  $\Psi$ -functions approach zero and the first and third of (2.52) reduce to the logarithmic profiles (2.40) and (2.43). It is also clear that, whenever  $\bar{u}_1$ ,  $\bar{\theta}_1$ , and  $\bar{q}_1$  refer to the surface values 0,  $\theta_s$ , and  $q_s$ , the dimensionless height  $\zeta_1$  must be taken as  $z_0/L$ ,  $z_{0h}/L$ , and  $z_{0v}/L$ , respectively (as can be seen for the analogous neutral case in (2.41) and (2.44)). In the present case (2.50), (2.51), and (2.52) assume then the form

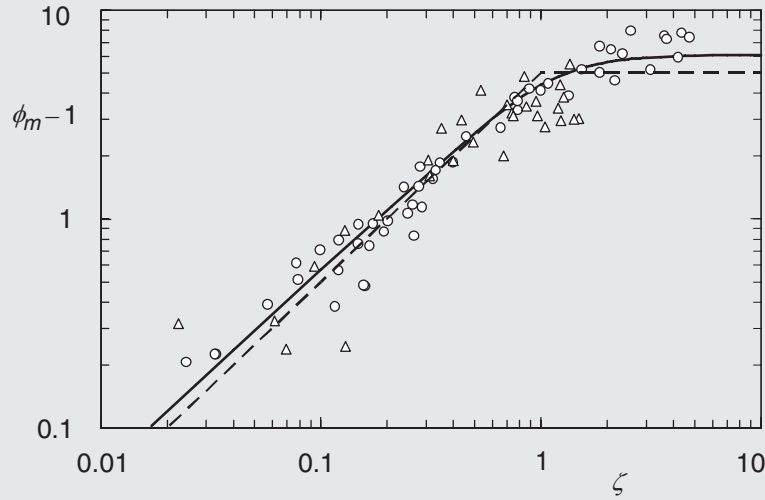
$$\bar{u} = \frac{u_*}{k} \left[ \ln \left( \frac{z-d_0}{z_0} \right) - \Psi_m \left( \frac{z-d_0}{L} \right) + \Psi_m \left( \frac{z_0}{L} \right) \right] \quad (2.54)$$

$$\theta_s - \bar{\theta} = \frac{H}{k u_* \rho} \left[ \ln \left( \frac{z-d_0}{z_{0h}} \right) - \Psi_h \left( \frac{z-d_0}{L} \right) + \Psi_h \left( \frac{z_{0h}}{L} \right) \right] \quad (2.55)$$

$$q_s - \bar{q} = \frac{E}{k u_* \rho} \left[ \ln \left( \frac{z-d_0}{z_{0v}} \right) - \Psi_v \left( \frac{z-d_0}{L} \right) + \Psi_v \left( \frac{z_{0v}}{L} \right) \right] \quad (2.56)$$

The profiles described by (2.54) and (2.56) are illustrated as the non-neutral, i.e. both stable and unstable curves in Figures 2.9 and 2.10, respectively.

The nature of the “universal” MOS functions, especially  $\phi_m$  and  $\phi_h$ , but less so  $\phi_v$ , has been the subject of much theoretical and experimental research. One of the earliest forms of these  $\phi$ -functions, intended for near-neutral conditions, i.e. small  $|\zeta|$ , was proposed by Monin and Obukhov (1954) simply by a series expansion and retention of the first term



**Figure 2.11** The dependence of  $(\phi_m - 1)$  and  $(\phi_h - 1)$  on  $\zeta$  under stable conditions, as determined in [Cheng and Brutsaert \(2005\)](#) from experimental wind profile data (circles) and temperature profile data (triangles) over a flat grassy surface ( $z_0 = 0.0219$  m,  $d_0 = 0.110$  m) in Kansas in October, 1999. The solid curve represents (2.60) and the dashed straight line segments represent (2.57).

only, or  $\phi = (1 + \beta_s \zeta)$ , in which  $\beta_s$  is a constant. Subsequent experimental investigations have revealed, however, that this form of  $\phi$  is applicable only under stable conditions, but not under unstable conditions. It was also observed later on (e.g. [Webb, 1970](#); [Kondo et al., 1978](#)) that this form can describe experimental data only over the range  $0 \leq \zeta \leq 1$  with a  $\beta_s$  value of the order of 5, but that  $\phi$  remains approximately constant for  $\zeta > 1$ . Accordingly, on the basis of the data then available (e.g. [Brutsaert, 1982](#)), for stable conditions the following was assumed:

$$\phi_m(\zeta) = \phi_h(\zeta) = \phi_v(\zeta) \begin{cases} = 1 + 5\zeta & \text{for } 0 \leq \zeta \leq 1 \\ = 6 & \text{for } \zeta > 1 \end{cases} \quad (2.57)$$

Equation (2.57) can be integrated with (2.53) to yield the stability correction functions  $\Psi$  needed for (2.50) through (2.52). These integral functions are

$$\Psi_m(\zeta) = \Psi_h(\zeta) = \Psi_v(\zeta) \begin{cases} = -5\zeta & \text{for } 0 \leq \zeta \leq 1 \\ = -5 - 5\ln \zeta & \text{for } \zeta > 1 \end{cases} \quad (2.58)$$

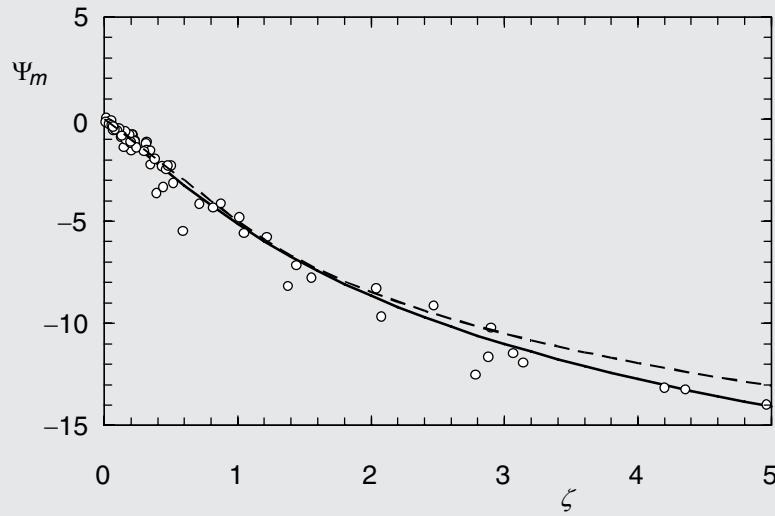
Equations (2.57) and (2.58) can be compared with some more recent experimental data in [Figures 2.11](#) and [2.12](#). With these same data, a single formulation was proposed by [Cheng and Brutsaert \(2005\)](#) to cover the entire stable range  $\zeta \geq 0$ , namely

$$\Psi_m(\zeta) = -a \ln \left[ \zeta + (1 + \zeta^b)^{1/b} \right] \quad (2.59)$$

in which  $a$  and  $b$  are constants, whose values were found to be  $a = 6.1$  and  $b = 2.8$ . Equation (2.59) is also illustrated in [Figure 2.12](#). It can be seen that (2.59) exhibits nearly the same behavior as the first of (2.58) for small  $\zeta$ , and nearly the same as the second for large values of  $\zeta$ . The corresponding  $\phi$ -function for the wind profile can be obtained by differentiation, as indicated by (2.53), to yield

$$\phi_m(\zeta) = 1 + a \frac{\zeta + \zeta^b (1 + \zeta^b)^{-1+1/b}}{\zeta + (1 + \zeta^b)^{1/b}} \quad (2.60)$$

As illustrated in [Figure 2.11](#), this equation behaves like  $(1 + a\zeta)$  for small values of  $\zeta$  and it approaches a constant  $(1 + a)$  for large  $\zeta$ , in accordance with (2.57). [Figure 2.11](#) also indicates that, although the  $\phi_h(\zeta)$  data points for temperature exhibit more scatter, (2.60) can represent these points practically as well as the  $\phi_m(\zeta)$  data points for wind speed; this



**Figure 2.12** The dependence of  $\Psi_m$  on  $\zeta$  under stable conditions, as determined in [Cheng and Brutsaert \(2005\)](#) from experimental wind profile data over grass ( $z_0 = 0.0219$  m,  $d_0 = 0.110$  m) in Kansas in October, 1999. The solid curve represents (2.59) and the dashed curve represents (2.58).

suggests that it is safe to assume that under stable conditions the ASL similarity functions for sensible heat and for momentum are the same. Moreover, experimental and theoretical evidence by [Dias and Brutsaert \(1996\)](#) supports the turbulence similarity of scalars under stable conditions. Thus the Reynolds analogy appears to be valid and one can put  $\phi_m(\zeta) = \phi_h(\zeta) = \phi_\lambda(\zeta)$  and  $\Psi_m(\zeta) = \Psi_h(\zeta) = \Psi_v(\zeta)$  for a stably stratified ASL.

For unstable conditions, [Kader and Yaglom \(1990\)](#) used a more fundamental approach; they reasoned, and were able to support with experimental evidence, that the surface layer can be subdivided into three sublayers, namely a dynamic, a dynamic-convective, and a convective sublayer, for each of which they derived simple power laws to describe the turbulence. However, the resulting  $\phi$ -functions cover only certain ranges, corresponding to these sublayers. Again, to cover the entire  $\zeta$ -range, an interpolation formulation should be developed; accordingly, in [Brutsaert \(1992; 1999\)](#) the functional behavior of  $\phi_m$  and  $\phi_h$  in each sublayer was combined, and this led to the following expression

$$\begin{aligned} \phi_m(\zeta) &= (a + by^{4/3}) / (a + y) & \text{for } y \leq b^{-3} \\ \phi_m(\zeta) &= 1.0 & \text{for } y > b^{-3} \end{aligned} \quad (2.61)$$

and

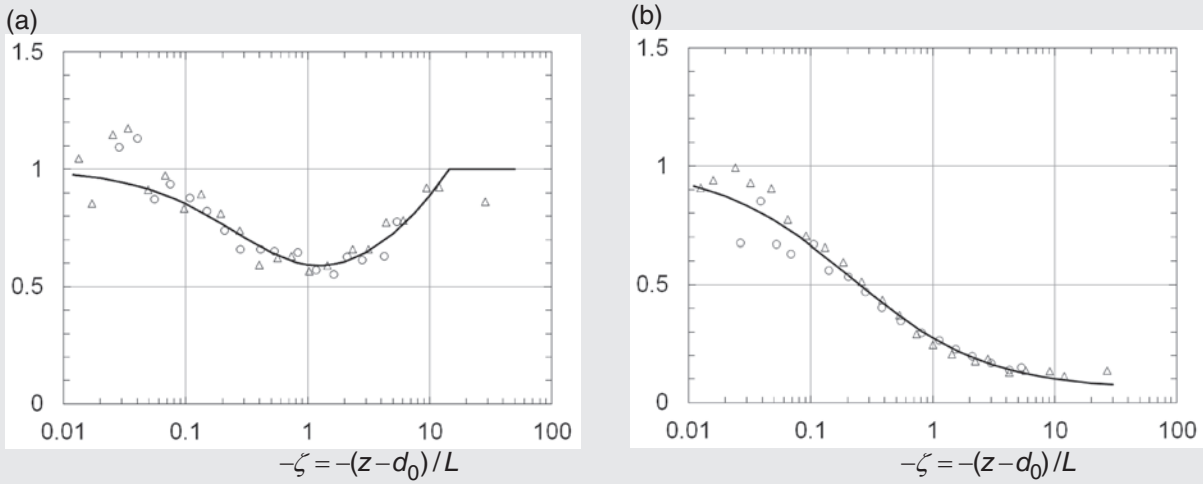
$$\phi_h(\zeta) = (c + dy^n) / (c + y^n) \quad (2.62)$$

in which  $y = -\zeta = -(z - d_0) / L$ , and  $a, b, c, d$ , and  $n$  are constants. After considering available data collections, the constants were assigned the following values  $a = 0.33$ ,  $b = 0.41$ ,  $c = 0.33$ ,  $d = 0.057$ , and  $n = 0.78$ . [Figure 2.13](#) shows these  $\phi$ -functions. The corresponding stability correction functions can be obtained in integral form by means of (2.53) as follows:

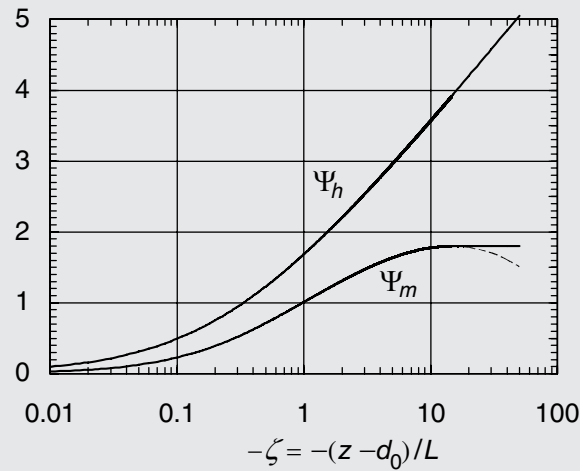
$$\begin{aligned} \Psi_m(-y) &= \ln(a + y) - 3by^{1/3} + \frac{ba^{1/3}}{2} \ln \left[ \frac{(1+x)^2}{(1-x+x^2)} \right] \\ &\quad + 3^{1/2}ba^{1/3} \tan^{-1}[(2x-1)/3^{1/2}] + \Psi_0 & \text{for } y \leq b^{-3} \\ \Psi_m(-y) &= \Psi_m(b^{-3}) & \text{for } y > b^{-3} \end{aligned} \quad (2.63)$$

and

$$\Psi_h(-y) = [(1-d)/n] \ln[(c + y^n)/c] \quad (2.64)$$



**Figure 2.13** (a) Flux-profile function for momentum  $\phi_m(\zeta)$  corresponding to Equation (2.61). The points are taken from the data collection of Kader and Yaglom (1990) and represent measurements at Tsimlyansk (triangles) and in Kansas, Minnesota, and Australia (open circles). (b) Same as (a) for the flux-profile function for sensible heat  $\phi_h(\zeta)$  corresponding to Equation (2.62).

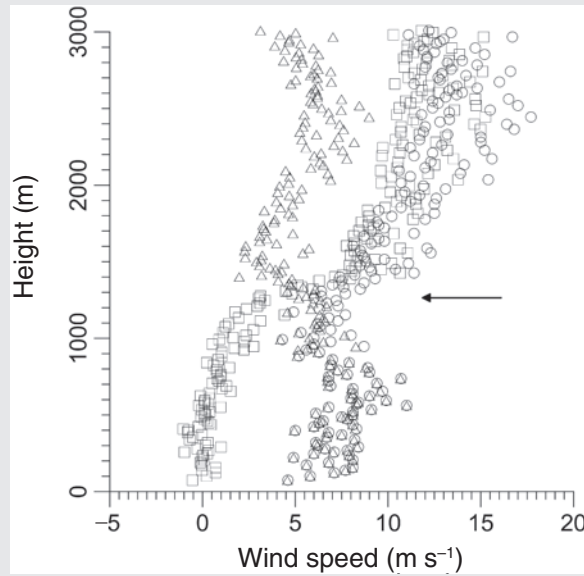


**Figure 2.14** Integral form of the flux-profile functions for momentum  $\Psi_m(\zeta)$  and for sensible heat  $\Psi_h(\zeta)$  under unstable conditions, as given by (2.63) and (2.64).

in which  $x = (y/a)^{1/3}$  and, as before,  $y = -\zeta = -(z-d_0)/L$  and  $a, b, c, d$ , and  $n$  are constants. The  $\Psi_0$  symbol is a constant of integration, given by  $\Psi_0 = (-\ln a + 3^{1/2} b a^{1/3} \pi / 6)$ ; in applications it is usually unimportant, because it cancels out in (2.50) and (2.54). Figure 2.14 shows (2.63) and (2.64) with the values of the constants given above behind (2.62). Also, for unstable conditions, it is usually assumed that  $\phi_h(\zeta) = \phi_v(\zeta)$  and  $\Psi_h(\zeta) = \Psi_v(\zeta)$ .

There is still no universal agreement on the vertical extent of the surface layer. However, numerous experimental observations (e.g. Brutsaert, 1998, 1999) mostly under neutral and unstable conditions suggest that the lower limit  $z_{sb}$  can be estimated from  $(z_{sb} - d_0) = \alpha_b z_0$ , in which  $\alpha_b$  is of the order of 50, ranging roughly between 40 and 60. Its upper limit  $z_{st}$  can be estimated by the rule of thumb, specifying that it is either at  $(z_{st} - d_0) = \alpha_t h_i$ , in which  $\alpha_t = 0.12$ , or at  $(z_{st} - d_0) = \beta_t z_0$ , in which  $\beta_t = 120$ , whichever is larger; the variable  $h_i$  is the height of the bottom of the inversion capping the atmospheric boundary layer. Note that the former value of  $z_{st}$  is for a moderately rough surface, whereas the latter is for very rough terrain; with a typical value of  $h_i = 1000$  m, the cut-off value between very rough and moderately rough terrain is around  $z_0 = (\alpha_t / \beta_t) h_i = 1$  m.

**Figure 2.15** Example of a wind profile measured above moderately hilly prairie terrain ( $z_0 = 1.05$  m,  $d_0 = 26.9$  m) in Kansas at 1500 CDST on August 14, 1987 by means of a radiosonde; the circles represent the wind speed, and the triangles, the  $x$  and the squares the  $y$  components of the wind velocity. The arrow indicates the height of the inversion. (From Brutsaert and Sugita, 1991.)



### Bulk ABL Similarity Formulation

As mentioned, the atmospheric surface layer typically occupies only the lowest 10% or so of the boundary layer. Numerous attempts have also been made to formulate similarity hypotheses for the entire boundary layer. In this approach, the surface fluxes are commonly related to “bulk” variables, namely values of the variables at the top and bottom of the ABL, or their averages over all or part of the ABL. The basic form of the equations is essentially similar to that of (2.50) through (2.52), or (2.54) through (2.56), but extended for larger heights above the surface layer. Ideas on the application of similarity to the entire ABL, including the outer region were put forth early on by Rossby and Montgomery (1935) and Lettau (1959), and subsequent developments can be traced through the work of Kazanski and Monin (1961), Clarke and Hess (1974), Zilitinkevich and Deardorff (1974), Yamada (1976), Garratt et al. (1982), Brutsaert (1982), Sugita and Brutsaert (1992), and Jacobs et al. (2000), among others. The various versions of this approach can be written in a general form as follows:

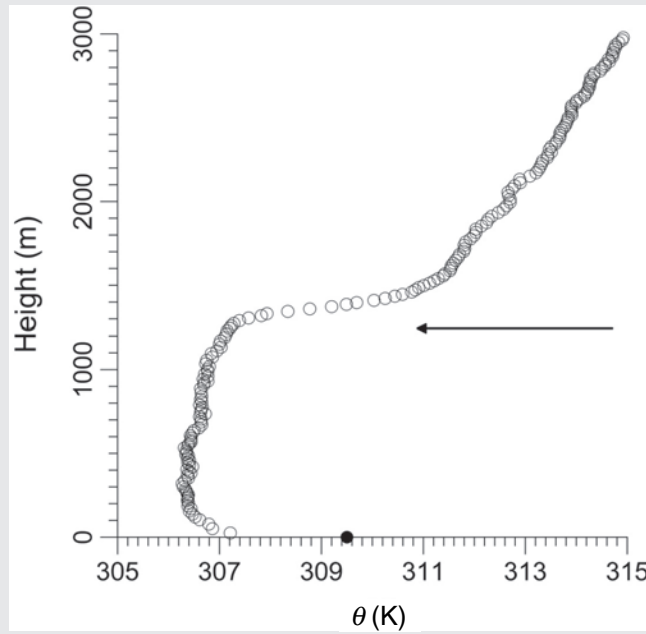
$$u_b = \frac{u_*}{k} \left[ \ln \left( (h_b - d_0) / z_0 \right) - B \right] \quad (2.65)$$

$$v_b = -\frac{u_*}{k} A$$

$$\theta_s - \theta_b = \frac{\overline{w'\theta'_0}}{k u_*} \left[ \ln \left( (h_b - d_0) / z_{0h} \right) - C \right] \quad (2.66)$$

where  $A$ ,  $B$ , and  $C$  are functions of a number of dimensionless variables that affect transport in the outer region and where the subscript  $b$  indicates bulk or characteristic scale variables of the ABL. Thus  $h_b$  denotes a characteristic thickness or height scale of the ABL; the variables  $u_b$  and  $v_b$  are characteristic horizontal wind velocity components in the  $x$  and  $y$  directions, respectively ( $x$  is the direction of the near-surface wind; because it may involve the Earth’s rotation, usually  $y$  points to the left of  $x$  in the northern hemisphere, and to the right in the southern), such that  $u_b^2 + v_b^2 = V_b^2$ , in which  $V_b$  is a characteristic wind speed aloft. These bulk variables have been given different definitions in the past, depending on the specific implementation of the approach. In the early applications  $u_b$ ,  $v_b$ , and  $\theta_b$  were taken as the values of these variables near the top of the ABL, in general, or just below the capping inversion, under unstable conditions.

The more recent implementations (e.g. Brutsaert, 1999) have been mostly for unstable conditions with mean values of the variables in the mixed layer, and with the wind speed as a scalar. The rationale for this choice of bulk variables is that, indeed, as illustrated in Figure 2.15, due to convection with vertical mixing the  $y$ -component of the velocity is nearly negligible, so that the  $x$ -component is practically equal to the wind speed; moreover, wind-speed measurements



**Figure 2.16** Example of the profile of potential temperature measured with the same radiosonde as that of the wind shown in Figure 2.15. The solid circle shows the median value of the surface temperature. The arrow indicates the height of the inversion. (From Brutsaert and Sugita, 1991.)

aloft can be noisy, so that a height-averaged value is likely to be more robust. Figure 2.16 shows the corresponding temperature profile. Thus, with this choice of variables, the formulation for momentum and sensible heat can be written as

$$V_m = \frac{a}{k} [\ln((h_i - d_0) / z_0) - B_w] \quad (2.67)$$

$$\theta_s - \theta_m = \frac{\overline{w'\theta'_0}}{k u_*} [\ln((h_i - d_0) / z_{0h}) - C] \quad (2.68)$$

in which  $V_m$  and  $\theta_m$  are the mean wind speed and potential temperature, respectively, in the mixed layer of the unstable ABL;  $h_i$  is the height of the top of the mixed layer, that is the bottom of the inversion above the ground, and  $B_w$  has been given a subscript to indicate that the wind speed,  $V$ , is used, instead of the wind velocity components  $u$  and  $v$ .

Until now, no general definitive form has been derived for these functions,  $B_w$  and  $C$ . An example of a formulation for unstable conditions, that has produced good results (Brutsaert, 1999), is summarized in what follows. It is based on the assumption of an ABL consisting of two layers, namely a surface layer, with profiles given by (2.63) and (2.64), and above it a mixed layer as a slab with uniform profiles; it is further also based on the assumption explained behind (2.64) regarding the position of the top of the surface layer, where it meets the mixed layer. For moderately rough terrain, i.e. when  $z_0 \leq (\alpha_t / \beta_t) h_i$ , the resulting functions are

$$\begin{aligned} B_w &= -\ln(\alpha_t) + \Psi_m(\alpha_t(h_i - d_0) / L) - \Psi_m(z_0 / L) \\ C &= -\ln(\alpha_t) + \Psi_h(\alpha_t(h_i - d_0) / L) - \Psi_h(z_{0h} / L) \end{aligned} \quad (2.69)$$

For very rough terrain, when  $z_0 > (\alpha_t / \beta_t) h_i$ , the functions are

$$\begin{aligned} B_w &= \ln((h_i - d_0) / (\beta_t z_0)) + \Psi_m(\beta_t z_0 / L) - \Psi_m(z_0 / L) \\ C &= \ln((h_i - d_0) / (\beta_t z_0)) + \Psi_h(\beta_t z_0 / L) - \Psi_h(z_{0h} / L) \end{aligned} \quad (2.70)$$

The similarity functions,  $B_w$  and  $C$ , given by (2.69) and (2.70) are illustrated in Figures 2.17 and 2.18. In the derivation of (2.69) and (2.70) it was assumed that the outer region is a perfectly mixed slab layer; this assumption has its